First-Order Ordinary Differential Equations

Maths Methods I

Week 7

First-Order Ordinary Differential Equations

- Classification of Differential Equations
- Analytical Solutions
- Methods for Solving 1st-order ODEs
  - Separation of variables
  - Integration factor technique
- Simple Examples in Earth Science

Differential Equations

In Earth Science we often want to solve a differential equation.

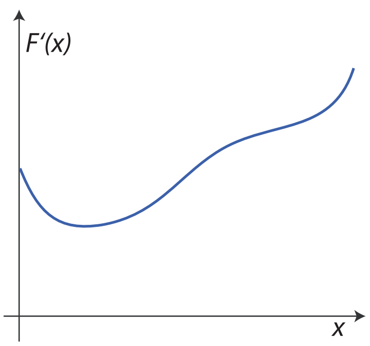
A differential equation is any equation involving a derivative. Perhaps the most familiar differential equation is:

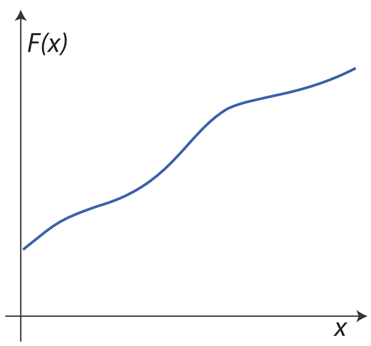
$$\frac{dy}{dx} = f(x)$$

The "solution" to this equation is a function $y(x)$, which we can often find by integration.

Integration Re-cap: The reverse of differentiation

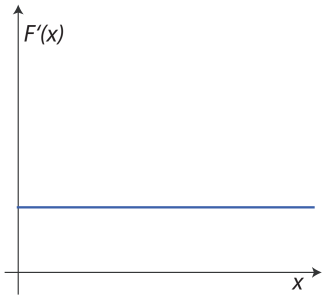
If we are given $\frac{dy}{dx}$ as a function of only x, which we might write as $\frac{dy}{dx} = F'(x) = f(x)$, we want to find $y = F(x)$

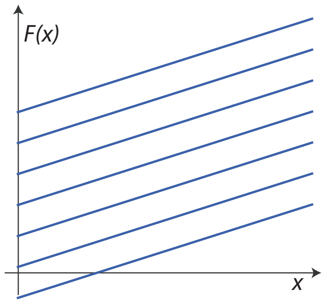




Integration Re-cap: The reverse of differentiation

We can find $F(x)$ from $F'(x)$ by integrating, which is the reverse of differentiation; however, the solution is one of many possible solutions...





This is because we can add any constant c to $F(x)$ and still get the same derivative:

$$\frac{d(F(x)+c)}{dx} = F'(x) = f(x)$$

The Indefinite Integral

We call $F(x) + c$ the general solution to the indefinite integral of $f(x)$ and write it as:

$$\int f(x)dx = F(x) + c$$

More information is needed to find the unique solution.

Example: Find $y(x)$, given $\frac{dy}{dx} = 2x$ and $y = 2$ when $x = 0$

The general form of $y(x)$ is given by the indefinite integral:

$$y = \int \frac{dy}{dx} dx = \int 2x dx = x^2 + c$$

Substituting in $x = 0$, $y = 2$ gives the particular solution:

$$2 = (0)^2 + c \qquad \Rightarrow c = 2$$
$$\Rightarrow y = x^2 + 2$$

Differential Equations

Differential equations like $\frac{dy}{dx} = f(x)$, where the derivative is a function of only the independent variable, are the simplest type of differential equation.

Much more important are equations where the derivative is a function of both the dependent and independent variable:

$$\frac{dy}{dx} = f(x, y)$$

Again the solution that we seek is a function $y(x)$ that satisfies the differential equation.

Differential Equations: Ordinary, Partial and Coupled

If the equation contains only one independent variable x and one dependent variable y, then the derivative is an ordinary derivative, and the equation is an ordinary differential equation (ODE).

$$\frac{dy}{dx} = f(x, y)$$

$$\begin{aligned} \frac{dP}{dt} &= \alpha P \\ \frac{dN}{dt} &= -\lambda N \end{aligned}$$

If there is more than one independent variable (x, t), the derivatives will be partial derivatives; in this case, the equation is a partial differential equation (PDE):

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = f(x, y, t)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + \kappa \frac{\partial^2 T}{\partial x^2} &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} &= 0 \end{aligned}$$

If there is more than one dependent variable (y, w), the equation is a coupled differential equation:

$$\frac{dy}{dx} + \frac{dw}{dx} = f(x, y, w)$$

$$k_p \frac{dp}{dx} + k_V \frac{dV}{dx} = j$$

For this course we will consider only ordinary differential equations.

Differential Equations: Order

The order of a differential equation is given by the the kind of derivative in the equation. First-order differential equations contain only first derivatives

$$\frac{dy}{dx} = f(x, y) \qquad\qquad \frac{dv}{dt} = -\eta v$$

Second-order differential equations contain a second derivative; they may or may not also contain a first derivative.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x, y) \qquad\qquad m\frac{d^2x}{dt^2} = -kx$$

An nth-order differential equation contains an nth derivative; it may or may not also contain lower-order derivatives

$$\frac{d^ny}{dx^n} + \frac{d^{n-1}y}{dx^{n-1}} + \cdots + \frac{dy}{dx} = f(x, y) \qquad\qquad \frac{d^4y}{dt^4} - \frac{d^2y}{dt^2} = 0$$

For this course we will consider only first-order ordinary differential equations.

