Topic 6: Digital Filtering

Contents:

Linear Digital Filtering

Main image processing operation, used for 90% of image processing operations.

Objective is to *Convolve* image $f(i,j)$, with filter function $h(i,j)$. In Real Space,

$$g(i,j) = h(i,j) \odot f(i,j)$$

or (by the convolution theorem) in Fourier Space

$$G(k,l) = H(k,l)F(k,l)$$

This operation can be performed in *Real* **OR** *Fourier* space. Mathematical operation is identical, but computational cost varies.

The operation of the filter is controlled by

$$\begin{aligned} h(i,j) &\rightarrow \text{In real space} \\ H(k,l) &\rightarrow \text{In Fourier space} \end{aligned}$$

Fourier Space Convolutions

Convolution can be written as,

$$g(i,j) = F^{-1}\{H(k,l)F(k,l)\}$$

Processing requires **TWO** DFTs and a complex multiply, (a third DFT required if $H(k,l)$ is formed from $h(i,j)$).

Note: DFTs and $\times$ must be performed in floating point format.

Computational time **independent** of filter type.

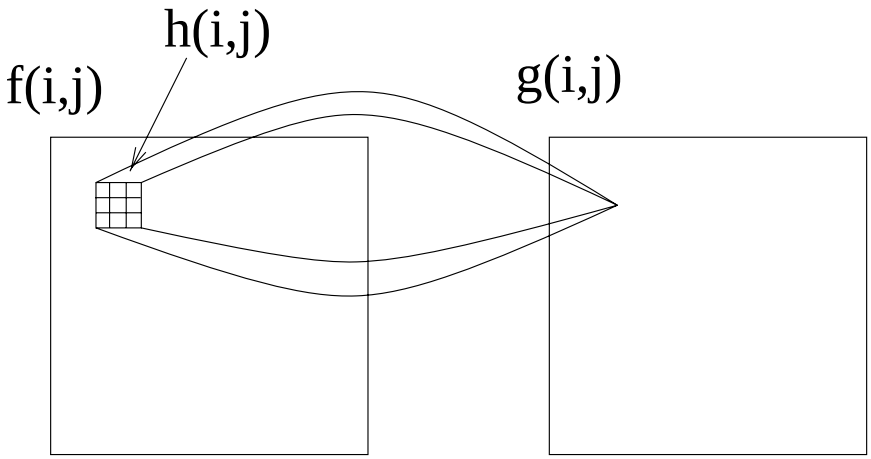
Time will depend on the computer system: for example 512×512 images, Pentium 200MMX, computational time about 2.3 seconds.

Real Space Convolutions

For filter $h(i,j)$ of size M by M,

$$g(i,j) = \sum_{m,n=-M/2}^{M/2} h(m,n)\, f(i-m, j-n)$$

"shift & multiply" scheme.



Computation $\propto M^2$

All calculations can be integer or byte.

Filter to 5×5 available in custom hardware at video rates.

For serial machine filters bigger that 9×9 are typically faster by Fourier space technique.

Fourier Space Filters

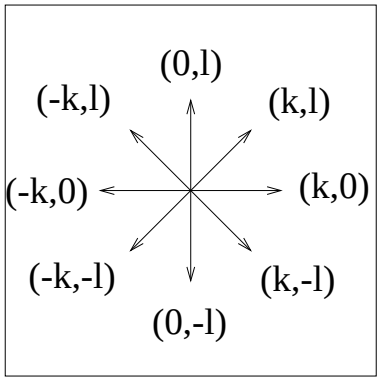
Filtering operation determined by $H(k,l)$, modified FT of image.

Most applications input & output images are **REAL**, so filter must preserve Symmetry conditions,

Real Part	$\Rightarrow$	Symmetric
Imaginary Part	$\Rightarrow$	Anti-symmetric

so $H(k,l)$ must obey these conditions.

In practice $H(k,l)$ is Real and Symmetric.



Low Pass Filter

Low pass filters allow *LOW* spatial frequencies to pass while attenuating or blocking *HIGH* spatial frequencies. Used extensively in the Reduction of Noise (see last lecture).

Ideal Low-Pass Filter
Block all frequencies greater than some limit,

$$\begin{aligned} H(k,l) &= 1 \qquad k^2+l^2 \leq w_0^2 \\ &= 0 \qquad \text{else} \end{aligned}$$

This Fourier transforms to give

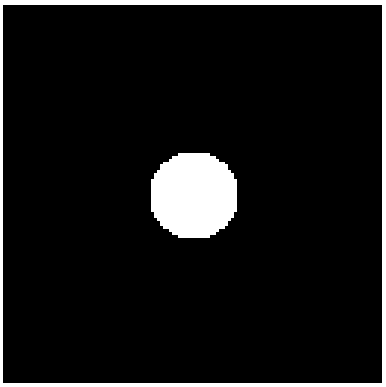
$$h(i,j) = \frac{J_1(r/w_0)}{r/w_0}$$

which results in "*ringing*" effects in the output image due to the lobes associated with $h(x,y)$.

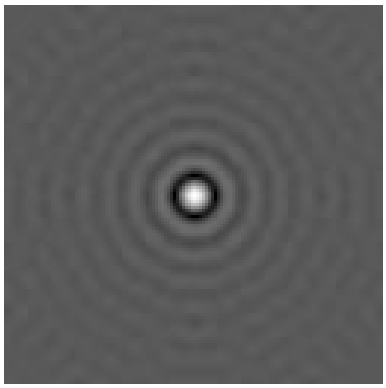
Not really useful.

