Vector transformation

- Vector magnitude and direction do not depend on basis
- When defined on an orthonormal basis, like rectangular Cartesian, the transformation to other orthonormal bases is simple, with real coefficients.

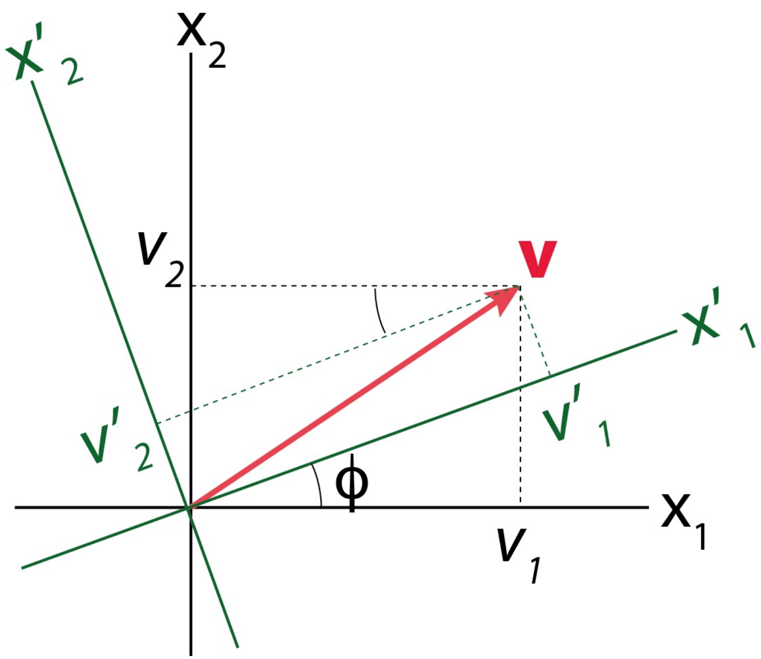
$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0 \qquad \text{if } i \neq j$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \left|\hat{\mathbf{e}}_i\right|^2 = 1$$

check out Khan Academy lectures on orthonormal bases

Vector transformation

physical parameters should not depend on coordinate frame



for vectors on an **orthonormal** basis, the transformed vector **v**' depends on **v**:

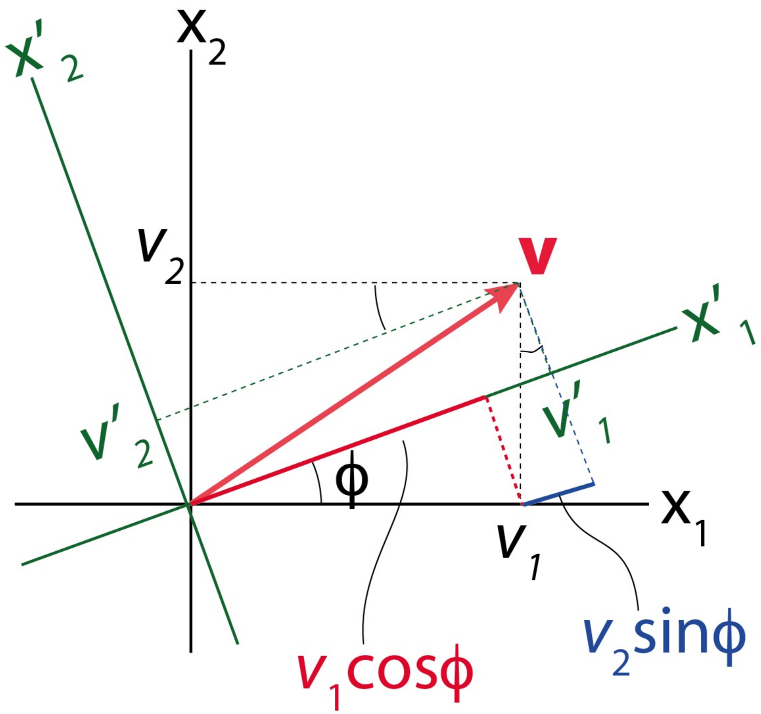
$$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$$
$$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$$

$$v'_1 = \cos\phi \; v_1 + \sin\phi \; v_2$$
$$v'_2 = -\sin\phi \; v_1 + \cos\phi \; v_2$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

