ENGG2400 Mechanics of Solids
Demonstration Notes

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Contents

1 Geometric Properties of Cross-sections

1.1 Centroid

A point where the weight of a body acts

$$\overline{x} = \frac{\int_A x dA}{\int_A dA} = \frac{\sum xA}{\sum A} \quad \text{and} \quad \overline{y} = \frac{\int_A y dA}{\int_A dA} = \frac{\sum yA}{\sum A} \tag{1}$$

1.2 Moment of Inertia

- Indication of bending resistance

$$I_{xx} = \int_A y^2 dA, \quad I_{yy} = \int_A x^2 dA \tag{2}$$

- Product Second Moment of Area, $I_{xy} = 0$ (i) at principal axes (ii) when an object is symmetric in either x axis or y axis.

$$I_{xy} = \int_A xy dA \tag{3}$$

- For a rectangular section about its centroidal axis,

$$I_{xx} = \frac{bd^3}{12}, \quad I_{xy} = 0 \tag{4}$$

- For a rectangular section about its bottom face,

$$I_{xx} = \frac{bd^3}{3} \tag{5}$$

1.3 Parallel Axis Theorem

$$I_{xx} = \sum \left(I_{x'x'} + d_y^2 A\right), \qquad I_{yy} = \sum \left(I_{y'y'} + d_x^2 A\right) \quad \text{and} \quad I_{xy} = \sum \left(I_{x'y'} + d_x d_y A\right) \tag{6}$$

where $d_y = \overline{y_i} - \overline{y}$, $d_y = \overline{x_i} - \overline{x}$ and d_x and d_y are distance between axis of rotation and centroid of a subdivided body

1.4 Rotation of Axes

$$\begin{aligned} I_{uu} &= I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - 2I_{xy}\sin\theta\cos\theta \\ I_{vv} &= I_{xx}\sin^2\theta + I_{yy}\cos^2\theta + 2I_{xy}\sin\theta\cos\theta \\ I_{uv} &= I_{xx}\sin\theta\cos\theta - I_{yy}\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}$$

1.5 Principal Second Moments of Area

$$I_{min/max} = \left(\frac{I_{xx} + I_{yy}}{2}\right) \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2} \tag{7}$$

Alternatively using **Principal Angle (θ_p)** which has two solutions, θ_{p1} and θ_{p2}, 90° apart,

$$\tan 2\theta_p = \frac{-2I_{xy}}{(I_{xx} - I_{yy})} \tag{8}$$

$$I_1 = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2}\cos 2\theta_{p1} - I_{xy}\sin 2\theta_{p1}$$

$$I_2 = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2}\cos 2\theta_{p1} + I_{xy}\sin 2\theta_{p1}$$

- The **"Inertia Tensor"** is a mathematical concept which prescribes the matrix of Second Moments of Area for a given shape:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix} \quad \text{and} \quad \mathbf{I} = \begin{bmatrix} I_{x'x'} & I_{x'y'} \\ I_{x'y'} & I_{y'y'} \end{bmatrix}$$

Invariants: trace and determinant

$$\text{trace}(\mathbf{I}) = I_{xx} + I_{yy} = I_{x'x'} + I_{y'y'} = I_1 + I_2$$

$$\det(\mathbf{I}) = \begin{vmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{vmatrix} = \begin{vmatrix} I_{x'x'} & I_{x'y'} \\ I_{x'y'} & I_{y'y'} \end{vmatrix} = I_{xx}I_{yy} - I_{xy}^2 = I_{x'x'}I_{y'y'} - I_{x'y'}^2$$

2 Concept of Stress and Transformation of Stress

2.1 Normal Stress

Axial forces in a two-force member cause normal stresses

$$P = \int_A \sigma dA, \quad \sigma_{ave} = \frac{P}{A} \tag{9}$$

Axial force resultant acts through the centroid.

