Contents

1979 IMO P1 (Pairing)

If p and q are natural numbers so that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319},$$

prove that p is divisible with 1979.

Solution.

$$\begin{aligned}\frac{p}{q} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1318} + \frac{1}{1319} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1318}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1318} + \frac{1}{1319} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{659}\right) \\ &= \frac{1}{660} + \frac{1}{661} + \cdots + \frac{1}{1319}\end{aligned}$$

Since

$$\frac{1}{660} + \frac{1}{1319} = \frac{1979}{660 \times 1319}, \frac{1}{661} + \frac{1}{1318} = \frac{1979}{661 \times 1318}, \cdots$$

$$\frac{p}{q} = \frac{1979}{660 \times 1319} + \frac{1979}{661 \times 1318} + \cdots + \frac{1979}{989 \times 990} = 1979\left(\frac{r}{s}\right)$$

Since 1979 is a prime and coprime to s, p is divisible by 1979.

1986 IMO P1 (Modular Contradiction, Parity)

Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Solution.

$$2 \times 5 - 1 = 3^2, 2 \times 13 - 1 = 5^2, 5 \times 13 - 1 = 8^2.$$

We will look for the non-perfect square in the set $\{2d - 1, 5d - 1, 13d - 1\}$.

For the sake of contradiction, assume that all these numbers are perfect squares.

$$2d - 1 = a^2 \quad (1)$$

$$5d - 1 = b^2 \quad (2)$$

$$13d - 1 = c^2 \quad (3)$$

Where $a, b, c \in \mathbb{N}$.

From (1), a is a odd number. Let $a = 2x + 1$.

From (1), $d = 2x(x + 1) + 1$

Since $x(x + 1)$ is always even, $d \equiv 1 \ (mod\ 4)$.

$\therefore$ b and c are even. Let $b = 2y, c = 2z$.

From (2) and (3), $8d = c^2 - b^2 = (2z)^2 - (2y)^2$

$$d = \frac{z^2 - y^2}{2} \equiv 0 \ (mod\ 4)(Contradiction)$$

■

1988 IMO P6 (Vieta Jumping)

If a, b are positive integers such that

$$\frac{a^2+b^2}{1+ab}$$

is an integer, then

$$\frac{a^2+b^2}{1+ab}$$

is a perfect square.

Solution.

For the sake of contradiction, we assume that

$$\frac{a^2+b^2}{1+ab} = k, k \neq square\ number. \tag{1}$$

For $a = b$, it is clear that k is a square number since

$$0 < \frac{2a^2}{1+a^2} = k < \frac{2a^2}{a^2} = 2$$

forces $k = 1$.

WLOG, it is assumed that $a > b$.

Let $(a, b) = (a_1, b_1)$ be a solution to Equation (1) where $a_1 + b_1$ has the smallest sum among all pairs (a, b) where $a > b$.

For the sake of minimality contradiction, we are to prove that there exists a solution $(a, b) = (a_2, b_1)$ such that $a_2 < a_1, a_2 > 0\ (i.e., a_2 \in \mathbb{N})$.

Equation (1) can be arranged into a quadratic in a.

$$a^2 - a(kb) + b^2 - k = 0$$

This equation has roots $a = a_1, a_2$, using Vietas,

$$a_1 + a_2 = kb_1 \tag{2}$$

$$a_1 a_2 = b_1^2 - k \tag{3}$$

From (3), if $a_2 = 0$, then $b_1^2 - k = 0$ which leads to $k = b_1^2 = square\ number$. (Contradiction)

Thus $a_2 \neq 0$. From (1) since $a_2^2 + b_1^2 = k(1 + a_2 b_1)$, then $k + a_2 b_1 k > 0$ which leads to $a_2 b_1 > -1$ ($\because k > 0$). As $a_2 \neq 0, b_1 > 0 \in \mathbb{N}$, then $a_2 > 0$. One part is finished. Another part is proved by

$$a_2 = \frac{b_1^2 - k}{a_1} < \frac{a_1^2 - k}{a_1} = a_1 - \frac{k}{a_1} < a_1.$$

■

1990 IMO P3 (Order, LTE)

Find all natural n such that

$$\frac{2^n+1}{n^2}$$

is an integer.

