3.17 Mathematical Methods IV: Partial Differential Equations (PDEs)

Matthew Piggott

Week 1

- Introductory comments
- Some notation, definitions and examples
- First-order problems
- Transport/advection: physics $\rightarrow$ PDE
- The direct integration method
- Characteristics

Contact details and lecture structure

- Email: m.d.piggott@imperial.ac.uk; Room: 4.82; Ext: 46396

Approximate lecture structure:

- Lecture for the first two hours, with breaks
- One hour exercise class at the end during which we will be joined by several demonstrators
- We will have "Class exercises" during the lectures that we will go through together, you will be given detailed solution sheets to these at the start of the exercise hour (notation: C2.3 will indicate class ex. 3 from week 2)
- You will also be set "Homework exercises" (e.g. H1.1) at the end of the lecture, you can start these during the final hour, ask for hints, and then complete at home (NB. these will not contribute to your final mark)
- Discuss both C & H exercises in the final hour, including the previous week's H exercises for which detailed solutions will also be provided
- Please feel free to hand in work by end of Thursday, so that it can be marked and feedback given at the following session (email a scan)
- Revision class planned

Books and resources

I will be making use of the following book for much of the course (for notation conventions and some, but not all, of the examples/exercises):

Walter A. Strauss, Partial Differential Equations: an Introduction, 2nd edition, John Wiley & Sons, New York, 2008.

The lecture notes and exercises will be self-contained — you don't need to buy the book.

A large number of other text books on PDEs and Engineering Maths (e.g. those that were used in your previous Maths Methods courses) will also cover the majority of the material in this course.

If you want to chat about PDEs over coffee!!! consider:
The (Unfinished) PDE Coffee Table Book, Lloyd N. Trefethen & Kristine Embree (eds), 2001:
`http://people.maths.ox.ac.uk/trefethen/pdectb.html`

Introduction: Use of PDEs

Partial differential equations (PDEs) are used widely to describe, predict and contribute to the fundamental understanding of processes in all areas of science and engineering. For example, in the flow of fluids, the transport of heat and chemical tracers, and the vibrations of solids.

Fluid dynamics – turbulence

Granular material – sand dunes/ripples

Phase change – water-ice

Pattern formation

Images taken from *Applied Partial Differential Equations: A Visual Approach, Peter Markowich.* E-book available via library.

Introduction: Solution methods for PDEs

In this course we will consider some of the most common and relevant examples of PDEs. The main *aim* of the course is to learn about some of the methods available for the solution of PDEs, and develop skills in their use.

These are generally termed *analytical methods* in that they allow us to find *analytical* or *closed-form* solutions to PDEs. This essentially means that we can write the solution down in terms of functions we know (e.g. sin, cos, exp, etc; although we will also learn something about functions that we perhaps haven't seen before).

Many interesting problems that people study with PDEs involve *coupled systems* of equations with *nonlinearities* and very *complex domains*, e.g. the climate system.

Each of these three attributes often impart significant complexities that mean that numerical solution techniques are necessary. These will be covered in other courses.

Some notation (partial differentiation)

The following shorthand for partial derivatives is common (assume $u \equiv u(x, y, t)$ is a function – I will use this equivalence as a shorthand to define the dependent and independent variables in our problem and hence the spatial dimension we're considering, whether it is time-varying etc):

- $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$, etc.

Index (or i, j, k) notation is also very useful. Assume $\mathbf{u} \equiv \mathbf{u}(\mathbf{x})$ where $\mathbf{u}$ (note bold font) is the vector $\mathbf{u} = (u, v, w)^T$, and $\mathbf{x} = (x, y, z)^T$, then we can write

- $\mathbf{u}_{i,j} = \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j}$, $i, j = 1, 2, 3$
- so for example $\mathbf{u}_2 = v$ and $\mathbf{u}_{2,3} = \frac{\partial v}{\partial z} = v_z$
- and you don't need an index before a comma: $w_{,i} = \frac{\partial w}{\partial \mathbf{x}_i}$, $v_{,2} = \frac{\partial v}{\partial y}$

In index notation the comma indicates that any index following it represents a (spatial) derivative.

Some notation (partial differentiation)

Note: We only use this comma (and also summation – on next slide) convention with "i, j, k notation" ($l, m, \ldots$ also used if i, j, k not sufficient, but never x, y, z, t as these have special meaning). Differentiation is implied with the coordinate subscripts x, y, z, i.e. no commas were necessary at the top of the page, and no summation is assumed.

The above index notation can be extended to higher order, so that

$$\mathbf{u}_{i,jk} = \frac{\partial^2 \mathbf{u}_i}{\partial \mathbf{x}_j \partial \mathbf{x}_k}, \quad \text{etc.}$$

More complex expressions is where this shorthand is clearly most useful.

Note that it is a common convention (often termed Einstein summation) with index notation that one assumes summation over any repeated indices (i.e. if an index i, j, k, l, ... appears twice). The summation occurs over all possible values of the index, i.e. typically the spatial dimension.

Some notation (partial differentiation)

Exercise C1.1: Assume $\mathbf{u}$ is a 3D vector function of 3 spatial coordinates (x, y, z). What is $\mathbf{u}_{i,i}$? What is another notational way of writing this (recall vector calculus: grad, div, curl etc)?

Exercise C1.2: Assume $u \equiv u(x, y, z)$. What is $u_{,ii}$? What is another notational way of writing this (recall the Laplacian operator: $\nabla^2 = \nabla \cdot \nabla$)?

Note: Any repeated index is termed a "dummy index" as we can change the letter we're actually using without changing the meaning of the expression, i.e. the following are all exactly equivalent $u_{,ii} = u_{,jj} = u_{,kk}$. Similar dummy *variables* appear in integrals – later lectures.

Partial differential equations (PDEs)

A PDE is a differential equation describing an unknown solution variable — the *dependent* variable (or variables for a PDE system), let's call it u.

The fact that it is a PDE means that it is an equation depending on multiple *independent* variables, and derivatives with respect to these, i.e. *partial* derivatives. NB. If there was only one independent variable then we would not need *partial* derivatives and hence the equation would simply be an ordinary differential equation (ODE).

Let's denote the independent variables x, y, ..., as these are often due to multiple spatial dimensions in our problem which are common to identify with these letters.

The solution u to the PDE is a function of these independent variables:

$$u \equiv u(x, y, \ldots).$$

As said above, this is simply notation allowing us to state that u is defined as a function of x, y, ..., once we have done this we can just use u in any equations we write down without needing the bracketed part all the time.

Definitions

Definition: The *order* of a PDE is the highest derivative that appears.

Definition: The *dimension* of a PDE usually refers to the number of spatial dimensions. "Usually", since sometimes we treat *time* within this definition, sometimes separately. To avoid confusion we will quote "spatial dimension", and then state whether time is present separately.

NB. Sometimes people call a problem *steady* (not dependent on time) or *unsteady* (dependent on time) to clarify the dependence on time.

NB. Sometimes an unsteady problem will *spin-up* to, or converge to, a steady state, in this case the problem is dependent on time, but asymptotes to something that is independent of time — the system has reached a steady state.

Partial differential equations (PDEs)

We can always write the PDE problem describing the solution u as a function of the independent variables, the dependent variable, and its partial derivatives in the following form:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0,$$

(assuming the PDE is first-order and we have two spatial dimensions and no time).

Example: The PDE

$$u_t = -cu_x,$$

can be written as

$$F(x, t, u(x, t), u_x(x, t), u_t(x, t)) = u_t + cu_x = 0.$$

Any function u (here $u \equiv u(x, t)$) that satisfies this equation is termed the solution to the PDE.

Q. What is the general form of a first-order equation with three independent variables, and a second-order PDE with two independent variables?

Partial differential equations (PDEs)

The most general second-order PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

We will come back to this equation in the following form in Lecture 3

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

Some physics! PDE example: transport/advection

Consider a fluid flowing at a constant velocity c in the x direction along a horizontal pipe of fixed cross section (so we can argue we can ignore y & z).

Suppose that there is a pollutant suspended in the water and let $u \equiv u(x,t)$ denote its concentration ($units : [u] = kg\, m^{-1}$) at location x and time t.

Consider the amount of pollutant initially at a point x_0 and at time t_0. At a later time, $t_0 + \Delta t$ assume that the same molecules of the pollutant have moved/transported/advected to the right a distance $c\Delta t$. We then must have

$$u(x_0 + c\Delta t, t_0 + \Delta t) = u(x_0, t_0),$$

$$\text{(or equivalently } u(x,t) = u(x - c\Delta t, t - \Delta t))\,[1].$$

Differentiating this relation with respect to (w.r.t.) the variable Δt, and then setting $\Delta t = 0$ yields

$$0 = cu_x(x,t) + u_t(x,t).$$

Exercise C1.3: Verify this. Hint: recall the chain rule for partial derivatives.

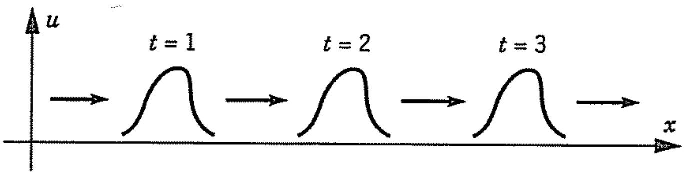
[1] NB. x_0 and x are both just arbitrary independent variables; note that these relations are independent of the actual one taken and so we can safely change between either. Ditto the t's.

PDE example: transport/advection

The governing equation of this transport/advection is thus: $\boxed{u_t + cu_x = 0}$.

Based on the later discussion on PDEs with *variable coefficients* we will see that the concentration must be a function of the combined quantity $(x - ct)$ only: $u(x,t) = f(x - ct)$.

I.e. the substance is transported to the right (think about the minus sign) at a fixed speed c, with $f(\cdot)$ (a function on one variable) describing the shape of the concentration – in the (x,t)-plane each particle moves along a so-called *characteristic* line – more later.



The extension to higher dimensions takes the following form, where the transporting velocity $\mathbf{c}$ is a constant vector:

$$u_t + \mathbf{c} \cdot \nabla u = 0.$$

In the case of a non-constant velocity $\mathbf{c}$ it takes the form[2]:

$$u_t + \nabla \cdot (\mathbf{c}u) = 0.$$

[2] NB. Also common to call c a 'concentration' and $\mathbf{u}$ the velocity, in which case the PDE would be: $c_t + \nabla \cdot (\mathbf{u}c) = 0$.

Further examples

Some examples of PDEs:

1. $u_x + u_y = 0$ (transport/advection)
2. $u_x + yu_y = 0$ (transport/advection)
3. $u_t + uu_x = 0$ (Burgers equation, shock waves, gas dynamics, traffic)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation — MM3)
5. $u_t - u_{xx} = 0$ (heat equation)
6. $u_{tt} - u_{xx} + u^3 = 0$ (wave equation with interaction term)
7. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
8. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
9. $\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \eta \nabla^2 \mathbf{u} + \rho \mathbf{g}$
   (Navier-Stokes momentum equation of fluid dynamics with buoyancy; NB. this needs other equations to close the system – how many equations/ unknowns are there here?)

Exercise C1.4: What is the order of each of these equations?

Linearity

Suppose we are able to write our PDE in the form $\mathcal{L}u = 0$ where $\mathcal{L}$ is an *operator* (e.g. $\mathcal{L} = \partial/\partial t + c\partial/\partial x$ implies that $\mathcal{L}u = u_t + cu_x$) then ...

Definition: A PDE is termed *linear* if it can be written in terms of an operator $\mathcal{L}$ that is itself linear, i.e. $\mathcal{L}$ satisfies

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \quad \mathcal{L}(\alpha u) = \alpha \mathcal{L}u$$

for any functions u and v, and for any constant α.

If a PDE cannot be written in this form in terms of an operator that is linear, then the PDE is said to be *nonlinear*.

Note that it is linearity in the dependent variables appearing in the PDE that defines linearity, e.g. if the independent var's (x,t etc) appear nonlinearly then the PDE can still be linear.

Exercise C1.5: Are the PDE examples on the previous slide linear or nonlinear?

Linearity of the PDE $\mathcal{L}u = 0$ is useful since if u and v are both solutions to the PDE then so is $(u + v)$. More generally, if $u_1, \ldots, u_n$ are all solutions then so is *any* linear combination $\sum_{i=1}^{n} \alpha_i u_i$ for any constants α_i.

Homogeneity

The general form of a linear PDE is actually $\mathcal{L}u = g$ where $g \equiv g(x, y)$ is a function of the independent variables only. Often termed a problem *source*.

Definition: The PDE $\mathcal{L}u = g$ is termed *homogeneous* if $g = 0$, otherwise ($g \neq 0$) it is termed *inhomogeneous*.

[NB. here we mean that the function g is identically equal to zero (could also write $g \equiv 0$), i.e. for all values of x and y, not just for certain values.]

Example: $(\cos xy^2)u_x - y^2u_y = \tan(x^2 + y^2)$ is an inhomogeneous linear PDE (with non-constant coefficients). The RHS is zero for some values of x, y, but it is not always (identically) equal to zero!

Note that if one adds to the solution of a linear inhomogeneous problem $\mathcal{L}u = g$ a solution from the corresponding homogeneous problem $\mathcal{L}v = 0$ then one gets another solution to the inhomogeneous problem.

Exercise C1.6: Demonstrate this.

Direct integration solution method

Sometimes PDEs take a (relatively simple) form that means we can 'integrate them up' using knowledge from (ordinary) differentiation, with the difference that arbitrary constants become arbitrary functions.

Exercise C1.7: Find all $u(x, y)$ that satisfy the PDE $u_{xx} = 0$.

Exercise C1.8: Solve the PDE $u_{xx} + u = 0$. [Hint: think of functions which when integrated twice are minus themselves.]

Exercise C1.9: Solve the PDE $u_{xy} = 0$.

Note: PDEs have arbitrary functions in their solutions, much like ODEs have arbitrary constants. As for ODEs we can try to use auxiliary conditions, such as boundary or initial conditions (which we will discuss later), to fix this arbitrariness. The order of the PDE implies the number of arbitrary functions, which implies the number of auxiliary conditions needed.

First-order linear PDEs

We'll now look at some very simple equations, and see how their solutions can be thought of as being quite geometric in nature. Consider the simplest possible PDE:

$$\frac{\partial u}{\partial x} = 0,$$

where we'll assume that we are in two spatial dimensions ($u \equiv u(x, y)$).

We can see immediately that the general solution to this PDE must be of the form $u(x, y) = f(y)$ (it clearly cannot depend on x from the PDE) where f is an arbitrary function of just one variable.

E.g. $u = y^2 - y$ and $u = e^y$ are both equally valid solutions to this PDE (we cannot be more specific about the precise solution (i.e. the form for $f(\cdot)$) until we prescribe additional constraints, or auxiliary conditions, on the problem).

Think about what all solutions look like when plotted in the (x, y)-plane ...

since the solutions $u(x, y)$ do not depend on x (& only depend on y), u must be constant on the lines $y = \text{const}$ in the (x, y)-plane. I.e. if we vary x then u is unchanged – a contour plot of the solution would therefore simply be a series of horizontal lines. For a constant contour interval the distance between these contours will depend on the exact form of f of course.

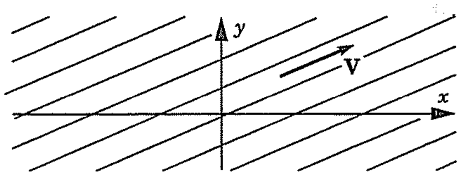
First-order linear PDEs with constant coefficients

Consider now the PDE $\boxed{au_x + bu_y = 0}$
where a and b are non-zero constants (if either was zero we are back to the case on the previous slide and the lines plotted would be horiz or vertical).

Notice that $au_x + bu_y$ is the *directional derivative* of u in the direction of the vector $\mathbf{V} = (a, b)^T = a\mathbf{i} + b\mathbf{j}$, where $\mathbf{i} = (1, 0)^T$ is the unit vector aligned with the x coordinate axis, etc.

Recall the directional derivative: $au_x + bu_y = \mathbf{V} \cdot \nabla u$. This is the rate of change of u in the direction $\mathbf{V}$ (other common notation: $\partial u / \partial V$).

By the PDE above, for any solution u, the derivative of u in direction $\mathbf{V}$ is always zero, i.e. $u(x, y)$ must be a constant in the direction of $\mathbf{V}$, the figure to the right is thus a contour plot of u.



The vector $(b, -a)^T$ is orthogonal to $\mathbf{V}$ (as can be seen from a dot product), and the lines parallel to $\mathbf{V}$ have the equations $bx - ay = \text{constant}$ (or equivalently $y = mx + const$, where slope $m = b/a$). The PDE solution u is const. along each such line, it only varies as we move orthogonally to the line.

First-order linear PDEs with constant coefficients

So we know that $u \equiv u(x, y)$ is constant along lines defined by $bx - ay = \text{const}$, i.e. u only changes value when the quantity $(bx - ay)$ changes value.

Therefore (as per the $b = 0$ case above) $u(x, y)$ can depend only on the single combined quantity $(bx - ay)$, and thus the PDE solution must take the form

$$\boxed{u(x, y) = f(bx - ay)}$$

for an arbitrary function f of one variable. [See also H1.5 for a 'non-geometric' derivation of this result].

Definition: These lines/curves on which the solution to the PDE is constant are called *characteristic lines/curves*. The quantity that defines the curve (here $bx - ay$) is termed a *characteristic variable*.

Exercise C1.10: Verify that the expression above is indeed a solution to the above PDE for any function f, by substitution. [Hint: Use the chain rule. You can use the notation f' for the derivative of the function f of one variable.]

Exercise C1.11: Solve the PDE $4u_x - 3u_y = 0$. Fix the arbitrariness of the solution using the auxiliary condition that $u(0, y) = y^3$.

Homework Exercises

H1.1 Show that the difference of two solutions of an inhomogeneous linear equation with the same right hand side (RHS) is a solution to the corresponding homogeneous problem.

H1.2 Verify that $u(x, y) = f(x)g(y)$ is a solution to the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable. You can use the notation f' and g' for the derivatives of f and g.

H1.3 Verify that $u_n(x, y) = \sin(nx)\sinh(ny)$ is a solution to the PDE $u_{xx} + u_{yy} = 0$ for every $n \neq 0$.

H1.4 Solve the PDE $2u_t + 3u_x = 0$ subject to the auxiliary condition $u = \sin x$ when $t = 0$.

H1.5 Consider the partial differential equation (PDE) $au_x + bu_y = 0$ where a and b are constants. Demonstrate that following the change of variables $\xi(x, y) = bx - ay$, $\eta(x, y) = ax + by$, the PDE transforms into one for the transformed function $U(\xi(x, y), \eta(x, y)) = u(x, y)$ that may be solved by the direct integration method. [Hint: substitute U into the PDE and use the chain rule to differentiate U w.r.t. x etc]. Hence, argue that the general solution to the problem must be $u(x, y) = f(bx - ay)$, where f is any arbitrary function of one variable.