Differential Equations: Linear and Nonlinear

A differential equation is linear if terms involving the dependent variable y and its derivatives do not occur as products, raised to powers or in nonlinear functions.

$$\frac{dy}{dx} + f(x)y = g(x) \qquad\qquad \frac{dy}{dx} + 3x^2y = \sin x$$

A differential equation that is not linear is called a nonlinear differential equation.

$$k\left(\frac{dy}{dx}\right) + f(x)h(y) = g(x) \qquad\qquad \left(\frac{dy}{dx}\right)^2 + f(x)e^y = g(x)$$

Note: here f, g, h, and k are functions, where h and/or k are non-linear.

For this course we will consider linear first-order ordinary differential equations and some very simple non-linear differential equations.

Differential Equations: Homogeneous & nonhomogeneous

It is common practice to arrange a differential equation so that all terms involving the dependent variable are on the left-hand side of the equation, and all the constant terms, or terms that only involve the independent variable, are on the right-hand side.

In this form, a differential equation is called homogeneous if the right-hand side is zero:

$$\frac{dy}{dx} + f(x)y = 0 \qquad\qquad m\frac{d^2x}{dt^2} + kx = 0$$

Otherwise, the differential equation is called nonhomogeneous:

$$\frac{dy}{dx} + f(x)y = g(x) \qquad\qquad m\frac{d^2x}{dt^2} + kx = \sin\omega t$$

When formulated this way, the left-hand side can be considered to represent the response of the system. If the differential equation is homogeneous the system is not being forced in any way; if the differential equation is nonhomogeneous the system is driven by $g(x)$, which is often referred to as the forcing term.

Solving Differential Equations

Differential equations describe the behaviour of many physical systems. By finding solutions to these equations we can model the physical process involved.

Even though the differential equation describing a process may appear quite concise and simple, the solutions of the equation are often very complex. In many cases, the only way to find a solution is through numerical modelling.

For linear, first-order, ordinary differential equations we can often find an exact mathematical equation for the solution $y(x)$. Such equations have many applications and are easy to analyse. Being able to solve differential equations is crucial for an Earth Scientist!

General and Particular Solutions

Recall that the solution (integral) of the differential equation

$$\frac{dy}{dx} = f(x),$$

is ambiguous; it involves an arbitrary constant of integration c:

$$y(x) = \int \frac{dy}{dx} dx = \int f(x) dx = F(x) + c.$$

We refer to this as the general solution.

To define a particular solution we require a further piece of information.

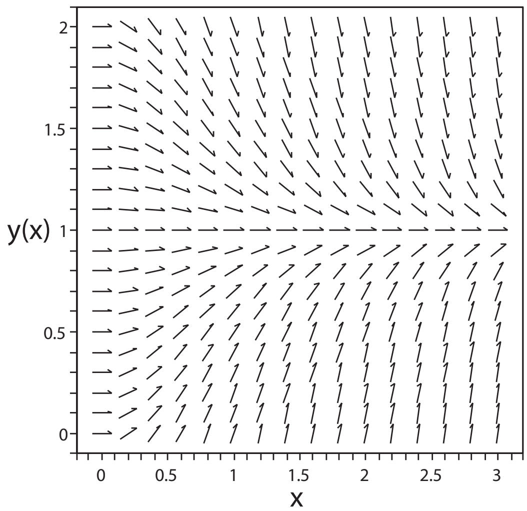
If the information we are given to define the particular solution is the initial value of y (i.e. $y(x = 0)$) the problem is called an initial value problem; otherwise it is called a boundary value problem.

Analytical Solutions

Consider a linear first-order ODE; for example,

$$\frac{dy}{dx} = 2x(1 - y)$$

The equation defines how y changes with respect to x for any combination of x and y. We can visualize this by plotting short line segments of slope $\frac{dy}{dx}$ at regular intervals of x and y.



The solution of the differential equation will be a curve that follows the directions of the line segments. The graph illustrates that not only does the position of the solution curve change depending on the initial condition, but so does the character of the solution.

Analytical Solutions

Sometimes we can solve differential equations using integration and write down an exact equation for $y(x)$; in this case, we have found an analytical solution.

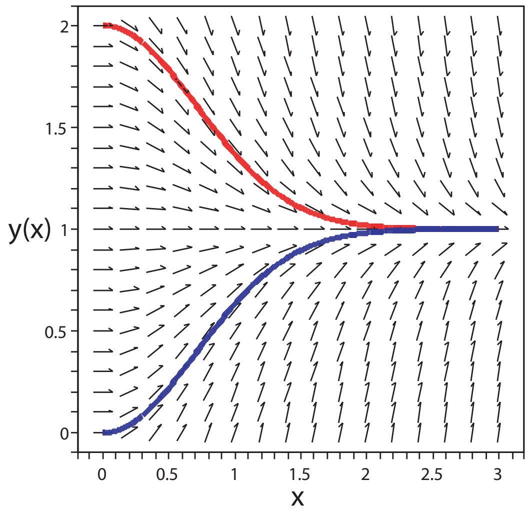
For our example, the analytical form of the general solution is:

$$y(x) = 1 + Ce^{-x^2}$$

The behaviour of this function will be different for different values of the constant C.

If $y(x=0) = 2,\ C = 1$ (red curve).

