NTR3 – Modular Arithmetic

Problem 1. How many prime numbers p are there such that $29^p + 1$ is a multiple of p?

Problem 2. Let $p \geq 7$ be a prime. Prove that the number

$$\underbrace{11 \ldots 1}_{p-1\ 1's}$$

is divisible by p.

Problem 3. Let p be a prime, and let $1 \leq k \leq p-1$ be an integer. Prove that

$$\binom{p-1}{k} \equiv (-1)^k \ (mod\ p).$$

Problem 4. (All Russian MO 2000). Evaluate the sum

$$\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor.$$

Problem 5. (PuMAC) Calculate the last 3 digits of

$$2008^{2007^{2006^{\cdots^{2^1}}}}.$$

[When we have to calculate $a\ (mod\ 1000)$ it is often more helpful to find $a\ (mod\ 8), a\ (mod\ 125)$ and then using the Chinese Remainder Theorem to find $a\ (mod\ 1000)$.

When a and n are relatively prime, we have $a^b \equiv a^{b\ (mod\ \phi(n))} (mod\ n)$

It then suffices to calculate $b\ (mod\ \phi(n))$.]

Problem 6. (Romania 2003) Consider the prime numbers $n_1 < n_2 < \cdots < n_{31}$. Prove that if 30 divides $n_1^4 + n_2^4 + \cdots n_{31}^4$, then among these numbers one can find three consecutive primes.

Problem 7. (St. Petersburg 2008) Given three distinct natural a, b, c show that

$$\gcd(ab+1, bc+1, ca+1) \leq \frac{a+b+c}{3}.$$

Problem 8. (IMO 2005) Consider the sequence $a_1, a_2, \ldots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1$$

For all positive integers n. Determine all positive integers that are relatively prime to every term of the sequence.