

# Introduction Tensors

- Tensors, generalisation of vectors to more dimensions
- Use when properties depend on direction in more than one way.
- A physical quantity that is independent of coordinate system used
- Derives from the word tension (= stress)
- Stress tensor as example
- *Not* just a multidimensional array

# Tensors

**Used in** Stress, strain, moment tensors

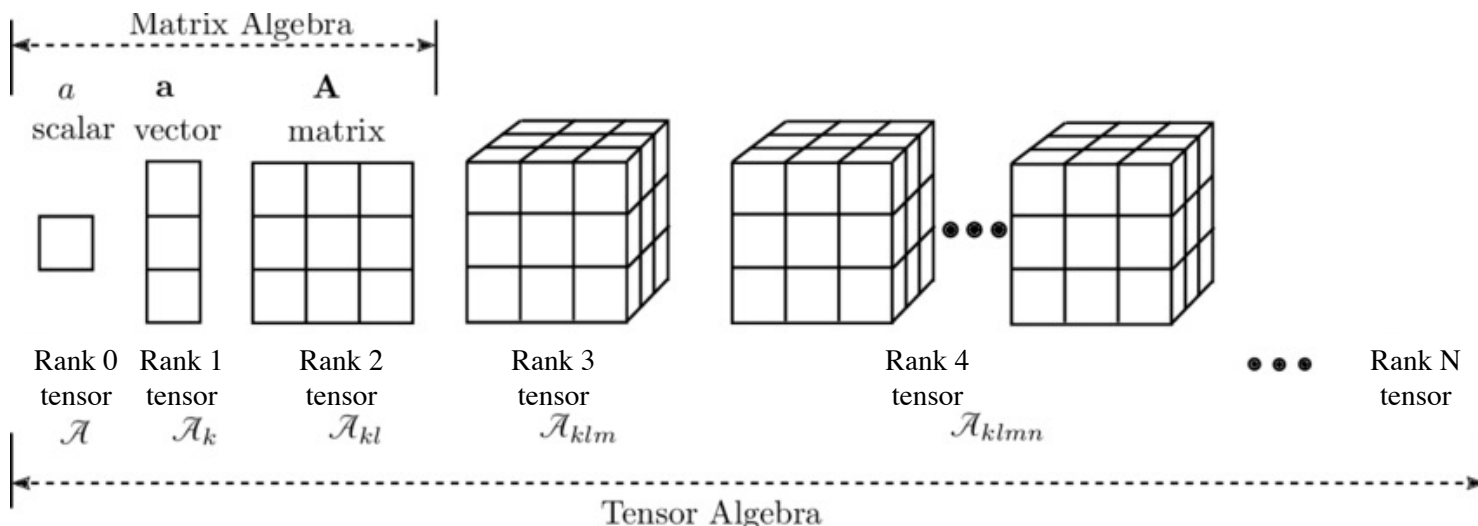
Electrostatics, electrodynamics, rotation, crystal properties

**Tensors describe properties that depend on direction**

Tensor rank 0 - scalar - independent of direction

Tensor rank 1 - vector - depends on direction in 1 way

Tensor rank 2 - tensor - depends on direction in 2 ways



# Notation

- Tensors as  $\mathbf{T}$
- for second order:  $\overline{\overline{T}}$  or  $\underline{\underline{T}}$
- Index notation  $T_{ij}$ ,  $i,j=x,y,z$  or  $i,j=1,2,3$
- But also higher order  $T_{ijkl}$