MM4 – solutions to class exercises – week 1

C1.1 Assume three dimensions ($d = 3$), then we have:

$$\mathbf{u}_{i,i} = \sum_{i=1}^{d} \mathbf{u}_{i,i} = \mathbf{u}_{1,1} + \mathbf{u}_{2,2} + \mathbf{u}_{3,3} = u_x + v_y + w_z = \nabla \cdot \mathbf{u}.$$

This is the the divergence of the vector $\mathbf{u}$.

C1.2 $u_{,ii} = u_{,11} + u_{,22} + u_{,33} = u_{xx} + u_{yy} + u_{zz} = \nabla^2 u$. The Laplacian of the scalar u.

C1.3 We had the following relation based on the transport/advection of a pollutant

$$u(x_0 + c\Delta t, t_0 + \Delta t) = u(x_0, t_0).$$

Recall the chain rule for partial derivatives: suppose that $u \equiv u(x, y)$, where we have further that x and y are themselves functions of two new independent variables, i.e. $x \equiv x(s, t)$, $y \equiv y(s, t)$ and $u \equiv u(x(s, t), y(s, t))$, then we can differentiate w.r.t. s or t in the following way:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

In the situation we are considering here we actually have three new independent variables:

$$u \equiv u(x(x_0, t_0, \Delta t), t(x_0, t_0, \Delta t)), \quad x(x_0, t_0, \Delta t) = x_0 + c\Delta t, \quad t(x_0, t_0, \Delta t) = t_0 + \Delta t.$$

Let's now use the above to differentiate our original expression w.r.t. Δt, noting that the right hand side is independent of Δt and so has zero derivative:

$$\begin{aligned} u(x_0 + c\Delta t, t_0 + \Delta t) &= u(x_0, t_0) \\ \implies \frac{\partial u}{\partial x}(x_0 + c\Delta t, t_0 + \Delta t)\frac{\partial (x_0 + c\Delta t)}{\partial \Delta t} + \frac{\partial u}{\partial t}(x_0 + c\Delta t, t_0 + \Delta t)\frac{\partial (t_0 + \Delta t)}{\partial \Delta t} &= 0 \\ \implies c\frac{\partial u}{\partial x}(x_0 + c\Delta t, t_0 + \Delta t) + \frac{\partial u}{\partial t}(x_0 + c\Delta t, t_0 + \Delta t) &= 0. \end{aligned}$$

Setting Δt (which is arbitrary so we can set to what we like and the above will still be true) to zero yields

$$c\frac{\partial u}{\partial x}(x_0, t_0) + \frac{\partial u}{\partial t}(x_0, t_0) = 0.$$

Finally, x_0 and t_0 are also arbitrary and so we can just call them x and t instead, giving

$$c\frac{\partial u}{\partial x}(x, t) + \frac{\partial u}{\partial t}(x, t) = 0,$$

or

$$u_t + cu_x = 0.$$

C1.4 1–3 are first order, 4–6 are second order, 7 is third order, 8 is fourth order and 9 is second order.

C1.5 In order to demonstrate that the PDEs are linear or nonlinear we need to consider the PDE in the form $\mathcal{L}u = 0$ (or more generally $\mathcal{L}u = g$ for an inhomogeneous PDE) and then check whether the following two conditions hold:

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \quad \mathcal{L}(\alpha u) = \alpha \mathcal{L}u$$

for any functions u and v, and any constant α. If we can not demonstrate this then the PDE is nonlinear.

Consider PDE 1: $u_x + u_y = 0$. In this case we can write

$$\mathcal{L} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Now consider two arbitrary functions u and u (they don't need to be solutions of the PDE for checking for linearity), we can see almost immediately in this simple case that

$$\begin{aligned}
\mathcal{L}(u+v) &= \frac{\partial(u+v)}{\partial x} + \frac{\partial(u+v)}{\partial y} \\
&= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{(simple property of derivatives)} \\
&= \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \text{(just rearranging)} \\
&= \mathcal{L}u + \mathcal{L}v,
\end{aligned}$$

and hence the first of the two properties holds. Now consider the second property, here we take an arbitrary function u and an arbitrary constant α, we have

$$\begin{aligned}
\mathcal{L}(\alpha u) &= \frac{\partial(\alpha u)}{\partial x} + \frac{\partial(\alpha u)}{\partial y} \\
&= \alpha\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial y} \quad \text{(simple property of derivatives)} \\
&= \alpha\mathcal{L}u,
\end{aligned}$$

and so we have demonstrated the second required property as well, and so this PDE is linear.

Consider PDE 2: $u_x + yu_y = 0$. We can write the associated operator as $\mathcal{L} = \partial/\partial x + y\partial/\partial y$. Proceeding as above,

$$\mathcal{L}(u+v) = (u+v)_x + y(u+v)_y = u_x + v_x + yu_x + yv_y = \mathcal{L}u + \mathcal{L}v$$

and

$$\mathcal{L}(\alpha u) = (\alpha u)_x + y(\alpha v)_y = \alpha u_x + \alpha y v_y = \alpha\mathcal{L}(u),$$

and hence we have demonstrated that the operator and hence the PDE is linear.

Consider PDE 3: $u_t + uu_x = 0$. Things get more complex now as it's non-obvious how to write the $\mathcal{L}$ operator – this should give a clue that the PDE could be nonlinear. We can write the operator in the following way:

$$\mathcal{L}(\cdot) = \frac{\partial}{\partial t}(\cdot) + (\cdot) \times \frac{\partial}{\partial x}(\cdot),$$

where we have added the brackets to indicate that there is a multiplier by the function being operated on. Let's now try to demonstrate our required properties:

$$\begin{aligned}
\mathcal{L}(u+v) &= \frac{\partial(u+v)}{\partial t} + (u+v)\frac{\partial(u+v)}{\partial x} \\
&= \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} + (u+v)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) \\
&= \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} \\
&= \mathcal{L}u + \mathcal{L}v + u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \\
&\neq \mathcal{L}u + \mathcal{L}v,
\end{aligned}$$

and so at least one of the two required properties does not hold (actually if you check you will see the other doesn't either as you will get an α^2 where you need just an α) and so this PDE is nonlinear.

Proceeding similarly it is relatively straightforward to demonstrate that PDEs 4, 5 and 8 are linear.

PDE 6 is nonlinear because of the final term: $(u+v)^3 \neq u^3+v^3$, as additional cross terms would be present when we expand the left hand side.

PDE 7 is nonlinear because of the second term: $(u+v)(u+v)_x \neq uu_x + vv_x$, as again there would be cross terms.

PDE 9 is nonlinear because of the $\mathbf{u}\cdot\nabla\mathbf{u}$ term, which is just a higher-dimensional form of the nonlinear term in PDE 7.

C1.6 Assume that u and v are solutions to a linear inhomogeneous and corresponding homogeneous PDE respectively:

$$\mathcal{L}u = g \quad \text{and} \quad \mathcal{L}v = 0,$$

then

$$\begin{aligned}\mathcal{L}(u+v) &= \mathcal{L}u + \mathcal{L}v \quad \text{(by linearity of } \mathcal{L}\text{)}\\ &= g + 0 \quad \text{(by PDEs that } u \text{ and } v \text{ satisfy)}\\ &= g,\end{aligned}$$

and hence $(u+v)$ is another solution to the inhomogeneous problem.

C1.7 Integrating once assuming that y is fixed: $u_{xx} = 0 \implies u_x = f(y)$, for an arbitrary function f of one variable. Integrating again while keeping y fixed gives $u(x,y) = xf(y) + g(y)$, where g is another arbitrary function of one variable. So the solution is in terms of two arbitrary functions of one variable.

C1.8 Considering y to be fixed we can see that the PDE $u_{xx}+u=0$ is equivalent to the ODE $v''+v=0$. We know that this second-order ODE has a general solution that involves two terms with two arbitrary constants, in this case this is $v(x) = C_1\sin(x) + C_2\cos(x)$, where C_1 and C_2 are the two arbitrary constants of integration. The solution to the PDE is therefore $u(x,y) = f(y)\sin(x) + g(y)\cos(x)$, where f and g are now arbitrary functions of one variable. As always, this can be checked of course by substituting back into the PDE.

C1.9 Proceeding as before, integrate up once w.r.t. y assuming x is fixed, this gives $u_x = f(x)$. Now integrating w.r.t. x assuming y is fixed we have $u(x,y) = F(x) + g(y)$ where $F' = f$. Note that since f was arbitrary this means that F is arbitrary as well. Hence we can just change the notation for the solution to be $u(x,y) = f(x) + g(y)$ where f and g are arbitrary functions of one variable. This can easily be verified through substitution into the PDE.

C1.10 $u(x,y) = f(bx-ay)$ and so differentiating we have $u_x(x,y) = bf'(bx-ay)$ and $u_y(x,y) = -af'(bx-ay)$. Therefore $au_x + bu_y = abf' - baf' = 0$.

C1.11 From lectures the PDE $4u_x - 3u_y = 0$ (i.e. $a=4$, $b=-3$) has the solution $u(x,y) = f(bx-ay) = f(-3x-4y)$ for an arbitrary function of one variable f. At $x=0$ we therefore know that $u(0,y) = f(-4y)$. We've done this since we've been given the additional information that $u(0,y) = y^3$, i.e. we've been told something about what happens when $x=0$. Combining these two pieces of information we know that the following must hold

$$f(-4y) = y^3.$$

You can now either jump to the final answer, or as an intermediate step make the substitution $w=-4y$ in order to see that

$$f(w) = (w/(-4))^3 = -w^3/64, \quad \text{or} \quad f(\cdot) = -(\cdot)^3/64.$$

I.e. the auxiliary information has allowed us to determine what the function f actually is – we have fixed the arbitrariness of the function f. Therefore, finally, the solution to the PDE is

$$u(x,y) = f(-3x-4y) = (3x+4y)^3/64.$$

MM4 – solutions to homework exercises – week 1

H1.1 Show that the difference of two solutions of an inhomogeneous linear equation with the same right hand side (RHS) is a solution to the corresponding homogeneous problem.

A1.1 Suppose that the functions u and v are solutions to the inhomogeneous linear equation, i.e. $\mathcal{L}u = g$ and $\mathcal{L}v = g$, then

$$\begin{aligned}\mathcal{L}(u-v) &= \mathcal{L}u - \mathcal{L}v \quad \text{(by linearity of the operator } \mathcal{L}\text{)}\\ &= g - g \quad \text{(as } u \text{ and } v \text{ are both solutions to the inhomogeneous PDE)}\\ &= 0\end{aligned}$$

and hence the function $(u-v)$ is a solution to the corresponding homogeneous problem $\mathcal{L}w = 0$.

H1.2 Verify that $u(x,y) = f(x)g(y)$ is a solution to the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable. You can use the notation f' and g' for the derivatives of f and g.

A1.2 Differentiating u gives $u_x = f'(x)g(y)$, $u_y = f(x)g'(y)$, $u_{xy} = f'(x)g'(y)$ and hence $uu_{xy} = fgf'g'$ and $u_x u_y = f'gfg'$ [strictly speaking we should write the (x) and (y)'s here, but they can be dropped without too much risk of confusion]. Therefore $uu_{xy} = u_x u_y$.

H1.3 Verify that $u_n(x,y) = \sin(nx)\sinh(ny)$ is a solution to the PDE $u_{xx} + u_{yy} = 0$ for every n.

[Note that n here is simply a label used to define the fact that there is a different function u_n for any choice of n. Could also have written this as $u^{(n)}$ say.]

Also note that the hyperbolic sin and cos functions are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2},$$

and so

$$\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x).$$

A1.3 Differentiating u once and twice you should find that

$$\frac{\partial u_n}{\partial x} = n\cos(nx)\sinh(ny), \quad \frac{\partial^2 u_n}{\partial x^2} = -n^2\sin(nx)\sinh(ny),$$

and

$$\frac{\partial u_n}{\partial y} = n\sin(nx)\cosh(ny), \quad \frac{\partial^2 u_n}{\partial y^2} = n^2\sin(nx)\sinh(ny).$$

Therefore we have demonstrated/verified that $u_{xx} + u_{yy} = 0$, for any choice of n, i.e. there are infinitely many solutions of this form.

H1.4 Solve the PDE $2u_t + 3u_x = 0$ subject to the auxiliary condition $u = \sin(x)$ when $t = 0$.

A1.4 From the lectures we know that the general solution of a PDE of the form $au_x + bu_y = 0$ is $u(x,y) = f(bx - ay)$.

The PDE we're considering here is clearly in this form just with the y dimension (or independent variable) replaced with t – time (just a different independent variable).

We can progress in two similar ways that will end in the same result:

(a) The first approach starts by observing that the x, y's in the PDE used in lectures can be swapped for t, x here to provide us with a general solution to this PDE [in this case $a = 2$, $x \to t$, $b = 3$, $y \to x$] we thus know that the solution to this problem is $u(x,t) = f(3t - 2x)$ where f is an arbitrary function of one variable.

We are given the additional information that $u(x, 0) = \sin(x)$. Combining this with the fact that we know $u(x,t)$ takes the form $f(3t - 2x)$ allows us to write

$$\sin(x) = u(x, 0) = f(-2x).$$

Purely to see what this tells us about the function $f(\cdot)$ let's introduce the change of (independent) variable $w = -2x$ (you could also do this in your head if confident enough). With this change we now have $f(w) = \sin(-w/2)$ [or equivalently $f(\cdot) = \sin(-(\cdot)/2)$], we have thus fixed the arbitrariness of the function f. Finally we therefore have that $u(x,t) = f(3t - 2x) = \sin(-(3/2)t + x) = \sin(x - (3/2)t)$.

(b) The second approach starts by observing that if we re-order the terms in the PDE (which changes nothing of course) we have $3u_x + 2u_t = 0$. This now looks a bit more like the PDE in the lecture where $a = 3$, x is still x and $b = 2$, $y \to t$. This approach therefore leads to the general solution $u(x,t) = f(2x - 3t)$ [i.e. the argument of the function is minus what it was in the first approach, <u>but the arbitrariness of f means that this does not matter</u> as we shall now see]. Given the additional information that $u(x, 0) = \sin(x)$. This tells us that $f(2x) = \sin(x)$. If we introduce the change of variable $w = 2x$ then we have that $f(w) = \sin(w/2)$, we have thus fixed the arbitrariness of the function f. Finally we therefore have that $u(x,t) = f(2x - 3t) = \sin(x - (3/2)t)$.

So both approaches end up at exactly the same place, even though they passed through a slightly different route – we shall see similar things in later lectures.

H1.5 Consider the PDE

$$au_x + bu_y = 0,$$

where a and b are constants. Demonstrate that following the change of variables

$$\xi(x, y) = bx - ay, \quad \eta(x, y) = ax + by,$$

the PDE transforms into one in terms of the transformed function $U(\xi(x, y), \eta(x, y)) = u(x, y)$ that may be solved by the direct integration method.

[Hint: substitute U into the PDE and use the chain rule to differentiate U w.r.t. x etc].

Hence, argue that the general solution to the problem must be $u(x, y) = f(bx - ay)$, where f is any arbitrary function of one variable.

A1.5 Defining

$$U(\xi(x, y), \eta(x, y)) = u(x, y)$$

we see from the chain rule that

$$u_x = U_\xi \xi_x + U_\eta \eta_x = bU_\xi + aU_\eta, \quad \text{and} \quad u_y = U_\xi \xi_y + U_\eta \eta_y = -aU_\xi + bU_\eta.$$

Hence, the PDE ($0 = au_x + bu_y$) transforms into the new PDE written in terms of the new variables

$$0 = aU_x + bU_y = a(bU_\xi + aU_\eta) + b(-aU_\xi + bU_\eta) = (a^2 + b^2)U_\eta.$$

a and b cannot both be zero otherwise the PDE would be meaningless, hence we must have

$$U_\eta = 0.$$

This PDE can be solved using the direct integration method:

$$U(\xi, \eta) = f(\xi),$$

where f is an arbitrary function of one variable. Hence, in the original variables we must have

$$u(x, y) = f(bx - ay).$$

This gives the same result as that we derived in lectures via the so-called *geometric* approach.

3.17 Mathematical Methods IV: Partial Differential Equations (PDEs)

Matthew Piggott

Week 2

- First-order PDEs with non-constant coefficients
- The derivation of some more common PDEs
- Initial conditions and boundary conditions
- Well-posedness

First-order linear PDEs with variable coefficients

Consider this example of a linear, homogeneous PDE with non-constant coefficients:

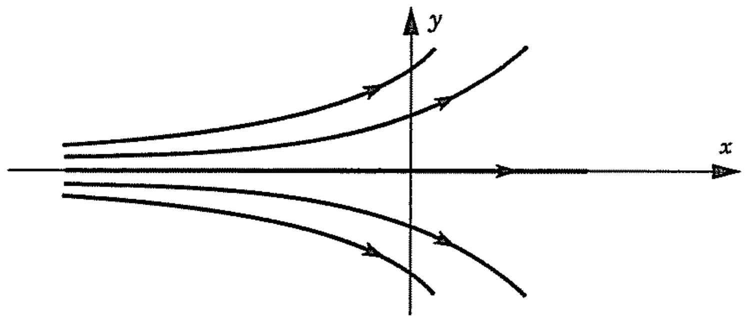
$$u_x + yu_y = 0.$$

As for the constant coefficient case last week, this tells us that solutions $u(x, y)$ are constant in the direction of the vector $\mathbf{V} = (1, y)^T$, i.e. $\mathbf{V} \cdot \nabla u = 0$.

The curves in the xy-plane with $\mathbf{V} = (1, y)^T$ as tangent vectors have slopes y:

$$\frac{dy}{dx} = \frac{y}{1} \quad \left(\text{following same argument as last week where slope} = \frac{b}{a}\right).$$

This is an ODE (as it defines y as a function of only one variable, x) that we know has the solution $y(x) = Ce^x$, for an arbitrary constant C (verify this). The solution $u(x, y)$ is constant along these so-called *characteristic curves*.



Exercise C2.1: Verify that this $u(x, y)$ is indeed constant on these characteristic curves.

First-order linear PDEs with variable coefficients

So on these characteristic curves the solution[1] is constant, i.e. as we move along a characteristic curve, u does not change its value.

Mathematically for the case above this means that the value of $u(x, y(x)) = u(x, Ce^x)$ is actually independent of x.

So it must be equal to the value when $x = 0$ say [we could progress this same argument making any fixed choice for x],

i.e. $u(x, y(x)) = u(0, C)$ for all x, where $C = y(0)$ is a constant.

Hence, the solution $u(x, y)$ can only be dependent on the value of the constant C which defines which curve we are on. This is completely consistent with the constant coefficient case we looked at last week.

[1] to the homogeneous problem – if the RHS of the PDE was non-zero things would be different, but we won't consider that case in this course.

First-order linear PDEs with variable coefficients

We can rearrange the definition of $y(x)$ on the characteristic curves for this particular example to yield $C = e^{-x}y$ [C can be thought of as a label for which curve we are on, and we know what value it takes for any given x & y].

