NTP1 – Euclidean and Division Algorithm

Homework code : NTP1

Issued on : 11th May 2023

Due date : 18th May 2023

Problems 1 to 10 are each worth 5 points.

Problem 1. Prove that $\sqrt{p}$ is irrational for any prime p.

Problem 2. Let n be an integer greater than 2. Prove that among the fractions

$$\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$$

an even number of them are irreducible.

Problem 3. Let $1, 4, \dots$ and $9, 16, \dots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

Problem 4. Let the integers a_n and b_n be defined by the relationship

$$a_n + b_n\sqrt{2} = \left(1 + \sqrt{2}\right)^n.$$

for all integers $n \geq 1$. Prove that $\gcd(a_n, b_n) = 1$ for all integers $n \geq 1$.

Problem 5. Prove that if k is odd, then 2^{n+2} divides

$$k^{2^n} - 1$$

for all natural numbers n.

Problem 6. Given a sequence of six strictly increasing positive integers such that each number (besides the first) is a multiple of the one before it and the sum of all six numbers is 79, what is the largest number in the sequence?

Problem 7. A positive integer is written on each face of a cube. Each vertex is then assigned the product of the numbers written on the three faces intersecting the vertex. The sum of the numbers assigned to all the vertices is equal to 1001. Find the sum of the numbers written on the faces of the cube.

Problem 8. Let m, n be positive integers such that

$$\gcd(m, n) + lcm(m, n) = m + n.$$

Show that one of the two numbers is divisible by the other.

Problem 9. Call a number *prime looking* if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

Problem 10. Prove that if p is a prime and $0 < k < p$, then $\binom{p}{k}$ is divisible by p.

Further Problems

Further Problems are each worth 5 special points.

F Problem 1. (USAMO 1978) An integer n will be called good if we can write

$$n = a_1 + a_2 + \cdots + a_k$$

where $a_1, a_2, \dots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given the information that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

F Problem 2. (MOSP 1998) Let $S(x)$ be the sum of the digits of the positive integer x in its decimal representation.

(a) Prove that for every positive integer x, $\frac{S(x)}{S(2x)} \leq 5$. Can this bound be improved?

(b) Prove that $\frac{S(x)}{S(3x)}$ is not bounded.

F Problem 3. (APMO) Are there distinct prime numbers a, b, c which satisfy

$$a|bc + b + c, b|ca + c + a, c|ab + a + b?$$

F Problem 4. (Iran 2005) Let $n, p > 1$ be positive integers and p be prime. Given that $n|p - 1$ and $p|n^3 - 1$, prove that $4p - 3$ is a perfect square.

F Problem 5. Most positive integers can be expressed as a sum of two or more consecutive positive integers. For example, $24 = 7 + 8 + 9$ and $51 = 25 + 26$. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all interesting integers?

F Problem 6. (China 2001) We are given three integers a, b and c such that $a, b, c, a + b - c, a + c - b, b + c - a$ and $a + b + c$ are seven distinct primes. Let d be the difference between the largest and smallest of these seven primes. Suppose that 800 is an element in the set $\{a+b, b+c, c+a\}$. Determine the maximum possible value of d.

F Problem 7. If p is an odd prime, and a, b are coprime, show that

$$\gcd\left(\frac{a^p+b^p}{a+b}, a+b\right) \in \{1, p\}$$

F Problem 8. (IMO 1986) Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab-1$ is not a perfect square.

F Problem 9. (IMO 1988) If a, b are positive integers such that $\frac{a^2+b^2}{1+ab}$ is an integer, then $\frac{a^2+b^2}{1+ab}$ is a perfect square.