NTR4 – Order, Arithmetic Functions

Problem 1. If a and b are positive integers relatively prime to m with $a^x \equiv b^x \ (mod\ m)$ and $a^y \equiv b^y \ (mod\ m)$, prove that

$$a^{\gcd(x,y)} \equiv b^{\gcd(x,y)} \ (mod\ m).$$

Problem 2. (MOSP 1997) Prove that the sequence $1, 11, 111, \ldots$ contains an infinite subsequence whose terms are pairwise relatively prime.

Problem 3. Choose arbitrarily some different integers between two adjacent perfect squares n^2 and $(n+1)^2$, prove that products of two of them are different mutually.

Problem 4. Determine the distinct numbers in the sequence

$$\left\lfloor \frac{1^2}{2005} \right\rfloor, \left\lfloor \frac{2^2}{2005} \right\rfloor, \ldots, \left\lfloor \frac{2005^2}{2005} \right\rfloor.$$

Problem 5. For a given positive integer n, show that

$$\left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor = \left\lfloor \sqrt{n - \frac{3}{4}} + \frac{1}{2} \right\rfloor.$$

Problem 6. Prove the Hermite's Identity.

Let x be a real number, and let n be a positive integer. Then

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor$$