Modelling and Numerical Methods

Lecture 3

Kinematics of Continua & Conservation Equations

Outline Lecture 3

Part 1: Kinematics

- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Strain Tensor

chapter3.ipynb

Part 2: Conservation Equations

- Conservation of Mass
- Conservation of Momentum

chapter4.ipynb

Learning Objectives
Kinematics

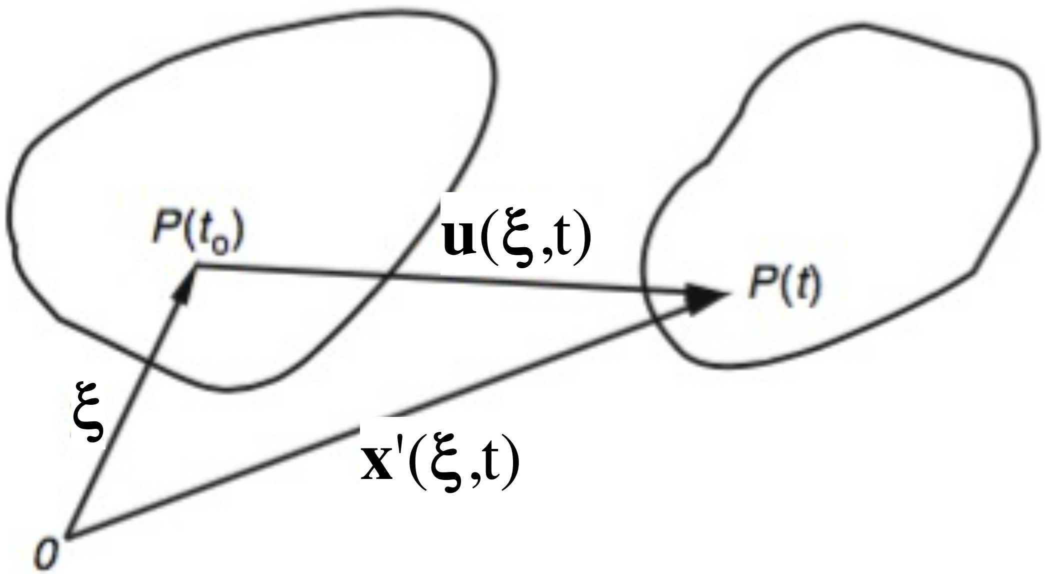
- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain

Displacement

Can result in

(a) Rigid body motion

(b) Deformation of the body



Rigid body motion

- Translation: $\mathbf{x}'=\xi+\mathbf{c}(t)$, with $\mathbf{c}(0)=\mathbf{0}$

  $\Rightarrow$ $\mathbf{u}=\mathbf{x}'-\xi$, each point same $\mathbf{u}(t)=\mathbf{c}(t)$

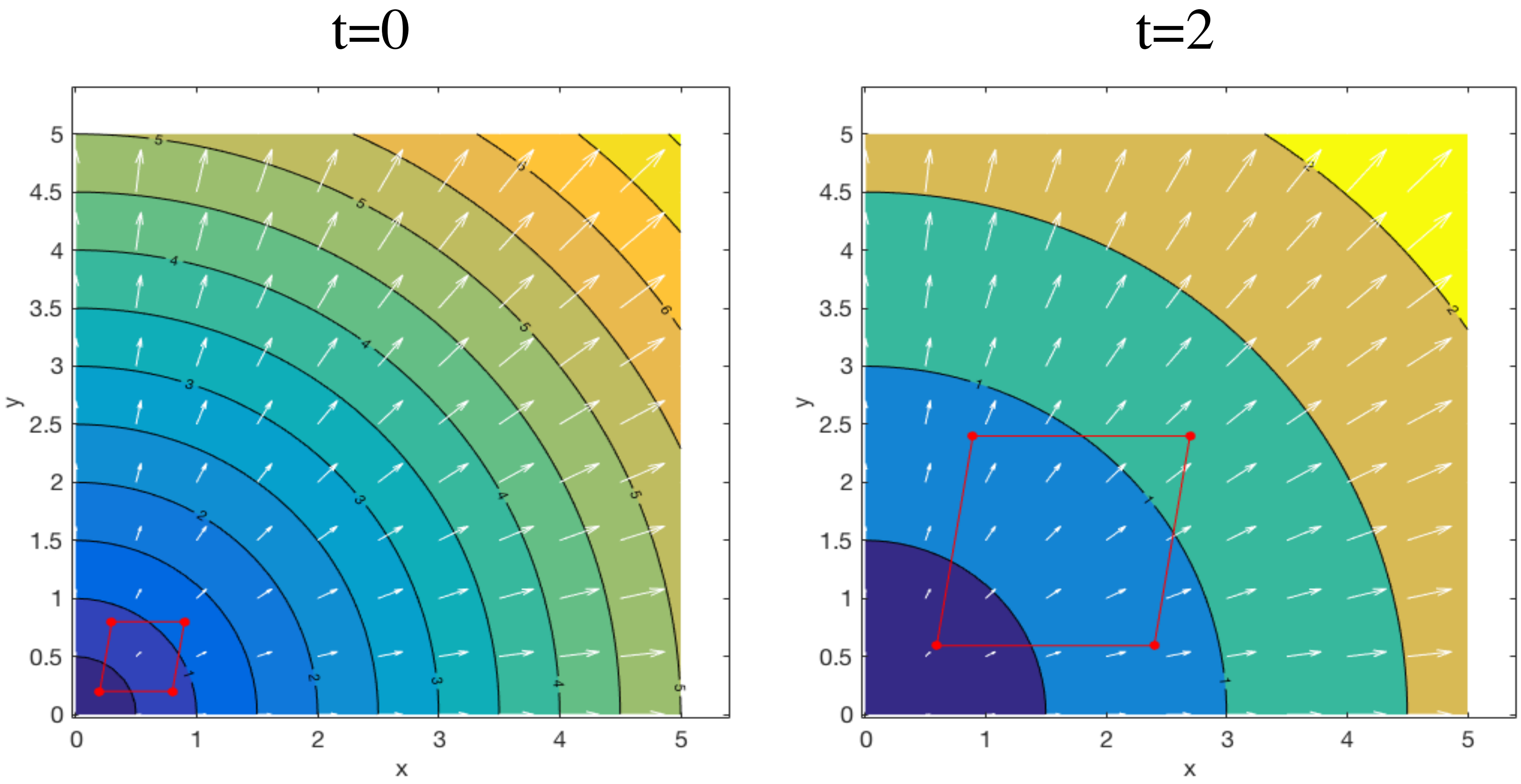
- Rotation: $\mathbf{x}'-\mathbf{b}=\mathbf{R}(t)(\boldsymbol{\xi}-\mathbf{b})$, where $\mathbf{R}(t)$ is rotation tensor, with $\mathbf{R}(0)=\mathbf{I}$, $\mathbf{b}$ is the point of rotation. $\mathbf{R}(t)$ is an orthogonal transformation (preserves lengths and angles, $\mathbf{R}^T\mathbf{R}=\mathbf{I}$, $\det(\mathbf{R})=1$)

If $\mathbf{u}$ depends on $\mathbf{x}$ and t, then internal deformation

Example:

$$v_i = \frac{kx_i}{1+kt}$$

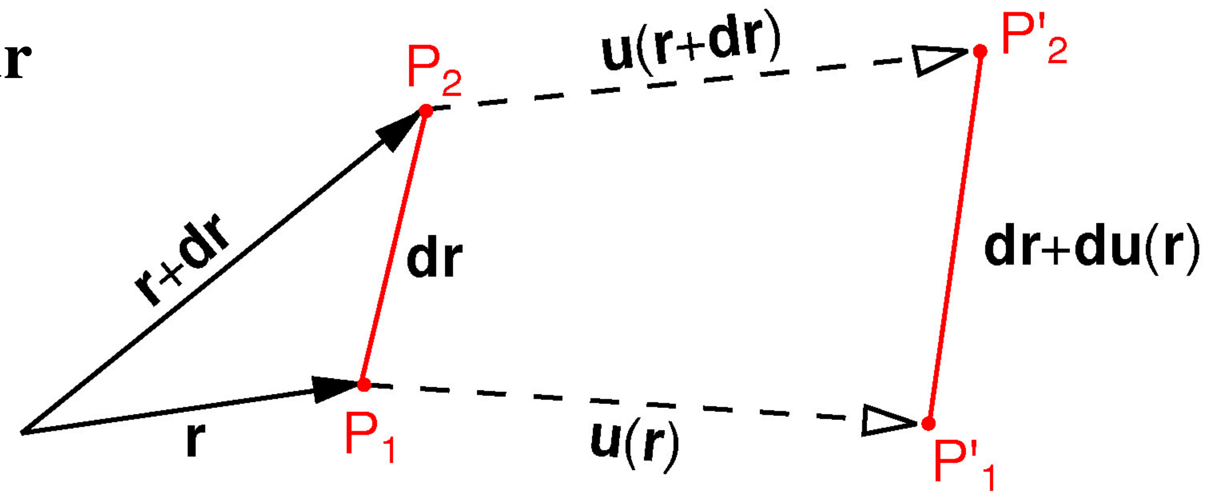
Displacement



translation &
deformation

Deformation tensor

For small **dr**



P_1 at $\mathbf{r}$ -> P'_1 at $\mathbf{r}+\mathbf{u}(\mathbf{r})$, P_2 at $\mathbf{r}+\mathbf{dr}$ -> P'_2 at $\mathbf{r}+\mathbf{dr}+\mathbf{u}(\mathbf{r}+\mathbf{dr})$.