# An example tensor

Gradient of velocity  
depends on  
direction in two  
ways

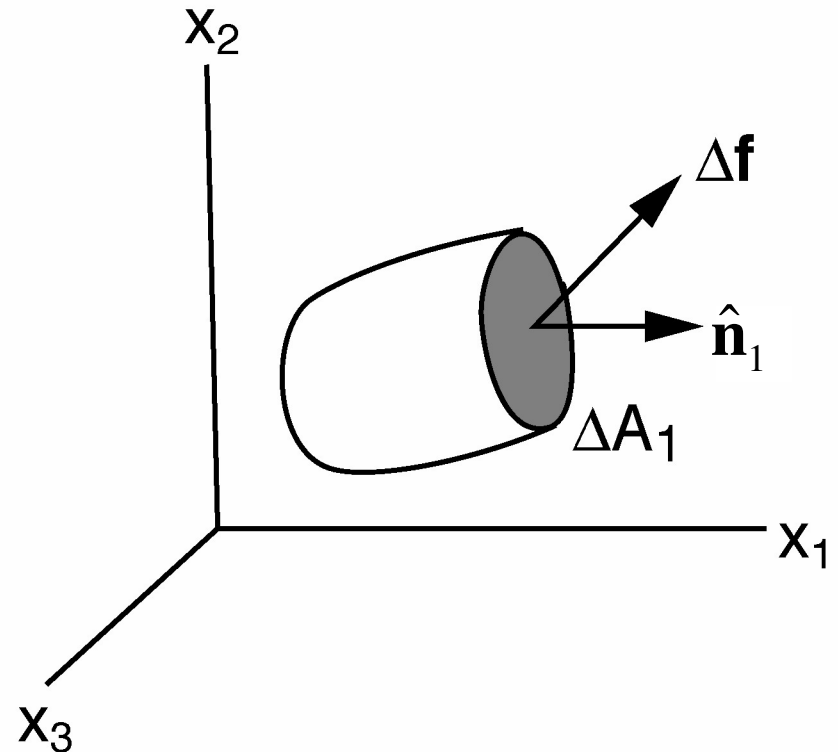
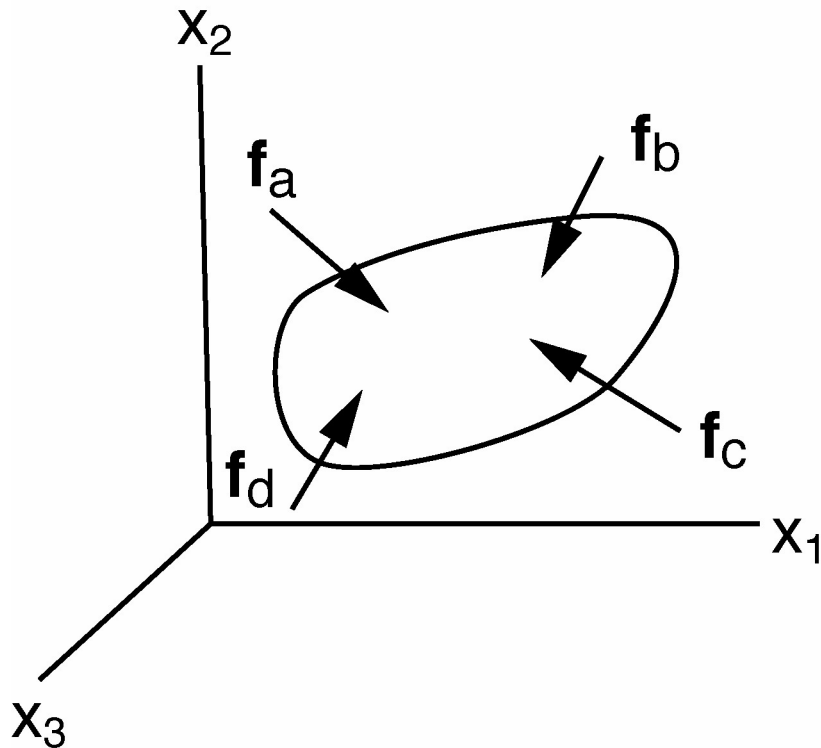
$$\nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Component of velocity

