Lecture 1 - Primes and Divisibility

1 What is Number Theory

Number theory is basically the study of integers. (a) natural numbers, (b) prime numbers, (c) composite numbers, (d) even numbers, (e) odd numbers, (f) perfect squares, (g) negative cubes, (h) powers of 2, (i) abundant numbers, (j) perfect numbers, (k) Fibonacci numbers, (l) numbers with bases other than 10, (m) residue class modulo 6, etc.

2 Primes and Composite Numbers

A prime number is a number that has exactly two positive divisors, 1 and itself. A composite number is a number that has more than two positive divisors. A common method usually used to identify primes is called the 'Sieve of Eratosthenes'. This is quite efficient compared to the brute force method especially for larger numbers. Its procedure can be summarized as follows:

1. Find a number k such that $k^2 > n$.
2. Divide n by each of the prime numbers smaller than k. If none of them divides n, then n is prime.

It is highly recommended to memorize first few primes as this can save some time during the contest: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37.

Example 1. Check whether the following numbers are primes or not.

(a) 137, (b) 437.

3 Divisibility

We say that a number b is *divisible by* a number a, if dividing b by a gives *an integer*. It is denoted by $a \mid b$ and is read "a divides b".In the following, we will use $\overline{abcde}$ to denote a five digit number with unit digit e, ten's digit d, and so on. This is to avoid confusion between $abcde$ which would be the product $a \times b \times c \times d \times e$.

4 Divisibility Rules

- Divisibility rule for 2: Last digit is even.
- Divisibility rule for 3: Sum of digits is divisible by 3.
- Divisibility rule for 4: Number formed by last two digits is divisible by 4.
- Divisibility rule for 5: Last digit is 0 or 5.
- Divisibility rule for 6: Divisible by both 2 and 3.

- Divisibility rule for 7: The difference between the number formed by the last three digits and the number formed by the preceding digits is divisible by 7. (OR) Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7
- Divisibility rule for 8: Number formed by last three digits is divisible by 8.
- Divisibility rule for 9: Sum of digits is divisible by 9.
- Divisibility rule for 10: Last digit is 0.
- Divisibility rule for 11: Calculate the sum of odd positioned digits (O) and even positioned digits (E). If $O - E$ is divisible by 11, then the number is also divisible by 11. Don't forget that $O - E$ can be negative.

Example 2. Find the sum of all four-digit natural numbers of the form $\overline{4AB8}$ which are divisible by $2, 3, 4, 6, 8$ and 9.

Example 3. Find the possible values of a in $\overline{3,333,33a,888,888}$ so that the number is divisible by 7.

Example 4. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

Example 5. Find all positive integers n such that $2n - 1 \mid -6n + 5$.

Example 6. The measure of each interior angle of a regular n-sided polygon is given by the formula $\frac{180(n-2)}{n}$. How many regular polygons (different values of n) have angles of integer measure?

Example 7. How many numbers from 1 to 2023 are divisible by either 2, 5 or 7?

Homework Problems

Homework code : **HWN101**
Issued on : 11th October 2023
Due date : 17th October 2023

Submit the solutions to at least 6 of the homework problems before the due date.
All of the problems are each worth 5 points.

1. The sum of four consecutive primes is itself prime. What is the largest of the four primes?

2. Find the value of $a+b+c$ if $\overline{173a}$ is divisible by 9, $\overline{173b}$ is divisible by 11 and $\overline{173c}$ is divisible by 6.

3. A number is called *flippy* if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy but 3883 and 1231123 are not. How many five-digit flippy numbers are divisible by 15?

4. A 6-digit number begins with the digit 7. The number is divisible by 9. All six digits of the number are different. Find the smallest possible value of this number.

5. A 6-digit number, $\overline{15abcd}$, is divisible by 36. If a, b, c and d have different values, find the values of a, b, c and d so that the number has the least value of quotient when it is divided by 36.

6. Find all positive integers n such that $3n-1 \mid 9n+5$.

7. 11 girls and n boys pick up mushrooms. All the children pick up n^2+9n-2 mushrooms in total, and every child picks up same number of mushrooms. Are girls more than boys or boys more than girls among these children? Give reasons for your answer.

8. How many numbers from 1 to 2023 are divisible by either 3, 5 or 11?

9. The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $\overline{pqrst}$. The three-digit number $\overline{pqr}$ is divisible by 4, the three-digit number $\overline{qrs}$ is divisible by 5, and the three-digit number $\overline{rst}$ is divisible by 3. Find the value of P.

10. The 7-digit numbers $\overline{74A52B1}$ and $\overline{326AB4C}$ are each multiples of 3. Find the sum of all possible values of C.