NTL6 – Lifting the Exponent

Example 1. For each non-negative integer n, compute

$$v_3\left(2^{3^n}+1\right).$$

Theorem 10.5 (Lifting the exponent/ LTE)

Let $p>2$ be a prime and $a,b\in\mathbb{Z}$ be coprime to p such that $p \mid a-b$. Suppose n is a positive integer.

$$v_p(a^n-b^n)=v_p(a-b)+v_p(n).$$

Note. Three particular conditions must be satisfied.

1. p must be odd. i.e., $p\neq 2$.

2. $\gcd(p,a)=\gcd(p,b)=1$. i.e., $p\dagger a,b$.

3. $p \mid a-b$, i.e., $v_p(a-b)\neq 0$

Example 2. Prove Theorem 10.5.

Example 3. (AIME 2018) Find the smallest positive integer n such that 3^n ends with 01 when written in base 143.

Corollary 10.5.1

Let $p>2$ be a prime and $a,b\in\mathbb{Z}$ be coprime to p such that $p \mid a+b$. Suppose n is an odd positive integer.

$$v_p(a^n+b^n)=v_p(a+b)+v_p(n)$$

Theorem 10.6 (Sad case when p=2/ LTE for p=2)

Let x,y be odd integers such that $2 \mid x-y$. Let n be an even integer. Then

$$v_2(x^n-y^n)=v_2(x^2-y^2)+v_2\left(\frac{n}{2}\right)=v_2(x-y)+v_2(x+y)+v_2(n)-1$$

Let x,y be integers such that $4 \mid x-y$. Let n be an even integer.

$$v_2(x^n-y^n)=v_2(x-y)+v_2(n)$$

Let x,y be integers such that $4 \mid x+y$. Let n be an even integer.

$$v_2(x^n-y^n)=v_2(x+y)+v_2(n)$$

Example 4. (1991 IMO Shortlist) Find the largest integer k for which 1991^k divides

$$1990^{1991^{1992}}+1992^{1991^{1990}}.$$