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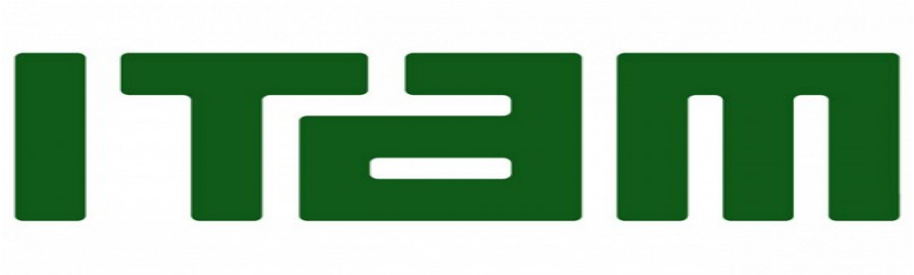

DISCRETIZACION DE LA ECUACION DE CALOR CON APLICACION A OPCIONES AMERICANAS

TESIS
QUE PARA OBTENER EL TITULO DE
MATEMATICAS APLICADAS
PRESENTA

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MEXICO, D.F. 2015

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Ana Cristina Perez-Gea González

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Resumen

En esta tesis se exploran métodos numéricos para aproximar la solución de la ecuación del calor. Consideramos los casos de valores iniciales suaves y no suaves. Los métodos que usamos se basan en diferencias finitas y reglas de cuadratura de distintos órdenes.

En seguida probamos Crank-Nicolson y Crank-Nicolson de orden cuatro con opciones americanas "call" que pagan dividendos. Construimos transformaciones de paralelogramo de manera que la solución puede ser obtenida en una malla uniforme y luego transformada a una malla que llega a la frontera libre. De esta manera, se calcula el valor de la opción americana al mismo tiempo que se aproxima la ubicación de la frontera libre.

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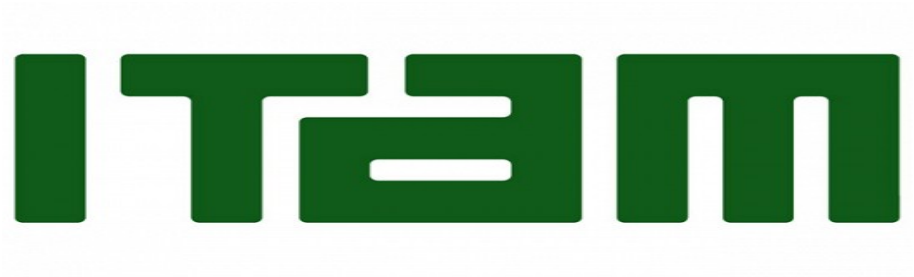

DISCRETIZATION OF THE HEAT EQUATION WITH APPLICATION TO AMERICAN OPTIONS

THESIS
FOR WHICH TO OBTAIN THE TITLE OF
APPLIED MATHEMATICS
PRESENTS

ANA CRISTINA PEREZ-GEA GONZALEZ

ADVISER: PROF. JUAN CARLOS AGUILAR

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Abstract

In this dissertation we explore numerical methods to approximate the solution of the heat equation. We consider the cases of smooth and non smooth initial values. The methods we consider are based on finite differences and quadrature rules of several orders.

We then test Crank-Nicolson and Crank-Nicolson of order four with American options with dividend-paying assets. We construct parallelogram transformations so that the solution can be obtained in a uniform grid and then transformed to a grid that goes up to the free boundary. In this fashion, we calculate the value of the American option while approximating the location of the free boundary.

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1 Introduction

The heat equation can model various kinds of problems. In particular, it can be used to model financial instruments because changes in their value, with respect to a collection of space and time variables, can be understood as a diffusion process. In this dissertation we will discuss the use of finite differences to approximate the solution of the heat equation for the cases of smooth and non smooth initial data. We will discuss how the algorithms can be modified to approximate the free boundary of American call options.

In the rest of the introduction we will explain the basic concepts of heat equation, which we will use throughout. In Chapter 2 we will discuss different kinds of finite difference methods and, in particular, the one we will be using with central differences. We will then review methods in Chapter 3 for solving equations adding a second variable: time. We will discuss how to construct different methods applied to the heat equation. For Chapter 4, we will experiment our method with simple examples that can easily be solved with Fourier series expansions. In Chapter 5 we will describe a discretization of the Black-Scholes equation for the case of American call options with constant dividends.

1.1 The Heat Equation

The heat equation is widely used in applied mathematics for the variety of its uses. It describes the variation of temperature in a certain region over time so it can be used, for example, in chemical engineering for heat transfer in certain materials. The heat equation is also called the diffusion equation in a more general version because it models diffusion processes of motions of molecules tending to level off differences in density or pressure in gases or liquids. The particle diffusion equation was originally derived by the German physician Adolf Eugen Fick in 1855.

The heat equation can also be modified for problems such as in satellite building, machine learning or, in our case, finance. In 1973, Fischer Black and Myron Scholes in [2] developed a formula for pricing options, which can be transformed into a heat equation problem.

The heat equation in one dimension which we will use throughout this work is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

which is a model of the flow of heat in a continuous medium. In the following chapters, we will focus on finding numerical solutions to this partial differential equation. There are different methods used in literature to approximate the solution to this equation such as Lattice methods and Monte Carlo methods. In this dissertation we will use Finite Difference methods.

1.2 Boundary Conditions

The heat equation has an initial temperature condition

$$u(x, 0) = f(x),$$

which is the steady state temperature distribution. Along with this, the values of x can be restricted to an interval $[a, b]$ which give us boundary conditions for u. The types of boundary conditions can be, but are not limited to, the following cases.

- **Neumann boundary conditions**. The first partial derivatives are given at the extremes $x = a$ and $x = b$ for any time $t \geq 0$

$$\frac{\partial u}{\partial x}(a, t) = g_1(t)$$
$$\frac{\partial u}{\partial x}(b, t) = g_2(t).$$

- **Dirichlet boundary conditions**. The solution u is given at the extremes $x = a$ and $x = b$ for any time $t \geq 0$ as shown in figure 1.1

$$u(a, t) = g_1(t)$$
$$u(b, t) = g_2(t).$$

There can also be mixed boundary conditions, which, as its name suggests, is a mixture of the above two conditions. We will be using the Dirichlet boundary conditions for this work.

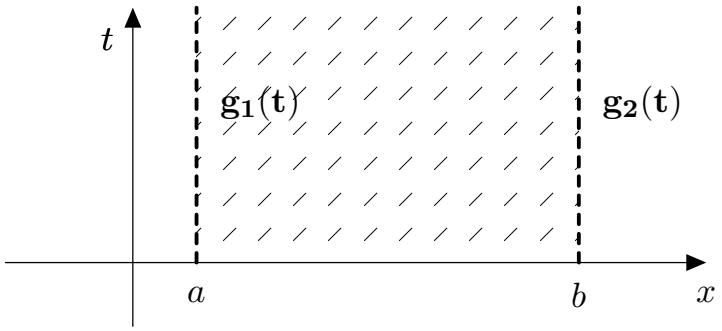


Figure 1.1: Dirichlet Boundary Conditions

2 Finite Difference Methods

Analytic solutions to general differential equations are quite difficult to find. Therefore, we are interested in numerical methods to approximate the solution of the heat equation. In this chapter we will discuss the finite difference method to approximate first and second derivatives. Finite differences approximate the derivative or higher-order derivatives of a function at a point by means of linear combinations of values of the function around that point. We will discuss centered differences of orders two and four for differentiation as well as the trapezoidal rule for integration, which we will use in further chapters.

To begin deriving the formulas for the approximations to the first and second derivatives, we first need to state Taylor's theorem.

Theorem 1. (**Taylor's Theorem**)
Let $k \geq 1$ be an integer and suppose that $f : \mathbb{R} \to \mathbb{R}$ is $k+1$ times differentiable at $x \in \mathbb{R}$. Then there exists a function $R_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \ldots + \frac{f^{(k)}(x)}{k!}h^k + R_k(h) \qquad (2.1)$$

and $\lim_{h\to 0} R_k(x)/h^k = 0$.

In the rest of the chapter, we will use Taylor's theorem to derive finite difference methods of different orders.

2.1 The Trapezoidal Rule

The trapezoidal rule is a numerical method to approximate definite integrals. Let us take a look at the integral of a function $f(x)$ in the region from a to b. We

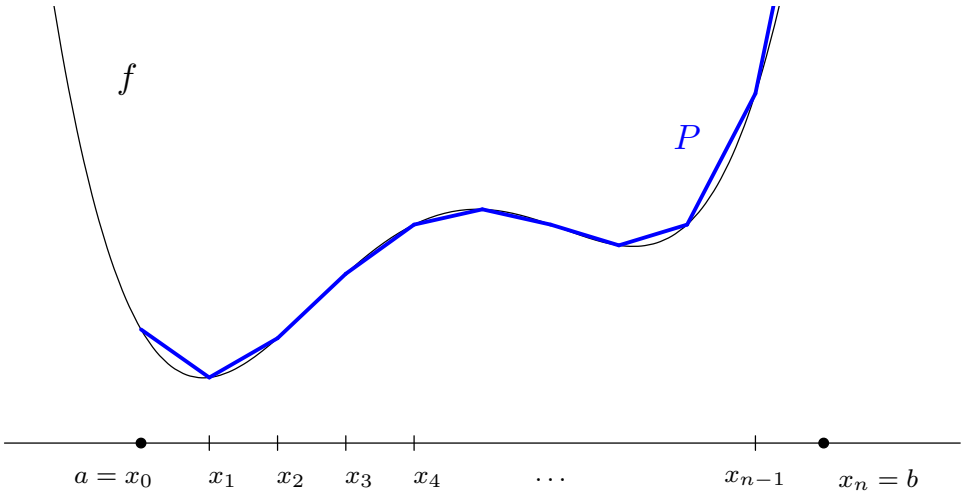


Figure 2.1: Trapezoidal Rule by Piecewise Linear Interpolation

have

$$\int_a^b f(x)dx$$

where we can calculate values of f along $[a, b]$. We can define $n+1$ evenly spaced points along $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. The size of each interval is $h = (b - a)/n$. If we take a look at any interval $[x_i, x_{i+1}]$, for $i = 0, \ldots n - 1$, we can construct a piecewise linear interpolating function P defined over the full interval $[a, b]$ as

$$P(x) = \frac{x_{i+1} - x}{h} f(x_i) + \frac{x - x_i}{h} f(x_{i+1})$$

for $x_i \leq x \leq x_{i+1}$.

And now integrating over the interval $[x_i, x_{i+1}]$ we find that

$$\int_{x_i}^{x_{i+1}} P(x)dx = \frac{h}{2} \left[f(x_i) + f(x_{i+1}) \right]$$

Adding the integrals over all subintervals $[x_i, x_{i+1}]$ we obtain

$$\begin{aligned} \int_a^b P(x)\, dx &= \sum_{i=0}^{n-1} \frac{h}{2} \left[f(x_i) + f(x_{i+1}) \right] \\ &= h \sum_{i=1}^{n-1} f(x_i) + \frac{h}{2} \left[f(x_0) + f(x_n) \right] \end{aligned}$$

and this leads to the following theorem.

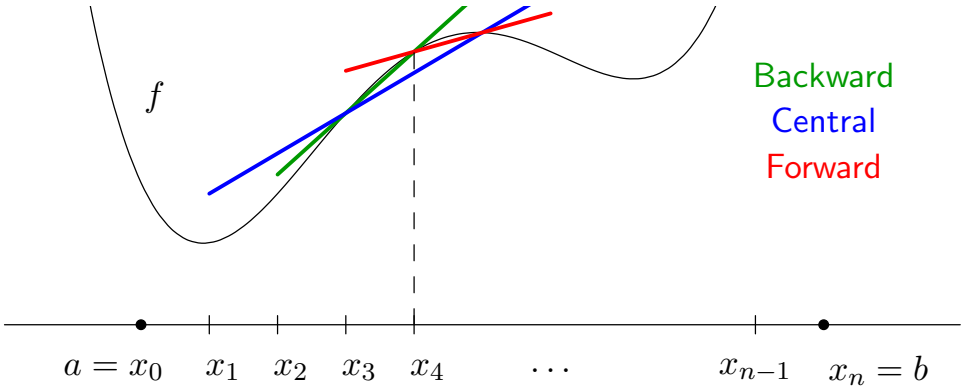


Figure 2.2: Finite Difference Methods

Theorem 2. (**Trapezoidal Rule**)
Assume f has a continuous second derivative f'' on $[a, b]$. If n is a positive integer, let $h = (b - a)/n$. Then we have

$$\int_a^b f(x)dx = h\sum_{i=1}^{n-1} f(x_i) + \frac{h}{2}\left[f(x_0) + f(x_n)\right] - h^2\frac{(b-a)}{12}f''(c) \tag{2.2}$$

for some c in (a, b).

The term $-f''(c)(b-a)h^2/12$ represents the error in approximating $\int_a^b f(x)\ dx$ by $\int_a^b P(x)\ dx$. The trapezoidal rule gives us an approximation of order two $O(h^2)$.

2.2 Finite Differences

In this chapter we will approximate derivatives using finite differences. As shown in figure 2.2, we can construct a finite difference method using either forward differences, central differences or backward differences. As their names suggest, forward differences use the next adjacent points and backward differences use the preceding points. We will be using centered differences which use both adjacent points for each point approximation. Central differences offer a more accurate way of approximating derivatives on a floating point arithmetic.

Let us derive the central difference approximation with the help of Taylor's theorem. When the second derivatives of $u : \mathbb{R} \to \mathbb{R}$ exist, we can write u as

$$u(x + h) = u(x) + u'(x)h + \frac{u''(x)}{2}h^2 + O(h^3).$$

And now we can substitute h by $-h$ so that we have

$$u(x-h) = u(x) - u'(x)h + \frac{u''(x)}{2}h^2 + O(h^3).$$

By subtracting the above equations and then dividing by $2h$, we get the desired central difference expression

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

Therefore, the derivative approximation with centered differences is given by

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h} \tag{2.3}$$

with $h > 0$, which is a second order finite difference.

Continuing on with the same idea, we can apply Taylor to u' and approximate u''. Following the previous result in equation 2.3, we obtain

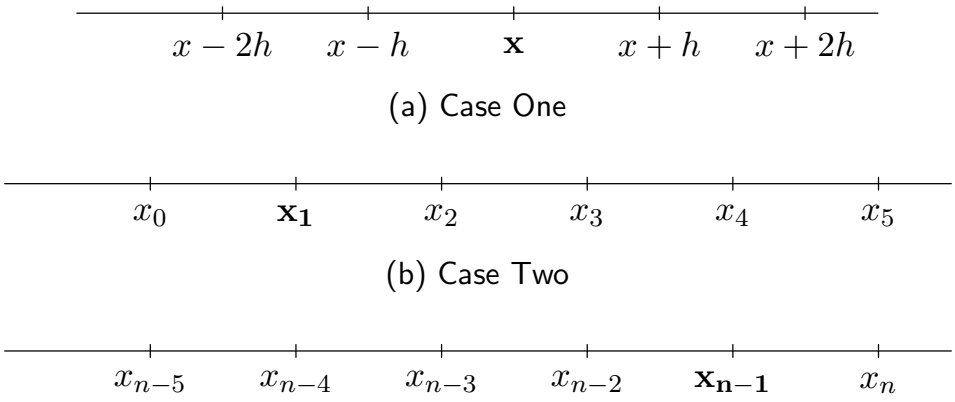
$$\begin{aligned} u''(x) &= \frac{u'(x+h) - u'(x-h)}{2h} + O(h^2) \\ &= \frac{u(x+2h) - u(x) - u(x) + u(x-2h)}{4h^2} + O(h^2) \end{aligned}$$

And therefore, without a loss of generalization, for the second derivative we get the expression

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

which is also of order two.

