# NTP5 – p-adic Valuation

*Homework code* : NTP5

*Issued on* : 8th June 2023

*Due date* : 15th June 2023

## Problems 1 to 5 are each worth 5 points.

Problem 1. Prove that $lcm[a, b, c]^2 \mid lcm[a, b]\ lcm[b, c]\ lcm[c, a]$ for any positive integers $a, b, c$.

Problem 2. Prove that there exists a constant c such that for any positive integers $a, b$ and $n > 1$ satisfying $a! \cdot b! \mid n!$, we have $a + b < n + c \log n$.

Problem 3. Show that if $n \geq 6$ is composite, then $n$ divides $(n - 1)!$.

Problem 4. (1975 USAMO P1) Prove that

$$\lfloor 5x \rfloor + \lfloor 5y \rfloor \geq \lfloor 3x + y \rfloor + \lfloor 3y + x \rfloor,$$

where $x, y \geq 0$. Using this or otherwise, prove that

$$\frac{(5m)!\,(5n)!}{m!\,n!\,(3m + n)!\,(3n + m)!}$$

is integral for all positive integers $m$ and $n$.

Hint: Introduce the fractional part.

Problem 5. Prove that for all integers $n \geq 1$,

$$C_n = \frac{1}{n + 1}\binom{2n}{n} \in \mathbb{Z}$$

(The number $C_n$ is called nth Catalan number.)

## Further Problems

## Further Problems are each worth 5 special points.

F Problem 1. (APMO 2017 P4) Call a rational number $r$ powerful if $r$ can be expressed in the form $p^k/q$ for some relatively prime positive integers $p, q$ and some integer $k > 1$. Let $a, b, c$ be positive rational number such that $a^x + b^y + c^z$ is an integer. Prove that $a, b, c$ are all powerful.

Hint: Consider a prime $p$ and show that if $v_p(a) > 0$, then it is divisible by some fixed integer $k > 1$, which is independent of $p$.