Problem I6: Fibonacci Sequence 2

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There are nine multiples of 4 in the first 50 Fibonacci numbers. These numbers are:

$$0, 8, 144, 2584, 46368, 832040, 14930352, 267914296, \text{ and } 4807526976.$$

About 33.333% of the first 10,000 Fibonacci numbers are even. When I looked at the indexes of the even numbers in the Fibonacci sequence, they were multiples of three, which means that every third number in the Fibonacci sequence is even (if we disregard the zeroth term, 0). This makes sense because the Fibonacci sequence is simply adding the two numbers before it. Consider the terms 1 and 1 in the Fibonacci sequence. Because they are odd, they will produce an even number, 2. Then, adding an odd number to an even number will produce an odd number for the next term, 3. For the next term, we are again adding an odd to an even, 2 to 3, which results in another odd number, 5. Finally, we add an odd to an odd, 3 to 5, resulting in an even number, 8. This pattern will continue throughout the sequence, and therefore every third number in the sequence will be even, which is why it makes sense that about 33% of all Fibonacci numbers will be even.

Theorem 1. Consider the Fibonacci sequence, with $f_0 = 0, f_1 = 1$, and $f_i = f_{i-2} + f_{i-1}$, where $i = 2, 3..., n$. Then f_{3k} (every third term) will be even.

Proof. We will prove Theorem 1 by mathematical induction.
First consider the base case, where $i = 1$. Then

$$\begin{aligned} f_{3i} &= f_3 \\ &= f_2 + f_1 \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

We know that 2 is even, so f_3 is even. Therefore, Theorem 1 holds true for $i = 1$.
Now assume that Theorem 1 is true for some $i = k$ so that $f_{3k} = 2m$, for some integer m.

Consider the term $f_{3(k+1)}$, which can be rewritten as

$$\begin{aligned} f_{3(k+1)} &= f_{3k+3} \\ &= f_{(3k+3)-2)} + f_{(3k+3)-1} \\ &= f_{3k+1} + f_{3k+2} \\ &= f_{3k+1} + [f_{(3k+2)-1} + f_{(3k+2)-2}] \\ &= f_{3k+1} + f_{3k+1} + f_{3k} \\ &= 2f_{3k+1} + f_{3k} \\ &= 2l + 2m, \text{ where } l = f_{3k+1} \in \mathbb{Z} \\ &= 2p, \text{ where } p = l + m. \end{aligned}$$

Because $f_{3(k+1)} = 2m$, and $m \in \mathbb{Z}$, $f_{3(k+1)}$ is even. So if Theorem 1 is true for $i = k$, it is also true for $i = k + 1$.
Therefore, by the Principle of Mathematical Induction, every third term of the Fibonacci sequence will be even. □

There definitely seems to be a pattern for the percentage of Fibonacci numbers that are multiples of m (for m other than 2), but it's not an obvious pattern. However, it is easy to see the relationship between the percentage of Fibonacci numbers that are multiples of m, and how often these numbers occur in the sequence. For example, 25% of Fibonacci numbers are multiples of three, and by looking at the index of the multiples in the sequence of numbers I created, I found that the multiples of three occur at every fourth number; so it makes sense that 25% of the Fibonacci numbers would end up being multiples of three.