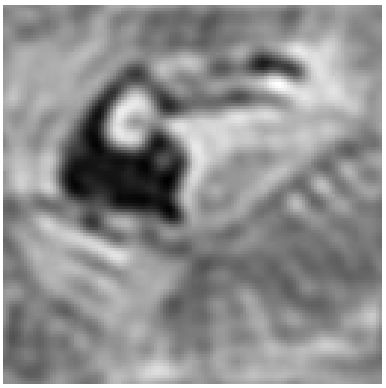
Digital Example

128×128 image with low-pass filter with $w_0 = 15$.



Input image Low-pass filter





Real space filter Filtered Image

Useful to reduce the effect of random noise, but too much “ringing” to be actually useful.

Gaussian Low Pass Filters

Filter profile in Fourier space is a two dimensional Gaussian of the type:

$$H(k,l) = \exp\left(-\frac{w}{w_0}\right)^2$$

where $w^2 = k^2 + l^2$ and w_0 is the $1/e$ point of the Gaussian.

$H(u,v)$ is *infinite*attenuates but does not remove high spatial frequencies.

In real space $h(i,j)$ is also Gaussian, begin, (see Fourier Booklet),

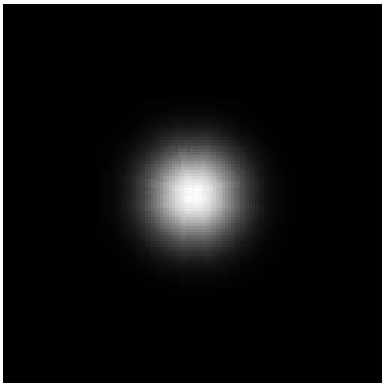
$$h(i,j) = \frac{\pi}{w_0^2}\exp\left(-\pi^2 w_0^2 r^2\right)$$

where $r^2 = i^2 + j^2$.

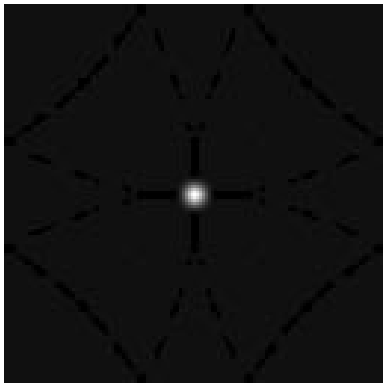
This gives a smooth filtered image without any ringing.

Digital Example

128×128 image with lowpass filter with $w_0 = 15$.



Input image Lowpass filter



Real space filter Filtered Image

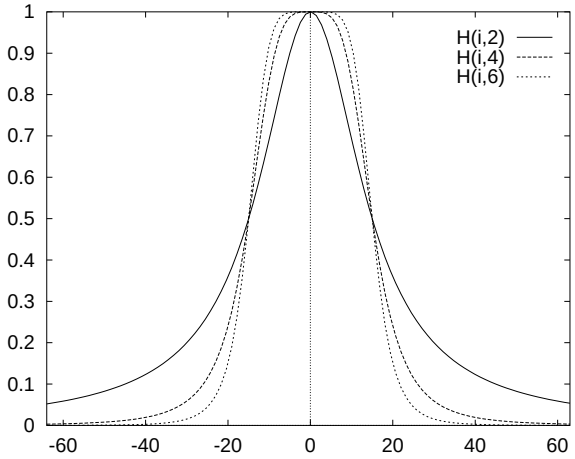
Very useful digital filter for noise reduction giving a very “smooth” filtered image. Tends to severely smooth edges making further “edge detection” difficult.

Other Smooth Low-Pass Filters

Butterworth Filter:

$$H(k,l) = \frac{1}{1+(\frac{w}{w_0})^n}$$

where w_0 is *half point* and n is the *order*.



Plot with $w_0 = 15$ and $n = 2,4,6$.

Very similar properties to Gaussian, filter inherited from analogue signal processing.

Trapezoidal filter:

$$\begin{aligned} H(k,l) \quad &= 1 \qquad && w < w_0 \\ &= \tfrac{w-w_1}{w_0-w_1} \qquad && w_0 \le w \le w_1 \\ &= 0 \qquad && w > w_1 \end{aligned}$$

This will exhibit more ringing than Gaussian or Butterworth, but less than ideal filter.

Use the `filtering` program to “play” with these.