Now we are ready to construct a method to approximate $u''(x)$ using centered finite differences of order four. In the next chapter we will use the finite differences to approximate derivatives with respect to the space variable x. We will be interested in the following problem. Assuming that the function u is defined on an interval $[a, b]$ and $a = x_0 < x_1 < x < x_{n-1} < x_n = b$ are $n+1$ equally distributed node points on $[a, b]$. We want to approximate $u''(x_i)$ for $i = 1, 2, \ldots n-1$ with finite differences of order $O(h^4)$ with $h = (b-a)/n$. Also, we need the finite differences to involve evaluations of u at node points only. For the case of approximating $u''(x_i)$ with $i = 2, 3 \ldots n-2$ we can use centered finite differences. For the case of $u''(x_1)$ and $u''(x_{n-1})$, however, the finite differences use points on both sides of x_1 or x_{n-1} respectively but the differences are not centered.



(a) Case One

(b) Case Two

(c) Case Three

Figure 2.3: Method of Order Four Points

We will be using

$$u''(x_i) \approx \frac{1}{12h^2}\left[16(u(x_{i+1}) + u(x_{i-1})) - (u(x_{i+2}) + u(x_{i-2})) - 30u(x_i)\right]. \tag{2.4}$$

The detailed procedure to achieve this formula can be found in [3]. The complicated part is now approximating $u''(x)$ when $x = x_1$ as shown in figure 2.3b and when $x = x_{n-1}$ as shown in figure 2.3c while keeping the error of order four. For doing this, we use

$$u''(x_1) \approx \frac{1}{h^2}\sum_{k=1}^{6} \alpha_k u(x_{k-1}) \tag{2.5}$$

and also

$$u''(x_{n-1}) \approx \frac{1}{h^2}\sum_{k=1}^{6} \alpha_k u(x_{n-k+1}) \tag{2.6}$$

with α_k the k-th component of the vector

$$\alpha = \frac{1}{12}\begin{pmatrix} 10 \\ -15 \\ -4 \\ 14 \\ -6 \\ 1 \end{pmatrix}.$$

By means of Taylor's theorem it can be shown that these approximations are of order four. We will be using this method throughout the next chapters. In the

next chapter we will use finite differences to approximate the solution to the heat equation.

3 Numerical Methods for the Heat Equation

The time variable in PDEs can be approached in different ways and in this work we will focus on fully discrete methods, meaning that we will discretize both time and space. This implies that we will be moving along a mesh of points using finite difference approximations. The accuracy of the solution will therefore be determined by the step sizes in both time and space.

In this section we will discretize the heat equation using finite differences as well as quadrature rules of different orders in time and space. For numerical stability reasons, all methods considered are implicit. In Chapter 5 we adapt some of these methods to approximate the solution of the Black-Scholes equation for American call options.

Let us recall the heat equation which we will discretize and is given by

$$\frac{\partial u}{\partial x}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{3.1}$$

for $t \in (0,T]$ and $x \in [a,b]$ with initial and boundary conditions given by

$$\begin{aligned} u(x,0) &= g_1(x) \\ u(a,t) &= g_2(t) \\ u(b,t) &= g_3(t) \end{aligned}$$

for $x \in [a,b]$ and $t > 0$. Let us begin by a definition which will help us understand these methods.

Definition 1. An **explicit method** calculates the state of a system at a later time from the state of the system at the current time, while an **implicit method** finds

a solution by solving an equation involving both the current state of the system and a later one.

The development of efficient and precise numerical methods that approximate the heat equation is important for many applications. In finance, we may only need a solution with digits up to the cents but the method needs to be able to make quick calculations every time an asset price changes. In applications such as satellite maintenance, on the other side, there may be more time to calculate these values while a very small approximation error can jeopardize the satellite's instruments. It is important to chose the right method for solving the problem at hand. In this section we will take a look at some methods that have different levels of both precision and efficiency.

3.1 Implicit Euler Method

Let us start off with the implicit Euler method which is a method of order one in time. Let $a = x_0 < x_1 < ... < x_n = b$ be $n+1$ uniformly distributed points on $[a, b]$ and let $0 = t_0 < ... < t_m = T$ be $m+1$ uniformly distributed times on $[0, T]$. We define $\Delta x = (b-a)/n$ and $\Delta t = T/m$. The implicit Euler method is implicit because it evaluates difference approximations to derivatives at the time step t_{j+1} that we are solving for. Setting $t = t_{j+1}$ in equation (3.1) and using central differences of order two, this means that our second partial derivative on x is

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1}) \approx \frac{u(x_{i+1}, t_{j+1}) - 2u(x_i, t_{j+1}) + u(x_{i-1}, t_{j+1})}{(\Delta x)^2}.$$

Using backward differences of order one we obtain the partial derivative on t as

$$\frac{\partial u}{\partial t}(x_i, t_{j+1}) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\Delta t}$$

for $i = 1, \dots n-1$ and $j = 0, \dots m-1$.

From the the heat equation (3.1), when $x = x_i$ and $t = t_{j+1}$ we obtain the approximation

$$u(x_i, t_{j+1}) - u(x_i, t_j) \approx \frac{\Delta t}{(\Delta x)^2} \left[u(x_{i+1}, t_{j+1}) - 2u(x_i, t_{j+1}) + u(x_{i-1}, t_{j+1}) \right].$$

By solving a linear system $A\tilde{u} = b$, where $\tilde{u}$ is the vector of unknowns that approximate the vector $(u(x_1, t_{j+1}), \dots u(x_{n-1}, t_{j+1}))$ we approximate the solution $u(x, t)$ at time t_{j+1}. The right hand side b contains all known values (or approximated values) at time t_j as well as the boundary values at time t_{j+1}. For this,

we separate variables in the equation above so that we obtain unknowns only on the left side of the equation. Therefore, with $z = \Delta t/(\Delta x)^2$ we define

$$A = \begin{pmatrix} 1+2z & -z & 0 & 0 & \dots & \\ -z & 1+2z & -z & 0 & \dots & \\ & & & \ddots & & \\ & \dots & 0 & -z & 1+2z & -z \\ & \dots & 0 & 0 & -z & 1+2z \end{pmatrix},$$

$$\tilde{u} \approx (u(x_1, t_{j+1}), u(x_2, t_{j+1}), \dots u(x_{n-1}, t_{j+1}))^\top$$

and

$$b = \begin{pmatrix} \tilde{u}(x_1, t_j) + z + u(x_0, t_{j+1}) \\ \tilde{u}(x_2, t_j) + z \\ \vdots \\ \tilde{u}(x_{n-2}, t_j) + z \\ \tilde{u}(x_{n-1}, t_j) + z + u(x_n, t_{j+1}) \end{pmatrix}$$

where $\tilde{u}(x_i, t_j)$ are previously computed approximations to $u(x_i, t_j)$. In vector b, the first element b_1 and the last b_n use the Dirichlet boundary conditions. We can then solve for $A\tilde{u} = b$ where we note that since A is a strictly diagonally dominant matrix and therefore it is not singular and the linear system has a unique solution. This matrix is also independent of the iteration so it depends only on the number of steps used to discretize. We will be solving the system of equations m times, the number of steps in time.

3.2 Crank-Nicolson Method

The Crank-Nicolson method is an implicit method of order two in time. It can be seen as an average of the implicit Euler method and the explicit one. It can also be seen as integrating in time by means of the trapezoidal rule.

Let x_i and t_j with $i = 0, 1, \dots n$ and $j = 0, 1, \dots m$ be as before with $\Delta x = (b-a)/n$ and $\Delta t = T/m$. Using the trapezoidal rule, we integrate both sides of the heat equation (3.1) with respect to t for $t \in [t_0, t_1]$ as

$$\int_{t_0}^{t_1} \frac{\partial u}{\partial t}(x,t)dt = \int_{t_0}^{t_1} \frac{\partial^2 u}{\partial x^2}(x,t)dt$$

then

$$u(x,t_1) - u(x,t_0) \approx \left(\frac{t_1 - t_0}{2}\right)\left[\frac{\partial^2 u}{\partial x^2}(x,t_0) + \frac{\partial^2 u}{\partial x^2}(x,t_1)\right].$$

And now for $x = x_i$ with $i = 1, 2, \ldots n-1$ we use the results in the previous chapter to write

$$u(x_i, t_1) - u(x_i, t_0) \approx \left(\frac{\Delta t}{2}\right) \left[\frac{u(x_{i+1}, t_0) - 2u(x_i, t_0) + u(x_{i-1}, t_0)}{(\Delta x)^2} + \frac{u(x_{i+1}, t_1) - 2u(x_i, t_1) + u(x_{i-1}, t_1)}{(\Delta x)^2}\right]$$

where we have $n-1$ unknowns and $n-1$ equations. Let

$$z = \frac{\Delta t}{2(\Delta x)^2}$$

then we can rewrite the formula above as

$$zu(x_{i+1}, t_1) + (2z+1)u(x_i, t_1) - zu(x_{i-1}, t_1) \approx zu(x_{i+1}, t_0) + (1-2z)u(x_i, t_0) + zu(x_{i-1}, t_0)$$

with $i = 1, 2, \ldots n-1$. The left side corresponds to the unknown terms and the right side to the known ones, even though in the particular cases of $i = 1$ and $i = n-1$ there is an additional known term that we can put in the right side.

For $i = 1$, we need to obtain

$$zu(x_2, t_1) + (2z+1)u(x_1, t_1) \approx zu(x_0, t_1) + zu(x_2, t_0) + (1-2z)u(x_1, t_0) + zu(x_0, t_0)$$

then for $i = 2$ we obtain

$$-zu(x_3, t_1) + (2z+1)u(x_2, t_1) - zu(x_1, t_1) \approx zu(x_3, t_0) + (1-2z)u(x_2, t_0) + zu(x_1, t_0)$$

then for $i = n-2$ we have

$$-zu(x_{n-1}, t_1) + (2z+1)u(x_{n-2}, t_1) - zu(x_{n-3}, t_1) \approx zu(x_{n-1}, t_0) + (1-2z)u(x_{n-2}, t_0) + zu(x_{n-3}, t_0)$$

and finally for $i = n-1$ we get

$$(2z+1)u(x_{n-1}, t_1) - zu(x_{n-2}, t_1) \approx zu(x_n, t_0) + (1-2z)u(x_{n-1}, t_0) + zu(x_{n-2}, t_0) + zu(x_n, t_1)$$

If we define $w = 2z + 1$ and $v = 1 - 2z$ then we can write the previous equations in matrix form as $Au \approx b$ with the definitions

$$A = \begin{pmatrix} w & -z & 0 & 0 & 0 & \dots & 0 \\ -z & w & -z & 0 & 0 & \dots & 0 \\ 0 & -z & w & -z & 0 & \dots & 0 \\ \vdots & & & & \ddots & & \\ 0 & \dots & 0 & 0 & -z & w & -z \\ 0 & \dots & 0 & 0 & 0 & -z & w \end{pmatrix},$$

$$u = \begin{pmatrix} u(x_1, t_1) \\ u(x_2, t_1) \\ u(x_3, t_1) \\ \vdots \\ u(x_{m-2}, t_1) \\ u(x_{m-1}, t_1) \end{pmatrix}$$

and

$$b = \begin{pmatrix} zu(x_0, t_1) + zu(x_2, t_0) + vu(x_1, t_0) + zu(x_0, t_0) \\ zu(x_3, t_0) + vu(x_2, t_0) + zu(x_1, t_0) \\ zu(x_4, t_0) + vu(x_3, t_0) + zu(x_2, t_0) \\ \vdots \\ zu(x_{m-1}, t_0) + vu(x_{m-2}, t_0) + zu(x_{m-3}, t_0) \\ zu(x_m, t_0) + vu(x_{m-1}, t_0) + zu(x_{m-2}, t_0) \end{pmatrix}.$$

Solving this system of equations, we get the approximation to the heat equation solution for the first step in time. We need to solve for the other $m - 1$ steps to be able to get the solution for all points of the space-time mesh. As in the case of the implicit Euler method, this A matrix is also only dependent on the step sizes (not on the iteration) and because it is strictly diagonally dominant, it is non singular and we can guarantee a unique solution.

3.2.1 Rannacher Method

As we will see in the next chapter, the Crank-Nicolson method may be less accurate when the initial values $u(x, 0)$ with $x \in [a, b]$ have a corner or a jump discontinuity than when the initial data is smooth. The accuracy in the maximum norm of the approximation may be poorer for the initial steps and improves as the method advances in time. Approximation of $\partial u/\partial x$ or $\partial^2 u/\partial x^2$ for $t > 0$ in the maximum

norm can be even poorer. One of the reasons for this behavior is that the Crank-Nicolson method requires the approximation of $\partial^2 u/\partial x^2$ at $t = t_0$ in order to approximate $u(x, t_1)$. Since the initial values may not be differentiable at t_0, this produces a larger error of approximation for times t near t_0. Some methods have been proposed to deal with this problem. The Rannacher method [10] consists in applying implicit Euler for a few beginning time steps and using Crank-Nicolson for the remainder steps. In Chapter 4 we present some numerical results for this method.

The Rannacher method that we will be using is done in the following fashion. To smooth out the solution for time t_1, we will subdivide the interval $[t_0, t_1]$ into q equally spaced intervals $t_0 = t_{0,0} < t_{0,1} \ldots < t_{0,q-1} < t_{0,q} = t_1$. We will then apply the implicit Euler method to the subinterval above, going from $t_{0,0}$ to $t_{0,q}$. We will use the value of t_1 to do the Crank-Nicolson method for the larger interval but now starting at t_1 for the initial boundary values. This second run of the method will therefore calculate the solution using the $m - 1$ points $t_1 < t_2 < \ldots < t_m$.

3.3 Simpson 3/8 Method

We will now describe a method of order four in time by using Simpson's rule and Simpson's 3/8 rule to approximate integrals. We describe how to approximate $u(x, t_1)$, $u(x, t_2)$ and $u(x, t_3)$ from values of $u(x, t_0)$ and boundary values of u at $t = t_1, t_2$ and t_3.

