Image Processing

(Year III, 2-nd semester)

Lecture 7:

Grayscale Images:

Convolution, Fourier Transform (II)

Notations and definitions

A continuous image is represented as a function of two independent variables: $f(x,y)$, $u(x,y)$, $v(x,y)$ and so on.

A sampled image is represented as a 2-dimensional sequence of real numbers: $f(i,j)$,$u(k,l)$, $v(m,n)$, and so on.

The symbols *i, j, k, l, m, n* will be integer indices of arrays and vectors.

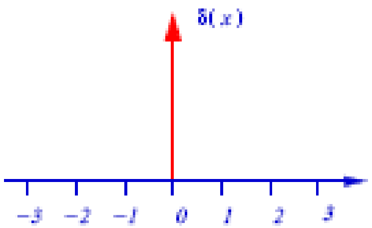
The symbol j represents $\sqrt{-1}$

Some well known one-dimensional functions that will be often used are **Dirac** and **Kronecker**. Their two-dimensional versions are functions of the separable form $f(x,y) = f_1(x)f_2(y)$:

- Two-dimensional continue **Dirac** delta function is defined by $\delta(x,y) = \delta(x)\,\delta(y)$.
- Two-dimensional discrete Kronecker delta function is defined by $\delta(m,n) = \delta(m)\delta(n)$.

Dirac Delta and Unit Impulse Function

The Dirac Delta Function can be thought of as a function on the real line which is zero everywhere except at the origin,



and which is also constrained to satisfy the identity:

$\delta(x) = 0$ for *x≠0;* $\int_{-\infty}^{+\infty} \delta(x) = 1$

This can be thought of as a very "*tall-and-thin" spike with unit area located at the origin, as* shown in figure.

The δ-functions should **not be considered to be an *infinitely high spike of zero width*** since it scales as: $\int_{-\infty}^{+\infty} a\delta(x)dx = a$

where a is a constant.

The delta function has **the fundamental property** that

$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$ and $\int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta(x)dx = f(0)$ for a=0, ε>0.

This property can be extended (**the shifting property**)

$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$ and $\int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta(x-a)dx = f(a)$ for ε>0.

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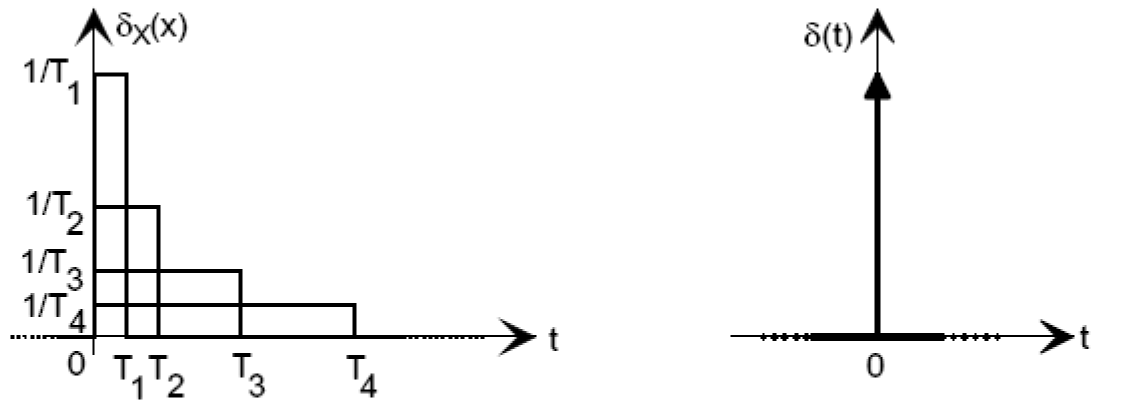
Dirac Delta and Unit Impulse Function



a) Unit pulses of different extents b) The impulse function

Figure shows a *unit pulse function $\delta_T(t)$, that is a brief rectangular pulse function of duration T, defined to have a constant amplitude 1/T over its extent, so that the area T × 1/T under the* pulse is unity:

$$\delta_T(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1/T & 0 < t \le T \\ 0 & \text{for } t > T \end{cases}$$

The Dirac Delta Function can be defined as the limiting form of the unit pulse $\delta_T(t)$ *as the duration T approaches zero. As the duration T of $\delta_T(t)$ decreases,* the amplitude of the pulse increases to maintain the requirement of unit area under the function, and

$$\delta(t) = \lim_{T \to 0} \delta_T(t)$$

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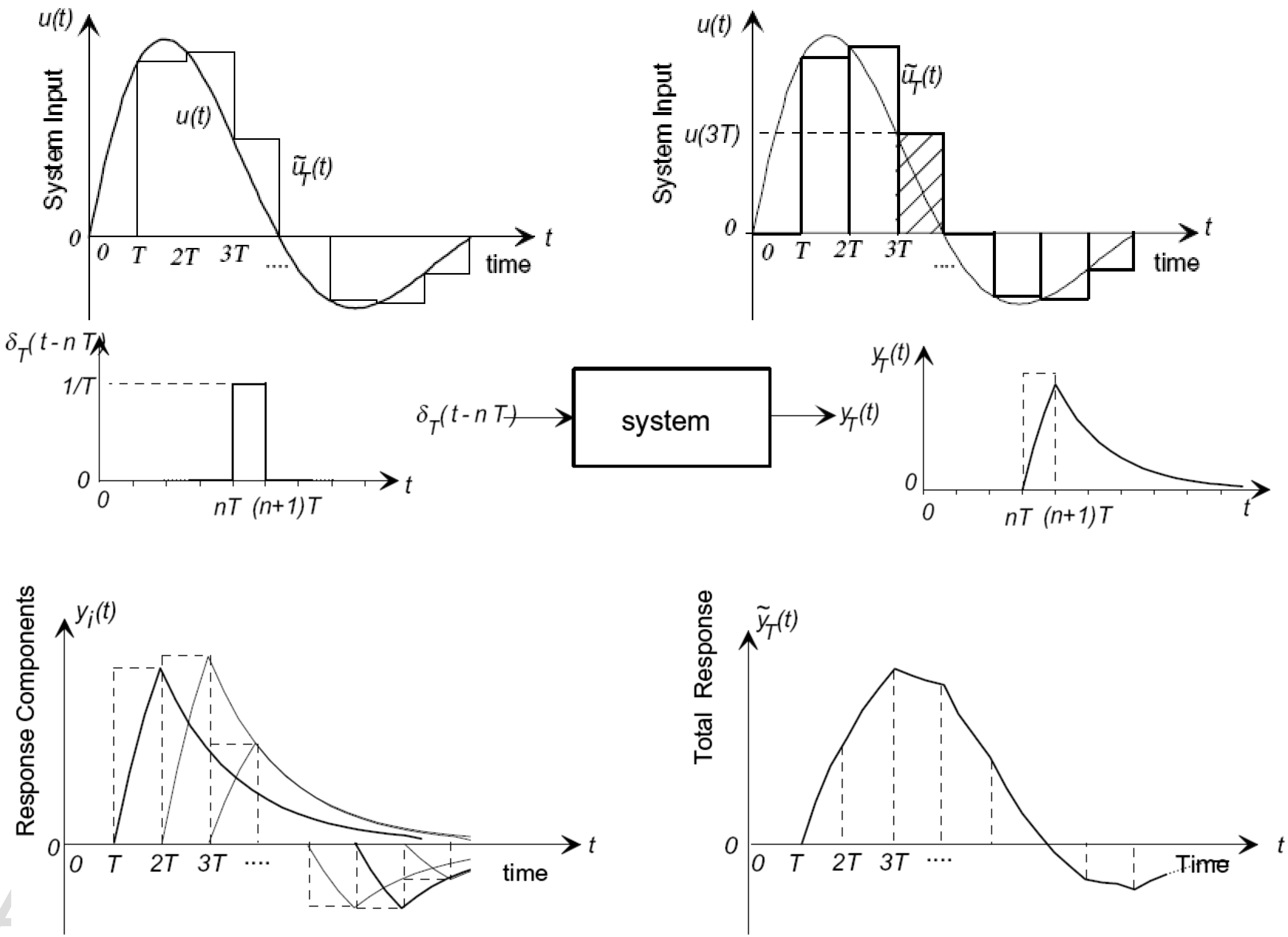
System response to individual pulses in the staircase approximation



