# PLANETARY DYNAMICS

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## 1 Introduction

After having detected exoplanets we need to visualize their orbits with the data we acquire. Our aim in this working group is to be able to determine the orbits of a 1-planet system and its evolution over time, and then extend it to an n-body problem, paying special attention to the orbit of the star around the center of mass of the system.
At the end, we will relate the radial velocity - which has been studied in the exoplanet detection part of the working group to the orbit of the star.

## 2 Orbit

From Kepler's first law we know that planets follow elliptical orbits around stars which are located at one focus of the ellipse. This means that the orbit of the planet can be described by the equations of an ellipse.
The general equation of an ellipse is the following:

$$a^2 = b^2 + c^2 \tag{2.1}$$

Where $a$ is the semi-major axis, $b$ the semi-minor axis and $c$ the distance from the center of the ellipse to a focus, as we can see in Figure 1.

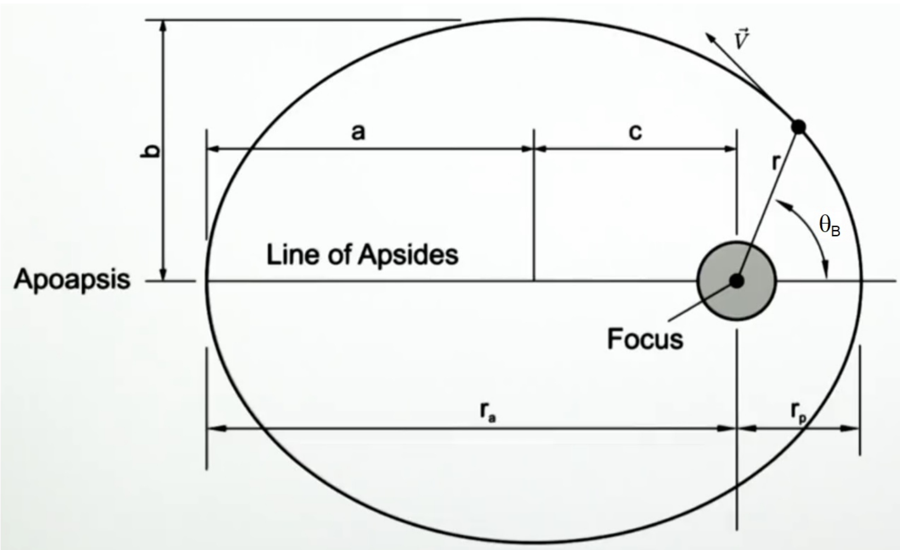


Figure 1: Parameters of the ellipse. $a$: semi-major axis, $b$: semi-minor axis, $c$: distance from the center to the focus, $r_a$: distance to the apoastris, $r_p$: distance to the periastris, $r$: distance from the center of mass to the position of the planet, $v$: velocity of the planet, $\theta_B$: angle from the focus occupied by the primary.

With these parameters we can know the eccentricity of the ellipse:

$$e = \frac{c}{a} \tag{2.2}$$

The eccentricity is a parameter that describes how much the orbit deviates from a circle. When the eccentricity is 0 the orbit is circular, and when it is 1, the orbit becomes a parabola.
The equation of an ellipse in which the focus is in the origin of coordinates and lying in the plane $z = 0$ is the following:

$$\vec{r} = \begin{pmatrix} a\cos E - c \\ b\sin E \\ 0 \end{pmatrix} \tag{2.3}$$

Where $E$ is the angle from the center of the ellipse (eccentric anomaly).
However, if we consider our viewing plane to be $z = 0$, in reality the orbits of planets are usually inclined.
Working in spherical coordinates, if we want to rotate a curve an angle $\theta$ with respect to the Z axis and an angle $\phi$ w.r.t. the X axis, the rotation matrix we have to apply is the following:

$$R = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \tag{2.4}$$

This way, we're considering all possible rotations of the ellipse. Applying the rotation matrix to the orbit of the planet, the equation of the general orbit of a planet is:

$$\vec{r} = \begin{pmatrix} (a\cos E - c)\cos\theta\cos\phi + b\sin E\cos\theta\sin\phi \\ -(a\cos E - c)\sin\phi + b\sin E\cos\phi \\ (a\cos E - c)\sin\theta\cos\phi + b\sin E\sin\theta\sin\phi \end{pmatrix} \tag{2.5}$$