Solution.

Let p be a prime divisor of n.

Since $2^n + 1$ is odd, then $n = odd$ which leads to $p = odd$.

Given that $2^n + 1 \equiv 0 \ (mod\ p)$ leading to $2^n \equiv -1 \ (mod\ p)$.

$$2^{2n} \equiv 1 \ (mod\ p)$$

By Fermat's Theorem,

$$2^{p-1} \equiv 1 \ (mod\ p)$$

Since $ord_p 2$ divides both $2n$ and $p-1$, it also divides $\gcd(2n, p-1)$.

As $v_2(2n) = 1(\because n = odd)$, $\gcd(2n, p-1) = 2$ if and only if $\gcd(n, p-1) = 1$.

Special choice for p.

So let p be the smallest prime divisor of n.

$ord_p 2 = 1$ is impossible. $\therefore ord_p 2 = 2$.

$$2^2 \equiv 4 \equiv 1 (mod\ p)$$

Smallest prime divisor of n, $p = 3$. Let $v_3(n) = k$, using LTE

$$v_3(2^n+1) = v_3(n) + v_3(2+1) = k+1 \geq v_3(n^2) = 2k$$

$$\therefore k = 1.$$

Let $n = 3n_1$. $n_1 = 1$ is a solution. Assume that $n_1 \neq 1$ and let the smallest prime divisor of $n_1 = q \neq 3$.

$$q \mid 2^{3n_1} + 1 = 8^{n_1} + 1 \mid 8^{2n_1} - 1$$

Similarly, since $\gcd(n_1, q-1) = 1$, $q \mid 8^{\gcd(2n_1, q-1)} - 1 = 8^2 - 1 = 63$

$$\therefore q = 7.$$

$$2^n + 1 = 8^{n_1} + 1 \equiv 1^{n_1} + 1 \equiv 2 \ (mod\ 7)(Contradiction)$$

The only solutions are $n = 1\ and\ 3$.

■

1991 IMO Shortlist (Lifting the Exponent)

Find the largest integer k for which 1991^k divides

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

Solution.

Let $N = 1990^{1991^{1992}} + 1992^{1991^{1990}} = A^{1991^{1990}} + B^{1991^{1990}}$

Where $A = 1990^{1991^2}, B = 1992.$

$$A + B \equiv (-1)^2 + 1 \equiv 0 \ (mod\ 1991)$$

Satisfying 3 conditions for LTE, i.e., $1991 \mid A + B, \gcd(1991, A) = \gcd(1991, B) = 1.$

Using LTE, $v_{1991}(N) = v_{1991}(A + B) + v_{1991}(1991^{1990}) = v_{1991}(A + B) + 1990$

Using Binomial Expansion,

$$\begin{aligned} A + B &= (1991 - 1)^{1991^2} + 1992 \\ &\equiv 0 + \cdots + \binom{1991^2}{1} 1991 - 1 + 1992 \ (mod\ 1991^2) \\ &\equiv 1991 \ (mod\ 1991^2) \end{aligned}$$

$$\therefore v_{1991}(A + B) = 1.$$

$$v_{1991}(N) = 1991.$$

■

1992 IMO P1 (Divisibility, Bounding, Factoring Eq.)

Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

divides $abc - 1$.

Solution.