Linear Systems and Shift Invariance

- Let f(m,n) be the **input sequence** and g(m,n) be the **output sequence** of a two-dimensional system $g(m,n) = H[f(m,n)]$
- The system is **linear** if and only if for arbitrary constants a_1 and a_2 it holds:

$$H[a_1 f_1(m,n)+a_2 f_2(m,n)] =$$

$$= a_1 H[f_1(m,n)]+a_2 H[f_2(m,n)]=a_1 g_1(m,n)+a_2 g_2(m,n)$$

- This is called **linear superposition property**

Linear Systems and Shift Invariance(2)

- The output g(m,n) of any linear system can be obtained as follows:

$$g(m,n) = \mathrm{H}[f(m,n)] = \mathrm{H}\left[\sum_{m'}\sum_{n'} f(m',n')\delta(m-m',n-n')\right]$$

$$= \sum_{m'}\sum_{n'} f(m',n')\mathrm{H}\left[\delta(m-m',n-n')\right]$$

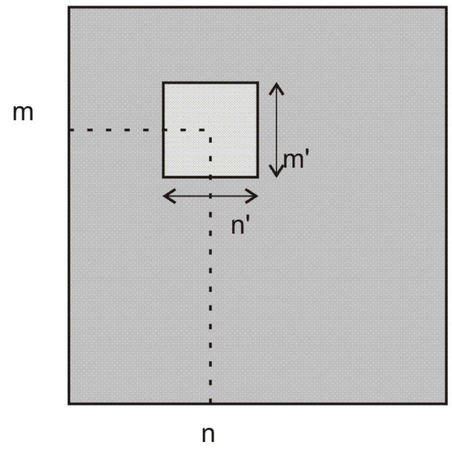
$$\Rightarrow g(m,n) = \sum_{m'}\sum_{n'} f(m',n')h(m,n;m',n')$$

where ***h(m,n;m',n')* is the impulse response of the system**.

- The impulse response *h(m,n;m',n')* is the output of the system H[·] at location *(m,n)*, when the input is the two-dimensional Kronecker delta function at location *(m',n')*
- The impulse response is called the **point spread function, PSF**, when the inputs and outputs represent a positive quantity e.g. the intensity of light in imaging systems. The term impulse response is more general, values may be negative and complex.

Linear Systems and Shift Invariance(3)

- The **region of support** of an impulse response is the smallest closed region in the *(m,n)* plane outside which the impulse response is zero.
- A system is said to be a:
  - **finite impulse response**, **FIR** system if its impulse response has finite region of support;
  - **infinite impulse response**, **IIR** system if its impulse response has infinite region of support.



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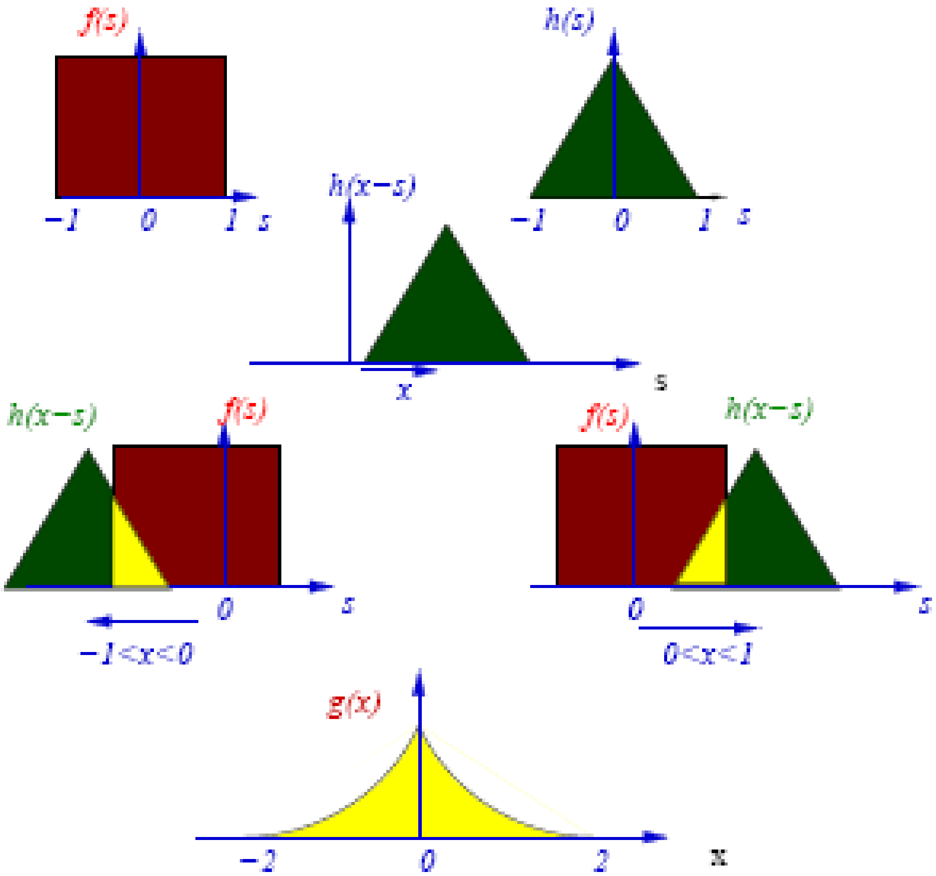
Linear Systems and Shift Invariance(4)

- A system is called **spatially invariant** or **shift invariant** if a translation of the input causes corresponding translation of the output.
- For shift invariant systems it holds: $h(m,n;m',n')=h(m-m', n-n')$
- The impulse response is a function of two variables only. The variables describe displacement.
- The shape of the impulse response does not change with the movements of the impulse in the *(m,n)* plane.
- For shift invariant systems, the output equals:

$$g(m,n) = \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} h(m-m', n-n') f(m', n')$$

which is called the **convolution** of the input with the impulse response.

Convolution



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Convolution

- We will use the symbol * to denote the convolution:

$$g(x,y) = h(x,y) * f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x',y')\, f(x-x',y-y')\,dx'\,dy'$$

$$g(x,y) = h(x,y) * f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x-x',y-y')\, f(x',y')\,dx'\,dy'$$

$$g(m,n) = h(m,n) * f(m,n) = \sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} h(m',n')\, f(m-m',n-n')$$

$$g(m,n) = h(m,n) * f(m,n) = \sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} h(m-m',n-n')\, f(m',n')$$

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Convolution

Commutativity: $f * g = g * f$

Associativity: $f * (g * h) = (f * g) * h$

Distributivity: $f * (g + h) = (f * g) + (f * h)$

Identity element: $f * \delta = \delta * f = f$

Associativity with scalar multiplication:

$$a(f * g) = (af) * g = f *(ag)$$

Differentiation rule: $D(f * g) = Df * g = f * Dg$

Convolution theorem: $F(f * g) = kF(f)F(g)$

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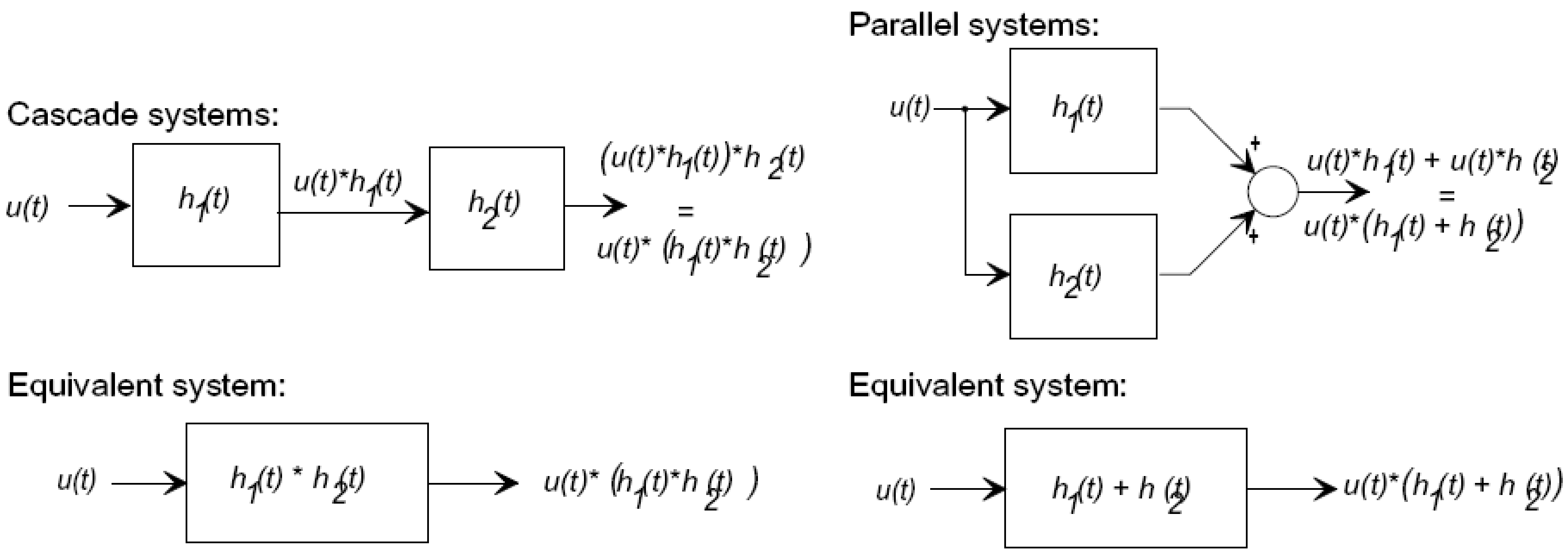
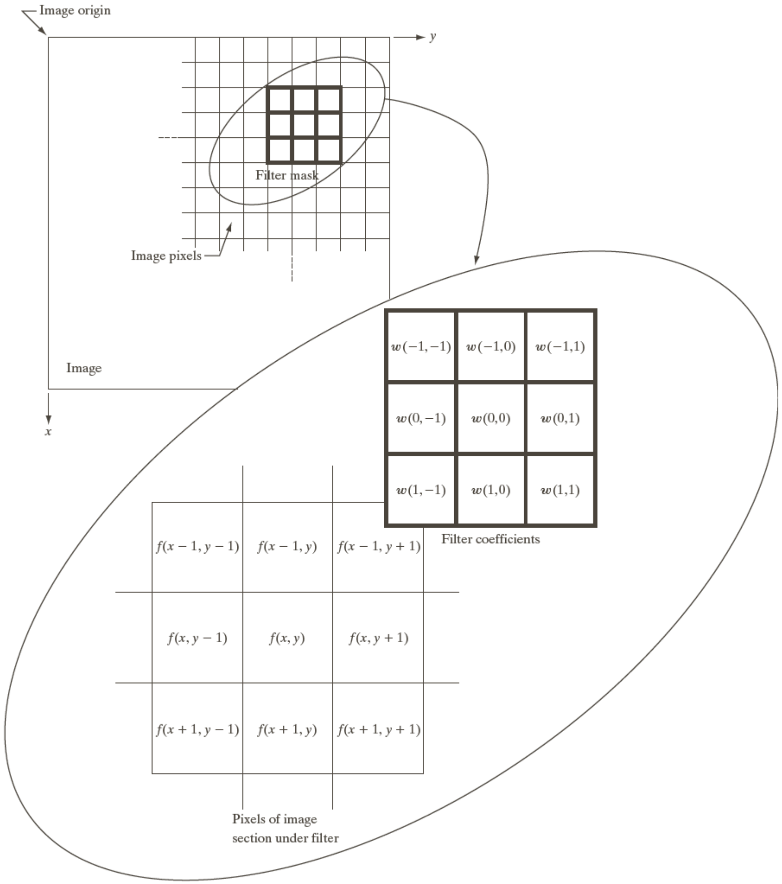
Convolution



Impulse response of series and parallel connected systems.

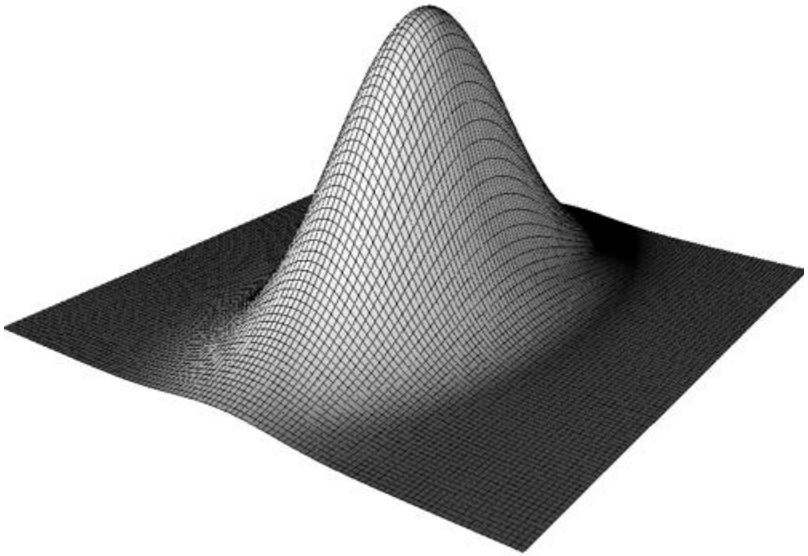
Convolution



$$g(x,y)=\sum_{s=-1}^{s=1}\sum_{t=-1}^{t=1} w(s,t)\,f(x+s,y+t)$$

Convolution

$$G_{2D}(x, y; \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



$\frac{1}{273}$

1	4	7	4	1
4	16	26	16	4
7	26	41	26	7
4	16	26	16	4
1	4	7	4	1

Discrete approximation of the Gaussian function with σ=1.0

The Fourier Transform

The Fourier Transform converts spatial image data f(x,y) into a frequency representation F(u,v). Both representations contain equivalent information. However, each representation has a set of strengths and weaknesses.

Spatial Domain

+Intuitive Representation of Image data
+Kernels applied directly to spatial data.
-Filtering with large Kernels may result in long processing times.

Frequency Domain

-Non-intuitive representation of image;
+Filtering with large Kernels can be performed very quickly;
-Image and Kernel must first be converted to frequency domain, modified, then reconverted;
+Engineering frequency filter somewhat easier

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The Fourier Transform

Image

– has been viewed as a spatial array of gray values,

– can be thought of as a spatially varying continuous (discrete) function.

Decompose the image into a set of orthogonal functions, called *basis functions*

– when basis functions are combined (linearly) the original function will be reconstructed.