Thus the general solution to the PDE has to be of the form

$$u(x, y) = u(0, C) = u(0, e^{-x}y).$$

Consider what this is telling us: the numerical value that $u(x, y)$ takes is only dependent on the single scalar quantity $e^{-x}y$. We can therefore conclude that the PDE solution has to be able to be written in the form

$$u(x, y) = f(e^{-x}y).$$

where f is an arbitrary function of one variable.

Exercise C2.2: Verify this by substituting this form for the solution into the PDE.

Exercise C2.3: Using the recipe above, solve the PDE $u_x + yu_y = 0$ subject to the auxiliary condition $u(0, y) = y^3$ [this allows you to fix the form of $f(\cdot)$].

First-order linear PDEs with variable coefficients

Note: This method (the geometric method) works for any PDE of the form

$$\boxed{a(x,y)u_x + b(x,y)u_y = 0}$$

It reduces the solution of the PDE to the solution of the ODE $\boxed{dy/dx = b/a}$. If we can solve this ODE, then we have a solution to the PDE following the above 'recipe'. Every solution to the PDE is constant on the solution curves of the ODE.

Exercise C2.4: Solve the PDE $xu_x + yu_y = 0$. What do the characteristics looks like? How (or rather where) could we define auxiliary information (think about where we would need this data in the xy-plane) in order to completely define the solution to this problem without an arbitrary function appearing? This has implications for the existence and uniqueness of solutions which will be discussed later.

Note: This method for solving first-order PDEs can easily be extended to inhomogeneous problems (now rather than the solution being constant along a characteristic it varies according to the RHS). We won't consider this further in this course.

A particular case: the transport/advection equation

Consider the PDE problem

$$u_t + c(x,t)u_x = 0,$$

then by the above, the quantity $u(x,t)$ is constant on the curves given by the solution to the ODE: $\dot{x} = dx/dt = c(x,t)$, where $x \equiv x(t)$.

The characteristic curves are the so-called *particle paths* of the flow that has velocity c, and the quantity u is advected around by this flow.

The *total (or Lagrangian) derivative* tells us how a quantity changes as we move with it within the flow (i.e. on the particle path / characteristic curve where x is a function of t and so with this restriction we can write $u(t) \equiv u(x(t),t)$):

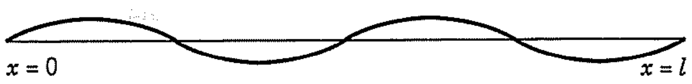
$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{using the chain rule}$$

i.e. the quantity u is constant following the flow. ($\partial u/\partial t$ is the derivative in time at a fixed x (the Eulerian viewpoint of transport) and is not necessarily constant here).

Aside: This is only strictly equivalent to the transport equation we considered earlier provided $c_x \equiv 0$. It's more interesting in higher dimensions and important in incompressible fluids.

Example: vibrating string (2nd-order wave equation)

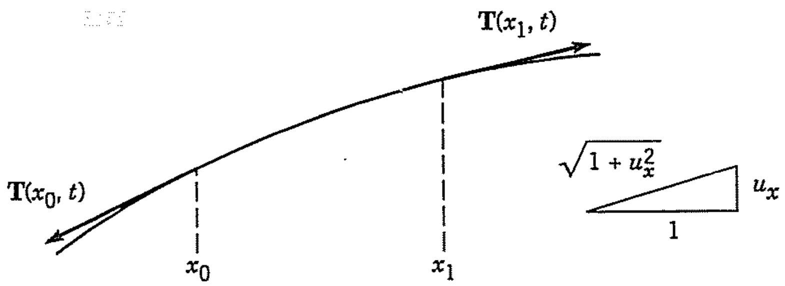
Consider a flexible, elastic homogeneous string of length l, e.g. a guitar string (ignore the boundaries for now)



Let ρ be the (assumed constant) density of the string.

Suppose that the string is perturbed (plucked) a *small* amplitude and let $u(x, t)$ be its displacement from its equilibrium position ($u = 0$).

The tension (force) is directed tangentially along the string, let $T(x, t)$ be the magnitude of the tension vector $\mathbf{T}(x, t)$.



How can we split this force into transverse (u, or 'vertical') and longitudinal (x-direction) components?

Example: vibrating string (2nd-order wave equation)

Newton's second law ($\mathbf{F} = ma$) for the string between any two points $x = x_0$ and $x = x_1$ in the transverse (u) direction can be written

$$\text{Net force} = \left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx = \text{Integrated mass}\times\text{acceleration.}$$

The right hand side (RHS) is the mass (spatial integral of density is mass) times the acceleration (u =displacement, u_t = velocity, u_{tt} = acceleration) integrated up over the length of string being considered; the LHS is the component of the force in the transverse (u) direction with u_x the slope of the string. In the longitudinal direction we can assume there is no net force or acceleration.

Now we said above that we will assume that the amplitude of the perturbation is small (this keeps the problem linear and helps us a lot in our derivation – the assumption of small perturbations in order to invoke linearity comes up regularly). Hence, we may assume that $|u_x|$ is small and Taylor series expansion tells us that $\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \ldots \approx 1$ ($|u_x|$ small implies u_x^2 very small!)

$$\implies \text{LHS} = \left. Tu_x\right|_{x_0}^{x_1} \quad \text{which is the same as} \quad \int_{x_0}^{x_1} (Tu_x)_x\, dx,$$

Example: vibrating string (2nd-order wave equation)

Setting this form of the LHS equal to the RHS and differentiating w.r.t. x [or simply noting that two integrals being equal for any integration limits must mean that the integrands are equal] yields

$$(Tu_x)_x = \rho u_{tt},$$

or, if we also assume that the tension T is constant which also follows from a small perturbation (and thinking about the longitudinal direction),

$$\boxed{u_{tt} = c^2 u_{xx}} \quad \text{where } c = \sqrt{\frac{T}{\rho}}.$$

This is the wave equation, c is the wave speed (or celerity).

Other variations (additional terms) of the wave equation result if we take into account things like air resistance, elasticity or externally applied forces.

The extension to higher dimensions takes the form

$$\boxed{u_{tt} = c^2 \nabla^2 u}$$

NB. You probably also saw a similar derivation of the wave eqn in the course Vibrations & Waves.

Aside: the Fundamental Theorem of Calculus

Theorem: Suppose $f(\cdot)$ is a sufficiently nice[2] function, and another function $F(\cdot)$ is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

[the 'x' in the integral limits is important and is the same x as the argument of the function on the LHS] then differentiating w.r.t. x yields

$$F'(x) = f(x),$$

(with prime indicates a derivative).

For a more detailed description and proof (both outside the scope of this course) consult a book on calculus, e.g. Calculus by Michael Spivak.

Note that a result of this is something you are familar with: if we term F the 'anti-derivative' of f then the above extends to

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

[2]the precise definition of 'nice' is related to smoothness of the function, it is outside the scope of this course, we will assume here that all functions are nice/smooth

Example: diffusion (our third key equation, after the advection & wave eqns)

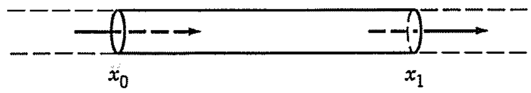
Consider a motionless liquid filling a straight tube or pipe with a substance such as dye diffusing through the liquid (cf. the advection example last week). The substance 'moves' from regions of higher concentration to regions of lower concentration (let's think about a Gaussian shaped concentration blob).

Fick's law of diffusion states that the rate of this (diffusive) motion is proportional[3] to the concentration gradient, i.e.

$$\text{'diffusive flux'} = -ku_x.$$

Let $u(x,t)$ be the concentration at position x and time t.

In the section of the pipe from x_0 to x_1 the total amount of substance present is:



$$M \equiv M(t) = \int_{x_0}^{x_1} u(x,t)\,dx, \qquad \text{hence} \qquad \frac{dM}{dt} = \int_{x_0}^{x_1} \frac{\partial u}{\partial t}(x,t)\,dx.$$

(if the integral limits are fixed in time, then the d/dt derivative becomes a $\partial/\partial t$ if we move it inside the integral, not possible if limits are evolving with time).

[3]with a constant of proportionality k (the "diffusivity"), and a minus sign if we assume the convention that $k > 0$

Example: diffusion

The amount of substance in this (or any) section of the pipe cannot change except by 'moving' through either of two ends of the section (assumed here to be x_0 and x_1). By Fick's law (and when only diffusion is acting) this gives

$$\begin{aligned}\frac{dM}{dt} &= \text{diffusive flux in} - \text{diffusive flux out}\\ &= (-ku_x(x_0,t)) - (-ku_x(x_1,t)) = k(u_x(x_1,t) - u_x(x_0,t)).\end{aligned}$$

Differentiating w.r.t. x_1 and using the fundamental theorem of calculus and the definition of dM/dt on the previous slide yields

$$\boxed{u_t = ku_{xx}}$$

(as before x_1 is an arbitrary independent variable so just replace it with x and on this slide we are assuming k is constant, it doesn't have to be).

This is the *diffusion equation*. The extension to higher dimensions takes the form

$$\boxed{u_t = k\nabla^2 u}$$

Exercise C2.5: What are the units of k?

Example: heat flow (via diffusion)

Suppose that the substance being diffused in the previous example was temperature, then we have the so-called *heat equation* which takes the form

$$\boxed{\rho c_p u_t = \nabla \cdot (\kappa \nabla u)}$$

κ here is the heat conductivity and we have not assumed that it is constant, hence it is within the divergence operator. ρ is the density of the material and c_p is its specific heat capacity.

Exercise C2.6: What are the units of all terms?

If we assume that the coefficients are constant then the heat equation reduces to the diffusion equation with *thermal diffusivity* given by

$$k = \frac{\kappa}{\rho c_p}.$$

Advice: Get into the habit of checking the dimensional consistency of equations. This will serve you well in all your courses which make use of PDEs, directly or indirectly.

Laplace's equation – waves and diffusion at steady state

In the preceding wave and diffusion problems, if we are at a steady state, i.e. our solution does not vary in time: $u_t \equiv u_{tt} \equiv 0$, then both equations reduce to the PDE

$$\boxed{\nabla^2 u = 0}$$

Recall that

$$\nabla^2 u = \Delta u = u_{xx} + u_{yy} + u_{zz} = u_{,ii} \quad \text{notation!}$$

This is *Laplace's equation*, our fourth key PDE.

Definition: Solutions to Laplace's eqn are called *harmonic functions*.

Note that if an applied force is acting then we get *Poisson's equation*

$$\boxed{\nabla^2 u = f}$$

(NB. it is also a very common notational convention to see this equation with a minus sign on the RHS).

Auxiliary conditions

We discussed earlier the fact that PDEs have a number of solutions (via the fact that we have arbitrary functions) and we can focus in on a single solution without the arbitrariness through the use of auxiliary conditions.

What we are trying to do is to establish a *unique* solution.

These auxiliary conditions are generally motivated by the physics of the problem being studied and are typically one or both of:

1. *initial conditions* – the value of the unknown solution for all spatial coordinate values, but at a fixed point in time, e.g. $u(x, t = 0)$;
2. *boundary conditions* – the value of the unknown solution for the spatial coordinate values that describe the boundary ∂D (and for all time if our problem is time dependent), i.e. $u(x \in \partial D, t)$

[initial conditions are of course just a special case of boundary conditions in the time (or temporal) dimension].

Initial conditions

An initial condition specifies the physical state at a particular time t_0 (often but not always $t_0 = 0$). For example, for the diffusion equation discussed above

$$u_t = ku_{xx},$$

we can associate an initial condition in the form

$$u(x, t_0) = \phi(x),$$

where ϕ is a given function, here it would be the initial concentration field of the quantity being diffused, and t_0 may well be 0.

For the wave equation

$$u_{tt} = c^2 u_{xx},$$

(as it has two time derivatives) we need two initial conditions to specify the dynamics:

$$u(x, t_0) = \phi(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, t_0) = \psi(x).$$

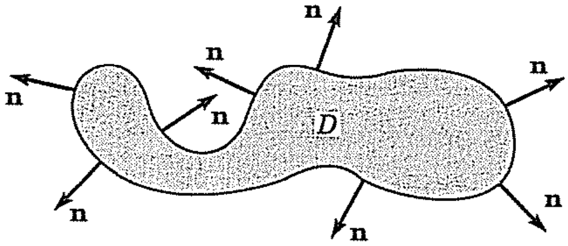
Physically, we need to know both the initial position ϕ as well as the initial velocity ψ to fully define u.

Domain (for the purposes of defining a boundary and associated BCs)

For each problem, and PDE, there is a domain D in which the PDE is valid/defined.

For our one-dimensional problems considered above that may be a finite (e.g. $[0, l]$), semi-infinite (e.g. $[0, \infty]$) or infinite ($[-\infty, \infty]$) interval. In the first case the boundary of the domain is just the set of points $x = 0$ and $x = l$ where boundary conditions may be applied.

For two-dimensional problems the domain can also be infinite. In the case when it is finite it is a plane region D and its boundary is a closed curve S.



We will often need to know the unit outward pointing normal to this boundary, this takes the notation **n**.

Note: Common notation to denote the boundary to D is ∂D.

Note: It is also very common notation to denote a domain as Ω and therefore its boundary as $\partial\Omega$.

Boundary conditions

The three most common types of boundary conditions are the following, where $g(\mathbf{x}, t)$ is a prescribed function, and $\mathbf{n}$ is the outward pointing normal to the domain (figure on previous slide).

Dirichlet condition – the value of the PDE's solution is specified on the boundary:

$$u(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{on} \quad \partial D.$$

Neumann condition – the value of the normal derivative of the solution to the PDE is specified on the boundary:

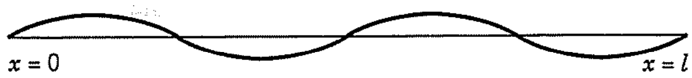
$$\mathbf{n} \cdot \nabla u = \frac{\partial u}{\partial n}(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{on} \quad \partial D.$$

Robin (or mixed or radiation) condition – a combination of the previous two, with $a(\mathbf{x}, t)$ another given function:

$$\frac{\partial u}{\partial n}(\mathbf{x}, t) + a(\mathbf{x}, t)u(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{on} \quad \partial D.$$

Note: If g is always zero ($g \equiv 0$) then we term the BC homogeneous, otherwise it is inhomogeneous. It's possible to have different types of BC applying to different parts of the domain boundary.

BC examples – wave equation



Consider the vibrating string example, leading to the wave equation. If the string is held fixed at both ends then we have homogeneous Dirichlet boundary conditions (on the displacement) at both ends: $u(0, t) = u(l, t) = 0$. If particular motion was imposed at a boundary then we would have an inhomogeneous Dirichlet condition.

If the right hand end ($x = l$) was free to move transversally without any resistance (e.g. along a frictionless track) then there is no vertical component of tension there and we have a homogeneous Neumann condition: $u_x(l, t) = 0$.

If the string were free to move transversally along the same track, but movement was damped as it was attached to a spring (obeying Hooke's law) then we would have a Robin condition, e.g. $u_x(l, t) = -\alpha u(l, t)$.

[Recalling that the transverse force was proportional to u_x, and with α a property of the spring ... a larger displacement u leads to a higher restorative force restricting motion.]

BC examples – diffusion and heat equations

Consider the diffusion equation in a 2D domain D with normal $\mathbf{n}$.

If D is some closed container which the diffusing substance cannot enter or leave, then the concentration gradient in the normal direction must vanish (recall Fick's law: $\mathbf{J} = -k\nabla u$) – the *flux*[4] of the substance is zero. Hence the appropriate boundary condition here would be a homogeneous Neumann condition ($\partial u/\partial \mathbf{n} \equiv \mathbf{n}.\nabla u = 0$).

If the container is permeable and is such that any substance that escapes is immediately washed away (i.e. the solution is forced to be zero on the boundary) then the appropriate BC is a homogeneous Dirichlet one ($u(x \in \partial D) = 0$).

Now consider the heat equation, the first case above is the case when the container is perfectly insulating, the second case would be when the container is immersed in a large reservoir of specified temperature (inhomogeneous Dirichlet if the reservoir's temperature is non-zero).

[4] the flux is the amount of substance crossing a unit area in unit time.

BC examples – diffusion and heat equations

Suppose now that we have a uniform (so can assume one spatial dimension x) rod insulated along its length $0 \le x \le l$, with the end $x = l$ immersed in a reservoir of temperature $g(t)$.

If heat were exchanged between the end of the rod and the reservoir so as to obey *Newton's law of cooling* (which state that heat flux is proportional (with a coefficient 'a') to the temperature difference; cf. *Fourier's law* which says basically the same thing) then:

$$\frac{\partial u}{\partial x}(l,t) = -a[u(l,t) - g(t)].$$

Rearranging we can see that this is an inhomogeneous Robin condition

$$\frac{\partial u}{\partial n}(l,t) + au(l,t) = ag(t).$$

Note: If you think about it the units of a must be m^{-1}; a is actually a heat transfer coefficient (units $Wm^{-2}K^{-1}$) divided by heat conductivity (units $Wm^{-1}K^{-1}$), cf. the non-dimensional Nusselt number.

Well-posed problems

Well-posed problems consist of a PDE in a domain together with a set of initial and/or boundary (or other auxiliary) conditions such that:

1. *Existence:* There exists at least one solution $u(x,t)$ satisfying the PDE and auxiliary conditions.
2. *Uniqueness:* There is at most one solution.
3. *Stability:* The unique solution u depends in a stable manner on the data of the problem (the prescribed functions: RHS, BCs, ICs etc). This means that if the data are changed a little, the corresponding solution changes only a little.

Well-posed problems

If there are too many auxiliary conditions it may be impossible to find a single solution that satisfies all conditions (*non-existence*), the problem is said to be over-determined.

Too few auxiliary conditions could result in more than one solution (*non-uniqueness*), the problem is said to be under-determined.

For physical systems we can often only measure things like initial conditions to a finite accuracy, we don't want any small errors in this to dramatically affect our solution (*instability*).

A job of the modeller is to formulate a PDE plus physically realistic auxiliary conditions such that the system is well-posed. Mathematicians often try to prove well-posedness for a given problem[5].

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[5] NB. Establishing the existence of (smooth) solutions to the Navier-Stokes equations governing fluid flow is an unsolved problem. Addressing this would win you \$1,000,000 (http://www.claymath.org/millennium/).

Update: 22 Jan 2014 New Scientist "Kazakh mathematician may have solved \$1 million puzzle":
http://www.newscientist.com/article/dn24915-kazak-mathematician-may-have-solved-1-million-puzzle.html

Update 2015: he was wrong!