Vector transformation

physical parameters should not depend on coordinate frame



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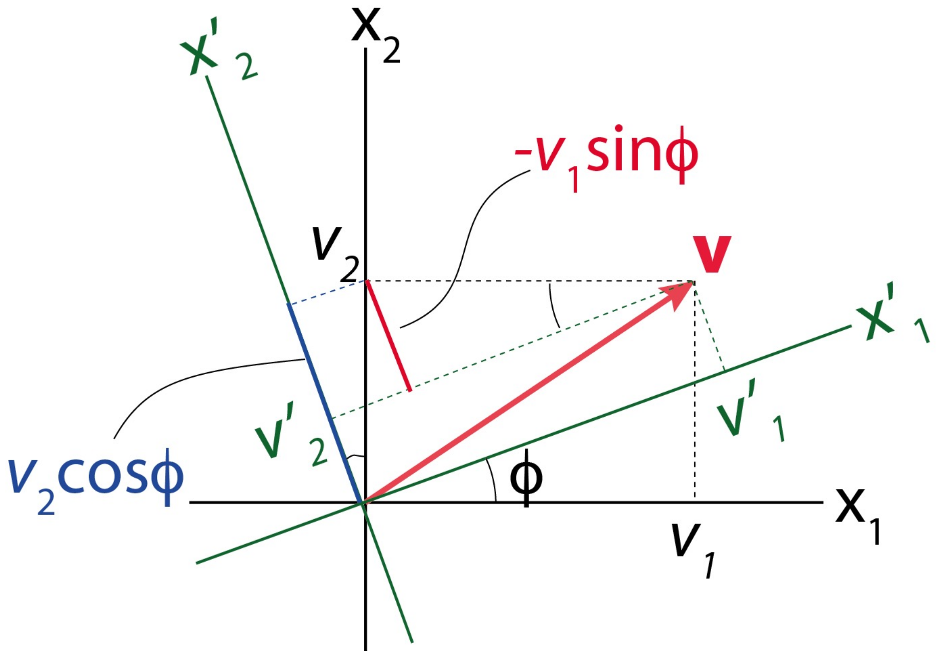

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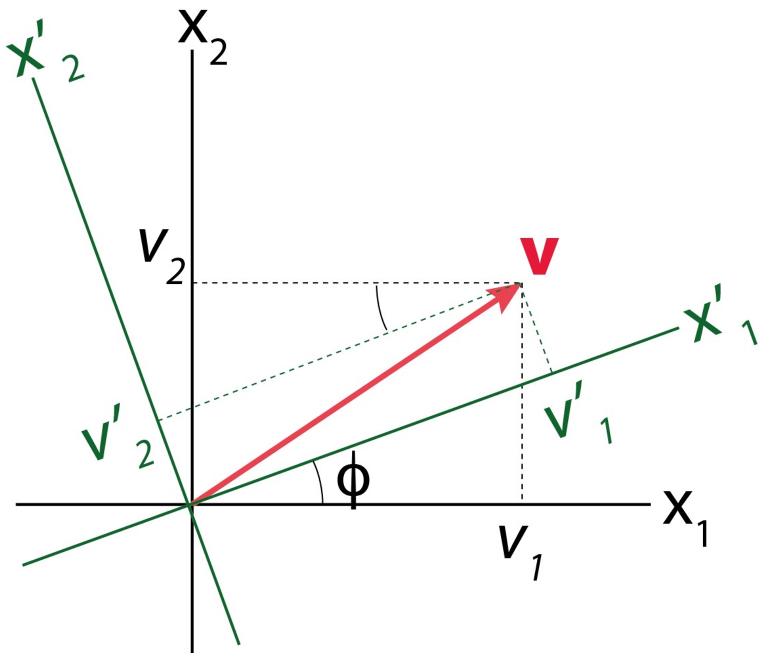