Let $x = a - 1, y = b - 1, z = c - 1$.

$$xyz \mid (x+1)(y+1)(z+1) - 1, where\ 0 < x < y < z$$

$$xyz \mid xyz + xy + yz + xz + x + y + z$$

$$xyz \mid xy + yz + xz + x + y + z$$

$$S = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} \in \mathbb{Z}$$

S is maximum when $x = 1, y = 2, z = 3$.

$$0 < S \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2\frac{5}{6}$$

$$\therefore S = 1\ or\ 2.$$

Let $(x, y, z) = (3,4,5)$.

$$S = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{12} + \frac{1}{15} + \frac{1}{20} = \frac{59}{60}$$

$$\therefore x = 1\ or\ 2$$

Case 1. $x = 1$,

$$S = 1 + \frac{1}{y} + \frac{1}{z} + \frac{1}{y} + \frac{1}{z} + \frac{1}{yz} = 1\ or\ 2.$$

$S = 1$ is impossible. $S = 2$,

$$\frac{2}{y} + \frac{2}{z} + \frac{1}{yz} = 1$$

$$(y-2)(z-2) = 5$$

It leads to $(x, y, z) = (1,3,7)$.

Case 2. $x = 2$, leads to $(x, y, z) = (2,4,14)$.

$$\therefore (a, b, c) = (2,4,8), (3,5,15)$$

■

1998 IMO P4 (Divisibility, Case work)

Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

Solution.

$$xy^2 + y + 7 \mid (x^2y + x + y)y - (xy^2 + y + 7)x = y^2 - 7x$$

Case 1. $y^2 - 7x > 0$,

$$xy^2 + y + 7 \leq y^2 - 7x$$

$$(x - 1)y^2 + y + 7x + 7 \leq 0\ (impossible)$$

Case 2. $y^2 - 7x = 0$

$$y = \sqrt{7x}$$

If $x = 7m^2$, then $y = 7m$. $(x, y) = (7m^2, 7m)$.

Case 3. $y^2 - 7x < 0$

$$\frac{7x - y^2}{xy^2 + y + 7} = k, where\ k \in \mathbb{N}$$

$$7x - y^2 = kxy^2 + ky + 7k$$

$$(7 - ky^2)x = y^2 + ky + 7k$$

If $y \geq 3$, then $LHS = (7 - ky^2)x < 0$ while $RHS > 0$ (Contradiction)

$$\therefore y = 1\ or\ 2.$$

Case (i) $y = 1, (7 - k)x = 1 + k + 7k$

$$(7 - k)(8 + x) = 57$$

Which leads to $(x, y) = (49,1), (11,1)$.

Case (ii) $y = 2, (7 - 4k)x = 4 + 2k + 7k$

$$(7 - 4k)(4x + 9) = 79$$

Which leads to no solution.

$\therefore$ The solutions are $(x, y) = (11,1), (49,1), (7m^2, 7m)$.

■

1999 IMO P4 (Order)

Find all pairs of positive integers (x, p) such that p is prime, $x \leq 2p$, and

$$x^{p-1} \mid (p-1)^x + 1.$$

Solution.

Obviously $(1, p)$ where p is an arbitrary prime is a solution.

Consider the case when $x, p \geq 3$. (i.e., $p\ is\ odd$)

Let q be a prime divisor of x which must also be odd.

[$\because$ $p-1$ is even, $(p-1)^x + 1$ is odd, x is odd.]

$$q \mid x \mid x^{p-1} \mid (p-1)^x + 1$$

$$(p-1)^x \equiv -1\ (mod\ q) \qquad (1)$$

By Fermat's Little Theorem,

$$(p-1)^{q-1} \equiv 1\ (mod\ q) \qquad (2)$$

Special choice for q

We want $\gcd(x, q-1) = 1$. So, let q be redefined as the smallest prime divisor of x.

We are going to prove $p = q$.

Alternative way 1

By Bezout's Identity, there exist two integers α and β such that

$$\alpha x - \beta(q-1) = 1. (\because \gcd(x, q-1) = 1)$$

α must be odd. [$\because$ $q-1$ is even]

