Physical Mechanics HW1

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(1). For the two vectors:
$\vec{A} = \hat{i} + 2\hat{j} - \hat{k}$
$\vec{B} = -2\hat{i} + 3\hat{j} + \hat{k}$

(a) $\vec{A} - \vec{B}$ and $|\vec{A} - \vec{B}|$

$$\vec{A} - \vec{B} = (1-(-2))\hat{i} + (2-3)\hat{j} + (-1-1)\hat{k} = \mathbf{3\hat{i} - \hat{j} - 2\hat{k}}$$
$$|\vec{A} - \vec{B}| = \sqrt{(3)^2 + (-1)^2 + (-2)^2} = \sqrt{9+1+4} = \boldsymbol{\sqrt{14}}$$

(b) The component of $\vec{B}$ along $\vec{A}$
$$\hat{A} = \frac{A}{|A|} = \frac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}} \rightarrow Comp_{\vec{B}}\vec{A} = \vec{B}\cdot\hat{A} = \frac{-2+6-1}{\sqrt{6}} = \boldsymbol{\frac{\sqrt{6}}{2}}$$

(c) The angle between $\vec{A}$ and $\vec{B}$
$$\vec{A}\cdot\vec{B} = |A||B|cos(\theta) \rightarrow \theta = cos^{-1}(\frac{\vec{A}\cdot\vec{B}}{|A||B|}) = cos^{-1}(\frac{-2+6-1}{\sqrt{6}\sqrt{14}}) = cos^{-1}(\frac{\sqrt{21}}{14}) = \mathbf{70.89°}$$

(d) $\vec{A} \times \vec{B}$

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix} = \hat{i}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} - \hat{j}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} + \hat{k}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = (2+3)\hat{i} - (1-2)\hat{j} + (3+4)\hat{k}$$

$$= \mathbf{5\hat{i} + \hat{j} + 7\hat{k}}$$

(e) $(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B})$

$$(\vec{A} - \vec{B}) = 3\hat{i} - \hat{j} - 2\hat{k}$$
$$\vec{A} + \vec{B} = (1+(-2))\hat{i} + (2+3)\hat{j} + (-1+1)\hat{k} = -\hat{i} + 5\hat{j}$$

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ -1 & 5 & 0 \end{bmatrix} = \hat{i}\begin{bmatrix} -1 & -2 \\ 5 & 0 \end{bmatrix} - \hat{j}\begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix} + \hat{k}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = (0+7)\hat{i} - (0-2)\hat{j} + (15-1)\hat{k}$$

$$= \mathbf{7\hat{i} + 2\hat{j} + 14\hat{k}}$$

(2). A particle moves in a plane elliptical orbit described by the position vector:

$$\vec{r} = 2bsin(\omega t)\hat{i} + bcos(\omega t)\hat{j}$$

(a) Find $\vec{v}, \vec{a}$, and the particle's speed.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[2bsin(\omega t)\hat{i} + bcos(\omega t)\hat{j}] = \boldsymbol{2b\omega cos(\omega t)\hat{i} - b\omega sin(\omega t)\hat{j}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[2b\omega cos(\omega t)\hat{i} - b\omega sin(\omega t)\hat{j}] = \boldsymbol{-2b\omega^2 sin(\omega t)\hat{i} - b\omega^2 cos(\omega t)\hat{j}}$$

$$|\vec{v}| = v = \sqrt{(2b\omega cos(\omega t))^2 + (-b\omega sin(\omega t))^2} = \sqrt{4b^2\omega^2 cos^2(\omega t) + b^2\omega^2 sin^2(\omega t)}$$

$$v = \boldsymbol{b\omega\sqrt{4cos^2(\omega t) + sin^2(\omega t)}}$$

(b) What is the angle between $\vec{v}$ and $\vec{a}$ at $t = \frac{\pi}{2\omega}$

$$\vec{v}(t) = 2b\omega cos(\omega t)\hat{i} - b\omega sin(\omega t)\hat{j} \rightarrow \vec{v}(\frac{\pi}{2\omega}) = 2b\omega cos(\frac{\pi}{2})\hat{i} - b\omega sin(\frac{\pi}{2})\hat{j}$$