2.2 Shear Stress

Transverse forces exerted on bolts and pins cause shear stresses

$$V = \int_A \tau dA, \quad \tau_{ave} = \frac{V}{A} \tag{10}$$

2.3 Differential Equations of Equilibrium

$$\frac{d\sigma_{xx}}{dx} + \frac{d\tau_{yx}}{dy} = 0, \quad \frac{d\sigma_{yy}}{dy} + \frac{d\tau_{xy}}{dx} = 0, \quad \tau_{xy} = \tau_{yx} \tag{11}$$

2.4 Transformation of Stresses

- Normal stress and shear stress transformation

$$\begin{aligned}
\sigma_{x'x'} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\
\sigma_{y'y'} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \\
\tau_{x'y'} &= -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\end{aligned}$$

- The principal stresses represent the **maximum** and **minimum** normal stress at the point.

- Principal angle: Orientation of the planes of maximum and minimum normal stress (principal planes). No shear stress acts on the principal plane.

$$\left.\frac{d\sigma_{x'}}{d\theta}\right|_{\theta=\theta_p} = \left(-\frac{\sigma_{xx} - \sigma_{yy}}{2}\right) 2 \sin 2\theta_p + 2\tau_{xy} \cos 2\theta_p = 0$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})/2}$$

- As $\tan 2\theta_p = \tan(180° + 2\theta_p)$, two solutions exist for the **principal angle**:

$$\theta_{p1} \quad \text{and} \quad \theta_{p2} = 90° + \theta_{p1} \tag{12}$$

- Principal in-plane normal stresses

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta_{p1} + \tau_{xy} \sin 2\theta_{p1} \tag{13}$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta_{p1} - \tau_{xy} \sin 2\theta_{p1} \tag{14}$$

  Alternatively

$$\sigma_{1,2} = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

- Orientation of the plane of maximum shear stress:

$$\left.\frac{d\tau_{x'y'}}{d\theta}\right|_{\theta=\theta_s} = \frac{\sigma_x - \sigma_y}{2} 2 \cos 2\theta_s - 2\tau_{xy} \sin 2\theta_s = 0 \tag{15}$$

$$\therefore \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}} = -\frac{1}{\tan 2\theta_p} \tag{16}$$

$$2\theta_s = 90^\circ + 2\theta_p \quad \Rightarrow \quad \theta_s = 45^\circ + \theta_p \tag{17}$$

- The plane for maximum shear stress is orientated 45° from principal planes.

2.5 Mohr's Circle for Stress Transformation

- Maximum shear stress or radius of Mohr's Circle

$$R = \tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2} \tag{18}$$

- Center of Mohr's Circle

$$C = \sigma_c = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \tag{19}$$

- Equation of Mohr's Circle

$$(x - C)^2 + y^2 = R^2 \tag{20}$$

  where, $x = \sigma_{x'x'}$ = normal stress and $y = \tau_{x'y'}$ = shear stress.

- Principal stresses

$$\sigma_1 = C + R, \quad \sigma_2 = C - R$$

- Sign Conventions for stress and angle
  1. Tensile stress is positive and compressive stress is negative.
  2. Two points on the circle are $(\sigma_{xx}, +\tau_{xy})$ and $(\sigma_{yy}, -\tau_{xy})$.
  3. A positive angle is clockwise on Mohr's circle. This is opposite to the rotation of axes.
  4. Rotating an angle θ of an element clockwise is equivalent to rotating the line in the Mohr's circle by 2θ anticlockwise.

- When $\sigma_{xx} = \sigma_{yy}$ and $\tau_{xy} = 0$, the Mohr's circle is a point and it is pure hydrostatic stress.

