Physical Mechanics HW2

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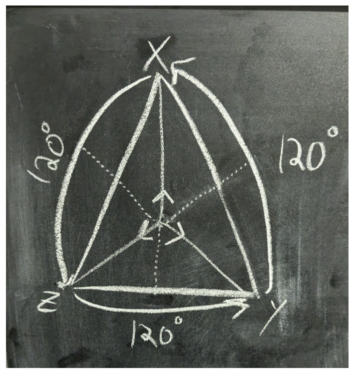
(1). Find the transformation matrix that rotates a rectangular coordinate system through an angle of $120°$ about an axis making equal angles with the original three coordinate axes



$$\vec{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} x & \rightarrow & z \\ y & \rightarrow & x \\ z & \rightarrow & y \end{matrix} \Longrightarrow \vec{A'} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using $\lambda_{ij} = Cos(X', X_j)$

$$\vec{A'} = \begin{pmatrix} Cos(x', x) & Cos(x', y) & Cos(x', z) \\ Cos(y', x) & Cos(y', y) & Cos(y', z) \\ Cos(z', x) & Cos(z', y) & Cos(z', z) \end{pmatrix} = \begin{pmatrix} Cos(90°) & Cos(90°) & Cos(0) \\ Cos(0) & Cos(90°) & Cos(90°) \\ Cos(90°) & Cos(0) & Cos(90°) \end{pmatrix}$$

$$\boldsymbol{\vec{A'}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

(2). Take $\vec{\lambda}$ to be a two-dimensional orthogonal transformation matrix. Show by direct expansion that $|\vec{\lambda}|^2 = 1$.

$$\vec{\lambda} = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{vmatrix} = (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) \rightarrow \vec{\lambda}^2 = (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})$$

$$\vec{\lambda}^2 = \lambda_{11}^2\lambda_{22}^2 + \lambda_{12}^2\lambda_{21}^2 - 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21}$$

Using

$$\Sigma_j^2 = \lambda_{ij}\lambda_{kj} = \delta_{ik} \rightarrow \lambda_{i1}\lambda_{k1} + \lambda_{i2}\lambda_{k2} = \delta_{ik}$$

$\mathscr{L}$et i=1,k=1

$$A = \lambda_{11}\lambda_{11} + \lambda_{12}\lambda_{12} = \delta_{11} = 1$$

$\mathscr{L}$et i=2,k=2

$$B = \lambda_{21}\lambda_{21} + \lambda_{22}\lambda_{22} = \delta_{22} = 1$$

$\mathscr{L}$et i=1,k=2

$$C = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = \delta_{12} = 0$$

$\mathscr{L}$et i=2,k=1

$$D = \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} = \delta_{21} = 0$$

From A,B,C,D,

$$AB = (\lambda_{11}^2 + \lambda_{12}^2)(\lambda_{21}^2 + \lambda_{22}^2) = (\lambda_{11}^2\lambda_{21}^2 + \lambda_{11}^2\lambda_{22}^2 + \lambda_{12}^2\lambda_{21}^2 + \lambda_{12}^2\lambda_{22}^2) = 1$$

$$CD = (\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22})(\lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12}) = (\lambda_{11}^2\lambda_{21}^2 + \lambda_{12}^2\lambda_{22}^2 + 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21}) = 0$$

$$AB - CD = (\lambda_{11}^2\lambda_{21}^2 + \lambda_{11}^2\lambda_{22}^2 + \lambda_{12}^2\lambda_{21}^2 + \lambda_{12}^2\lambda_{22}^2) - (\lambda_{11}^2\lambda_{21}^2 + \lambda_{12}^2\lambda_{22}^2 + 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21}) = 1 - 0$$
$$= \lambda_{11}^2\lambda_{22}^2 + \lambda_{12}^2\lambda_{21}^2 - 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21} = 1$$

$$\mathbf{AB - CD = |\tilde{\lambda}^2| = 1}$$

(3). Show that $\Sigma_i \lambda_{ij} \lambda_{ik} = \delta_{jk}$ can be obtained by using the requirement that the transformation leaves unchanged the length of a line segment.