The Fourier basis functions are sinusoids

Changes in image intensity → changes in spatial frequency

We write:

– f(x,y) = ∑ *Weighted basis functions*

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The Fourier Transform

- The forward Fourier transform of a complex function f(x) is defined as:

$$F(u) \stackrel{\Delta}{=} \mathrm{F}[f(x)] = \int_{-\infty}^{\infty} f(x)\exp(-j2\pi ux)dx$$

$$F(u) \stackrel{\Delta}{=} \mathrm{F}[f(x)] = \int_{-\infty}^{\infty} f(x)(\cos 2\pi ux - j\sin 2\pi ux)dx$$

- The inverse Fourier transform of F(u) is:

$$f(x) \stackrel{\Delta}{=} \mathrm{F}^{-1}[F(u)] = \int_{-\infty}^{\infty} F(u)\exp(j2\pi ux)du$$

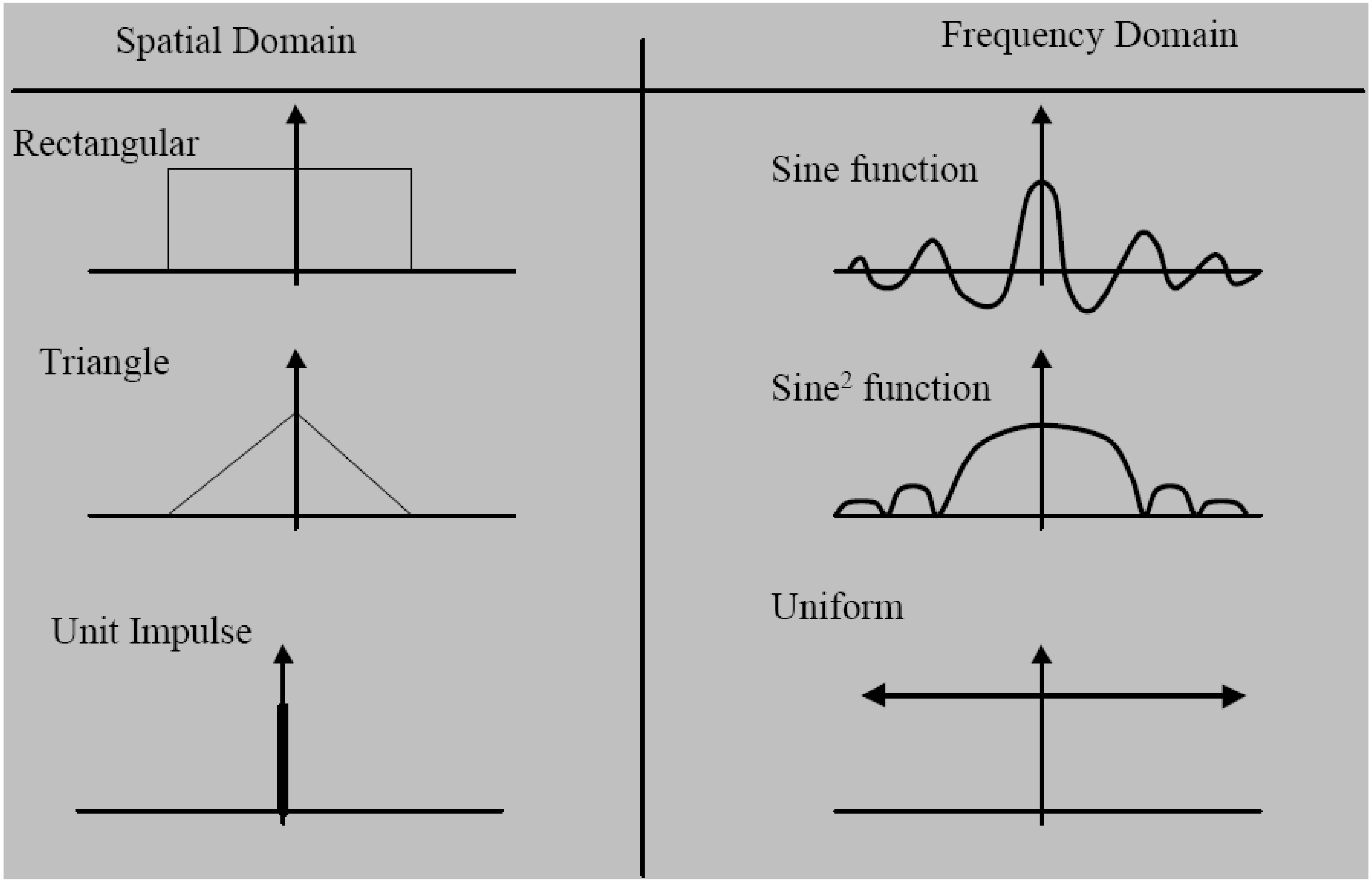
$$f(x) \stackrel{\Delta}{=} \mathrm{F}^{-1}[F(u)] = \int_{-\infty}^{\infty} F(u)(\cos 2\pi ux + j\sin 2\pi ux)du$$

- Two dimensional case:

$$F(u,v) \stackrel{\Delta}{=} \mathrm{F}[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-j2\pi(ux+vy))dxdy$$

$$f(x,y) \stackrel{\Delta}{=} \mathrm{F}^{-1}[F(u,v)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp(j2\pi(ux+vy))dudv$$

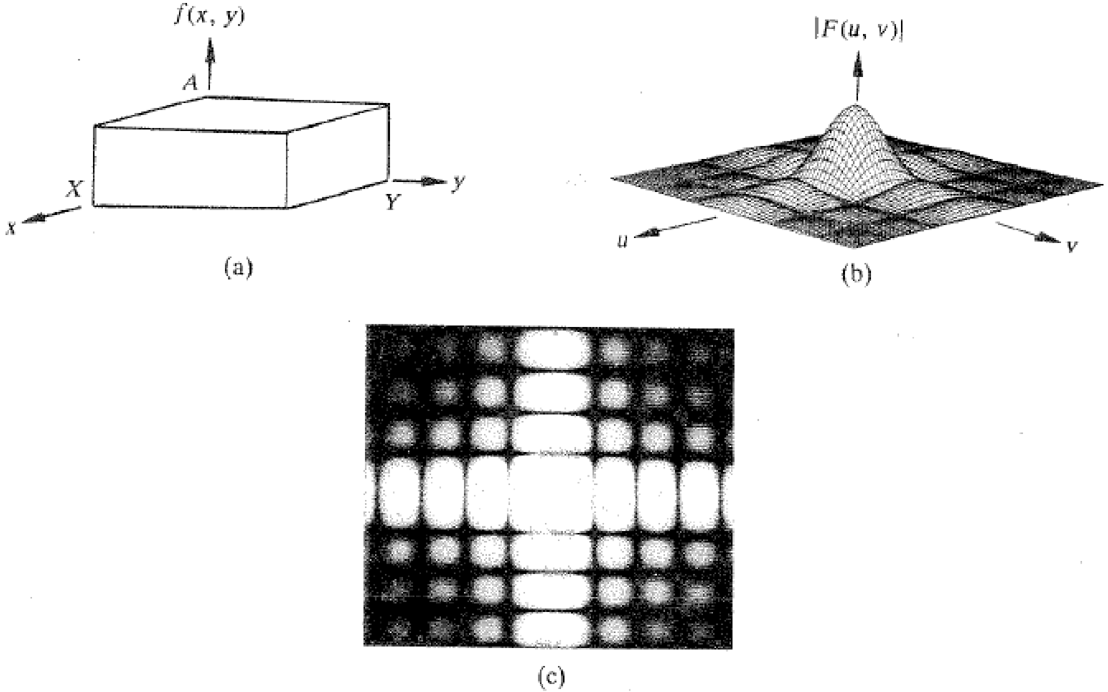
The Fourier Transform



sinc=sin(x)/x

The Fourier Transform

FT of a 2D rectangle function



Discrete Fourier Transform

- The Discrete Fourier Transform (DFT) is defined as:
- Forward DFT

$$F(u,v) \stackrel{\Delta}{=} \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp\left[-j2\pi\left(ux/M + vy/N\right)\right]$$

$(u = 0, 1, \ldots, M\text{-}1,\ v = 0, 1, \ldots, N\text{-}1)$

- Inverse DFT

$$f(x,y) \stackrel{\Delta}{=} \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp\left[j2\pi\left(ux/M + vy/N\right)\right]$$

$(x = 0, 1, \ldots, M\text{-}1,\ y = 0, 1, \ldots, N\text{-}1)$

Spatial frequencies: If *f(x,y)* is luminance and *x,y* the spatial coordinates, then *u, v* are spatial frequencies representing luminance changes with respect to spatial distances. The units of *u, v* are the reciprocals of *x* and *y*.

Uniqueness: For continuous functions, *f(x,y)* and *F(u, v)* are unique with respect to one another.