$$(p-1)^{x\alpha} \equiv (p-1)^{\beta(q-1)+1}\ (mod\ q)$$

$$(-1)^{\alpha} \equiv ((p-1)^{q-1})^{\beta} \cdot (p-1)\ (mod\ q)$$

$$-1 \equiv p-1\ (mod\ q)$$

$$p \equiv 0\ (mod\ q)$$

Alternative way 2

By squaring equation 1 and 2,

$$(p-1)^{2x} \equiv 1\ (mod\ q)$$

$$(p-1)^{2(q-1)} \equiv 1\ (mod\ q)$$

$$ord_q(p-1)^2 \mid x, ord_q(p-1)^2 \mid q-1$$

$$ord_q(p-1)^2 \mid \gcd(x, q-1) = 1$$

$$(p-1)^2 \equiv 1 \ (mod\ q)$$
$$p(p-2) \equiv 0 \ (mod\ q)$$

If $p - 2 \equiv 0 (mod\ q)$,

$$(p-1)^x + 1 \equiv 1^x + 1 \equiv 2 \not\equiv 0 \ (mod\ q)$$

This contradicts to $q \mid (p-1)^x + 1$.

$$\therefore p \equiv 0 \ (mod\ q)$$

Since p and q are both primes, $p = q$.

We are going to prove that $p > 3$ is impossible.

Alternative way 1

Since $p = q, q \mid x$, then $p \mid x$. But $x \le 2p$.

For all $x, p \ge 3$, $x = p$.

$$p^{p-1} \mid (p-1)^p + 1 = p^p - \binom{p}{1} p^{p-1} + \cdots - \binom{p}{p-2} p^2 + \binom{p}{p-1} p - 1 + 1$$
$$= p^2 \left(p^{p-2} - \binom{p}{1} p^{p-3} + \cdots - \binom{p}{p-2} + 1\right)$$

Since the expression in the parentheses is not divisible by p, $p - 1 \le 2, p \le 3$.

Alternative way 2

For odd primes,

$$x^{p-1} \mid (p-1)^x + 1$$

By lifting the exponent, (satisfying 3 conditions, $p \ne 2, \gcd(p, p-1) = \gcd(p, 1) = 1, p \mid (p-1) + 1$

For $x \le 2p, p \ne 2, v_p(x^{p-1}) = p - 1$

$$v_p((p-1)^x + 1) = 1 + v_p(x) \ge p - 1$$

$$v_p(x) \ge p - 2$$
$$x \ge p^{p-2} > 2p \ for\ p > 3$$

[$\because p^{p-2} > 2p$ for $p > 3$]

This contradicts to $x \le 2p$. So, $p > 3$ is impossible.

Using manual case work,

$p = 2$ gives $x \mid 2$ or $x = 1,2$.

$p = 3$ gives $x^2 \mid 2^x + 1$ where $x \le 6$. We have $x = 1,3$.

The only solutions are hence $(x, p) = (1, p), (2,2), (3,3)$.

$\blacksquare$

2005 IMO P4 (Fermat's Theorem, Construction)

Consider the sequence $a_1, a_2, \ldots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1$$

for all positive integers n. Determine all positive integers that are relatively prime to every term of the sequence.

Solution.

By Fermat's Theorem, $2^{p-1} \equiv 3^{p-1} \equiv 6^{p-1} \equiv 1 \ (mod\ p)$ for $p \geq 5$

$$a_{p-1} \equiv 2^{p-1} + 3^{p-1} + 6^{p-1} - 1 \equiv 2 \ (mod\ p) \ [not\ desired]$$

$$a_{p-2} = 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 = 2^{p-1} \cdot 2^{-1} + 3^{p-1} \cdot 3^{-1} + 6^{p-1} \cdot 6^{-1} + 1$$
$$\equiv 2^{-1} + 3^{-1} + 6^{-1} - 1 \equiv \frac{3+2+1}{6} - 1 \equiv 0 \ (mod\ p)$$

a_{p-2} is divisible by p for $p \geq 5$.

$$a_1 = 10, a_2 = 48, a_3 = 250$$

a_2 is divisible by $p = 2,3$.

∴ There is no such positive integer that are relatively prime to every term of the sequence.

■

2007 IMO Shortlist 2 (p-adic Valuation, Division Algorithm)

Let $b, n > 1$ be integers. For all $k > 1$, there exists an integer a_k so that $k \mid (b - a_k^n)$. Prove that $b = m^n$ for some integer m.