$$L = \sqrt{\Sigma_i X_i^2}, L' = \sqrt{\Sigma_i X'^2_i}$$

$$L = L' \implies \Sigma_i X_i^2 = \Sigma_i X'^2_i = X'_i = \Sigma_j \lambda_{ij} X_j$$

$$\Sigma_i X_i^2 = \Sigma_i [(\Sigma_k \lambda_{ik} X_k)(\Sigma_l \lambda_{ik} X_l)] = \Sigma_i X_k X_l (\Sigma_{k,l} \lambda_{ik} \lambda_{il})$$

For $\Sigma_i X_i^2 = \Sigma_i X_k X_l (\Sigma_{k,l} \lambda_{ik} \lambda_{il})$ to hold true, $\Sigma_{k,l} \lambda_{ik} \lambda_{il} = \delta_{kl}$

Which implies that $\boldsymbol{\Sigma_i \lambda_{ij} \lambda_{ik} = \delta_{jk}}$

(4). Consider the following matrices:

$$\vec{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \vec{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \vec{C} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}$$

Find the following:

(a): $|\vec{\boldsymbol{A}}\vec{\boldsymbol{B}}|$

(b): $\vec{\boldsymbol{A}}\vec{\boldsymbol{C}}$

(c): $\vec{\boldsymbol{A}}\vec{\boldsymbol{B}}\vec{\boldsymbol{C}}$

(d): $\vec{\boldsymbol{A}}\vec{\boldsymbol{B}} - \vec{\boldsymbol{A}}^{\boldsymbol{T}}\vec{\boldsymbol{B}}^{\boldsymbol{T}}$

(a)

$$\vec{A}\vec{B} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2-1 & 1-2-1 & 4-3 \\ 1 & -3+1 & 6+3 \\ 4+1 & 2+1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix}$$

$$|\tilde{\mathbf{A}}\tilde{\mathbf{B}}| = 1(-6-27) - (-2)(3-45) + (3+10) = -33 - 84 + 13 = \mathbf{-104}$$

(b)

$$\vec{\boldsymbol{A}}\vec{\boldsymbol{C}} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2+8-1 & 1+6 \\ 12+1 & 9 \\ 4+1 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{9} & \mathbf{7} \\ \mathbf{13} & \mathbf{9} \\ \mathbf{5} & \mathbf{2} \end{pmatrix}$$

(c)

$$\vec{\boldsymbol{A}}\vec{\boldsymbol{B}}\vec{\boldsymbol{C}} = (\vec{A}\vec{B})\vec{C} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{-5} & \mathbf{-5} \\ \mathbf{3} & \mathbf{-5} \\ \mathbf{25} & \mathbf{14} \end{pmatrix}$$

(d)

$$\vec{A}^T = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \vec{B}^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \rightarrow \vec{B}^T\vec{A}^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \\ -2 & -2 & 3 \\ 1 & 9 & 3 \end{pmatrix}$$

$$\vec{\boldsymbol{A}}\vec{\boldsymbol{B}} - \vec{\boldsymbol{B}}^{\boldsymbol{T}}\vec{\boldsymbol{A}}^{\boldsymbol{T}} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 5 \\ -2 & -2 & 3 \\ 1 & 9 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{-4} \\ \mathbf{3} & \mathbf{0} & \mathbf{6} \\ \mathbf{4} & \mathbf{-6} & \mathbf{0} \end{pmatrix}$$

(5). Find the value(s) of α needed to make the following transformation matrix orthogonal.

$$\vec{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix} \rightarrow \vec{A}\vec{A}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & -\alpha & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2+\alpha^2 & \alpha^2-\alpha^2 \\ 0 & \alpha^2-\alpha^2 & \alpha^2+\alpha^2 \end{pmatrix}$$

$$\vec{A}\vec{A}^T = \vec{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2+\alpha^2 & \alpha^2-\alpha^2 \\ 0 & \alpha^2-\alpha^2 & \alpha^2+\alpha^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \alpha^2+\alpha^2 = 1, \alpha^2-\alpha^2 = 0 \boldsymbol{\alpha = \pm\frac{\sqrt{2}}{2}}$$

$$2\alpha^2 = 1 \implies \boldsymbol{\alpha = \pm\frac{\sqrt{2}}{2}}$$