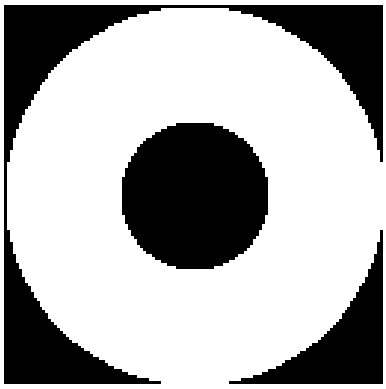
High Pass Filter

High pass filters allow *HIGH* spatial frequencies to pass while attenuating or blocking *LOW* spatial frequencies. User for the enhancement of high frequencies (and thus edges)

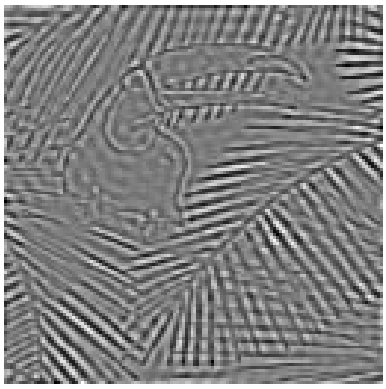
Ideal High-Pass Filter
Block all frequencies less than some limit,

$$\begin{aligned} H(l,k) &= 0 \qquad l^2+k^2 \leq w_0^2 \\ &= 1 \qquad \text{else} \end{aligned}$$

This filter suffers from such sever ringing artifacts that it is rarely used and much better operations can be obtained from the smooth high pass filters.



Filter



Output Image

Example with $w_0 = 25$.

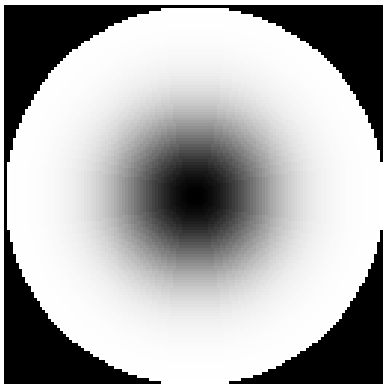
Gaussian High Pass Filter

We want a smooth reduction of *Low* spatial frequencies while the high spatial frequencies are pass unaltered.

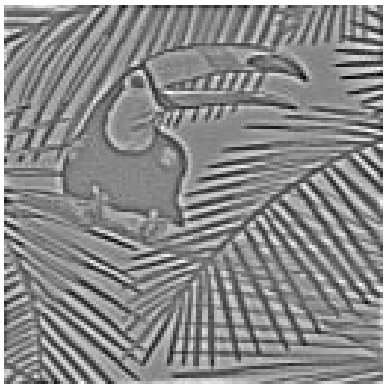
$$H(k,l) = 1 - \exp\left(-\frac{w}{w_0}\right)^2$$

again with $w^2 = k^2 + l^2$, which is a smooth filter in Fourier space.

This gives a smooth $h(i,j)$ in real space, so enhances edges without introduction of "ringing"



Filter



Output Image

Example with $w_0 = 25$.

In practice often combined with the Gaussian Lowpass filter to form a composite Difference of Gaussians (DOG) filter.

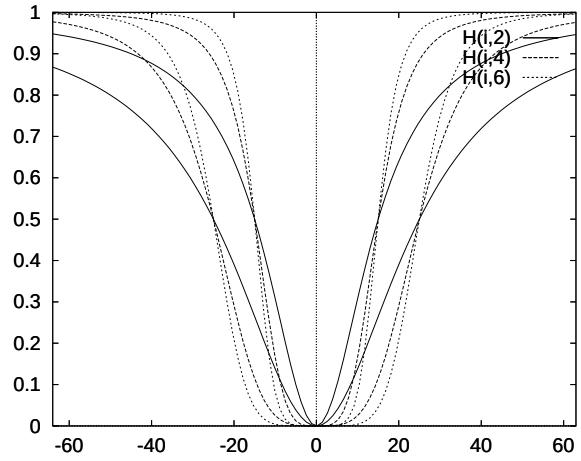
Other Smooth High Pass Filters

We can modify the smooth low pass filters to give High Pass, for example:

Highpass Butterworth:

$$H(k,l) = 1 - \frac{1}{1+(\frac{w}{w_0})^n} = \frac{1}{1+(\frac{w_0}{w})^n}$$

where w_0 is *half point* and n is the *order*.



Plot with $w_0 = 25$ and $n = 2, 4, 6$. This give almost identical results to the highpass Gaussian.

Use the `filtering` program to "play" with these. Filters offered are:

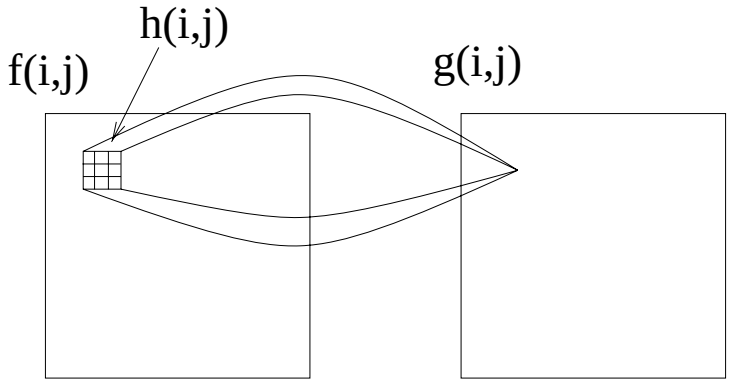
Lowpass: Ideal, Gaussian, Butterworth.

Highpass: Ideal, Gaussian, Butterworth.

Bandpass: Difference of Gaussians (DOG) filter.

Real Space Filters

Filter is specified in real space by the mask $h(i, j)$ of finite size, typically 3×3, 5×5 or sometimes 7×7.



The filter operation is then specified by the mask elements $h(i, j)$.

- Mask elements are Real, usually integer.
- Able to use integer, or fixed point arithmetic.
- For masks bigger that $\approx 7 \times 7$, faster to use Fourier technique.