Homework exercises

- H2.1 Solve the first-order constant coefficient PDE $3u_t = 5u_x$ subject to the auxiliary condition $u(x, 0) = \sin(x)$. [Hint: this is just the case we saw in Lecure 1 where x is now t and y is now x, i.e. we have $au_t + bu_x = 0$. Consider also the Note on the geometric method on slide 4 of this lecture.]
- H2.2 Solve the PDE $u_x + 2xy^2u_y = 0$.
- H2.3 Solve the PDE $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves. [Hint: to find the characteristic curves you might find the following useful: $d(\tan^{-1}(z))/dz = 1/(1 + z^2)$.]
- H2.4 Consider the wave equation $u_{tt} = c^2u_{xx}$ in the infinite domain, where c is a constant. Using the transformation $\xi = x + ct$, $\eta = x - ct$, and $U(\xi(x, t), \eta(x, t)) = u(x, t)$ show that the original PDE can be written as $U_{\xi\eta} = 0$, hence what is the general solution of the problem in the original variables? [Hint: use the chain rule to differentiate $u(x, t) = U(\xi(x, t), \eta(x, t))$ w.r.t. x and t and substitute into the original PDE.]

Homework exercises

H2.5 Suppose that some particles are suspended in a liquid sink with velocity $V > 0$ because of gravity. Extending the previous derivation of the diffusion equation, find the so-called advection-diffusion equation governing the concentration of particles. You may assume homogeneity in the x and y directions, and consider the z direction to be pointing upwards.

H2.6 Consider the PDE $u_x + yu_y = 0$ with boundary condition $u(x, 0) = \phi(x)$. What is the general solution, do you have existence and uniqueness in the cases $\phi(x) \equiv x$ and $\phi(x) \equiv 1$?

H2.7 Consider the PDE problem and Neumann boundary condition

$$\nabla^2 u = f(\mathbf{x}) \quad \text{in} \quad D$$
$$\frac{\partial u}{\partial n} = g(\mathbf{x}) \quad \text{on} \quad \partial D.$$

Does this problem have a unique solution? [Hint: what would happen if you added an arbitrary constant to a solution, would this new function also be a solution to the problem?]

MM4 – solutions to class exercises – week 2

C2.1 On characteristic curves $u(x, y) = u(x, Ce^x)$. We can see from this that on the characteristic curve there is really only one independent variable (x) that matters as y is no longer independent, it is a function of x: $y \equiv y(x)$ on the characteristic curve (i.e. on this curve $u \equiv u(x, y(x))$).

The *total* derivative w.r.t. x (notation D/Dx, this is the derivative where we don't assume the other independent variables are held constant, as in partial differentiation, we consider how they vary with x, and make use of the chain rule[1]) is

$$\frac{Du}{Dx} \equiv \frac{D}{Dx}u(x, y(x)) = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = u_x.1 + u_y Ce^x = u_x + yu_y = 0.$$

This is the rate of change of u as we move along the characteristic curve and the fact that it is zero tells us that the value of u does not vary along the characteristic curve.

C2.2 $u(x, y) = f(e^{-x}y)$, differentiating using the chain rule: $u_x = f'(e^{-x}y) \times (-e^{-x}y)$ and $u_y = f'(e^{-x}y) \times (e^{-x})$. Hence,

$$u_x + yu_y = -e^{-x}yf'(e^{-x}y) + ye^{-x}f'(e^{-x}y) = 0.$$

So we have successfully demonstrated that this is indeed a solution to the PDE.

C2.3 Now the auxiliary condition tells us that

$$u(0, y) = f(e^{-0}y) = f(y) = y^3,$$

thus we have found what the function f is – we have used the extra information to remove the arbitrariness. Substituting in this form for f yields the solution to the problems

$$u(x, y) = (e^{-x}y)^3 = e^{-3x}y^3.$$

This is the unique solution that satisfies both the PDE and the auxiliary solution.

C2.4 Solve the PDE $xu_x + yu_y = 0$. This is a first-order PDE with non-constant coefficients, we know from lectures that we need to consider the characteristic curves for this problem.

Before we do this we need to think about what happens at the origin ($x = y = 0$). When $x = 0$ and $y \neq 0$ then the PDE tells us that $u_y = 0$. When $y = 0$ and $x \neq 0$ then the PDE tells us that $u_x = 0$. Both of these are telling us something about the behaviour of the solution at these locations (i.e. on the y-axis and on the x-axis respectively). However note that the PDE tells us nothing about the solution at the origin: $x = y = 0$. In what follows therefore we are forced to avoid this location.

Recall that the characteristic curves satisfy (or are defined by)

$$\frac{dy}{dx} = \frac{y}{x}.$$

This separable ODE problem can be written as

$$\frac{dy}{y} = \frac{dx}{x},$$

integrating both sides gives $\log y = \log x + C_1$, where C_1 is an arbitrary constant of integration, or $y(x) = \exp(\log(x) + C_1) = \exp(\log(x))\exp(C_1) = x\exp(C_1) = Cx$, where C is just another

[1]
$$\frac{D}{Ds}u(x(s), y(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

arbitrary constant of integration (just notational convenience to change from C_1 to C). As these are the characteristic curves to the problem we know (from lectures) that $u(x, y)$ is constant on them.

Thus, $u(x, y) = f(C)$ where f is an arbitrary function, and from the characteristics here $C = y/x$ — as long as both x and y aren't zero then this defines a value for C (which is 0 on the x-axis and ∞ on the y-axis, both as long as we are away from the origin), which we can consider as a 'label' for the characteristic curve we are on.

Hence, the solution to the PDE is

$$u(x, y) = f\left(\frac{y}{x}\right).$$

You could (and probably always should) verify that this is a solution by substituting it back into the PDE and checking that LHS=RHS.

Let's think about the characteristics for a second. For this problem a plot of these curves would actually be straight lines fanning out from the origin (but not actually including the origin based on the discussions above) in xy-space. Notice that there are therefore characteristic curves that sit exactly on the x and the y axes, i.e. characteristics don't cut across these lines. Therefore, for this problem, if we were to define an auxiliary condition of the form $u(x, 0) = \ldots$ or $u(0, y) = \ldots$ (i.e. along these axes), as we have done previously, then this wouldn't define our arbitrary f for us.

This hopefully makes sense since as we know that characteristics lie along these lines then the solution must be a constant on them, knowing this constant doesn't define the value of u on **every** characteristic, this is what we need the auxiliary condition to tell us in order to uniquely solve the problem.

[This is why for the example considered at the start of this lecture (recall the figure on slide 2 of exponential characteristics) specifying $u(0, y)$ uniquely solved our problem (because the line in phase space $(0, y)$ *cuts across* all of the characteristics). But defining $u(x, 0)$ would not have solved our problem (as the line $(x, 0)$ does **not** cross all characteristics – indeed it sits on top of a single one), and any choice of auxiliary condition other than specifying a constant for $u(x, 0)$ would have been inconsistent with the PDE.]

If we did specify an auxiliary condition that specified a varying value for u along a characteristic line (where we know u should be constant) then this problem would fail the "existence" of solutions test – as no function would be able to satisfy both the PDE and the auxiliary condition.

We can rectify this easily for this problem just be specifying the auxiliary condition over a line that cuts all characteristics, e.g. $u(x, 1) = \ldots$ or $u(2, y) = \ldots$ would both uniquely define the solution in this case.

C2.5 Use the notation that square brackets indicate units, e.g. $[t] = [T] = s$ (seconds), $[x] = [L] = m$ (metres), etc. Then the equation

$$u_t = k u_{xx},$$

implies that $[u][t]^{-1} = [k][u][x]^{-2}$. The units of u cancel, so we can just ignore u whatever it is representing in our PDE (heat, pollutant etc). Hence, $[k] = [x]^2[t]^{-1} = m^2\, s^{-1}$.

The **diffusive flux**

$$J := -k u_x,$$

where u is a concentration (i.e. an amount of stuff per unit volume) of some substance then has the units

$$[k u_x] = [k][u_x] = [k][u][x]^{-1} = m^2\, s^{-1} \times [\text{units of "stuff"}] m^{-3} \times m^{-1} = [\text{units of "stuff"}] m^{-2}\, s^{-1},$$

i.e. J is the amount of stuff (diffusing or "flowing") per unit area per unit time.

C2.6 The units of density are $[\rho] = [M\, L^{-3}] = [M][L]^{-3} = kg\, m^{-3}$, of specific heat capacity $[c_p] = J\, K^{-1}\, kg^{-1}$ and diffusivity $[k] = m^2\, s^{-1}$. Hence $[\kappa] = [\rho][c_p][k] = W\, K^{-1}\, m^{-1}$.

MM4 – solutions to homework exercises – week 2

Q1. Solve the first-order constant coefficient PDE $3u_t = 5u_x$ subject to the auxiliary condition $u(x,0) = \sin(x)$. [Hint: this is just the case we saw in Lecure 1 where x is now t and y is now x, i.e. we have $au_t + bu_x = 0$.]

A1. First rewrite the PDE in the form $3u_t - 5u_x = 0$. From lecture 1 or 2, we know that we can solve this type of PDE by first computing the characteristic curves, i.e. solving the ODE

$$\frac{dx}{dt} = \frac{b}{a} = \frac{-5}{3} \implies x(t) = \frac{-5}{3}t + C,$$

or $3x + 5t = C$ (a different C, but it's still arbitrary at this stage so we can safely use the same notation). The general solution to the PDE is thus $u(x,t) = f(3x + 5t)$ for an arbitrary function f of one variable.

Now consider what the auxiliary condition tells us about f:

$$\sin(x) = u(x,0) = f(3x) \implies f(z) = \sin(z/3).$$

This defines what the function f must be, and so the solution to the PDE which satisfies the auxiliary condition is

$$u(x,t) = f(3x + 5t) = \sin\left(\frac{3x + 5t}{3}\right).$$

Q2. Solve the PDE $u_x + 2xy^2u_y = 0$.

A2. The characteristic curves satisfy the ODE

$$\frac{dy}{dx} = \frac{2xy^2}{1} = 2xy^2.$$

This separable problem can be written as

$$\frac{dy}{y^2} = 2x\,dx,$$

hence we know that $-1/y = x^2 - C$, where C is an arbitrary constant of integration (the minus is here just for later convenience). Hence, $y(x) = (C - x^2)^{-1}$. As these are the characteristic curves to the problem, $u(x,y)$ is constant on them. Thus, $u(x,y) = f(C)$ where f is an arbitrary function. Hence, the solution to the PDE is

$$u(x,y) = f\left(x^2 + \frac{1}{y}\right).$$

Note that as the constant C and function f are arbitrary we have taken the liberty of changing signs to simplify our presentation. The arbitrariness of the function f means that we can make changes to the precise form of the argument, as long as the fundamental relationship between the independent variables remains the same, e.g. minus, or one over, etc are allowed since we can account for these in the arbitrariness of f. Given auxiliary information to fix f would always result in the same final answer (cf. the discussion in H1.4).

Q3. Solve the PDE $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves. [Hint: you might find the following useful: $d(\tan^{-1}(z))/dz = 1/(1 + z^2)$.]

A3. The characteristic curves for this problem are given by the solution to the ODE

$$\frac{dy}{dx} = \frac{1}{1 + x^2},$$

which has the solution $y(x) = \tan^{-1}(x) + C$ where C is an arbitrary coefficient. Therefore the solution to the PDE is $u(x,y) = f(y - \tan^{-1}(x))$ for an arbitrary function of one variable f. The sketch of the characteristic curves will just look like the standard arctan (= $\tan^{-1}$ =atan) plot with other representative lines translated up and down the y axes (different choices of the constant C).

Q4. Consider the wave equation $u_{tt} = c^2 u_{xx}$ in the infinite domain, where c is a constant. Using the transformation $\xi = x + ct$, $\eta = x - ct$, and $U(\xi(x,t), \eta(x,t)) = u(x,t)$ show that the original PDE can be written as $U_{\xi\eta} = 0$, hence what is the general solution of the problem in the original variables? [Hint: use the chain rule to differentiate $u(x,t) = U(\xi(x,t), \eta(x,t))$ w.r.t. x and t and substitute into the original PDE.]

Aside: the chain rule

If we had the function $u(x) = f(g(x))$ then we can take the derivative in the following way using the chain rule:

$$\frac{du}{dx}(x) = u'(x) = f'(g(x))g'(x) = \frac{df}{dg}(g(x))\frac{dg}{dx}(x).$$

Let's extend this in several ways requiring the use of partial derivatives (cf. MM1 - Lecture 2):

$$u(x) = f(g(x), h(x)) \implies \frac{du}{dx}(x) = \frac{\partial f}{\partial g}(g(x), h(x))\frac{dg}{dx}(x) + \frac{\partial f}{\partial h}(g(x), h(x))\frac{dh}{dx}(x),$$

$$\begin{aligned} u(x,y) = f(g(x,y)) \implies & \frac{\partial u}{\partial x}(x,y) = \frac{df}{dg}(g(x,y))\frac{\partial g}{\partial x}(x,y) \\ \& \quad & \frac{\partial u}{\partial y}(x,y) = \frac{df}{dg}(g(x,y))\frac{\partial g}{\partial y}(x,y), \end{aligned}$$

$$\begin{aligned} u(x,y) = f(g(x,y), h(x,y)) \implies & \frac{\partial u}{\partial x}(x,y) = \frac{\partial f}{\partial g}(g(x,y), h(x,y))\frac{\partial g}{\partial x}(x,y) + \frac{\partial f}{\partial h}(g(x,y), h(x,y))\frac{\partial h}{\partial x}(x,y) \\ \& \quad & \frac{\partial u}{\partial y}(x,y) = \frac{\partial f}{\partial g}(g(x,y), h(x,y))\frac{\partial g}{\partial y}(x,y) + \frac{\partial f}{\partial h}(g(x,y), h(x,y))\frac{\partial h}{\partial y}(x,y). \end{aligned}$$

Taking this last case as an example we can drop the arguments to the functions, hopefully without the risk of confusion, and just write

$$\begin{aligned} u(x,y) = f(g(x,y), h(x,y)) \implies & \frac{\partial u}{\partial x} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial x} \\ \& \quad & \frac{\partial u}{\partial y} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial y}. \end{aligned}$$

To take higher order derivatives we just have to iteratively apply the above rules, and in addition as now we have functions multiplying each other (from the first application of the chain rule) we need to make use of the product rule, e.g.

$$\begin{aligned} & u(x,y) = f(g(x,y), h(x,y)) \implies \\ & \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial x}\right) \\ & \quad = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial g}\right)\frac{\partial g}{\partial x} + \frac{\partial f}{\partial g}\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial h}\right)\frac{\partial h}{\partial x} + \frac{\partial f}{\partial h}\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) \\ & \quad = \dots \end{aligned}$$

where the first term, recognising that $\partial f/\partial g$ is itself a function of $(g(x,y),h(x,y))$, can be evaluated using the chain rule to give:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial g}\right)=\frac{\partial}{\partial g}\left(\frac{\partial f}{\partial g}\right)\frac{\partial g}{\partial x}+\frac{\partial}{\partial h}\left(\frac{\partial f}{\partial g}\right)\frac{\partial h}{\partial x},$$

and similarly for the other terms which I won't write out.

A4. Under this transformation we have, via the chain rule, that (using subscript notation for derivatives)

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta,$$

and so differentiating again

$$u_{xx} = U_{\xi x} + U_{\eta x} = U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta},$$

and

$$u_t = U_\xi \xi_t + U_\eta \eta_t = cU_\xi - cU_\eta,$$
$$u_{tt} = cU_{\xi t} - cU_{\eta t} = cU_{\xi\xi}\xi_t - cU_{\xi\eta}\eta_t - cU_{\eta\xi}\xi_t + cU_{\eta\eta}\eta_t = c^2U_{\xi\xi} - 2c^2U_{\xi\eta} + c^2U_{\eta\eta}.$$

The original PDE can now be written as

$$0 = u_{tt} - c^2u_{xx} = -4c^2U_{\xi\eta}, \quad \text{or} \quad U_{\xi\eta} = 0 \quad \text{as we can assume that } c \neq 0.$$

We know from last week's classroom exercises that this PDE has the general solution $U(\xi,\eta) = f(\xi)+g(\eta)$ where f and g are arbitrary functions of one variable. Finally, in terms of the original variables, the general solution to the problem is

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are arbitrary functions of one variable. Physically this indicates that the general solution is composed of two independent waves travelling left and right at speed c and maintaining their profiles.

N.B. A similar procedure can be used to demonstrate the solution to the advection equation is in the form $u(x,t) = f(x-ct)$: $\xi = x - ct$, $\eta = t$, $U(\xi,\eta) = u(x,t)$, ... cf. last week.

Q5. Suppose that some particles are suspended in a liquid *sink* with velocity $V > 0$ because of gravity. Extending the previous derivation of the diffusion equation, find the so-called advection-diffusion equation governing the concentration ($u \equiv u(x,t)$) of particles. You may assume homogeneity in the x and y directions, and consider the z direction to be pointing upwards.

A5. From the derivation of the diffusion equation in lecture 2 we have for the region $z_0 < z < z_1$:

$$M(t) = \int_{z_0}^{z_1} u(z,t)\,dz, \qquad \frac{dM}{dt} = \int_{z_0}^{z_1} u_t(z,t)\,dz.$$

As before, the concentration of particles in this region cannot change except by flowing through the ends. By Fick's law, as per lectures, the concentration gradient contributes a term $k(u_z(z_1,t)-u_z(z_0,t))$, while the motion of the particles contributes a term $Vu(z_1,t)-Vu(z_0,t)$ which is a combination of those fluxing in at the top minus those fluxing out at the bottom due to advection. Thus when diffusion and advection are both present we have

$$\int_{z_0}^{z_1} u_t(z,t)\,dz = \frac{dM}{dt} = k(u_z(z_1,t) - u_z(z_0,t)) + Vu(z_1,t) - Vu(z_0,t).$$

In a similar mannet to the diffusion-only case considered in lectures, let's differentiate both sides w.r.t. z_1. By the fundamental theorem of calculus (lecture 2), as we are differentiating w.r.t. a variable that is a limit within the integral, the left-hand side just becomes

$$u_t(z_1, t)$$

Now consider the right-hand side – differentiating w.r.t. to z_1 means that the terms that contain z_0 (and hence do not depend on z_1) disappear. We now need to think what the z_1 derivative is of the terms that contain z_1 as an argument. To do this we can use the chain rule:

$$\frac{\partial}{\partial z_1} k u_z(z_1, t) = k \frac{\partial}{\partial z_1} u_z(z_1, t) = k \frac{\partial}{\partial z} u_z(z_1, t) \frac{\partial z}{\partial z_1} = k u_{zz}(z_1, t) \frac{\partial z}{\partial z_1},$$

and similarly for the other term containing z_1 on the RHS. But z_1 is just a possible value of the variable z (they're essentially the same thing, have the same units etc) and so $\partial z / \partial z_1 = 1$. We are therfore left with

$$u_t(z_1, t) = k u_{zz}(z_1, t) + V u_z(z_1, t).$$

Now finally note that z_1 is completely arbitrary – the above equation is true no matter what value of z_1 we substitute, hence it is a dummy (independent) variable, and so we can just replace it with a different symbol, e.g. z. Therefore we can equyivalently write

$$u_t(z, t) = k u_{zz}(z, t) + V u_z(z, t),$$

or just

$$u_t = k u_{zz} + V u_z.$$

This is an advection-diffusion equation which has the general 3D form

$$u_t + \nabla \cdot (\mathbf{c} u) = k \nabla^2 u,$$

where $\mathbf{c}$ is a velocity vector. The concentration in space and time varies both through the diffusion and the advection of particles.