$\mathbf{dr}' = P'_2 - P'_1 = \mathbf{dr} + [\mathbf{u}(\mathbf{r}+\mathbf{dr}) - \mathbf{u}(\mathbf{r})] = \mathbf{dr}+\mathbf{du}(\mathbf{r}) = \mathbf{dr} + \mathbf{dr} \cdot \nabla \mathbf{u}(\mathbf{r})$

deformation of P_2-P_1 described by: $\mathrm{du}_i = \mathrm{dx}_j \dfrac{\partial \mathrm{u}_i}{\partial \mathrm{x}_j}$

$\mathbf{du} = \mathbf{dr} \cdot \nabla \mathbf{u} = \nabla^T \mathbf{u} \cdot \mathbf{dr}$

$$du_i = dx_j \frac{\partial u_i}{\partial x_j} \quad : \qquad \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij}}$$

$$\nabla \mathbf{u}^T = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T)$$

Total deformation is:

- rigid body translation - $\mathbf{u}(\mathbf{r})$
- rigid body rotation - $\boldsymbol{\omega} \cdot \mathbf{dr}$
- internal deformation, strain - $\boldsymbol{\varepsilon} \cdot \mathbf{dr}$ - result of stresses

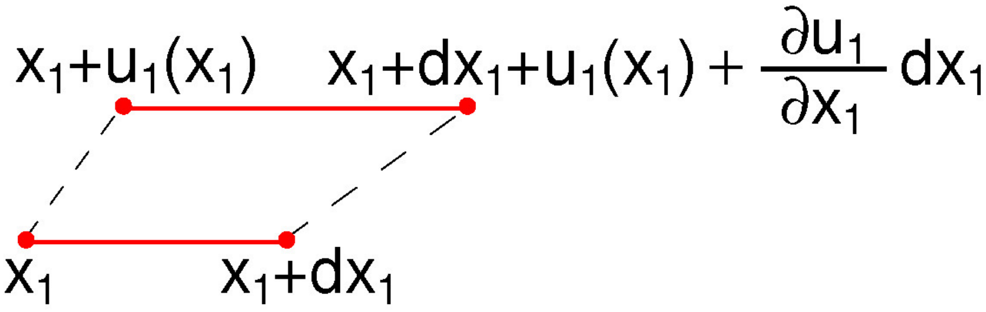
Infinitesimal strain and rotation tensors

$$\varepsilon = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix}$$

<u>diagonal</u> infinitesimal strain tensor elements

For a line segment
dr = $(dx_1,0,0)$
deforming
in displacement
field $\mathbf{u}=(u_1,0,0)$:

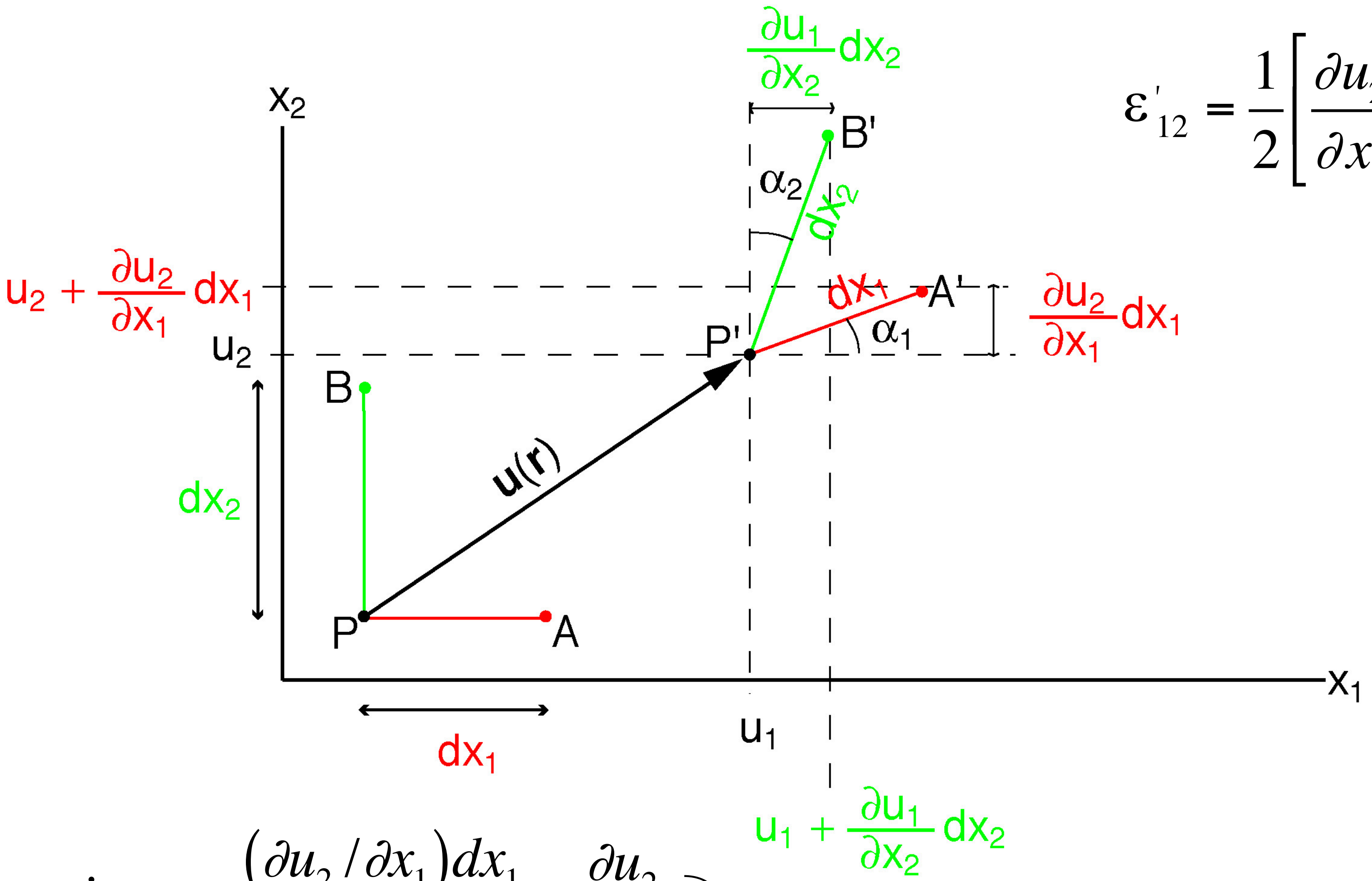


the new length $dx'_1 \approx dx_1 + (\partial u_1/\partial x_1)dx_1 = dx_1+\varepsilon_{11}dx_1$

$\Rightarrow$ $\varepsilon_{11}=[dx'_1 - dx_1]/dx_1$ = the relative change in length of a line element, originally in x_1 direction.

The relative change in volume $(V'-V)/V$ of a cube $V=dx_1dx_2dx_3$
$\approx \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \varepsilon_{ii} = tr(\boldsymbol{\varepsilon}) = \nabla\cdot\mathbf{u}$.