Recall that Simpson's rule is of the form

$$\int_{t_0}^{t_2} f(t)dt \approx \frac{\Delta t}{3}\left[f(t_0) + 4f(t_1) + f(t_2)\right]$$

where $t_1 = (t_0 + t_2)/2$ and $\Delta t = t_1 - t_0$ while Simpson's 3/8 rule is given by

$$\int_{t_0}^{t_3} f(t)dt \approx \frac{3}{8}\Delta t\left[f(t_0) + 3f(t_1) + 3f(t_2) + f(t_3)\right].$$

Integrating both sides of equation (3.1) with respect to $t \in [t_0, t_2]$ and using Simpson's rule on the right side we obtain

$$\begin{aligned} u(x, t_2) - u(x, t_0) &= \int_{t_0}^{t_2} \frac{\partial^2 u}{\partial x^2}(x, t)dt \\ &\approx \frac{\Delta t}{3}\left[\frac{\partial^2 u}{\partial x^2}(x, t_0) + 4\frac{\partial^2 u}{\partial x^2}(x, t_1) + \frac{\partial^2 u}{\partial x^2}(x, t_2)\right] \end{aligned}$$

Integrating both sides of the equation now with respect to t on $[t_1, t_2]$ and analogously using Simpson's rule we obtain

$$\begin{aligned} u(x,t_3) - u(x,t_1) &= \int_{t_1}^{t_3} \frac{\partial^2 u}{\partial x^2}(x,t)dt \\ &\approx \frac{\Delta t}{3}\left[\frac{\partial^2 u}{\partial x^2}(x,t_1) + 4\frac{\partial^2 u}{\partial x^2}(x,t_2) + \frac{\partial^2 u}{\partial x^2}(x,t_3)\right] \end{aligned}$$

Finally, integrating both sides with respect to t on $[t_0, t_3]$ and this time using Simpson's 3/8 rule we get

$$\begin{aligned} u(x,t_3) - u(x,t_0) &= \int_{t_0}^{t_3} \frac{\partial^2 u}{\partial x^2}(x,t)dt \\ &\approx \frac{3}{8}\Delta t\left[\frac{\partial^2 u}{\partial x^2}(x,t_0) + 3\frac{\partial^2 u}{\partial x^2}(x,t_1) + 3\frac{\partial^2 u}{\partial x^2}(x,t_2) + \frac{\partial^2 u}{\partial x^2}(x,t_3)\right]. \end{aligned}$$

Discretizing $\partial^2 u/\partial x^2$ using centered finite differences of order two we obtain

$$\begin{aligned} u(x_i,t_2) - u(x_i,t_0) \approx \frac{\Delta t}{3(\Delta x)^2}\,&[u(x_{i-1},t_0) - 2u(x_i,t_0) + u(x_{i+1},t_0) + 4u(x_{i-1},t_1) \\ &-u(x_i,t_1) + u(x_{i+1},t_1) + u(x_{i-1},t_2) - 2u(x_i,t_2) + u(x_{i+1},t_2)] \\ u(x_i,t_3) - u(x_i,t_1) \approx \frac{\Delta t}{3(\Delta x)^2}\,&[u(x_{i-1},t_1) - 2u(x_i,t_1) + u(x_{i+1},t_1) + 4u(x_{i-1},t_2) \\ &-8u(x_i,t_2) + 4u(x_{i+1},t_2) + u(x_{i-1},t_3) - 2u(x_i,t_3) + u(x_{i+1},t_3)] \\ u(x_i,t_3) - u(x_i,t_0) \approx \frac{3\Delta t}{8(\Delta x)^2}\,&[u(x_{i-1},t_0) - 2u(x_i,t_0) + u(x_{i+1},t_0) + 3u(x_{i-1},t_1) \\ &-6u(x_i,t_1) + 3u(x_{i+1},t_1) + 3u(x_{i-1},t_2) - 6u(x_i,t_2) + 3u(x_{i+1},t_2) \\ &+u(x_{i-1},t_3) - 2u(x_i,t_3) + u(x_{i+1},t_3)] \end{aligned} \tag{3.2}$$

for all $i = 1, 2, \ldots n-1$. Setting up a system of equations with the above approximations we get a vector with entries that approximate the values of $u(x_i, t_1)$, $u(x_i, t_2)$ and $u(x_i, t_3)$ for $i = 1, 2, \ldots n-1$. The initial and boundary conditions are $u(x_i, t_0) = g_1(x_i)$ for $i = 0, 2, \ldots n$ and $u(x_0, t_j) = g_2(t_j)$ and $u(x_n, t_j) = g_3(t_j)$ for $j = 0, 1, \ldots m$. This makes $3(n-1)$ unknowns and solves for three steps in time. Because of the three steps in time for each iteration, we will be be doing $\lceil m/3 \rceil$ iterations to move m steps in time.

The order of the unknown vector entries is important when setting up the

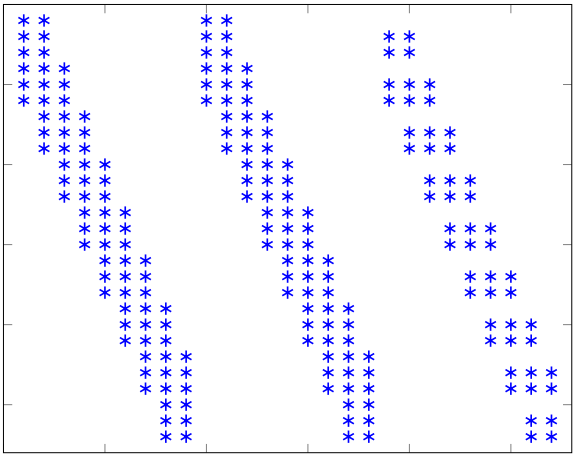


Figure 3.1: Simpson 3/8 Method's A Matrix

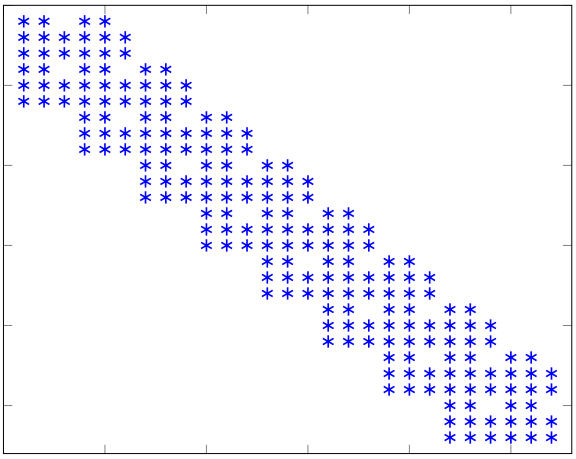


Figure 3.2: Simpson 3/8 Method's $\tilde{A}$ Matrix

system of equations. When the unknown vector approximates

$$\begin{aligned}&(u(x_1,t_1),u(x_2,t_1),\dots u(x_{n-1},t_1),\\ &u(x_1,t_2),u(x_2,t_2),\dots u(x_{n-1},t_2),\\ &u(x_1,t_3),u(x_2,t_3),\dots u(x_{n-1},t_3))^\top\end{aligned}$$

and the order of the equations is given by equations (3.2) for $i = 1, 2, \dots n-1$ we obtain a system $Au = b$ where the sparse matrix $A \in \mathbb{R}^{(3n-3)\times(3n-3)}$ is shown in figure 3.1. If we alter the unknown vector so that it approximates

$$\begin{aligned}&(u(x_1,t_1),u(x_1,t_2),u(x_1,t_3),\\ &u(x_2,t_1),u(x_2,t_2),u(x_2,t_3),\dots\\ &u(x_{n-1},t_1),u(x_{n-1},t_2),u(x_{n-1},t_3))^\top\end{aligned}$$

we obtain a redefined system $\tilde{A}\tilde{u} = b$ where $\tilde{A}$ has the structure shown in figure 3.2 and $u = P\tilde{u}$ where P is a permutation matrix.

Matrix $\tilde{A}$ can be described as follows. Let

$$\begin{aligned}z_1 &= \frac{h}{3(\Delta x)^2}\\ z_2 &= \frac{3h}{8(\Delta x)^2}\\ k_1 &= -2z_1 + 1\\ k_2 &= -2z_2 - 1\\ k_3 &= -2z_1 - 1\end{aligned}$$

Then, in terms of the above variables, we define the matrices

$$A_1 = \begin{pmatrix} -8z_1 & k_3 & 0 & 4z_1 & z_1 & 0\\ k_1 & -8z_1 & k_3 & z_1 & 4z_1 & z_1\\ -6z_2 & -6z_2 & k_2 & 3z_2 & 3z_2 & z_2\end{pmatrix}$$

$$B_1 = \begin{pmatrix} 4z_1 & z_1 & 0 & -8z_1 & k_3 & 0 & 4z_1 & z_1 & 0\\ z_1 & 4z_1 & z_1 & k_1 & -8z_1 & k_3 & z_1 & 4z_1 & z_1\\ 3z_2 & 3z_2 & z_2 & -6z_2 & -6z_2 & k_2 & 3z_2 & 3z_2 & z_2\end{pmatrix}$$

$$C_1 = \begin{pmatrix} 4z_1 & z_1 & 0 & -8z_1 & k_3 & 0\\ z_1 & 4z_1 & z_1 & k_1 & -8z_1 & k_3\\ 3z_2 & 3z_2 & z_2 & -6z_2 & -6z_2 & k_2\end{pmatrix}$$

And finally $\tilde{A}$ is defined as

$$\tilde{A} = \begin{pmatrix} A_1 & & & & & \\ B_1 & & & & & \\ & B_1 & & & & \\ & & \ddots & & & \\ & & & B_1 & & \\ & & & & B_1 & \\ & & & & C_1 & \end{pmatrix}$$

3.4 Crank-Nicolson Method of Order Four

A Crank-Nicolson method of order four in the spatial variable x can be obtained by approximating $\partial^2 u/\partial x^2(x_i, t_j)$ using finite differences of order four. Since we need such approximation for $i = 1, 2, \ldots n-1$, we can start by using centered differences as we did in Section 2.2 for points x_i with $i = 2, 3, \ldots n-2$.

Let us integrate both sides of the heat equation (3.1) with respect to t on the interval $[t_0, t_1]$ as

$$\int_{t_0}^{t_1} \frac{\partial u}{\partial t}(x,t)dt = \int_{t_0}^{t_1} \frac{\partial^2 u}{\partial x^2}(x,t)dt.$$

Then we approximate the right side with the trapezoidal rule in equation (2.2) and set $x = x_i$ so that we get

$$u(x_i, t_1) - u(x_i, t_0) \approx \frac{\Delta t}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, t_0) + \frac{\partial^2 u}{\partial x^2}(x_i, t_1)\right] \tag{3.3}$$

for all $i = 1, 2, \ldots n-1$.

For simplicity, we define the constants $\tilde{\alpha}_k$ for $k = 1, 2, \ldots 6$ as the elements of the vector

$$\tilde{\alpha} = \begin{pmatrix} 10 \\ -15 \\ -4 \\ 14 \\ -6 \\ 1 \end{pmatrix}.$$

Let us start with the case where $i = 1$. The equation for moving in time after the first step in space looks like

$$u(x_1, t_1) - u(x_1, t_0) \approx \frac{\Delta t}{24(\Delta x)^2}\left[\sum_{k=1}^{6} \tilde{\alpha}_k u(x_{k-1}, t_0) + \sum_{k=1}^{6} \tilde{\alpha}_k u(x_{k-1}, t_1)\right]$$

We can now reduce the equation and separate all unknown terms on one side and the known values in the other side. We also define $z = \Delta t/24(\Delta x)^2$ and finally get

$$(1 - z\tilde{\alpha}_2)u(x_1, t_1) - z\sum_{k=2}^{5} \tilde{\alpha}_{k+1} u(x_k, t_1)$$

$$\approx z\tilde{\alpha}_1 u(x_0, t_1) + z\tilde{\alpha}_1 u(x_0, t_0) + (z\tilde{\alpha}_2 + 1)u(x_1, t_0) + z\sum_{k=2}^{5} \tilde{\alpha}_{k+1} u(x_k, t_0)$$

Now for the case of the second step in x, when $i = 2$, we obtain

$$\begin{aligned} u(x_2, t_1) - u(x_2, t_0) \approx \\ \frac{\Delta t}{2}\left[\frac{16(u(x_3, t_0) + u(x_1, t_0)) - (u(x_4, t_0) + u(x_0, t_0)) - 30u(x_2, t_0)}{12(\Delta x)^2}\right] \\ + \frac{\Delta t}{2}\left[\frac{16(u(x_3, t_1) + u(x_1, t_1)) - (u(x_4, t_1) + u(x_0, t_1)) - 30u(x_2, t_1)}{12(\Delta x)^2}\right] \end{aligned}$$

and then analogously reducing the expression and setting the unknowns in one side and the knowns in the other, we get

$$\begin{aligned} -16zu(x_1, t_1) + (30z + 1)u(x_2, t_1) - 16zu(x_3, t_1) + zu(x_4, t_1) \\ \approx -zu(x_0, t_0) + 16zu(x_1, t_0) - (30z - 1)u(x_2, t_0) \\ + 16zu(x_3, t_0) - zu(x_4, t_0) - zu(x_0, t_1) \end{aligned}$$

Let us now take a look at the middle terms. For $i = 3, 4, \ldots n - 3$ we have the equation

$$\begin{aligned} u(x_i,t_1) - u(x_i, t_0) \approx \\ \frac{\Delta t}{2}\left[\frac{16(u(x_{i+1}, t_0) + u(x_{i-1}, t_0)) - (u(x_{i+2}, t_0) + u(x_{i-2}, t_0)) - 30u(x_i, t_0)}{12(\Delta x)^2}\right] \\ + \frac{\Delta t}{2}\left[\frac{16(u(x_{i+1}, t_1) + u(x_{i-1}, t_1)) - (u(x_{i+2}, t_1) + u(x_{i-2}, t_1)) - 30u(x_i, t_1)}{12(\Delta x)^2}\right] \end{aligned}$$

and simplifying we get

$$\begin{aligned} zu(x_{i-2}, t_1) - 16zu(x_{i-1}, t_1) + (30z + 1)u(x_i, t_1) - 16zu(x_{i+1}, t_1) + zu(x_{i+2}, t_1) \\ \approx -zu(x_{i-2}, t_0) + 16zu(x_{i-1}, t_0) - (30z - 1)u(x_i, t_0) \\ + 16zu(x_{i+1}, t_0) - zu(x_{i+2}, t_0) \end{aligned}$$

We have now defined how to move in time for the first $n-3$ steps in space. The last two equations for $i = n-2$ and $n-1$ are defined similar to the first two. For x_{n-2} we have

$$\begin{aligned}
&u(x_{n-2}, t_1) - u(x_{n-2}, t_0) \approx \\
&\quad \frac{\Delta t}{2}\left[\frac{16(u(x_{n-1}, t_0) + u(x_{n-3}, t_0)) - (u(x_n, t_0) + u(x_{n-4}, t_0)) - 30u(x_{n-2}, t_0)}{12(\Delta x)^2}\right] \\
&+\frac{\Delta t}{2}\left[\frac{16(u(x_{n-1}, t_1) + u(x_{n-3}, t_1)) - (u(x_n, t_1) + u(x_{n-4}, t_1)) - 30u(x_{n-2}, t_1)}{12(\Delta x)^2}\right]
\end{aligned}$$

and simplifying we obtain

$$\begin{aligned}
zu(x_{n-4}, t_1) - 16zu(x_{n-3}, t_1) + (30z+1)u(x_{n-2}, t_1) - 16zu(x_{n-1}, t_1) \\
\approx -zu(x_{n-4}, t_0) + 16zu(x_{n-3}, t_0) - (30z-1)u(x_{n-2}, t_0) \\
+ 16zu(x_{n-1}, t_0) - zu(x_n, t_0) - zu(x_n, t_1)
\end{aligned}$$

And finally, for the last term with $i = n-1$ we have

$$\begin{aligned}
&u(x_{n-1}, t_1) - u(x_{n-1}, t_0) \approx \\
&\qquad \frac{\Delta t}{24(\Delta x)^2}\left[\sum_{k=n-5}^{n} \tilde{\alpha}_{n-k+1} u(x_k, t_0) + \sum_{k=n-5}^{n} \tilde{\alpha}_{n-k+1} u(x_k, t_1)\right]
\end{aligned}$$

and simplifying we obtain

$$\begin{aligned}
(1 - z\tilde{\alpha}_2)u(x_{n-1}, t_1) - z\sum_{k=n-5}^{n-2} \tilde{\alpha}_{n-k+1} u(x_k, t_1) \\
\approx z\tilde{\alpha}_1 u(x_n, t_1) + z\tilde{\alpha}_1 u(x_n, t_0) + (z\tilde{\alpha}_2 + 1)u(x_{n-1}, t_0) \\
+ z\sum_{k=n-5}^{n-2} \tilde{\alpha}_{n-k+1} u(x_k, t_0)
\end{aligned}$$

With these equations, we can construct the problem in the form $Au = b$, where $u = (u(x_1, t_j), u(x_2, t_j), \ldots u(x_{n-1}, t_j))^\top$ for all $j = 1 \ldots m$. The resulting A matrix has the shape

$$A = \begin{pmatrix}
* & * & * & * & * & & & & & \\
* & * & * & * & & & & & & \\
* & * & * & * & * & & & & & \\
 & * & * & * & * & * & & & & \\
 & & * & * & * & * & * & & & \\
 & & & & & \ddots & & & & \\
 & & & & & & * & * & * & * & * \\
 & & & & & & * & * & * & * & * \\
 & & & & & & * & * & * & * & * \\
 & & & & & & & * & * & * & * \\
 & & & & & & * & * & * & * & *
\end{pmatrix}$$

with $*$ representing a non zero entry. This system of equations needs to be solved m times, which represents the number of steps that we will be moving in time.

