NTP4 – Order, Arithmetic Functions

Homework code : NTP4

Issued on : 1st June 2023

Due date : 8th June 2023

Problems 1 to 10 are each worth 5 points.

Problem 1. Let p be an odd prime, and let q and r be primes such that p divides $q^r + 1$. Prove that either $2r \mid p - 1$ or $p \mid q^2 - 1$.

Problem 2. Compute sum of the greatest odd divisor of each of the numbers $2006, 2007, \dots, 4012$.

Problem 3. Prove that for all positive integers $a > 1$ and n, we have $n \mid \phi(a^n - 1)$.

Problem 4. For real numbers x and y, prove that

$$\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor$$

Problem 5. (AIME 1991) Suppose that r is a real number for which

$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \cdots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$

Find $\lfloor 100r \rfloor$.

Problem 6. (AIME 1995) Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

Problem 7. (AIME 2001). How many positive integer multiples of 1001 can be expressed in the form $10^j - 10^i$, where i and j are integers and $0 \leq i < j \leq 99$?

Problem 8. (ARML 2002) Let a be the integer such that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{23} = \frac{a}{23!}$$

Compute the remainder when a is divided by 13.

Hint: Use Wilson's Theorem.

Problem 9. Show that for all positive integers n,

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor = \left\lfloor \sqrt{4n+3} \right\rfloor.$$

Problem 10. (RMO 2018, P5) Find all natural numbers n such that $1 + \left\lfloor \sqrt{2n} \right\rfloor$ divides $2n$.

Hint: Substitute $k = \left\lfloor \sqrt{2n} \right\rfloor$. Find $2n$ in terms of k and another integer x, defining the limits of x.

Further Problems

Further Problems are each worth 5 special points.

F Problem 1. Prove that any number consisting of 2^n identical digits has at least n distinct prime factors.

F Problem 2. (IMO 1968, P6) Let x be a real number. Prove that

$$\sum_{k=0}^{\infty} \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor = \lfloor x \rfloor.$$

F Problem 3. (USA TST 2003) Find all ordered prime triples (p, q, r) such that $p \mid q^r + 1, q \mid r^p + 1$, and $r \mid p^q + 1$.

Hint: Consider the case $p, q, r \neq 2$ and you will find that there are no such values.

F Problem 4. The sequence

$$\{a_n\}_{n=1}^{\infty} = \{2,3,5,6,7,8,10,\dots\}$$

consists of all the positive integers that are not perfect squares. Prove that

$$a_n = n + \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor.$$

Hint: Define b_n such that $a_n - b_n = n$ and prove that $b_n = \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor$.

F Problem 5. (China 2003) The sides of a triangle have integer lengths k, m and n.
Assume that $k > m > n$ and

$$\left\{\frac{3^k}{10^4}\right\} = \left\{\frac{3^m}{10^4}\right\} = \left\{\frac{3^n}{10^4}\right\}.$$

Determine the minimum value of the perimeter of the triangle.