Solution.

To prove $n \mid v_p(b)$.

For the sake of contradiction, it is assumed that n does not divide $v_p(b)$.

Using division algorithm, we can let $b = p^{e_1 n + f_1} b_1$

where $\gcd(p, b_1) = 1, p = prime, 1 \leq f_1 \leq n - 1$.

Given that $k \mid (b - a_k^n), \therefore v_p(k) \leq v_p(b - a_k^n)$.

$$v_p(b - a_k^n) = \begin{cases} v_p(b) \; if \; v_p(b) < v_p(a_k^n) \\ v_p(a_k^n) \; if \; v_p(a_k^n) < v_p(b) \end{cases}$$

If $v_p(a_k) \geq e_1 + 1$, then $v_p(b) < v_p(a_k^n)$ which will lead to

$$v_p(k) \leq v_p(b - a_k^n) = v_p(b) = e_1 n + f_1$$

If $v_p(a_k) \leq e_1$, then $v_p(a_k^n) < v_p(b)$ which will lead to

$$v_p(k) \leq v_p(b - a_k^n) = v_p(a_k^n) = n v_p(a_k) \leq n e_1$$

Special choice for k, i.e., $v_p(k)$ such that there is contradiction in both cases.

Let $v_p(k) = n(e_1 + 1)$. Then the two cases

$$v_p(k) \leq e_1 n + f_1$$

$$v_p(k) \leq n e_1$$

become contradiction ($\because f_1 < n$).

■

2007 IMO P5 (Vieta Jumping)

Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Solution.

Firstly, we are going to simplify $(4a^2-1)^2$ by subtracting suitable terms.

Since $\gcd(b, 4ab-1)=1$,

$$4ab-1 \mid (4a^2-1)^2 \Leftrightarrow 4ab-1 \mid b^2(4a^2-1)^2$$

$$4ab \equiv 1 (mod\ 4ab-1)$$

$$b^2(4a^2-1)^2 = 16a^4b^2 - 8a^2b^2 + b^2 = (4ab)^2a^2 - (4ab)(2ab) + b^2 \equiv a^2 - 2ab + b^2 \equiv (a-b)^2 \equiv 0\ (mod\ 4ab-1)$$

Since $4ab-1$ divides $(a-b)^2$,

$$\frac{(a-b)^2}{4ab-1} = k, where\ k\ is\ a\ positive\ integer \quad (1)$$

WLOG, assume $a > b$.

Let $(a,b)=(a_1,b_1)$ be a solution to the above equation (1) with $a_1 > b_1$.

Assume that $a_1 + b_1$ has the smallest sum among all pairs (a,b) with $a > b$.

For the sake of contradiction, we are going to prove there exists another solution $(a,b)=(a_2,b_1)$ with a smaller sum, i.e., $a_2 < a_1, a_2 > 0\ (i.e., a_2 \in \mathbb{N})$.

See the equation 1 as a quadratic in a,

$$\frac{(a-b_1)^2}{4ab_1-1} = k, a^2 - a(2b_1 + 4b_1k) + b_1^2 + k = 0$$

This equation has roots $a = a_1, a_2$, using Vietas,

$$a_1 + a_2 = 2b_1 + 4b_1k \quad (2)$$

$$a_1a_2 = b_1^2 + k \quad (3)$$

From (3), since $b_1^2 + k > 0$, $a_2 > 0$. We are to show $a_2 < a_1$.

$$a_2 < a_1$$

$$\Leftrightarrow \frac{b_1^2 + k}{a_1} < a_1$$

$$\Leftrightarrow b_1^2 + k < a_1^2$$

$$\Leftrightarrow b_1^2 + \frac{(a_1-b_1)^2}{4a_1b_1-1} < a_1^2$$

$$\Leftrightarrow \frac{(a_1-b_1)^2}{4a_1b_1-1} < (a_1-b_1)(a_1+b_1)$$

$$\Leftrightarrow \frac{a_1-b_1}{4a_1b_1-1} < (a_1+b_1)\ (\because a_1 > b_1)$$

The last inequality is true because $4a_1b_1 - 1 > 1$. Therefore $a > b$ is impossible. Similarly, it is impossible to have $b > a$. $\therefore a = b$.

