Lecture 3 - GCD and LCM

1 Greatest Common Divisor (GCD)

The Greatest Common Divisor (GCD) of two or more integers (which are not all zero) is the largest positive integer that divides each of the integers. This is also known as GCF (Greatest Common Factor), and the terms GCF and GCD are often used interchangeably.

Concept. Greatest common divisor of m and $n = gcd(m, n)$ can be found by taking the lowest prime exponents from the prime factorizations of m and n. GCD is used to equally distribute two or more sets of items into their largest possible grouping.

2 Least Common Multiple (LCM)

The Least Common Multiple (LCM) of two or more integers (which are not all zero) is the smallest positive integer that is divisible by both the numbers.

Concept. Least common multiple of m and $n = lcm(m, n)$ can be found by taking the highest prime exponents from the prime factorizations of m and n. LCM is used to figure out when something will happen again at the same time.

3 Coprime numbers (Relatively prime numbers)

Two numbers are called relatively prime, or coprime, if their greatest common divisor equals 1. For instance, $\gcd(9, 28) = 1$. So, 9 and 28 are relatively prime. A fraction is irreducible or in lowest terms or reduced form when the numerator and denominator are relatively prime.

Example 1. Group 6, 8, 10, 15, 21 and 25 into three relatively prime pairs of integers.

Example 2.
(a) Find $\gcd(80, 144)$ and $\text{lcm}[80, 144]$.
(b) Find $\gcd(160, 288)$ and $\text{lcm}[160, 288]$.
(c) Find $\gcd(400, 720)$ and $\text{lcm}[400, 720]$.
(d) Find $\gcd(80n, 144n)$ and $\text{lcm}[80n, 144n]$ for any positive integer n.

Example 3. A box contains gold coins. If the coins are equally divided among six people, four coins are left over. If the coins are equally divided among five people, three coins are left over. If the box holds the smallest number of coins that meets these two conditions, how many coins are left when equally divided among seven people?

Example 4.
(a) Find $\gcd(18, 30)$ and $\text{lcm}[18, 30]$.
(b) Use the relationships between $18, 30, \gcd(18, 30)$ and $\text{lcm}[18, 30]$ to create a single equation that uses all four numbers and no others.

4 GCD and LCM Product

The product of two natural numbers, m, n, is equal to the product of their GCD and LCM.

$$m \times n = \gcd(m, n) \times \operatorname{lcm}[m, n].$$

5 More Properties on GCD and LCM

- For any positive integers a, b and c,

$$\gcd(ac, bc) = c \times \gcd(a, b),$$

$$\operatorname{lcm}[ac, bc] = c \times \operatorname{lcm}[a, b].$$

- If a number is divisible by two numbers a and b, it will also be divisible by $\operatorname{lcm}(a, b)$.

Example 5. What is the largest three-digit number divisible by both 7 and 8?

Example 6. How many four-digit perfect squares are multiples of 7?

Example 7.

(a) A 4-digit number has a remainder of 6 when divided by 7, has a remainder of 7 when divided by 8, has a remainder of 8 when divided by 9. How many such possible 4-digit numbers are there?

(b) A 4-digit number has a remainder of 1 when divided by 7, has a remainder of 1 when divided by 8, has a remainder of 1 when divided by 9. How many such possible 4-digit numbers are there?

Example 8. The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c?

Example 9. Tayza writes the cubes of three positive integers on a piece of paper. Thura points out that each is a multiple of 20. Tayza then points out that the GCD of all three perfect cubes is n. Find the smallest possible value of n.

Example 10. How many distinct pairs of positive integers are there which have a GCD of 6 and LCM of 600?

Homework Problems

Homework code : **HWN103**
Issued on : 23rd October 2023
Due date : 30th October 2023

Submit the solutions to at least 6 of the homework problems before the due date. All of the problems are each worth 5 points.

1. Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1086 and 828. Find the largest possible value of the common width of the rectangles.

2. Find the five smallest multiples of 18 and 30 that are both perfect squares and perfect cubes.

3. When a three-digit number is divided by 2, 3, 4, 5 and 7, the remainders are all 1. Find the minimum and maximum values of such three-digit numbers.

4. How many four-digit integers are multiples of 15, 20, 25?

5. If the GCD of 2 numbers is 12 and the LCM of the same 2 numbers is 240, find the product of the numbers.

6. The least common multiple of 12, 15, 20 and k is 420. What is the least possible value of k?

7. Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

8. A group of 10 friends were discussing a large positive integer. "It can be divided by 1," said the first friend. "It can be divided by 2," said the second friend. "And by 3," said the third friend. "And by 4," added the fourth friend. This continued until everyone had made such a comment. If exactly two friends were incorrect, and those two friends said consecutive numbers, what was the least possible integer they discussed?

9. Three numbers a, b, and c satisfy the conditions that $\gcd(a, b) = 4, \gcd(b, c) = 18$, and $\operatorname{lcm}(a, c) = 144$. Find the value of $a + c$.

10. How many positive integers are multiples of 2013 and have exactly 2013 divisors (including 1 and the number itself)?