CVEN2002 Engineering Computations (Numerics)
Demonstration Notes

Phyoe Min Khant

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Contents

1 Math Review

1.1 Matrix Operations

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & x & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

1.1.1 Matrix Transpose: A^T

The transpose of matrix A switches rows and columns:

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & x & 8 \\ 3 & 6 & 9 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

1.1.2 Matrix Addition: $A + B$

For matrix addition, corresponding elements are added:

$$A + B = \begin{pmatrix} 1+1 & 2+2 & 3+1 \\ 4+2 & x+1 & 6+2 \\ 7+2 & 8+3 & 9+1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 \\ 6 & x+1 & 8 \\ 9 & 11 & 10 \end{pmatrix}$$

1.1.3 Matrix Multiplication: AB

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second.

$$\boxed{(AB)_{ij} = \sum_k A_{ik} B_{kj}}$$

$$\begin{aligned}
AB &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & x & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1+4+6 & 2+2+9 & 1+4+3 \\ 4+2x+12 & 8+x+18 & 4+2x+6 \\ 7+16+18 & 14+8+27 & 7+16+9 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 13 & 8 \\ 16+2x & 26+x & 10+2x \\ 41 & 49 & 32 \end{pmatrix} \\
AC &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & x & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+4+9 \\ 4+2x+18 \\ 7+16+27 \end{pmatrix} = \begin{pmatrix} 14 \\ 22+2x \\ 50 \end{pmatrix} \\
C^T A &= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & X & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 4 + 3\cdot 7 & 1\cdot 2 + 2\cdot X + 3\cdot 8 & 1\cdot 3 + 2\cdot 6 + 3\cdot 9 \end{pmatrix} \\
&= \begin{pmatrix} 30 & 26+2X & 42 \end{pmatrix} \\
C^T C &= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1+4+9 = 14
\end{aligned}$$

1.1.4 Determinant: $\det(B)$

For a 3×3 matrix, the determinant is:

$$\begin{aligned}\det(B) &= 1 \cdot \det\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix} - 2 \cdot \det\begin{pmatrix}2 & 2\\ 2 & 1\end{pmatrix} + 1 \cdot \det\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}\\ &= 1(1-6) - 2(2-4) + 1(6-2)\\ &= -5 + 4 + 4 = 3\end{aligned}$$

1.2 Derivatives and Integrals

Derivative of $f(x) = \cos x \sin x$

$$\frac{df}{dx} = \overbrace{\frac{d}{dx}(\cos x \sin x) = (-\sin x)(\sin x) + (\cos x)(\cos x)}^{\boxed{\text{Product rule: } (uv)' = u'v + uv'}} = -\sin^2 x + \cos^2 x$$

Or using double angle identity: $\cos x \sin x = \frac{1}{2}\sin(2x)$

$$\frac{df}{dx} = \frac{1}{2} \cdot 2\cos(2x) = \cos(2x)$$

Derivative of $g(x) = \cosh(\sinh x)$

$$\frac{dg}{dx} = \overbrace{\frac{d}{dx}[\cosh(\sinh x)] = \sinh(\sinh x) \cdot \frac{d}{dx}(\sinh x)}^{\boxed{\text{Chain rule: } (f(g(x)))' = f'(g(x)) \cdot g'(x)}} = \sinh(\sinh x) \cdot \cosh x$$

Partial Derivatives of $h(x, y) = e^{2x+y} + y^2$

$$\begin{aligned}\frac{\partial h}{\partial x} &= 2e^{2x+y}, \quad \frac{\partial h}{\partial y} = e^{2x+y} + 2y\\ dh(x,y) &= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy = 2e^{2x+y}dx + (e^{2x+y} + 2y)dy\end{aligned}$$

$\int (x^2 + 5)dx$

$$\int (x^2 + 5)dx = \frac{x^3}{3} + 5x + C$$

$\int h(x, y)dx$ **(treating y as constant)**

$$\int h(x,y)dx = \int (e^{2x+y} + y^2)dx = \int e^{2x+y}dx + \int y^2 dx = \frac{e^{2x+y}}{2} + y^2 x + C(y)$$

$\int h(x, y)dy$ **(treating x as constant)**

$$\int h(x,y)dy = \int (e^{2x+y} + y^2)dy = \int e^{2x+y}dy + \int y^2 dy = e^{2x+y} + \frac{y^3}{3} + C(x)$$

1.3 Volume of Revolution

Given region bounded by:

- $y = 3\sin x$ (upper boundary)
- $y = 0$ (x-axis, lower boundary)
- $x = 0$ and $x = \frac{\pi}{2}$ (left and right boundaries)

is rotated about the x-axis to form a solid. Calculate the volumne of the solid by analytical and numerical methods.