■

2008 IMO Shortlist 1 (Parity Contradiction)

Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa,$$

then $a = b = c$.

Solution.

If two of a, b, c are equal, say $b = c$

$$a^n + pb = b^n + pb = b^n + ap$$

Then all of them are equal.

So, we assume that $a \neq b \neq c$.

After some manipulations,

$$\frac{a^n - b^n}{b - c} = -p, \frac{b^n - c^n}{c - a} = -p, \frac{c^n - a^n}{a - b} = -p$$

Similarly,

$$\frac{a^n - b^n}{a - b} \cdot \frac{b^n - c^n}{b - c} \cdot \frac{c^n - a^n}{c - a} = -p^3 \tag{1}$$

If $n = odd$, $a^n - b^n$ and $a - b$ have the same sign, leading to $\frac{a^n - b^n}{a - b} = positive$.

The left side is then positive and the right side is negative (Contradiction)

$$\therefore n = even$$

Method 1

Let $d = \gcd(a - b, b - c, c - a)$ such that

$$a - b = du, b - c = dv, c - a = dw, \gcd(u, v, w) = 1$$

$$\therefore u + v + w = 0.$$

From the given equation,

$$a^n - b^n = p(c - b)$$

$\therefore a-b \mid p(c-b)$ which corresponds to $du \mid\ p(-dv)$. Then, $u \mid pv, v \mid pw, w \mid pu$

Assume that $\gcd(u,p)=\gcd(v,p)=\gcd(w,p)=1$, then $u \mid v, v \mid w, w \mid u$.

Since $\gcd(u,v,w)=1$,then $|u|=|v|=|w|=1$ and therefore $u+v+w\neq 0$ (Contradiction)

$\therefore p$ divides at most one of u,v,w. Say $p \mid u, u=pu_1 . \gcd(p,v)=\gcd(p,w)=1$.

Similarly, as before, $u_1 \mid v, v \mid w, w \mid u$ and $|u_1|=|v|=|w|=1$.

$pu_1+v+w=0,(v+w)\in\{-2,0,2\}. \therefore p$ must be even, $p=2$.

Since $v=w=\pm 1$, then $u=\pm 2v$, $a-b=\pm 2(b-c)=even$

$$n=even=2k, a^n-b^n=-p(b-c), p=2$$

$$\left(a^k+b^k\right)\left(a^k-b^k\right)=\pm(a-b)$$

Since $a-b$ divides a^k-b^k, then $\left(a^k+b^k\right)=\pm 1$ and exactly one of a and b must be odd.

(Contradiction to $a-b=even$)

Method 2

Suppose that p is odd.

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1}=odd$$

$$n=2k, \frac{a^{2k}-b^{2k}}{a-b}=\underbrace{a^{2k-1}+a^{2k-2}b+\cdots+ab^{2k-2}+b^{2k-1}}_{2k\ terms}$$

If both a and b are even or odd, then the sum is even. (Contradiction)

Therefore, a and b have different parities. The same goes for b and c and for c and a. So, a,b,c and a alternate in their parities. (Contradiction)

Thus $p=2$. Testing different kinds of parties in the original equation, it can be concluded that a,b,c must be of the same parity. Dividing (1) by $p^3=8$,

$$\frac{a^k+b^k}{2}\cdot\frac{a^k-b^k}{a-b}\cdot\frac{b^k+c^k}{2}\cdot\frac{b^k-c^k}{b-c}\cdot\frac{c^k+a^k}{2}\cdot\frac{c^k-a^k}{c-a}=-1 \qquad (2)$$

Each one of the factors must be equal to ± 1. $a^k+b^k=\pm 2$.