3 Concept of Strain and Transformation of Strain

3.1 Normal Strain

Normal strain causes a change in volume of a rectangular element

$$\varepsilon = \frac{de}{dx}, \quad e = \int_0^L \varepsilon dx, \quad \varepsilon_{ave} = \frac{e}{l} \tag{21}$$

3.2 Shear Strain

Shear strain causes a change in shape

$$\gamma = \frac{\pi}{2} - \theta \tag{22}$$

3.3 Transformation of Strains

Note: $\sigma_{xx} \Leftrightarrow \varepsilon_{xx}, \quad \sigma_{yy} \Leftrightarrow \varepsilon_{yy}, \quad \tau_{xy} \Leftrightarrow \frac{\gamma_{xy}}{2}$

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'y'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Principal Strain

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{1,2} = \left(\frac{\varepsilon_{xx} + \varepsilon_{yy}}{2}\right) \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Maximum In-Plane Shear Strain

$$\tan 2\theta_s = -\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}$$

3.4 Mohr's Circle for Strain Transformation

- Maximum shear strain or radius of Mohr's Circle

$$R = \frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \frac{\varepsilon_1 - \varepsilon_2}{2} \tag{23}$$

- Center of Mohr's Circle

$$C = \varepsilon_c = \frac{\varepsilon_1 + \varepsilon_2}{2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \tag{24}$$

- Equation of Mohr's Circle
$$(x - C)^2 + y^2 = R^2 \tag{25}$$
where, $x = \varepsilon_{x'x'}$ = normal strain and $y = \frac{\gamma_{x'y'}}{2}$ = shear strain.

- Principal strains
$$\varepsilon_1 = C + R, \quad \varepsilon_2 = C - R$$

- Sign Conventions for strain and angle
  1. Tensile strain is positive and compressive strain is negative.
  2. Two points on the circle are $\left(\varepsilon_{xx}, +\frac{\gamma_{xy}}{2}\right)$ and $\left(\varepsilon_{yy}, -\frac{\gamma_{xy}}{2}\right)$.
  3. A positive angle is clockwise on Mohr's circle. This is opposite to the rotation of axes.

3.5 Application - Measurement of Stresses

Strain Rosettes If "Material Properties" are known, then we can calculate the stresses from the strains.

$$\begin{aligned}
\varepsilon_a &= \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \\
\varepsilon_b &= \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b \\
\varepsilon_c &= \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c
\end{aligned}$$

Strain rosettes are often arranged in 45° or 60° patterns. For the 45° or "rectangular" strain rosette, with angles: $\theta_a = 0°, \quad \theta_b = 45°, \quad \theta_c = 90°$, the resulting strain components are:

$$\begin{aligned}
\varepsilon_x &= \varepsilon_a \\
\varepsilon_y &= \varepsilon_c \\
\gamma_{xy} &= 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)
\end{aligned}$$

For the 60° strain rosette with angles: $\theta_a = 0°, \quad \theta_b = 60°, \quad \theta_c = 120°$, the strain components become:

$$\begin{aligned}
\varepsilon_x &= \varepsilon_a \\
\varepsilon_y &= \frac{1}{3}\left(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a\right) \\
\gamma_{xy} &= \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c)
\end{aligned}$$

4 Axial Deformations

4.1 Mechanical Properties

- Young's Modulus (Hooke's Law) Slope of stress-strain curve in elastic region

$$\sigma_{xx} = E\varepsilon_{xx}$$

- Poisson's Ratio

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}} = -\frac{d\varepsilon_y}{d\varepsilon_x} = -\frac{d\varepsilon_z}{d\varepsilon_x}$$

- Shear Modulus

$$G = \frac{E}{2(1+\nu)}$$

4.2 Generalized Hooke's Law for Isotropic Bodies (3D)

Strains

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \frac{\nu\ \sigma_{yy}}{E} - \frac{\nu\ \sigma_{zz}}{E} \\
\varepsilon_{yy} &= -\frac{\nu\ \sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \frac{\nu\ \sigma_{zz}}{E} \\
\varepsilon_{zz} &= -\frac{\nu\ \sigma_{xx}}{E} - \frac{\nu\ \sigma_{yy}}{E} + \frac{\sigma_{zz}}{E} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\gamma_{zx} &= \frac{\tau_{zx}}{G}
\end{aligned}$$