When a region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is rotated about the x-axis:

$$V = \pi \int_a^b [f(x)]^2\, dx$$

Analytical Method. For our problem: $f(x) = 3\sin x$, $a = 0$, $b = \frac{\pi}{2}$

$$\begin{aligned} V &= \pi \int_0^{\pi/2} (3\sin x)^2 dx = 9\pi \int_0^{\pi/2} \sin^2 x\, dx = 9\pi \int_0^{\pi/2} \frac{1-\cos(2x)}{2} dx \\ &= \frac{9\pi}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^{\pi/2} = \frac{9\pi}{2}\left[\left(\frac{\pi}{2} - \frac{\sin(\pi)}{2}\right) - \left(0 - \frac{\sin(0)}{2}\right)\right] \\ &= \frac{9\pi}{2}\left[\frac{\pi}{2} - 0 - 0 + 0\right] = \frac{9\pi^2}{4} \approx 22.21 \end{aligned}$$

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Numerical Methods. *Upper Bound:* The solid fits inside a cylinder of radius 3 and height $\frac{\pi}{2} \approx 1.5$. Using $\pi \approx 3$, its volume is

$$V_{\text{cyl}} = \pi r^2 h \approx 3 \cdot 9 \cdot 1.5 = 40.5.$$

Lower Bound: The solid contains a cone of the same base and height:

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h \approx \frac{1}{3} \cdot 3 \cdot 9 \cdot 1.5 = 13.5.$$

Improved Upper Bound: Drawing a tangent cone at the origin produces a new solid which is part cone and part cylinder. Slope at $x = 0$ is $3\cos(0) = 3$. Approximating the solid with a cone (height 1) and a cylinder (height $\frac{\pi}{2} - 1 \approx 0.5$):

$$V \approx \frac{1}{3}\pi \cdot 9 \cdot 1 + \pi \cdot 9 \cdot 0.5 = 3\pi + 4.5\pi = 7.5\pi \approx 23.6.$$

Conclusion: The volume lies between 13.5 and 23.6. A rough average gives ≈ 18.5, close to the exact value $\frac{9\pi^2}{4} \approx 22.21$.

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2 Introduction to MATLAB

2.1 Plotting

Plotting a Circle and a Parabola

To draw a unit circle:

```
theta = linspace(0, 2*pi, 100);
x1 = cos(theta);
y1 = sin(theta);
plot(x1, y1)
```

To draw a parabola $y = x^2$ for $0 \leq x \leq 1$:

```
theta = linspace(0, 1, 100);
x2 = theta;
y2 = theta .* theta;
plot(x2, y2)
```

To plot both on the same figure:

```
plot(x1, y1)
hold on
plot(x2, y2)
```

2.2 Functions with Multiple Outputs

Defining and Calling a Function

The following function returns both the sum and difference of two input arguments:

```
function [x, y] = Myfunction(a, b)
    x = a + b;
    y = a - b;
end
```

Calling the function:

```
[var_1, var_2] = Myfunction(1, 2)
```

The output will be: `var_1 = 3, var_2 = -1`

2.3 Matrix Construction with For Loop

Creating a matrix with defined values

The following code creates an $s \times s$ matrix H, where each element is given by:

$$H(r, c) = \frac{1}{r + c - 1}$$

```
s = 10;
H = zeros(s);

for c = 1:s
    for r = 1:s
        H(r,c) = 1/(r + c - 1);
    end
end
```

For example, when `s = 3`, the resulting matrix H is:

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.5000 & 0.3333 \\ 0.5000 & 0.3333 & 0.2500 \\ 0.3333 & 0.2500 & 0.2000 \end{bmatrix}$$

2.4 While Loop

Finding the First Index Where `x(i)` $\geq$ `y(i)`

To find the first index `i` such that `x(i)` is not less than `y(i)`:

```
x = [3 7 1 6 2];
y = [8 9 5 4 6];
i = 1;
while x(i) < y(i)
    i = i + 1;
end
```

In this case, the loop stops at `i = 4` because `x(4) = 6` is not less than `y(4) = 4`.

2.5 If Statement

Conditional Assignment

This example shows a basic use of `if` to conditionally assign values:

```
var_A = 0;
var_B = 1;

if var_A < var_B
    var_C = 5;
else
    var_C = 10;
end
```

Since $0 < 1$, the result is `var_C = 5`.

2.6 For Loop with If Statement

2.6.1 Simulating a Probability Experiment

How many heads would you expect if you tossed a fair coin 1000 times?

This can be simulated by generating 1000 random numbers between 0 and 1, and counting how many of them are less than 0.5—since a fair coin has a 50% chance of landing heads, each number below 0.5 corresponds to a "head".

```
below = 0;
above = 0;

for i = 1:1000
    if rand(1,1) < 0.5
        below = below + 1;
    else
        above = above + 1;
    end
end

count = below;
sum = below + above;
```

Sample output for the first few iterations:

- `i = 1`, `rand = 0.32` $\Rightarrow$ `below = 1`
- `i = 2`, `rand = 0.61` $\Rightarrow$ `above = 1`
- `i = 3`, `rand = 0.44` $\Rightarrow$ `below = 2`

Comment

- The expression `a = a + 1` is not a mathematical equality but an assignment statement. It updates the value of `a` by incrementing it by 1.
- `rand(x, y)` generates an `x`-by-`y` matrix of random numbers drawn from a uniform distribution over the interval $(0, 1)$.

Instead of using loops, MATLAB allows vectorised operations, which are often more efficient. For example, to count how many random numbers are below 0.5:

```
x = rand(1,1000);
indices = find(x < 0.5);
count = length(indices);
```

This can be further simplified into a one-liner using logical indexing:

```
count = sum(rand(1,1000) < 0.5);
```

We can combine conditions using logical operators. For example, to count values less than 0.1 and greater than 0.9:

```
x = rand(1,1000);
count = length(find(x < 0.1 & x > 0.9));
```

Since no number satisfies this condition, the answer will be zero.