$$\vec{v}(\frac{\pi}{2\omega}) = \sqrt{2}b\omega\hat{i} - b\omega\frac{\sqrt{2}}{2}\hat{j} = b\omega 2\sqrt{2}((2\hat{i} - \hat{j})$$

$$|\vec{v}(\frac{\pi}{2\omega})| = \sqrt{8b^2\omega^2(4+1)} = 2b\omega\sqrt{10}$$

$$\vec{a}(t) = -2b\omega^2 sin(\omega t)\hat{i} - b\omega^2 cos(\omega t)\hat{j} \rightarrow \vec{a}(\frac{\pi}{2\omega}) = -2b\omega^2 sin(\frac{\pi}{2})\hat{i} - b\omega^2 cos(\frac{\pi}{2})\hat{j}$$

$$\vec{a}(\tfrac{\pi}{2\omega}) = -2b\omega^2\sqrt{2}(2\hat{i} + \hat{j})$$

$$|\vec{a}(\frac{\pi}{2\omega})| = \sqrt{8b^2\omega^4(4+1)} = 2b\omega^2\sqrt{10}$$

$$\vec{a} \cdot \vec{v} = (b\omega 4\sqrt{2} * -b\omega^2 4\sqrt{2}) + (-b\omega 2\sqrt{2} * -b\omega^2 2\sqrt{2}) = (-32b^2\omega^3 + 8b^2\omega^3) = -26b^2\omega^3$$

$$\vec{a} \cdot \vec{v} = |\vec{a}||\vec{v}|cos(\theta) \rightarrow \theta = cos^{-1}(\frac{\vec{a} \cdot \vec{v}}{|\vec{a}||\vec{v}|})$$

$$\theta = cos^{-1}(\frac{-26b^2\omega^3}{2b\omega^2\sqrt{10} * 2b\omega\sqrt{10}}) = cos^{-1}(\frac{-26}{40}) = \mathbf{130.542°}$$

(3). $\vec{X}$ is an unknown vector satisfying the following relations involving the known vectors $\vec{A}$ and $\vec{B}$ and the scalar ϕ. Express $\vec{X}$ in terms of $\vec{A}, \vec{B}, \phi, |A|$

$$\vec{A} \times \vec{X} = \vec{B}$$

$$\vec{A} \cdot \vec{X} = \phi$$

$$\vec{A} \times \vec{X} = \vec{B} \rightarrow \vec{A} \times (\vec{A} \times \vec{X}) = \vec{A} \times \vec{B}$$

Using

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{A} \times (\vec{A} \times \vec{X}) = \vec{A} \times \vec{B} = \vec{A}(\vec{A} \cdot \vec{X}) - \vec{X}(\vec{A} \cdot \vec{A}) = \vec{A} \times \vec{B}$$

$$\vec{A}\phi - \vec{X}A^2 = \vec{A} \times \vec{B}$$

$$\boldsymbol{\vec{X} = \frac{\vec{A}\phi - \vec{A} \times \vec{B}}{A^2}}$$

(4) Recover the cosine law of plane trigonometry (a.k.a. the law of cosines) by interpreting the product $(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B})$ and the expansion of the product.

$$\ell et$$
$$\vec{C} = \vec{A} - \vec{B}$$

$$\vec{A} - \vec{B} \cdot \vec{A} - \vec{B} = \vec{C} \cdot \vec{C} = C^2$$

$$C^2 = (\vec{A} - \vec{B}) \cdot \vec{C} = (\vec{A} \cdot \vec{C}) - (\vec{B} \cdot \vec{C})$$

$$C^2 = (\vec{A} \cdot (\vec{A} - \vec{B})) - (\vec{B} \cdot (\vec{A} - \vec{B})$$

$$C^2 = \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$

$$C^2 = A^2 + B^2 - (\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A})$$

$$C^2 = A^2 + B^2 - (ABcos(\theta) + BAcos(\theta))$$

$$\boldsymbol{C^2 = A^2 + B^2 - 2ABcos(\theta)}$$