Modeling Paul's Principle

1 Model Overview

We implement a continuous-time Hidden Markov Model (HMM) to test Paul's Principle: verbs with higher irregularity (both vowel and consonant alternations) regularize slower than those with lower irregularity. The model has:

- **Hidden states**: 4 states representing presence/absence of alternations: (1,1), (1,0), (0,1), (0,0)
- **Time span**: Middle High German to Early New High German
- **Key hypothesis**: Transitions from (1,1) $\rightarrow$ other states occur slower than from (1,0)/(0,1) $\rightarrow$ (0,0)

2 Data Structure

- N_{verbs}: Number of verb forms
- N_{obs}: Total observations
- $N_{\text{states}} = 4$: Hidden states
- `form`: Verb form ID
- `time`: Observation time
- `obs_v`, `obs_c`: Observed vowel/consonant alternations (binary)
- `freq`: Verb frequency
- `dialect_id`: Dialect identifier
- `lemma_id`: Verb lemma identifier
- `principal_part_id`: Grammatical form (e.g., infinitive, past)

3 Irregularity Index Calculation

Precomputed irregularity index measures paradigm diversity:

$$\text{Irregularity} = \alpha \cdot \frac{|V|-1}{m-1} + (1-\alpha) \cdot \frac{|C|-1}{m-1}$$

- $|V|$, $|C|$: Distinct vowel/consonant patterns
- m: Observed principal parts
- $\alpha = 0.5$: Balancing weight

Forward-filled when data is missing. Stored in matrices:

- `irregularity_index`: $N_{\text{lemmas}} \times n_{\text{time\_points}}$
- `m_values`: Number of observed principal parts

4 Parameters

Transition dynamics

`log_lambda` $\in R^{4\times4}$	(Baseline log transition rates)
`beta_trans` $\in R^{4\times4}$	(Frequency effect)
$\gamma_{\text{irreg}} \in R$	(Paul's Principle effect)

Emission model

$\beta_v^{\text{true1}}, \beta_v^{\text{true0}} \in R$	(Vowel emission)
$\beta_c^{\text{true1}}, \beta_c^{\text{true0}} \in R$	(Consonant emission)
$\beta_{\text{dialects}} \in R^{n_{\text{dialects}}}$	(Dialect effects)

Paradigm interactions

`paradigm_influence` $\in R^{4\times4}$	(Cross-part influence)
$\sigma_{\text{adj}} \in R^+$	(Adjustment SD)
$\delta_l \in R^+$	(Lemma-specific adjustments)

5 Model Components

5.1 State-Space Encoding

States represent alternation patterns:

State	Vowel	Consonant
1	1	1
2	1	0
3	0	1
4	0	0

5.2 Adjusted Irregularity Index

Accounts for missing principal parts:

$$\text{true\_irregularity}_{l,t} = \sigma\left(\text{irregularity\_index}_{l,t} + \delta_l \cdot \frac{N_{\text{pp}} - m_{l,t}}{N_{\text{pp}}}\right)$$

where σ is logistic function, $N_{\text{pp}} = 4$ (principal parts), and $\delta_l \sim \mathcal{N}^+(0, \sigma_{\text{adj}})$.

TODO: I realize there probably is an error and sigmoid function should only be applied to the adjustment part, not the whole one.

5.3 Transition Rate Matrix

Instantaneous rate Q_{ij} from state i to j ($i \neq j$):

$$Q_{ij} = \exp\left(\underbrace{\texttt{log\_lambda}_{ij}}_{\text{baseline}} + \underbrace{\texttt{beta\_trans}_{ij} \cdot \text{freq}}_{\text{frequency}} - \underbrace{\gamma_{\text{irreg}} \cdot I_{\text{reg}}(i,j) \cdot \text{true\_irregularity}}_{\text{Paul's effect}} + \underbrace{\text{influence}_j}_{\text{paradigm}}\right)$$

- $I_{\text{reg}}(i,j) = 1$ if transition reduces alternations (regularization)
- Diagonal: $Q_{ii} = -\sum_{j \neq i} Q_{ij}$
- It probably makes sense to have a single log_lambda_{ij} for all transitions? Then, all the difference in transitions would be produced due to other variables

5.4 Paradigm Influence

Influence from other principal parts:

$$\text{influence}_j = \sum_{k \neq l} \texttt{paradigm\_influence}_{l,k} \cdot P(s_k = j \mid \mathbf{obs}_k)$$

where $P(s_k = j \mid \mathbf{obs}_k)$ is estimated from most recent observation of part k.

5.5 Emission Probabilities

Observation likelihood given state s:

$$P(\texttt{obs\_v} = 1 \mid s) = \sigma\left(\beta_v^{\text{state}} + \beta_{\text{dialects}}[d]\right)$$

$$\beta_v^{\text{state}} = \begin{cases} \beta_v^{\text{true1}} & \text{if } s \in \{1,2\} \\ \beta_v^{\text{true0}} & \text{otherwise} \end{cases}$$

(Similarly for consonants)

6 Prior Distributions

$$
\begin{aligned}
&\texttt{log\_lambda}, \texttt{beta\_trans} \sim \mathcal{N}(0,1)\\
&\beta_v^{\text{true}*}, \beta_c^{\text{true}*} \sim \mathcal{N}(0,1)\\
&\gamma_{\text{irreg}} \sim \mathcal{N}(0,1)\\
&\texttt{paradigm\_influence} \sim \mathcal{N}(0.5, 0.5)\\
&\sigma_{\text{adj}} \sim \text{Exponential}(2)\\
&\delta_l \sim \mathcal{N}^+(0, \sigma_{\text{adj}})\\
&\text{Initial state} \sim \text{Dirichlet}([3,2,2,1])
\end{aligned}
$$

7 Likelihood Computation

Using forward algorithm for each verb:

Algorithm 1 Forward Algorithm

```
Initialize log_forward_1 = log(initial_probs) + log P(obs_1 | s)
for t = 2 to T do
    Compute Q matrix at time t using current irregularity and paradigm influence
    P_Δt = exp(Q · Δt)
    log_forward_t = log(exp(log_forward_{t-1}) · P_Δt) + log P(obs_t | s)
end for
Marginal likelihood = log Σ exp(log_forward_T)
```

8 Testing Paul's Principle

The key test is the sign of γ_{irreg}:

- $\gamma_{\text{irreg}} > 0$: Supports Paul (irregularity slows regularization)
- $\gamma_{\text{irreg}} < 0$: Contradicts Paul

Directly output as `paradigm_effect` in generated quantities.