Note : Convolution can be easily implemented by a parallel computer system, or custom parallel hardware.

Real Space Averaging

Replace each pixel by the average of its neighbours, has the effect of "*Low pass*" filtering, so used to reduce the effect of noise.

5 Pixel Average

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

gives a filter with an effective radius of 1 pixel.

9 Pixel Average

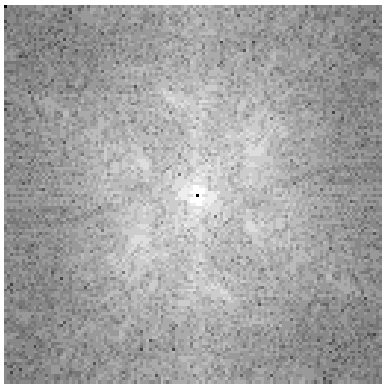
$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

This in equivalent of multiplying with $H(k,l)$ in Fourier space, (see tutorial for detail).

Effect or reducing high spatial frequencies, but not removing them, (actually removes a range of spatial frequencies).

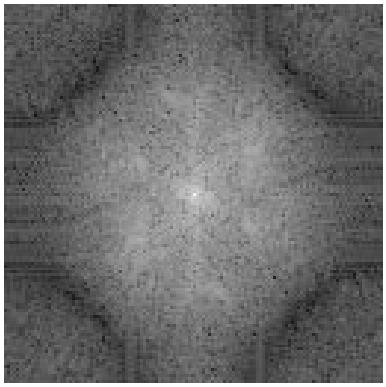
Digital Example

Input Image



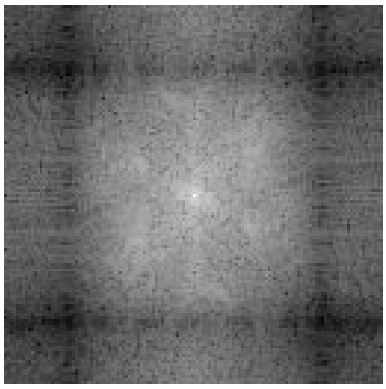
Fourier Transform

5 point ave



Fourier Transform

9 point ave



Fourier Transform

Real Space Differentiation

For a one-dimensional continuous function we have the definition of differentiation being:

$$\frac{df(x)}{dx} = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

Discrete case of $\delta = 1$,

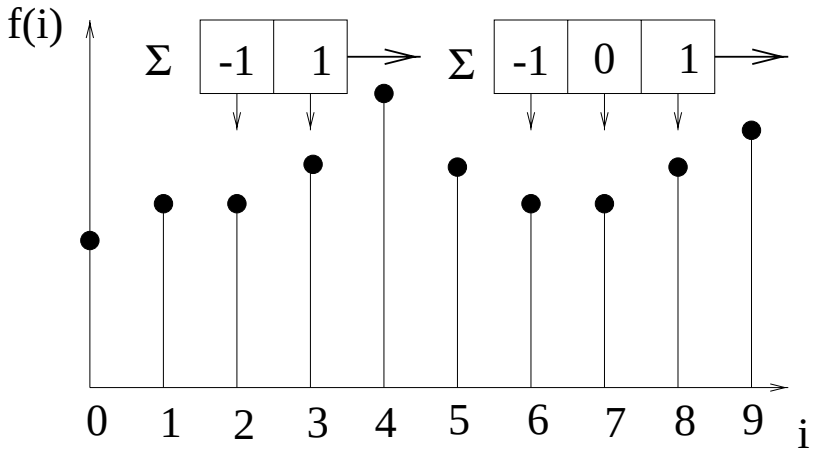
$$\frac{df(i)}{di} = f(i+1) - f(i)$$

which if we consider convolution as "Shift-Fold-Multiply-Add" then differentiation can be written as:

$$\frac{df(i)}{di} = [-1 \quad 1] \odot f(i)$$

Similarly with $\delta = 2$ we get

$$\frac{df(i)}{di} = [-1 \quad 0 \quad 1] \odot f(i)$$



Either of these convolutions can be used to approximate the first order differential of a sampled function.

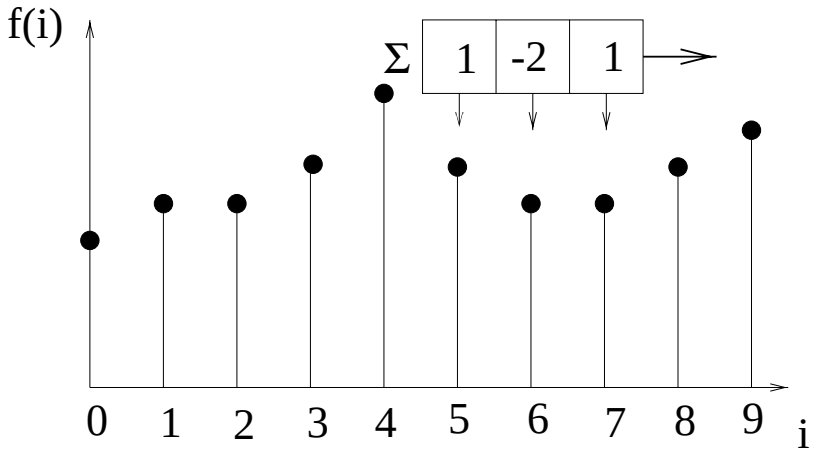
Second Order Differentials

Similarly the second order differential is given by,

$$\frac{\mathrm{d}^2 f(i)}{\mathrm{d}i^2} = f(i+1) - 2f(i) + f(i-1)$$

which can be written as

$$\frac{\mathrm{d}^2 f(i)}{\mathrm{d}i^2} = [1 \quad -2 \quad 1] \odot f(i)$$



Note also that

$$[1 \quad -2 \quad 1] = [-1 \quad 1] \odot [-1 \quad 1]$$

as would be expected since convolution is a linear operation.