We said at the beginning of this section that a planet is orbiting a star which is at a focus of its elliptical orbit. However, we can't suppose that the star is stuck at the focus, as the only way we can study the orbit of a planet is by observing the orbit of the star.
The planet and the star are going to have elliptical orbits sharing the same focus, which is the centre of mass of the two bodies. Setting the centre of mass as the origin, the equation of the orbit of the star is:

$$\vec{r} = \begin{pmatrix} a_s\cos E\cos\theta\cos\phi + c_s\cos\theta\cos\phi + b_s\sin E\cos\theta\sin\phi \\ -a_s\cos E\sin\phi - c_s\sin\phi + b_s\sin E\cos\phi \\ a_s\cos E\sin\theta\cos\phi + c_s\sin\theta\cos\phi + b_s\sin E\sin\theta\sin\phi \end{pmatrix} \tag{2.6}$$

In a 2-body case, the eccentricity of the planet's orbit is the same as the eccentricity of the star's orbit around the center of mass. We can get the semi-major axis of the star's orbit with this relationship:

$$m \cdot a_p = M \cdot a_s \tag{2.7}$$

Where $m$ and $a_p$ are the mass and semi-major axis of the planet and $M$ and $a_s$ the mass and semi-major axis of the star.
With the eccentricity and the semi-major axis of the star we can easily get the semi-minor axis and the focal distance to the center of mass, so that we can also represent the orbit of the star.

## 2.1 Orbit simulation

We have simulated the orbit of a star and a planet. We're assuming that we know the orbital period (which is the same for both), the eccentricity of the orbit (which is the same for both too), the mass of both the planet and the star and the angle of inclination, which is also the same in a 2-body problem.

We can obtain the semi-major axis of the planet with this equation:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{2.8}$$

Where $\mu \approx GM$, $G$ is the gravitational constant and $M$ is the mass of the star. This approximation will only work when the mass of the planet is negligible compared to the mass of the star.
The rest of the parameters we need can be obtained with equations (2.1) and (2.2). This way, we just have to plot equations (2.5) and (2.6) with $\alpha$ between 0 and $2\pi$.
In general, the orbit of the star around the center of mass is very small compared to the orbit of the planet, and on a small scale it is almost impossible to notice that the star is moving. Because of this in our program we're choosing a large eccentricity and the masses of the star and the planet differ only by one order of magnitude.
We can see the result in Figure 2: the planet draws out a big ellipse around the centre of mass (which is the origin point) and the star makes another small ellipse around the same point.
The motion of the 2 bodies is such that both are at the periapsis at the same time, and likewise for the apoapsis.

# 3 Time dependence

Now that we know what the orbits are like, the next step is determining the positions of the two bodies in their orbits at each instant.
Going back to Figure 1, we'll now focus on the distance from the center of mass to the position of the planet, $r$. The following equation gives us the distance $r$, when we only know the angle, $\theta_B$:

$$r = \frac{a(1-e^2)}{1+e\cos\theta_B} \tag{3.1}$$

Notice that $\theta_B$ is the angle of the position of the planet with respect to the center of mass (the focus), and not the angle from the origin, $E$.
As our program in python works with angles from the center of the ellipse, we need to be able to change from one angle to another. We can do this with the equation:

$$\cos E = \frac{e+\cos\theta_B}{1+e\cos\theta_B} \tag{3.2}$$

We can calculate the distance from the centre of mass to the star with this relationship:

$$m \cdot r_p = M \cdot r_s \tag{3.3}$$

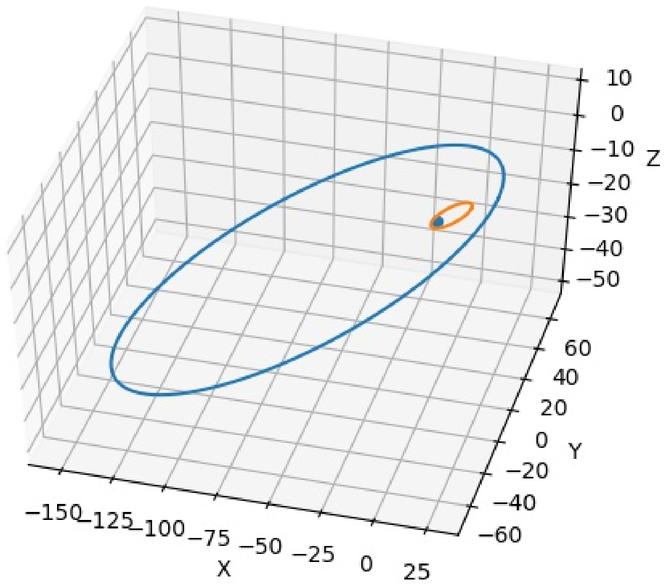


Figure 2: Orbit of a planet and a star. Parameters: $T = 10^5$, $M = 5 \cdot 10^{15}$, $m = 5 \cdot 10^{14}$, $e = 0.7$, $\theta = \pi/10$. Arbitrary units.