4 Experiments

In this chapter we will test the numerical methods we described in Chapter 3. We will test the methods with examples where the solution can be written in terms of a Fourier series expansion. The tests include examples with smooth and non smooth initial values.

In Chapter 5 we will consider an example of continuous peicewise smooth initial data that appears in American call options with dividends. We will start by recalling some basic elements of Fourier series.

4.1 Fourier Series for the Heat Equation

The Fourier series expansion uses trigonometric functions to rewrite a function with possibly infinite linear combinations of sines and cosines. Intuitively, it means that the heat wave signal can be decomposed in a sum of simpler wave signals.

Definition 2. A function f is **periodic** if there is some constant $p > 0$ such that $f(x + p) = f(x)$ for all x. The smallest such p for a given function is called the **period** of the function.

Both sine and cosine functions are periodic of period 2π. Representing a function with a possibly infinite linear combination of sines and cosines decomposes the function into its components of various frequencies. The coefficients of the sines and cosines represent those frequencies and in what amounts.

If we consider the functions of period p, we have an inner product given by

$$\langle f, g \rangle = \frac{2}{p} \int_0^p f(x)g(x)dx.$$

Using this product, this space has an orthonormal basis given by the functions

$$\frac{1}{\sqrt{2}}, \quad \sin\left(\frac{2\pi kx}{p}\right), \quad \cos\left(\frac{2\pi lx}{p}\right)$$

with $k, l \in \mathbb{N}$. The Fourier series expansion arises from writing a function in terms of this basis and using the above inner product.

Definition 3. (**Fourier series**)
The Fourier series of a function $f(x)$ of period p is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi kx}{p}\right) + b_k \sin\left(\frac{2\pi kx}{p}\right) \right] \tag{4.1}$$

where the Fourier coefficients are

$$a_k = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{2\pi kx}{p}\right) dx$$

$$b_k = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{2\pi kx}{p}\right) dx$$

Now, we need to make sure that $f(x)$ can be written in terms of this Fourier series.

Theorem 3. If $f(x)$ is an absolutely integrable function[1] of period p which is piecewise smooth on the interval $[a, b]$, then for all x in $a < x < b$ the Fourier series of $f(x)$ converges to $f(x)$ at points of continuity and to the value

$$\frac{f^+(x) + f^-(x)}{2}$$

at points of discontinuity.

Note that this theorem may fail at the endpoints a and b because we lack left and right derivatives. If the interval $[a, b]$ is of length p, however, we can do a periodic extension of f to the whole real axis and therefore the theorem applies for all x. The proof of this theorem can be found in [12].

We will now go on with a simple case of the heat equation to exemplify the use of the Fourier series expansion. Let us recall the heat equation which for $x \in \mathbb{R}$ and $t > 0$ states

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

[1] An absolutely integrable function is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x)| dx < \infty$.

with the initial value $u(x,0) = g_1(x)$ with $g_1(x)$ a function of period p but without any other boundary conditions. For each $t \geq 0$, we need to find the solution to the heat equation $u = u(x,t)$ such that it is periodic, of period p.

We know from theorem 3 that u can be written as its Fourier series expansion

$$u(x,t) = \frac{\tilde{a}_0(t)}{2} + \sum_{k=1}^{\infty} \left[\tilde{a}_k(t) \cos\left(\frac{2\pi kx}{p}\right) + \tilde{b}_k(t) \sin\left(\frac{2\pi kx}{p}\right) \right].$$

Taking the partial derivative with respect to t we get

$$\frac{\partial u}{\partial t} = \frac{\tilde{a}_0'(t)}{2} + \sum_{k=1}^{\infty} \left[\tilde{a}_k'(t) \cos\left(\frac{2\pi kx}{p}\right) + \tilde{b}_k'(t) \sin\left(\frac{2\pi kx}{p}\right) \right]$$

and with respect to x twice

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=1}^{\infty} \left[-\tilde{a}_k(t) \left(\frac{4\pi^2 k^2}{p^2}\right) \cos\left(\frac{2\pi kx}{p}\right) - \tilde{b}_k(t) \left(\frac{4\pi^2 k^2}{p^2}\right) \sin\left(\frac{2\pi kx}{p}\right) \right].$$

Since the heat equation is satisfied, we can equate the two expressions above and obtain

$$\tilde{a}_0'(t) = 0 \tag{4.2}$$

$$\tilde{a}_k'(t) = \frac{-4\pi^2 k^2}{p^2} \tilde{a}_k(t) \tag{4.3}$$

$$\tilde{b}_k'(t) = \frac{-4\pi^2 k^2}{p^2} \tilde{b}_k(t) \tag{4.4}$$

with $\tilde{a}_0(0) = a_0$, $\tilde{a}_k(0) = a_k$ and $\tilde{b}_k(0) = b_k$ for all $k \geq 1$.

From (4.2), we get $\tilde{a}_0(t) = a_0$ for all t. Then from (4.3) and using $\tilde{a}_k(0) = a_k$ and $\tilde{b}_k(0) = b_k$, for all $t \geq 0$ we get

$$\tilde{a}_k(t) = e^{-\frac{4\pi^2 k^2}{p^2} t} a_k.$$

$$\tilde{b}_k(t) = e^{-\frac{4\pi^2 k^2}{p^2} t} b_k$$

for all $k \geq 1$ where a_k and b_k are the Fourier series coefficients for $g_1(x)$. And therefore

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{p^2} t} \left[a_n \cos\left(\frac{2\pi nx}{p}\right) + b_n \sin\left(\frac{2\pi nx}{p}\right) \right] \tag{4.5}$$

4.1.1 Dirichlet Boundary Conditions

We will now consider this problem with the Dirichlet boundary conditions. We are looking for a solution of u with x in $[a, b]$ and t in $[0, T]$. To simplify our calculations of the Fourier series, we define the period $p = b - a$ and we will consider x in $[0, p]$ which can then be transformed to $[a, b]$. The heat equation in one dimension looks like

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{4.6}$$

and the the Dirichlet boundary conditions are $u(0, t) = g_2(t)$ and $u(p, t) = g_3(t)$. The initial temperature distribution is given by $u(x, 0) = g_1(x)$. The following calculations are valid when we keep $g_2(t)$ and $g_3(t)$ at constant values $g_1(0)$ and $g_1(p)$.

Let $\tilde{u} = u(x, t) - l(x)$ where $l(x)$ is a linear function such that $\tilde{u}(0, t) = 0$ and $\tilde{u}(p, t) = 0$ for all $t > 0$. We define

$$l(x) = \left[g_1(0) + \frac{x}{p} \left(g_1(p) - g_1(0) \right) \right]$$

so that

$$\tilde{u}(x, t) = u(x, t) - \left[g_1(0) + \frac{x}{p} \left(g_1(p) - g_1(0) \right) \right]$$

where $\tilde{u}$ satisfies the heat equation (4.6) with initial conditions $\tilde{u} = g_1(x) - l(x)$ and boundary values $\tilde{u}(0, t) = 0$ and $\tilde{u}(p, t) = 0$ for all $t \geq 0$. We define

$$g(x) = \tilde{u}(x, 0),$$

which satisfies $g(0) = 0$ and $g(p) = 0$ so it can be written in its sine series expansion (see [12]) as

$$g(x) = \sum_{n=0}^{\infty} b_n \sin \left(\frac{\pi n x}{p} \right),$$

where the coefficients b_n are defined by the formula

$$b_n = \frac{2}{p} \int_0^p g(x) \sin \left(\frac{\pi n x}{p} \right) dx.$$

We will now search the general solution in the form of the series with time-dependent coefficients $c_k(t)$. We write $\tilde{u}$ as its sine series expansion

$$\tilde{u}(x, t) = \sum_{k=0}^{\infty} c_k(t) \sin \left(\frac{\pi k x}{p} \right).$$

The boundary conditions $\tilde{u}(0,t) = 0$ and $\tilde{u}(p,t) = 0$ are satisfied for all times $t > 0$. The initial conditions for $c_k(t)$ are $c_k(0) = b_k$ for $k \in \mathbb{N} \cup \{0\}$. We substitute these expressions into the heat equation (4.6) and get

$$\sum_{k=0}^{\infty} \frac{dc_k(t)}{dt} \sin\left(\frac{k\pi x}{p}\right) = -\sum_{k=0}^{\infty} \left(\frac{k^2\pi^2}{p^2} c_k(t)\right) \sin\left(\frac{k\pi x}{p}\right).$$

We then multiply both sides of the last expression by $\sin(m\pi x/p)$ and integrate on the interval $[0,p]$ using the orthogonality relations

$$\frac{2}{p}\int_0^p \sin\left(\frac{k\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)dx = \begin{cases} 0, & m \neq k \\ 1, & m = k \end{cases}$$

so that we get

$$\sum_{k=0}^{\infty} \frac{dc_k(t)}{dt} \int_0^p \sin\left(\frac{k\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)dx$$
$$= -\sum_{k=0}^{\infty} \frac{k^2\pi^2}{p^2} c_k(t) \int_0^p \sin\left(\frac{k\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)dx$$

which we reduce to

$$\frac{dc_k(t)}{dt} = -\frac{k^2\pi^2}{p^2} c_k(t).$$

Solving the ordinary differential equation for $c_k(t)$, we obtain

$$c_k(t) = A\exp\left(-\frac{k^2\pi^2}{p^2}t\right)$$

where $A = c_k(0)$ is a constant dependent on the initial conditions. Taking into account that $c_k(0) = b_k$, we get the solution for $c_k(t)$ in the form

$$c_k(t) = b_k \exp\left(-\frac{k^2\pi^2}{p^2}t\right).$$

Hence, the final solution for the heat equation is expressed through the formula

$$u(x,t) = l(x) + \sum_{k=0}^{\infty} b_k \exp\left(-\frac{k^2\pi^2}{p^2}t\right)\sin\left(\frac{\pi k x}{p}\right)$$
$$= g_1(0) + \frac{x}{p}\left[g_1(p) - g_1(0)\right] + \sum_{k=0}^{\infty} b_k \exp\left(-\frac{k^2\pi^2}{p^2}t\right)\sin\left(\frac{\pi k x}{p}\right).$$

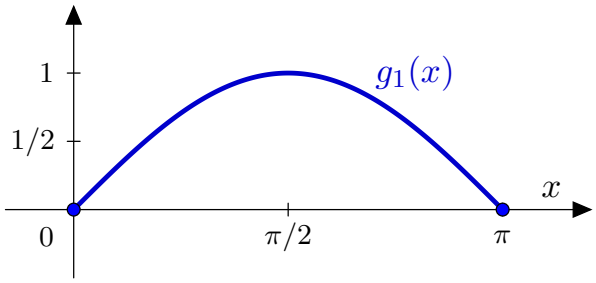


Figure 4.1: Smooth Example Function

In the next sections we will construct examples of different cases of the initial boundary $g_1(x) = u(x, 0)$. We will start both time and space variables at zero and end them at 1 and p respectively. With this said, we will use the boundary conditions $u(0, t) = g_2(t) = 0$ and $u(p, t) = g_3(t) = 0$ for our examples.

4.2 Smooth Initial Values

In this section, we will test our methods with smooth initial values. For a numerical example, let us consider

$$u(x, t) = e^{-t} \sin(x)$$

with smooth initial values. We will set boundaries at $a = 0$ and $b = \pi$ so that we can write the boundary conditions as

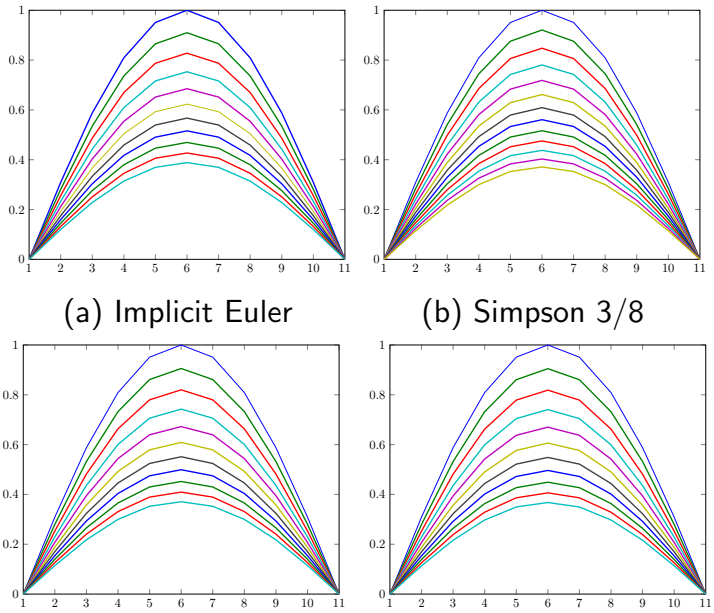
$$\begin{aligned} g_1(x) &= u(x, 0) = \sin(x) \\ g_2(t) &= u(0, t) = e^{-t} \sin(0) = 0 \\ g_3(t) &= u(\pi, t) = e^{-t} \sin(\pi) = 0. \end{aligned}$$

Figure 4.1 shows the behavior of our initial value function $g_1(x)$, with $x \in [0, \pi]$.