Two-Dimensional Differentials

In Two Dimensions we have

$$\frac{\partial f(i,j)}{\partial i} = [-1 \quad 0 \quad 1] \odot f(i,j)$$

and similarly

$$\frac{\partial f(i,j)}{\partial j} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \odot f(i,j)$$

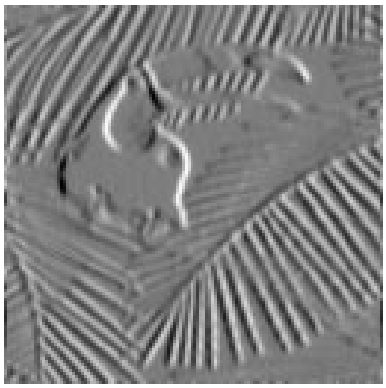
However to reduce the effect of noise, it is conventional to "average" the differential over 3 rows/columns respectively to give,

$$\frac{\partial f(i,j)}{\partial i} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \odot f(i,j)$$

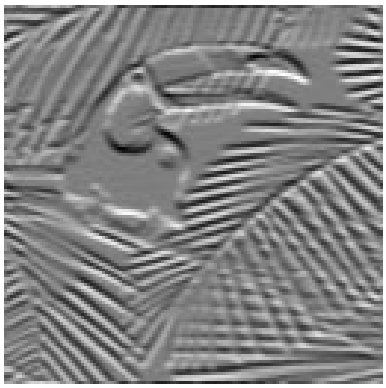
and

$$\frac{\partial f(i,j)}{\partial j} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \odot f(i,j)$$

which will enhance the vertical and horizontal edges respectively.



x-differential



y-differential

Fourier Space Differentials

From equation 15 of Fourier Transform Booklet, we have that

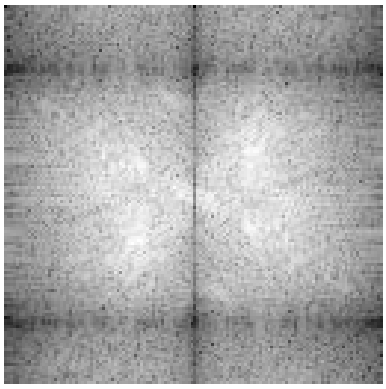
$$F\left\{\frac{\partial f(x,y)}{\partial x}\right\} = \imath 2\pi u F(u,v)$$

and

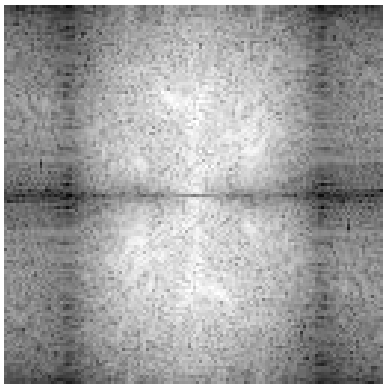
$$F\left\{\frac{\partial f(x,y)}{\partial y}\right\} = \imath 2\pi v F(u,v)$$

so that differential is equivalent to Fourier space multiplication by $\imath 2\pi u/v$.

This has the effect of enhancing high frequency at the expense of low frequencies, so is essentially a “high-pass” filter.



x-differential (FT)



y-differential (FT)

Note: the Fourier transforms show zero through vertical/horizontal as expected, plus addition line zeros due to averaging effect.

Second Order Differentials

For the second order differentials,

$$\frac{\partial^2 f(i,j)}{\partial i^2} = [1 \quad -2 \quad 1] \odot f(i,j)$$

and

$$\frac{\partial^2 f(i,j)}{\partial j^2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \odot f(i,j)$$

so that the Laplacian,

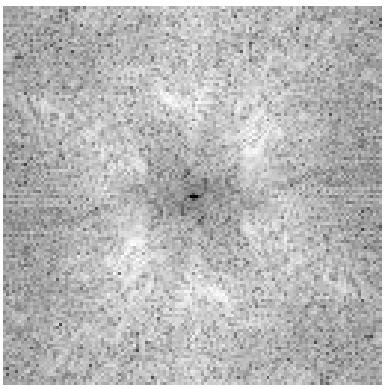
$$\nabla^2 f(i,j) = \frac{\partial^2 f(i,j)}{\partial i^2} + \frac{\partial^2 f(i,j)}{\partial j^2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \odot f(i,j)$$

which forms the Laplacian of the 2-dimensional image.

While in Fourier space, we have from *Fourier Transform* booklet equation (17) we have that:

$$F\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v)$$

where $w^2 = u^2 + v^2$, giving



Enhances edges in all directions.

Laplacian Variations

Alternative Filter:
We can form an 8 point Laplacian by using,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

which also takes the Laplacian, but is less sensitive to noise.