2.6.2 Recurrence Function or Summing Consecutive Terms

Slow and Ugly Version (Using Loop):

```
a = rand(1,100);
b = zeros(1,100);
for n = 1:100
    if n == 1
        b(n) = a(n);
    else
        b(n) = a(n-1) + a(n);
    end
end
```

Efficient Vectorised Version:

```
a = rand(1,100);
b = [0 a(1:end-1)] + a;
```

- `a(1:end-1)` returns all elements of `a` except the last.
- `[0 a(1:end-1)]` shifts the vector to the right and adds a 0.
- Adding this to `a` produces:

$$b(n) = a(n) + a(n-1), \quad \text{with } b(1) = a(1)$$

3 Introduction to Numerical Methods

Numerical methods give approximate solutions. Analytical methods give exact solutions.

3.1 Introduction to Error

- **Modeling Error**
  - **Blunders**: gross errors caused by human mistakes, equipment malfunctions, or procedural faults.
    - Misreading instruments
    - Recording incorrect values
    - Applying wrong formulas
    - Using improperly calibrated equipment
  - **Formulation Errors**: mathematical model fails to accurately capture the behavior of the physical system.
    - Neglecting air resistance in projectile motion
    - Assuming ideal gas behavior under high pressure
    - Ignoring friction in mechanical systems
    - Using linear models for inherently nonlinear phenomena
  - **Data Uncertainty**: limitations in the accuracy or completeness of input parameters.
    - Instrument precision limits
    - Statistical variability in experimental data
    - Incomplete or imprecise system characterization
    - Environmental factors affecting measurements
- **Numerical Error**
  - **Truncation Error**: an infinite process is approximated by a finite one.
    - Truncated Taylor series expansions: $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
    - Numerical integration with finite step sizes
    - Premature termination of iterative methods
    - Finite difference approximations of derivatives
  - **Roundoff Error**
    - Quantization Error: $1/3 = 0.3333\ldots$ becomes 0.333333 in finite precision
    - Numerical Manipulations
      - Loss of significance when subtracting nearly equal numbers
      - Accumulation of small errors in iterative methods

3.2 Taylor Series

The concept of a Taylor series was formulated by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715. The whole idea is that we can approximate a non-polynomial function with a polynomial function.

For example, let's approximate $\cos(x)$ using a second-degree polynomial.

$$P(x) = a_1 + a_2 x + a_3 x^2$$

The coefficients a_1, a_2, a_3 are determined from the following conditions:

$$P(0) = \cos(0), \quad P'(0) = \cos'(0), \quad P''(0) = \cos''(0)$$

Geometrically,

- Matching the function value ensures that the polynomial passes through the same point as $\cos(x)$ at $x = 0$.
- Matching the first derivative ensures that the polynomial has the same slope at that point.
- Matching the second derivative ensures that the polynomial has the same curvature (i.e., concavity) at that point.

Step 1: Compute derivatives of $\cos(x)$ at $x = 0$

$$\begin{aligned} f(x) &= \cos(x) &,\quad f(0) &= 1 \\ f'(x) &= -\sin(x) &,\quad f'(0) &= 0 \\ f''(x) &= -\cos(x) &,\quad f''(0) &= -1 \end{aligned}$$

Step 2: Compute derivatives of $P(x)$ at $x = 0$

$$\begin{aligned} P(x) &= a_1 + a_2 x + a_3 x^2 &,\quad P(0) &= a_1 \\ P'(x) &= a_2 + 2a_3 x &,\quad P'(0) &= a_2 \\ P''(x) &= 2a_3 &,\quad P''(0) &= 2a_3 \end{aligned}$$

Step 3: Match function values and derivatives at $x = 0$

$$a_1 = 1, a_2 = 0, a_3 = -\frac{1}{2}$$

So, the second-degree polynomial approximation of $\cos(x)$ around $x = 0$ is

$$P(x) = 1 - \frac{1}{2}x^2$$

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Comment. Looking back at the example, we observe that $P(x)$ can be written as

$$P(x) = P(0) + P'(0)x + \frac{P''(0)}{2!}x^2.$$

This resembles the beginning of the Taylor series expansion. The general form of the Taylor series for a function $f(x)$ about the point a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(4)}(a) + \cdots$$

This series provides an approximation of the function $f(x)$ near the point a. In particular, when $a = 0$, the approximation matches the value and derivatives of the function at $x = 0$, ensuring that the polynomial closely reflects the behavior of $f(x)$ at that point.

Adding more terms to the polynomial increases the accuracy of the approximation. If including an additional term does not change the result beyond a certain number of decimal places, the approximation can be considered sufficiently accurate, and further terms may be omitted.

Q1. Let $f(x) = \sin x$.

(a) Write the Taylor series for $\sin(x)$, expanding about $x = \frac{\pi}{4}$, with at least 5 terms.

(b) Evaluate $\sin(35°)$, considering that the required accuracy is 4 decimal places, using $a = \frac{\pi}{4}$.

(c) Evaluate $\sin(35°)$, considering that the required accuracy is 4 decimal places, using $a = 30°$.

(d) How many terms of the Taylor series are necessary to obtain the required accuracy?