off-diagonal infinitesimal strain tensor elements



$$\varepsilon'_{12} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right]$$

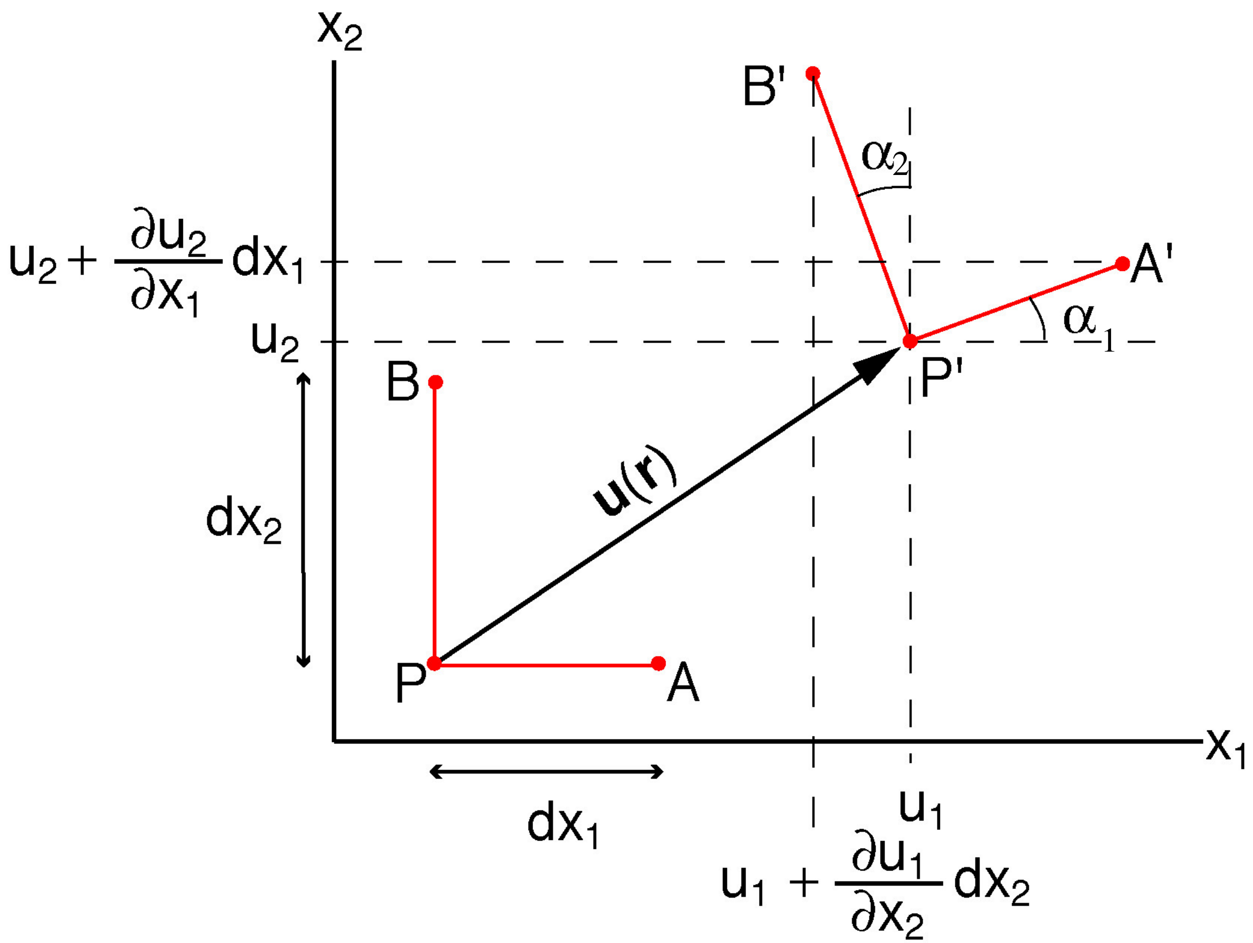
$$\alpha_1 \approx \sin\alpha_1 = \frac{(\partial u_2/\partial x_1)dx_1}{dx_1} = \frac{\partial u_2}{\partial x_1}$$

$$\alpha_2 \approx \frac{(\partial u_1/\partial x_2)dx_2}{dx_2} = \frac{\partial u_1}{\partial x_2}$$

$$\boxed{\varepsilon_{12}=\varepsilon_{21}=(\alpha_1+\alpha_2)/2}$$

$2\varepsilon_{12}$ is the change in angle of an originally 90^o angle between dx_1 and dx_2

infinitesimal rotation tensor elements



$$\omega_{12}=-\omega_{21}= [(\partial u_2/\partial x_1)-(\partial u_1/\partial x_2)]/2=(\alpha_1-\alpha_2)/2$$

ω_{12} is common rigid rotation angle of vectors in the dx_1 - dx_2 plane (around x_3)

Extra: Rotation tensor and rotation vector

For any antisymmetric tensor **W**, a corresponding ***dual*** or ***axial vector*** **w** can be found so that

$$\mathbf{W}\cdot\mathbf{a}=\mathbf{w}\times\mathbf{a}$$

Vector **w** relates to the components of **W** as:

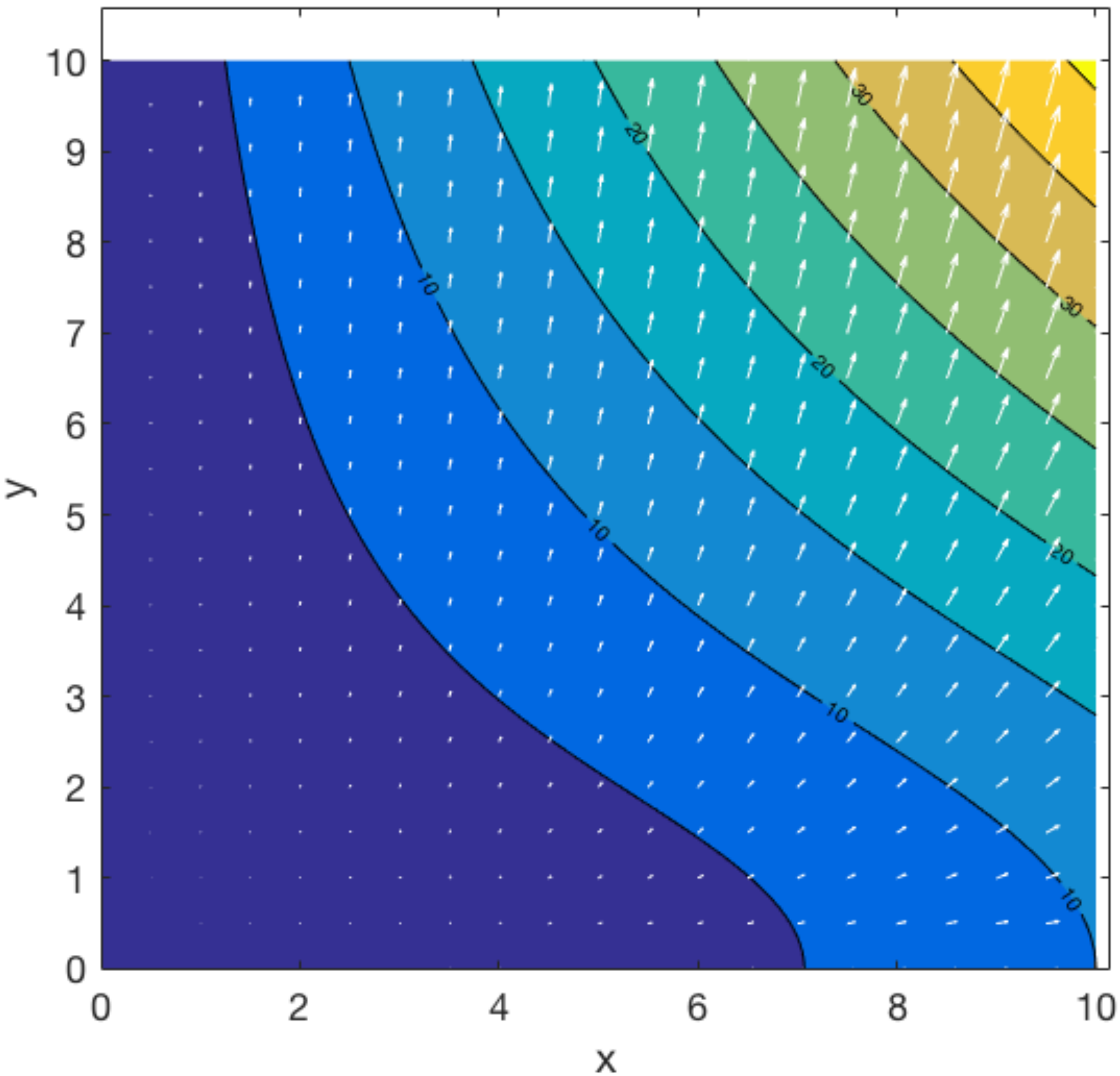
$$\mathbf{w}= -W_{23}\hat{\mathbf{e}}_1+W_{13}\hat{\mathbf{e}}_2-W_{12}\hat{\mathbf{e}}_3$$

For the rotation tensor, an equivalent rotation vector exists:

$$\boldsymbol{\omega}\cdot\mathbf{dx}=r_\omega\times\mathbf{dx} \qquad \text{where:} \qquad r_\omega=\frac{1}{2}\nabla\times\mathbf{u}$$

Note that $\boldsymbol{\omega}$ only describes the overall rigid body rotation, not the total rotation of each individual segment **dx**, which is also influenced by $\boldsymbol{\varepsilon}$

Example displacement – infinitesimal strain



displacement

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Please take a break

And then look at the "Class Exercise" on Displacement and Strain fields and Exercise 8