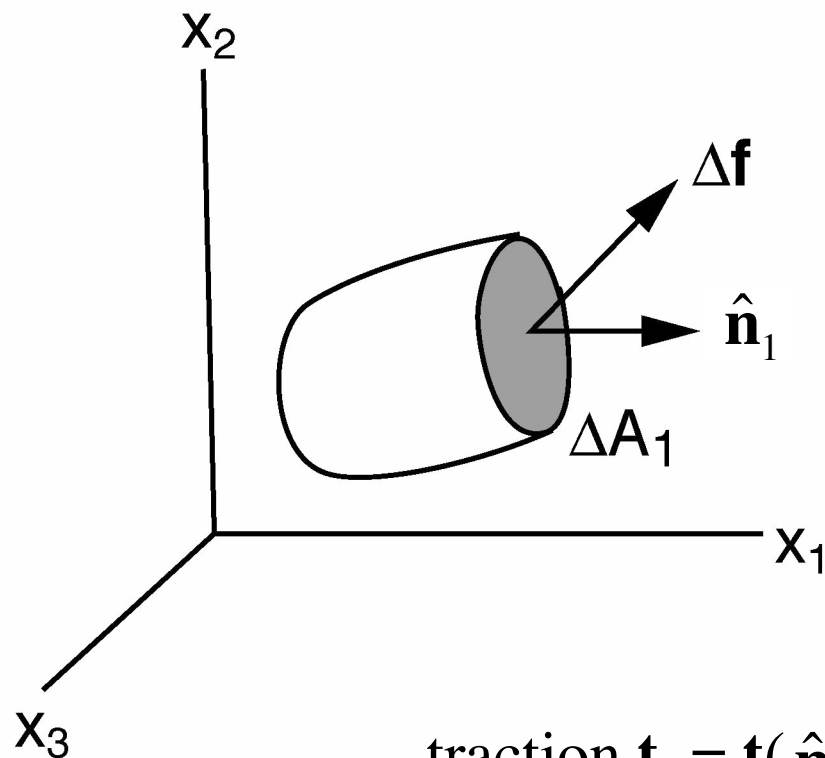
Spatial variation in this direction

# Another example: Stress

- *Body forces* - depend on volume, e.g., gravity
- *Surface forces* - depend on surface area, e.g., friction



forces introduce a state of stress in a body



- $\Delta \mathbf{f}$  necessary to maintain equilibrium depends on orientation of the plane,  $\hat{\mathbf{n}}_1$

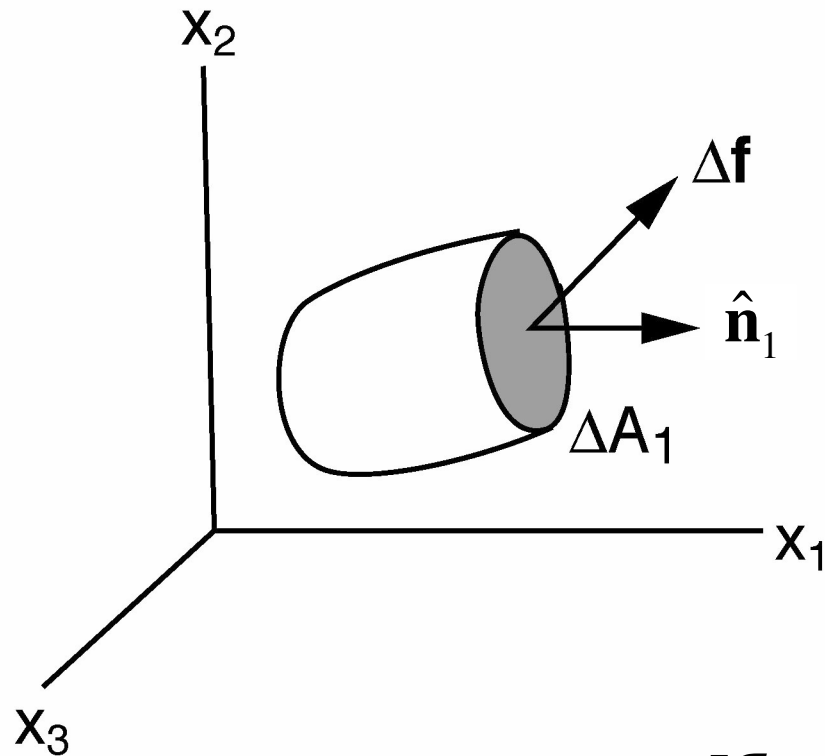
$$\text{traction } \mathbf{t}_1 = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \rightarrow 0} \Delta \mathbf{f} / \Delta A_1$$

$$\mathbf{t}_1 = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

$$\sigma_{11} = \lim_{\Delta A_1 \rightarrow 0} \Delta \mathbf{f}_1 / \Delta A_1$$

$$\sigma_{12} = \lim_{\Delta A_1 \rightarrow 0} \Delta \mathbf{f}_2 / \Delta A_1$$

$$\sigma_{13} = \lim_{\Delta A_1 \rightarrow 0} \Delta \mathbf{f}_3 / \Delta A_1$$



Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$  for  $\Delta A_1$

$\sigma_{22}, \sigma_{21}, \sigma_{23}$  for  $\Delta A_2$

$\sigma_{33}, \sigma_{31}, \sigma_{32}$  for  $\Delta A_3$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

first index = orientation of plane  
second index = orientation of force

## **Difference between a tensor and its matrix**

Tensor – physical quantity that is independent of coordinate system used

Matrix of a tensor – contains components of that tensor in a particular coordinate frame

*Could test* that indeed tensor addition and multiplication satisfy transformation laws



# Summation (Einstein) convention

When an index in a single term is a duplicate, dummy index, summation implied without writing summation symbol

$$a_1v_1 + a_2v_2 + a_3v_3 = \sum_{i=1}^3 a_i v_i = a_i v_i$$

$$\begin{aligned} \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} x_i y_j &= a_{ij} x_i y_j = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{13}x_1y_3 \\ &\quad + a_{21}x_2y_1 + a_{22}x_2y_2 + a_{23}x_2y_3 \\ &\quad + a_{31}x_3y_1 + a_{32}x_3y_2 + a_{33}x_3y_3 \end{aligned}$$

