Euclidiad Olympiad Training LEVEL 1
Day 19 - A Few Notes
Prime Factorization and Divisor Problems

1 What is Number Theory?

Number theory is basically the study of integers. Some examples of different types of integers are: (a) natural numbers, (b) prime numbers, (c) composite numbers, (d) even numbers, (e) odd numbers, (f) perfect squares, (g) negative cubes, (h) powers of 2, (i) abundant numbers, (j) perfect numbers, (k) Fibonacci numbers, (l) numbers with bases other than 10, (m) residue class modulo 6, etc.

2 Primes and Composite Numbers

A prime number is a number that has exactly two positive divisors, 1 and itself. A composite number is a number that has more than two positive divisors. A common method usually used to identify primes is called the 'Sieve of Eratosthenes'. This is quite efficient compared to the brute force method especially for larger numbers. Its procedure can be summarized as follows:

1. Find a number k such that $k^2 > n$.
2. Divide n by each of the prime numbers smaller than k. If none of them divides n, then n is prime.

Examples Given in Class

Example 1. Check whether the following numbers are primes or not.

(a) 137, (b) 337, (c) 507

3 Prime Factorization

The prime factorization of an integer is the product of primes equal to that integer.

Example 2. Find the prime factorization of the following numbers.

(a) 450, (b) 1200, (c) 132^2, (d) 14^3, (e) 18^8, (f) 201201.

Example 3. The product of 693 and a number n is a square number. What is the smallest possible value of n?

Example 4.
(a) Find the smallest positive integer m such that the product of m and 2016 is a cube.
(b) Find the smallest positive integer n such that when 2016 is divided by n, the quotient is a square.

4 Number of Positive Divisors

For a natural number n whose prime factorization is

$$n = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_m^{e_m},$$

the total number of positive divisors of n is given by

$$\tau(n) = (e_1 + 1)(e_2 + 1)(e_3 + 1) \cdots (e_m + 1).$$

A positive integer is a perfect square if and only if it has an odd number of positive divisors.

Example 5. Find the number of positive divisors of the following numbers:

(a) 200, (b) 1200, (c) 504.

5 Sum of Positive Divisors

For a natural number n whose prime factorization is

$$n = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_m^{e_m},$$

the sum of positive divisors of n is

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{e_1})(1 + p_2 + p_2^2 + \cdots + p_2^{e_2}) \cdots (1 + p_m + p_m^2 + \cdots + p_m^{e_m}).$$

Example 6. Find the sum of positive divisors of the following numbers:

(a) 12, (b) 100, (c) 72.

6 Product of positive divisors

For a natural number n whose prime factorization is

$$n = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_m^{e_m},$$

the product of positive divisors of n is

$$P_n = n^{\frac{\tau(n)}{2}}.$$

Example 7. Find the product of positive divisors of the following numbers.

(a) 18, (b) 81, (c) 110.

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Day 19 - Homework
Prime Factorization and Divisor Problems

Homework code : **HWN101**
Issued on : 6th July 2021
Due date : 20th July 2021

Submit the solutions to at least 6 of the homework problems before the due date. Problems 1-10 are each worth 5 points. Challenge problems are worth 10 points each.

1. Check whether the following numbers are primes or not.

   (a) 271, (b) 437, (c) 741.

2. Find the prime factorization of the following numbers.

   (a) 360, (b) 1800, (c) 6^5, (d) 108^3, (e) 72^2.

3. The product of 2860 and m is a square number. Find the smallest possible value of m.

4. When 1540 is divided by n, the quotient is a square. Find the smallest possible value of n.

5. What is the greatest prime divisor of the following arithmetic series

$$1 + 2 + 3 + \cdots + 70?$$

6. Find the number of positive divisors of the following numbers.

   (a) 81, (b) 1440, (c) 999.

7. Find the sum of positive divisors of the following numbers.

   (a) 40, (b) 216, (c) 220.

8. Find the product of positive divisors of the following numbers.

   (a) 128, (b) 801, (c) 225.

9. Tony places 360 marbles into m total boxes such that each box contains an equal number of marbles. There is more than one box, and each box contains more than one marble. For how many values of m can this be done?

10. 108 chickens are kept in n cages such that each cage contains the same number of chickens. What is the product of possible values of n?

Challenge Problems

11. What is the first year in the twenty-first century that is a prime number? Give reasons for your answer.

12. Find the prime number p such that there are integers a, b, c and d, which satisfy the system of equations:

$$\begin{aligned} a+b+c &= p-3, \\ a+b+d &= p+1, \\ a+c+d &= 2p+2, \\ b+c+d &= p. \end{aligned}$$

13. A new school has exactly 900 lockers and 900 students. On the first day of school, the first student enters the school and opens all the lockers. The second student then enters and closes every locker with an even number. The third student will 'reverse' every third locker (If the locker is closed, it will be opened and vice versa.) The fourth student will reverse every fourth locker and so on, until all 900 students have entered and reversed the respective lockers. Which lockers will be open at the end?