If $y(x=0) = 0,\ C = -1$ (blue curve).



Simple Methods for Solving First-Order ODEs

There are two simple techniques for solving first-order ODEs.

If the function $f(x, y)$ in the first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is such that the equation can be rearranged into the form:

$$g(y)\frac{dy}{dx} = h(x)$$

the solution is found using the method of separation of variables.

If, on the other hand, the function $f(x, y)$ in the first-order linear ODE is such that the equation can be rearranged into the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

the solution is found by the integration factor method.

Solving Differential Equations: Examples

Example: Which method should be used to solve the following equations?

$$\frac{dy}{dx} = 2x - y\sin x$$

Rearranging:

$$\frac{dy}{dx} + y\sin x = 2x$$

$\Rightarrow$ *integration factor method*

$$x^2\frac{dy}{dx} = 2y$$

Rearranging:

$$\frac{1}{2y}\frac{dy}{dx} = \frac{1}{x^2}$$

$\Rightarrow$ *separation of variables*

$$\frac{dy}{dx} = 2x - 2xy$$

Rearranging:

$$\frac{dy}{dx} + 2xy = 2x$$

$\Rightarrow$ *integration factor method*

Or:

$$\begin{aligned} \frac{dy}{dx} &= 2x(1-y) \\ \frac{1}{1-y}\frac{dy}{dx} &= 2x \end{aligned}$$

$\Rightarrow$ *separation of variables*

Solving Differential Equations: Separation of variables

Separation of Variables

For first-order ODEs of the form

$$g(y)\frac{dy}{dx} = h(x)$$

We can integrate both sides with respect to x, giving:

$$\int g(y)\frac{dy}{dx}dx = \int h(x)dx$$

Separation of variables implies:

$$\int g(y)dy = \int h(x)dx$$

Solving Differential Equations: Separation of Variables

Proof.

If y is a function of x, and

$$g(y)\frac{dy}{dx} = h(x) \qquad (1)$$

Let us define,

$$G(y) = \int g(y)dy \Rightarrow g(y) = \frac{dG}{dy}$$

and

$$H(x) = \int h(x)dx \Rightarrow h(x) = \frac{dH}{dx}$$

Substituting these into Eq. (1):

$$\begin{aligned} \frac{dG}{dy}\frac{dy}{dx} &= \frac{dH}{dx} \\ \frac{dG}{dx} &= \frac{dH}{dx} \end{aligned}$$

Integrating both sides with respect to x

$$G(y) = H(x) + c$$

which implies that

$$\int g(y)dy = \int h(x)dx + c$$

and, since this must be true for any value of c,

$$\int g(y)dy = \int h(x)dx$$

$\square$

Solving Differential Equations: Examples

Example: Find the solution to the differential equation $\frac{dy}{dx} = y\cos x$, given the initial condition $y(x=0) = 10$.

Rearranging, gives:

$$\frac{1}{y}\frac{dy}{dx} = \cos x$$

Hence, we use *separation of variables*.

Integrating both sides with respect to x:

$$\begin{aligned} \int \frac{1}{y}\frac{dy}{dx}dx &= \int \cos x dx \\ \int \frac{1}{y}dy &= \int \cos x dx \\ \ln y &= \sin x + c \end{aligned}$$

Taking the exponent of both sides:

$$\begin{aligned} y &= e^{\sin x + c} \\ y &= e^c e^{\sin x} \\ y &= Ae^{\sin x} \end{aligned}$$

Inserting the initial condition:

$$\begin{aligned} y &= Ae^{\sin x} \\ 10 &= Ae^{\sin 0} = A.1 \\ \Rightarrow A &= 10 \end{aligned}$$

Hence, the particular solution is:

$$y = 10e^{\sin x}$$

Solving Differential Equations: Integration Factor Method

Integration Factor Technique

For first-order linear ODEs of the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

The solution $y(x)$ is given by:

$$y(x) = \frac{1}{g(x)} \int q(x)g(x)dx$$

Where $g(x)$ is the integration factor given by:

$$g(x) = e^{\int p(x)dx}$$

Note: in defining $g(x)$ we ignore the constant of integration from $\int p(x)dx$.

Solving Differential Equations: Integration Factor Method

Proof.

If our ODE is of the form

$$\frac{dy}{dx} + p(x)y(x) = q(x)$$

and we define the integration factor:

$$g(x) = e^{\int p(x)dx}$$

Multiplying each term in the ODE by the integration factor:

$$\frac{dy}{dx}g(x) + p(x)g(x)y(x) = q(x)g(x)$$

Now, since:

$$\frac{dg}{dx} = p(x)e^{\int p(x)dx} = p(x)g(x)$$

We see that

$$\frac{dy}{dx}g(x) + \frac{dg}{dx}y(x) = q(x)g(x)$$

The left-hand side of this equation is the derivative of the product $g(x)y(x)$; hence,

$$\frac{d[g(x)y(x)]}{dx} = q(x)g(x)$$

Integrating both sides with respect to x:

$$g(x)y(x) = \int q(x)g(x)dx$$

and, therefore,

$$y(x) = \frac{1}{g(x)} \int q(x)g(x)dx$$

□

Solving Differential Equations: Examples

Example: Find the solution to the differential equation $\frac{dy}{dx} = 2x(1-y)$, given the initial condition $y(x=0) = 2$.