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$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$

$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$

$$\Rightarrow \quad \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v}$$

$$\alpha_{11} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 \quad \alpha_{12} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2$$

$$\alpha_{21} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 \quad \alpha_{22} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2$$

A word of caution!

vector represented in new basis

$$\mathbf{v}' = \mathbf{A}\mathbf{v}$$

vector represented in old basis

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v}$$

Clockwise rotation by ϕ

Matrix describing change of basis

New basis vectors are **rows**

new basis vector

$$\hat{\mathbf{e}}' = \mathbf{A}^T \hat{\mathbf{e}}$$

old basis vector

$$\hat{\mathbf{e}}' = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \hat{\mathbf{e}}$$

Anticlockwise rotation by ϕ

Matrix describing basis transformation

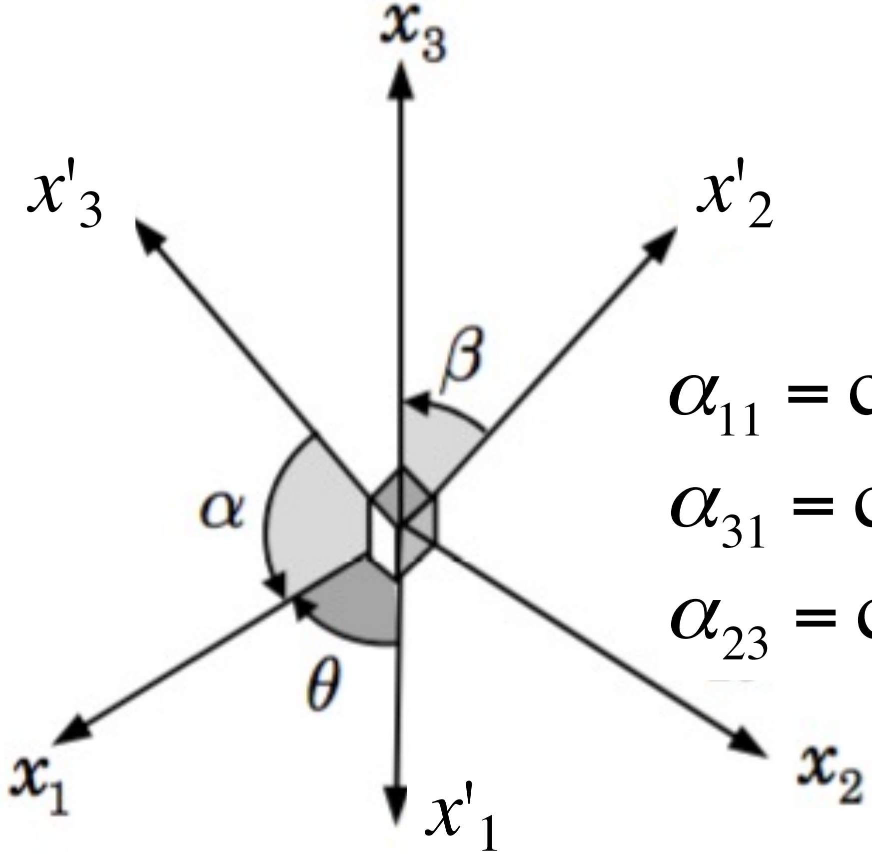
New basis vectors are **columns**

Transformation orthonormal bases

$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij} \hat{\mathbf{e}}_j$$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$