Note that the sign of V, our convention for which direction increasing z is in, and the definition of sinking (or rising), means that you have to think carefully about the sign ($\pm$) in front of V – we used a more normal convention when writing the 3D form and this is why the sign of the advection term has changed (equivalently we have moved it over to the LHS), here $\mathbf{c}$ is the velocity in the direction of the coordinate axes (in the 1D example V was a sinking velocity but z was pointing up, so they were acting in different directions).

Q6. Consider the PDE $u_x + y u_y = 0$ with boundary condition $u(x, 0) = \phi(x)$. What is the general solution, do you have existence and uniqueness in the cases $\phi(x) \equiv x$ and $\phi(x) \equiv 1$?

A6. From lectures we know that the general solution is of the form $u(x, y) = f(e^{-x} y)$ for some arbitrary function of one variable f. Suppose that the boundary condition is $u(x, 0) = x$, then from the form of the general solution this would imply that $x = u(x, 0) = f(0)$ but $f(0)$ is a constant and so cannot be made to equal the variable x, in this case no solution can exist to the problem. In the second case infinitely many functions f can be chosen such that $f(0) = 1$ (e.g. $f \equiv 1$, $f(w) = \cos(w)$, $f(w) = \exp(w)$, $f(w) = w^2 + 1$) and so we do not have uniqueness in this case.

Think about what the plot of the characteristics looks like and the discussion as part of C2.4.

Q7. Consider the PDE problem and Neumann boundary condition

$$\begin{aligned} \nabla^2 u &= f(\mathbf{x}) \quad \text{in} \quad D \\ \frac{\partial u}{\partial n} &= g(\mathbf{x}) \quad \text{on} \quad \partial D. \end{aligned}$$

Does this problem have a unique solution? [Hint: what would happen if you added an arbitrary constant to a solution?]

A7. Suppose that we have a solution $v(x,y)$ to the above problem, i.e. the Laplacian applied to v is equal to $f(x)$, and the normal derivative of v on the boundary is equal to $g(x)$. Now suppose that we add any constant to this solution, i.e. consider $u(x,y) = v(x,y) + C$, then linearity of the Laplacian operator tells us that $\nabla^2 u = \nabla^2 v + \nabla^2 C = f(x) + 0 = f(x)$, as $\nabla^2 v = f$ (as v is a solution to the above PDE) and $\nabla^2 C = 0$ (as the derivative of a constant is zero. Hence, $v + C$ is also a solution to the above Poisson problem. Similarly, $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} + \frac{\partial C}{\partial n} = \frac{\partial v}{\partial n} = g$, as again the derivative of C is zero. So $v + C$ also satisfies the BCs. Hence, we can add an arbitrary constant onto any solution of the problem above and get another solution. We therefore do not have uniqueness in this problem.

3.17 Mathematical Methods IV: Partial Differential Equations (PDEs)

Matthew Piggott

Week 3

- Second-order problems
- The separation of variables solution method

Second-order PDEs

Consider the second-order PDE in two independent variables in the following form

$$F(\ldots) = a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

This is a linear equation of order two in two variables with six real constant coefficients.

[NB. for convenience some people introduce a factor 2 before the mixed derivative term ($a_{12}u_{xy}$) and this changes the definition of the discriminant on the next slide.]

Classification

The type of equation we are dealing with is dependent on the sign of the *discriminant* $a_{12}^2 - 4a_{11}a_{22}$:

1. $a_{12}^2 - 4a_{11}a_{22} < 0$: In this case the PDE is termed *elliptic*.
2. $a_{12}^2 - 4a_{11}a_{22} > 0$: In this case the PDE is termed *hyperbolic*.
3. $a_{12}^2 - 4a_{11}a_{22} = 0$: In this case the PDE is termed *parabolic*.

[NB. If we had have introduced the "2" then we wouldn't have a "4" in the above.]

Note that when the coefficients are not constant the PDEs may still be classified as above, but that classification, and it turns out the behaviour of solutions, may change in different regions of the domain or with time as the value of the discriminant changes.

NB. We can come up with a similar classification for problems involving any number of independent variables – but we won't in this course.

Classification

Exercise C3.1: Classify the following PDEs:

(a) $u_{xx} - 5u_{xy} = 0.$

(b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0.$

(c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0.$

(d) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

(e) $yu_{xx} - 2u_{xy} + xu_{yy} = 0.$

The expected behaviour of solutions to *elliptic* problems

Solutions to elliptic equations are "smooth", i.e. they vary in a sedate manner. The level of smoothness is partly dictated by the coefficients in the PDE. Solutions may be rough at boundaries.

If one of the independent variables is time, then solutions will tend to become smoother with time.

If both of the independent variables are space coordinates, then the solution will tend to be increasingly smooth in the interior of the region away from the boundaries.

We shall see some examples of this behaviour later when we compute some solutions to elliptic problems.

Laplace's equation is an example of an elliptic problem.

The expected behaviour of solutions to *Parabolic* problems

Parabolic equations can always be transformed into a form equivalent to the diffusion equation by a suitable change of variables.

In this form, the solutions always become smoother as the transformed time variable increases (which makes sense if you think about diffusion physically!).

Again, we shall see some examples of this later.

The diffusion, or heat, equation is an example of a parabolic problem.

The expected behaviour of solutions to *hyperbolic* problems

Hyperbolic equations have solutions that do not get smoother with time. Instead discontinuities in the initial conditions tend to propagate across space and time.

Recall also the discussion on the development of a shock in Burgers equation even starting from a smooth initial condition in week 1.

Hyperbolic equations are often difficult to deal with because at large times and distances their solutions are still complicated.

Again, we shall see some examples of this later.

The wave equation is an example of a hyperbolic problem.

The separation of variables solution method

This is a series solution method, the 'recipe' for which is the following:

1. Use 'separation of variables' to convert the PDE into two (or more) ODEs (cf. as was our typical approach for first-order PDEs as well).
2. Find general solutions for these ODEs.
3. Use some of the boundary conditions (the pair of homogeneous ones) to restrict the possible solutions.
4. Write the solution as a linear combination of the solutions computed above.
5. Apply the remaining boundary conditions (or initial conditions) to find the coefficients of the infinite series (recall Fourier series from MM2 and MM3), thus giving the unique solution to the problem.

NB. Depending on the exact nature of the problem and auxiliary conditions applied the details may change slightly but the underlying idea/recipe is always the same.

Separation of variables

The two factors that make the technique successful are:

1. The *linearity* of the PDE since this allows superposition of the individual candidate solutions into an infinite series (step 4 above).

2. A *pair of homogeneous BCs* (required in step 3) since these give rise to a so-called *Sturm-Liouville* ODE problem and also allows superposition (Why?).

Provided that these two factors are present, separation of variables is a possible method of solution. Check the problem before attempting to solve using this approach!

and think about which coordinate the pair of homogeneous BCs match with — sketch the problem domain and add in the BCs.

Sturm-Liouville problems

A Sturm-Liouville problem is a particular type of ODE plus BCs which has a constant but unspecified constant λ appearing.

The solution to the problem involves determining values for λ (the 'eigenvalues') for which non-trivial (non-identically-zero) solutions (the 'eigenfunctions') to the Sturm-Liouville ODE + BCs exists.

Solving Sturm-Liouville problems is a crucial step in the process of separation of variables.

Exercise C3.2: Find all the (in this course we will always assume real-valued) eigenvalues (λ) and eigenfunctions ($X \equiv X(x)$, $0 < x < l \neq 0$) of the following problems:

(a) $X'' - \lambda X = 0$, with $X(0) = X(l) = 0$.

(b) $X'' - \lambda X = 0$, with $X'(0) = X'(l) = 0$.

Separation of variables – Laplace's eqn with Dirichlet BCs

Exercise C3.3: Solve the following problem using separation of variables:

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{in} \quad 0 < x < a,\ 0 < y < b,$$

with the Dirichlet boundary conditions

$$\begin{aligned} &u(0,y) = u(a,y) = 0 \quad \text{for} \quad 0 < y < b, \\ &u(x,0) = 0 \quad \text{for} \quad 0 < x < a, \\ &u(x,b) = \phi(x) \quad \text{for} \quad 0 < x < a. \end{aligned}$$

Compute the solution in the case where $\phi(x) = x$ and $a = \pi$.

[Hint: note that $\frac{d}{dz}(-z\cos(nz)/n + \sin(nz)/n^2) = z\sin(nz)$.]

Compute the solution in the case where $a = b = 1$ and $\phi(x) = \sin(2\pi x)$.

Separation of variables – wave equation

Exercise C3.4: Use separation of variables to solve the wave equation

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 \leq x \leq l,$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0,$$

and some general initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

Homework exercises

H3.1. Find all the (real-valued) eigenvalues (λ) and eigenfunctions ($X(x)$) of the following problems:
(a) $X'' - \lambda X = 0$, in $-\pi < x < l \neq \pi$, with $X(-\pi) = X(\pi)$, $X'(-\pi) = X'(\pi)$.
(b) $X'' - \lambda X = 0$, in $0 < x < l \neq 0$ with $X'(0) = X(l) = 0$.
(c) $X'' - \lambda X = 0$, in $0 < x < l \neq 0$ $X(0) = X'(l) = 0$.

H3.2. Use separation of variables to find the solution to Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{in} \quad 0 < x < a,\ 0 < y < b,$$

with mixed Dirichlet and Neumann boundary conditions

$$\begin{aligned} u(0,y) &= 1 \quad \text{for} \quad 0 < y < b, \\ u(a,y) &= \cos(\pi y/b) \quad \text{for} \quad 0 < y < b, \\ u_y(x,0) = u_y(x,b) &= 0 \quad \text{for} \quad 0 < x < a. \end{aligned}$$

What is the solution when we instead have the BC $u(a,y) = \cos^2(\pi y/b)$?

[Hint: $cos^2(z) = (1 + \cos(2z))/2$; the two solutions you should find are:

$$u(x,y) = 1 - \frac{x}{a} + \frac{\sinh\left(\frac{\pi x}{b}\right)\cos\left(\frac{\pi y}{b}\right)}{\sinh\left(\frac{\pi a}{b}\right)}, \quad u(x,y) = 1 - \frac{x}{2a} + \frac{\sinh\left(\frac{2\pi x}{b}\right)\cos\left(\frac{2\pi y}{b}\right)}{2\sinh\left(\frac{2\pi a}{b}\right)}.$$

]

Homework exercises

H3.3. Use separation of variables to find the solution to the diffusion equation

$$u_t = k u_{xx} \quad \text{for} \quad 0 \leq x \leq l,\ 0 \leq t \leq \infty \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0, \tag{2}$$

and the initial conditions

$$u(x,0) = \phi(x). \tag{3}$$

Find the solution in the case where the domain is of two units in size in the x direction ($l = 2$) and the initial condition is given by

$$u(x,0) = \phi(x) \equiv 1.$$

Hint: I make the solution:

$$u(x,t) = \sum_{n(odd)=1}^{\infty} \frac{4}{n\pi} \exp\left(-\left(\frac{n\pi}{2}\right)^2 kt\right) \sin\left(\frac{n\pi x}{2}\right).$$

This summation is only over odd values of n.

Try using Python to evaluate your solution and plot at a number of time levels to demonstrate its behaviour for different values of k.

Homework exercises

H3.4. Calculate the solution to the problem

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 \leq x \leq l,$$

with homogeneous Dirichlet boundary conditions

$$u(0, t) = u(l, t) = 0,$$

and some general initial conditions

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),$$

when $l = 1$ and in the two cases where $\phi(x) = x(1 - x)$, $\psi \equiv 0$ and $\phi(x) = \sin(5\pi x) + 2\sin(7\pi x)$, $\psi \equiv 0$.

[Hint: Use the general solution to this problem found in class as your starting point. I make the solutions:

$$u(x, t) = \sum_{n(odd)=1}^{\infty} \frac{8}{n^3\pi^3} \sin(n\pi x)\cos(cn\pi t),$$

$$u(x, t) = \sin(5\pi x)\cos(5c\pi t) + 2\sin(7\pi x)\cos(7c\pi t).$$

[Hint: For the first problem note that $\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + C$, $\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$. You can derive these using integration by parts. For the second (easier) case you should be able to figure out the Fourier constants directly without the need to compute the integrals].

Try plotting your solutions.

MM4 – A reminder on Fourier Series

We know that some functions may be expanded in a power series, e.g.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

However, if we have a periodic function, it is often more useful to expand in terms of simpler periodic functions (of the same period) such as sine and/or cosine. Suppose $f(x)$ is a periodic function of period 2π, e.g.

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 \le x \le \pi \end{cases}, \qquad f(x + 2\pi) = f(x).$$

Note that the function does not need to be continuous in order for us to extend its definition through periodicity (draw this function for a domain wider that $[-\pi, \pi]$).

Now some simple functions we know to be periodic with period 2π are $\sin(nx)$ and $\cos(nx)$ for any non-negative integer n. Fourier series represent functions such as f in terms of an infinite series of such functions:

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}.$$

The set of functions $\{1, \cos(nx), \sin(nx)\}$ is *orthogonal* on any interval of width 2π, e.g. $[-\pi, \pi]$. Because of this it can be shown that the coefficients in the Fourier series are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad (n \ge 0),$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \quad (n \ge 1).$$

These are the Fourier coefficients of $f(x)$.

Notice that if f is an even function (i.e. $f(-x) = f(x)$, e.g. x^2 or $\cos(x)$) defined on $x \in [-\pi, \pi]$ then $b_n = 0$ for $n \ge 1$, and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \quad (n \ge 0).$$

If f is an odd function (i.e. $f(-x) = -f(x)$, e.g. x^3 or $\sin(x)$) defined on $x \in [-\pi, \pi]$ then $a_n = 0$ for $n \ge 0$, and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \quad (n \ge 1).$$

So we can simplify the full Fourier series to a so-called *Fourier cosine series* or a *Fourier sine series* if we know that we have an even or odd function respectively.

Suppose we have a function $f(x)$ defined only on $x \in [0, \pi]$, then we can *choose to extend this to either an even or an odd function* on $[-\pi, \pi]$:

$$f_e(x) = \begin{cases} f(-x), & -\pi \le x \le 0 \\ f(x), & 0 \le x \le \pi \end{cases}, \qquad f_o(x) = \begin{cases} -f(-x), & -\pi \le x \le 0 \\ f(x), & 0 \le x \le \pi \end{cases}.$$

f_o is called the *odd extension* of f, and f_e is called the *even extension*.

The full Fourier series of f_e is called the *Fourier cosine series* of f:

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{1}$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \quad (n \geq 0). \tag{2}$$

The full Fourier series of f_o is called the *Fourier sine series* of f:

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin(nx), \tag{3}$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \quad (n \geq 1). \tag{4}$$

Note that if our interval had been $[0, l]$ instead of $[0, \pi]$ then we would have

$$f(x) \approx \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right),$$

where

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \quad (n \geq 0).$$

and a similar generalisation for the sine series.

What you need to remember for MM4:

When conducting **separation of variables**, by considering your final boundary condition(s) you will typically end up with something that looks like either (1) or (3) (or the generalisation to a different interval given directly above) where you know what f is from the specified boundary condition function.

You just need to remember that, for this given f, formulae (2) and (4) (or the generalisation directly above) can be used to compute the final coefficients you need in each of these two cases.

MM4 – solutions to class exercises – week 3

C3.1. Classify the following PDEs:

(a) $u_{xx} - 5u_{xy} = 0$.

(b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$.

(c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$.

(d) $u_{xx} - u_{xy} + 2u_y + 2u_{yy} - 3u_{yx} + 4u = 0$.

(e) $yu_{xx} - 2u_{xy} + xu_{yy} = 0$.

(a) For this problem $a_{12}^2 - 4a_{11}a_{22} = (-5)^2 - 4 \times 1 \times 0 = 25 > 0$ and hence this problem is of hyperbolic type.

(b) For this problem $a_{12}^2 - 4a_{11}a_{22} = (-12)^2 - 4 \times 4 \times 9 = 0$ and hence this problem is of parabolic type.

(c) For this problem $a_{12}^2 - 4a_{11}a_{22} = (6)^2 - 4 \times 4 \times 9 = -108 < 0$ and hence this problem is of elliptic type.

(d) For this problem $a_{12}^2 - 4a_{11}a_{22} = (-4)^2 - 4 \times 1 \times 1 = 12 > 0$ and hence this problem is of hyperbolic type.

(e) For this problem $a_{12}^2 - 4a_{11}a_{22} = (-2)^2 - 4 \times x \times y = 4(1 - xy)$. Therefore the problem is parabolic on the line $xy = 1$, it is elliptic in the regions where $xy > 1$ and hyperbolic in the region where $xy < 1$.

C3.2. Find all the (real-valued) eigenvalues (λ) and eigenfunctions ($X(x)$, $0 < x < l \neq 0$) of the following problems:

(a) $X'' - \lambda X = 0$, with $X(0) = X(l) = 0$.

(b) $X'' - \lambda X = 0$, with $X'(0) = X'(l) = 0$.

(a) For Sturm-Liouville problems considered alone, or as part of a solution to a separation of variables problem, we always follow the same procedure – we consider three cases based on the sign on λ and consider whether any solutions that satisfy the BCs and are non-zero are possible. These three cases are (i) $\lambda = 0$, (ii) $\lambda > 0$, (iii) $\lambda < 0$.

$\lambda = 0$ gives $X'' = 0 \implies X(x) = Ax + B$, for constants A and B. The BCs then tell us that $0 = X(0) = B$ and $0 = X(l) = Al + B \implies A = 0$ as $B = 0$ and $l \neq 0$. So there are **no non-zero solutions in this case**.

Aside: solution to simple second-order ODEs

Consider an unknown function of one variable: $u \equiv u(x)$. Suppose it satisfies the ODE

$$u^{''} + u = \frac{d^2u}{dx^2} + u = 0$$

This is equivalent to

$$u^{''} = -u$$

i.e. I am looking for functions that when they are differentiated twice, take the same form, but with an extra minus sign. As this is a second order problem I know I am looking for a general solution with two terms and two arbitrary constants. I can spot immediately that two functions which fill this criteria are sin and cos, and the general solution is

$$u(x) = A\cos(x) + B\sin(x)$$

where A and B are arbitrary constants. Now suppose that the ODE had been $u^{''} + \lambda u = 0$ or $u^{''} = -\lambda u$, i.e. now when I differentiate twice I still get a minus sign but also a factor of λ. I should be able to spot that I would get this if I update my previous solution to now read $u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ – if you don't see why this works, substitute it into the ODE to confirm it works.

