Lecture 2 - Prime Factorization and Factors

1 Prime Factorization

Prime factorization of an integer n is the breaking down of n into the products of the powers of prime numbers.

Example 1. Find the smallest positive integer that is neither prime nor square and that has no prime factor less than 50.

Example 2.
(a) Find the prime factorization of each of $15^2, 16^2, 17^2, 18^2$.
(b) What do the prime factorization of all perfect squares have in common?
(c) Find the prime factorization of each of $19^3, 20^3, 21^3, 24^3$.
(d) What do the prime factorization of all perfect cubes have in common?

Example 3.
(a) Suppose n is a positive integer such that the number $2016 \cdot n$ is a perfect cube. Find the smallest possible value of n.
(b) Suppose m is a positive integer such that the number $\frac{2016}{m}$ is a perfect square. Find the smallest possible value of m.

2 Number of divisors

If the prime factorization of a number, n, is expressed as

$$n = p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of (positive) divisors of the number is

$$\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$

Note. Basically in order to find the number of positive divisors of a number,

1. Find the prime factorization of the number.
2. Add 1 to all of the exponents.
3. Multiply them together.

Example 4. Mg Ka Gyi places 360 marbles into m total boxes such that each box contains an equal number of marbles. There is more than one box, and each box contains more than one marble. For how many values of m can this be done?

Example 5.
(a) Find the number of positive divisors of 23^2, 24^2 and 63^2.
(b) Why is the number of divisors of a perfect square always an odd number?

Example 6. How many positive factors does the number 144 have that are perfect square?

Example 7. A new school has exactly 2023 lockers and 2023 students. On the first day of school, the first student enters the school and opens all the lockers. The second student then enters and closes every locker with an even number. The third student will reverse every third locker. (If the locker is closed, it will be opened and vice versa.) The fourth student will reverse every fourth locker and so on, until all 2023 students have entered and reversed the respective lockers. Which lockers will be open at the end?

3 Sum of divisors

If the prime factorization of a number, n, is expressed as

$$n = p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the sum of (positive) divisors of the number is

$$S(n) = (1+p_1^1+p_1^2+\cdots+p_1^{e_1-1}+p_1^{e_1})(1+p_2^1+p_2^2+\cdots+p_2^{e_2-1}+p_2^{e_2})\cdots(1+p_k^1+p_k^2+\cdots+p_k^{e_k-1}+p_k^{e_k})$$

Example 8. Find the sum of positive divisors of (a) 45, (b) 1200, (c) 28^2.

4 Product of divisors

The product of (positive) divisors of a number n where it has f (positive) divisors is $n^{f/2}$.

Example 9. 108 chickens are kept in n cages such that each cage contains the same number of chickens. What is the product of the possible values of n?

Homework Problems

Homework code : **HWN102**
Issued on : 15th October 2023
Due date : 18th October 2023

Submit the solutions to at least 6 of the homework problems before the due date.
All of the problems are each worth 5 points.

1. For how many positive integer values of n is the value of $4000 \cdot (\frac{2}{5})^n$ an integer?

2. Ko Shine writes down the smallest positive multiple of 20 that is a perfect square, the smallest positive multiple of 20 that is a perfect cube and all the multiples of 20 between them. How many integers are in Ko Shine's list?

3. When 1540 is divided by n, the quotient is a square. Find the smallest possible value of n.

4. Find the least possible whole number that can be multiplied by 200 such that the product is a perfect cube.

5. A positive integer q is the product of a prime number and a perfect square. Additionally, q is the product of a different prime number and a perfect cube. What is the least possible value of q?

6. Let n be a natural number with exactly 7 positive divisors. How many positive divisors does n^2 have?

7. How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

8. For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M. What is the largest integer n for which 5^n is a factor of the sum $98! + 99! + 100!$?

9. Ko Phoe Tote writes the integers from 1 to 2023 on a piece of paper. He erases all the multiples of 3, then all the multiples of 5, and so on, erasing all the multiples of each odd prime. How many numbers are left when he finishes?

10. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?