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

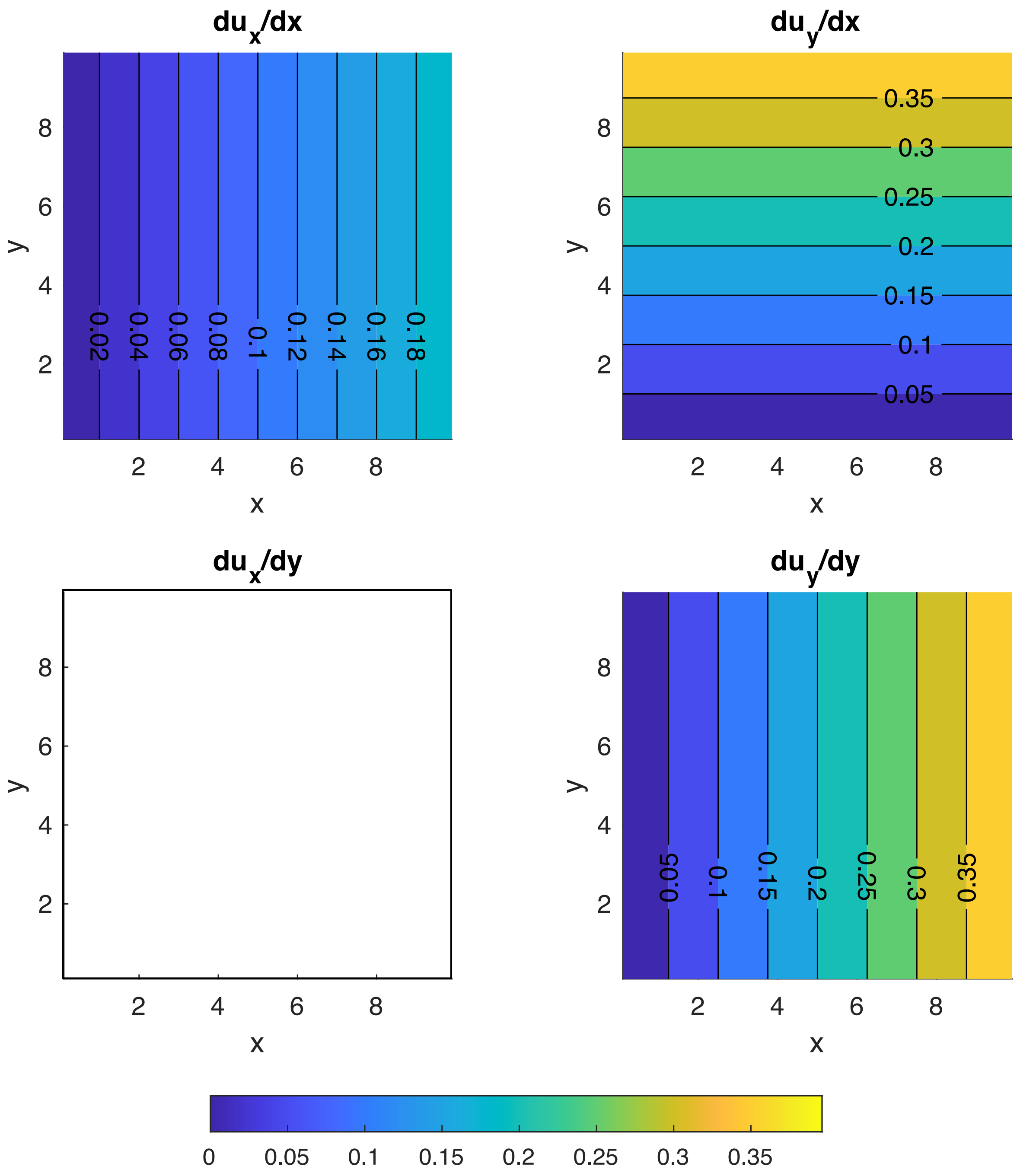
Example displacement gradient

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\frac{\partial u_x}{\partial y} = 0$$

$$\frac{\partial u_y}{\partial y} = 0.4x$$



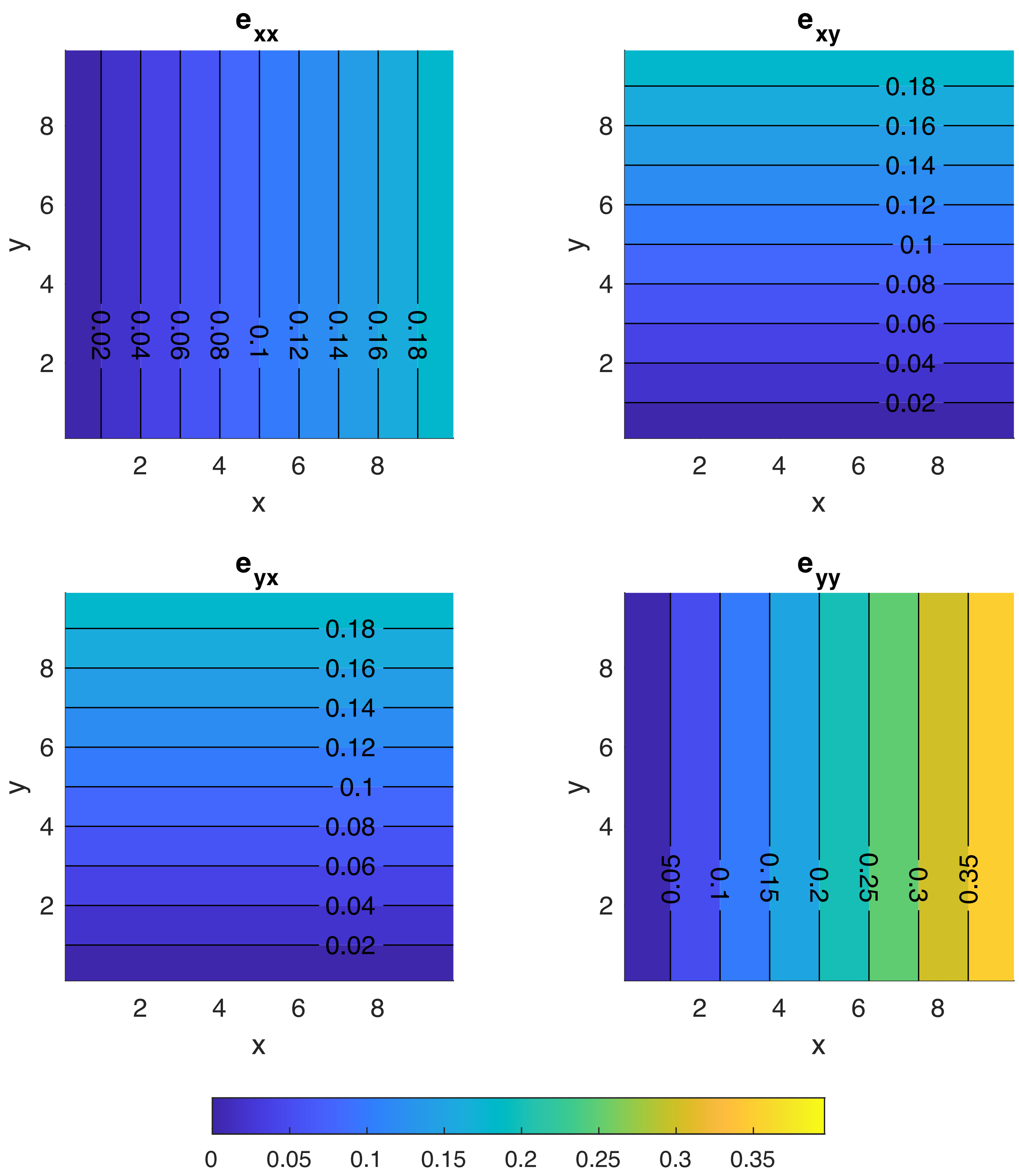
Example infinitesimal strain

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$



$$\frac{\partial u_x}{\partial y} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\varepsilon_{yx} = \varepsilon_{xy}$$

