# NTL8 – Diophantine Equations (2)

## Modular Contradictions

Example 1. Find all pairs of integers $(x, y)$ that satisfy the equation

$$x^2 - y! = 2001.$$

Example 2. (2021 JBMO Shortlist) Find all positive integers $a, b, c$ such that

$$ab + 1, bc + 1, ca + 1$$

are all equal to factorial of some positive integers.

Example 3. (2002 IMO Shortlist 1) Find the smallest positive integer $t$ such that there exist integers $x_1, x_2, \dots, x_t$ with

$$x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002}.$$

## Infinite Descent

Example 4. (Fermat) Show that the only solution to the equation

$$x^3 + 2y^3 + 4z^3 = 0$$

in integers is (0,0,0).

## Vieta Jumping

Example 5. (2007 IMO P5) Let $a$ and $b$ be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

Example 6. (1988 IMO P6) If $a, b$ are positive integers such that $\frac{a^2+b^2}{1+ab}$ is an integer, then $\frac{a^2+b^2}{1+ab}$ is a perfect square.

## Vieta Jumping Practice Problems

MONT Eg 4.7.2 Let $a$ and $b$ be positive integers such that $ab$ divides $a^2 + b^2 + 1$. Show that

$$\frac{a^2 + b^2 + 1}{ab} = 3.$$

MONT Eg 4.7.4 Let $k$ be a positive integer not equal to 1 or 3. Prove that the only solution to

$$x^2 + y^2 + z^2 = kxyz$$

over integers is (0,0,0).

Example 5. (2007 IMO P5) Let $a$ and $b$ be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Solution.

Firstly, we are going to simplify $(4a^2-1)^2$ by subtracting suitable terms.

Since $\gcd(b, 4ab-1)=1$,

$$4ab-1 \mid (4a^2-1)^2 \Leftrightarrow 4ab-1 \mid b^2(4a^2-1)^2$$

$$4ab \equiv 1 (mod\ 4ab-1)$$

$$b^2(4a^2-1)^2 = 16a^4b^2 - 8a^2b^2 + b^2 = (4ab)^2a^2 - (4ab)(2ab) + b^2 \equiv a^2 - 2ab + b^2 \equiv (a-b)^2$$
$$\equiv 0\ (mod\ 4ab-1)$$

Since $4ab-1$ divides $(a-b)^2$,

$$\frac{(a-b)^2}{4ab-1} = k, where\ k\ is\ a\ positive\ integer \qquad (1)$$

WLOG, assume $a > b$.

Let $(a,b)=(a_1,b_1)$ be a solution to the above equation (1) with $a_1 > b_1$.

Assume that $a_1 + b_1$ has the smallest sum among all pairs $(a,b)$ with $a > b$.

For the sake of contradiction, we are going to prove there exists another solution $(a,b)=(a_2,b_1)$ with a smaller sum, i.e., $a_2 < a_1, a_2 > 0$.

See the equation 1 as a quadratic in $a$,

$$\frac{(a-b_1)^2}{4ab_1-1} = k, a^2 - a(2b_1 + 4b_1k) + b_1^2 + k = 0$$

This equation has roots $a = a_1, a_2$, using Vietas,

$$a_1 + a_2 = 2b_1 + 4b_1k \qquad (2)$$

$$a_1a_2 = b_1^2 + k \qquad (3)$$

From (3), since $b_1^2 + k > 0$, $a_2 > 0$. We are to show $a_2 < a_1$.

$$a_2 < a_1$$

$$\Leftrightarrow \frac{b_1^2 + k}{a_1} < a_1$$

$$\Leftrightarrow b_1^2 + k < a_1^2$$

$$\Leftrightarrow b_1^2 + \frac{(a_1 - b_1)^2}{4a_1b_1 - 1} < a_1^2$$

$$\Leftrightarrow \frac{(a_1 - b_1)^2}{4a_1b_1 - 1} < (a_1 - b_1)(a_1 + b_1)$$

$$\Leftrightarrow \frac{a_1 - b_1}{4a_1b_1 - 1} < (a_1 + b_1)\ (\because a_1 > b_1)$$

The last inequality is true because $4a_1b_1 - 1 > 1$. Therefore $a > b$ is impossible. Similarly, it is impossible to have $b > a$. $\therefore a = b$. ■