Solution.

$$
\begin{aligned}
f(x) &= \sin x, & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, & f(30^\circ) &= \frac{1}{2} \\
f'(x) &= \cos x, & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, & f'(30^\circ) &= \frac{\sqrt{3}}{2} \\
f''(x) &= -\sin x, & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}, & f''(30^\circ) &= -\frac{1}{2} \\
f^{(3)}(x) &= -\cos x, & f^{(3)}\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}, & f^{(3)}(30^\circ) &= -\frac{\sqrt{3}}{2} \\
f^{(4)}(x) &= \sin x, & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, & f^{(4)}(30^\circ) &= \frac{1}{2}
\end{aligned}
$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f^{(3)}(a) + \frac{(x-a)^4}{4!}f^{(4)}(a) + \dots$$

Taylor series about $a = \frac{\pi}{4}$:

$$f(x) = \frac{\sqrt{2}}{2} + \left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{24}\left(x - \frac{\pi}{4}\right)^4\left(\frac{\sqrt{2}}{2}\right) + \dots$$

Actual value:

$$\sin(35^\circ) = 0.5736 \quad \text{(4 d.p.)}$$

Approximation using $a = \frac{\pi}{4}$:

$$f(35^\circ) = 0.707106 + (-0.123413) + (-0.010770) + 0.000627 = 0.5736$$

Approximation using $a = \frac{\pi}{6} = 30^\circ$:

$$\sin(35^\circ) \approx 0.5 + 0.075575 + (-0.0019\dots) + (-0.00010\dots) = 0.5736$$

Terms	Match at $x = \frac{\pi}{4}$	Match at $x = \frac{\pi}{6}$
1 term	0.7071	0.5
2 terms	0.5837	0.5756
3 terms	0.5730	0.5737
4 terms	0.5736	0.5736

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Q2. Find a Taylor series for $\sin^2(x)$, expanding at $x = 0$, to give a representation up to $\mathcal{O}(x^4)$.

Solution.

$$
\begin{aligned}
f(x) &= \sin^2 x, & f(0) &= 0 \\
f'(x) &= 2\sin x\cos x, & f'(0) &= 0 \\
f''(x) &= 2(\cos^2 x - \sin^2 x), & f''(0) &= 2 \\
f^{(3)}(x) &= 2(-2\cos x\sin x - 2\sin x\cos x) = -8\sin x\cos x,, & f^{(3)}(0) &= 0
\end{aligned}
$$

Taylor expansion about $a = 0$:

$$
\begin{aligned}
f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \dots \\
&= 0 + 0 + \frac{x^2}{2}\cdot 2 + 0 + \dots = x^2 + \mathcal{O}(x^4)
\end{aligned}
$$

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4 Finding Roots of Equations

Bisection Method: Also known as "binary chopping," "interval halving," or "Bolzano's method," the bisection method relies on the Intermediate Value Theorem. From the graphical method, we observe that if the function $f(x)$ changes sign across an interval $[x_l, x_u]$, i.e., $f(x_l)f(x_u) < 0$, then there is at least one real root within that interval. The bisection method incrementally searches for this root by repeatedly halving the interval and checking the sign of the function at the midpoint. This process continues until the desired level of accuracy is achieved.

$$f(x_l)f(x_u) < 0 \quad \Rightarrow \quad x_r = \frac{x_l + x_u}{2}$$

Newton-Raphson Method: This method uses the first derivative of the function to iteratively approximate the root:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

One limitation of the Newton-Raphson method is its dependence on the analytical derivative $f'(x)$, which may be difficult or inconvenient to compute for some functions. In such cases, the derivative can be approximated using the backward finite divided difference:

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

Secant Method: The secant method is similar to Newton-Raphson but does not require the analytical derivative. Instead, it uses two initial estimates to construct a secant line and iteratively update the root estimate:

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

Relative Error: To assess convergence and determine when to stop iterations, the relative error at each step is commonly used:

$$\varepsilon_i = \left| \frac{x_i - x_{i-1}}{x_i} \right| \times 100\%$$

This provides a percentage measure of how much the current approximation has changed compared to the previous one.

5 Simultaneous Linear Equations and Matrix Inversion

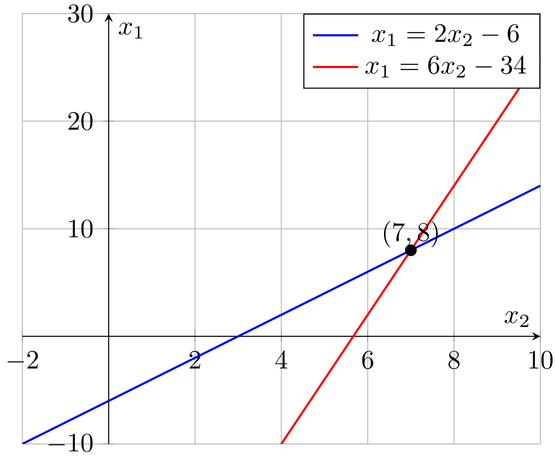
5.1 Graphical Method

Solve the system by graphical method:

$$\begin{aligned} 4x_1 - 8x_2 &= -24 \\ -x_1 + 6x_2 &= 34 \end{aligned}$$

Solution. Rewriting into slope-intercept form:

$$\begin{aligned} x_1 - 2x_2 &= -6 && \Rightarrow \quad x_1 = 2x_2 - 6 \\ -x_1 + 6x_2 &= 34 && \Rightarrow \quad x_1 = 6x_2 - 34 \end{aligned}$$