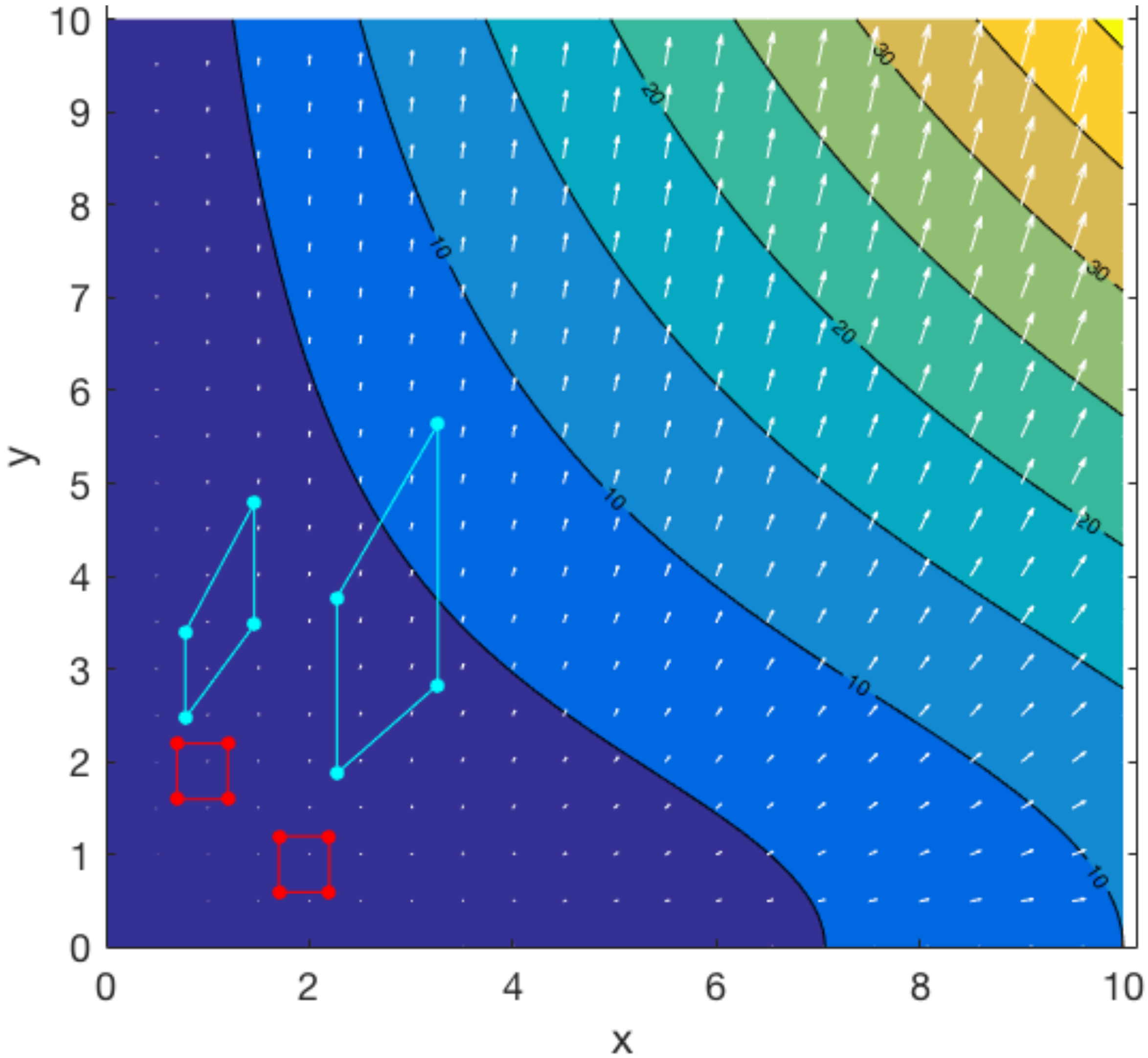
$$\frac{\partial u_y}{\partial y} = 0.4x$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

Deformation after finite strain

original shape

shape at later time



displacement

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

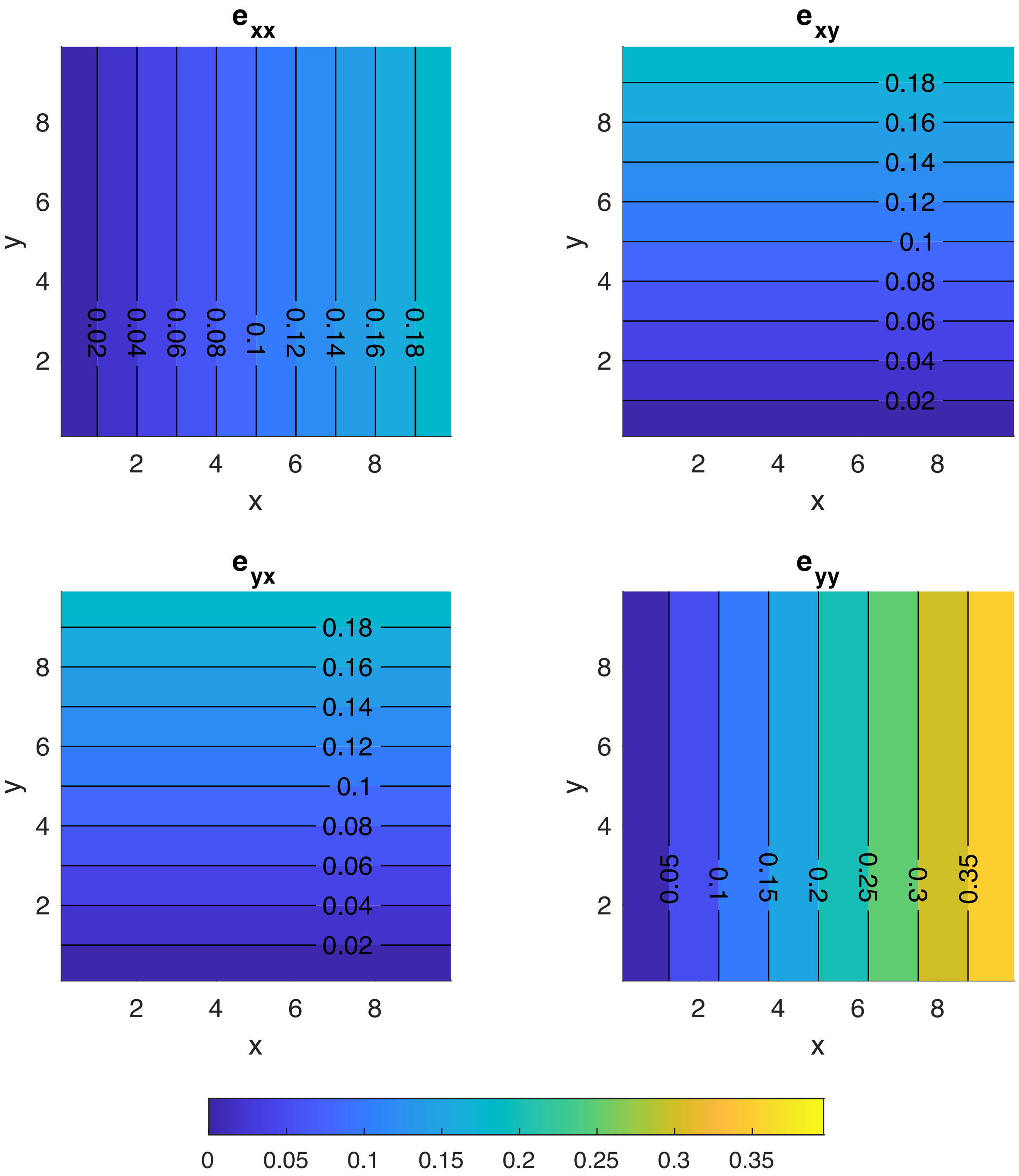
Example infinitesimal strain

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$



$$\frac{\partial u_x}{\partial y} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\varepsilon_{yx} = \varepsilon_{xy}$$

$$\frac{\partial u_y}{\partial y} = 0.4x$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

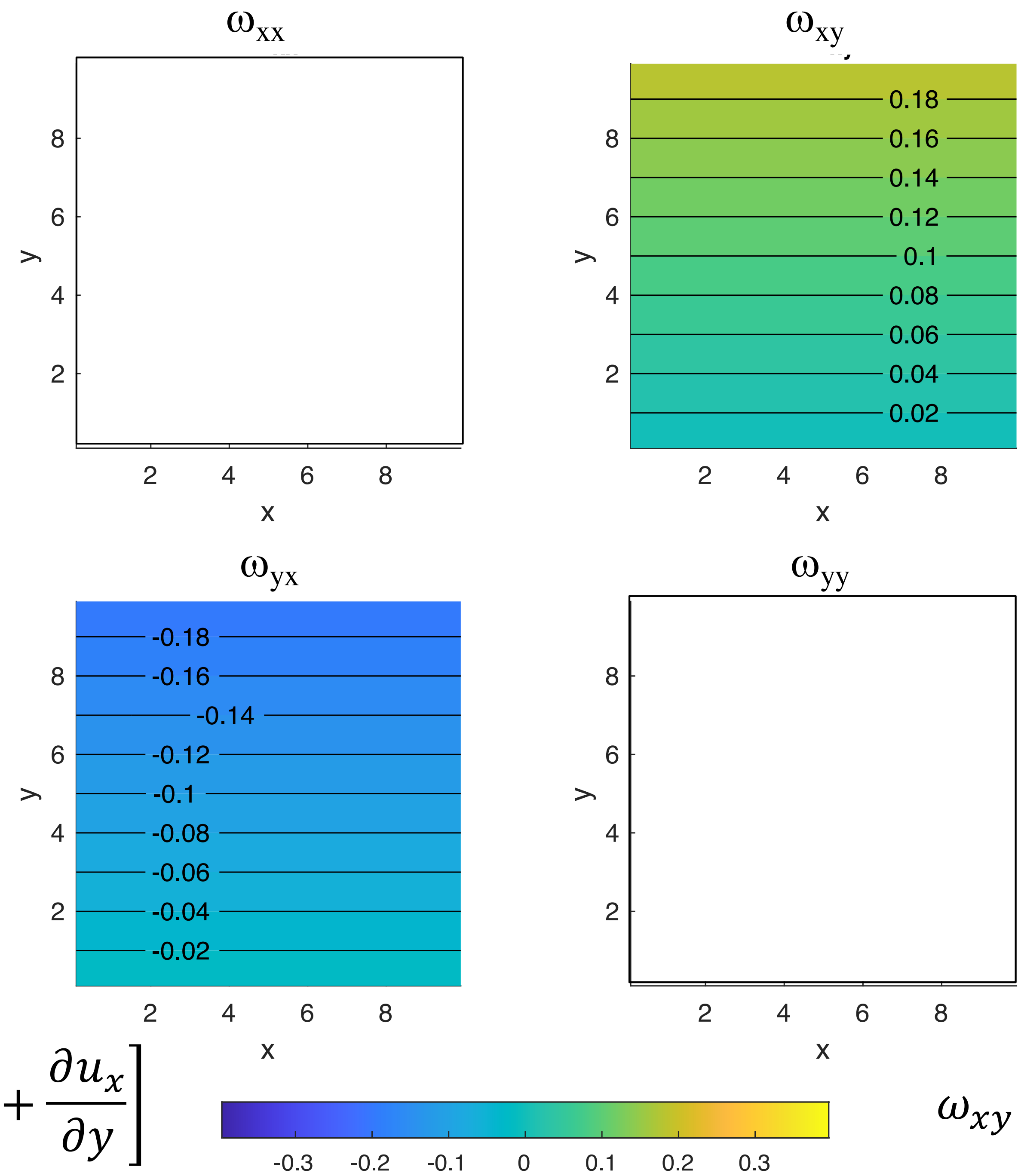
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Example infinitesimal rotation