Figure 4.2 shows the behavior of the solution with $n = 10$ steps in space and $m = 5$ steps in time for t in $[0, 1]$. We do not use the Rannacher method for this example because there is no need for smoothing the data.

We now compute the relative error of the approximations with the different methods. If u is the solution to the heat equation using Fourier series expansion and v is the approximation using any of the different methods we are testing, then the relative error ϵ is given by the formula

$$\epsilon = \frac{\|v - u\|}{\|u\|}.$$



(a) Implicit Euler (b) Simpson 3/8

(c) Crank-Nicolson (d) Crank-Nicolson 4

Figure 4.2: Smooth Function One Comparison

Method	Relative Error	
	t_1	t_m
Implicit Euler	1.2561×10^{-5}	2.5153×10^{-3}
Simpson 3/8	1.0230×10^{-7}	2.0562×10^{-5}
Crank-Nicolson	9.2391×10^{-8}	1.8478×10^{-5}
Crank-Nicolson 4	1.0413×10^{-8}	2.0827×10^{-6}

Table 4.1: Example One Relative Error, $n = 200$, $m = 200$

Method	Relative Error	
	t_1	t_m
Implicit Euler	3.2976×10^{-3}	4.0297×10^{-2}
Simpson 3/8	1.6162×10^{-6}	1.9955×10^{-5}
Crank-Nicolson	4.6558×10^{-5}	5.5855×10^{-4}
Crank-Nicolson 4	4.8274×10^{-5}	5.7914×10^{-4}

Table 4.2: Example One Relative Error, $n = 200$, $m = 12$

Method	Relative Error	
	t_1	t_m
Implicit Euler	2.2680×10^{-5}	6.2860×10^{-1}
Simpson 3/8	1.0221×10^{-5}	2.0566×10^{-3}
Crank-Nicolson	1.0262×10^{-5}	2.0545×10^{-3}
Crank-Nicolson 4	3.7251×10^{-8}	4.0201×10^{-6}

Table 4.3: Example One Relative Error, $n = 20$, $m = 200$

Table 4.1 shows the relative error of the different methods at t_1 and t_m the first and final time steps for t in $[0, 1]$. For this comparison, we use 200 steps for both time and space. We can see that the Crank-Nicolson of order four method does best for both first and last steps. The Crank-Nicolson and Simpson 3/8 methods do no lag behind but the implicit Euler method has significantly more error than the others. To take advantage of the higher order methods, let us test the outcome when we reduce time and then space steps. Table 4.2 shows only 12 time steps with the same 200 space steps. We can see that Simpson 3/8 is the one with the least error because of its high order in time. Table 4.3 shows a reduction to 20 steps in space while keeping the 200 steps in time. In this last case, Crank-Nicolson of order four is far best than the others.

4.3 Non Smooth Initial Values

In this section we are interested in testing out the numerical methods when the values initial values are piecewise continuously differentiable. We will look at three examples: two where the function is continuous and the other with a jump discontinuity. We will construct initial functions $g_1(x)$ so that we are able to obtain an analytical form of its Fourier series. For this, we need a periodic function and for convenience we will make all our non smooth initial functions of period two.

4.3.1 Example Two

Our second example has an initial function defined as

$$g_1(x) = \begin{cases} x, & x \in [0, 1) \\ 2 - x, & x \in [1, 2] \end{cases} \tag{4.7}$$

for all $x \in [0, 2]$ as depicted in figure 4.3.

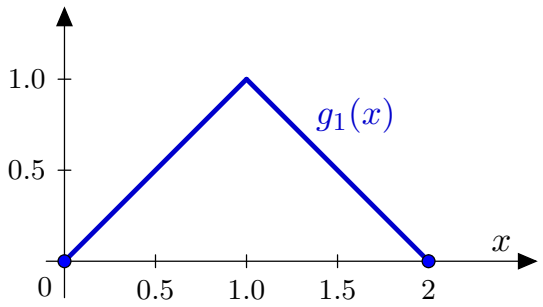
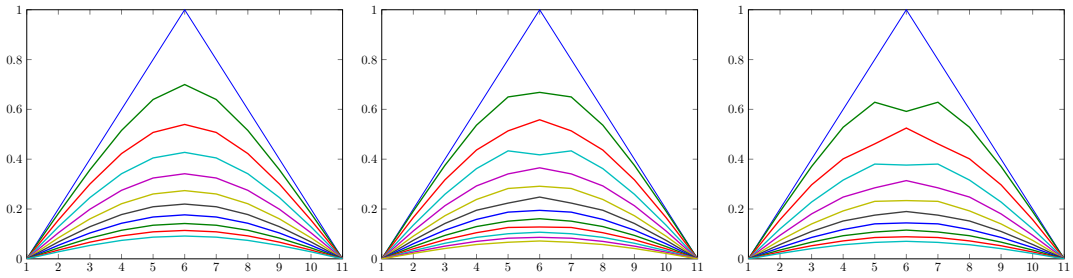
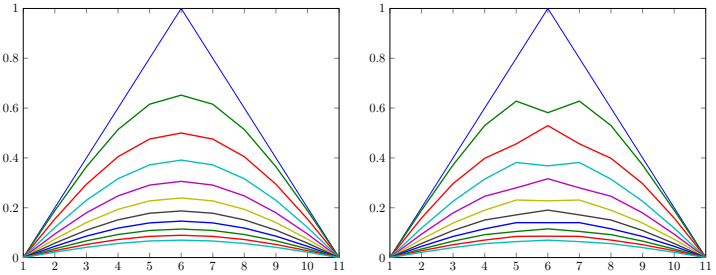


Figure 4.3: Non Smooth Example Two Function



(a) Implicit Euler (b) Simpson 3/8 (c) Crank-Nicolson



(d) Rannacher (e) Crank-Nicolson 4

Figure 4.4: Non Smooth Function Two Comparison

Figure 4.4 shows the resulting approximations of the methods for $m = 10$ steps in time and $n = 10$ steps in space. We can see how at $x = 1$, the initial function $g_1(x)$ is not differentiable and therefore the solution at further steps is not accurate around that point. The initial steps for the Rannacher method seem to make a significant correction since they smooth out the initial step. The Crank-Nicolson of order four, though, is the same as the one of order two because of the symmetry of the initial value function.

We will now solve the heat equation with a Fourier series expansion for this particular initial value function. First we get the coefficient b_n as follows

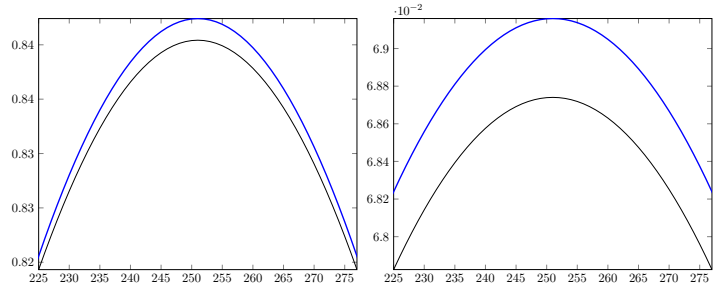
$$\begin{aligned} b_n &= \frac{2}{p}\left[\int_0^p f(x)\sin\left(\frac{\pi n x}{p}\right)dx\right] \\ &= \int_0^1 x\sin\left(\frac{\pi n x}{2}\right)dx + \int_1^2 (2-x)\sin\left(\frac{\pi n x}{2}\right)dx \\ &= \frac{1}{\pi^2 n}\left(4\sin\left(\frac{\pi n}{2}\right) - 2\pi n\cos\left(\frac{\pi n}{2}\right)\right) \\ &\quad + \frac{2}{\pi^2 n^2}\left(2\sin\left(\frac{\pi n}{2}\right) - 2\sin(\pi n) + \pi n\cos\left(\frac{\pi n}{2}\right)\right) \\ &= \frac{4}{\pi^2 n^2}\left[2\sin\left(\frac{\pi n}{2}\right) - \sin(\pi n)\right] \end{aligned}$$

With which we can now calculate the solution as

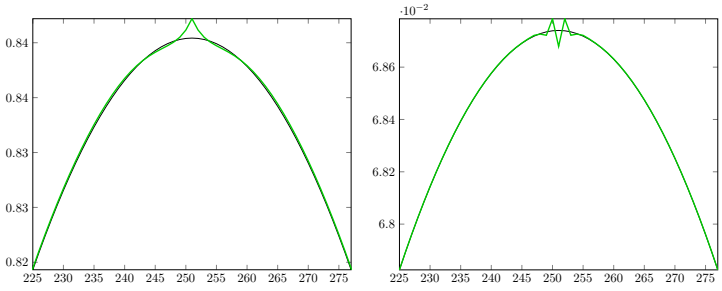
$$\begin{aligned} u(x,t) &= g_2 + (g_3 - g_2)\frac{x}{2} + \sum_{n=0}^{\infty} b_n \exp\left(-\frac{n^2\pi^2}{p^2}t\right)\sin\left(\frac{\pi n x}{p}\right) \\ &= \sum_{n=0}^{\infty} b_n \exp\left(-\frac{n^2\pi^2}{4}t\right)\sin\left(\frac{\pi n x}{2}\right). \end{aligned}$$

We can now compare the analytic solution to the solution obtained with our methods. Each method's solution will be shown in its own color and the analytic solution will be shown in black.

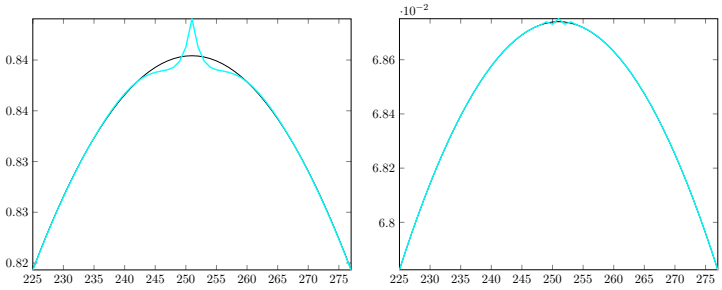
Figure 4.5 shows method comparisons versus the Fourier series solution. These figures are close-ups since we took 500 steps both in time and space. The y axis shows the solution value while the horizontal axis shows the step in space. There are two figures for each method, one for the tenth step in time $t = t_{10} = 10\Delta t$ and the second for the last step $t = T = 1$. We can see that all methods except for implicit Euler and Rannacher show a spike (or more than one spikes) at the non differentiable middlepoint. This happens because the methods depend on the derivative approximation to calculate next steps. The Rannacher method in figures



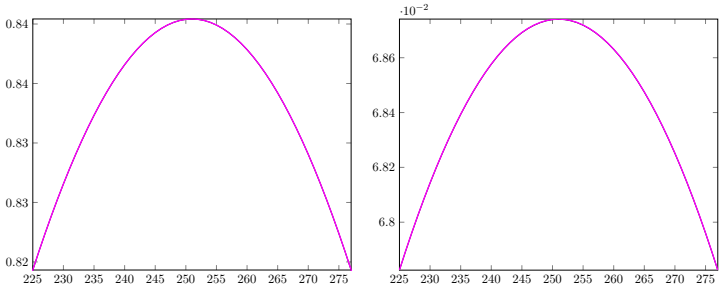
(a) Implicit Euler, t_{10} (b) Implicit Euler, T



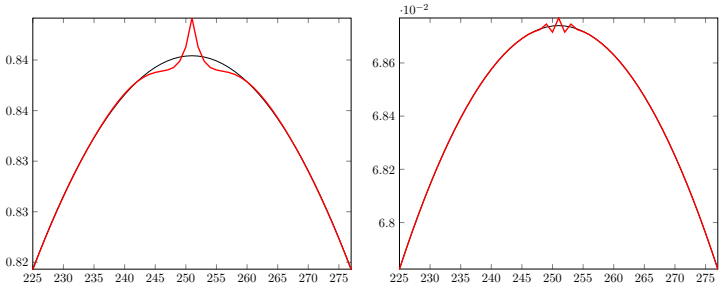
(c) Simpson 3/8, t_{10} (d) Simpson 3/8, T



(e) Crank-Nicolson, t_{10} (f) Crank-Nicolson, T



(g) Rannacher, t_{10} (h) Rannacher, T



(i) Crank-Nicolson 4, t_{10} (j) Crank-Nicolson 4, T

Figure 4.5: Non Smooth Function Two Fourier Series Comparison

Method	$n = 201$, $m = 201$		$n = 201$, $m = 12$		$n = 20$, $m = 201$	
	t_1	t_m	t_1	t_m	t_1	t_m
Implicit Euler	$2.8 \cdot 10^{-3}$	$1.5 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$	$2.5 \cdot 10^{-1}$	$8.8 \cdot 10^{-3}$	$2.2 \cdot 10^{-2}$
Simpson 3/8	$3.1 \cdot 10^{-4}$	$1.1 \cdot 10^{-4}$	$1.2 \cdot 10^{-2}$	$7.9 \cdot 10^{-2}$	$5.0 \cdot 10^{-3}$	$7.2 \cdot 10^{-3}$
Crank-Nicolson	$6.0 \cdot 10^{-4}$	$1.7 \cdot 10^{-5}$	$3.7 \cdot 10^{-2}$	$5.5 \cdot 10^{-2}$	$4.3 \cdot 10^{-3}$	$7.1 \cdot 10^{-3}$
Rannacher	$3.4 \cdot 10^{-5}$	$2.8 \cdot 10^{-5}$	$9.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$6.2 \cdot 10^{-3}$	$7.1 \cdot 10^{-3}$
Crank-Nicolson 4	$6.4 \cdot 10^{-4}$	$5.9 \cdot 10^{-5}$	$3.6 \cdot 10^{-2}$	$5.5 \cdot 10^{-2}$	$3.7 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$
Rannacher 4	$3.0 \cdot 10^{-5}$	$2.2 \cdot 10^{-5}$	$9.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$6.2 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$

Table 4.4: Example Two Relative Error

4.5g and 4.5h, though, used only five initial steps in time using implicit Euler to smooth out the solution. This smoothing got rid of the spikes and therefore the Rannacher method looks most accurate specially around that point. The implicit Euler method has no spikes but it looks the least accurate (except for middlepoint vicinity) in both figures 4.5a and 4.5b.

Let us look at the relative errors for the methods above. We use the same values as done for the previous example for t in $[0, 1]$. We again reduce time or space steps to see how well the methods approximate the solution with the different orders. Table 4.4 shows that results with this non smooth function are poorer than in the previous smooth example. As it can be seen from the analytical form of the Fourier series expression, the solution $u(x, t)$ is smooth for $t > 0$. The last row in table 4.4 shows a smoothing method analogous to the Rannacher method but applied to the Crank-Nicolson of order four. We can see that the Rannacher method improves the Crank-Nicolson solution at small times but it keeps about the same error at further steps in time. Having a smooth approximation for $t > 0$ has advantages in certain applications such as in American options.

4.3.2 Example Three

Our last initial time example function $g_1(x)$ has a jump discontinuity as shown in figure (4.6) and it is defined as

$$g_1(x) = \begin{cases} x, & x \in [0, 1) \\ 0, & x \in [1, 2] \end{cases} \tag{4.8}$$

In Figure 4.7 we can see how that jump discontinuity in $g_1(x)$ makes a lot of noise in the solution. The implicit Euler method and Rannacher method seem to be the best approximation methods.