The lines intersect at the point $(x_2, x_1) = (7, 8)$. So the solution to the system is:

$$\boxed{x_1 = 8, \quad x_2 = 7}$$

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5.2 Cramer's Rule

Derivation of Cramer's Rule for a 2x2 System Consider the linear system:

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

Matrix form:

$$A\mathbf{x} = \mathbf{b}, \quad \text{where} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The solution is given by:

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{bmatrix}.$$

Thus,

$$x = \frac{a_{22}b_1 - a_{12}b_2}{\det(A)} = \frac{1}{\det(A)} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \frac{\det(A_1)}{\det(A)},$$

$$y = \frac{-a_{21}b_1 + a_{11}b_2}{\det(A)} = \frac{1}{\det(A)} \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \frac{\det(A_2)}{\det(A)}.$$

Here, A_1 is the matrix obtained by replacing the first column of A with $\mathbf{b}$, and A_2 is obtained by replacing the second column of A with $\mathbf{b}$:

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}.$$

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5.2.1 Worked Example

Solve the system using Cramer's Rule:

$$\begin{aligned} 2x_2 + 5x_3 &= 9 \\ 2x_1 + x_2 + x_3 &= 9 \\ 3x_1 + x_2 &= 10 \end{aligned}$$

Solution. Matrix form:

$$A = \begin{bmatrix} 0 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 9 \\ 9 \\ 10 \end{bmatrix}$$

Step 1: Compute $\det(A)$

$$\det(A) = 0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} + 5 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 0 + 6 - 5 = \boxed{1}$$

Step 2: Compute $\det(A_1)$ (replace 1st column with $\vec{b}$). Similarly, compute $\det(A_2)$ and $\det(A_3)$ (replace 2nd and 3rd column with $\vec{b}$)

$$A_1 = \begin{bmatrix} 9 & 2 & 5 \\ 9 & 1 & 1 \\ 10 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 9 & 5 \\ 2 & 9 & 1 \\ 3 & 10 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 2 & 9 \\ 2 & 1 & 9 \\ 3 & 1 & 10 \end{bmatrix}$$

$$\det(A_1) = 9 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 9 & 1 \\ 10 & 0 \end{vmatrix} + 5 \cdot \begin{vmatrix} 9 & 1 \\ 10 & 1 \end{vmatrix} = 9(-1) + 2(10) + 5(-1) = -9 + 20 - 5 = \boxed{6}$$

$$\det(A_2) = 0 - 9 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} + 5 \cdot \begin{vmatrix} 2 & 9 \\ 3 & 10 \end{vmatrix} = -9(-3) + 5(-7) = 27 - 35 = \boxed{-8}$$

$$\det(A_3) = 0 - 2 \cdot \begin{vmatrix} 2 & 9 \\ 3 & 10 \end{vmatrix} + 9 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2(7) + 9(-1) = 14 - 9 = \boxed{5}$$

Thus,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{6}{1} = \boxed{6}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-8}{1} = \boxed{-8}, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{5}{1} = \boxed{5}$$

Verification by Substitution

$$\begin{aligned} &\text{Eq(1): } 2(-8) + 5(5) = -16 + 25 = 9 \quad \checkmark \\ &\text{Eq(2): } 2(6) + (-8) + 5 = 12 - 8 + 5 = 9 \quad \checkmark \\ &\text{Eq(3): } 3(6) + (-8) = 18 - 8 = 10 \quad \checkmark \end{aligned}$$

■

5.3 Gauss Elimination and LU Decomposition

Given:

$$A\mathbf{x} = \mathbf{b}, \quad \text{with} \quad A = LU$$

Substitute:

$$LU\mathbf{x} = \mathbf{b}$$

Let:

$$L\mathbf{d} = \mathbf{b} \implies U\mathbf{x} = \mathbf{d}$$

Thus:

1. Solve $L\mathbf{d} = \mathbf{b}$ via forward substitution.
2. Solve $U\mathbf{x} = \mathbf{d}$ via back substitution.

Finding L and U for a 3×3 System: Consider a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The entries are found as follows:

1. Compute the first row of U:
$$u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$$
2. Compute the first column of L:
$$\ell_{21} = \frac{a_{21}}{u_{11}}, \quad \ell_{31} = \frac{a_{31}}{u_{11}}$$
3. Compute the second row of U:
$$u_{22} = a_{22} - \ell_{21}u_{12}, \quad u_{23} = a_{23} - \ell_{21}u_{13}$$
4. Compute ℓ_{32}:
$$\ell_{32} = \frac{a_{32} - \ell_{31}u_{12}}{u_{22}}$$
5. Finally, compute u_{33}:
$$u_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23}$$

The diagonal entries of L are always 1 by definition.

5.3.1 Worked Example

Solve the system by LU Decomposition:

$$\begin{aligned} 10x_1 + 2x_2 - x_3 &= 27 \\ -3x_1 - 6x_2 + 2x_3 &= -61.5 \\ x_1 + x_2 + 5x_3 &= -21.5 \end{aligned}$$

Solution. Matrix form:

$$A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

LU Decomposition Using Naïve Gauss Elimination

First, eliminate a_{21} using:

$$l_{21} = \frac{-3}{10} = -0.3$$

Next, eliminate a_{31} using:

$$l_{31} = \frac{1}{10} = 0.1$$

After these row operations, the matrix becomes:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix}$$

Now, eliminate a_{32} using:

$$l_{32} = \frac{0.8}{-5.4} = -0.14815$$

The updated matrix is:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Thus, the LU decomposition is:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Note: Multiplying $L \cdot U$ reconstructs the original matrix A, confirming the decomposition.