$\omega_{xx} = 0$

$\omega_{xy} = \varepsilon_{xy}$



$\omega_{yx} = -\varepsilon_{yx}$

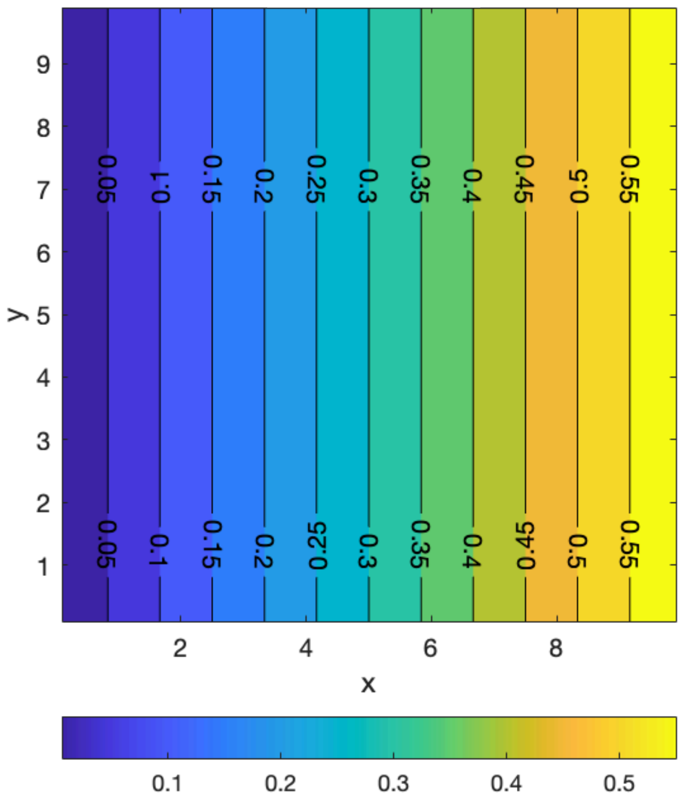
$\omega_{yy} = 0$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

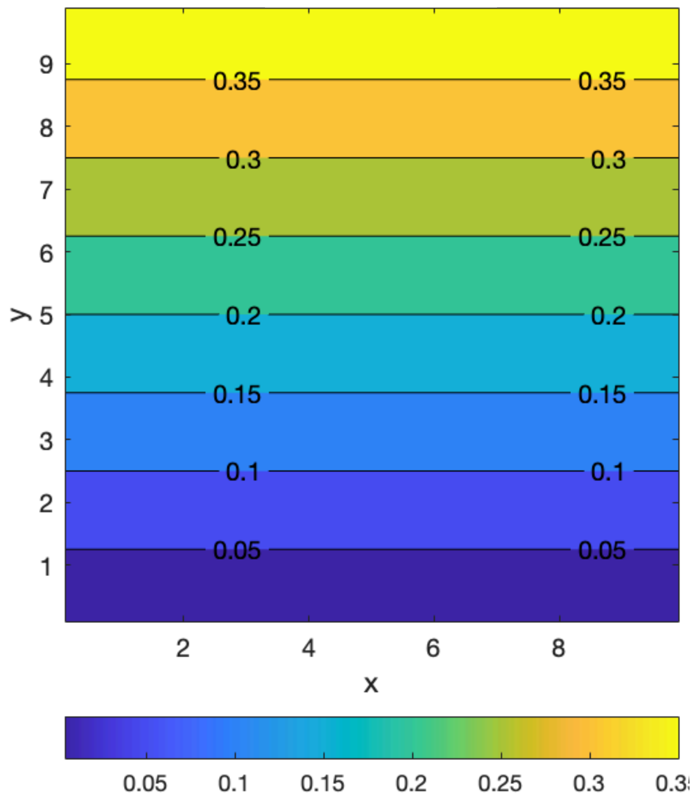
$$\omega_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right]$$

Example displacement – infinitesimal strain

div(**u**)



curl(**u**)



$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}$$

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{\mathbf{e}}_z$$

infinitesimal strain tensor properties

transform to arbitrary *coordinate frame*:

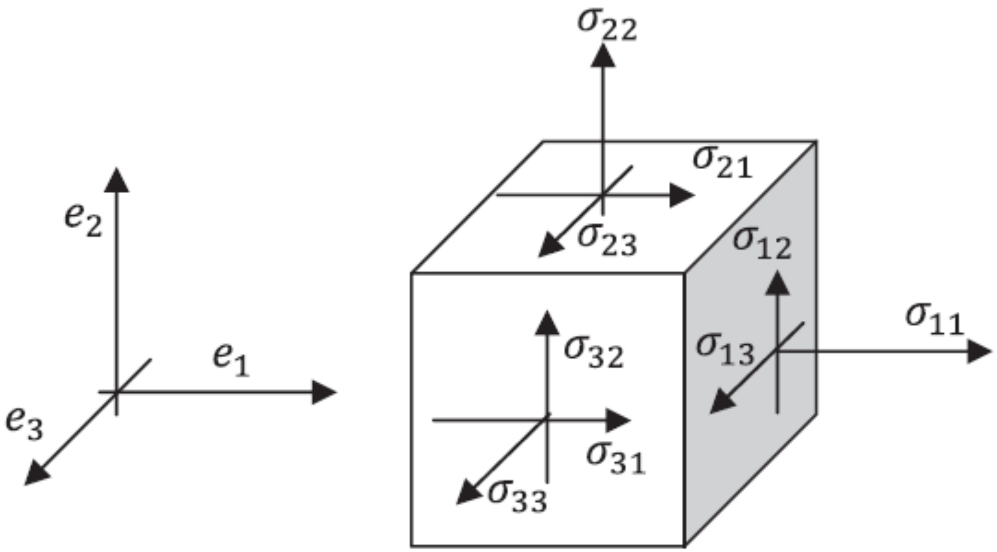
$\varepsilon_{nn} \quad = \varepsilon_{11}\cos^2\phi + \varepsilon_{21}\sin\phi\cos\phi + \varepsilon_{12}\sin\phi\cos\phi + \varepsilon_{22}\sin^2\phi$

$\varepsilon_{ns} \quad = \varepsilon_{11}\sin\phi\cos\phi + \varepsilon_{21}\sin^2\phi - \varepsilon_{12}\cos^2\phi - \varepsilon_{22}\sin\phi\cos\phi$

$\varepsilon_1, \varepsilon_2, \varepsilon_3$ - *principal strains* : minimum, maximum and intermediate fractional length changes

isotropic, *deviatoric* strain: $\varepsilon_{ij} = (\theta/3)\delta_{ij} + \varepsilon'_{ij}$

- $tr(\boldsymbol{\varepsilon}) = \theta$ = sum of normal strains = volume change
- ε'_{ij} is deviatoric strain, change in shape; no change in volume
- $tr(\boldsymbol{\varepsilon}') = 0$, does not imply $\varepsilon'_{ij} = 0$ for $i = j$
- $\varepsilon_{ij} = 0$ for $i \neq j$ does not ensure no shape change