Now we proceed with the Fourier series expansion. The series coefficients b_k

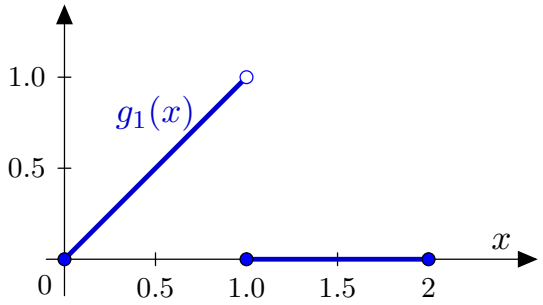
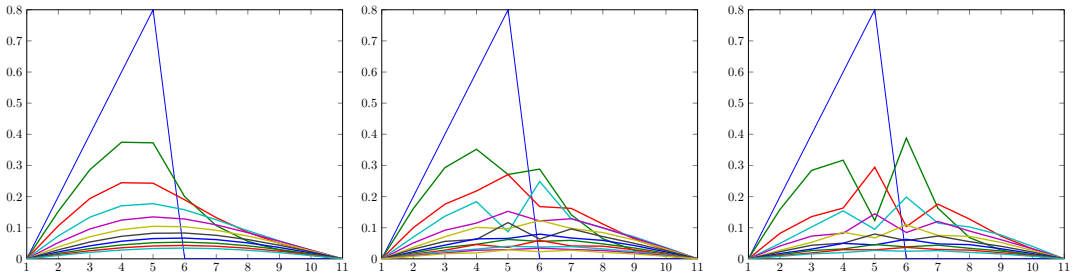
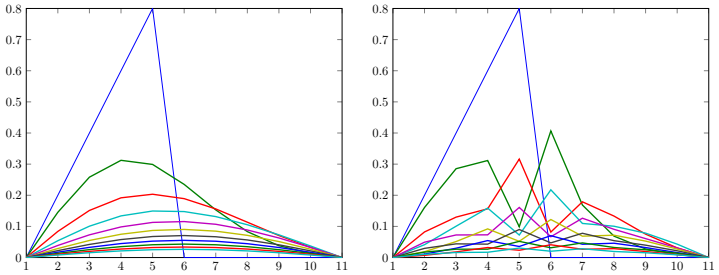


Figure 4.6: Non Smooth Example Three Function



(a) Implicit Euler (b) Simpson 3/8 (c) Crank-Nicolson



(d) Rannacher (e) Crank-Nicolson 4

Figure 4.7: Smooth Function Three Comparison

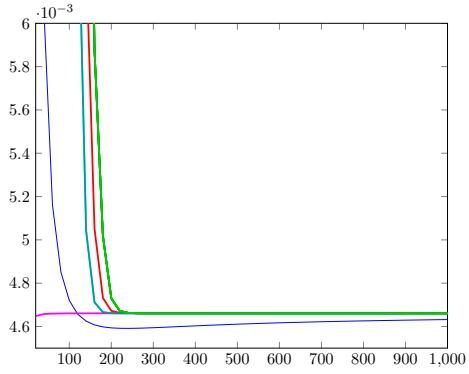
are defined as

$$\begin{aligned} b_k &= \frac{2}{p}\int_0^p f(x)\sin\left(\frac{\pi k x}{p}\right)dx \\ &= \frac{2}{2}\int_0^1 x\sin\left(\frac{\pi k x}{2}\right)dx \\ &= \frac{2}{\pi^2 k^2}\left[2\sin\left(\frac{\pi k}{2}\right) - \pi k\cos\left(\frac{\pi k}{2}\right)\right] \end{aligned}$$

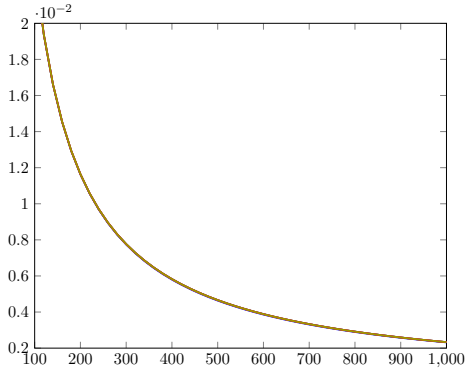
And therefore the solution to the heat equation becomes

$$\begin{aligned} u(x,t) &= g_2 + (g_3 - g_2)\frac{x}{p} + \sum_{k=0}^{\infty} b_k \exp\left(-\frac{k^2\pi^2}{p^2}t\right)\sin\left(\frac{\pi k x}{p}\right) \\ &= \sum_{k=0}^{\infty} b_k \exp\left(-\frac{k^2\pi^2}{2^2}t\right)\sin\left(\frac{\pi k x}{2}\right) \end{aligned}$$

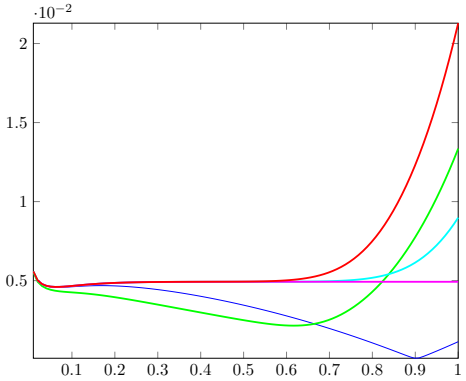
In figure 4.8 we can see the relative errors with our last example function for initial values. We use 500 constant steps for time and space and set $T = 0.1$ when they are not evaluated. The horizontal axis is how many steps we use for t or x and the final time used for T respectively. The vertical axis is the relative error for the last step t_m. Rannacher uses four initial time steps.



(a) Steps in Time t



(b) Steps in Space x



(c) Values of Final Time T

Figure 4.8: Example Three Relative Error

5 Option Theory

In this chapter we will adapt some of the methods in Chapter 3 to approximate the solution and free boundary of American options with dividend-paying assets. This problem has a non smooth free boundary at the intersection of the initial data and the free boundary.

To help us present the problem, we will start off giving some option theory definitions and then explaining the Black-Scholes equation. We will proceed by explaining how we applied the Crank-Nicolson method of orders two and four to discretize the Black-Scholes equation. We will conclude by stating our results.

5.1 Basic Definitions

A stock, share, or equity is a tool that companies use in order to raise money and allows individuals (shareholders) to invest in such companies. Thus, each shareholder owns a part of the company, depending on how many stocks are acquired. Now there exists a complex and immense theory behind buying and selling these stocks. As part of the company, the shareholders expect to gain profits, in the form of dividends. The assets are traded publicly in a securities exchange market and their value reflect the expectations of the growth of the company and the likely value of its future dividends.

Financial markets have become more sophisticated and they go further than buying and selling basic securities. As an example, derivatives are financial instruments in which the deals for buying and selling can be done according to the investor's needs. Common forms of underlying assets include: stocks, bonds, commodities, currencies, interest rates, and market indexes. A form of derivative in which an investor is given the option to either buy or sell an asset at a deter-

Notation	Meaning
V	option value
$C(S,t)$	call value
$P(C,t)$	put value
S	value of the underlying asset
t	time
σ	volatility of the underlying asset
E	exercise price
T	time to expiry
r	interest rate
D_0	dividend yield

Table 5.1: Terminology

mined time and for a specific price is known as an option. We will use this type of derivative for our purposes.

Definition 4. An **option** is a derived financial instrument established as a contract that gives its buyer the right, but not the obligation, to buy or sell goods or values at a predetermined **exercise price** or **strike price**, at or before an **expiry date**. A buying option is a **call** and a selling option is a **put**.

There are different kinds of options like barrier options, Asian options, and lookback options. The value of these derivatives is derived from the price of underlying or basic assets. The simplest type of option is the European option in which the holder buys or sells the asset at the expiry date. A central problem in financial mathematics is how much a holder should pay for a given option. The real value of the option changes as the asset value changes over time. This value is a function of the expectations on the potential economic benefit caused by the difference of its strike price and its price at expiry. For the European call option, the value of the option at the expiry date T is given by

$$C(S,T) = \max(S - E, 0)$$

where S is the underlying asset price and E is the exercise price. The European call option writer has the obligation of selling the asset if the holder decides to buy it at the expiration date. For the holder, the contract is a right and for the writer it is an obligation. A put option is the right to sell an asset and has payoff properties opposite of those described for the call. In a put option, the holder buys the right to sell the asset and the writer is obliged to buy it.

Since the value of the European option at its expiry date is known and therefore we need to back step through time to get its current value. A solution to this problem is given by using a binomial model or in its continuous form by the commonly used Black-Scholes Model.

Definition 5. An **American option** is that which can be exercised at any point in time prior to expiry.

The American option will be the one we will use throughout this dissertation. We will analyze these options since we not only need to find the value of the asset, but also the best time to exercise the option. The valuation of American options is more complicated than European options since at each time, we need to determine not only the option price, but if it has to be exercised or not. The value of this option can therefore be interpreted as a free boundary problem. At each time t, there is a particular asset price S which sets the boundary between where the option should be exercised and where it should not. We call this boundary value $S_f(t)$ and will reflect the optimal exercise price.

One of the most important concepts of option pricing is that of arbitrage. The maximum risk-free profit that anyone can have comes from government bonds or equivalent deposits. Arbitrage tells us that there is no opportunity to make additional profit without any risk. As mentioned before, the Black-Scholes model provides a fair theoretical estimate for European option prices but it can also be used in a modified version to value American options. The Black-Scholes formula is derived from the assumption of no arbitrage.

We will next define our problem as one for American options with dividends. In table 5.1 we establish the notation that we will continue using throughout the chapter.

5.2 American Options with Dividends

We now consider the American option with a dividend-paying asset. The value of the call $C(S,t)$ when exercise is not optimal satisfies the modified Black-Scholes equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S\frac{\partial C}{\partial S} - rC = 0 \tag{5.1}$$

where D_0 is the dividend yield. The payoff condition for a call value is

$$C(S,T) = \max(S - E, 0)$$

and because the American option can be exercised at any time, at every point in time the call value satisfies

$$C(S,t) \geq \max(S-E,0).$$

If there is an optimal exercise boundary $S = S_f(t)$, then we have

$$C(S_f(t),t) = S_f(t) - E$$

and

$$\frac{\partial C}{\partial S}(S_f(t),t) = 1.$$

Since the optimal exercise price does exist, the Black-Scholes equation 5.1 is only true while $C(S,t) > \max(S-E,0)$. However, since $\max(S-E,0)$ is not a solution to 5.1 it can be replaced by the inequality

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S\frac{\partial C}{\partial S} - rC \leq 0.$$

Where equality holds true only when $C(S,t) > \max(S-E,0)$ since if early exercise is optimal, then it is because the option would be less valuable if it were held than if it were exercised immediately.

By introducing the change of variables $S = Ee^x$ and $t = T - 2\tau/\sigma^2$, and replacing the function $u = u(x,\tau)$ related to the call value $C(S,t)$ by the equation $C(S,t) = S - E + Eu(x,\tau)$, the Black-Scholes equation 5.1 can be written in relation to the new variables x and τ and in terms of u as

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k_1 - 1)\frac{\partial u}{\partial x} - ku + f(x), \tag{5.2}$$

where $k = \dfrac{2r}{\sigma^2}$, $k_1 = 2(r - D_0)/\sigma^2$, and $f(x) = (k_1 - k)e^x + k$. The variable x is defined in the interval $(-\infty, \infty)$, and τ in $[0, \sigma^2 T/2]$. When $\tau = 0$, the function u satisfies $u(x,0) = \max(1 - e^x, 0)$. Moreover, the condition $C(0,t) = 0$ implies that $\lim_{x\to-\infty} u(x,\tau) = 1$ for any τ. With these new variables, the free boundary is therefore an unknown function x_f dependent on τ with the properties

$$u(x_f(\tau),\tau) = 0,$$

$$\frac{\partial u}{\partial x}(x_f(\tau),\tau) = 0,$$

and

$$x_f(0) = \ln(r/D_0).$$

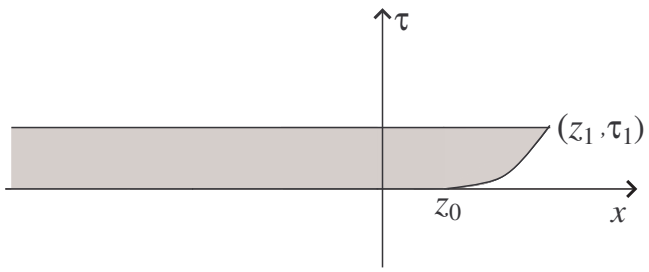


Figure 5.1: Domain of the Solution u in Equation (5.2).

5.3 Algorithm Based on the Crank-Nicolson Method

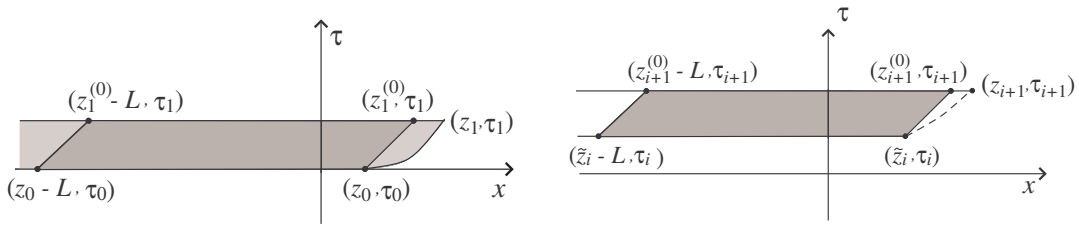
We will now describe how to approximate the free boundary and then use it to calculate the solution $u = u(x, \tau)$ of equation (5.2). Let $0 = \tau_0 < \tau_1 < \ldots < \tau_n = \sigma^2 T/2$ be a partition of $[0, \sigma^2 T/2]$. For τ in $(\tau_0, \tau_1]$, the domain where $u = u(x, \tau)$ satisfies equation (5.2) has the shape shown in figure 5.1.

The points $(z_i, \tau_i) = (x_f(\tau_i), \tau_i)$ on the free boundary curve are not known for any $i > 0$, but we do know the value of the point (z_0, τ_0) with $z_0 = \ln(r/D_0)$. Since $\lim_{x \to -\infty} u(x, \tau) = 1$ for any τ, let $L > 0$ be a constant large enough so that $u(z_i - L, \tau_i) \approx 1$ for all $i = 0, 1, \ldots n$. We construct a parallelogram P with vertices $(z_i - L, \tau_i)$, (z_i, τ_i), (z_{i+1}, τ_{i+1}) and $(z_{i+1} - L, \tau_{i+1})$ where the function u satisfies equation (5.2) together with the boundary conditions $u(z_{i+1}, \tau_{i+1}) = 0$, $u(z_{i+1} - L, \tau_{i+1}) \approx 1$ and $u(x, \tau_0) = \max(1 - e^x, 0)$ for all x in $[z_0 - L, z_0]$. Furthermore, we have the partial derivative on x satisfies $(\partial u/\partial x)(z_{i+1}, \tau_{i+1}) = 0$.

