PHYS 331 Computational Project Proposal: Stellar Modeling: Polytropes

Erin Conn   Project Partner: Matthew Hurley

1 Background

Understanding stellar mechanics requires the use of mathematical models of the internal structure of stars. We understand stars to be nearly spherical collections of hot, compressed gas (fluid), so using what we know of stellar composition and Newtonian and fluid mechanics, we can create models for the pressure, density, temperature, and luminosity of stars. One of the simpler, but powerful, such models is the polytrope.[2]

In astrophysics, *polytropes* are solutions to the Lane-Emden equation[3] for the gravitational potential of a spherically symmetric distribution of Newtonian, polytropic[1] fluid under its own gravitation[4]:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi) \tag{1}$$

The Lane-Emden equation is a dimensionless form of Poisson's equation for the divergence of a field:

$$\nabla^2\Phi = f \tag{2}$$

where Φ is a scalar potential function and f is a position-dependent density function. The Poisson equation is familiar in electrostatics[1] where it is commonly used to find the electric potential for a charge distribution.

In the derivation in the next subsection, we get Poisson's equation in this form:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r) \tag{3}$$

where ρ is the density of a fluid, P is the pressure, G is Newton's gravitational constant, and r is the radial distance from the center of the distribution.

This equation does not explicitly depend on the thermal structure and properties of the distribution (star), but the temperature dependence is embedded in the derivation of this equation - eq. 5 below depends on temperature through an equation of state $P(\rho, T, X_i)$, for example.

Therefore in general Poisson's equation (eq. 3) cannot be solved independently of other energy-transport and conservation equations that are coupled through the thermal dependence. However, in a special case of the equation of state for P in which the pressure is independent of temperature,

[1] obeys the relation $PV^n = C$, where P = pressure, V = volume, n = polytropic index, C = constant

the mechanical and thermal structures of the star can be decoupled. This equation of state is usually written[4, p.177]:

$$P(r) = K\rho_c^{\frac{n+1}{n}}\theta^{n+1} \tag{4}$$

As we shall see in the following derivation, using this state equation and a little more mathematical manipulation yields the Lane-Emden equation(eq. 1) above.

1.1 Derivation of the Lane-Emden Equation[4, pp.176–179]

The Lane-Emden equation can be derived multiple ways; one way is from the equations for hydrostatic equilibrium and mass conservation:

$$\frac{dP(r)}{dr} = -\frac{\rho(r)GM(r)}{r^2} \tag{5}$$

$$dM(r) = 4\pi r^2\rho(r)dr \rightarrow \frac{dM(r)}{dr} = 4\pi r^2\rho(r) \tag{6}$$

where $P(r)$ is the gas pressure as a function of radial distance from the center of the distribution, $\rho(r)$ is the gas density, G is Newton's gravitational constant, and $M(r)$ is the mass enclosed within a sphere of radius r.

These equations can be related by multiplying eq. 5 by r^2/ρ:

$$\frac{r^2}{\rho(r)}\frac{dP(r)}{dr} = -\frac{r^2}{\rho(r)}\frac{\rho(r)GM(R)}{r^2}$$

then differentiating with respect to r:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\frac{dM(r)}{dr}$$

Substituting in eq. 6 we obtain Poisson's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r) \tag{7}$$

Now, using the polytropic state equation:

$$P = K\rho^{\frac{n+1}{n}} \tag{8}$$

where n is called the *polytropic index* and K is a constant, and defining a dimensionless function $\theta(r)$:

$$\rho(r) = \rho_c\theta^n r \tag{9}$$

where ρ_c is the central density of the star, we can rewrite the pressure as a function of $\theta(r)$:

$$P(r) = K\rho_c^{\frac{n+1}{n}}\theta^{n+1}(r) = P_c\theta^{n+1}(r)$$

where $P_c = K\rho_c^{\frac{n+1}{n}}$ is the central pressure of the star. Substituting this into eq. 3:

$$K\rho_c^{\frac{n+1}{n}} \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n(r)}\frac{d\theta^{n+1}(r)}{dr}\right) = -4\pi G\rho_c\theta^n(r)$$

This can be simplified a bit by realizing that $\frac{d\theta^{n+1}(r)}{dr} = (n+1)\theta^n(r)\frac{d\theta(r)}{dr}$:

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r) \tag{10}$$

Since we defined $\theta(r)$ as a dimensionless function, this equation requires that $\frac{(n+1)P_c}{4\pi G\rho_c^2}$ has the dimension of length squared. For further simplification, we can define a new variable α that depends on the polytropic index n:

$$\alpha^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2} \tag{11}$$

and a new dimensionless radius ξ:

$$\xi = \frac{r}{\alpha} \tag{12}$$

Substituting this ξ into eq. 10 we finally obtain the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi) \tag{13}$$

which can be rearranged into a 2^{nd} *order homogeneous differential equation* for $\theta(\xi)$:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0 \tag{14}$$

This equation is only linear (and analytically solvable) for the case where $n = 0$ or $n = 1$, (an analytic solution exists for $n = 5$ as well, using some clever trickery), so we will be using numerical methods to find solutions to this equation.

2 Usefulness of the model

While it appears that the polytropic model sacrifices a great deal of information in the name of simplicity since no dependency on the star's thermal properties is assumed, these models have uses in interpreting stellar structures and are remarkably accurate for stars in hydrostatic equilibrium, such as main-sequence stars and white dwarfs, given their simplicity[2, p.332].

Carl Hansen *et al* provide detailed examples of such applications in chapter 7 of *Stellar Interiors*[2] for the historically interesting Eddington standard model, and for approximations of white dwarfs, leading us to our proposal.

3 Proposal

We propose to use numerical methods covered in class to find solutions to the Lane-Emden equation˜refeq:laneemden to create models of white dwarf stars and compare them with observed data from known dwarfs.

White dwarfs are particularly good candidates for polytropic modeling because they are composed of degenerate electrons[2]. It is natural, and not wholly inaccurate, to assume they are composed of a completely degenerate[3] gas, which implies an equation of state[2, pp.163-166]:

$$P = K\rho^{\gamma} \tag{15}$$

where $\gamma = 5/3$ for a non-relativistic gas and $\gamma = 4/3$ for a completely relativistic gas.

It is trivial to show that γ can be expressed in the form $\dfrac{n+1}{n}$ and we obtain $n = 1.5$ and $n = 3$ polytropes for the non-relativistic and relativistic cases respectively.

Hansen outlines several methods for solving the Lane-Emden equation numerically, two of which use methods we have covered in class[2, pp.338-351]: Fourth-order Runge-Kutta, which requires recasting the problem as a set of two first-order differential equations, or Newton-Raphson, which can be applied after linearizing the differential equation on a lattice. In fact, any of our techniques for solving systems of linear equations can be applied after linearizing the equations.

References

[1] D. J. Griffiths. *Introduction to Electrodynamics.* Pearson Education Inc., Upper Saddle River, NJ 07458, USA, 4th edition, 2013.

[2] C. J. Hansen, S. D. Kawaler, and V. Trimble. *Stellar interiors: physical principles, structure, and evolution.* Springer-Verlag, New York, 2nd edition, 2004.

[3] J. H. Lane. On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment. *The American Journal of Science and Arts*, 50:57–74, 1870.

[4] F. LeBlanc. *An introduction to stellar astrophysics.* John Wiley & Sons Ltd, The Atrium, Southerngate, Chichester, West Sussex, PO19 8SQ, United Kingdom, 1st edition, 2010.

[2] all in the same energy state

[3] behaves as if the temperature is 0K