Algorithm for applying a domain decomposition to unstructed FEM data.

1. Read in the velocity field on the unstructured mesh from the vtu file.
2. Create a number of non-overlapping structured grids whose union covers the domain. (This is the strategy we will start with at least, for the slug flow test case we will probably have overlapping grids.)
3. For each grid:
   (a) map the velocity field on the unstructured mesh to the grid;
   (b) reshape to a vector (when applying CAEs, do not reshape);
4. assemble the snapshots matrix;
5. apply POD and obtain a number of POD basis functions (collected in the matrix $\boldsymbol{R}$).

The code `dd_nirom_orig.py` goes through the steps above. It then maps the POD coefficients back to 'full space' (the space of the structured grids) and finally interpolates back to the unstructured mesh:

$$\textit{Mesh} \longrightarrow \textit{Grids} \longrightarrow \textit{Reduce dimension} \longrightarrow \textit{Reconstruct to Grids} \longrightarrow \textit{Mesh.}$$

After this process the error can be calculated, which includes errors from the interpolation steps and a truncation error relating to the dimensionality reduction. The error can be written as follows:

$$\boldsymbol{u}_{\text{FEM}}^{t_k} - \bigcup_{i=1}^{\text{nG}} \mathcal{F}^{G_i 2M} \left(\boldsymbol{R}\boldsymbol{R}^T \, \mathcal{F}^{M2G} \left(\boldsymbol{u}_{\text{FEM}}^{t_k} \right) \right) \tag{1}$$

where (for the 2D case)

- $\boldsymbol{u}_{\text{FEM}}^{t_k} \in \mathbb{R}^{\text{nNodes} \times \text{nDim}}$ is the solution on the finite element (FE) mesh at time level t_k,
- $\mathcal{F}^{M2G} : \boldsymbol{u}_{\text{FEM}}^{t_k} \to \boldsymbol{u}_{\text{G}_i}^{t_k} \in \mathbb{R}^{N^x N^y \, \text{nDim}}$ is a function which interpolates the solution on a mesh to the ith grid G_i,
- $R \in \mathbb{R}^{N^x N^y \, \text{nDim} \times \text{nPOD}}$,
- $\mathcal{F}^{G_i 2M}$ interpolates from the ith grid to the mesh,
- $\bigcup_{i=1}^{\text{nG}} \mathcal{F}^{G_i 2M}$ takes the union of the interpolated solutions (in the code, nodes that have solutions on two grids are detected and the solution is adjusted here).

The second term in the error (the reconstructed solution), shows the stages that will be passed through when doing reduced-order modelling. The term $\boldsymbol{R}^T \mathcal{F}^{M2G}\left(\boldsymbol{u}_{\text{FEM}}^{t_k}\right)$ finds the compressed or POD coefficients of the kth snapshot. The POD coefficients can then be used to train a neural network. Once trained, an initial condition is set and POD coefficients are predicted. The operation $\mathcal{F}^{G_i 2M}\boldsymbol{R}$ is applied to the predicted POD coefficients to reconstruct the solution on the unstructured mesh (although the solution on the grids could be sufficient).

This explanation does not follow the code unfortunately ☹, which involes some reshaping of arrays.

nNodes	number of nodes on the finite element (FE) mesh
nDim	number of physical dimensions
nPOD	number of POD functions
N^x	number of nodes in the x direction on a grid
nG	number of Grids

Table 1: Definition of some symbols.