Polytropic Models of White Dwarfs

UNC PHYS 331 Project

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Table of Contents

Table of Contents

What are polytropes?

Solutions to...

The Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n(\xi)$$

A dimensionless, 2nd order nonlinear differential equation relating the pressure of a spherically-symmetric gas distribution to the radius.

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- Easier to solve than full equations of stellar structure
- Require less computational effort - some analytic solutions even exist!

Definitions

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Poisson's equation Relates a force density function to a potential field

$$\nabla^2\Phi = f$$

Derivation 1: Poisson Equation

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$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r)$$

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Table of Contents

Placeholder

Something about degenerate matter and how polytropic models suit it?

Placeholder

Relativistic vs. Non-Relativistic?

Table of Contents

Alternate form of the Lane-Emden Equation

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n(\xi)$$

Translating to a system of 1st order equations

$$\begin{cases} \phi &= \frac{d\theta}{d\xi} \\ \frac{d\phi}{d\xi} &= -\frac{2}{\xi}\phi - \theta^n \end{cases}$$

Boundary Values

Obtained from central density and hydrostatic equation

$$\begin{aligned} \xi &= 0 \\ \theta &= 1 \\ \frac{d\theta}{d\xi} &= 0 \end{aligned}$$

Runge-Kutta Solution

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Singularity at $\xi_0 = 0$:

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- Offset the starting point: $0 < \xi_0 \ll 1$

Table of Contents

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Table of Contents

Questions?