If I don't want to keep writing square root signs I could instead note that the solution to $u^{''} + \beta^2 u = 0$ is $u(x) = A\cos(\beta x) + B\sin(\beta x)$. Again if you can't see this, just substitute it into the ODE to convince yourself that it works.

Similarly, what is the solution to the ODE

$$u^{''} - \beta^2 u = 0 \iff u^{''} = \beta^2 u$$

Again I am looking for two functions to combine with two arbitrary constants to form my general solution. These functions need to look like themselves when differentiated twice (without the minus sign this time so sin and cos won't work) and we need an extra factor of β^2 to appear. I should be able to spot that $\exp(\beta x)$ and $\exp(-\beta x)$ fit this bill. Therefore the general solution to this ODE is

$$u(x) = A\exp(\beta x) + B\exp(-\beta x)$$

where A and B are arbitrary constants.

$\lambda < 0$ i.e. $\lambda = -\beta^2$ for some real non-zero β. Hence, we have $X'' + \beta^2 X = 0$ which has the general solution $X(x) = A\cos(\beta x) + B\sin(\beta x)$, for constants A and B. The BCs now tell us that $0 = X(0) = A$ and $0 = X(l) = B\sin(\beta l)$. So **we have a non-zero solution** ($B \neq 0$) if $\beta l = n\pi$ for integer n as then $\sin(\beta l) = 0$, i.e. the **eigenvalues** are $\lambda = \lambda_n = -(n\pi/l)^2$ and the corresponding **eigenfunctions** are $X_n(x) = B_n \sin(n\pi x/l)$, for $n = 1, 2, 3, \ldots$, where B_n is an arbitrary constant.

$\lambda > 0$ i.e. $\lambda = \beta^2$ for some real non-zero β. Hence, we have $X'' - \beta^2 X = 0$ which has the general solution $X(x) = A\exp(\beta x) + B\exp(-\beta x)$, for constants A and B. The BCs now tell us that $0 = X(0) = A + B \implies B = -A$, and $0 = X(l) = A\exp(\beta l) + B\exp(-\beta l) = A(\exp(\beta l) - \exp(-\beta l))$. Hence, $A = B = 0$ since $(\exp(\beta l) - \exp(-\beta l)) \neq 0$ as $\beta l \neq 0$. So there are **no non-zero solutions in this case**.

Therefore, the only non-zero solutions are those listed above for $\lambda < 0$. NB. we have a whole family of solution indexed by n as described above.

(b) Let's consider the three cases:

$\lambda = 0$ gives $X'' = 0 \implies X(x) = Ax + B$, for constants A and B. The BCs then tell us that $0 = X'(0) = A$ and $0 = X'(l) = A$. Hence, we do have the non-zero solution $X(x) = B$ an arbitrary constant, for $\lambda = 0$.

$\lambda < 0$ i.e. $\lambda = -\beta^2$ for some real non-zero β. Hence, we have $X'' + \beta^2 X = 0$ which has the general solution $X(x) = A\cos(\beta x) + B\sin(\beta x)$, for constants A and B. The BCs now tell us that $0 = X'(0) = \beta B \implies B = 0$ as $\beta \neq 0$, and $0 = X'(l) = -A\beta\sin(\beta l)$. So we have a non-zero solution if $\beta l = n\pi$, i.e. the eigenvalues are $\lambda = \lambda_n = -(n\pi/l)^2$ and the corresponding eigenfunctions are $X_n(x) = A_n\cos(n\pi x/l)$, for $n = 1, 2, 3, \ldots$, where A_n is an arbitrary constant.

$\lambda > 0$ i.e. $\lambda = \beta^2$ for some real non-zero β. Hence, we have $X'' - \beta^2 X = 0$ which has the general solution $X(x) = A\exp(\beta x) + B\exp(-\beta x)$, for constants A and B. The BCs now tell us that $0 = X'(0) = A\beta - B\beta \implies B = A$ and $0 = X'(l) = A\beta\exp(\beta l) - B\beta\exp(-\beta l) = A\beta(\exp(\beta l) - \exp(-\beta l))$. Hence, $A = B = 0$ since $(\exp(\beta l) - \exp(-\beta l)) \neq 0$ as $\beta l \neq 0$.

Therefore, the only non-zero solutions are those listed above for $\lambda = 0$ and $\lambda < 0$.

C3.3: Separation of variables applied to Laplace's equation with Dirichlet BCs

Recall that in two-dimensions, in rectangular Cartesian coordinates, Laplace's equation for the unknown variable $u \equiv u(x,y)$ is

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{in} \quad 0 < x < a,\ 0 < y < b. \tag{1}$$

Note that since time does not enter into the problem, there are no initial conditions, and the solution is fully defined by the conditions on the spatial boundaries. Let's suppose that the boundary conditions take the form

$$u(0,y) = u(a,y) = 0 \quad \text{for} \quad 0 < y < b, \tag{2}$$
$$u(x,0) = 0 \quad \text{for} \quad 0 < x < a, \tag{3}$$
$$u(x,b) = \phi(x) \quad \text{for} \quad 0 < x < a, \tag{4}$$

The solution method termed *separation of variables* attempts to build up the general solution to the problem as a linear combination (so we're obviously applying separation of variables to linear problems only!) of special ones that are easy to find.

We start by considering a *separated solution* of the form

$$u(x,y) = X(x)Y(y). \tag{5}$$

We're using the notational convention that the lower case letter is the independent variable appearing in the original problem, and the upper case letter is the associated function of, this, one variable which we are going to try to find. NB. when proposing a particular form for a solution to a problem you sometimes hear the term *solution ansatz*. Let's also assume that we discount the zero solution $u \equiv 0$, which would satisfy the PDE but which would almost certainly not satisfy one or more of the BCs/ICs for interesting problems. This means that we must also discount $X \equiv 0$ and $Y \equiv 0$.

We start by looking for as many separated solutions as possible, we will then use our auxiliary conditions to get rid of some of them. Plugging the proposed form for the solution (5) into the original problem (1) yields

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

(NB. X and Y are functions of one variable and so we can safely use a prime ($'$) to indicate differentiation w.r.t this single variable) which can be rearranged to

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} =: \lambda. \tag{6}$$

(NB. The notation := means define to be equal to). Now notice that the first expression is independent of y, as λ is equal to this it also can't depend on y. Similarly the the second expression tells us that λ is also independent of x, hence λ must be independent of both variables, i.e. it must be a constant. We have thus reduced our PDE to the following two separate ODEs which we will now solve

$$X'' - \lambda X = 0,$$
$$Y'' + \lambda Y = 0.$$

It can be shown, but we will always just assume it here, that λ is a real number. Depending on the sign of λ we can solve the ODEs for a number of solutions that are candidates to be summed to form the general solution to the PDE, but we will be able to discount some due to the boundary conditions.

The next step is to consider the solutions that satisfy the *pair of homogeneous* boundary conditions (as the sum of these solutions will still satisfy both the *linear* PDE and the *homogeneous* BCs – lecture 1), this is the Sturm-Liouville part of the process.

We progress by noting that in this case (be careful as sometimes it will be the Y equation) it is the X ODE that possesses the pair of homogeneous BCs (to satisfy (2) we must have $X(0) = X(a) = 0$ as the alternative is that $Y \equiv 0$, i.e.$Y(y) = 0$, $\forall y$ which would mean $u \equiv 0$ which we discounted – explained again in a few lines) and consider three potential sets of values for λ:

1. Suppose that $\lambda = 0$, then the ODE $X'' - \lambda X = 0$ becomes $X'' = 0$ which has the general solution $X(x) = C + Dx$, where C and D are two arbitrary constants of integration.

   But our homogeneous boundary conditions (2): $u(0, y) = u(a, y) = 0$ implies that

   $$X(0)Y(y) = X(a)Y(y) = 0 \quad \text{for all values of} \quad y \qquad (\text{notation: } \forall \lambda)$$

   hence we must have either that $X(0) = X(a) = 0$, OR $Y(y) = 0 \ \forall y$. The second option of course also implies that $u \equiv 0$, but we have specifically said we're looking for non-trivial solutions, and hence we must have the case that $X(0) = X(a) = 0$.

   Substituting this into $X(x) = C + Dx$ leads to the conclusion (as $a > 0$) that $C = D = 0$, i.e. $X \equiv 0$. But we discounted the zero solution and hence λ cannot take the value 0.

2. Now let's suppose that $\lambda > 0$, i.e. we can write $\lambda = \beta^2$ for a real valued $\beta \neq 0$. Then our ODE for X takes the form $X'' - \beta^2 X = 0$ which has the general solution $X(x) = C\exp(\beta x) + D\exp(-\beta x)$, where C and D are two arbitrary constants of integration. Then the homogeneous boundary conditions (2) tell us that $0 = X(0) = C + D$ and $0 = X(a) = C\exp(\beta a) + D\exp(-\beta a)$. Combining these two gives $C(\exp(\beta a) - \exp(-\beta a)) = 0$, so either $C = 0$ or $\exp(\beta a) = \exp(-\beta a)$. The latter can only be true if β or a are zero, but we know that neither of these is the case. Hence $C = D = 0$, i.e. $X \equiv 0$ is the only option which we don't want, and thus we also have to discount the case $\lambda > 0$.

3. Now suppose that $\lambda < 0$, and so we can write $\lambda = -\beta^2$ for some real valued $\beta > 0$. Then the ODE for X takes the form $X'' + \beta^2 X = 0$, which has the general solution

   $$X(x) = A\cos(\beta x) + B\sin(\beta x),$$

   where A, B are arbitrary constants. But now the homogeneous boundary conditions (2) tell us that

   $$\begin{aligned} 0 &= X(0) = A\cos(0) + B\sin(0) = A, \\ 0 &= X(a) = A\cos(\beta a) + B\sin(\beta a) \implies B\sin(\beta a) = 0, \end{aligned}$$

   and hence we only have non-zero (X) solutions when $\sin(\beta a) = 0$, i.e. when $\beta a = n\pi$ for some integer n. Since β, l, π are all positive we can also conclude that n must be a positive integer: $n = 1, 2, 3, \ldots$. So for any integer n, $\lambda = -\beta^2 = -(n\pi/a)^2$ admits a non-identiacally zero solution X, we can index this family of λ's and X's with a subscript n. These are the eigenvalues/functions for this Sturm-Liouville problem.

Notation

Recall that separation of variables is a *series solution* method. This means that we are doing to find as many potential solutions that we can, and then by linearity and homogeneity (of the PDE as well as some of the BCs considered upto the point at which we sum) add them all up to form a potential general solution to our problem.

From above we know (at the end of the Sturm-Liouville stage) that we have infinitely many potential X's for this problem corresponding to different values of λ. To keep track of these, and to make use of them below, we introduce the notation of a subscript n – for this problem:

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{7}$$

where each of these solutions X_n may also be multiplied by an arbitrary constant and still satisfy the X ODE. We can safely drop this constant if we wish at this point, as arbitrary constants can be included when we form our infinite sum.

Note that for some cases you might get possible solutions for more than just one of the λ options above. In particular you may find that you also get a possible non-trivial solution for $\lambda = 0$, in case we would supplement the above with

$$\lambda_0 = 0, \quad X_0(x) = \ldots.$$

So up to this point we have solved the Sturm-Liouville problem for one of the ODEs, this told us the potential λ (which we labelled with a subscript n to keep track of them all) that provide non-trivial X's (and hence u's). We now take each of these λ_n's, substitute into the Y ODE and solve it to find the corresponding Y_n.

Now, for $\lambda = \lambda_n$ the ODE for $Y = Y_n$ is

$$0 = Y_n'' + \lambda_n Y_n = Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n,$$

The general solution to this ODE is

$$Y_n(y) = C \exp\left(\frac{n\pi y}{a}\right) + D \exp\left(-\frac{n\pi y}{a}\right),$$

for arbitrary constants C and D. The homogeneous boundary condition (3) tells us that

$$0 = u(x, 0) = X(x)Y(0), \quad \text{for all} \quad 0 < x < b,$$

and hence we must have that

$$0 = Y_n(0) = C + D \implies D = -C,$$

and so

$$Y_n(y) = C\left(\exp\left(\frac{n\pi y}{a}\right) - \exp\left(-\frac{n\pi y}{a}\right)\right) = 2C \sinh\left(\frac{n\pi y}{a}\right).$$

(NB. C is already an arbitrary constant and so we can just drop the 2.) You could leave things in terms of exp's rather than rewriting as a hyperbolic function if you wish.

At the stage we therefore have an infinite number of separated solutions to (1) of the form (recall the homework exercise from week 1 – H1.4)

$$u_n(x, y) = X_n(x)Y_n(y) = E_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \quad \text{for } n = 1, 2, 3, \ldots, \tag{8}$$

where E_n are arbitrary constants. Any linear combination of solutions of this type is also a solution to the linear PDE (1) satisfying the homogeneous boundary conditions (2) and (3), i.e.

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{9}$$

We now need to consider our inhomogeneous boundary condition (4). We know that $u(x, b) = \phi(x)$, hence using (9) we have

$$\phi(x) = u(x, b) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \quad \text{for} \quad 0 < x < a. \tag{10}$$

Notice that this is the form of a Fourier sine series of an odd periodic function $\phi(x)$ with a period of $2a$. Thus we know that for a given function ϕ we can compute the constants E_n via

$$E_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{n\pi x}{a}\right) dx, \quad n = 1, 2, 3, \ldots$$

Finally, the solution to the PDE and all associated BCs is thus (9) with the values for the constants E_n implied by this:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \left[\int_0^a \phi(z) \sin\left(\frac{n\pi z}{a}\right) dz\right] \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

where we've changed the independent, dummy variable in the integral to a z to avoid confusion with the independent variables x and y.

For the case where $\phi(x) = x$ (equivalently $\phi(z) = z$) and $a = \pi$ we simply need to evaluate the integrals in the above (ignore the sinh part of the constant for now):

$$
\begin{aligned}
G_n :&= \frac{2}{a}\int_0^a \phi(z)\sin\left(\frac{n\pi z}{a}\right)\,dz \\
&= \frac{2}{\pi}\int_0^\pi z\sin(nz)\,dz \\
&= \frac{2}{\pi}\left[-\frac{z\cos(nz)}{n} + \frac{\sin(nz)}{n^2}\right]_0^\pi \qquad \text{from hint; recall that } n \neq 0 \\
&= \frac{2}{\pi}\left[-\frac{(-1)^n\pi}{n} + 0 + 0 - 0\right] \qquad n \text{ is an integer} \implies \cos(n\pi) = (-1)^n,\ \cos(n\pi) = 0 \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}
$$

The solution in this case is therefore

$$
u(x,y) = \sum_{n=1}^{\infty}\left\{\frac{2(-1)^{n+1}}{n\sinh(nb)}\right\}\sin(nx)\sinh(ny).
$$

This is plotted in figure 1. We can see here the strange behaviour at the top right corner. If you consider the BCs they were actually inconsistent at this point. We posed this problem so that the solution could be written in a relatively nice form, and it also displays the property of elliptic problems where non-smooth behaviour at boundaries is rapidly "smoothed" in the interior.

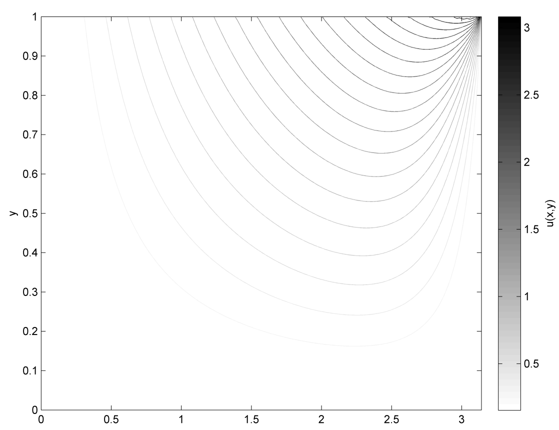


Figure 1: Contour plot of the solution to Laplace's equation with boundary condition $\phi(x) = x$ at $y = b$ (plotted with the assumption that $b = 1$).

Consider now the case that $a = b = 1$ and $\phi(x) = \sin(2\pi x)$, then (10) tells us that

$$
\sin(2\pi x) = \sum_{n=1}^{\infty} E_n \sinh(n\pi)\sin(n\pi x) \quad \text{for} \quad 0 < x < 1.
$$

By observing that the sin of something times x on the LHS, looks similar to the sin's in the sum on the RHS we can conclude immediately what the constants E_n must be without doing the Fourier integral. We can expand the RHS:

$$
E_1\sinh(\pi)\sin(\pi x) + E_2\sinh(2\pi)\sin(2\pi x) + E_3\sinh(3\pi)\sin(3\pi x) + \ldots,
$$

and by matching up terms with what's on the LHS note that the choices $E_1 = 0$, $E_2 = 1/\sinh(2\pi)$, $E_3 = E_4 = \ldots = 0$ will mean this sum equals the LHS we want above.

Plugging these constants back into our general soln (9) yields the unique solution for this BC to be

$$u(x,y) = \frac{\sin(2\pi x)\sinh(2\pi y)}{\sinh(2\pi)}.$$

This solution is plotted below. Again the solution becomes smoother in the interior. This is the pattern of temperature you would see at steady state if the four boundaries had prescribed values for temperature.

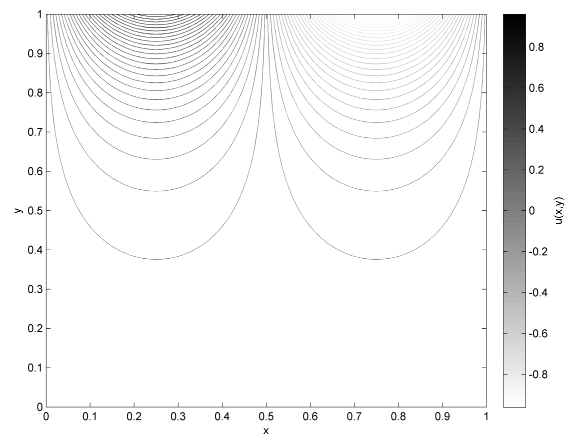


Figure 2: Contour plot of the soln to Laplace's eqn with BC $\phi(x) = \sin(2\pi x)$ at $y = 1$.

C3.4: Separation of variables applied to the wave equation

Consider the wave equation

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 \le x \le l, \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0, \tag{2}$$

and some initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \tag{3}$$

The solution method termed *separation of variables* attempts to build up the general solution to the problem as a linear combination (so we're obviously applying separation of variables to linear problems only!) of special ones that are easy to find.

We start by considering a *separated solution* of the form

$$u(x,t) = X(x)T(t). \tag{4}$$

We're using the notational convention that the lower case letter is the independent variable appearing in the original problem, and the upper case letter is the associated function of, this, one variable. NB. when proposing a particular form for a solution to a problem you sometimes hear the term *solution ansatz*. Let's also assume that we discount the zero solution $u \equiv 0$, we may be forced to do this due to non-zero initial or boundary conditions for example.