Stresses

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}\right] \\
\tau_{xy} &= G\gamma_{xy} \\
\tau_{yz} &= G\gamma_{yz} \\
\tau_{zx} &= G\gamma_{zx}
\end{aligned}$$

4.3 Plane Stress and Plane Strain Problems (2D)

These expressions are derived by applying the conditions to the generalized 3D forms

Plane Stress

(conditions: $\sigma_{zz} = \tau_{yz} = \tau_{zx} = 0$)

Strains

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \\
\varepsilon_{yy} &= \frac{1}{E}\left(-\nu\sigma_{xx} + \sigma_{yy}\right) \\
\varepsilon_{zz} &= -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) \neq 0 \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}$$

Stresses

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left(\nu\varepsilon_{xx} + \varepsilon_{yy}\right) \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}$$

Plane Strain

(conditions: $\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$)

Strains

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right) \\
\varepsilon_{yy} &= \frac{1-\nu^2}{E}\left(-\frac{\nu}{1-\nu}\sigma_{xx} + \sigma_{yy}\right) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}$$

Stresses

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}\right] \\
\sigma_{zz} &= \nu(\sigma_{xx} + \sigma_{yy}) = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) \neq 0 \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}$$

4.4 Axial Deformation

Elongation

$$e = \int_0^L \varepsilon dx$$

Elongation due to axial force,

$$e = \int_0^L \frac{P(x)}{E(x)A(x)} dx = \sum \frac{PL}{AE}$$

Elongation due to thermal stress

$$e = \alpha l \Delta T$$

Sign convention: elongation is positive and contraction is negative

It is convenient to use N and mm for elongation calculations. Unit conversion: $kN/m = N/mm, N/mm^2 = MPa$

4.5 Statically Indeterminate Structures

In statically indeterminate structures, draw **deformed shape** to write compatibility equations.

No. of compatibility equations = no. of unknown reactions − no. of equilibrium equations

5 Stresses due to Bending

5.1 Elastic Bending

Curvature

$$\kappa = \frac{M}{EI} = \frac{d\theta}{dx} = \frac{1}{\rho} = \frac{\varepsilon_{max}}{c} = \frac{\varepsilon}{y}$$

where ρ = radius of curvature, ε_{max} = maximum strain at outmost fiber, c = distance between neutral axis and outermost fiber

Bending stress under linear elastic condition (Triangular stress block)

$$\sigma = -\frac{My}{I}$$

where y is positive above neutral axis (compression) and negative below neutral axis (tension).

Elastic section modulus

$$Z = \frac{I}{y_{max}}$$

5.2 Stresses in composite sections

Cross sections are transformed. Modular ratio,

$$n = \frac{E_1}{E_2}$$

For a rectangular cross section, width is transformed during the calculation of centroid,

$$\bar{b} = nb$$

Stresses in transformed cross section,

$$\bar{\sigma} = n\sigma$$

5.3 Inelastic bending

- The flexure formula only allows us to determine normal stress due to bending if the material behaves in a linear elastic manner.
- If the applied moment causes the material to yield, a "plastic analysis" must then take place to determine the stress distribution
- When fully elastic, the neutral axis is always located at the centroidal axis. If the cross-section is non-symmetric, or if the stress-strain curve is different in tension vs compression, the neutral axis will not be located at the centroidal axis when inelastic.
- Elastic moment: the moment that will first cause yield in the material (the outermost fiber); **triangular stress block**. The neutral axis is the centroidal axis.

  $$M_Y = \sum M_{NeutralAxis} = \sigma_y \left(\frac{I}{y_{max}}\right) = f_y Z$$

  where $\sigma_y = f_y$ = yield stress, Z = elastic section modulus

- Plastic moment: the moment that will cause the entire cross section to be plastic, that is, the whole section reaches yield stress; **rectangular stress block**. The neutral axis is located where the area above and below are the same; that is, the cross-sectional area below and above the neutral axis must be equal.