Solving the System Using LU Decomposition

First, solve $L\vec{d} = \vec{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

Forward substitution gives:

$$d_1 = 27, \quad d_2 = -61.5 + 0.3(27) = -53.4, \quad d_3 = -21.5 - (0.1)(27) + (0.14815)(-53.4) = -32.1111$$

Next, solve $U\vec{x} = \vec{d}$:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -53.4 \\ -32.1111 \end{bmatrix}$$

Back substitution gives:

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = \frac{-53.4 + 10.2}{-5.4} = 8$$

$$x_1 = \frac{27 - 2(8) - (-1)(-6)}{10} = \frac{27 - 16 - 6}{10} = 0.5$$

Thus, the solution is:

$$\boxed{x_1 = 0.5, \quad x_2 = 8, \quad x_3 = -6}$$

Solving with an Alternative Right-Hand Side: $\vec{b'} = \begin{bmatrix} 12 \\ 18 \\ -6 \end{bmatrix}$

First, solve $L\vec{d'} = \vec{b'}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{bmatrix} d'_1 \\ d'_2 \\ d'_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \\ -6 \end{bmatrix}$$

Forward substitution gives:

$$d'_1 = 12, \quad d'_2 = 18 + 0.3(12) = 21.6, \quad d'_3 = -6 - (0.1)(12) + (0.14815)(21.6) = -4$$

Next, solve $U\vec{x'} = \vec{d'}$:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 21.6 \\ -4 \end{bmatrix}$$

Back substitution gives:

$$x'_3 = \frac{-4}{5.351852} = -0.7474$$

$$x'_2 = \frac{21.6 - 1.7(-0.7474)}{-5.4} = -4.2353$$

$$x'_1 = \frac{12 - 2(-4.2353) - (-1)(-0.7474)}{10} = 1.9723$$

Thus, the solution for the alternative right-hand side is:

$$\boxed{x'_1 \approx 1.9723, \quad x'_2 \approx -4.2353, \quad x'_3 \approx -0.7474}$$

■

6 Numerical Integration: Newton-Cotes Method

The Newton-Cotes method is a widely used approach for numerical integration. The fundamental idea is to replace a complicated or tabulated function with a simpler interpolating polynomial, which is easy to integrate exactly.

- **Closed Formulas**: Include data points at both ends of the integration interval.
- **Open Formulas**: Do not include data points at the boundaries; the integration limits extend beyond the available data points.

In this course, we focus on **closed formulas**.

- **Single Application** : The entire integration interval $[a, b]$ is approximated using a single interpolating polynomial (e.g., one trapezoid or one parabola).
- **Multiple Applications**: The integration interval $[a, b]$ is divided into several sub-intervals, and the integration rule is applied to each sub-interval. This significantly improves accuracy, especially for functions that are not well-approximated by a low-degree polynomial over a large interval.

6.1 Trapezoidal Rule (Linear Approximation)

Single Application. For $f(x)$ over $[a, b]$ with $h = b - a$:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a) + f(b)\right)$$

Multiple Application. Divide $[a, b]$ into n equal sub-intervals of width $h = \frac{b-a}{n}$:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$$

6.2 Simpson's 1/3 Rule (Quadratic Approximation)

Single Application. For three equally spaced points with $h = \frac{b-a}{2}$:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4f\left(a+h\right) + f(b)\right)$$

Multiple Application. Divide $[a, b]$ into an even number n of sub-intervals, $h = \frac{b-a}{n}$:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_0) + 4\sum_{i=1,\text{odd}}^{n-1} f(x_i) + 2\sum_{i=2,\text{even}}^{n-2} f(x_i) + f(x_n)\right)$$

6.3 Simpson's 3/8 Rule (Cubic Approximation)

Single Application. For four equally spaced points with $h = \frac{b-a}{3}$:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left(f(a) + 3f(a+h) + 3f(a+2h) + f(b)\right)$$

Multiple Application. Divide $[a, b]$ into n equal sub-intervals, where n is a multiple of 3, and $h = \frac{b-a}{n}$:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left(f(x_0) + 3\sum_{\substack{i=1 \\ i \bmod 3 \neq 0}}^{n-1} f(x_i) + 2\sum_{\substack{i=3 \\ i \bmod 3 = 0}}^{n-3} f(x_i) + f(x_n)\right)$$

6.4 Worked Examples for Numerical Integration

Q1. Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} (6 + 3\cos x)\ dx$$

(a) Analytically.

(b) Single application of the trapezoidal rule.

(c) Multiple application of the trapezoidal rule with $n = 2$ and $n = 4$.

(d) Single application of Simpson's 1/3 rule.

(e) Multiple application of Simpson's 1/3 rule with $n = 4$.

(f) Single application of Simpson's 3/8 rule.

(g) Multiple application of Simpson's 1/3 and 3/8 rules with a total number of intervals $n = 5$.

For each of the methods above, also calculate the relative error compared to the analytical solution in part (a).

