Physical Mechanics Homework 7

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1. TM 6-3; Show that the shortest distance between two points in three-dimensional space is a straight line.

$$L = \int_{x_1}^{x_2} dS = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2 + dz^2} = \int_{x_1}^{x_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

$$f(x, \dot{x}, y, \dot{y}, z, \dot{z}, t) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

From this functional, three Euler-Lagrange equations will be needed. To satisfy these equations, I will be using $\hat{e}$ in place of x, y, z

$$\frac{\partial f}{\partial \hat{e}} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\hat{e}}} = 0$$

$$\frac{\partial f}{\partial \hat{e}} = \frac{\partial}{\partial \hat{e}} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = 0$$

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{\hat{e}}} = 0 \implies \frac{\partial f}{\partial \dot{\hat{e}}} = constant$$

$$\frac{\partial f}{\partial \dot{\hat{e}}} = \frac{\partial}{\partial \dot{\hat{e}}} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{\dot{\hat{e}}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = constant$$

From the previous notation that $\hat{e}$ represents three equations for x, y, z, we will expand it back out to three equations by the radical.

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_x \qquad \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_y \qquad \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_z$$

For the shortest time length, dt, the radical is equivalent, so these three equations can be equated.

$$\frac{\dot{x}}{C_x} = \frac{\dot{y}}{C_y} = \frac{\dot{z}}{C_z}$$

With some calculus, this simplifies to the equation of a straight line in three dimensions.

$$\frac{dx}{dtC_x} = \frac{dy}{dtC_y} = \frac{dz}{dtC_z}$$

$$\int_{t_0}^{t} \frac{dx}{C_x} dt = \int_{t_0}^{t} \frac{dy}{C_y} dt = \int_{t_0}^{t} \frac{dz}{C_z} dt$$

$$(t - t_0)\frac{dx}{C_x} = (t - t_0)\frac{dy}{C_y} = (t - t_0)\frac{dz}{C_z}$$

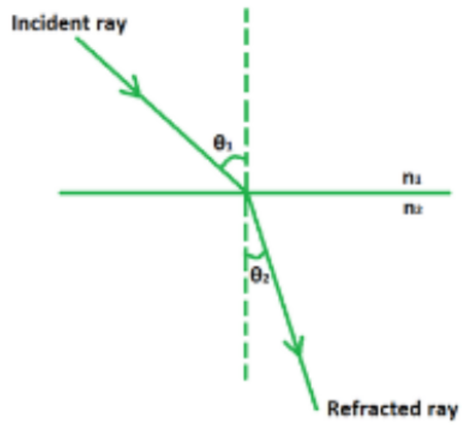
$$\int_{x_0}^{x} \frac{dx}{C_x} = \int_{y_0}^{y} \frac{dy}{C_y} = \int_{z_0}^{z} \frac{dz}{C_z}$$

Where the equation of a straight line between three dimensions is represented by:

$$\boldsymbol{\frac{x - x_0}{C_x} = \frac{y - y_0}{C_y} = \frac{z - z_0}{C_z}}$$

2. TM 6-7; Consider light passing from one medium with index of refraction n_1 to another medium with index of refraction n_2 (illustrated below for the scenario $n_2 > n_1$). Use Fermat's Principle—that the travel time is minimized—and derive the law of refraction:

$$n_1 sin(\theta_1) = n_2 sin(\theta_2).$$



Let the incident ray begin at the point (x_1, y_1) and end at the point $(0, 0)$

Let the refracted ray begin at the point $(0, 0)$ and end at the point (x_2, y_2)

Finally, consider a point on the x-axis, x_c, such that x_c can be varied to separate the incident and refracted rays by the x-axis

These definitions imply the length of each ray, with the incident ray being denoted ℓ_1 and the refracted ray being denoted ℓ_2

$$\ell_1 = \sqrt{(x_1 - x_c)^2 + (y_1 - 0)^2}$$

$$\ell_2 = \sqrt{(x_2 - x_c)^2 + (y_2)^2}$$

From the definition of velocity, the total time it takes to go from the start of the incident ray, (x_1, y_1) to (x_2, y_2) can be found.

$$v = \frac{x}{t} \implies t = \frac{x}{v} \implies t_{tot} = \frac{\ell_1}{v_1} + \frac{\ell_2}{v_2}$$

The speed at which light takes to travel from the start of the medium defined above and below the x-axis is equal to the index of refraction, n divided by the speed of light.

$$v = \frac{c}{n} \implies t_{tot} = \frac{n_1 \ell_1}{c} + \frac{n_2 \ell_2}{c}$$

We want to minimize the total time, so that would imply

$$\frac{dt}{dx_c} = 0$$

$$\frac{d}{dx_c}(\frac{n_1\ell_1}{c} + \frac{n_2\ell_2}{c}) = 0$$

To further reduce the equation, the speed of light can be factored out

$$\frac{d}{dx_c}(n_1\sqrt{(x_1 - x_c)^2 + (y_1)^2} + n_2\sqrt{(x_2 - x_c)^2 + (y_2)^2}) = 0$$

After applying the chain rule, the two parts of the equation can be given as:

$$n_1(\frac{1}{2}\frac{-2(x_1 - x_c)}{\sqrt{(x_1 - x_c)^2 + (y_1)^2}}) + n_2(\frac{1}{2}\frac{-2(x_2 - x_c)}{\sqrt{(x_2 - x_c)^2 + (y_2)^2}}) = 0$$

Which then reduces to:

$$n_1(\frac{x_1 - x_c}{\sqrt{(x_1 - x_c)^2 + (y_1)^2}}) = n_2(\frac{x_2 - x_c}{\sqrt{(x_2 - x_c)^2 + (y_2)^2}})$$

The parts of the equation in the parenthesis are the definition of sine, which then after reductionl becomes Snell's law.

$$\boldsymbol{n_1 sin(\theta_1) = n_2 sin(\theta_2)}$$