Edge Enhancement:
Edges of an image may be enhanced by the subtraction of the Laplacian from and image, which can be formed by,

$$f(i,j) - \nabla^2 f(i,j) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix} \odot f(i,j)$$

giving:

Input image

Edge Enhanced

Use of Linear Filters

- **Low Pass Filters:** are used to smooth images and reduce the effect of noise, in particular used to smooth image prior to edge detection.
- **High Pass Filter:** (also differentiations filters), have the effect of enhancing high frequencies and thus edges.

Filters can be combined to form *Bandpass* that attenuates both low & high spatial frequencies allowing middle frequencies to pass..

Due to linear nature, filters can be combined in Fourier space by $\times$ **or** is real space by $\odot$ operation.

Examples: Try filters with different images by using `convolve` program. Filter elements are numbered:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

Check the Fourier Transform of the results by saving `pgm` files and using `fourier` program.

Non-Linear Real Space Filters

The real space "shift & multiply" operation can be modified to

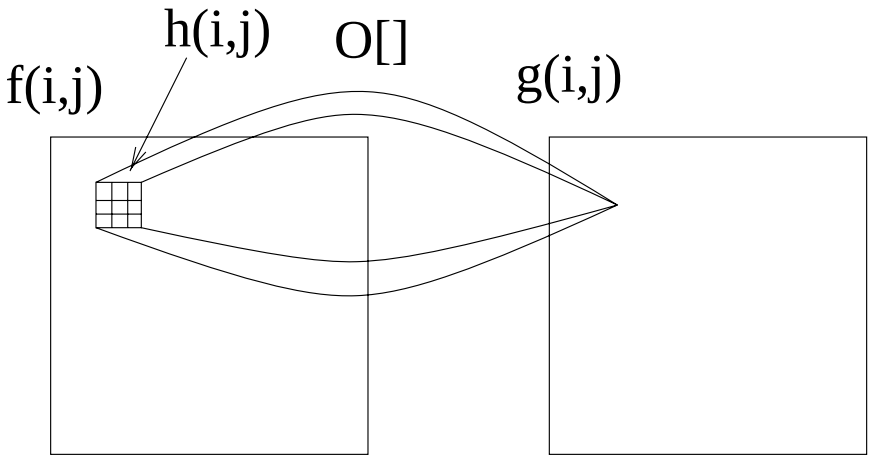
$$g(i,j) = O_{m,n \in w}[h(m,n)\, f(i-m, j-n)]$$

range of $h(m,n)$ defined by w. Operation now defined by mask $h(i,j)$ and operator $O[]$

In **most** non-linear filters we have

$$h(i,j) = 1 \quad i,j \in w$$

with the operation of the filter controlled by $O[]$ and the size of w only.



(In theory you can alter $h(i,j)$ as well, but it becomes very difficult to predict what will happen).

Shrink & Expand Filters

Taking

$$O[] = \mathrm{Min}[]$$

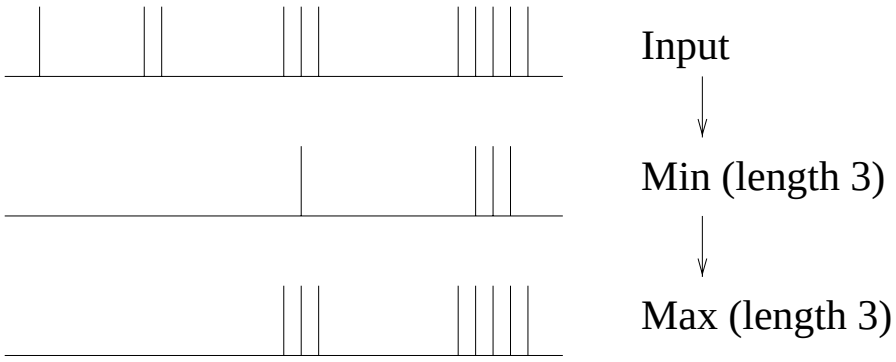
the operator will act as a *Shrink* operation with bright objects reduced in size by approximately the "size" of the filter.

Taking

$$O[] = \mathrm{Max}[]$$

the operator will act as an *Expand* filter, with bright objects increasing in size by approximately the "size" of the filter.

There operators typically used as a *pair* on binary image to remove small, isolated regions.



Note: These filters NOT commutative, ie.

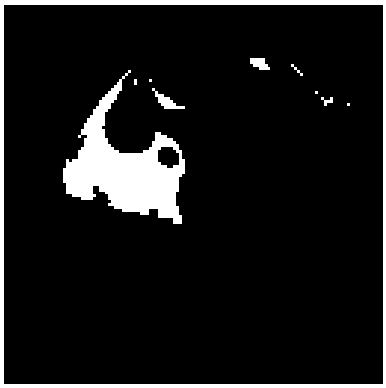
$$E[S[f(i,j)]] \neq S[E[f(i,j)]]$$

Two-Dimensional Case

In two dimensions the *Min* and *Max* occurs over a two dimensional window, so will selectively remove small **bright** objects.