Rearranging, gives:

$$\begin{aligned} \frac{dy}{dx} &= 2x - 2xy \\ \frac{dy}{dx} + 2xy &= 2x \\ \Rightarrow\ p(x) = 2x \qquad & q(x) = 2x \end{aligned}$$

We use the *integration factor method*, with the integration factor:

$$g(x) = e^{\int p(x)dx} = e^{\int 2xdx} = e^{x^2}$$

Inserting $g(x)$ into the formula:

$$\begin{aligned} y(x) &= \frac{1}{g(x)} \int q(x)g(x)dx \\ \Rightarrow\ y(x) &= \frac{1}{e^{x^2}} \int 2xe^{x^2} dx \end{aligned}$$

Evaluating the integral with respect to x:

$$\begin{aligned} y(x) &= \frac{1}{e^{x^2}} \left(e^{x^2} + C\right) \\ y(x) &= \frac{e^{x^2}}{e^{x^2}} + \frac{C}{e^{x^2}} \\ y(x) &= 1 + Ce^{-x^2} \end{aligned}$$

Inserting the initial condition:

$$\begin{aligned} 2 &= 1 + Ce^{-(0)^2} = 1 + C.1 \\ \Rightarrow\ C &= 1 \end{aligned}$$

Hence, the particular solution is:

$$y = 1 + e^{-x^2}$$

Solving Differential Equations: Examples

Example: Find the solution to the differential equation $\frac{dy}{dx} = 2x(1-y)$, given the initial condition $y(x=0) = 2$.

Rearranging, gives:

$$\begin{aligned} \frac{dy}{dx} &= 2x - 2xy \\ \frac{dy}{dx} + 2xy &= 2x \\ \Rightarrow\ p(x) = 2x \qquad & q(x) = 2x \end{aligned}$$

We use the *integration factor method*, with the integration factor:

$$g(x) = e^{\int p(x)dx} = e^{\int 2xdx} = e^{x^2}$$

Multiplying through by $g(x)$:

$$\begin{aligned} e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y &= 2xe^{x^2} \\ \Rightarrow\ \frac{d\left(ye^{x^2}\right)}{dx} &= 2xe^{x^2} \end{aligned}$$

Integrating both sides with respect to x:

$$\begin{aligned} ye^{x^2} &= \int 2xe^{x^2} dx \\ ye^{x^2} &= e^{x^2} + C \\ y &= 1 + Ce^{-x^2} \end{aligned}$$

Inserting the initial condition:

$$\begin{aligned} 2 &= 1 + Ce^{-(0)^2} = 1 + C.1 \\ \Rightarrow\ C &= 1 \end{aligned}$$

Hence, the particular solution is:

$$y = 1 + e^{-x^2}$$

Earth Science Application: Radioactive Decay

Let us consider how we might derive and solve a differential equation for a real physical system in Earth Science: the radioactive decay of Carbon 14 (C_{14}).

If N is the number of C_{14} atoms in a sample, and λ is the probability that any given atom decays in the next second, then the total number of C_{14} atoms D that will decay during the interval of time Δt is given by:

$$D = \lambda N \Delta t$$

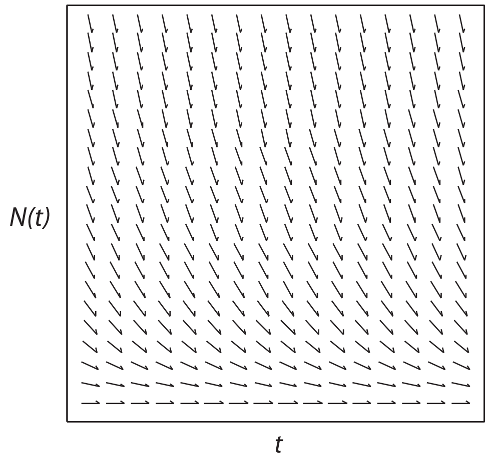
Hence, if N is the number of atoms of C_{14} at time t, the number of atoms remaining at time $t + \Delta t$ is:

$$N(t + \Delta t) = N - D = N - \lambda N \Delta t$$

Earth Science Application: Radioactive Decay

The rate of change of number of atoms of C_{14} with respect to time, is:

$$\begin{aligned} \frac{dN}{dt} &= \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{N - \lambda N \Delta t - N}{\Delta t} \\ \Rightarrow \frac{dN}{dt} &= -\lambda N \end{aligned}$$



We have derived a linear first-order ODE to describe the radioactive decay of C_{14}:

$$\frac{dN}{dt} = -\lambda N,$$

which states that the rate of change of the number of C_{14} atoms in the sample with time is proportional to the number of atoms of C_{14}. The dependent variable is N, which appears in the derivative and on the right-hand side; the independent variable t appears only in the derivative.