Separability: The Fourier transform kernel is separable. Hence two dimensional Fourier transform can be realized by two one-dimensional transforms along the spatial coordinates.

Properties of the Fourier Transform

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}e^{-j2\pi(\frac{ux}{N})}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(\frac{vy}{N})}$$

$$\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(\frac{vy}{N})}=F(x,v)$$

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}e^{-j2\pi(\frac{ux}{N})}F(x,v)$$

Properties of the Fourier Transform (2)

Linearity: The Fourier transform is a linear operation so that the Fourier transform of the sum of two functions is given by the sum of the individual Fourier transforms.

$$F\{a\,f(x,y)+bg(x,y)\} = aF(u,v)+bG(u,v)$$

Complex Conjugate: The Fourier transform of the Complex Conjugate of a function is given by

$$F\{f^*(x,y)\} = F^*(-u,-v)$$

where *F(u,v) is the Fourier transform of f (x,y).*

Forward and Inverse: We have that

$$F\{F(u,v)\} = f(-x,-y)$$

so that if we apply the Fourier transform twice to a function, we get a spatially reversed version of the function.

Similarly with the inverse Fourier transform we have that,

$$F^{-1}\{f(x,y)\} = F(-u,-v)$$

so that the Fourier and inverse Fourier transforms differ only by a sign.

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Properties of the Fourier Transform (3)

Scaling: f(a*x) <--> (1/|a|) * F(u/a)

This means that if you make the function wider in the x-direction, it's spectrum will become smaller in the x-direction, and vice versa. The amplitude will also be changed.

Time Shifting (Translation in the spatial domain):

$$f(x-x_0, y-y_0) <---> F(u,v)e^{-j2\pi(\frac{ux_0+vy_0}{N})}$$

The only thing that happens if you shift the time, is a multiplication of the Fourier Transform with the exponential of an imaginary number, you won't see the difference of a time shift in the amplitude of the spectrum, only in the phase.

Frequency Shifting (Translation in the frequency domain):

$$f(x,y)\exp\left[-j2\pi\left(u_0x/N+v_0y/N\right)\right] <--> F(u-u_0, v-v_0)$$

This is the dual of the time shifting.

Properties of the Fourier Transform (4)

Differentials: The Fourier transform of the derivative of a functions is given by: $F\{d f(x)/dx\} = j2\pi u F(u)$

and the second derivative is given by:

$$F\{d^2 f(x)/dx^2\} = -(2\pi u)^2 F(u)$$

$$F\{df(x,y)/dx = j2\pi u F(u,v)$$

$$F\{d f(x,y)/dy\} = j2\pi v F(u,v)$$

$$F\{\Delta f(x,y)\} = -(2\pi w)^2 F(u,v) \qquad \text{where } w^2=u^2+v^2$$

Convolution theorem: g(x,y)=h(x,y)*f(x,y) G(u,v)=H(u,v)F(u,v)

Inner product preservation: The inner product of two functions is equal to the inner product of their Fourier transforms. From this we obtain the Parceval energy conservation formula:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x,y)|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(u,v)|^2 dudv$$

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Fourier Transform

Fourier coefficients are generally complex numbers expressed as two components:

$F(u,v)=R(u,v) + jI(u,v)$

Typically these are converted into equivalent vectors represented by magnitude and phase angle:

$F(u,v)=|F(u,v)|e^{j\theta(u,v)}$

$|F(u,v)| = sqrt(R(u,v)^2+I(u,v)^2)$ – Fourier spectrum

$\Theta(u,v)=tan^{-1}(I(u,v)/R(u,v))$ – Phase angle

$|F(u,v)|^2=R(u,v)^2+I(u,v)^2=P(u,v)$ – Power spectrum of f(x,y)

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Fourier Transform

The N x N DFT is a cyclic of period N, in both u and v directions,

F(u + nN, v+ mN) = F(u,v)

Most algorithms locate F(0, 0) at top/left. So we can shift the F(u,v) terms in two dimensions to give: F(u,v) for u & v = -N/2, …, 0, …,N/2-1

This allows us to have F(0,0) to appear at the centre of the Fourier array.

For that a translation in the frequency domain has to be done for u_0=N/2 and v_0=N/2. From:

$$e^{j2\pi\left(\frac{\frac{N}{2}x+\frac{N}{2}y}{N}\right)} = e^{j\pi(x+y)} = (-1)^{x+y}$$

$$f(x,y)(-1)^{x+y} <\!\!-\!\!-\!\!> F(u-N/2,\, v-N/2)$$

Typically we display |F(u,v)| but the dynamic range is to high.

The solution is to scale the signal:

D(u,v)= c log (1 + |F(u,v)|)

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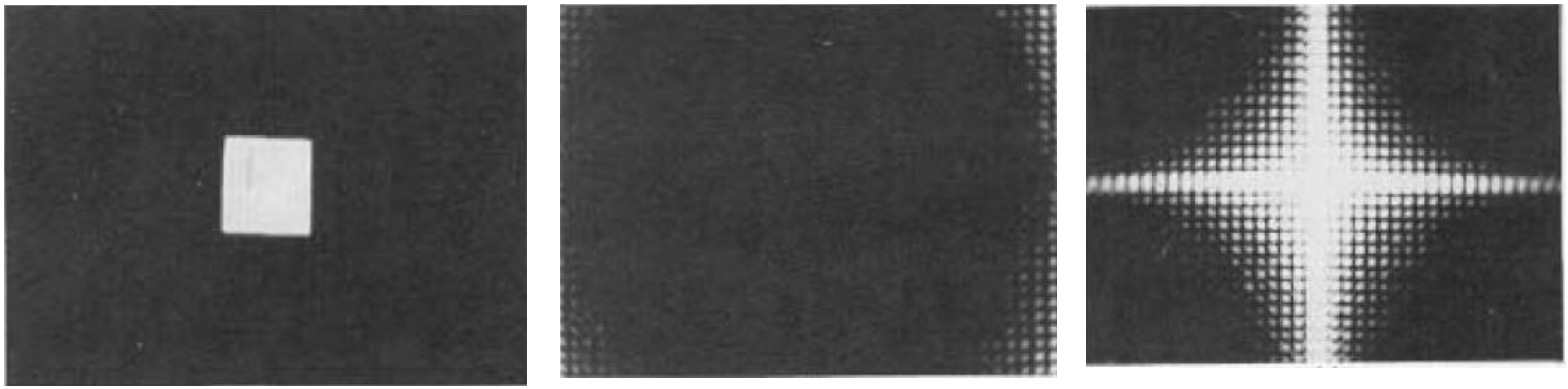
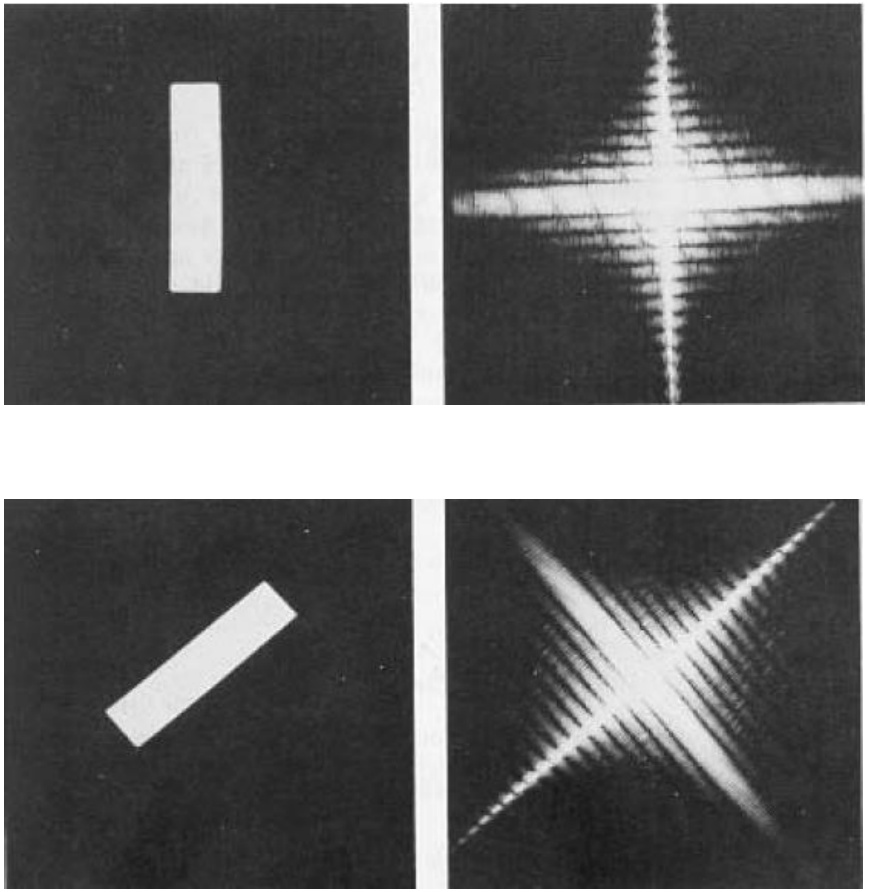
Fourier Transform