If $k = even$, this becomes $a^k + b^k = 2$ and thus $|a| = |b| = 1$ which corresponds to $a^k - b^k = 0$. (Contradicting to Eq. (2))

If $k = odd$, $a^k + b^k = \pm 2$ and has a factor $a + b$. Since a and b are of same parity, $a + b = \pm 2$. Similarly, $b + c = \pm 2, c + a = \pm 2$. In two of these equations, the signs must coincide and hence some two of a, b, c are equal. (Contradiction)

■

2010 IMO Shortlist 1 (Bounding, Construction)

Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\ldots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

Solution.

$$\frac{51}{2010}=\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\ldots\left(1-\frac{1}{s_n}\right)\geq\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n+1}\right)$$
$$=\frac{1}{2}\times\frac{2}{3}\times\ldots\times\frac{n}{n+1}=\frac{1}{n+1}$$

$$\frac{17}{670}\geq\frac{1}{n+1}$$

Leads to $n \geq 38.41 \ldots$, $i.e., n \geq 39$.

We will find whether there is a case for $n = 39$.

Prime factors required to be in numerator = 17

Prime factors required to be in denominator = 67

$$\underbrace{\frac{1}{2}\times\frac{2}{3}\times\ldots\times\frac{32}{33}}_{32\ terms}\times\underbrace{\frac{34}{35}\times\ldots\times\frac{39}{40}}_{6\ terms}\times\frac{66}{67}=\frac{17}{670}$$

Which means the case for $n = 39$ exists.

■

2019 IMO P4 (p-adic Valuation, Bounding)

Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \dots (2^n - 2^{n-1}).$$

Solution.

$$\begin{aligned} Let\ P &= (2^n - 1)(2^n - 2)(2^n - 4) \dots (2^n - 2^{n-1}) \\ &= 2^{0+1+2+\cdots+n-1}(2^n - 1)(2^{n-1} - 1)(2^{n-2} - 1) \dots (2 - 1) \end{aligned}$$

$$v_2(P) = \frac{n(n-1)}{2},$$

By Legendre's Formula, $v_2(k!) = k - s_2(k) < k$

$$\frac{n(n-1)}{2} < k$$

Finding the upper bound for P, $P < (2^n)^n = 2^{n^2}$

$$P < 2^{n^2} < \frac{n(n-1)}{2}! < k!$$

The values of n which satisfy $2^{n^2} < \frac{n(n-1)}{2}!$ will lead to a contradiction.

We claim that

$$2^{n^2} < \frac{n(n-1)}{2}! \text{ for n} \geq 6.$$

For $n = 6$, $2^{6^2} \approx 6.4 \times 10^{10} < 15! \approx 1.3 \times 10^{12}$. For $n > 6$,

$$\begin{aligned} \frac{n(n-1)}{2}! = 15! \cdot 16 \cdot 17 \dots \frac{n(n-1)}{2} &> 2^{36} \cdot 16^{\frac{n(n-1)}{2} - 15} \\ &= 2^{2n(n-1)-24} = 2^{n^2} \cdot 2^{n(n-2)-24} > 2^{n^2} \end{aligned}$$

Hence $n \geq 6$ is not possible. Checking manually the cases $n \leq 5$,

The only such pairs are (1,1) and (3,2).

2019 IMO Shortlist 2 (Bounding, Factoring Equations)

Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

Solution.

WLOG, assume that $a \geq b \geq c$.

We are going to bound a from two sides.

$$3a^3 \geq a^3 + b^3 + c^3 = (abc)^2$$
$$3a \geq b^2c^2$$

Since $a^2 \mid (abc)^2 = a^3 + b^3 + c^3$, then $a^2 \mid b^3 + c^3$.

$$a^2 \leq b^3 + c^3$$

$$\therefore \left(\frac{b^2c^2}{3}\right)^2 \leq a^2 \leq b^3 + c^3$$
$$\frac{b^4c^4}{9} \leq b^3 + c^3$$
$$\frac{bc^4}{9} \leq 1 + \left(\frac{c}{b}\right)^3 \leq 2$$
$$bc^4 \leq 18.$$

