NTP2 – Bezout's Theorem and Fundamental Theorem of Arithmetic

Homework code : NTP2

Issued on : 18th May 2023

Due date : 25th May 2023

Problems 1 to 10 are each worth 5 points.

Problem 1. Prove that if $\sqrt[3]{a}$ is rational then $\sqrt[3]{a}$ is an integer.

Problem 2. Prove that there is one natural number n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

Problem 3. (AIME 1998) For how many values of k is 12^{12} the least common multiple of $6^6, 8^8$ and k?

Problem 4. (AIME 1987) Let $[r, s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples a, b, c such that $[a, b] = 1000, [b, c] = 2000, [c, a] = 2000$.

Problem 5. Let m, n be two natural numbers, $mn | (m^2 + n^2)$, then $m = n$.

Problem 6. Let n be a positive integer. Prove that

$$\gcd(n! + 1, (n + 1)! + 1) = 1.$$

Problem 7. Let k be a positive odd number. Prove that $1 + 2 + \cdots + n$ divides $1^k + 2^k + \cdots + n^k$.

Problem 8. Prove that for positive integers $a, b > 2$, we cannot have $2^b - 1 \mid 2^a + 1$.

Problem 9. (Putnam 2000) Prove that the expression

$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Problem 10. (St. Petersburg 1996). Find all positive integers n such that

$$3^{n-1} + 5^{n-1} \mid 3^n + 5^n.$$

Further Problems

Further Problems are each worth 5 special points.

F Problem 1. Prove that if p is an odd prime and if

$$\frac{a}{b} = 1 + \frac{1}{2} + \cdots + \frac{1}{p-1},$$

then p divides a.

F Problem 2. Suppose that the greatest common divisor of the positive integers a, b and c is 1, and

$$\frac{ab}{a-b} = c.$$

Prove that $a - b$ is a perfect square.

F Problem 3. (APMO 2002).

Find all pairs of positive integers a, b such that

$$\frac{a^2+b}{b^2-a} \ and \ \frac{b^2+a}{a^2-b}$$

are both integers.

F Problem 4. Prove that for different choices of signs $+$ and $-$, the expressions

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm (4n+1)$$

yields all odd positive integers less than or equal to $(2n+1)(4n+1)$.

F Problem 5. (St. Petersburg 2001) For all positive integers $m > n$, prove that

$$lcm[m,n] + lcm[m+1, n+1] > \frac{2mn}{\sqrt{m-n}}.$$

F Problem 6. (USAMO 1973) Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression.

F Problem 7. (Iran 1998) Suppose that a and b are natural numbers such that

$$p = \frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$$

is a prime number. Find all possible values of a, b, p.