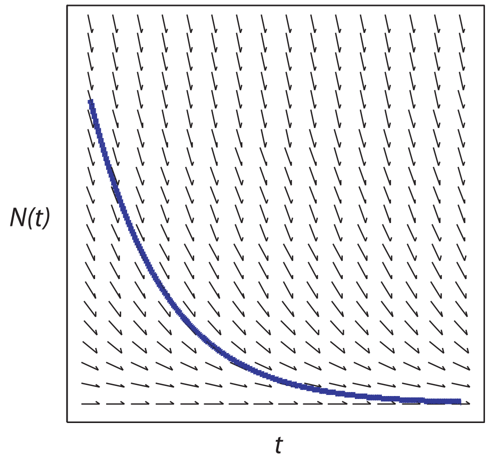
Earth Science Application: Radioactive Decay

We can easily rearrange the equation into the form:

$$\frac{1}{N}\frac{dN}{dt} = -\lambda$$

Hence, the equation is separable and we can solve to find the general solution for $N(t)$ by integrating both sides with respect to t.

$$\begin{aligned}
\int \frac{1}{N}\frac{dN}{dt}dt &= \int -\lambda dt \\
\int \frac{1}{N}dN &= -\lambda \int dt \\
\Rightarrow \ln N &= -\lambda t + c \\
\Rightarrow N &= e^{-\lambda t + c} \\
N &= e^{c}e^{-\lambda t} \\
N &= Ae^{-\lambda t}
\end{aligned}$$



Earth Science Application: Radioactive Decay

If we are given the initial condition $N(t = 0) = N_0$, we can define the particular solution:

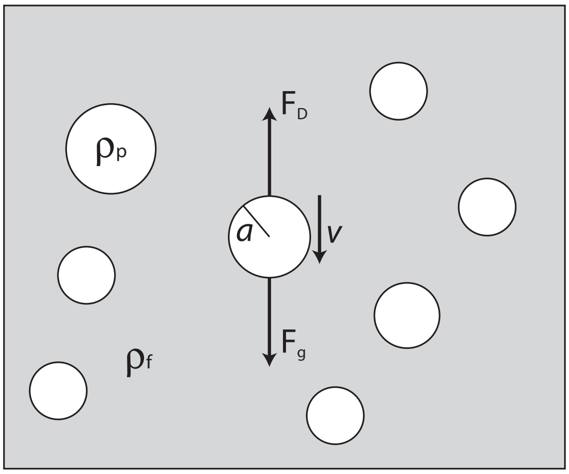
$$\begin{aligned}
N_0 &= Ae^{-\lambda(0)} \\
\Rightarrow A &= N_0 \\
\Rightarrow N &= N_0 e^{-\lambda t} = \frac{N_0}{e^{\lambda t}}
\end{aligned}$$

What can we say about this solution? We can see that as $t \to \infty$ the number of atoms of C_{14} decay to zero. We can also define the "half-life" of C_{14}, by finding $t = \tau$ when $N = N_0/2$:

$$\begin{aligned}
\frac{N_0}{2} &= N_0 e^{-\lambda\tau} \\
\frac{1}{2} &= \frac{1}{e^{\lambda\tau}} \\
\Rightarrow e^{\lambda\tau} &= 2 \\
\Rightarrow \lambda\tau &= \ln 2 \\
\Rightarrow \tau &= \frac{\ln 2}{\lambda}
\end{aligned}$$

Earth Science Application: Particle Settling

Now let us consider how we might derive and solve a differential equation for another physical system in Earth Science: a particle settling through an atmosphere or ocean.



Newton's second law tells us that the acceleration of the particle is proportional to the resultant force on the particle:

$$\sum F = m\frac{dv}{dt}$$

where m is the mass of the particle and $\frac{dv}{dt}$ is the acceleration of the particle; that is, the rate of change of the particle velocity with time t.

Earth Science Application: Particle Settling

Consider the forces on the particle. The gravitational force F_g pulls the particle downwards; it depends on the density of the particle ρ_p, the density of the surrounding fluid ρ_f, the gravitational acceleration g and the volume of the spherical particle $\frac{4\pi}{3}a^3$:

$$F_g = (\rho_p - \rho_f)g\frac{4\pi}{3}a^3.$$

The drag force of the fluid on the particle F_D, on the other hand, slows the particle's descent; it depends on the viscosity of the fluid η, and the radius a and velocity v of the particle:

$$F_D = -6\pi\eta a v.$$

Combining these, and noting that $m = \frac{4\pi}{3}\rho_p a^3$, gives:

$$\begin{aligned} m\frac{dv}{dt} &= \sum F \\ \frac{4\pi}{3}\rho_p a^3\frac{dv}{dt} &= (\rho_p - \rho_f)g\frac{4\pi}{3}a^3 - 6\pi\eta a v \\ \Rightarrow \frac{dv}{dt} &= \left(\frac{\rho_p - \rho_f}{\rho_p}\right)g - \left(\frac{9\eta}{2a^2\rho_p}\right)v \end{aligned}$$

Earth Science Application: Particle Settling

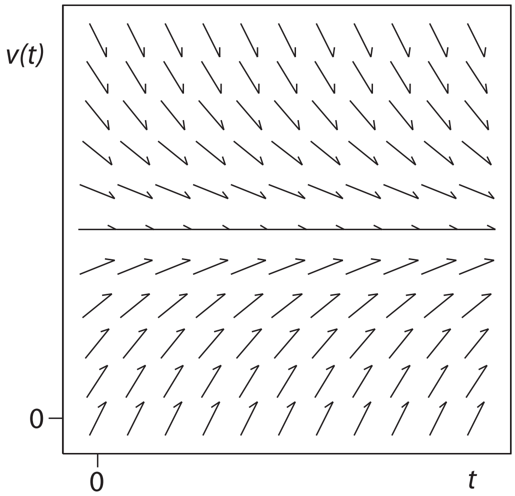
Thus, we have derived a linear first-order ODE to describe the settling of a particle through a fluid:

$$\frac{dv}{dt} = \left(\frac{\rho_p - \rho_f}{\rho_p}\right) g - \left(\frac{9\eta}{2a^2\rho_p}\right) v$$

The dependent variable is v, which appears in the derivative and on the right-hand side; the independent variable t appears only in the derivative. Hence, the equation has the form:

$$\frac{dv}{dt} = \alpha - \beta v$$

where α and β are constants.



