NTP3 – Modular Arithmetic

Homework code : NTP3

Issued on : 25th May 2023

Due date : 1st June 2023

Problems 1 to 10 are each worth 5 points.

Problem 1. Give an example of 11 consecutive positive integers the sum of whose squares is a perfect square.

Problem 2. Let a, b, c and d be positive integers. Prove that $a^{4b+d} - a^{4c+d}$ is divisible by 240.

Problem 3. (HMMT 2009) Find the last two digits of 1032^{1032}. Express your answer as a two-digit number.

Problem 4. (Senior Hanoi Open MO 2006) Calculate the last three digits of

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Problem 5. (Freshman's Dream) Let a, b be integers and p be a prime. Prove that

$$(a+b)^p \equiv a^p + b^p \ (mod\ p)$$

Problem 6. Let p be a prime. Prove that p divides $ab^p - ba^p$ for all integers a and b.

Problem 7. Prove that in sequence 1, 31, 331, 3331, … there are infinitely many composite numbers.

Problem 8. (PuMAC 2008) Define $f(x) = x^{x^{x^x}}$. Find the last two digits of

$$f(17) + f(18) + f(19) + f(20).$$

Problem 9. If $a \equiv b \ (mod\ n)$, show that $a^n \equiv b^n (mod\ n^2)$. Is the converse true?

Problem 10. (AIME 1994) The increasing sequence 3, 15, 24, 48, … consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994th term of the sequence is divided by 1000?

Further Problems

Further Problems are each worth 5 special points.

F Problem 1. (Balkan). Let n be a positive integer with $n \geq 3$. Show that

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

F Problem 2. (USAMO 1991) Show that, for any fixed integer $n \geq 1$, the sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots (mod\ n)$$

is eventually constant.

[The tower of exponents is defined by $a_1 = 2, a_{i+1} = 2^{a_i}$. Also $a_i\ (mod\ n)$ means the remainder which results from dividing a_i by n.]

F Problem 3. (Canada 2003) Find the last 3 digits of $2003^{2002^{2001}}$.

F Problem 4. Let m and n be integers greater than 1 such that $\gcd(m, n-1) = \gcd(m, n) = 1$. Prove that the first $m-1$ terms of the sequence $n_1, n_2, \ldots$, where $n_1 = mn + 1$ and $n_{k+1} = n \cdot n_k + 1, k \geq 1$, cannot be all primes.

F Problem 5. (IMO 1979, P1) If p and q are natural numbers so that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319},$$

prove that p is divisible with 1979.