Stress components

Reminder

traction on a plane $$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

what is $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$?

t_1 on plane with normal $\hat{\mathbf{n}}$

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_1$? σ_{11}

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_2$? σ_{21}

Strain components

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\hat{\mathbf{e}}_1 \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{e}}_1 = \varepsilon_{11}$$

$$\hat{\mathbf{e}}_1 \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{e}}_2 = \varepsilon_{12}$$

$\boldsymbol{\varepsilon} \cdot \widehat{\mathbf{p}} = \mathbf{dp}'$ the change in unit vector $\hat{\mathbf{p}}$ after deformation by $\boldsymbol{\varepsilon}$

$\widehat{\mathbf{p}} \cdot \boldsymbol{\varepsilon} \cdot \widehat{\mathbf{p}} =$ elongation by $\boldsymbol{\varepsilon}$ of unit vector $\hat{\mathbf{p}}$ in direction $\hat{\mathbf{p}}$

$= \widehat{\mathbf{p}} \cdot \mathbf{dp}' = |\mathbf{dp}'| \cos \alpha$

Where α is angle between $\mathbf{dp}'$ and $\widehat{\mathbf{p}}$

Strain Rate Tensor

In similar way as strain tensor, a tensor that describes the rate of change of deformation can be defined from **velocity gradient**:

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathbf{dr} = \nabla\mathbf{v}$$

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

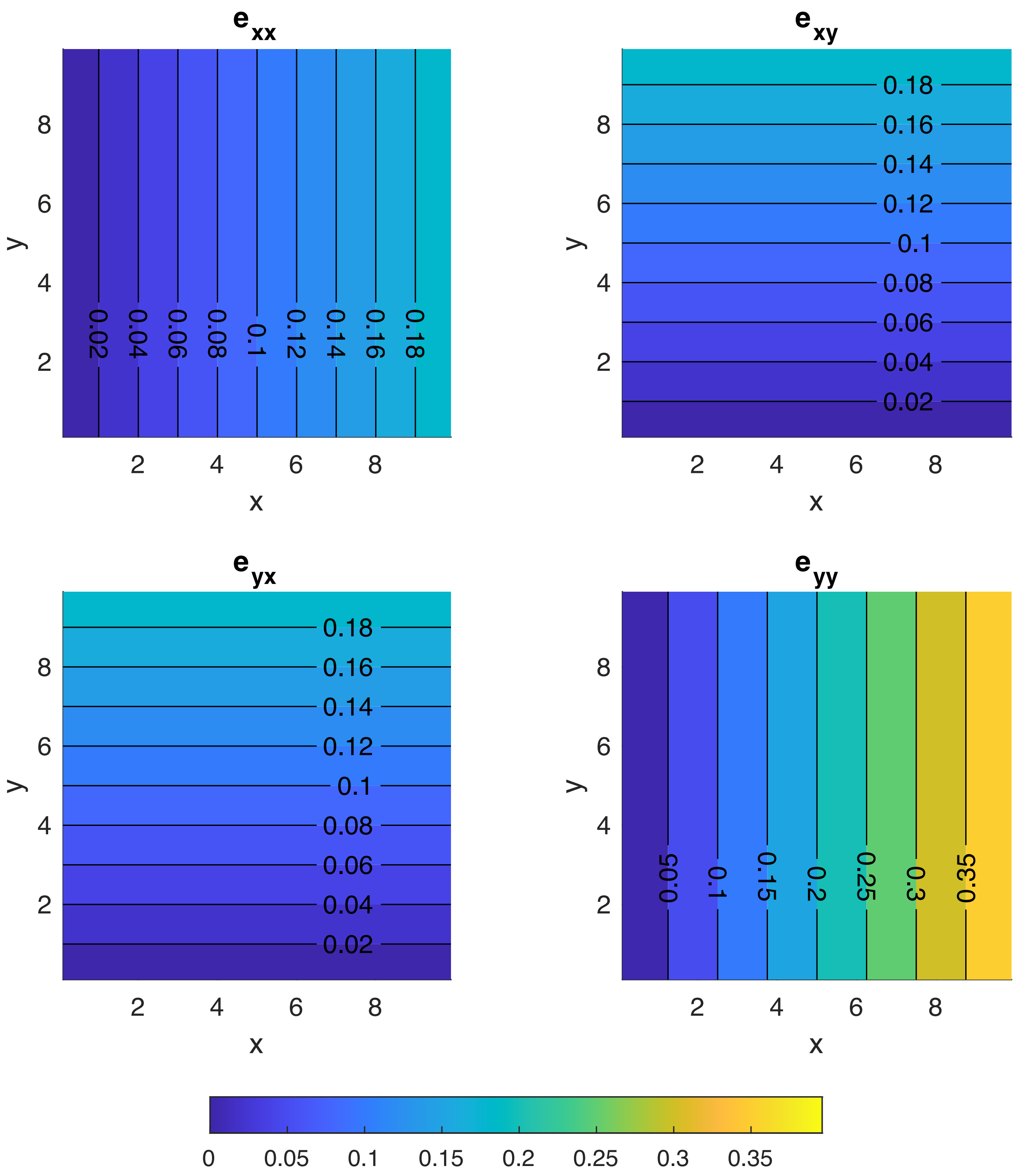
$$\nabla\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) + \frac{1}{2}(\nabla\mathbf{v} - \nabla\mathbf{v}^T)$$

$$\nabla\mathbf{v} = \mathbf{D} + \mathbf{W}$$

Velocity gradient tensor is the sum of **strain rate** and **vorticity** tensors

Infinitesimal strain

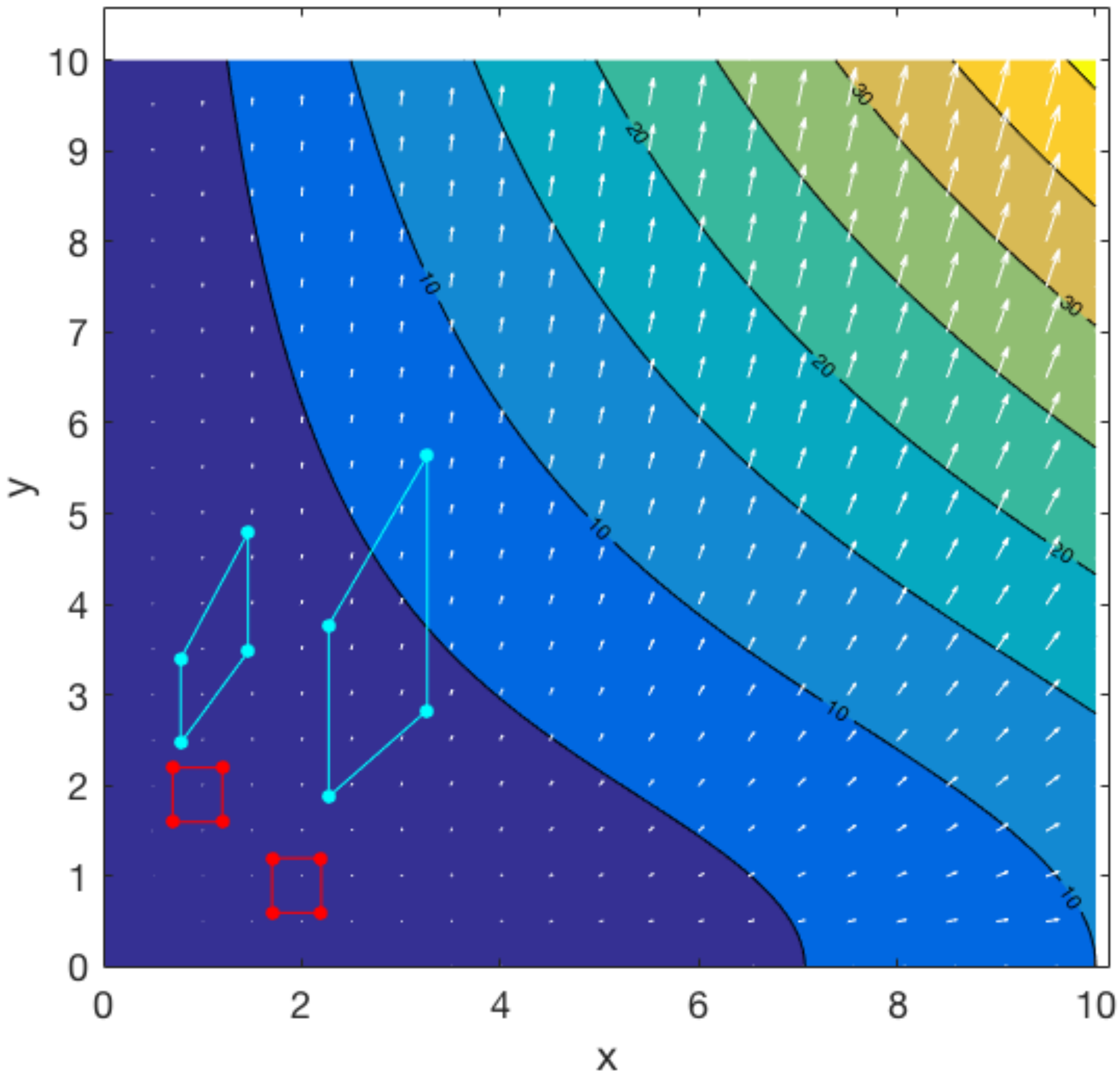
Over small increments, can assume constant displacement gradient encountered