Earth Science Application: Particle Settling

How do we solve this equation? We can rearrange it to give:

$$\frac{dv}{dt} + \beta v = \alpha$$

So we can use the integration factor technique. The integration factor is:

$$e^{\int \beta dt} = e^{\beta t}$$

Multiplying through by the integration factor:

$$\frac{dv}{dt} e^{\beta t} + \beta v e^{\beta t} = \alpha e^{\beta t}$$

Hence,

$$\frac{d\left(ve^{\beta t}\right)}{dt} = \alpha e^{\beta t}$$

implying

$$\begin{aligned} ve^{\beta t} &= \int \alpha e^{\beta t} dt \\ ve^{\beta t} &= \frac{\alpha}{\beta} e^{\beta t} + C \\ v &= \frac{\alpha}{\beta} + Ce^{-\beta t} \end{aligned}$$

Earth Science Application: Particle Settling

If we are given the initial condition $v(t=0)=0$, we see that:

$$\begin{aligned} 0 &= \frac{\alpha}{\beta} + Ce^{-\beta(0)} \\ \Rightarrow C &= -\frac{\alpha}{\beta} \end{aligned}$$

Hence, the particular solution is given by:

$$v = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

Plugging back in α and β:

$$v = \left(\frac{\rho_p - \rho_f}{\rho_p}\right) g \frac{2a^2\rho_p}{9\eta}\left(1 - e^{-\frac{9\eta t}{2a^2\rho_p}}\right)$$

$$\Rightarrow \quad v = \frac{2(\rho_p - \rho_f)ga^2}{9\eta}\left(1 - e^{-\frac{9\eta t}{2a^2\rho_p}}\right)$$

Earth Science Application: Particle Settling

What happens as $t \to \infty$ here?

$$v(t \to \infty) = \frac{2(\rho_p - \rho_f)ga^2}{9\eta}$$

Note that we have not imposed this condition; it emerges from our analysis and tells us something about about the modelled phenomenon.

Consider again the forces acting on the particle. When the gravitational force and drag force are equal the velocity will be constant:

$$\begin{aligned} \sum F = 0 &= (\rho_p - \rho_f)g\frac{4\pi}{3}a^3 - 6\pi\eta a v \\ \Rightarrow 6\pi\eta a v &= \frac{4\pi(\rho_p - \rho_f)ga^3}{3} \\ \Rightarrow v &= \frac{4\pi(\rho_p - \rho_f)ga^3}{18\pi\eta a} = \frac{2(\rho_p - \rho_f)ga^2}{9\eta} \end{aligned}$$

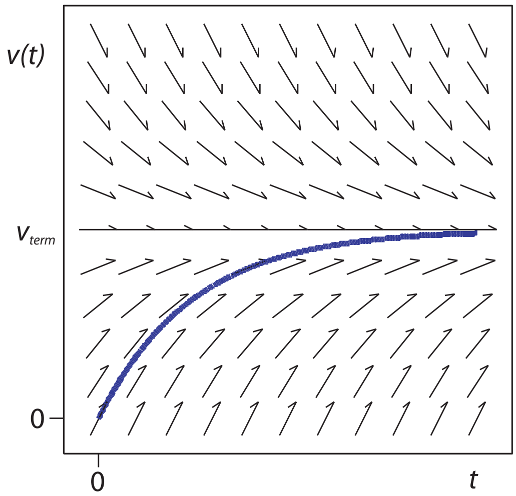
Earth Science Application: Particle Settling

This velocity is the terminal velocity of the settling particle.

$$v_{\text{term}} = \frac{2(\rho_p - \rho_f)ga^2}{9\eta}$$

Hence, our solution predicts that, if starting at rest, the particle accelerates until it reaches terminal velocity.

$$v = v_{\text{term}}\left(1 - e^{-\frac{9\eta t}{2a^2\rho_p}}\right)$$



Analysing this equation tells us that the terminal velocity is larger for larger diameter particles and for a lower viscosity fluid.

Summary

You should be able to:

- Find the general solution of a linear first-order ordinary differential equation using (where appropriate)
  - Separation of variables
  - Integration factor method
- Find the particular solution given an initial condition
- Formulate and solve a first-order differential equation from a simple word description of a process

Linear First-order Ordinary Differential equations

Maths Methods I Practice Questions

Questions

1. **Classification of differential equations** For each of the following equations state which of the terms are independent variables and which are dependent variables. Classify the equations by their *order* (first, second, etc.), and whether they are *linear or nonlinear*, and whether they are *ordinary, partial or coupled* differential equations (*do not solve the equations*).

   a) $\dfrac{dP}{dt} + \alpha P = 0,$ where α is a constant.

   b) $\dfrac{d^2\rho}{dx^2} + \beta \dfrac{d\rho}{dx} = \gamma\left(\rho - \rho_0\right),$ where β, γ and ρ_0 are constants.

   c) $\dfrac{\partial^2 s}{\partial x^2} - \phi \dfrac{\partial^2 s}{\partial t^2} = \dfrac{\kappa}{s^2}$ where ϕ and κ are constants.

   d) $\dfrac{dP}{dz} + \psi \dfrac{dT}{dz} = 0$ where ψ is a constant.