We start by looking for as many separated solutions as possible, we will then use our auxiliary conditions to get rid of some. Plugging this form for the solution into the original problem yields

$$X(x)T''(t) = c^2 X''(x)T(t),$$

(NB. X and T are functions of one variable and so we use a prime ($'$) to indicate differentiation w.r.t this single variable) which can be rearranged to

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} =: \lambda. \tag{5}$$

We have introduced the minus sign here just so that the later discussions make use of a positive λ, but this is not important to achieving the same final answer. Now since the first expression is independent of x this implies that λ must also be independent of x, but the second expression tells us that λ is also independent of t, hence λ must be a constant. We have this reduced our PDE to two ODEs which we will now solve.

It can be shown, but we will always just assume it here, that λ is a real number. Depending on the sign of λ we can solve the ODEs for a number of solutions that are candidates to be summed to form the general solution to the PDE, but we will be able to discount some due to the boundary conditions.

The next step is to consider the solutions that satisfy the *homogeneous* boundary conditions (as the sum of these solutions will still satisfy both the *linear* PDE and the *homogeneous* BCs – lecture 1), this is the Sturm-Liouville part of the process.

1. Suppose that $\lambda = 0$, then the solution to the ODE $X'' + \lambda X = 0$ is obviously $X(x) = C + Dx$, but our homogeneous boundary conditions $u(0,t) = u(l,t) = 0$ implies that $X(0)T(t) = X(l)T(t) = 0 \; \forall t$, i.e. $X(0) = X(l) = 0$ (as per the argument used in C3.3) and so $C = D = 0$. But we discounted the zero solution and hence λ cannot take the value 0.

2. Now let's suppose that $\lambda < 0$, i.e. we can write $\lambda = -\beta^2$ for any real value of β. Then our ODE for X takes the form $X'' = \beta^2 X$ which has the general solution $X(x) = C\exp(\beta x) + D\exp(-\beta x)$. Then the homogeneous boundary conditions tell us that $0 = X(0)T(t) = X(l)T(t)$ for all t, and hence we must have $0 = X(0) = X(l)$, i.e. $0 = X(0) = C + D$ and $0 = X(l) = C\exp(\beta l) + D\exp(-\beta l)$. But these two conditions can only be satisfied if $C = D = 0$. So again we can discount the case $\lambda < 0$.

3. Now suppose that $\lambda > 0$, and so we can write $\lambda = \beta^2$ for some $\beta > 0$. Then we have a pair of separate ODEs for $X(x)$ and $T(t)$:

$$X'' + \beta^2 X = 0, \quad T'' + c^2\beta^2 T = 0. \tag{6}$$

We know that problems of this form have the general solutions

$$X(x) = C\cos(\beta x) + D\sin(\beta x), \tag{7}$$
$$T(t) = A\cos(\beta ct) + B\sin(\beta ct), \tag{8}$$

where A, B, C, D are constants.

Consider now the boundary conditions (2), and use these to restrict the possible X solutions. In this case this yields

$$u(0,t) = 0 \implies C = 0 \quad \text{and} \quad u(l,t) = 0 \implies D\sin(\beta l) = 0.$$

We are not interested in the solution that also has $D = 0$ and hence we must have $\sin(\beta l) = 0$, or $\beta l = n\pi$ for some integer n. Since β, l, π are all positive we can also conclude that n must be a positive integer: $n = 1, 2, 3, \ldots$.

So we have the following set of solutions to the X part of (5)

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{9}$$

where each of these solutions may also be multiplied by an arbitrary constant.

At the stage we therefore have an infinite number of separated solutions to (1) of the form

$$u_n(x,t) = X_n(t)T(t) = \sin\left(\frac{n\pi x}{l}\right)\left(A_n\cos\left(\frac{n\pi ct}{l}\right) + B_n\sin\left(\frac{n\pi ct}{l}\right)\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{10}$$

where A_n and B_n are arbitrary constants. Any linear combination of solutions of this type is also a solution to the linear PDE (1) satisfying the homogeneous boundary conditions (2), e.g.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left(A_n\cos\left(\frac{n\pi ct}{l}\right) + B_n\sin\left(\frac{n\pi ct}{l}\right)\right). \tag{11}$$

We now need to consider the initial conditions for the problem, (3). We know that $u(x,0) = \phi(x)$, hence using (11) we have (as $\cos(0) = 1$ and $\sin(0) = 0$)

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right). \tag{12}$$

Recall that this is the form of a Fourier sine series of an odd periodic function $\phi(x)$ with a period of $2l$. Thus we know that for a given function ϕ we can compute the constants A_n as

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx, \quad n = 1, 2, 3, \ldots$$

We also know that $u_t(x,0) = \psi(x)$, so differentiating (11) w.r.t. t and substituting $t = 0$ gives us

$$\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right). \tag{13}$$

Again we can use what we know about Fourier series to find the constants B_n for a given function ψ:

$$\frac{n\pi c}{l}B_n = \frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$$
$$\implies B_n = \frac{2}{n\pi c}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx, \quad n = 1, 2, 3, \ldots$$

MM4 – solutions to homework exercises – week 3

H3.1. Find all the (real-valued) eigenvalues (λ) and eigenfunctions ($X(x)$) of the following problems:

(a) $X'' - \lambda X = 0$, in $-\pi < x < \pi$, with $X(-\pi) = X(\pi),\ X(-\pi)' = X'(\pi)$.
(b) $X'' - \lambda X = 0$, in $0 < x < l$, with $X'(0) = X(l) = 0$.
(c) $X'' - \lambda X = 0$, in $0 < x < l$, with $X(0) = X'(l) = 0$.

H3.1.(a) Answer: Let's consider the three cases:

$\lambda = 0$ gives $X'' = 0 \implies X(x) = Ax + B$, for constants A and B. The BCs then tell us that $X(-\pi) = X(\pi) \implies -A\pi + B = A\pi + B \implies A = 0$. The other BC does not give us any further information, and so we have the solution $X(x) = B$ an arbitrary constant, for $\lambda = 0$.

$\lambda < 0$ i.e. $\lambda = -\beta^2$ for some real non-zero β. Hence, we have $X'' + \beta^2 X = 0$ which has the general solution $X(x) = A\cos(\beta x) + B\sin(\beta x)$, for constants A and B. The BCs now tell us that $X(-\pi) = X(\pi) \implies A\cos(-\beta\pi) + B\sin(-\beta\pi) = A\cos(\beta\pi) + B\sin(\beta\pi) \implies B\sin(\beta\pi) = 0$ ($\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$), hence $B = 0$ or $\beta = n$ for some integer n. The other BC tells us similarly that $A = 0$ or $\beta = n$ for some integer n. So we have a non-zero solution only if $\beta = n$, for $n = 1, 2, 3, \ldots$, i.e. the eigenfunctions are $\lambda = \lambda_n = -n^2$ and the corresponding eigenfunctions are $X_n(x) = A_n\cos(nx) + B_n\sin(nx)$, for $n = 1, 2, 3, \ldots$, where A_n and B_n are arbitrary constants.

$\lambda > 0$ i.e. $\lambda = \beta^2$ for some real non-zero β. Hence, we have $X'' - \beta^2 X = 0$ which has the general solution $X(x) = A\exp(\beta x) + B\exp(-\beta x)$, for constants A and B. The BCs now tell us that $X(-\pi) = X(\pi) \implies A\exp(\beta\pi) + B\exp(-\beta\pi) = A\exp(-\beta\pi) + B\exp(\beta\pi)$ and $X'(-\pi) = X'(\pi) \implies A\beta\exp(\beta\pi) - B\beta\exp(-\beta\pi) = A\beta\exp(-\beta\pi) - B\beta\exp(\beta\pi)$. Adding β times the first expression to the second tells us that $2A\beta(\exp(\beta\pi) - \exp(-\beta\pi)) = 0$ and hence $A = 0$ as $\beta \neq 0$, therefore we must also have $B = 0$ and there are no non-zero solutions.

Therefore, the only non-zero solutions are those listed above for $\lambda = 0$ and $\lambda < 0$.

H3.1.(b) Answer: Let's consider the three cases:

$\lambda = 0$ gives $X'' = 0 \implies X(x) = Ax + B$, for constants A and B. The BCs then tell us that $0 = X'(0) = A$ and $0 = X(l) = Al + B \implies A = B = 0$. So there are no non-zero solutions in this case.

$\lambda < 0$ i.e. $\lambda = -\beta^2$ for some real non-zero β. Hence, we have $X'' + \beta^2 X = 0$ which has the general solution $X(x) = A\cos(\beta x) + B\sin(\beta x)$, for constants A and B. The BCs now tell us that $0 = X'(0) = \beta B \implies B = 0$ as $\beta \neq 0$, and $0 = X(l) = A\cos(\beta l)$. So we have a non-zero solution if $\beta l = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$, i.e. the eigenvalues are $\lambda = \lambda_n = -((n + \frac{1}{2})\pi/l)^2$ and the corresponding eigenfunctions are $X_n(x) = A_n\cos((n + \frac{1}{2})\pi x/l)$, for $n = 0, 1, 2, \ldots$, where A_n is an arbitrary constant.

$\lambda > 0$ i.e. $\lambda = \beta^2$ for some real non-zero β. Hence, we have $X'' - \beta^2 X = 0$ which has the general solution $X(x) = A\exp(\beta x) + B\exp(-\beta x)$, for constants A and B. The BCs now tell us that $0 = X'(0) = A\beta - B\beta \implies B = A$ and $0 = X(l) = A(\exp(\beta l) + \beta\exp(-\beta l))$. Hence, $A = B = 0$ since $(\exp(\beta l) + \exp(-\beta l)) \neq 0$ as both terms are positive.

Therefore, the only non-zero solutions are those listed above for $\lambda < 0$.

H3.1.(c) Answer: Let's consider the three cases:

$\lambda = 0$ gives $X'' = 0 \implies X(x) = Ax + B$, for constants A and B. The BCs then tell us that $0 = X(0) = B$ and $0 = X'(l) = A$. So there are no non-zero solutions in this case.

$\lambda < 0$ i.e. $\lambda = -\beta^2$ for some real non-zero β. Hence, we have $X'' + \beta^2 X = 0$ which has the general solution $X(x) = A\cos(\beta x) + B\sin(\beta x)$, for constants A and B. The BCs now tell us that $0 = X(0) = A$, and $0 = X'(l) = B\beta\cos(\beta l)$. So we have a non-zero solution if $\beta l = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$, i.e. the eigenvalues are $\lambda = \lambda_n = -((n + \frac{1}{2})\pi/l)^2$ and the corresponding eigenfunctions are $X_n(x) = B_n \sin((n + \frac{1}{2})\pi x/l)$, for $n = 0, 1, 2, \ldots$, where A_n is an arbitrary constant.

$\lambda > 0$ i.e. $\lambda = \beta^2$ for some real non-zero β. Hence, we have $X'' - \beta^2 X = 0$ which has the general solution $X(x) = A\exp(\beta x) + B\exp(-\beta x)$, for constants A and B. The BCs now tell us that $0 = X(0) = A + B \implies B = -A$ and $0 = X'(l) = A\beta(\exp(\beta l) - \beta\exp(-\beta l))$. Hence, $A = B = 0$ since $(\exp(\beta l) - \exp(-\beta l)) \neq 0$ as $\beta l \neq 0$.

Therefore, the only non-zero solutions are those listed above for $\lambda < 0$.

H3.2. & H3.3. See separate sheets.

H3.4. Calculate the solution to the problem

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 \leq x \leq l,$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0,$$

and some general initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),$$

when $l = 1$ and in the two cases where $\phi(x) = x(1-x)$, $\psi \equiv 0$ and $\phi(x) = \sin(5\pi x) + 2\sin(7\pi x)$, $\psi \equiv 0$.

H3.4. Answer: From lectures we know that the general solution to the wave equation is (substituting $l = 1$)

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(A_n \cos(n\pi ct) + B_n \sin(n\pi ct)\right), \tag{1}$$

where

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)\,dx \quad n = 1, 2, 3, \ldots,$$

and

$$B_n = \frac{2}{n\pi c}\int_0^1 \psi(x)\sin(n\pi x)\,dx \quad n = 1, 2, 3, \ldots.$$

In the case where $\phi(x) = x(1-x),\ \psi \equiv 0$ we can see that $B_n = 0$ for $n = 1, 2, 3, \ldots$, and

$$A_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = 2\int_0^1 x\sin(n\pi x)\,dx - 2\int_0^1 x^2\sin(n\pi x)\,dx \quad n = 1, 2, 3, \ldots,$$

By integration by parts we can show that

$$\begin{aligned}\int x\sin(x)\,dx &= -x\cos(x) + \int \cos(x)\,dx \\ &\qquad \text{(int. by parts with: } u = x,\ dv = \sin(x) \implies du = dx,\ v = -\cos(x)) \\ &= -x\cos(x) + \sin(x).\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^1 x\sin(n\pi x)\,dx &= \left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^1 \\ &= -\frac{1}{n\pi}\cos n\pi + \frac{\sin(n\pi)}{(n\pi)^2} + \frac{0}{n\pi}\cos 0 - \frac{\sin(0)}{(n\pi)^2} \\ &= -\frac{1}{n\pi}\cos n\pi + 0 + 0 - 0.\end{aligned}$$

Similarly

$$\begin{aligned}\int x^2\sin(n\pi x)\,dx &= -\frac{x^2\cos(n\pi x)}{n\pi} + \int 2x\frac{\cos(n\pi x)}{n\pi}\,dx \\ &\qquad \text{(int. by parts with: } u = x^2,\ dv = \sin(n\pi x) \implies du = 2xdx,\ v = -\cos(n\pi x)/n\pi)\end{aligned}$$

and we can apply integration by parts to the second term in the above

$$\begin{aligned}\int 2x\frac{\cos(n\pi x)}{n\pi}\,dx &= \frac{2x\sin(n\pi x)}{(n\pi)^2} - \int 2\frac{\sin(n\pi x)}{(n\pi)^2}\,dx \\ &\qquad \text{(int. by parts with: } u = 2x,\ dv = \cos(n\pi x)/n\pi \implies du = 2dx,\ v = \sin(n\pi x)/(n\pi)^2) \\ &= \frac{2x\sin(n\pi x)}{(n\pi)^2} + \frac{2\cos(n\pi x)}{(n\pi)^3},\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 x^2\sin(n\pi x)\,dx &= \left[-\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2x\sin(n\pi x)}{(n\pi)^2} + \frac{2\cos(n\pi x)}{(n\pi)^3}\right]_0^1 \\ &= \cos(n\pi)\frac{2 - n^2\pi^2}{(n\pi)^3} - \frac{2}{(n\pi)^3}.\end{aligned}$$

Hence,

$$
\begin{aligned}
A_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx &= 2\int_0^1 x\sin(n\pi x)\,dx - 2\int_0^1 x^2\sin(n\pi x)\,dx \\
&= -\frac{2}{n\pi}\cos n\pi - \cos(n\pi)\frac{4-2n^2\pi^2}{(n\pi)^3} + \frac{4}{(n\pi)^3} \\
&= \frac{4}{(n\pi)^3}(1-\cos(n\pi)) \\
&= \begin{cases} 0 & n \text{ even} \\ \dfrac{8}{(n\pi)^3} & n \text{ odd} \end{cases}, \quad n = 1, 2, 3, \ldots,
\end{aligned}
$$

And so finally we have

$$u(x,t) = \sum_{n(odd)=1}^{\infty} \frac{8}{n^3\pi^3}\sin(n\pi x)\cos(cn\pi t).$$

The next case is far easier. From the form of the function ϕ we can see that

$$A_n = \begin{cases} 1 & \text{if } n = 5 \\ 2 & \text{if } n = 7 \\ 0 & \text{otherwise} \end{cases},$$

and so the solution is

$$u(x,t) = \sin(5\pi x)\cos(5c\pi t) + 2\sin(7\pi x)\cos(7c\pi t).$$

H3.2: Separation of variables applied to Laplace's equation with mixed Dirichlet and Neumann BCs

Consider Laplace's equation for the unknown variable $u \equiv u(x, y)$

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{in} \quad 0 < x < a,\ 0 < y < b. \tag{1}$$

Subject to the boundary conditions

$$u(0, y) = 1 \quad \text{for} \quad 0 < y < b, \tag{2}$$
$$u(a, y) = \cos(\pi y/b) \quad \text{for} \quad 0 < y < b, \tag{3}$$
$$u_y(x, 0) = u_y(x, b) = 0 \quad \text{for} \quad 0 < x < a. \tag{4}$$

The solution method termed *separation of variables* attempts to build up the general solution to the problem as a linear combination (so we're obviously applying separation of variables to linear problems only!) of special ones that are easy to find.

We start by considering a *separated solution* of the form

$$u(x, y) = X(x)Y(y). \tag{5}$$

We're using the notational convention that the lower case letter is the independent variable appearing in the original problem, and the upper case letter is the associated function of, this, one variable. NB. when proposing a particular form for a solution to a problem you sometimes hear the term *solution ansatz*. Let's also assume that we discount the zero solution $u \equiv 0$, we may be forced to do this due to non-zero initial or boundary conditions for example.

We start by looking for as many separated solutions as possible, we will then use our auxiliary conditions to get rid of some. Plugging this form for the solution into the original problem yields

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

(NB. X and Y are functions of one variable and so we use a prime ($'$) to indicate differentiation w.r.t this single variable) which can be rearranged to

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} =: \lambda \implies X'' - \lambda X = 0 \ \ \& \ \ Y'' + \lambda Y = 0. \tag{6}$$

Now since the first expression is independent of y this implies that λ must also be independent of y, but the second expression tells us that λ is also independent of x, hence λ must be a constant. We have this reduced our PDE to two ODEs which we will now solve.

It can be shown, but we will always just assume it here, that λ is a real number. Depending on the sign of λ we can solve the ODEs for a number of solutions that are candidates to be summed to form the general solution to the PDE, but we will be able to discount some due to the boundary conditions.

The next step is to consider the solutions that satisfy the *homogeneous* boundary conditions (as the sum of these solutions will still satisfy both the *linear* PDE and the *homogeneous* BCs – lecture 1), this is the Sturm-Liouville part of the process.

Notice that the problem associated with the pair of homogeneous BCs is $Y'' + \lambda Y = 0$ subject to $Y'(0) = Y'(b) = 0$. This is equivalent to a Sturm-Liouville problem we considered in class (C3.2(b)). To see this note that the BCs

$$u_y(x, 0) = u_y(x, b) = 0 \quad \text{for} \quad 0 < x < a \implies X(x)Y'(0) = X(x)Y'(b) = 0 \quad \text{for} \quad 0 < x < a.$$

We don't want the trivial solution which would result from $X \equiv 0$ and hence we must have $Y'(0) = Y'(b) = 0$.

Therefore it has the solutions: $Y_0(y) = A_0$ for $\lambda_0 = 0$, and $Y_n(y) = A_n \cos(n\pi y/b)$ for $\lambda_n = (n\pi/b)^2$ for $n = 1, 2, 3, \ldots$, where A_0 and A_n $n = 1, 2, 3, \ldots$ are arbitrary constants. Note that there has been a change of sign in λ from how we considered it in class, coming from the precise form of (6), but $\cos(-x) = \cos(x)$.