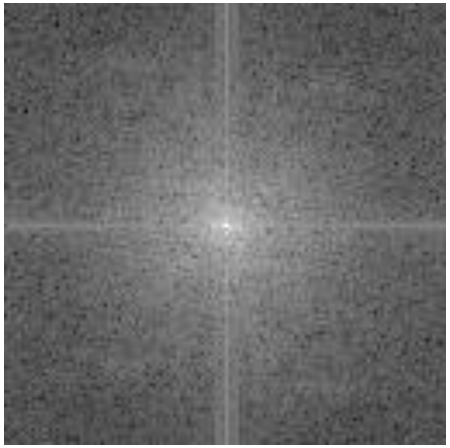


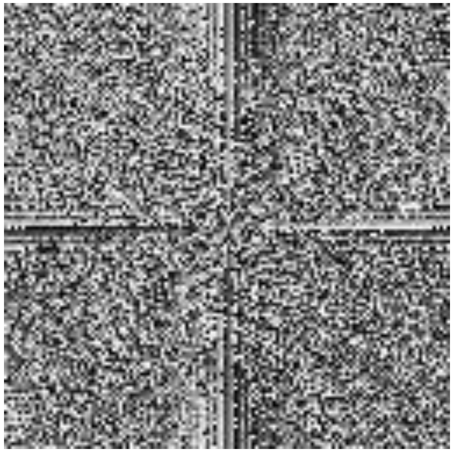
Fourier Transform



Fourier Transform

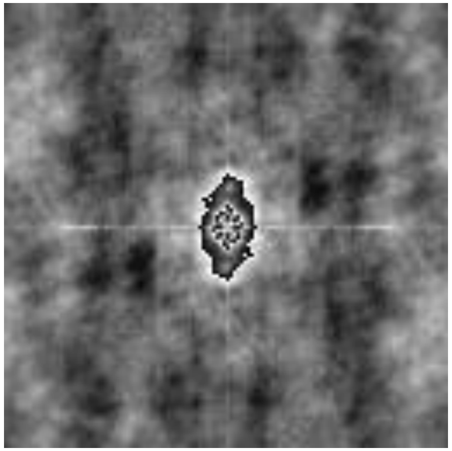
Direct Fourier Transform





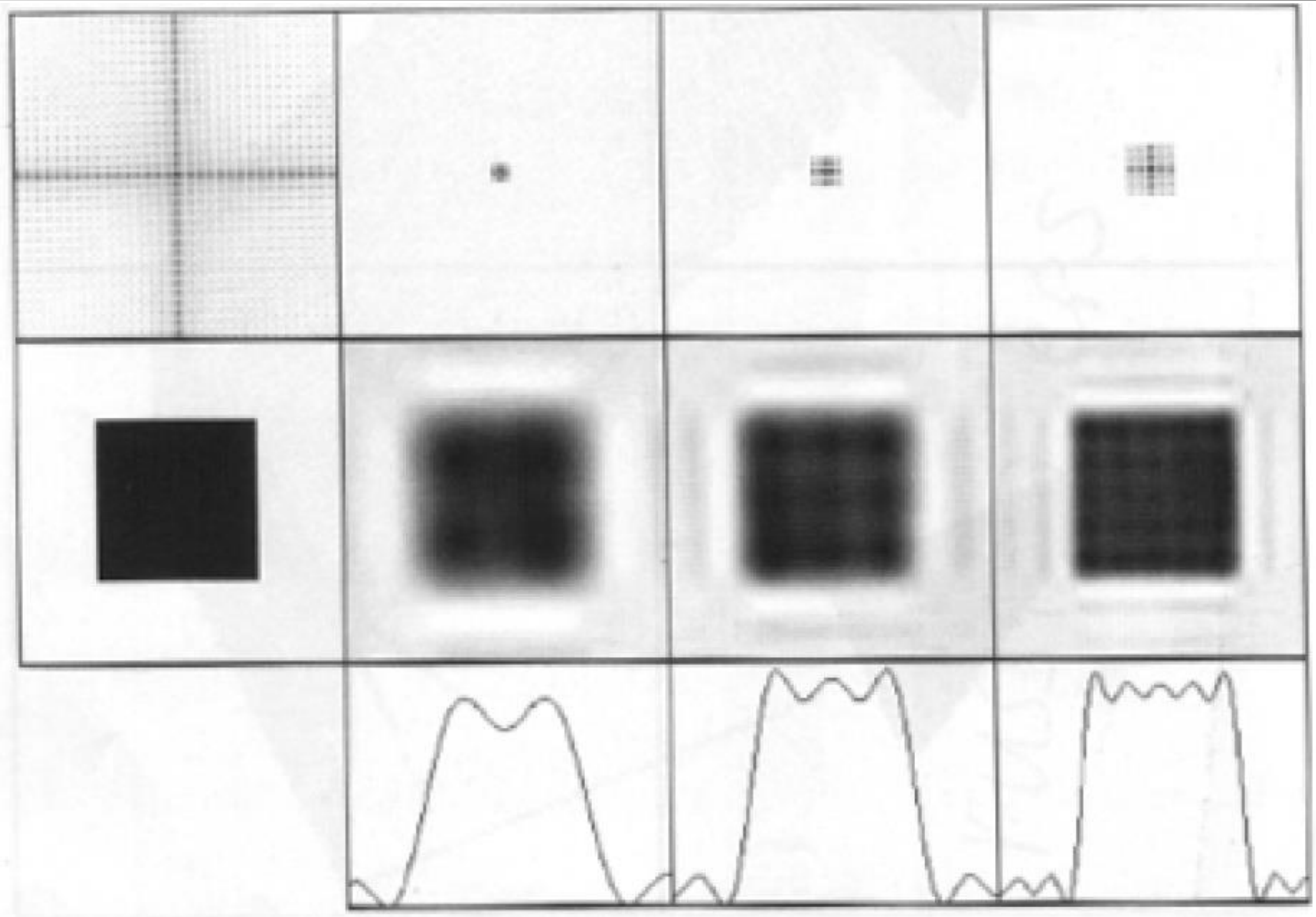
Original $\log(1+|F(u,v)|)$ $\phi(u,v)$

Invers Fourier Transform



$\phi(u,v)=0$ $|A(u,v)|=0$

Fourier Transform



Images in the spatial domain are in the middle row, and their frequency space are shown on the top row. The bottom row shows the varying brightness of the horizontal line through the center of an image. The low frequency terms are on the center of the square, and terms with higher magnitude are on the outer edges.