2. **Solving differential equations** For the following differential equations state whether separation of variables or the integrating factor technique should be used to find the solution; use this technique to find the general solution; and use the initial/boundary value information to find the particular solution.

   a) $\dfrac{dy}{dx} = \dfrac{2x+1}{3y^2}$ where $y(x=0) = 3.$

   b) $\dfrac{dz}{dt} - \dfrac{z}{t} = t^2 - 3$ where $z(t=1) = -1.$

   c) $\dfrac{dy}{dt} = \cos t - \dfrac{2y}{t}$ where $y(t=\pi) = 1.$

3. **Application: Population growth** The growth of a population of trilobites can be approximately described as *the rate of change of the number of organisms, with respect to time, is proportional to the number of organisms.* Write down the corresponding differential equation. If a population of trilobites starts at 500 at the time of the Cambrian Explosion, and 2 million years later has a population of 10000, what is the constant of proportionality? How long will it take for the population to reach 500 million?

Solutions

1. (a) Dependent variable is P, independent variable is t; First Order; Linear; Ordinary differential equation.

   (b) Dependent variable is ρ, independent variable is x; Second Order; Linear; Ordinary differential equation.

   (c) Dependent variable is s, independent variables are x, t; Second Order; Non-linear; Partial differential equation.

   (d) Dependent variables are P, T, independent variable is z; First Order; Linear; Coupled differential equation.

2. (a) Can be rearranged to the form: $3y^2\frac{dy}{dx} = 2x + 1$, so we can use *separation of variables*:

$$\begin{aligned}
\int 3y^2\frac{dy}{dx}dx &= \int 2x+1\ dx \\
\int 3y^2 dy &= \int 2x+1\ dx \\
\frac{3y^3}{3} &= \frac{2x^2}{2} + x + C \\
y^3 &= x(x+1) + C \\
y &= \sqrt[3]{x\left(x+1\right) + C}
\end{aligned}$$

   Using the initial condition, $y = 3$ when $x = 0$, we can see that $\sqrt[3]{C} = 3$, so the constant $C = 27$. Hence, the particular solution is: $y = \sqrt[3]{x\left(x+1\right)+27}$.

   (b) Using the *integration factor method*:

$$\begin{aligned}
\text{Let } g(t) = e^{\int -1/t\ dt} &= e^{-\ln t} = \frac{1}{t} \\
z = t\int \frac{t^2-3}{t}\ dt &= t\int t - \frac{3}{t}\ dt \\
z &= t\left(\frac{t^2}{2} - 3\ln t + C\right) \\
z &= \frac{t^3}{2} - 3t\ln t + Ct \\
z = -1 \text{ when } t = 1, \text{ so: } C &= \ -1 - \frac{1^3}{2} + 3\ln 0 = -\frac{3}{2} \\
\text{and the particular solution is: } z &= \frac{t^3}{2} - 3t\ln t - \frac{3t}{2}
\end{aligned}$$

(c) Can be rearranged to: $\frac{dy}{dt} + \frac{2y}{t} = \cos t$, so we use *integration factor method*.

$$\text{Let } g(t) = e^{\int 2/t dt} = e^{2\ln t} = t^2$$

$$y = \frac{1}{t^2}\int t^2 \cos t \; dt$$

Using integration by parts:

$$y = \frac{1}{t^2}\left[t^2 \sin t - 2\int t \sin t \; dt\right]$$

Using integration by parts again:

$$y = \frac{1}{t^2}\left[t^2 \sin t - 2\left(-t\cos t - \int -\cos t \; dt\right) + C\right]$$

$$y = \frac{1}{t^2}\left[t^2 \sin t - 2\left(-t\cos t + \sin t\right) + C\right]$$

$$y = \left(1 - \frac{2}{t^2}\right)\sin t + \frac{2}{t}\cos t + \frac{C}{t^2}$$

Using the boundary condition:

$$1 = \frac{2}{\pi}\cos\pi + \frac{C}{\pi^2}$$

$$C = \pi(\pi + 2) = 16.153$$

and the particular solution is: $y = \left(1 - \frac{2}{t^2}\right)\sin t + \frac{2}{t}\cos t + \frac{\pi(\pi+2)}{t^2}$

3. The population growth equation can be written as:

$$\frac{dN}{dt} = kN,$$

where N is the number of organisms at time t, and k is a constant.

We can find the particular solution to the differential equation above by using the separation of variables technique:

$$\begin{aligned}
\frac{dN}{dt} &= kN \\
\int \frac{1}{N}\frac{dN}{dt}\ dt &= \int k\ dt \\
\ln N &= kt + C \\
N &= e^{kt}e^{C} \\
N &= N_0 e^{kt} \\
\text{at } t = 0, N = 500, \text{so } 500 = N_0 e^0, N_0 &= 500 \\
\text{and therefore } N &= 500e^{kt}.
\end{aligned}$$

To find the constant of proportionality, k, insert the conditions $N = 10000, t = 2 \times 10^6$

$$\begin{aligned}
10000 &= 500e^2 \times 10^6 k \\
k &= \frac{\ln(20)}{2 \times 10^6} = 1.5 \times 10^{-6}.
\end{aligned}$$

