Euclidiad Olympiad Training LEVEL 1
Day 15 - Factorization Formulae of Polynomial Expressions

- A polynomial in one variable has the expression $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$, where x is the variable and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants for each individual term, also known as coefficients. The degree of the polynomial is given by the highest power of the variables.
- An expression is a sentence with a minimum of two numbers and at least one math operation.

1 Basic Formulae

$$\begin{aligned}
a^2-b^2 &= (a+b)(a-b)\\
(a+b)^2 &= a^2+2ab+b^2\\
(a-b)^2 &= a^2-2ab+b^2\\
(a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ca\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b)\\
(a-b)^3 &= a^3-3a^2b+3ab^2-b^3 = a^3-b^3-3ab(a-b)\\
a^3+b^3 &= (a+b)(a^2-ab+b^2)\\
a^3-b^3 &= (a-b)(a^2+ab+b^2)
\end{aligned}$$

2 Derived Basic Formulae

$$\begin{aligned}
a^2+b^2 &= (a+b)^2-2ab\\
a^2+b^2 &= (a-b)^2+2ab\\
(a+b)^2-(a-b)^2 &= 4ab\\
a^3+b^3 &= (a+b)^3-3ab(a+b)\\
a^3-b^3 &= (a-b)^3+3ab(a-b)\\
a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)
\end{aligned}$$

Examples Given in Class

Example 1. Prove that $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$.

Example 2. Prove that $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, and hence, if $a+b+c=0$, prove that $a^3+b^3+c^3 = 3abc$.

Example 3. If $a+b=3$ and $ab=4$, find the value of a^2+b^2, a^3+b^3 and a^4+b^4.

Example 4. It is given that $(x+y)^2=64$ and $(x-y)^2=4$. Find the value of $\frac{x}{y}+\frac{y}{x}$.

Example 5. It is given that $x^2-5x+1=0$. Find the value of $x^2+\frac{1}{x^2}$.

Example 6. Evaluate the expression

$$(2+1)(2^2+1)(2^4+1)\cdots(2^{2^{10}}+1).$$

Example 7. Find all possible values of $x^3+\frac{1}{x^3}$, given that $x^2+\frac{1}{x^2}=7$.

Example 8. If q is an integer that can be expressed as the sum of two integer squares, show that both $2q$ and $5q$ can also be expressed as the sum of two integer squares.

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Day 15 - Homework

Homework code : **HWA106**
Issued on : 21st June 2021
Due date : 4th July 2021

Submit the solutions to at least 6 of the homework problems before the due date. Problems 1-10 are each worth 5 points. Challenge problems are worth 10 points each.

1. Prove that $(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$.

2. Find $\frac{1}{a} + \frac{1}{b}$ if $a + b = 6$ and $ab = 3$.

3. It is given that $m + \frac{1}{m} = 4$. Find the value of $m^4 + \frac{1}{m^4}$.

4. It is given that $a - b = 4$. Find the value of $a^3 - b^3 - 12ab$.

5. It is given that $a + b = 4$ and $a^3 + b^3 = 28$. Find the value of $a^2 + b^2$.

6. It is given that $x - \frac{1}{x} = 5$. Find the value of $x + \frac{1}{x}$.

7. It is given that $(x + \frac{1}{x})^2 = 3$. Find the value of $x^3 + \frac{1}{x^3}$.

8. It is given that $\frac{x}{2} + \frac{2}{x} = 3$. Find the value of $\frac{x^2}{2} + \frac{8}{x^2}$.

9. If $a^3 - b^3 = 24$ and $a - b = 2$, then find all possible values of $a + b$.

10. For integers a, b, c and d, rewrite the expression $(a^2 + b^2)(c^2 + d^2)$ as a sum of squares of two integers.

Challenge Problems

11. Given that $a - b = 2$ and $b - c = 4$, find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

12. If $a + b = 1$, $a^2 + b^2 = 2$, find the value of $a^7 + b^7$.

13. Given that the real numbers a, b, c satisfy the system of equations

$$\begin{aligned} a+b+c&=6,\\ a^2+b^2+c^2&=26,\\ a^3+b^3+c^3&=90, \end{aligned}$$

Find the values of abc and $a^4+b^4+c^4$.