Euclidiad Introduction to Number Theory (Short Course)
Lecture 1 - Prime Factorization

Number theory is basically the study of integers.

Terminology

Integers: $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots$
Positive integers: $1, 2, 3, 4, 5, 6, \ldots$
Natural numbers: $1, 2, 3, 4, 5, 6, \ldots$
Perfect square (or just "square"): $1, 4, 9, 16, 25, 36, \ldots$
Perfect cube (or just "cube"): $1, 8, 27, 64, 125, \ldots$
Prime number: $2, 3, 5, 7, \ 11, 13, 17, 19, \ldots$
(An integer that has exactly two factors, 1 and itself)
Composite number: $4, 6, 8, 9, 10, 12, 14, 15, \ldots$
(An integer that has more than two factors)

Example 1. Determine which of the following are prime numbers: (i) 72, (ii) 71, (iii) 379, (iv) 299.

Prime Factorization

Key terms: Factor, divisor, divisible

Do you remember how to compute the g.c.d (greatest common divisor), for example, $\gcd(24, 30)$, the greatest common divisor of 16 and 24?

The prime factorization of an integer n is nothing but just the breaking down of n into the product of powers of prime numbers.

Example 2. Find the prime factorization of the following numbers: (i) 45, (ii) $4 \cdot 45$, (iii) 28^2, (iv) $\frac{150^3}{6}$.

Example 3. What is the smallest positive integer that is neither prime nor square and that has no prime factor less than 50?

Example 4. For how many positive integer values of n is the value $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer?

Example 5. Thura calculates $1 \times 2 \times 3 \times 4 \times \ldots \times 14 \times 15$ with his scientific calculator. Then, he divides the result by 2 again and again and stops as soon as the number is no longer an integer. How many times did he divide with 2?

Example 6.
(a) Find the prime factorizations of each of $14^2, 15^2, 16^2, 17^2, 18^2$.
(b) What do the prime factorizations of all perfect squares have in common.

Example 7. What do the prime factorizations of all perfect cubes have in common?

Example 8. Suppose n is a positive integer such that the number $2016 \cdot n$ is a perfect cube. Find the smallest possible value of n.

Example 9. Suppose n is a positive integer such that the number $\frac{2016}{n}$ is a perfect square. Find the smallest possible value of n.

Number of divisors

If

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_n^{a_n},$$

then the number of divisors of n is

$$\tau(n) = (a_1 + 1)(a_2 + 1)(a_3 + 1) \ldots (a_n + 1).$$

Example 10. Calculate the number of divisors of (i) 45, (ii) 180, (iii) 28^2.

Example 11. Why is the number of divisors of a perfect square always an odd number?

Lecture 1 - Homework Problems

Homework code	:	HW1
Issued on	:	22^{nd} March 2022
Due date	:	29^{th} March 2022

Submit the solutions to at least 4 of the homework problems before due date.
Problems 1 to 7 are each worth 5 points. Challenge problem is 10 points worth.

Problem 1. Find the five smallest positive multiples of 8 that are perfect squares.

Problem 2. Find the six smallest positive integers that are both perfect squares and perfect cubes.

Problem 3. The sum of four consecutive primes is itself prime. What is the largest of the four primes?

Problem 4. Ko Shine writes down the smallest positive multiple of 20 that is a perfect square, the smallest positive multiple of 20 that is a perfect cube and all the multiples of 20 between them. How many integers are in Ko Shine's list?

Problem 5. What is the least possible whole number that can be multiplied by 200 such that the product is a perfect cube?

Problem 6. How many possible values are there for the sum $a + b + c$ if $a, b,$ and c are positive integers and $abc = 72$?

Problem 7. Let n be a natural number with exactly 7 positive divisors. How many positive divisors does n^2 have?

Challenge Problem

Problem 8. Find the number of positive integers n, $1 \leq n \leq 1000$, such that n^n is a perfect square.