Assignment 4

Reading Assignment:

1. Chapter 5: Discrete Random Variables.

Problems:

1. **A two-envelopes puzzle.** You are handed two envelopes, and you know that each contains a positive integer dollar amount and that the two amounts are different. The values of these two amounts are modeled as constants that are unknown. Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. A friend claims that the following strategy will increase above $1/2$ your probability of ending up with the envelope with the larger amount: toss a coint repeatedly, let X be equal to $1/2$ plus the number of tosses required to obtain heads for the first time, and switch if the amount in the envelope you selected is less than the value of X. Is your friend correct?

2. **Using a biased coin to make an unbiased decision.** Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased and lands heads with probability p. Since they both know p, whomever calls the toss in the air can bias the decision towards their preference. Design a coin-toss experiment, which does not depend on p, so that they can use the biased coin to make a decision so that either option (opera or movies) is equally likely?

3. **The birthday problem.** Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year, independently of everyone else, and ignore the additional complication presented by leap years (i.e., nobody is born on February 29). What is the probability that each person has a distinct birthday?

4. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximate by using the Poisson PMF. (For simplicity, exclude birthdays on February 29.)

5. An internet service provider uses 50 dial-up modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independently of the other customers.

   (a) What is the PMF of the number of modems in use at a given time?

   (b) Repeat part (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.

   (c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact, as well as an approximate formula based on the Poisson approximation of part (b).

6. **From the Poisson PMF.** Let X be a Poisson random variable with parameter λ. Show that the PMF $p_X(k)$ increases monotonically with k up to the point where k reaches the largest integer not exceeding λ, and after that point decreases monotonically with k.