$$M_P = \sum M_{NeutralAxis} = \sum F_i y_i$$

- Shape Factor: ratio of plastic to elastic moment

$$k = \frac{M_P}{M_Y}$$

  For a rectangular cross-section, this shape factor is 1.5

- In either elastic or inelastic or partially inelastic bending, the location of the neutral axis can be found by equating the compressive forces and tensile forces. The force is the volume of the stress prism, that is, it can be determined by multiplying the area under the stress diagram by the width of the section.

6 Deflections due to Bending

6.1 Moment-Curvature Relationship

For small deflections,

$$\begin{aligned}
\text{Slope of elastic curve,} \quad & \frac{dv}{dx} = \tan\theta \approx \theta \\
& \frac{d^2v}{dx^2} = \frac{d\theta}{dx} = \frac{M}{EI} \quad \iff \quad M = EI\frac{d^2v}{dx^2} \\
\text{Slope of elastic curve,} \quad & \frac{dv}{dx} = \theta(x) = \int \frac{M(x)}{EI}dx + C_1 \\
\text{Deflection of elastic curve,} \quad & v(x) = \int \theta(x)dx = \iint \frac{M(x)}{EI}dxdx + C_1x + C_2
\end{aligned}$$

6.2 Boundary conditions

Compatibility equations from boundary conditions are used to solve indeterminate beams.

- Roller and pinned support

$$\text{Deflection,} \quad v = 0$$

- Fixed support

$$\text{Deflection,} \quad v = 0 \quad \text{and slope,} \quad \frac{dv}{dx} = 0$$

6.3 Step function

When the moment function is discontinuous, piecewise functions can be combined to form a single expression using step functions.

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases}$$

$$\int_{-\infty}^{x} \langle x-a\rangle^n \, dx = \frac{1}{n+1}\langle x-a\rangle^{n+1}$$

Alternatively, the Heaviside function,

$$H(x-a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x \geq a \end{cases}$$

- Take section at the farthest interval to write the moment function.
- Downward uniformly distributed (UD) load
  In moment function, start; subtract $\frac{w}{2}\langle x-L_1\rangle^2$. End; add $\frac{w}{2}\langle x-L_2\rangle^2, L_2 > L_1$.

$$\begin{aligned}
M = EI\frac{d^2v}{dx^2} &= -\frac{w}{2}\langle x-L_1\rangle^2 + \frac{w}{2}\langle x-L_2\rangle^2 \\
\theta = EI\frac{dv}{dx} &= -\frac{w}{6}\langle x-L_1\rangle^3 + \frac{w}{6}\langle x-L_2\rangle^3 + C_1 \\
\Delta = EIv &= -\frac{w}{24}\langle x-L_1\rangle^4 + \frac{w}{24}\langle x-L_2\rangle^4 + C_1x + C_2
\end{aligned}$$

- Downward concentrated point load

  Replace $-P$ with $+R$ for upward reaction forces.

$$
\begin{aligned}
M = EI\frac{d^2v}{dx^2} &= -P\langle x - L_1\rangle \\
\theta = EI\frac{dv}{dx} &= -\frac{P}{2}\langle x - L_1\rangle^2 + C_1 \\
\Delta = EIv &= -\frac{P}{6}\langle x - L_1\rangle^3 + C_1x + C_2
\end{aligned}
$$

- Superposition can be used for combining the above scenarios.
- When $x < L_1$, the term $\langle x - L_1\rangle$ becomes 0.
- Sometimes, it's easier to begin writing the moment function from the free end or roller support rather than from the fixed support, since starting from the fixed end introduces an additional unknown, the fixed-end moment.