Solution. (a) Analytical Solution

$$\int_0^{\frac{\pi}{2}} (6 + 3\cos x)\, dx = [6x + 3\sin x]_0^{\frac{\pi}{2}} = 6 \cdot \frac{\pi}{2} + 3 \cdot \sin\left(\frac{\pi}{2}\right) = 3\pi + 3 \approx 12.424$$

(b) Single Application of the Trapezoidal Rule Let $n = 1$, so $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{1} = \frac{\pi}{2}$.

$$I \approx \frac{h}{2}\left(f(0) + f\left(\frac{\pi}{2}\right)\right) = \frac{\pi}{4}(6+9) = \frac{15\pi}{4} \approx 11.781$$

$$\varepsilon_r = \frac{|12.424 - 11.781|}{12.424} \times 100\% = 5.18\%$$

(c) Multiple Applications of the Trapezoidal Rule For $n = 2$, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$.

$$I \approx \frac{h}{2}\left(f(0) + 2f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right) = \frac{\pi}{8}\left(6 + 2(8.812) + 9\right) \approx 12.269$$

$$\varepsilon_r = \frac{|12.424 - 12.269|}{12.424} \times 100\% = 1.25\%$$

For $n = 4$, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$.

$$I \approx \frac{h}{2}\left(f(0) + 2f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right) \approx 12.386$$

$$\varepsilon_r = \frac{|12.424 - 12.386|}{12.424} \times 100\% = 0.311\%$$

(d) Single Application of Simpson's 1/3 Rule Let $n = 2$, so $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$.

$$I \approx \frac{h}{3}\left(f(0) + 4f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right) = \frac{\pi}{12}\left(6 + 4(8.812) + 9\right) \approx 12.431$$

$$\varepsilon_r = \frac{|12.431 - 12.424|}{12.424} \times 100\% = 0.055\%$$

(e) Multiple Applications of Simpson's 1/3 Rule ($n = 4$) $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$.

$$I \approx \frac{h}{3}\left(f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{4}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right) \approx 12.425$$

$$\varepsilon_r = \frac{|12.425 - 12.424|}{12.424} \times 100\% = 0.003\%$$

(f) Single Application of Simpson's 3/8 Rule Let $n = 3$, so $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$.

$$I \approx \frac{3h}{8}\left(f(0) + 3f\left(\frac{\pi}{6}\right) + 3f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)\right) \approx 12.428$$

$$\varepsilon_r = \frac{|12.428 - 12.424|}{12.424} \times 100\% = 0.024\%$$

(g) Combined Simpson's 1/3 and 3/8 Rules ($n = 5$) $h = \frac{b-a}{n} = \frac{\pi}{10}$.

$$I \approx \frac{h}{3}\left(f(0) + 4f\left(\frac{\pi}{10}\right) + f\left(\frac{2\pi}{10}\right)\right) + \frac{3h}{8}\left(f\left(\frac{2\pi}{10}\right) + 3f\left(\frac{3\pi}{10}\right) + 3f\left(\frac{4\pi}{10}\right) + f\left(\frac{5\pi}{10}\right)\right) \approx 12.425$$

$$\varepsilon_r = \frac{|12.425 - 12.424|}{12.424} \times 100\% = 0.002\%$$

■

Q2. Evaluate the integral of the following tabular data using: (a) Trapezoidal rule, (b) Simpson's rule.

x	-2	0	2	4	6	8	10
$f(x)$	35	5	-10	2	5	3	20

Solution.

(a) Trapezoidal Rule ($n = 6$)

$$h = \frac{b-a}{n} = \frac{10-(-2)}{6} = 2$$

$$I \approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right) = \frac{2}{2}\left(35 + 2(5 - 10 + 2 + 5 + 3) + 20\right) = 65$$

(b) Simpson's 1/3 Rule ($n = 6$**, even number required)**

$$\begin{aligned} I &\approx \frac{h}{3}\left(f(x_0) + 4\sum_{\text{odd } i} f(x_i) + 2\sum_{\text{even } i} f(x_i) + f(x_n)\right) \\ &= \frac{h}{3}(f(-2) + 4f(0) + 2f(2) + 4f(4) + 2f(6) + 4f(8) + f(10)) \\ &= \frac{2}{3}\left(35 + 4(5 + 2 + 3) + 2(-10 + 5) + 20\right) = 56.667 \end{aligned}$$

■

7 Ordinary Differential Equations (ODE)

Numerical Solution of One-Dimensional Steady-State Heat Conduction Equation Using Finite Difference Method

A long rod of length $L = 10\,\mathrm{m}$ experiences steady-state heat conduction with internal heat generation. The temperature distribution $T(x)$ along the rod satisfies the differential equation:

$$\frac{d^2T}{dx^2} - h_1(T - T_a) = 0,$$

where:

- $h_1 = 0.01\,\mathrm{m}^{-2}$ is the internal heat transfer coefficient,
- $T_a = 20°\mathrm{C}$ is the ambient temperature.

The boundary conditions are:

$$T(0) = 40°\mathrm{C}, \quad T(10) = 200°\mathrm{C}.$$

Find the temperature at the middle of the rod using a numerical method with a step size $h = 0.5\,\mathrm{m}$.