**Invalid**, indices repeated more than twice

$$\sum_{i=1}^3 a_i b_i v_i \neq a_i b_i v_i$$

# Notation conventions

index notation

$$\alpha_{ij}x_iy_j=$$

matrix-vector notation

$$\mathbf{x}^T \mathbf{A} \mathbf{y} =$$

$$(x_1 \quad x_2 \quad x_3) \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

other versions index notation

$$\alpha_{ij}x_iy_j= x_i\alpha_{ij}y_j=$$

$$\alpha_{ij}y_jx_i$$

# Dummy vs free index

$$a_1v_1 + a_2v_2 + a_3v_3 = \sum_{i=1}^3 a_i v_i = \sum_{k=1}^3 a_k v_k$$

- i,k – dummy index – appears in duplicates and can be substituted without changing equation

$$F_j = A_j \sum_{i=1}^3 B_i C_i \Rightarrow \begin{aligned} F_1 &= A_1 (B_1 C_1 + B_2 C_2 + B_3 C_3) \\ F_2 &= A_2 (B_1 C_1 + B_2 C_2 + B_3 C_3) \\ F_3 &= A_3 (B_1 C_1 + B_2 C_2 + B_3 C_3) \end{aligned}$$

- j – free index, appears once in each term of the equation

# Addition and subtraction of tensors

$$\mathbf{W} = a\mathbf{T} + b\mathbf{S}$$

add each component:  $W_{ijkl} = aT_{ijkl} + bS_{ijkl}$

$\mathbf{T}$  and  $\mathbf{S}$  must have same rank, dimension and units

$\mathbf{W}$  has same rank, dimension and units as  $\mathbf{T}$  and  $\mathbf{S}$

$\mathbf{T}$  and  $\mathbf{S}$  are tensors  $\Rightarrow \mathbf{W}$  is a tensor

commutative, associative

*This is same as how vectors and matrices are added.*

# Multiplication of tensors

Inner product = dot product

$$\mathbf{W} = \mathbf{T} \cdot \mathbf{S}$$

involves contraction over 1 index:  $W_{ik} = T_{ij} S_{jk}$

As normal matrix and matrix-vector multiplication

$\mathbf{T}$  and  $\mathbf{S}$  can have different rank, but same dimension  
 $\text{rank } \mathbf{W} = \text{rank } \mathbf{T} + \text{rank } \mathbf{S} - 2$ , dimension as  $\mathbf{T}$  and  $\mathbf{S}$ ,  
units as product of units  $\mathbf{T}$  and  $\mathbf{S}$

$\mathbf{T}$  and  $\mathbf{S}$  are tensors  $\Rightarrow \mathbf{W}$  is a tensor

Examples:  $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \text{ or } \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \text{ (Hooke's law)}$$

# Multiplication of tensors

Tensor product = outer product = dyadic product  
 $\neq$  cross product

$\mathbf{W} = \mathbf{T}\mathbf{S}$  often written as  $\mathbf{W} = \mathbf{T} \otimes \mathbf{S}$

no contraction:  $W_{ijkl} = T_{ij}S_{kl}$

$\mathbf{T}$  and  $\mathbf{S}$  can have different rank, but same dimension  
 $\text{rank } \mathbf{W} = \text{rank } \mathbf{T} + \text{rank } \mathbf{S}$ , dimension as  $\mathbf{T}$  and  $\mathbf{S}$ ,  
units as product of units  $\mathbf{T}$  and  $\mathbf{S}$

$\mathbf{T}$  and  $\mathbf{S}$  are tensors  $\Rightarrow \mathbf{W}$  is a tensor

Examples:  $\nabla \mathbf{v}$  (gradient of a vector)  $\neq \nabla \cdot \mathbf{v}$  (divergence)

*remember gradient is a vector*  $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$

# Multiplication of tensors

For both multiplications

Distributive:  $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$

Associative:  $\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$

Not commutative:  $\mathbf{TS} \neq \mathbf{ST}, \mathbf{T} \cdot \mathbf{S} \neq \mathbf{S} \cdot \mathbf{T}$

but:  $\mathbf{T} \cdot \mathbf{S} = \mathbf{S}^T \cdot \mathbf{T}^T$

and:  $\mathbf{ab}=(\mathbf{ba})^T$  but only for rank 2

Remember transpose:  $\mathbf{a} \cdot \mathbf{T} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{T}^T \cdot \mathbf{a} \Rightarrow T_{ji} = T^T_{ij}$