7 Shear Stress and Shear Flow

- Shear stress in a cross section, assumed to be constant and averaged across the width,

$$\tau = \frac{VQ}{Ib}$$

$$\text{and shear flow,} \quad q = \frac{VQ}{I} = \tau b = \frac{\Delta F}{\Delta x}$$

where, V = internal resultant shear force, determined from the method of sections, I = moment of inertia of the whole cross section, b = width at the point to be determined, $Q = \int y dA = \overline{y}' A'$ and A' = cross sectional area above the section plane which is above the neutral axis and below the section plane which is below the section plane.

- For I sections and box sections, maximum shear stress occurs at the neutral axis where the first moment of area Q is the largest. If the width varies along length, maximum shear stress takes place at a point where its first derivative $\frac{d\tau}{dy} = 0$.

- **Shear center** is the point on a cross-section through which a transverse shear force must act to produce no twisting of the section when taking moments of the forces about the centroid.

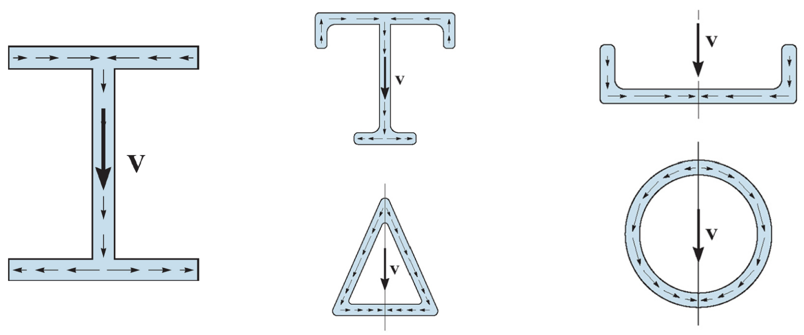
$$\sum M_{centroid} = Ve = \sum (\text{Horizontal shear force } \times \text{ moment arm})$$

Horizontal shear force can be found by integration of shear flow along the length and moment arm is the distance between horizonal shear force and the centroid. For a channel,

$$\sum M_{centroid} = Ve = F_f h$$

F_f = coupled force in flanges, h = distance between the force couple

- The directional sense of q is such that, the shear appears to "flow" through the cross section "inward" at the top flange-combining-then flowing downward through the web-separating and flowing "outward" at the bottom flange.



- Shear force is the rate of change of moment with respect to distance.

$$V = \frac{dM}{dx} = \frac{\Delta M}{\Delta x}$$

- Shear force in the web with constant width can be calculated by

$$V = \int \tau dA = \int \frac{VQ}{Ib} b dx = \int \frac{VQ}{I} dx$$

- Fastener spacing,

$$s = \frac{F}{q}$$

F = capacity of 1 fastener to resist shear,

8 Torsion

- Shear stress at any radial distance ρ due to torsion T,

$$\tau_\rho = \frac{T\rho}{J} \quad \text{and} \quad \tau_{max} = \frac{Tc}{J}$$

  where, polar moment of inertia, $J = \int_A \rho^2 dA$

- For a solid circular shaft with radius c,

$$J = \frac{\pi}{2}c^4 = \frac{\pi}{32}D^4$$

- For a hollow circular shaft,

$$J = \frac{\pi}{2}(c_0 - c_i)^4 = \frac{\pi}{32}(D_0 - D_i)^4$$

- Shear strain,

$$\gamma = \frac{\tau}{G} = \frac{T\rho}{GJ} = \frac{\rho}{c}\gamma_{max}$$

- Angle of twist (in radian),

$$\phi = \int_L \frac{T(x)}{G(x)J(x)}dx = \sum \frac{TL}{GJ}$$

  Take sections to use the internal torque T. Direction and units should be used consistently. For example, $MPa = N/mm^2$ for G, mm for L and J and Nmm for T. Radian can be converted to degree by multiplying by $\frac{180}{\pi}$.

- Write compatibility equations using angle of twist for indeterminate problems.

- Displacement due to twist,

$$s = \phi r$$

  where, r distance from the centre.