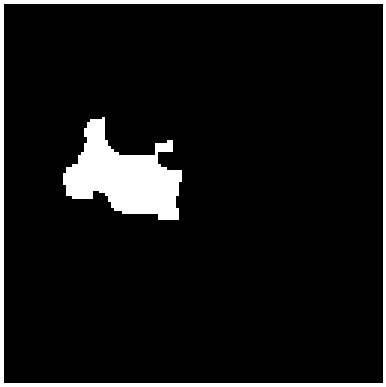
Very useful in “cleaning-up” isolated points in a binary thresholded image

Input



Binary Threshold

Binary Shrink



Binary Expand

Example with 4×4 shrink/expand filters.

Can be used with Grey Scale image, but you tend to get funny results.

Threshold Average Filter

For "data-dropout" noise we have isolated "noise points" that differ from the neighbour pixels, so compares each pixel with average of neighbours and smoothes only if pixel deviates significantly

For each point form

$$A = \sum_{m,n=-M/2}^{M/2} h(m,n) f(i-m, j-n)$$

for 3 by 3 filter

$$h(i,j) = \begin{bmatrix} k & k & k \\ k & 0 & k \\ k & k & k \end{bmatrix}$$

where $k = 1/(M^2 - 1) = 0.125$, then output is

$$\begin{array}{lll} g(i,j) & = A & |A - f(i,j)| > T \\ & f(i,j) & \text{else} \end{array}$$

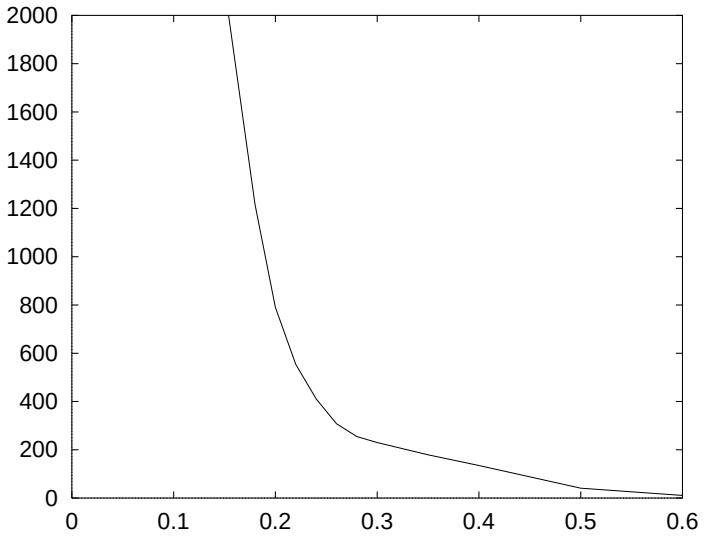
Selectively removes points that differ from neighbours.

Threshold T set by "trial-and-error". Usually specified as *fraction of (Maximun - Minimum)* value of image.

Random Bit Error Example

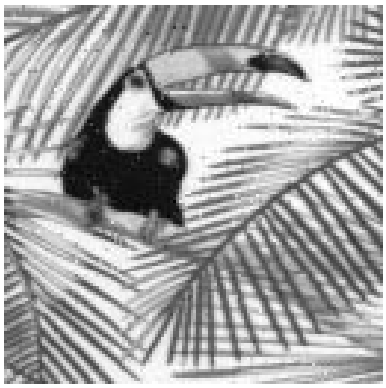
8 bit image and we corrupt 1:50 “bits”. Large corruption when *most significant bit* is corrupted.

For 128×128 pixel image “expect” about 325 seriously corrupted pixels. Apply *Average Threshold Filter* and count number of changed “noise” pixels.



Shows rapid rise at about 0.25.

1:50 Bit Error



Threshold of 0.26

Median Filter

The Median filter is formed by setting

$$O[] = \text{Median}[]$$

where the median is defined as the *middle* value. Eg. for the 5 values

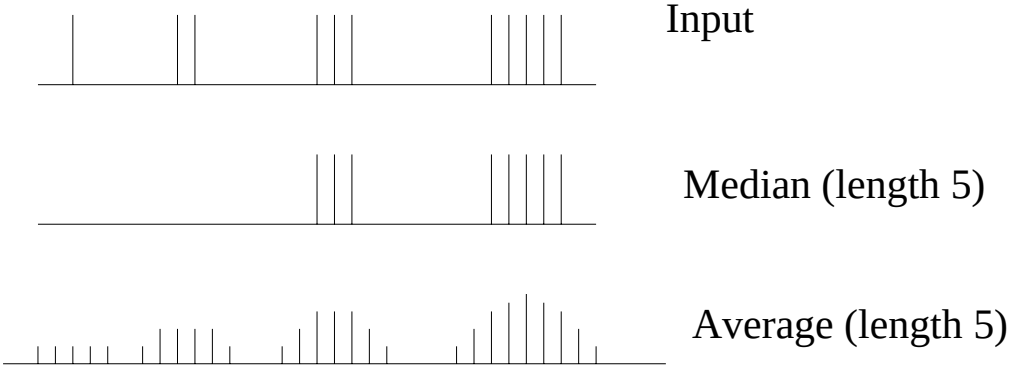
$$f(i) = 61, 10, 9, 11, 9$$

then

$$\text{Median}[f(i)] = 10$$

Note: is effectively "ignores" the out-of-place large value, so removes "noise" points.