Derive the finite difference approximation to discretize the governing equation using a central difference scheme with uniform spacing h. Write the general form of the discretized equation at an internal node x_i. Then, formulate the resulting linear system $A\mathbf{T} = \mathbf{b}$ corresponding to the discretized equations, explicitly stating the structure of matrix A and vector $\mathbf{b}$, including the effects of the boundary conditions.

Finally, provide the exact analytical solution to the differential equation and briefly explain how you would compare it with the numerical result.

Solution. We consider a one-dimensional steady-state heat conduction equation with internal heat generation:

$$\frac{d^2T}{dx^2} - h_1(T - T_a) = 0,$$

defined over the domain $0 \le x \le 10$,
The boundary conditions are:

$$T(0) = T_0, \quad T(10) = T_L.$$

An exact solution is given by:

$$T_{\text{exact}}(x) = 73.45238e^{0.1x} - 53.4523e^{-0.1x} + 20.$$

Numerical Method. We discretize the domain using a uniform step size $h = 0.5$, leading to $n = \frac{10}{h} = 20$ intervals.

To approximate the second derivative $\frac{d^2T}{dx^2}$, we use Taylor series expansions about the point x_i:

$$T_{i+1} = T_i + hT_i' + \frac{h^2}{2}T_i'' + \frac{h^3}{6}T_i^{(3)} + \frac{h^4}{24}T_i^{(4)} + \dots,$$
$$T_{i-1} = T_i - hT_i' + \frac{h^2}{2}T_i'' - \frac{h^3}{6}T_i^{(3)} + \frac{h^4}{24}T_i^{(4)} - \dots.$$

Adding these two expressions:

$$T_{i+1} + T_{i-1} = 2T_i + h^2T_i'' + \frac{h^4}{12}T_i^{(4)} + \dots$$

Rearranging gives:

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} = T_i'' + \mathcal{O}(h^2)$$

Thus, the central difference formula:

$$\frac{d^2T}{dx^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}$$

is a second-order accurate approximation.

Using this finite difference approximation for the second derivative, the governing equation becomes:

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} - h_1(T_i - T_a) = 0.$$

Multiplying through by h^2, we obtain the discretized form:

$$-T_{i-1} + (2 + h^2 h_1)T_i - T_{i+1} = h^2 h_1 T_a.$$

This gives a system of linear equations of the form $A\mathbf{T} = \mathbf{b}$, where:

- A is a tridiagonal coefficient matrix,
- $\mathbf{T}$ is the vector of unknown nodal temperatures,
- $\mathbf{b}$ is the right-hand side vector.

Matrix Assembly in MATLAB. The MATLAB code to construct the matrix A and vector '$\mathbf{b}$ = rhs' is:

```
for i = 2:n  % internal points
    A(i, i-1) = -1;
    A(i, i)   = 2 + h^2 * h1;
    A(i, i+1) = -1;
end

for j = 2:n  % internal points
    rhs(j) = h^2 * h1 * Ta;
end
```

This corresponds to the finite difference form:

$$-T_{i-1} + (2 + h^2 h_1)T_i - T_{i+1} = h^2 h_1 T_a, \quad \text{for } i = 1, 2, \ldots, n-1.$$

The coefficients:

- $A(i, i-1) = -1$: coefficient of T_{i-1},
- $A(i, i) = 2 + h^2 h_1$: coefficient of T_i,
- $A(i, i+1) = -1$: coefficient of T_{i+1}.

The right-hand side vector $\mathbf{b}$ at interior nodes is:

$$b_i = h^2 h_1 T_a, \quad \text{for } i = 2, \ldots, n.$$

Boundary Conditions The boundary conditions are enforced as:

$$T_0 = 40, \quad T_L = 200.$$

These are applied in the matrix system by modifying the first and last rows of A and $\mathbf{b}$:

$$A(1,1) = 1, \quad \text{rhs}(1) = T_0, \quad A(n+1, n+1) = 1, \quad \text{rhs}(n+1) = T_L.$$

■

8 Partial Differential Equations (PDE)

A partial differential equation (PDE) is classified as **parabolic** if it is of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

and the discriminant satisfies:

$$B^2 - 4AC = 0$$

Rather than focusing on the steady-state distribution in two spatial dimensions, the problem shifts to determining how the one-dimensional spatial distribution changes as a function of time.

Numerical Example

An example of a parabolic PDE is the **heat-conduction equation**:

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

where $T(x,t)$ is the temperature as a function of space and time, and k is the thermal diffusivity constant.

First derivative in time:

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

Second derivative in space: Finite Difference Approximations

- Explicit Method

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{\Delta x^2}$$

$$T_i^{l+1} = T_i^l + \lambda\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right), \quad \lambda = \frac{k\Delta t}{\Delta x^2}$$

- Implicit Method

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}$$

Rearranging:

$$-\lambda T_{i-1}^{l+1} + (1+2\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l, \quad \lambda = \frac{k\Delta t}{\Delta x^2}$$

- Crank–Nicolson Method

This method takes the average of the spatial second derivatives at time levels l and $l+1$:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{1}{2}\left(\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{\Delta x^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}\right)$$

After simplification:

$$-\lambda T_{i-1}^{l+1} + (2+2\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^l + (2-2\lambda)T_i^l + \lambda T_{i+1}^l, \quad \lambda = \frac{k\Delta t}{\Delta x^2}$$

Note. Lower indices - space, upper indices - time.