The corresponding solution to the other ODE in these cases are: $X_0(x) = C_0 + D_0 x$ and $X_n(x) = C_n \exp(n\pi x/b) + D_n \exp(-n\pi x/b)$, where C_0, C_n and D_n for $n = 1, 2, 3, \ldots$ are arbitrary constants.

The most general solution to the PDE and the homogeneous BCs is therefore

$$u(x, y) = (E_0 + F_0 x) + \sum_{n=1}^{\infty} (E_n \exp(n\pi x/b) + F_n \exp(-n\pi x/b)) \cos(n\pi y/b),$$

where the E's and F's are arbitrary constants made up from the A's, C's and D's.

We now need to consider the inhomogeneous BCs. (2) tells us that

$$1 = u(0, y) = E_0 + \sum_{n=1}^{\infty} (E_n + F_n) \cos(n\pi y/b), \quad \text{for} \quad 0 < y < b.$$

This is a Fourier cosine series and we notice that we can straight away conclude that $E_0 = 1$ and $F_n = -E_n$ for $n = 1, 2, 3, \ldots$, so that all other terms cancel. Always try to see if you can conclude directly what the coefficients in the Fourier series are before computing the integrals. Our solution (to (1), (2) and (4), but not (3) and that hasn't yet been considered) now takes the form

$$u(x, y) = (1 + F_0 x) + \sum_{n=1}^{\infty} (G_n \sinh(n\pi x/b)) \cos(n\pi y/b),$$

(from the relationship between the E's and F's, and recalling the definition of the hyperbolic function $\sinh(x) = (\exp(x) - \exp(-x))/2)$), for arbitrary constants F_0 and G_n, $n = 1, 2, 3, \ldots$.

We now consider the final remaining BC. (3) tells us that

$$\cos(\pi y/b) = u(a, y) = (1 + F_0 a) + \sum_{n=1}^{\infty} (G_n \sinh(n\pi a/b)) \cos(n\pi y/b), \quad \text{for} \quad 0 < y < b.$$

Again, this is a Fourier cosine series and we can spot immediately (from the particular form the LHS takes here – always see if you can spot this type of shortcut) that we must have

$$\begin{aligned} 0 = 1 + F_0 a &\implies F_0 = -\frac{1}{a}, \\ 1 = G_1 \sinh(\pi a/b) &\implies G_1 = \frac{1}{\sinh(\pi a/b)}, \\ 0 = G_n \sinh(n\pi a/b) &\implies G_n = 0, \ n = 2, 3, 4, \ldots. \end{aligned}$$

Finally, the solution to the PDE and associated BCs is

$$u(x, y) = 1 - \frac{x}{a} + \frac{\sinh\left(\frac{\pi x}{b}\right) \cos\left(\frac{\pi y}{b}\right)}{\sinh\left(\frac{\pi a}{b}\right)}.$$

Now consider the case $u(a, y) = \cos^2(\pi y/b)$. Noting that[1] this can also be written as

$$u(a, y) = \frac{1}{2}(1 + \cos(2\pi y/b)),$$

[1] $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b) \implies \cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1$

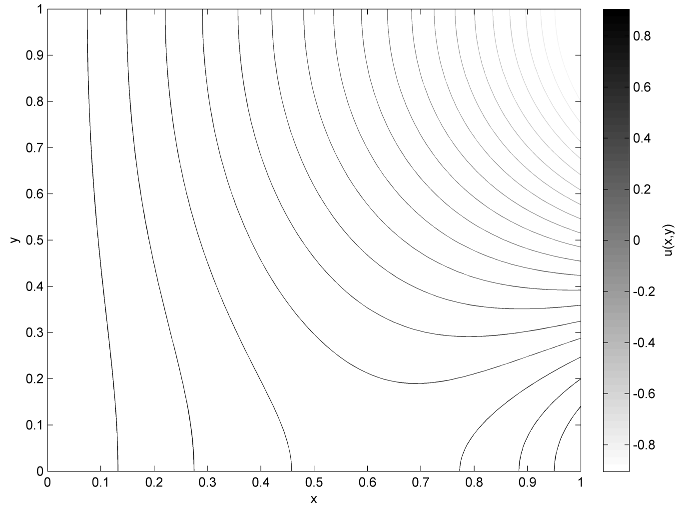


Figure 1: Contour plot of the first computed solution. Can you interpret what the contours are doing at the boundaries in terms of the boundary conditions applied?

we can see from the Fourier series above that

$$\frac{1}{2} = 1 + F_0 a \implies F_0 = -\frac{1}{2a},$$
$$\frac{1}{2} = G_2 \sinh(2\pi a/b) \implies G_2 = \frac{1}{2\sinh(2\pi a/b)},$$
$$0 = G_n \sinh(n\pi a/b) \implies G_n = 0,\ n = 1, 3, 4, \ldots,$$

and the final solution is therefore

$$u(x,y) = 1 - \frac{x}{2a} + \frac{\sinh\left(\frac{2\pi x}{b}\right)\cos\left(\frac{2\pi y}{b}\right)}{2\sinh\left(\frac{2\pi a}{b}\right)}.$$

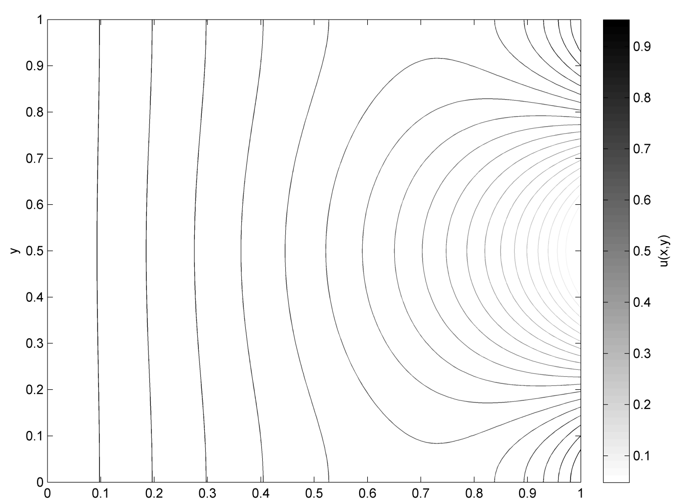


Figure 2: Contour plot of the second computed solution.

H3.3: Separation of variables applied to the diffusion equation

Consider the diffusion equation

$$u_t = ku_{xx} \quad \text{for} \quad 0 \leq x \leq l,\ 0 \leq t \leq \infty \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0, \tag{2}$$

and the initial conditions

$$u(x,0) = \phi(x). \tag{3}$$

The solution method termed *separation of variables* attempts to build up the general solution to the problem as a linear combination (so we're obviously applying separation of variables to linear problems only!) of special ones that are easy to find.

We start by considering a *separated solution* of the form

$$u(x,t) = X(x)T(t). \tag{4}$$

We're using the notational convention that the lower case letter is the independent variable appearing in the original problem, and the upper case letter is the associated function of, this, one variable. NB. when proposing a particular form for a solution to a problem you sometimes hear the term *solution ansatz*. Let's also assume that we discount the zero solution $u \equiv 0$, we may be forced to do this due to non-zero initial or boundary conditions for example.

We start by looking for as many separated solutions as possible, we will then use our auxiliary conditions to get rid of some. Plugging this form for the solution into the original problem yields

$$X(x)T'(t) = kX''(x)T(t),$$

(NB. X and T are functions of one variable and so we use a prime ($'$) to indicate differentiation w.r.t this single variable) which can be rearranged to

$$-\frac{T'(t)}{kT(t)} = -\frac{X''(x)}{X(x)} =: \lambda. \tag{5}$$

We have introduced the minus sign here just so that the later discussions make use of a positive λ, but this is not important to achieving the same final answer. Now since the first expression is independent of x this implies that λ must also be independent of x, but the second expression tells us that λ is also independent of t, hence λ must be a constant. We have this reduced our PDE to two ODEs which we will now solve.

It can be shown, but we will just assume here, that λ is a real number. Depending on the sign of λ we can solve the ODEs for a number of solutions that are candidates to be summed to form the general solution to the PDE, but we will be able to discount some due to the boundary conditions.

The next step is to consider the solutions that satisfy the *homogeneous* boundary conditions (as the sum of these solutions will still satisfy both the *linear* PDE and the *homogeneous* BCs – lecture 1).

1. Suppose that $\lambda = 0$, then the solution to the ODE $X''+\lambda X = 0$ is obviously $X(x) = C+Dx$, but our homogeneous boundary conditions $u(0,t) = u(l,t) = 0$ implies that $X(0)T(t) = X(l)T(t) = 0\ \forall t$, i.e. $X(0) = X(1) = 0$ (as per the argument used in C3.3), and so $C = D = 0$. But we discounted the zero solution and hence λ cannot take the value 0.
2. Now let's suppose that $\lambda < 0$, i.e. we can write $\lambda = -\beta^2$ for some $\beta > 0$. Then our ODE for X takes the form $X'' = \beta^2 X$ which has the general solution $X(x) = C\exp(\beta x) + D\exp(-\beta x)$. Then the homogeneous boundary conditions tell us that $0 = X(0)T(t) = X(l)T(t)$ for all t, and hence we must have $0 = X(0) = X(l)$, i.e. $0 = X(0) = C + D$ and $0 = X(a) = C\exp(\beta a) + D\exp(-\beta a)$. But these two conditions can only be satisfied if $C = D = 0$. So again we can discount the case $\lambda < 0$.

3. Now suppose that $\lambda > 0$, and so we can write $\lambda = \beta^2$ for some $\beta > 0$. Then we have the ODE for $X(x)$:
$$X'' + \beta^2 X = 0. \tag{6}$$
We know that problems of this form have the general solutions
$$X(x) = A\cos(\beta x) + B\sin(\beta x), \tag{7}$$
where A, B are constants. Consider now the boundary conditions (2), and use these to restrict the possible X solutions. In this case this yields
$$X(0) = 0 \implies A = 0 \quad \text{and} \quad X(l) = 0 \implies B\sin(\beta l) = 0.$$
We are not interested in the solution that also has $B = 0$ and hence we must have $\sin(\beta l) = 0$, or $\beta l = n\pi$ for some integer n. Since β, l, π are all positive we can also conclude that n must be a positive integer: $n = 1, 2, 3, \ldots$.

So we have the following set of solutions to the X part of (5)
$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{8}$$
where each of these solutions may also be multiplied by an arbitrary constant.

Now, for $\lambda = \lambda_n$ the ODE for $T = T_n$ is
$$0 = T_n' + \lambda_n k T_n = T_n' + \left(\frac{n\pi}{l}\right)^2 k T_n,$$
The general solution to this ODE is
$$T_n(t) = C_n \exp\left(-\left(\frac{n\pi}{l}\right)^2 kt\right),$$
for arbitrary constants C_n.

At the stage we therefore have an infinite number of separated solutions to (1) of the form
$$u_n(x,t) = X_n(t)T_n(t) = \exp\left(-\left(\frac{n\pi}{l}\right)^2 kt\right)\sin\left(\frac{n\pi x}{l}\right), \quad \text{for } n = 1, 2, 3, \ldots. \tag{9}$$
Any linear combination of solutions of this type is also a solution to the linear PDE (1) satisfying the homogeneous boundary conditions (2), e.g.
$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\left(\frac{n\pi}{l}\right)^2 kt\right)\sin\left(\frac{n\pi x}{l}\right), \tag{10}$$
for arbitrary constants A_n.

We now need to consider the initial conditions for the problem (3). We know that $u(x,0) = \phi(x)$, hence using (10) we have
$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right). \tag{11}$$
Recall that this is the form of a Fourier series of an odd periodic function $\phi(x)$ with a period of $2l$. Thus we know that for a given function ϕ we can compute the constants A_n as
$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \quad n = 1, 2, 3, \ldots$$

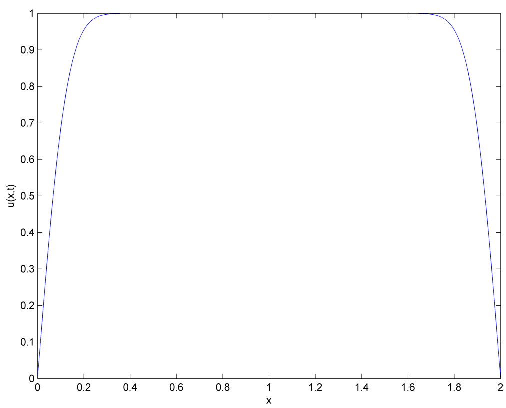
Let's consider a simple problem where the domain is of two units in size in the x direction ($l = 2$) and the initial condition is given by
$$u(x,0) = \phi(x) \equiv 1.$$

Then the constants in the solution to the problem (10) are given by

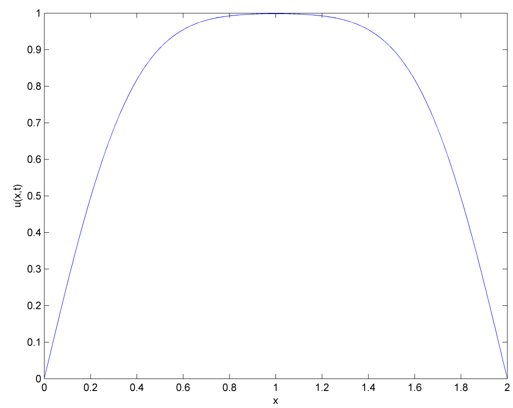
$$
\begin{aligned}
A_n &= \int_0^2 \sin\left(\frac{n\pi x}{2}\right)\, dx \\
&= \left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 \\
&= \frac{2}{n\pi}\left\{1-\cos\left(n\pi\right)\right\} \\
&= \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases}
\end{aligned}
$$

Therefore, the solution to this problem is given by

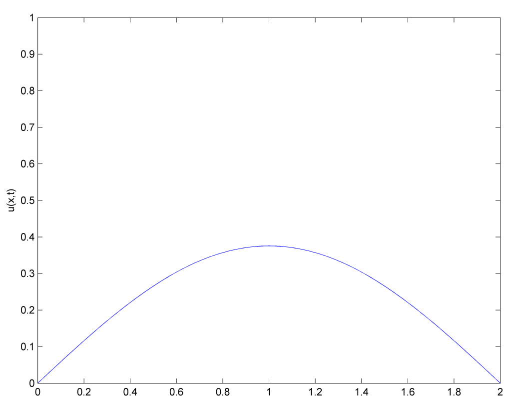
$$
u(x,t) = \sum_{n(odd)=1}^{\infty} \frac{4}{n\pi} \exp\left(-\left(\frac{n\pi}{2}\right)^2 kt\right) \sin\left(\frac{n\pi x}{2}\right). \tag{12}
$$



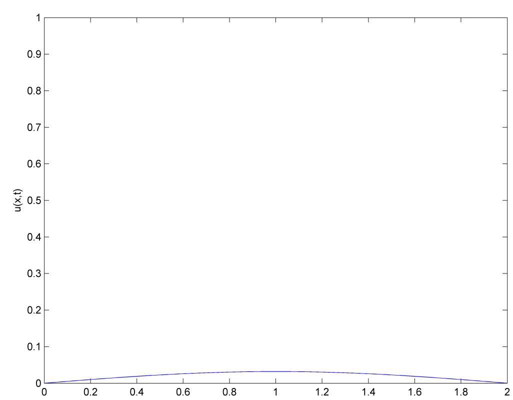
(a) $t = t_1$



(b) $t = t_2$

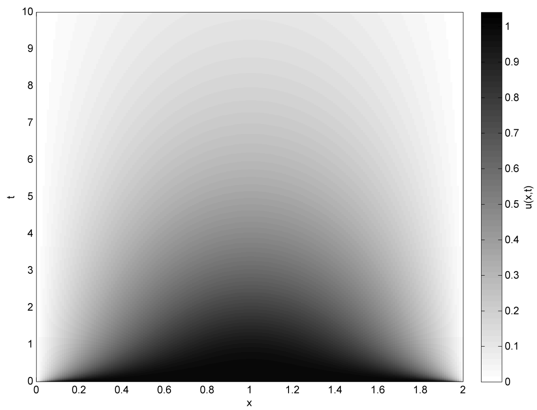


(c) $t = t_3$

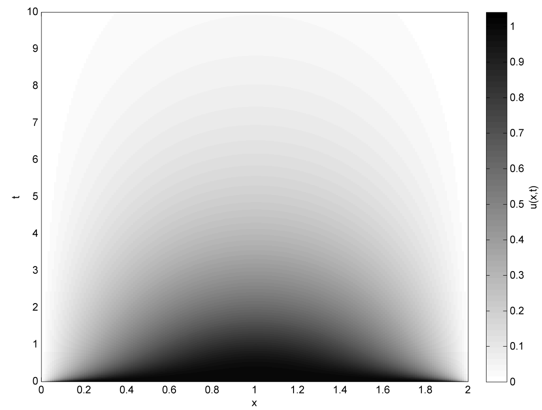


(d) $t = t_4$

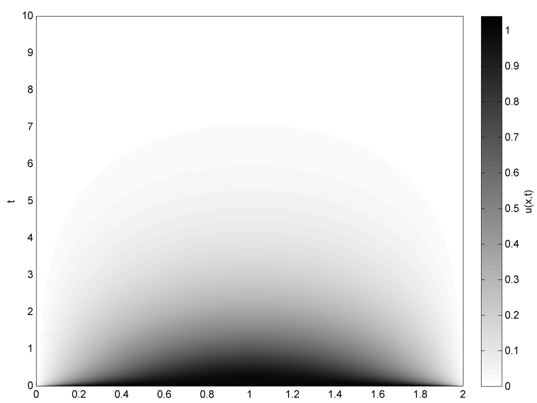
Figure 1: Plots of the solution (12) at a succession of increasing times.



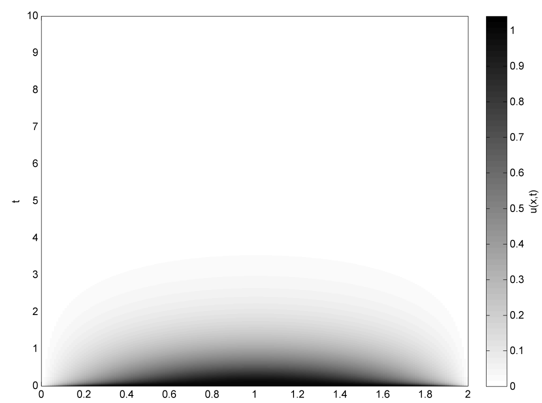
(a) $k = 1.0 \times 10^{-1}$



(b) $k = 1.5 \times 10^{-1}$



(c) $k = 2.5 \times 10^{-1}$



(d) $k = 5.0 \times 10^{-1}$

Figure 2: Contour plots of the solution (12) for the case of four different values of the diffusivity (bigger diffusivity diffuses faster!.