N.1a: Collatz Conjecture

Amy Creel
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Conjecture 1: For all $n \in \mathbb{Z}$, with $n \geq 1$, the sequence starting with 2^n will have $n+1$ terms in its sequence after convergence.

Proof. Let $n \in \mathbb{Z}$. Define a sequence by $a_0 = 2^n$, and

$$a_{i+1} = \begin{cases} a_i/2 & \text{if } a_i \text{ is even} \\ 3a_i + 1 & \text{if } a_i \text{ is odd} \end{cases}$$

We will prove this theorem using mathematical induction.

We first consider the base case, where $n = 1$. Then the first term of our sequence, a_0, is:

$$\begin{aligned} a_0 &= 2^1 \\ &= 2. \end{aligned}$$

Because a_0 is even, the second term of our sequence is defined as $a_1 = a_0/2$, so we have:

$$\begin{aligned} a_1 &= a_0/2 \\ &= 2/2 \\ &= 1. \end{aligned}$$

We have reached one, so our sequence has converged. Our final sequence is $\{2, 1\}$, which contains 2 terms. Note that $n + 1 = 1 + 1 = 2$. Thus, the theorem holds when $n = 1$.

Now, assume that the theorem is true for some $k \in \mathbb{Z}$, so that the sequence with the initial term $a_0 = 2^k$ contains $k + 1$ terms after converging to one. Note that because 2 raised to any power greater than or equal to one is even, the next term in the sequence will be the previous term divided by two. Thus this sequence will be of the form:

$$\{2^k, 2^{k-1}, 2^{k-2}, ..., 2, 1\}$$

Now consider the sequence starting with initial term $a_0 = 2^{k+1}$. Note that the initial term is a power of two, so it will be even. Then the second term of our sequence is:

$$\begin{aligned} a_1 &= 2^{k+1}/2 \\ &= 2^{k+1-1} \\ &= 2^k. \end{aligned}$$

We can continue to calculate the terms of this sequence to end up with a sequence of the form:

$$\{2^{k+1}, 2^k, 2^{k-1}, 2^{k-2}, ..., 2, 1\}$$

Note that the terms following 2^k will be the same as the terms of the sequence with initial term $a_0 = 2^k$. We know that there are $k+1$ terms in the sequence with initial term $a_0 = 2^k$,

$$\{2^{k+1}, \underbrace{2^k, 2^{k-1}, 2^{k-2}, ..., 2, 1}_{k+1 \text{ terms}}\}$$

So we can see that the sequence 2^{k+1} will have $(k+1)+1$ terms. Therefore, if the theorem is true for k, it will also be true for $k+1$.

We have shown that the theorem holds true for $n = 1$, and that if the theorem is true for $n = k$, it will also be true for $n = k+1$. Therefore, by the Principal of Mathematical Induction, for all $n \in \mathbb{Z}$, with $n \geq 1$, the sequence starting with 2^n will have $n+1$ terms in its sequence after convergence. $\square$

Conjecture 2: For all $n \in \mathbb{Z}$, the terms in the sequence starting with 3^n will eventually reach 16 and then begin to converge, because $16 = 2^4$.

After making the conjecture above and looking at sequences created by having an initial term that is a power of three ($3^1, 3^2, 3^3$, etc.), there seemed to be a pattern where the sequence would eventually reach 16, which is a power of 2, and so once it reached 16, it converged to one the same way every time. For example,

initial term: $3^1 = 3 \to \{3, 10, 5, 16, 8, 4, 2, 1\}$
initial term: $3^2 = 9 \to \{9, 28, 16, 8, 4, 2, 1\}$
initial term: $3^3 = 27 \to \{27, 82, 41, 124, ..., 16, 8, 4, 2, 1\}$
(not all terms are listed because there are 112 terms in the sequence).

I think the best way to attempt to prove this conjecture would be through mathematical induction, similar to conjecture one. However, I'm not sure if that would work (since this conjecture isn't as straightforward as the first conjecture, and I actually haven't tried to prove it using this method), and if it did work, it could be an extremely tedious process. I also noticed that this pattern held for powers of 5, so it could be that this conjecture could be generalized to all odd numbers, or maybe all prime numbers, but I haven't investigated it any further so I can't make a conclusion about that.

Conjecture 3: For any $k \in \mathbb{N}$, and $n \in \mathbb{Z}$, where k is not a power of 2, at least 60% of the terms of the sequence of terms starting with initial term $a_0 = k^n$ will be even numbers.

After looking at numbers of terms in a sequence (conjecture one), and how a sequence could converge (conjecture two), I wanted to look at the actual contents of a converging sequence. I decided to look at the ratio of even and odd numbers in various sequences. I specifically looked at sequences whose initial terms were powers of three, powers of five, and powers of six, and I noticed that in each case, at least 60% of the terms in the sequence were odd numbers. Some examples are:

$$\text{initial term: } 3^1 = 3 \rightarrow 62.5\% \text{ even}, 37.5\% \text{ odd}$$
$$\text{initial term: } 3^2 = 9 \rightarrow 65\% \text{ even}, 35\% \text{ odd}$$
$$\text{initial term: } 3^3 = 27 \rightarrow 62.5\% \text{ even}, 37.5\% \text{ odd}$$
$$...$$
$$\text{initial term: } 5^1 = 5 \rightarrow \approx 66.6\% \text{ even}, \approx 33.3\% \text{ odd}$$
$$\text{initial term: } 5^2 = 25 \rightarrow \approx 66.6\% \text{ even}, \approx 33.3\% \text{ odd}$$
$$\text{initial term: } 5^3 = 125 \rightarrow \approx 63.3\% \text{ even}, \approx 36.7\% \text{ odd}$$
$$...$$
$$\text{initial term: } 6^1 = 6 \rightarrow \approx 66.6\% \text{ even}, \approx 33.3\% \text{ odd}$$
$$\text{initial term: } 6^2 = 36 \rightarrow \approx 68\% \text{ even}, \approx 32\% \text{ odd}$$
$$\text{initial term: } 6^3 = 216 \rightarrow \approx 63.5\% \text{ even}, \approx 36.5\% \text{ odd}$$
$$...$$

Whenever an odd number occurs in the sequence, the next number will be even, because for an odd number $2k+1$, the next term in the sequence will be $3(2k+1)+1 = 6k+4 = 2(3k+2)$, which is even. Then an even number divided by 2 could be odd, but could also be another even number. So if we have an odd number, we are guaranteed that the next term will be even, but if we have an even number, the next term could be either even or odd. Therefore, having more even numbers than odd numbers in these sequences makes sense.