If $c \geq 2$, then $b \geq 2, bc^4 \geq 32$. (Contradiction)

$$\therefore c = 1.$$

In original equation, $a^3 + b^3 + 1 = a^2b^2$

$$b^2(a^2 - b) = a^3 + 1$$

Clearly, $(a, b) = (1,1)$ is not a solution. $\therefore a > 1$,

$$a^2 - b \mid a^3 + 1$$

To reduce the right side,

$$a^2 - b \mid a^3 + 1 - a(a^2 - b) = ab + 1$$

If $a = b$, the original equation: $2a^3 + 1 = a^4$

$$a^3(a - 2) = 1 \ (impossible)$$

$$\therefore a > b, a \geq b + 1$$

Since $ab + 1$ is divisible by $a^2 - b$,

$$ab + 1 \geq a^2 - b \geq a(b + 1) - b = ab + a - b \geq ab + 1.$$

Since the leftmost and the rightmost are equal, the terms must be strictly equal.

$$\therefore a = b + 1.$$

In original equation,

$$
\begin{aligned}
a^3 + (a-1)^3 + 1 &= a^2(a-1)^2 \\
a^3 + 1 &= (a-1)^2(a^2 - a + 1) \\
(a+1)(a^2 - a + 1) &= (a-1)^2(a^2 - a + 1) \\
(a^2 - a + 1)a(a-3) &= 0
\end{aligned}
$$

$$\therefore a = 3$$

$(a, b, c) = (3,2,1)$ and its permutations.

■

2004 APMO P4 (Parity, Wilson's Theorem)

Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Solution

Let $x = \left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$. For $1 \leq n \leq 5, x = 0$. So, we will consider $n \geq 6$.

We will apply Wilson's theorem.

So, there will be three cases, (i) $both\ n\ and\ n+1 = composite$, (ii) $n = prime$, (iii) $n+1 = prime$

(i) If $both\ n\ and\ n+1 = composite$

$$gcd(n, n+1) = 1, n(n+1)\ divides\ (n-1)!$$

$$v_2(n-1)! > v_2(n(n+1))$$

$$x = \left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor = even$$

(ii) If $n = p$ is a prime,

$$x = \left\lfloor \frac{(p-1)!}{p(p+1)} \right\rfloor$$

Obviously $p+1$ is even and divides $(p-1)!$

$$v_2(p-1)! > v_2(p+1)$$

$$Let\ k = \frac{(p-1)!}{p+1} = even$$

By Wilson's theorem

$$k = \frac{(p-1)!}{p+1} \equiv -\frac{1}{1} \equiv -1\ (mod\ p)$$

$\therefore p \mid k+1$ which is odd integer.

$$\frac{k+1}{p} = odd$$

$$\frac{\frac{(p-1)!}{p+1} + 1}{p} = \frac{(p-1)!}{p(p+1)} + \frac{1}{p} = odd$$

$$\therefore x = \left\lfloor \frac{(p-1)!}{p(p+1)} \right\rfloor = even$$

(iii) If $p = n+1$ is a prime,

$$x = \left\lfloor \frac{(p-2)!}{(p-1)p} \right\rfloor$$

Obviously $p-1$ is even and divides $(p-2)!$

$$v_2(p-2)! > v_2(p-1)$$

$$Let\ k' = \frac{(p-2)!}{p-1} = even$$

By Wilson's theorem

$$k' = \frac{(p-2)!}{p-1} \equiv \frac{(p-1)!}{(p-1)^2} \equiv -\frac{1}{1} \equiv -1\ (mod\ p)$$

$\therefore p \mid k'+1$ which is odd integer.

$$\frac{k'+1}{p} = odd$$

$$\frac{\frac{(p-2)!}{p-1}+1}{p} = \frac{(p-2)!}{(p-1)p} + \frac{1}{p} = odd$$

$$\therefore x = \left\lfloor \frac{(p-2)!}{(p-1)p} \right\rfloor = even$$

■