We will now describe how to approximate $z_1, z_2, \ldots z_m$. Let us define $\tilde{z}_0 = z_0$ a known value and suppose an approximation $\tilde{z}_i$ to z_i has already been calculated for any $i = 1, 2, \ldots m$. Let us also define $\tilde{u}(x, \tau_0) = \max(1 - e^x, 0)$ for x in $[\tilde{z}_0 - L, \tilde{z}_0]$ and also assume that $\tilde{u}(x, \tau_i)$ has been calculated for the same i and x in $[\tilde{z}_i - L, \tilde{z}_i]$. To compute an approximation $\tilde{z}_{i+1}$ to z_{i+1}, we construct a sequence of q numbers $z_{1+1}^{(1)}$, $z_{i+1}^{(2)}, \ldots z_{i+1}^{(q)}$ that approximate z_{i+1} by means of Newton's method starting with an initial approximation $z_{i+1}^{(0)}$ which we take as $\tilde{z}_i$ for $i > 0$, and for $i = 0$ we take $z_1^{(0)} = z_0 + 0.9034\sqrt{\tau_1}$[1]. We will finally take the final value $z_{i+1}^{(q)}$ as the approximation $\tilde{z}_{i+1}$ to the value of z_{i+1}.

In what follows, we will describe how to compute $\tilde{z}_{i+1}$ and how to then approximate $u(x, \tau_{i+1})$ for x in the interval $[\tilde{z}_{i+1} - L, \tilde{z}_{i+1}]$. We first construct a sequence of parallelograms $A_0, A_1, \ldots$ to help us approximate $\tilde{z}_{i+1}$. For the initial approximation $z_{i+1}^{(0)}$ to the value of z_{i+1}, we define the parallelogram A_0 with

[1]The value is based on the expansion of the free boundary for small values of τ.



(a) Approximation of z_1 (b) Approximation of z_{i+1}

Figure 5.2: Starting Parallelogram A_0

vertices $(\tilde{z}_i - L, \tau_i)$, $(\tilde{z}_i, \tau_i)$, $(z_{i+1}^{(0)}, \tau_{i+1})$ and $(z_{i+1}^{(0)} - L, \tau_{i+1})$ as shown in figure 5.2. Then A_0 is used to obtain a new approximation $z_{i+1}^{(1)}$ to the same value z_{i+1}, and this $z_{i+1}^{(1)}$ is consequently used to define the parallelogram A_1 with vertices $(\tilde{z}_i - L, \tau_i)$, $(\tilde{z}_i, \tau_i)$, $(z_{i+1}^{(1)}, \tau_{i+1})$ and $(z_{i+1}^{(1)} - L, \tau_{i+1})$. In general, for all p in $\mathbb{N} \cup \{0\}$ the parallelogram A_p has vertices at $(\tilde{z}_i - L, \tau_i)$, $(\tilde{z}_i, \tau_i)$, $(z_{i+1}^{(p)}, \tau_{i+1})$ and $(z_{i+1}^{(p)} - L, \tau_{i+1})$.

We will now describe how to obtain $z_{i+1}^{(p+1)}$ from $z_{i+1}^{(p)}$. Let $u_i^{(p)}$ be a solution of equation (5.2) on the parallelogram A_p with boundary conditions $u_i^{(p)}(z_{i+1}^{(p)} - L, \tau_{i+1}) = 1$, $u_i^{(p)}(z_{i+1}^{(p)}, \tau_{i+1}) = 0$ and initial data $u_i^{(p)}(x, \tau_i) = \tilde{u}(x, \tau_i)$ for $x \in [\tilde{z}_i - L, \tilde{z}_i]$. Once $u_i^{(p)}$ has been computed, $z_{i+1}^{(p+1)}$ is defined as

$$z_{i+1}^{(p+1)} = z_{i+1}^{(p)} - \frac{\frac{\partial u_i^{(p)}}{\partial x}(z_{i+1}^{(p)}, \tau_{i+1})}{\frac{\partial^2 u_i^{(p)}}{\partial x^2}(z_{i+1}^{(p)}, \tau_{i+1})}. \tag{5.3}$$

Since $u_i^{(p)}$ will be approximated numerically, the partial derivatives $(\partial u_i^{(p)}/\partial x)(z_{i+1}^{(p)}, \tau_{i+1})$ and $(\partial^2 u_i^{(p)}/\partial x^2)(z_{i+1}^{(p)}, \tau_{i+1})$ are approximated through finite differences.

We now turn to the problem of approximating the function $u_i^{(p)}$. By definition, the function $u_i^{(p)}$ satisfies the partial differential equation

$$\frac{\partial u_i^{(p)}}{\partial \tau} = \frac{\partial^2 u_i^{(p)}}{\partial x^2} + (k_1 - 1)\frac{\partial u_i^{(p)}}{\partial x} - k u_i^{(p)} + f(x) \tag{5.4}$$

on the parallelogram A_p. In addition, $u_i^{(p)}$ satisfies the boundary conditions $u_i^{(p)}(z_{i+1}^{(p)} - L, \tau_{i+1}) = 1$ and $u_i^{(p)}(z_{i+1}^{(p)}, \tau_{i+1}) = 0$ with initial values $u_i^{(p)}(x, \tau_i) = \tilde{u}(x, \tau_i)$ for x in $[\tilde{z}_i - L, \tilde{z}_i]$. We then introduce a new variable ω related to the

variables x and τ through the formula

$$\omega = x + s_{i,p}(\tau - \tau_i) + R_i$$

where $s_{i,p} = (\tilde{z}_i - z_{i+1}^{(p)})/(\tau_{i+1} - \tau_i)$ and $R_i = z_0 - \tilde{z}_i$. In the $\omega - \tau$ plane, a point (ω, τ) on parallelogram A_p belongs to a rectangle with vertices $(z_0 - L, \tau_i), (z_0, \tau_i), (z_0 - L, \tau_{i+1})$ and (z_0, τ_{i+1}).

We then define a function $\tilde{v}$ as

$$\tilde{v}(\omega, \tau) = \begin{cases} \max(1 - e^{\omega}, 0), & \tau = \tau_0 \\ \tilde{u}(\omega - R_i, \tau_i), & \tau = \tau_i \end{cases}$$

for $i \in \mathbb{N}$ with $\omega \in [z_0 - L, z_0]$. Let $v_i^{(p)}$ be a function given by

$$v_i^{(p)}(\omega, \tau) = u_i^{(p)}(x, \tau) = u_i^{(p)}(\omega - s_{i,p}(\tau - \tau_i) - R_i, \tau).$$

Since $u_i^{(p)}$ is a solution to equation (5.4), it follows that $v_i^{(p)}$ satisfies the constant coefficient differential equation

$$\frac{\partial v_i^{(p)}}{\partial \tau} = \frac{\partial^2 v_i^{(p)}}{\partial \omega^2} + (k_1 - 1 - s_{i,p})\frac{\partial v_i^{(p)}}{\partial \omega} - k v_i^{(p)} + g_{i,p}(\omega, \tau) \tag{5.5}$$

where $g_{i,p}(\omega, \tau) = f(\omega - s_{i,p}(\tau - \tau_i) - R_i)$.

Equation (5.5) together with the boundary conditions $v_i^{(p)}(z_0 - L, \tau_{i+1}) = 1$, $v_i^{(p)}(z_0, \tau_{i+1}) = 0$ and initial values $v_i^{(p)}(w, \tau_i) = \tilde{v}(w, \tau_i)$ for $w \in [z_0 - L, z_0]$ can be discretized using the Crank-Nicolson method obtaining $m + 1$ approximation values $v_0,\ v_1, \ldots v_m$ of the function $v_i^{(p)}$, with $v_i^{(p)}(w_j, \tau_{i+1}) \approx v_j$ and where $z_0 - L = w_0 < w_1 < w_2 < \ldots < w_m = z_0$ are $m + 1$ equally spaced numbers on $[z_0 - L, z_0]$. Therefore we have $u_i^{(p)}(x_j, \tau_{i+1}) \approx v_j$ where $x_j = w_j - s_{i,p}(\tau_{i+1} - \tau_i) - R_i$ for $j = 0, 1, \ldots m$ are $m + 1$ equally distributed numbers on the interval $[z_{i+1}^{(p)} - L, z_{i+1}^{(p)}]$. The values $v_0 = 1$ and $v_m = 0$ are given as boundary conditions while $v_1, \ldots v_{m-1}$ are calculated by solving a tridiagonal linear system of equations $A\vec{v} = \vec{b}$, where $A \in R^{(m-1)\times(m-1)}$ is defined as

$$A = \begin{pmatrix} \alpha_2 & \alpha_3 & & & & \\ \alpha_1 & \alpha_2 & \alpha_3 & & & \\ & \alpha_1 & \alpha_2 & \alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_1 & \alpha_2 & \alpha_3 \\ & & & & \alpha_1 & \alpha_2 \end{pmatrix}$$

with $\Delta\tau_{i+1} = \tau_{i+1} - \tau_i$ and $\Delta w = L/m$ which help us define

$$\alpha_1 = \frac{\Delta\tau_{i+1}}{2(\Delta w)^2} - (k_1 - 1 - s_{i,p})\frac{\Delta\tau_{i+1}}{4\Delta w}$$
$$\alpha_2 = -\frac{\Delta\tau_{i+1}}{(\Delta w)^2} - k\frac{\Delta\tau_{i+1}}{2} - 1$$
$$\alpha_3 = \frac{\Delta\tau_{i+1}}{2(\Delta w)^2} + (k_1 - 1 - s_{i,p})\frac{\Delta\tau_{i+1}}{4\Delta w}.$$

The vector $\vec{b} = (b_1, \dots b_{m-1})^\top$ on the right side of the equation is defined with

$$b_1 = \sum_{j=1}^{3} \beta_j \tilde{v}_{j-1} - \frac{\Delta\tau_1}{2}(g(w_1, \tau_i) + g_{i,p}(\omega_1, \tau_{i+1})) - \alpha_1$$
$$b_r = \sum_{j=1}^{3} \beta_j \tilde{v}_{r+j-2} - \frac{\Delta\tau_{i+1}}{2}(g_{i,p}(\omega_{r+j-1}, \tau_i) + g_{i,p}(\omega_{r+j-1}, \tau_{i+1}))$$
$$b_{m-1} = \sum_{j=1}^{3} \beta_j \tilde{v}_{m+j-3} - \frac{\Delta\tau_{i+1}}{2}(g_{i,p}(\omega_{m-1}, \tau_i) + g_{i,p}(\omega_{m-1}, \tau_{i+1})) - \alpha_3$$

for $r = 2, \dots m-2$ and where $\tilde{v}_j = \tilde{v}(w_j, \tau_i)$ and

$$\beta_1 = -\alpha_1$$
$$\beta_2 = -\alpha_2 - 2$$
$$\beta_3 = -\alpha_3.$$

Having approximated u_i^p, the value of $z_{i+1}^{(p+1)}$ is then calculated with equation (5.3) and with the use of finite differences. The process is done to calculate all values $z_{i+1}^{(1)}, \dots z_{i+1}^{(q)}$[2]. We then define $\tilde{z}_{i+1} = z_{i+1}^{(q)}$ as our approximation to z_{i+1} and we also define the function $\tilde{v}$ at τ_{i+1} as

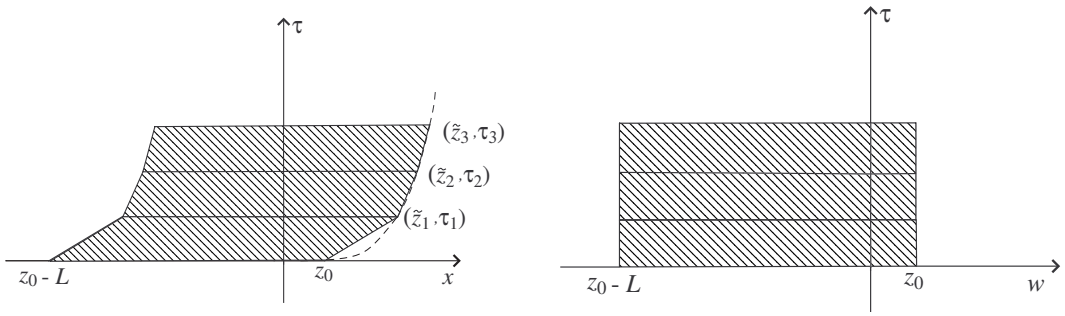
$$\tilde{v}(w, \tau_{i+1}) = v_i^{(q)}(w, \tau_{i+1}).$$

We finally set the values $\tilde{u}(x, \tau_{i+1}) = u_i^p(x, \tau_{i+1})$ for $x \in [\tilde{z}_{i+1} - L, \tilde{z}_{i+1}]$[3].

The final domain where the solution $\tilde{u}$ is approximated is shown in figure 5.3a, which is a union of parallelograms with the free boundary represented by a dashed

[2]We used the value $q = 11$ for numerical tests.

[3]The domain of x is actually restricted to a set of $m+1$ equally distributed points x_i such that $\tilde{z}_{i+1} - L = x_0 < x_1 < \ldots < x_m = \tilde{z}_{i+1}$.



(a) Domain of $\tilde{u}$

(b) Domain of $\tilde{v}$

Figure 5.3: Parallelogram Transformation

curve. The domain of function $\tilde{v}$ is a union of rectangles shown in figure 5.3b. The solution u of the Black-Scholes equation (5.2) is therefore approximated by $u(x_j, \tau_i) \approx \tilde{u}(x_j, \tau_i)$ for each $i = 1, 2, \ldots n$ and where $\tilde{z}_i - L = x_0 < x_1 < \ldots < x_m = \tilde{z}_i$, with $x_j = \tilde{z}_i - L + jL/m$ for $j = 0, 1, \ldots m$.

5.4 Algorithm Based on the Crank-Nicolson Method of Order Four

In this section we will construct a Crank-Nicolson method of order four and apply it to the equation of American options with dividends. As we saw in the previous section, the equation for American options with dividends can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c_1 \frac{\partial u}{\partial x} + c_2 u + f(x,t) \tag{5.6}$$

where $u = u(x,t)$, $x \in [a,b]$ and $t \in [0,T]$. We need to discretize the equation 5.6 to be able to move both in time and in space. When $t = 0$, we define u by $u(x,0) = g_1(x)$, for all $x \in [a,b]$. When $t = t_j$, the values of u in the boundary are known and we define them as $u(a,t_j) = \alpha_j$ and $u(b,t_j) = \beta_j$.

We shall now partition the domain in x and in t. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be the number of subintervals in x and in t respectively. Let us define $a = x_0 < x_1 < \ldots < x_n = b$ a regular partition of $[a,b]$ and $0 = t_0 < t_1 < \ldots < t_n = T$ a regular partition of the domain of t. The sizes of the subintervals are then $\Delta x = (b-a)/n$ and $\Delta t = T/m$. It then holds that $x_i = a + i\Delta x$ with $i = 0, 1, \ldots n$. Next we wish to take the first step in time and approximate the value of $u(x_i, t_1)$ for $i = 1, 2, \ldots n-1$ using the given values $u(x_i, t_0) = g_1(x_i)$ for $i = 0, 1, \ldots n$,

$u(x_0, t_1) = \alpha_1$ and $u(x_n, t_1) = \beta_1$. Latter steps in time are calculated analogously by approximating $u(x_i, t_j)$ with $i = 0, 1, \ldots n$ for every $j = 1, 2, \ldots m$ using the values of $u(x_i, t_{j-1})$ for $i = 0, 1, \ldots n$, $u(x_0, t_{j-1}) = \alpha_{j-1}$ and $u(x_n, t_{j-1}) = \beta_{j-1}$.

