Physical Mechanics Homework 5

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(1). Two masses $m_1 = 100g$ and $m_2 = 200g$ slide freely on a horizontal frictionless track and are connected by a spring whose force constant is $k = 0.5N/m$. Find the frequency of oscillatory motion for this system. (Hint: assume that the natural length of the spring is ℓ. How does that relate to the positions of the masses at any given time?)

$$\vec{F} = m\ddot{x} = -kx$$

$$m_1\ddot{x_1} = -k(x_1 - x_2 + \ell)$$

$$m_2\ddot{x_2} = -k(x_2 - x_1 - \ell) \rightarrow x_1 = \frac{m_2\ddot{x_2} + kx_2 - k\ell}{k}$$

Substituting x_1 into the first equation for x_1 gives

$$m_1\ddot{x_1} = -k(x_2 - x_2 + \ell - \ell) - m_2\ddot{x_2} \rightarrow m_1\ddot{x_1} = -m_2\ddot{x_2}$$

$$x_1 = \frac{m_2\ddot{x_2} + kx_2 - k\ell}{k} \implies \ddot{x_1} = \frac{d^2}{dt^2}\left(\frac{m_2\ddot{x_2} + kx_2 - k\ell}{k}\right)$$

Substituting $\ddot{x_1}$ into $m_1\ddot{x_1} = -m_2\ddot{x_2}$ gives

$$m_1\frac{d^2}{dt^2}\left(\frac{m_2\ddot{x_2} + kx_2 - k\ell}{k}\right) = -m_2\ddot{x_2}$$

$m_1, m_2, k,$ and ℓ are constants, thus the equation can be rewritten as

$$\frac{m_1m_2}{k}\frac{d^2}{dt^2}(\ddot{x_2}) + m_1\ddot{x_2} + m_2\ddot{x_2} = 0$$

$$\frac{d^2}{dt^2}\left(\ddot{x_2}\left(\frac{m_1m_2}{k}\right) + x_2(m_1 + m_2)\right) = 0 \implies \ddot{x_2}\left(\frac{m_1m_2}{k}\right) + x_2(m_1 + m_2) = 0$$

From this, the equation can be written into the form of the differential equation

$$\ddot{x} + \omega_0^2x = 0 \rightarrow \ddot{x} + \frac{k}{m}x = 0$$

with m being the sum of the inverse masses

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1m_2}$$

$$\therefore \omega^2 = \frac{k(m_1 + m_2)}{m_1m_2} \implies \omega = \sqrt{\frac{k(m_1 + m_2)}{m_1m_2}}$$

$$\boldsymbol{\omega} = \sqrt{\frac{0.5\frac{N}{m}(0.1kg + 0.2kg)}{0.1kg * 0.2kg}} = \mathbf{2.74}\boldsymbol{Hz}$$

(2). On the surface of the Moon, the acceleration of gravity is about one sixth that on the surface of Earth. What is the period of a simple pendulum of length $1m$ on the Moon?

$$F = m\ddot{x} = -mgsin(\theta)$$

$$\ddot{x} = -gsin(\theta)$$

Using $\ell\theta = s \implies \ell\ddot{\theta} = \ddot{s}$

$$\ddot{s} + gsin(\theta) = 0$$

$$\ell\ddot{\theta} + gsin(\theta) = 0$$

For $\theta < 20°$, $sin(\theta) \approx \theta$ thus,

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

This differential equation can be mapped to $\ddot{x} + \omega^2 x = 0$, thus

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \iff \ddot{x} + \omega^2 x = 0$$

$$\omega^2 = \frac{g}{\ell}$$

$$\omega_0 = \sqrt{\frac{g}{\ell}}$$

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

$$\boldsymbol{T} = 2\pi\sqrt{\frac{1m}{9.81\frac{m}{s^2}/6}} = \mathbf{4.91}\boldsymbol{s}$$

(3). Two springs with stiffnesses k_1 and k_2 are used in a vertical position to support a single object of mass m.

(a). Show that the angular speed of oscillation is $\omega_0 = \sqrt{\frac{k_1+k_2}{m}}$ if the springs are connected in parallel.

(b). Show that the angular speed of oscillation is $\omega_0 = \sqrt{\frac{k_1 k_2}{(k_1+k_2)m}}$ if the springs are connected in series.

For a system of springs connected in parallel, $k_{eq} = k_1 + k_2$, thus

$$F = m\ddot{x} = -kx = -x(k_1 + k_2)$$

$$\ddot{x} + x(\frac{k_1 + k_2}{m}) = 0$$

Which is equivalent to the differential equation

$$\ddot{x} + x(\frac{k_1 + k_2}{m}) = 0 \iff \ddot{x} + \omega_0^2 x = 0$$

$$\therefore \omega_0^2 = \frac{k_1 + k_2}{m} \implies \boldsymbol{\omega_0 = \sqrt{\frac{k_1 + k_2}{m}}}$$

For a system of springs connected in series, $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \implies k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$$F = m\ddot{x} = -kx = -x(\frac{k_1 k_2}{k_1 + k_2})$$

$$m\ddot{x} + x(\frac{k_1 k_2}{k_1 + k_2}) = 0$$

$$\ddot{x} + x(\frac{k_1 k_2}{m(k_1 + k_2)}) = 0$$

This equation is equivalent to the differential equation

$$\ddot{x} + x(\frac{k_1 k_2}{m(k_1 + k_2)}) = 0 \iff \ddot{x} + \omega_0^2 x = 0$$

This implies that

$$\omega_0^2 = \frac{k_1 k_2}{m(k_1 + k_2)} \implies \boldsymbol{\omega_0 = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}}$$