Euclidiad Olympiad Training LEVEL 1
Day 21 - GCD and LCM

1 Greatest Common Divisor (GCD)

- The greatest common divisor (GCD) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers.
- For two integers x and y, the greatest common divisor of x and y is denoted by $\gcd(x, y)$. For example, the GCD of 8 and 12 is 4, that is, $\gcd(8, 12) = 4$.

2 Coprime Numbers

- Two numbers are called relatively prime, or coprime, if their greatest common divisor equals 1. For example, $\gcd(9, 28) = 1$, so 9 and 28 are relatively prime.
- A fraction is in lowest terms or in *reduced form* when the numerator and the denominator are relatively prime integers.

3 Euclidean Algorithm

- For any integers m, n,
$$\gcd(m, n) = gcd(m - n, n).$$
- Let m and n be integers such that $m = qn + r$, where $0 \leq r < n$, then
$$\gcd(m, n) = \gcd(r, n).$$

4 Least Common Mulitple (LCM)

- The least common multiple (LCM), also known as the lowest common multiple, of two (or more) integers a and b is the smallest positive integer that is divisible by both a and b.
- For two integers x, y, the least common multiple of x and y is denoted by $\text{lcm}[x, y]$. For example, the LCM of 8 and 12 is 24, that is $\text{lcm}[8, 12] = 24$.

5 Relationship between GCD and LCM

- The product of two natural numbers, m, n, is equal to the product of their GCD and LCM.
$$m \times n = \gcd(m, n) \times \text{lcm}[m, n].$$
- For any positive integers a, b and c,
$$\gcd(ac, bc) = c \times \gcd(a, b),$$
$$\text{lcm}[ac, bc] = c \times \text{lcm}[a, b].$$

- GCD is used to equally distribute two or more sets of items into their largest possible grouping.
- LCM is used to figure out when something will happen again at the same time.

Examples Given in Class

Example 1. (a) By listing all the factors of 24 and 32, find $\gcd(24, 32)$.

(b) By using prime factorization, find $\gcd(24, 32)$.

Example 2. Jon splits 42 blue marbles into piles containing n marbles each. Jon then divides 28 green marbles into n total piles with an equal number of marbles in each pile. What is the largest possible value of n?

Example 3. What is the smallest perfect square that is divisible by both 2 and 3?

Example 4. Group 6, 8, 10, 15, 21 and 25 into three relatively prime pairs of integers.

Example 5. Find $\gcd(42, 12)$ and $\gcd(1086, 828)$ using the Euclidean Algorithm.

Example 6. How many positive divisors do 8400 and 7560 have in common?

Example 7. By using prime factorization, find $\text{lcm}[24, 32]$.

Example 8. How many three-digit integers are divisible by both 12, 14 and 21?

Example 9. The remainder is 1 when a certain number isi divided by 2. The remainder is also 1 when the divisors are 3, 4, 5 and 6 respectively. What is the smallest possible value of this number?

Example 10. A number has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3 and a remainder of 4 when divided by 5. How many such 3-digit numbers exist?

Example 11. The GCD of 70 and some natural number n is 10. Their LCM is 210. Find n.

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Day 21 - Homework Problems

Homework code : **HWN102**
Issued on : 13th July 2021
Due date : 27th July 2021

Submit the solutions to at least 6 of the homework problems before the due date. Problems 1-10 are each worth 5 points. Challenge problems are worth 10 points each.

1. Ying and Catherine walked to the store together to buy some pencils. Ying bought 40 penciles and Catherine bought 24. If each package of pencils sold at the store contain the same number of pencils, what is the largest possible number of pencils in a package?

2. The GCD of n and 216 is 24 and their LCM is 864. Find n.

3. The smallest positive integer greater than 1 that leaves a remainder of 1 when divided by 4, 5 and 6 lies between which of the following pairs of numbers?

   (a) 2 and 19, (b) 20 and 39, (c) 40 and 59, (d) 60 and 79, (e) 80 and 124.

4. What is the smallest counting number that is divisible by all of the first ten counting numbers?

5. Wanda, Darren, Beatrice and Chi are tutors in the school math lab. Their schedules are as follows: Darren works every third school day, Wanda works every fourth school day, Beatrice works every sixth school day, and Chi works every seventh school day. Today they are all working in the math lab. In how many days from today will they be together tutoring in the lab again?

6. What is the ratio of the least common mulitple of 180 and 594 to the greatest common factor of 180 and 594?

7. How many 4-digit positive integers are divisble by 4, 5, 6 and 7?

8. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her grandchildren?

9. A positive integer n leaves a remainder of 1 when divided by 4, a remainder of 2 when divided by 5 and remainder of 3 when divided by 6. If $n \leq 1000$, how many possible values of n are there?

10. Find the greatest common divisor of 6432 and 132 using the Euclidean Algorithm.

Challenge Problems

11. Janet writes down cubes of three different positive integers. Rob points out that all three cubes are multiples of 18. Janet finds out that GCD of three cubes is n. What is the smallest possible value of n?

12. Jay writes the integers from 1 to 2000 on a piece of paper. He erases all the multiples of 3, then all the multiples of 5, and so on, erasing all the multiples of each odd prime. How many numbers are left when he finishes?

13. How many distinct pairs of positive integers are there which have a GCD of 6 and LCM of 600?