# Euclidiad Introduction to Number Theory (Short Course)
## Lecture 5 – Modular Arithmetic II

Today's topics:
(i) Multiplication and Exponentiation in Modular Arithmetic
(ii) Patterns and Exploration

Example 1.
Let $a_1, a_2$ be integers such that $a_1 \equiv a_2 (mod\ 6)$ and let $b$ be an integer.
Show that $a_1 b \equiv a_2 b\ (mod\ 6)$.

Example 2.
Let $a_1, a_2, b_1, b_2$ be integers such that

$$a_1 \equiv a_2\ (mod\ m)$$
$$b_1 \equiv b_2\ (mod\ m)$$

Show that $a_1 b_1 \equiv a_2 b_2\ (mod\ m)$.

Example 3.
The remainders when two natural numbers are divided by 12 are 5 and 9.
(a) Find the remainder when their product is divided by 12.
(b) Find the remainder when their product is divided by 4.

Example 4.
Kyi Pyar's teacher asked her to find the remainder when the sum of the following 201-term arithmetic series is divided by 5:

$$2 + 7 + 12 + \cdots + 1002$$

(a) At first, Kyi Pyar began to apply his knowledge to sum the series. Then she realized that each of the terms in the arithmetic progression is congruent to 2 (mod 5). This allowed her to find the answer more quickly. How did she do it?
(b) Later that day, she realized that she could also solve the problem quickly by applying modular arithmetic on a formula for the sum of an arithmetic progression:

$$a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}.$$

How did she do it?

Example 5. Let $a_1 \equiv a_2\ (mod\ m)$ and $n$ be a natural number. Show that $a_1^n \equiv a_2^n\ (mod\ m)$.

Example 6. Is $21^{100} - 12^{100}$ a multiple of 11?

In modular arithmetic, we usually work with residues because they typically make the arithmetic easiest. However, negative integers are easier to work with in some cases. It's always good to keep an open mind to the possibility of simpler solutions.

Example 7. Find the remainder when
(a) $514 \cdot 891$ is divided by 11.

(b) $317 \cdot 5^{51}$ is divided by 6.
(c) $24^{50} - 15^{50}$ is divided by 13.

Example 8. Find the remainder when $5^{2005}$ is divided by 7.
(Hint: Find the remainders when $5^2, 5^3, 5^4, 5^5, 5^6, 5^7, 5^8$ are divided by 7.)

Example 9. Find the remainder when $4^{18} \cdot 19^{80}$ is divided by 9.

Example 10. The square of a positive integer leaves a remainder of 1 when divided by 5. What are the possible remainders when the integer itself is divided by 5.

Example 11. Find the units digit of $7^{7^7}$.

Example 12. Show that a natural number is congruent to the sum of its digits modulo 3.

## Summary

- Let $a_1, a_2, b_1,$ and $b_2$ be integers such that

$$a_1 \equiv a_2 \ (mod\ m),$$
$$b_1 \equiv b_2 \ (mod\ m),$$

  then each of the following must be true

$$a_1 + b_1 \equiv a_2 + b_2 \ (mod\ m)$$
$$a_1 b_1 \equiv a_2 b_2 \ (mod\ m)$$

  For any positive integer $n$,

$$a_1^n \equiv a_2^n \ (mod\ m).$$

# Lecture 5 - Homework Problems

| | | |
|---|---|---|
| *Homework code* | : | HW5 |
| *Issued on* | : | $19^{th}$ April 2022 |
| *Due date* | : | $25^{th}$ April 2022 |

Submit the solutions to at least 4 of the homework problems before due date. Problems 1 to 7 are each worth 5 points. Challenge problem is 10 points worth.

Problem 1. For how many values of $n$, where $40 \leq n \leq 80$ is $n \equiv -n \ (mod\ 12)$?

Problem 2. Find the remainder when $9^{42} - 5^{42}$ is divided by 7.

Problem 3. Prove that if $a \equiv 19 \ (mod\ 30)$, then $3a \equiv 7 \ (mod\ 10)$.

Problem 4. Find the remainder when $7^{255}$ is divided by 11.

Problem 5. Find the smallest natural number $n$ such that $617n \equiv 943n \ (mod\ 18)$.

Problem 6. Show that a natural number is congruent to the sum of its digits modulo 9.

Problem 7. Find the remainder when $2019^1 + 2019^2 + 2019^3 + \cdots + 2019^{2019}$ is divided by 101.

## Challenge Problem

Problem 8. Find the remainder when

$$10^{10^1} + 10^{10^2} + 10^{10^3} + \cdots + 10^{10^{10}}$$

is divided by 7.