**Special tensor:**  
**Kronecker delta  $\delta_{ij}$**

$$\delta_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$$

$$\delta_{ij} = 1 \text{ for } i=j, \delta_{ij} = 0 \text{ for } i \neq j$$

In 3-D:

$$\delta = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T} \cdot \delta = \mathbf{T} \cdot \mathbf{I} = \mathbf{T} \quad \text{or} \quad T_{ij} \delta_{jk} = T_{ik}$$

$\delta$  is isotropic:  $\delta_{ij} = \delta'_{ij}$  upon coordinate transformation

can be used to write dot product:  $T_{ij} S_{jl} = T_{ij} S_{kl} \delta_{jk}$

can be used to write trace:  $A_{ii} = A_{ij} \delta_{ij}$

orthonormal transformation:  $\alpha_{ij} \alpha_{jk}^T = \delta_{ik}$

Isotropic tensors,  
invariant upon  
coordinate  
transformation

- Scalars
- **0** vector
- $\delta_{ij}$



## Special tensor: Permutation symbol $\epsilon_{ijk}$

$$\epsilon_{ijk} = (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k$$

$\epsilon_{ijk} = 1$  if  $i,j,k$  an even permutation of 1,2,3

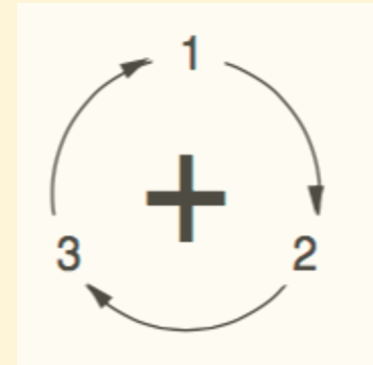
$\epsilon_{ijk} = -1$  if  $i,j,k$  an odd permutation of 1,2,3

$\epsilon_{ijk} = 0$  for all other  $i,j,k$

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$$

$$\epsilon_{111} = \epsilon_{112} = \epsilon_{222} = \dots = 0$$



Note that  $\epsilon_{ijk} \mathbf{a}_i \mathbf{b}_j \hat{\mathbf{e}}_k$  where  $\hat{\mathbf{e}}_k$  is the unit vector in  $k$  direction is index notation for cross product  $\mathbf{a} \times \mathbf{b}$

**Exercise:** useful identity  $\epsilon_{ijm} \epsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$

# Vector derivatives - curl

Curl of a vector:  $\nabla \times \mathbf{v} = \varepsilon_{ijk} \frac{\partial}{\partial x_i} v_j \hat{\mathbf{e}}_k = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}$

In index notation, using special tensor

# Some tensor calculus

Gradient of a vector is a tensor:

$$\nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Such that the change  $d\mathbf{v}$  in field  $\mathbf{v}$  in direction  $d\mathbf{x}$  is:  $d\mathbf{v} = \nabla \mathbf{v} \cdot d\mathbf{x}$

Divergence of a vector:  $\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$

$$\nabla \cdot \mathbf{v} = \text{tr}(\nabla \mathbf{v})$$

Trace of a tensor is the sum of diagonal elements

# Some tensor calculus

Divergence of a tensor:

$$\nabla \cdot T = \frac{\partial T_{ij}}{\partial x_j} = \begin{pmatrix} \frac{\partial T_{1j}}{\partial x_j} \\ \frac{\partial T_{2j}}{\partial x_j} \\ \frac{\partial T_{3j}}{\partial x_j} \end{pmatrix} = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{pmatrix}$$

Laplacian =  $\text{div}(\text{grad } f)$ , where  $f$  is a scalar function

$$\nabla \cdot \nabla f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

# Objectives

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention

# Summary

- **Vectors**
  - Addition, linear independence
  - Orthonormal Cartesian bases, transformation
  - Multiplication
  - Derivatives, del, div, curl
- **Tensors**
  - Tensors, rank, stress tensor
  - Index notation, summation convention
  - Addition, multiplication
  - Special tensors,  $\delta_{ij}$  and  $\varepsilon_{ijk}$
  - Tensor calculus: gradient, divergence, curl, ..

*Further reading/studying e.g: **Reddy** (2013) 2.2.1-2.2.3, 2.2.5, 2.2.6, 2.4.1, 2.4.4, 2.4.5, 2.4.6, 2.4.8 (not co/contravariant), **Lai, Rubin, Kremple** (2010): 2.1-2.13, 2.16, 2.17, 2.27-2.32, 4.1-4.3, **Khan Academy** – linear algebra, multivariate calculus*