Deformation after finite strain

original shape

shape at later time



displacement

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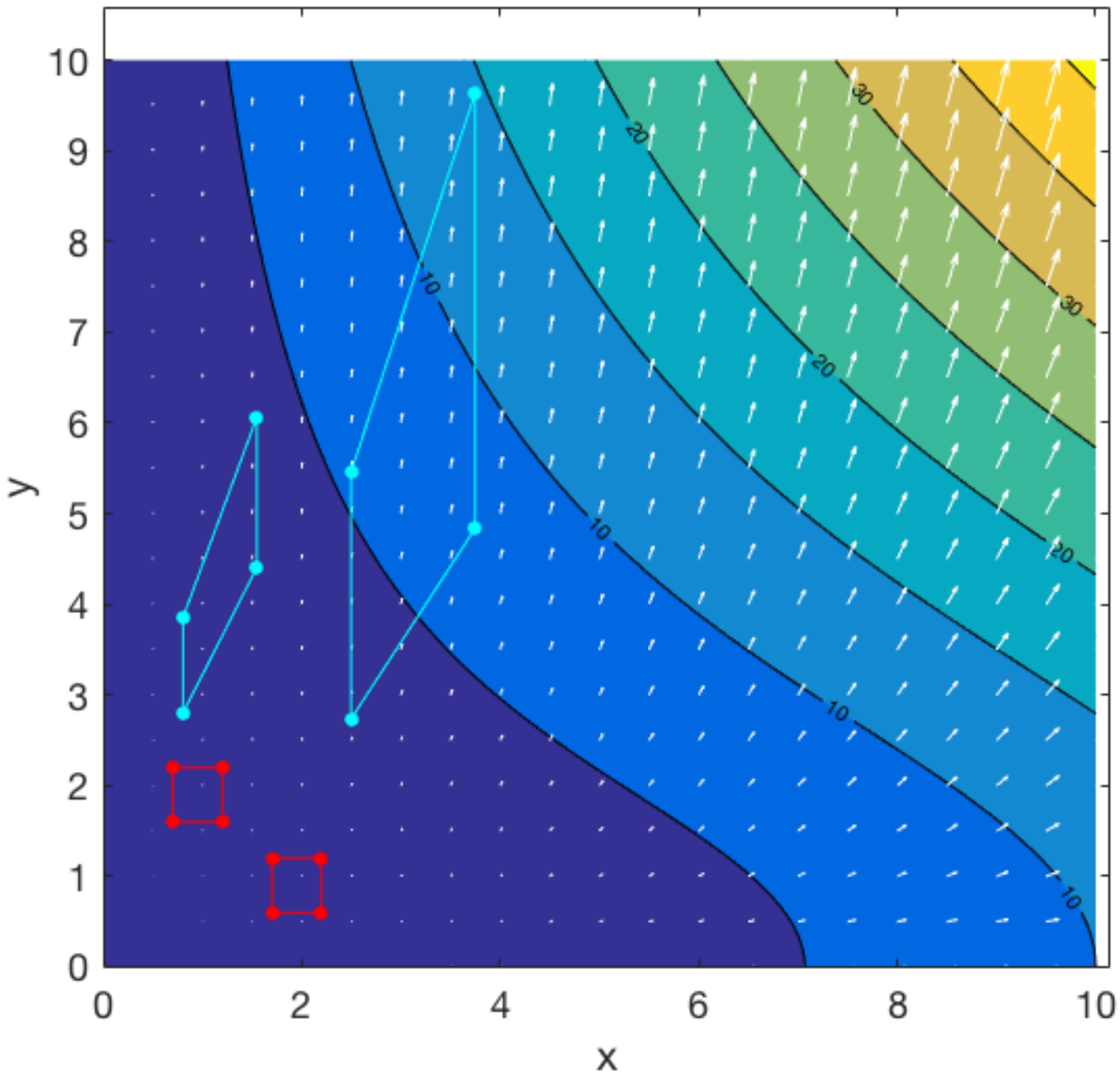
Deformation after finite strain



original shape

shape at even later time

displacement

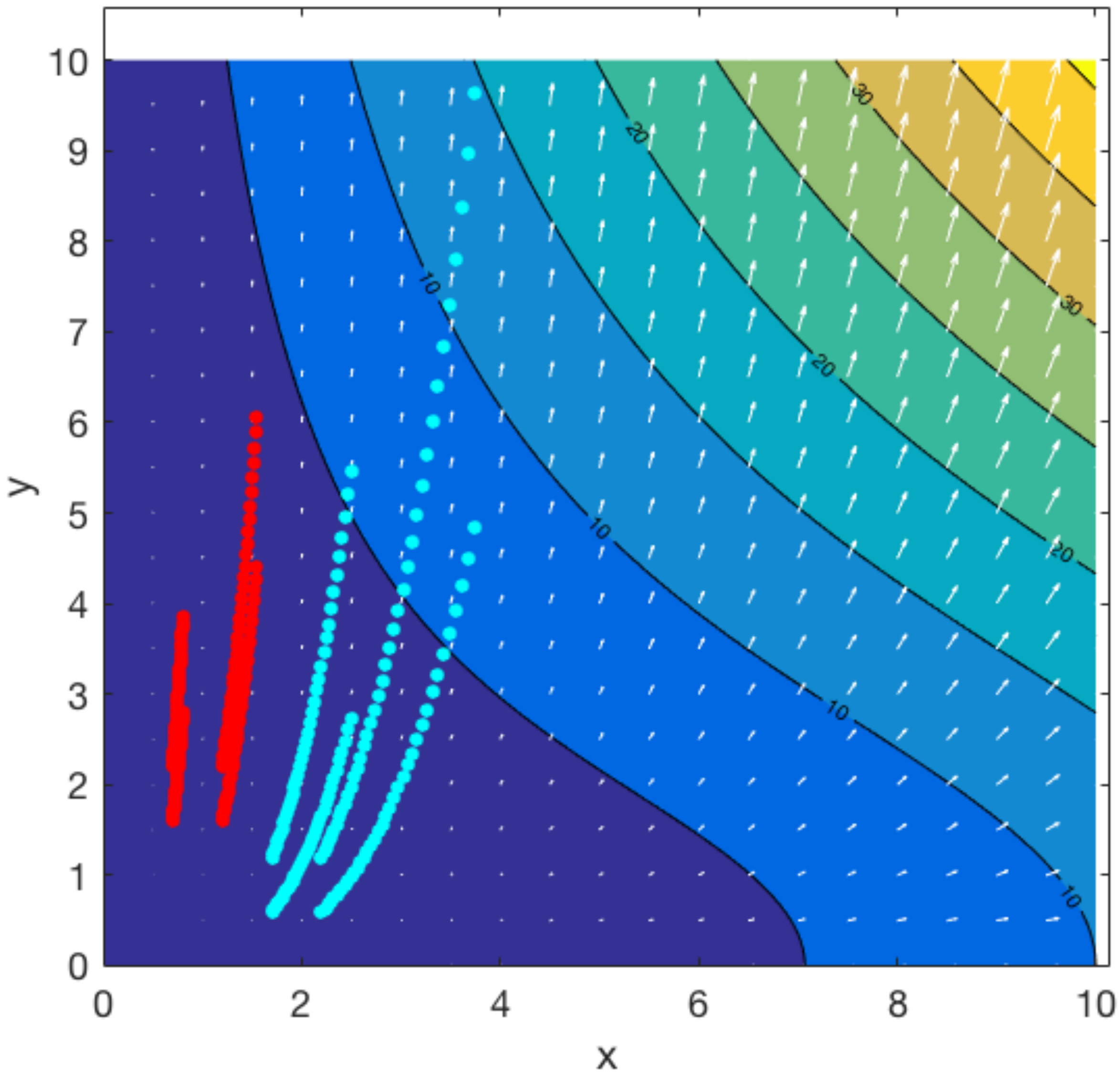
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Deformation after finite strain

points shape 1

points shape 2



displacement

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Learning Objectives

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain

Outline Lecture 3

- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Deformation

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 3-1 through 3-15 and we covered some of the basics discussed in 3-20 to 3-26

Try yourself

Complete any of the remaining exercises in *chapter3.ipynb*:

Exercise 1, 2, 5, 7, 9a

- Additional practise: **Exercise 3, 6, 8**
- Advanced practise: **Exercise 4, 9b, 10**