POLYTROPIC MODELS OF WHITE DWARFS UNC PHYS 331 PROJECT

Erin Conn Matthew Hurley

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Polytropes White Dwarfs



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WHAT ARE POLYTROPES?

Solutions to...

The Lane-Emden Equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n(\xi)$$

A dimensionless, 2nd order nonlinear differential equation relating the pressure of a spherically-symmetric gas distribution to the radius.

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- ► Easier to solve than full equations of stellar structure
- ▶ Require less computational effort some analytic solutions even exist!

DEFINITIONS

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Poisson's equation Relates a force density function to a potential field

$$\nabla^2 \Phi = f$$

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$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r)$$



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DERIVATION 2: WORKING TOWARDS A DIMENSIONLESS FORM

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$$\frac{(n+1)P_c}{4\pi G\rho_c^2} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta(r)}{dr} \right) = -\theta^n(r)$$

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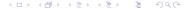
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PLACEHOLDER

Something about degenerate matter and how polytropic models suit it?

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PLACEHOLDER

Relativistic vs. Non-Relativistic?



METHODS



$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi} \frac{d\theta}{d\xi} - \theta^n(\xi)$$

Translating to a system of 1st order **EQUATIONS**

$$\begin{cases} \phi = \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} = -\frac{2}{\xi}\phi - \theta^n \end{cases}$$

BOUNDARY VALUES

Obtained from central density and hydrostatic equation

$$\xi = 0 \\
\theta = 1 \\
\frac{d\theta}{d\xi} = 0$$

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$$\phi' = -\left(\frac{2}{\xi_0}\phi\right) - \theta_0^n$$

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- ▶ Taylor expand at $\xi = 0$ and take limit as $\xi \to 0$: $\phi' \to -\frac{1}{3}$
- Offset the starting point: $0 < \xi_0 \ll 1$



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