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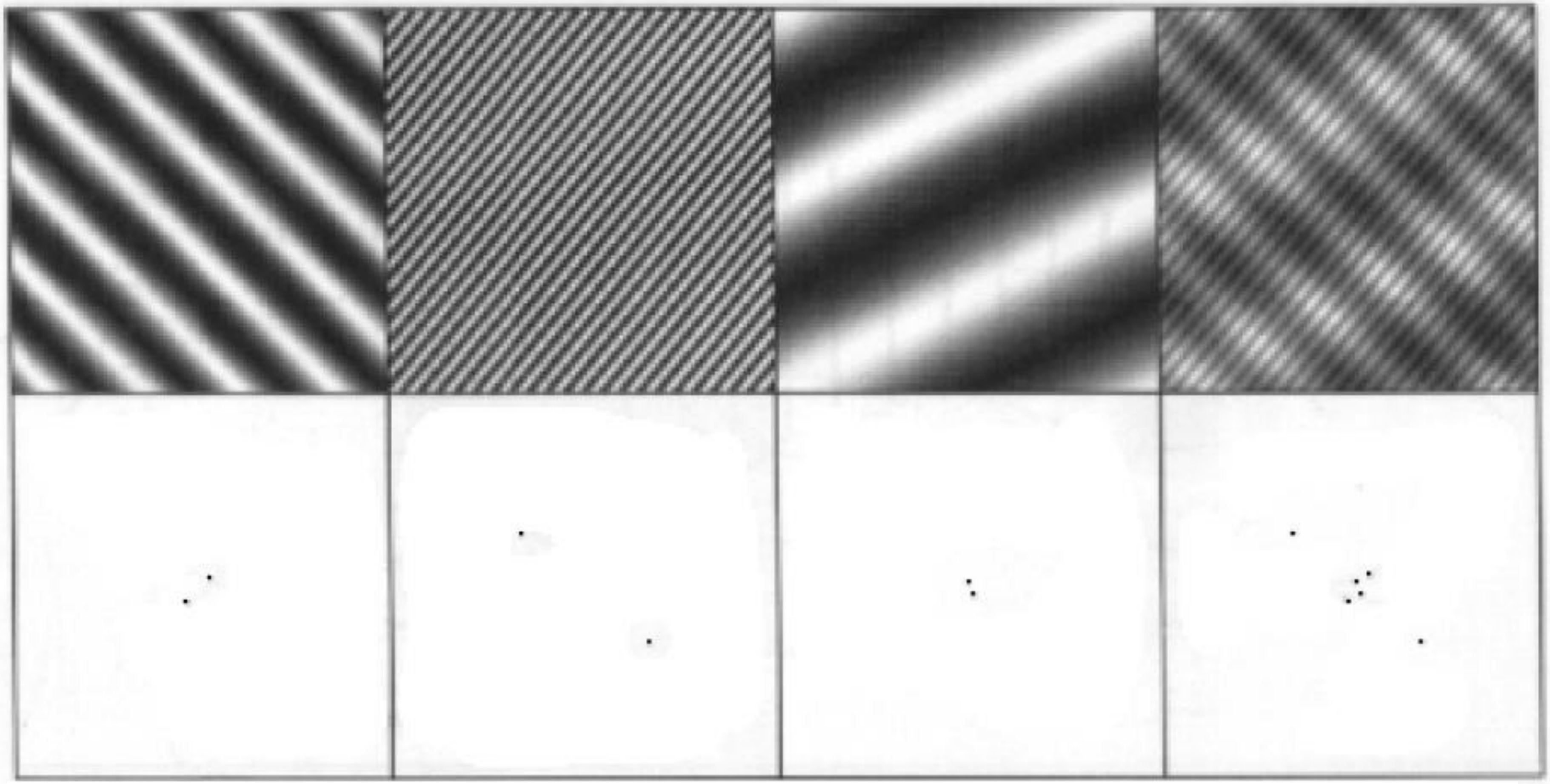
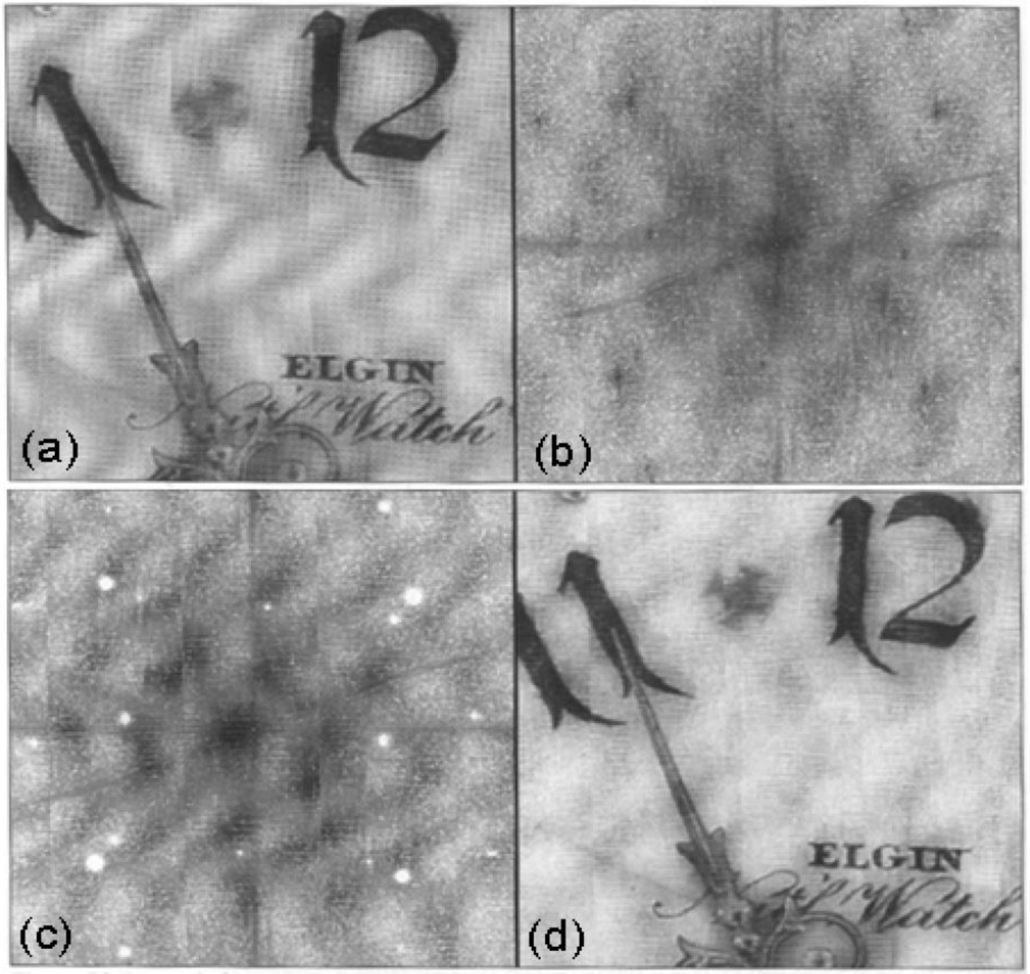
Fourier Transform