The velocity of the planet at every point of the orbit can be calculated with this equation:

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \tag{3.4}$$

Notice that this equation is only valid for the velocity of the planet, and not for the star. We can obtain the velocity of the star at each point with this equation:

$$m \cdot v_p = M \cdot v_s \tag{3.5}$$

We can calculate the time since periapsis (which is the same for both the star and the planet) with this equation:

$$t = T + \sqrt{\frac{a^3}{\mu}} \left[ 2 \arctan \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta_B}{2} \right) - e\sqrt{1-e^2} \frac{\sin \theta_B}{1 + e \cos \theta_B} \right] \tag{3.6}$$

With these equations, we have all we need to know all the orbital parameters of the planet and the star.

## 3.1 Time dependent simulation

We have simulated the position of a planet and a star, given a particular time from periapsis. We obtained the orbits in Figure 3.

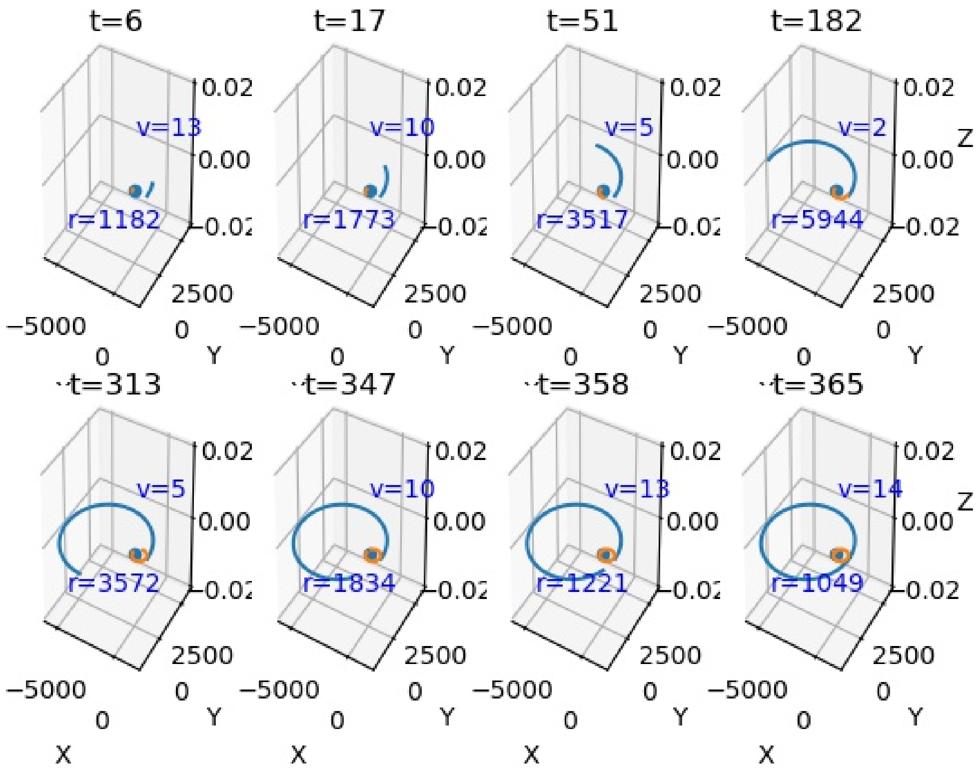


Figure 3: Orbit of a planet and a star. Parameters: $T = 365$, $M = 1.9 \cdot 10^{15}$, $m = 3 \cdot 10^{14}$, $e = 0.7$. Arbitrary units.

# 4 n-bodies

Now that we know how to solve the problem for a planet orbiting a star, we can extend it to the n-body problem.
Given the orbits of n planets around a star, we can determine the orbit of the star around the center of mass if we approximate the problem to a superposition of 2-body problems. The procedure is the following: we solve the problem individually for each planet and we determine the orbit of the star in each case. Then, the final orbit of the star is the addition of the position vectors of the star (obtained for each planet) at every instant.
The procedure for solving this problem is the same for satellites orbiting a planet, and we have simulated in this case the orbit of Pluto and its satellites around their center of mass. Using the real data of period, mass and eccentricity of the satellites, we have obtained the orbit in Figure 4.

