Euclidiad Introduction to Number Theory (Short Course)
Lecture 2 – GCD and LCM

Quiz: Find gcd (34417,34959).

Example 1.
What is the smallest positive integer N such that the value $7 + 30N$ is not a prime number?

Example 2.
The largest prime factor of 16384 is 2 because $16384 = 2^{14}$. What is the sum of the digits of the largest prime factor of 16383?

Example 3.
A 3-digit number has exactly 9 positive factors. Find the maximum value of this 3-digit number.

Terminology

Greatest Common Divisor (GCD)
GCD of two or more integers is the largest positive integer that divides each of the integers.
For example, $\gcd(8,12) = 4$

Least Common Multiple (LCM)
LCM of two (or more) integers, a and b, is the smallest positive integer that is divisible by both a and b.
For example, $lcm[8,12] = 24$

Coprime Numbers (Relatively Prime Numbers)
Two Numbers are called relatively prime, or coprime, if their greatest common divisor is 1.
For example, $\gcd(9,28) = 1$. So 9 and 28 are relatively prime.
A fraction is irreducible or in lowest terms or reduced form when the numerator and denominator are relatively prime.

Example 4.
(a) Find gcd $(80, 144)$ and $lcm[80, 144]$.
(b) Find gcd $(160, 288)$ and $lcm[160, 288]$.
(c) Find gcd $(240, 432)$ and $lcm[240, 432]$.
(d) Find gcd $(400, 720)$ and $lcm[400, 720]$.
(e) Find gcd $(80n, 144n)$ and $lcm[80n, 144n]$ for any positive integer n.

Example 5.
(a) Find gcd $(18, 30)$ and $lcm[18, 30]$.
(b) Use the relationships between $18, 30, \gcd(18,30)$, $lcm[18,30]$ to create a single equation that uses all four numbers and no others.

Relationships between GCD and LCM

The product of two natural numbers, m, n, is equal to the product of their GCD and LCM.

$$m \times n = \gcd(m, n) \times lcm[m, n]$$

For any positive integers a, b, and c,

$$\gcd(ac, bc) = c \times gcd(a, b)$$
$$lcm[ac, bc] = c \times lcm[a, b]$$

Example 6. A box contains gold coins. If the coins are equally divided among six people, four coins are left over. If the coins are equally divided among five people, three coins are left over. If the box holds the smallest number of coins that meets these two conditions, how many coins are left when equally divided among seven people?

Example 7. The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c?

Example 8.
(a) A 4-digit number has a remainder of 6 when divided by 7, has a remainder of 7 when divided by 8, has a remainder of 8 when divided by 9. How many such possible 4-digit numbers are there?
(b) A 4-digit number has a remainder of 1 when divided by 7, has a remainder of 1 when divided by 8, has a remainder of 1 when divided by 9. How many such possible 4-digit numbers are there?

Example 9. Tayza writes the cubes of three positive integers on a piece of paper. Thura points out that each is a multiple of 20. Tayza then points out that the GCD of all three perfect cubes is n. Find the smallest possible value of n.

Euclidean Algorithm

For any integers m, n,

$$\gcd(m, n) = \gcd(m - n, n)$$

Let m and n be integers such that $m = qn + r$, where $0 \le r < n$, then

$$\gcd(m, n) = \gcd(r, n)$$

Lecture 2 - Homework Problems

Homework code	:	HW2
Issued on	:	29th March 2022
Due date	:	4th April 2022

Submit the solutions to at least 4 of the homework problems before due date.
Problems 1 to 7 are each worth 5 points. Challenge problem is 10 points worth.

Problem 1. Find the five smallest multiples of 18 and 30 that are both perfect squares and perfect cubes.

Problem 2. The least common multiple of $12, 15, 20$ and k is 420. What is the least possible value of k?

Problem 3. When a three-digit number is divided by 2, 3, 4, 5 and 7, the remainders are all 1. Find the minimum and maximum values of such three-digit numbers.

Problem 4. Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

Problem 5. A group of 10 friends were discussing a large positive integer. "It can be divided by 1," said the first friend. "It can be divided by 2," said the second friend. "And by 3," said the third friend. "And by 4," added the fourth friend. This continued until everyone had made such a comment. If exactly two friends were incorrect, and those two friends said consecutive numbers, what was the least possible integer they discussing?

Problem 6. Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1086 and 828. Find the largest possible value of the common width of the rectangles.

Problem 7. Prove that for all positive integers n, the fraction $\frac{21n+4}{14n+3}$ is irreducible.

Challenge Problem

Problem 8. How many positive integers are multiples of 2013 and have exactly 2013 divisors (including 1 and the number itself)?