To formulate the equations of order four in space x, we use centered differences except for the first and last points. For the first point x_1, we will use the known value x_0 and the three following points x_2, x_3 and x_4. For the point x_{n-1}, we will be doing the analogous with the three preceding points and the last one x_n. With this procedure, we will use the following finite difference formulas of order four to approximate $\frac{\partial^2 u}{\partial x^2}(x_i, t)$ and $\frac{\partial u}{\partial x}(x_i, t)$ respectively. We have three cases for the formulas.

For the first step in space x_1 we have

$$\frac{\partial^2 u}{\partial x^2}(x_1, t) \approx \frac{1}{(\Delta x)^2} \sum_{k=0}^{5} a_{k+1} u(x_k, t),$$

for the middle points in time x_i with $1 < i < n - 1$ we approximate

$$\frac{\partial^2 u}{\partial x^2}(x_i, t) \approx \frac{1}{(\Delta x)^2} \sum_{k=1}^{5} b_k u(x_{i+k-3}),$$

and finally for x_{n-1} we get

$$\frac{\partial^2 u}{\partial x^2}(x_{n-1}, t) \approx \frac{1}{(\Delta x)^2} \sum_{k=0}^{5} a_{k+1} u(x_{n-k}, t)$$

where the constants a_k and b_k are the k-th elements of the vectors

$$a = \frac{1}{12} \begin{pmatrix} 10 \\ -15 \\ -4 \\ 14 \\ -6 \\ 1 \end{pmatrix}$$

and

$$b = \frac{1}{12} \begin{pmatrix} -1 \\ 16 \\ -30 \\ 16 \\ -1 \end{pmatrix}$$

respectively.

And now, for the case of $\frac{\partial u}{\partial x}(x_i, t)$ we get the following approximations of order four for the same three cases. For x_1 we have

$$\frac{\partial u}{\partial x}(x_1, t) \approx \frac{1}{\Delta x} \sum_{k=0}^{4} d_{k+1} u(x_k, t)$$

then for $1 < i < n-1$ we have

$$\frac{\partial u}{\partial x}(x_i, t) \approx \frac{1}{\Delta x} \sum_{k=1}^{5} e_k u(x_{k+i-3}, t)$$

and finally for x_{n-1} we have

$$\frac{\partial u}{\partial x}(x_{n-1}, t) \approx \frac{-1}{\Delta x} \sum_{k=0}^{4} d_{k+1} u(x_{n-k}, t)$$

where

$$d = \frac{1}{12} \begin{pmatrix} -3 \\ -10 \\ 18 \\ -6 \\ 1 \\ 0 \end{pmatrix}$$

and

$$e = \frac{1}{12} \begin{pmatrix} 1 \\ -8 \\ 0 \\ 8 \\ -1 \end{pmatrix}.$$

Making $x = x_i$ and integrating both sides of the equation with respect to t, for t in $[t_0, t_1]$ we get

$$\begin{aligned} &u(x_i, t_1) - u(x_i, t_0) \\ &\qquad = \int_{t_0}^{t_1} \left[\frac{\partial^2 u}{\partial x^2}(x_i, t) + c_1 \frac{\partial u}{\partial x}(x_i, t) + c_2 u(x_i, t) + f(x_i, t) \right] dt. \end{aligned}$$

Approximating the integral with the trapezoid rule,

$$\int_{t_0}^{t_1} g(t) dt \approx \frac{\Delta t}{2} \left[g(t_1) + g(t_0) \right]$$

and we get the approximation

$$\begin{aligned} u(x_i,t_1)-u(x_i,t_0) \approx & \frac{\Delta t}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i,t_0)+\frac{\partial^2 u}{\partial x^2}(x_i,t_1)\right] \\ & + c_1\frac{\Delta t}{2}\left[\frac{\partial u}{\partial x}(x_i,t_0)+\frac{\partial u}{\partial x}(x_i,t_1)\right] \\ & + c_2\frac{\Delta t}{2}\left[u(x_i,t_0)+u(x_i,t_1)\right] \\ & + \frac{\Delta t}{2}\left[f(x_i,t_0)+f(x_i,t_1)\right] \end{aligned}$$

Taking into consideration that the approximation formulas are different for the cases $i=1$, $i=n-1$ and $1<i<n-1$, we obtain an approximation of the form $A\tilde{u}\approx l$, where A is an $n-1$ squared matrix, l is a vector in $\mathbb{R}^{n-1}$, $\tilde{u}\in\mathbb{R}^{n-1}$ is the vector defined as $(u(x_1,t_1),u(x_2,t_1),\dots u(x_{n-1},t_1))^\top$. The sparse matrix A has its non-zero entries defined as follows.

$$A=\begin{pmatrix} z_2 & z_3 & z_4 & z_5 & z_6 & & & \\ k_2 & k_3 & k_4 & k_5 & & & & \\ k_1 & k_2 & k_3 & k_4 & k_5 & & & \\ & k_1 & k_2 & k_3 & k_4 & k_5 & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & k_1 & k_2 & k_3 & k_4 & k_5 \\ & & & & k_1 & k_2 & k_3 & k_4 \\ & & & \tilde{z}_6 & \tilde{z}_5 & \tilde{z}_4 & \tilde{z}_3 & \tilde{z}_2 \end{pmatrix}$$

where

$$\begin{aligned} z_i &= a_i\frac{\Delta t}{2(\Delta x)^2}+c_1d_i\frac{\Delta t}{2\Delta x}-\left[1-c_2\frac{\Delta t}{2}\right]\mathbb{I}_{i=2} \\ \tilde{z}_i &= a_i\frac{\Delta t}{2(\Delta x)^2}-c_1d_i\frac{\Delta t}{2\Delta x}-\left[1-c_2\frac{\Delta t}{2}\right]\mathbb{I}_{i=2} \\ k_j &= b_j\frac{\Delta t}{2(\Delta x)^2}+c_1e_j\frac{\Delta t}{2\Delta x}-\left[1-c_2\frac{\Delta t}{2}\right]\mathbb{I}_{j=3} \end{aligned}$$

for $i=1,2,\dots 6$ and $j=1,2,\dots 5$.

The right side of the system $A\tilde{u}=l$ takes the form $l=\tilde{l}-\omega$, where for

$j = 3, 4, \ldots n-3$ we have

$$\begin{aligned}
\tilde{l}_1 =& - z_1(\alpha + u_0(x_0)) - (z_2+2)u_0(x_1) - z_3u_0(x_2) \\
& - z_4u_0(x_3) - z_5u_0(x_4) - z_6u_0(x_5) \\
\tilde{l}_2 =& - k_1(\alpha + u_0(x_0)) - k_2u_0(x_1) - (k_3+2)u_0(x_2) \\
& - k_4u_0(x_3) - k_5u_0(x_4) \\
\tilde{l}_j =& - k_1u_0(x_{j-2}) - k_2u_0(x_{j-1}) - (k_3+2)u_0(x_i) \\
& - k_4u_0(x_{j+1}) - k_5u_0(x_{j+2}) \\
\tilde{l}_{n-2} =& - k_1u_0(x_{n-4}) - k_2u_0(x_{n-3}) - (k_3+2)u_0(x_{n-2}) \\
& - k_4u_0(x_{n-1}) - k_5(u_0(x_n) + \beta) \\
\tilde{l}_{n-1} =& - \tilde{z}_6u_0(x_{n-5}) - \tilde{z}_5u_0(x_{n-4}) - \tilde{z}_4u_0(x_{n-3}) - \tilde{z}_3u_0(x_{n-2}) \\
& - (\tilde{z}_2+2)u_0(x_{n-1}) - \tilde{z}_1(u_0(x_n) + \beta)
\end{aligned}$$

Furthermore, for $j = 1, 2, \ldots n-1$ we define

$$\omega_j = \frac{\Delta t}{2}\left[f(x_i, t_0) + f(x_i, t_1)\right]$$

so that we can now solve the system of equations. We finally then use the parallelogram transformation described in the previous section to be able to move sideways and approximate the free boundary.

5.5 Numerical Results

In this section we provide numerical tests of the algorithm to approximate the free boundary for several values of time to expiry. We use 16 initial steps Rannacher timesteppping for all runs. For each example, we report results for both the Crank-Nicolson method and the Crank-Nicolson method of order four. The algorithms calculate the free boundary x_f but we will report the value of $S_f = S(x_f) = E\exp(x_f)$.

Since we do not have an analytical expression of the exact solution, the estimated relative error of approximation ε was calculated with respect to a separate approximation of the solution obtained with larger values of time steps, more mesh points in space, and a larger interval for the space variable.

Time to Expiry	Crank-Nicolson		Crank-Nicolson 4	
	S_f	ε	S_f	ε
1 year	2.8095181	6.4×10^{-6}	2.8095104	3.7×10^{-6}
6 months	2.4420121	1.2×10^{-6}	2.4420211	2.5×10^{-6}
3 months	2.1113699	3.1×10^{-5}	2.1114235	6.1×10^{-6}

Table 5.2: Option Example One Values

5.5.1 Option Example One

In this example we use $\sigma = 0.8$, $r = 0.25$, $D_0 = 0.2$ and $E = 1$. Table 5.2 shows the resulting free boundary value S_f and relative error for three different times to expiry. Note that at one year to expiry we have $T = 1$ which translates to $\tau = 0.32$. We use 800 time mesh points to calculate the free boundary at time to expiry of one year, 400 for six months, and 200 for three months. The left side of the table shows results for Crank-Nicolson of order two in space. The values tabulated were calculated using 2,000 points to discretize the x variable on the interval $[-15, x_0]$, where $x_0 = \ln(r/D_0) \approx 0.22314355$. The right hand side of table 5.2 shows the results for the method of order four. For this, we use 1,000 steps in space for x in $[-15, x_0]$.

For the approximation of the relative error ε, we compare the results with another approximation that uses 60,000 steps in space for the interval $[-25, x_0]$ and 3,200 steps in time.

5.5.2 Option Example Two

In this example we use $\sigma = 0.2$, $r = 0.1$, $D_0 = 0.05$, and $E = 1$. The free boundary is calculated for the same time to expiry periods as in the previous example but now at one year to expiry we translate $T = 1$ to $\tau = 0.02$. We used 200 time steps to calculate the free boundary at one year to expiry, 100 time steps for 6 months, and 50 time steps for 3 months. The values tabulated for the left side of table 5.3 were calculated using 2,000 points to discretize the x variable on the interval $[-15, x_0]$, where $x_0 = \ln(r/D_0) \approx 0.6931471805$. On the right side of the table we have the results for the fourth order method which uses 1,000 steps on x.

For the approximation of the relative error ε, we compare the results with another approximation that uses 60,000 steps in space for the interval $[-25, x_0]$ and 1,000 steps in time.

The following tables show the relative errors calculated for iterations two through five of Newton's method. Tables 5.4 and 5.5 are of the first example

Time to Expiry	Crank-Nicolson		Crank-Nicolson 4	
	S_f	ε	S_f	ε
1 year	2.2377065	2.9×10^{-5}	2.23764119	2.2×10^{-7}
6 months	2.1725228	3.8×10^{-5}	2.17243864	3.5×10^{-7}
3 months	2.1240261	5.3×10^{-5}	2.12391171	1.3×10^{-6}

Table 5.3: Option Example Two Values

Iteration	Crank-Nicolson	Rannacher	Crank-Nicolson 4
2	9.7×10^{-1}	9.6×10^{-1}	9.9×10^{-1}
3	3.7×10^{-3}	5.0×10^{-3}	3.8×10^{-3}
4	4.9×10^{-5}	3.2×10^{-4}	5.1×10^{-5}
5	1.2×10^{-5}	6.5×10^{-6}	4.9×10^{-6}

Table 5.4: Option Example One Relative Errors at One Year

for one year and three months to expiry respectively. Table 5.6 corresponds to the second example for one year to expiry. We can see how the realative errors calculated with five iterations are approximately the same to the ones calculated with the original 11 iterations with a few exceptions. The Rannacher method reported in these tables uses one starting implicit Euler step. In some cases the Rannacher method has a slightly smaller relative error than Crank-Nicolson but the inverse is true in other cases. In the shown calculations the Crank-Nicolson method of order four used 1,000 points in the space variable while the other two methods used 2,000 steps.

Iteration	Crank-Nicolson	Rannacher	Crank-Nicolson 4
2	6.6×10^{-1}	5.6×10^{-1}	7.8×10^{-1}
3	6.2×10^{-3}	8.3×10^{-3}	6.4×10^{-3}
4	1.2×10^{-4}	3.6×10^{-4}	7.2×10^{-5}
5	2.2×10^{-5}	2.0×10^{-4}	2.1×10^{-5}

Table 5.5: Option Example One Relative Errors at Three Months

Iteration	Crank-Nicolson	Rannacher	Crank-Nicolson 4
2	2.4×10^{-4}	2.5×10^{-4}	3.3×10^{-4}
3	2.5×10^{-5}	1.3×10^{-5}	1.1×10^{-6}
4	2.8×10^{-5}	1.9×10^{-5}	2.4×10^{-7}
5	2.8×10^{-5}	1.9×10^{-5}	2.1×10^{-7}

Table 5.6: Option Example Two Relative Errors at One Year

6 Conclusions

In this dissertation we have worked different methods that approximate the solution to the heat equation. We have tested the familiar implicit Euler and Crank-Nicolson methods as well as described how to construct methods using Simpson 3/8 and Crank-Nicolson of order four. The Rannacher method uses additional Euler startup steps for Crank-Nicolson [10].

In Chapter 4, we tested the methods with simple examples. We saw how with smooth initial values, we can take advantage of the higher order methods and use only a few steps in time or space to approximate the solution. With non smooth initial values, however, we get a dent that spikes the error for initial steps with only the implicit Euler method as an exception. The dent in the approximations smoothes out over time so if we take several steps in time, both Crank-Nicolson methods or Simpson 3/8 give a good estimation. For few time steps, we can use implicit Euler to smooth out the first step and then continue on with Crank-Nicolson, the Rannacher timestepping. This method has been analyzed in [6] for Black-Scholes applications. For the more radical jump discontinuities in initial values, all methods behave approximately the same in terms of the relative error.

In Chapter 5 we explained some option theory concepts and how the Black-Scholes equation can be modified for American options with dividend paying assets and solved as the heat equation. We modify the Crank-Nicolson method and the Crank-Nicolson of order four method specifically for the value of American options with dividends. We used a parallelogram transformation for the data and approximated the free boundary with Newton's method. For each of these Crank-Nicolson methods, we finally give two examples from different American options that show promising results.

In conclusion, this dissertation has explored different methods of approximation to the solution of heat equation which all have their own advantages and

disadvantages. The heat equation has many applications and depending on the properties of the problem, we can use an adequate method which can add to accuracy and save computation time.

All code is available on GitHub `https://github.com/acpg`.

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