To find the time it takes to reach 500 million, insert this value for N into the equation:

$$\begin{aligned}
5 \times 10^8 &= 500e^{1.5 \times 10^{-6} t} \\
t &= \frac{\ln(10^6)}{1.5 \times 10^{-6}} = 9.2 \times 10^6,
\end{aligned}$$

So, the population will reach 500 million trilobites 9.2 million years after the Cambrian Explosion.

```python
#!/usr/bin/env python

## Maths Methods 1
## Lecture 7 (ODEs)

import numpy
import pylab
from math import pi, exp
from sympy import sin, cos, Function, Symbol, diff, integrate, dsolve, checkodesol,
solve, ode_order, classify_ode, pprint

###### ORDER OF AN ODE ######
###### Lecture 7, slide 9 ######
t = Symbol('t') # Independent variable
eta = Symbol('eta') # Constant
v = Function('v') # Dependent variable v(t)
ode = diff(v(t),t) + eta*v(t) # The ODE we wish to solve. Make sure the RHS is equal to
zero.
print "ODE #1: \n"
pprint(ode)
print "The order of ODE #1 is %d\n" % ode_order(ode, v(t))

x = Function('x') # Dependent variable x(t)
m = Symbol('m') # Constant
k = Symbol('k') # Constant
ode = m*diff(x(t),t,2) + k*x(t)
print "ODE #2: \n"
pprint(ode)
print "The order of ODE #2 is %d\n" % ode_order(ode, x(t))

y = Function('y') # Dependent variable y(t)
ode = diff(y(t),t,4) - diff(y(t),t,2)
print "ODE #3: \n"
pprint(ode)
print "The order of ODE #3 is %d\n" % ode_order(ode, y(t))


###### ANALYTICAL SOLUTIONS ######
###### Lecture 7, slide 14 ######
x = Symbol('x') # Independent variable
y = Function('y') # Dependent variable y(x)

# The ODE we wish to solve. Make sure the RHS is equal to zero.
ode = diff(y(x),x) - 2*x*(1-y(x))
solution = dsolve(ode, y(x)) # Solve the ode for function y(x).
print "ODE #4: \n"
pprint(ode)
print "The solution to ODE #4 is: ", solution

# This function checks that the result of dsolve is indeed a solution
# to the ode. Basically it substitutes in 'solution' into 'ode' and
# checks that the RHS is zero. If it is, the function returns 'True'.
print "\nChecking solution using checkodesol..."
check = checkodesol(ode, y(x), solution)
if(check[0] == True):
   print "y(x) is indeed a solution to ODE #4.\n"
else:
   print "y(x) is NOT a solution to ODE #4.\n"

# The mpmath module can handle initial conditions (x0, y0) when solving an
# initial value problem, using the odefun function. However, this will
# not give you an analytical solution to the ODE, only a numerical
# solution. The print statement below compares the numerical solution
# with the values of the (already known) analytical solution between x=0 and x=10.
# Remember to uncomment the lines below if you have the mpmath module installed.
#import mpmath
#f = mpmath.odefun(lambda x, y: 2*x*(1-y), x0=0, y0=2)
#for x in numpy.linspace(0, 10, 100):
   #print(f(x), 1.0 + exp(-x**2))

###### SEPARATION OF VARIABLES ######
```

```python
###### Lecture 7, slide 20 ######
x = Symbol('x') # Independent variable
y = Function('y') # Dependent variable y(x)
# The ODE we wish to solve.
ode = (1.0/y(x))*diff(y(x),x) - cos(x)
print "\nODE #5:"
pprint(ode)
# Solve the ode for function y(x).using separation of variables.
# Note that the optional 'hint' argument here has been used
# to tell SymPy how to solve the ODE. However, it is usually
# smart enough to work it out for itself.
solution = dsolve(ode, y(x), hint='separable')
print "The solution to ODE #5 is: ", solution

###### INTEGRATION FACTOR ######
###### Lecture 7, slide 23 ######
x = Symbol('x') # Independent variable
y = Function('y') # Dependent variable y(x)
# The ODE we wish to solve.
ode = diff(y(x),x) - 2*x + 2*x*y(x)
print "\nODE #6:"
pprint(ode)
# Solve the ode for function y(x).using separation of variables
solution = dsolve(ode, y(x))
print "The solution to ODE #6 is: ", solution


###### APPLICATION: RADIOACTIVE DECAY ######
###### Lecture 7, slide 26 ######
t = Symbol('t') # Independent variable
N = Function('N') # Dependent variable N(t)
l = Symbol('l') # Constant
# The ODE we wish to solve.
ode = diff(N(t),t) + l*N(t)
print "\nODE #7:"
pprint(ode)
solution = dsolve(ode, N(t))
print "The solution to ODE #7 is: ", solution


###### APPLICATION: PARTICLE SETTLING ######
###### Lecture 7, slide 31 ######
t = Symbol('t') # Independent variable - time
v = Function('v') # Dependent variable v(t) - the particle velocity
# Physical constants
rho_f = Symbol('rho_f') # Fluid density
rho_p = Symbol('rho_p') # Particle density
eta = Symbol('eta') # Viscosity
g = Symbol('g') # Gravitational acceleration
a = Symbol('a') # Particle radius
# The ODE we wish to solve.
ode = diff(v(t),t) - ((rho_p - rho_f)/rho_p)*g + (9*eta/(2*(a**2)*rho_p))*v(t)
print "\nODE #8:"
pprint(ode)
solution = dsolve(ode, v(t))
print "The solution to ODE #8 is: ", solution
```