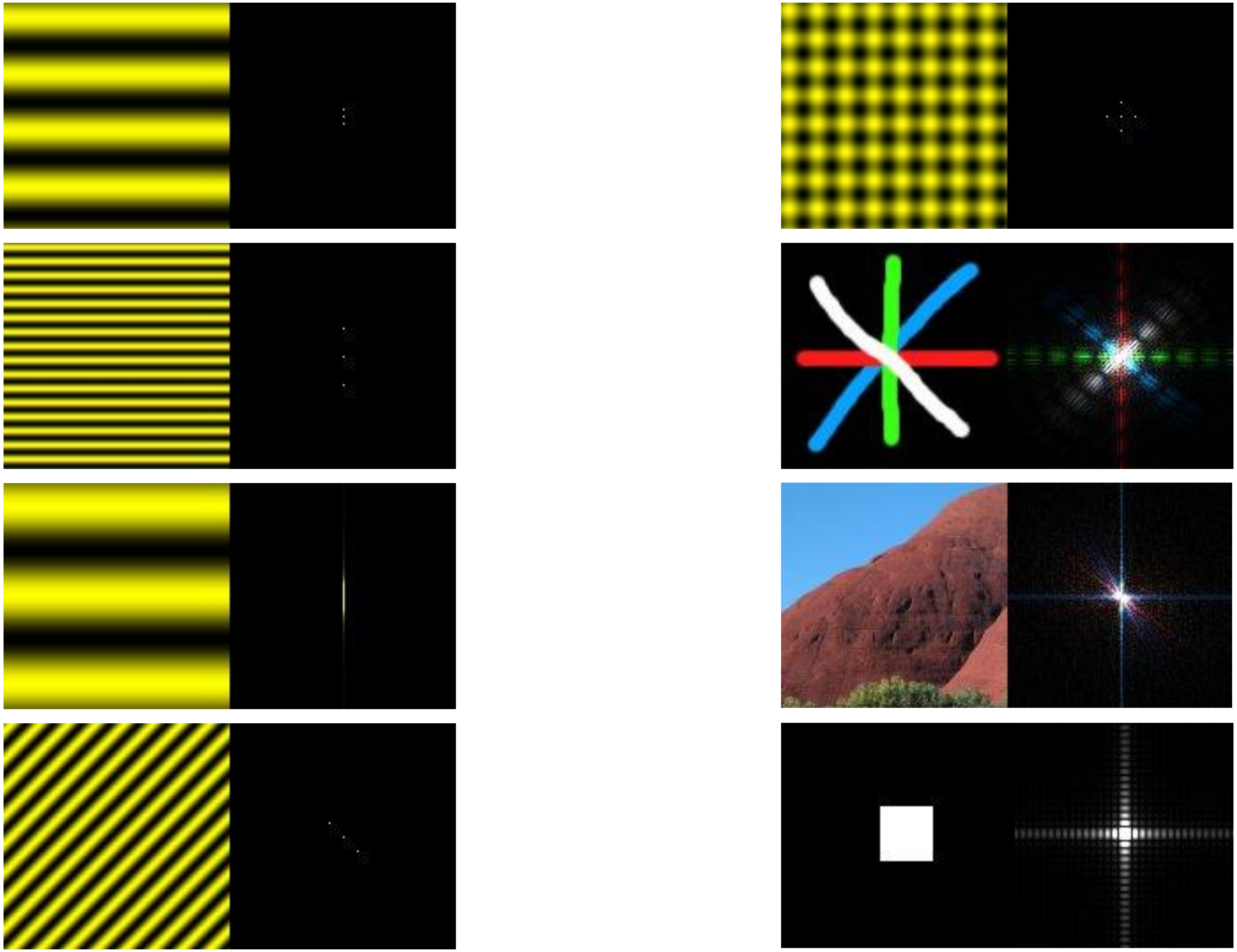
Images with perfectly sinusoidal variations in brightness: The first three images are represented by two dots. You can easily see that the position and orientation of those dots have something to do with what the original image looks like. The 4th image is the sum of the first three.

Fourier Transform



(a) Dirty looking photo copied image. (b) The representation of image in the frequency space, i.e. the star diagram. c) Those stars, however, do no good to the image, so we rub them out. (d) Reconstruct the image using (c) and those dirty spots on the original image are gone!

Fourier Transform



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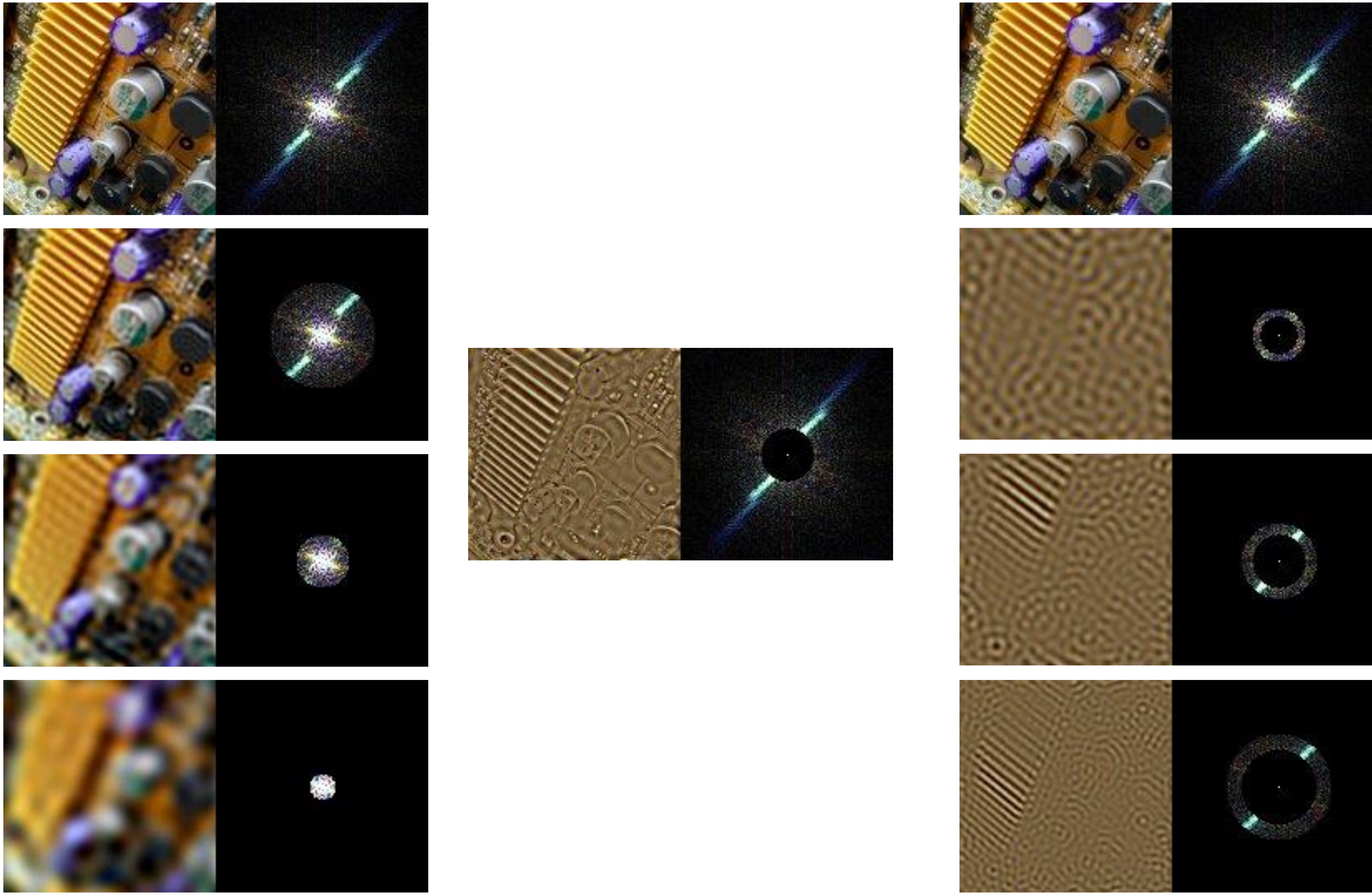
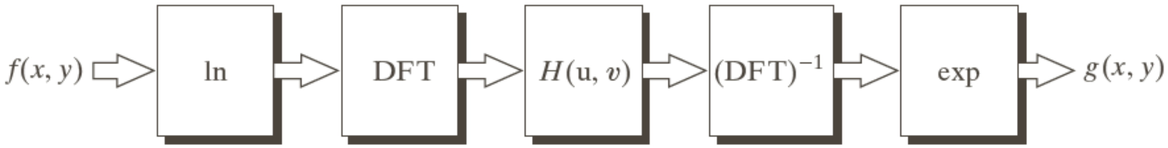
Fourier Transform



Fourier Transform

FIGURE 4.60 Summary of steps in homomorphic filtering.



Fourier Filtering Methods

Convert the spatial image into a frequency signal via the Fourier transform.

Modify frequencies as desired using a frequency filter.

Convert frequency image back to a spatial image using the inverse Fourier transform.

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