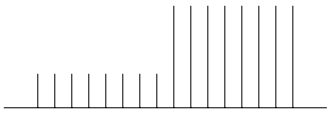
In one-dimensions the median filter removes all features of less than $M/2+1$ in size but preserves all other features.



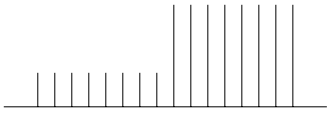
Similar to Shrink/Expand, but is also valid for Grey Level images.

Edge Preserving Properties

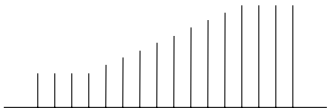
The *most* useful feature of the Median filter is its property at edges:



Input



Median (any length)



Average (length 5)

In Two-dimensions it removes all feature of size $< M^2/2 - 1$ while retaining all other features, **and** retaining edges.

3×3 Median

5×5 Median

Very useful noise reduction filter used throughout image processing.

Noise Reduction Properties

Filter effectively "smoothes" the image into regions of constant intensity but retains edges. So acts as a "selective" Low-Pass filter.

(See tutorial questions, and solutions for qualitive effect of Median Filter on Fourier Transform of image).

Implementation:
To calculate "Median" over each window the data must be (partly) sorted. Computationally expensive, and typically 5×5 Median filter about the same computational time as DFT.

Aside: Medians of large arrays are very slow to calculated by "thick" (SelectSort) way. For fast technique, see Numerical Recipes in `<language>` *section 8.5*

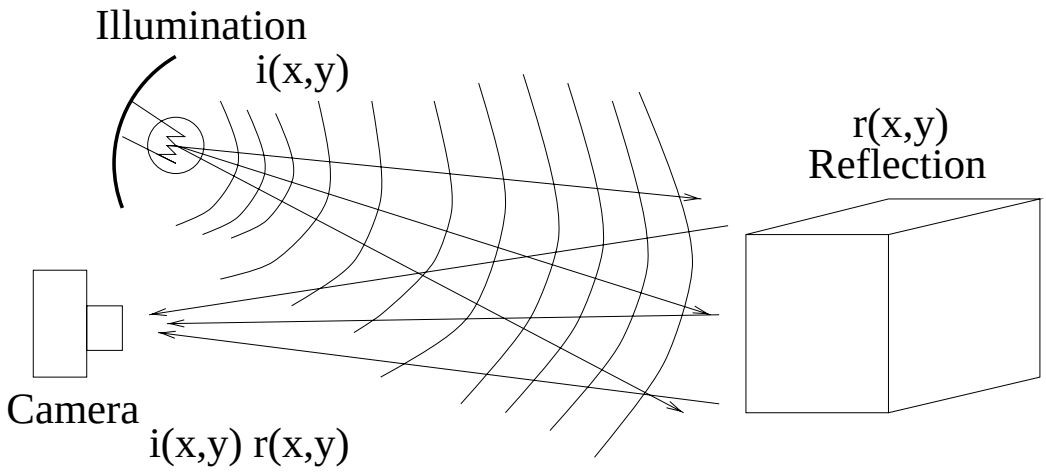
Homomorphic Filtering

For the case of a multiplicative process in Real space,

$$f(x,y) = i(x,y)r(x,y)$$

where

$$\begin{aligned} i(x,y) &= \text{Illumination} \\ r(x,y) &= \text{Reflectance} \end{aligned}$$



Form $\ln[]$ to separate terms go give,

$$z(x,y) = \ln(i(x,y)) + \ln(r(x,y))$$

Fourier transform this to give,

$$Z(u,v) = F\ \{\ln(i(x,y))\} + F\ \{\ln(r(x,y))\}$$

known as the *Cepstrum* of $f(x,y)$.

Filtering

Consider the frequency characteristics of each term:

$$\begin{aligned} i(x,y)\text{smooth} &\Rightarrow \ln(i(x,y))\text{smooth} \\ r(x,y)\text{rough} &\Rightarrow \ln(r(x,y))\text{rough} \end{aligned}$$

Filter $Z(u,v)$ to get

$$Y(u,v) = Z(u,v)H(u,v)$$

where

$$\begin{aligned} \text{High pass} &\Rightarrow \text{Reduces } i(x,y) \\ \text{Low pass} &\Rightarrow \text{Reduces } r(x,y) \end{aligned}$$

then reform "filtered" image by,

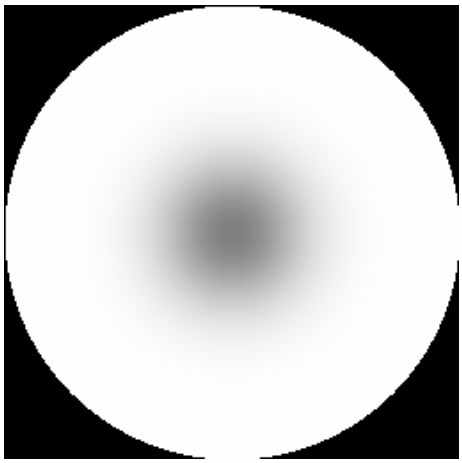
$$g(x,y) = \exp[F^{-1}\{Y(u,v\}]$$

Typically used to correct for illumination effect, but can also be used for deal with multiplicative noise.

Homomorphic Example

Input Image

Log of Input



Filter

Output

Low frequency variation in illumination has been (partially) removed.

See Gonzalez & Woods for better example