| Object | Mass($10^{19} kg$) | Semi-major axis(km) | Orbital period(days) | Eccentricity | Inclination |
|---|---|---|---|---|---|
| Pluto | 1305 | 2035 | 6.39 | 0.0022 | 0.001 |
| Charon | 158.7 | 17 536 | 6.39 | 0.0022 | 0.001 |
| Styx | 0.00075 | 42 656 | 20.1 | 0.0058 | 0.81 |
| Nix | 0.005 | 48 694 | 24.9 | 0.002036 | 0.133 |
| Kerberos | 0.0016 | 57 783 | 32.2 | 0.00328 | 0.389 |
| Hydra | 0.005 | 64 738 | 38.2 | 0.005862 | 0.242 |

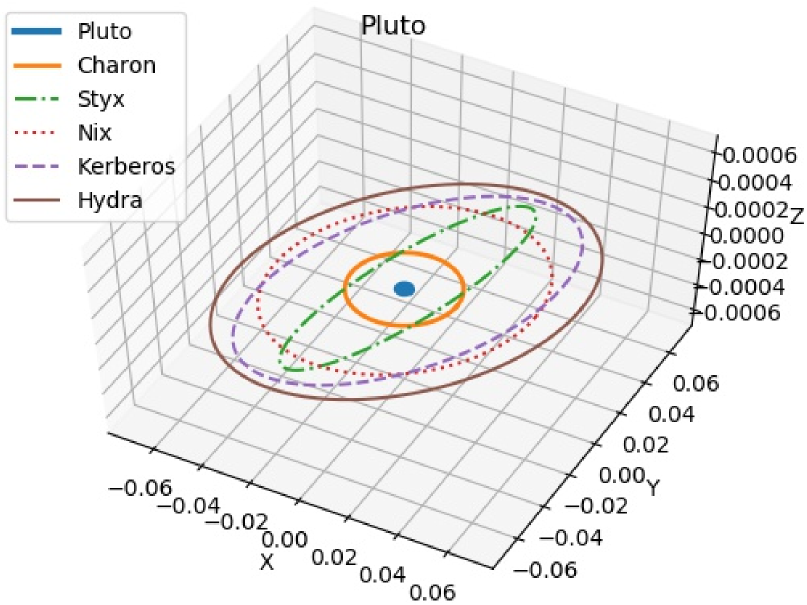


Figure 4: Orbit of Pluto and its satellites. Arbitrary units.

# 5 Radial velocity

The radial velocity of an object with respect to a given point (usually the observer) is the rate of change of the distance between the object and the point. That is, the radial velocity is the component of the object's velocity that points in the direction of the line connecting the object and the point, i.e. the speed with which the object moves away from or approaches the observer.
When only a single planet is orbiting, the central star mirrors its motion in a scale proportional to the masses of both bodies (see equation 2.7). Therefore, a more massive planet will have much larger effect than a smaller one. In case of more planets orbiting the same star, the final motion is a more complex superposition.
We consider here the effect of one planet only and the observer to be in an inertial system. The equation for the radial velocity (with respect to the observer) of an object orbiting a distant body is:

$$v_r = \frac{2\pi}{T} \frac{a_s \cos i}{\sqrt{1-e^2}} \left[e \cos\omega + \cos(\omega + \theta_B)\right] \tag{5.1}$$

Where $i$ is the inclination of the orbital plane with respect to the plane of sight and $\omega$ is the argument of periapsis, which is the angle formed by the ascending node and the semi-major axis above the viewing plane, as we can see in Figure 5.

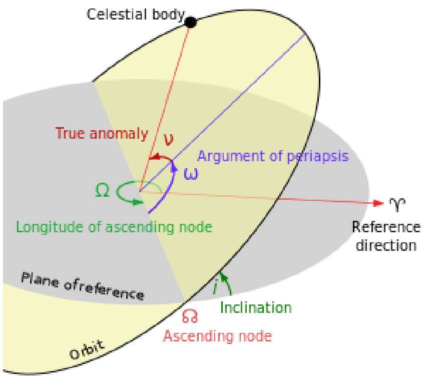


Figure 5: orbital parameters. Ascending node: intersection of the orbit and the reference plane.

## 6 Conclusion

At the beginning of our project, we set out to simulate exoplanet orbits. In order to reach our goal, we had to review orbital theory. We dove deeper into elliptical trajectories and dynamics, culminating in a time-dependent orbital simulation. Then, we took a break from distant exoplanets and alien star systems and focused on the outer parts of our Solar System, where we visualized Pluto and its moons as an example of the n-body problem. To finish, we discussed celestial mechanics and radial velocity with respect to an observer back on Earth.

## 7 Acknowledgments

**Melvin**, thank you for your help, for your patience, and for reminding us that we could do it. You've been the light in our darkness.
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