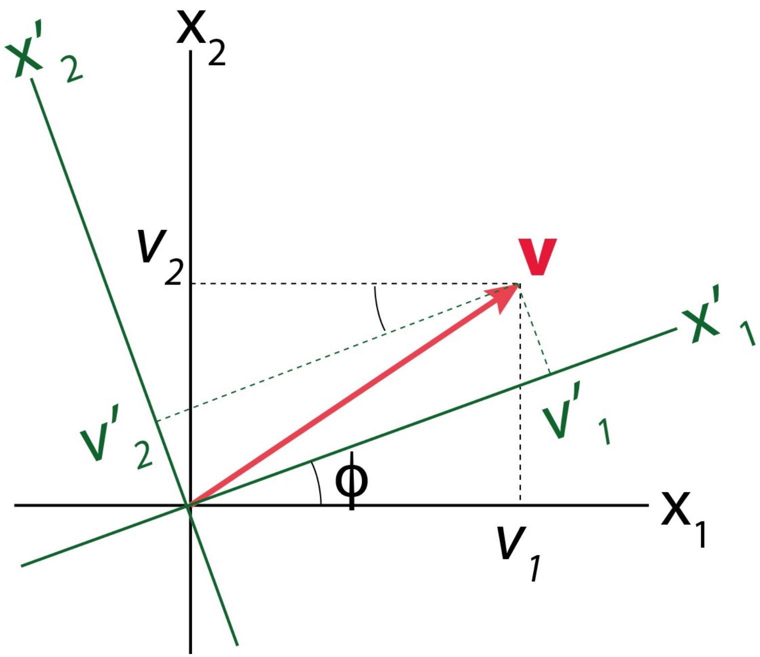
$$\alpha_{11} = \cos\theta$$
$$\alpha_{31} = \cos\alpha$$
$$\alpha_{23} = \cos\beta$$

Transformation orthonormal bases

$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij}\hat{\mathbf{e}}_j$$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$



angle x'$_1$ and x$_2$

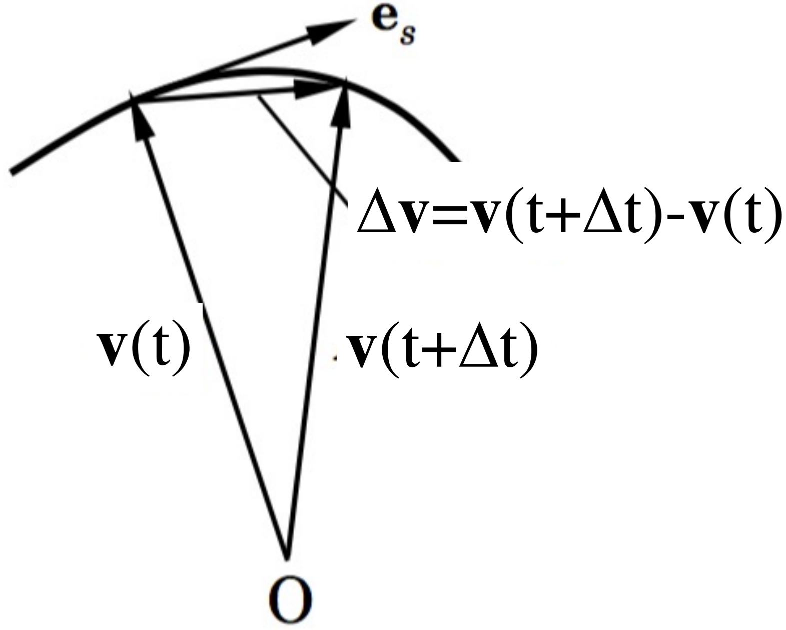
$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\mathbf{v} \qquad \mathbf{v}' = \begin{bmatrix} \cos\phi & \cos(90-\phi) \\ \cos(90+\phi) & \cos\phi \end{bmatrix}\mathbf{v}$$

angle x'$_2$ and x$_1$

Vector derivatives
Scalar: e.g. time

$$\boxed{\frac{d\mathbf{v}}{dt} = \begin{pmatrix} \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dt} \end{pmatrix}}$$

$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$

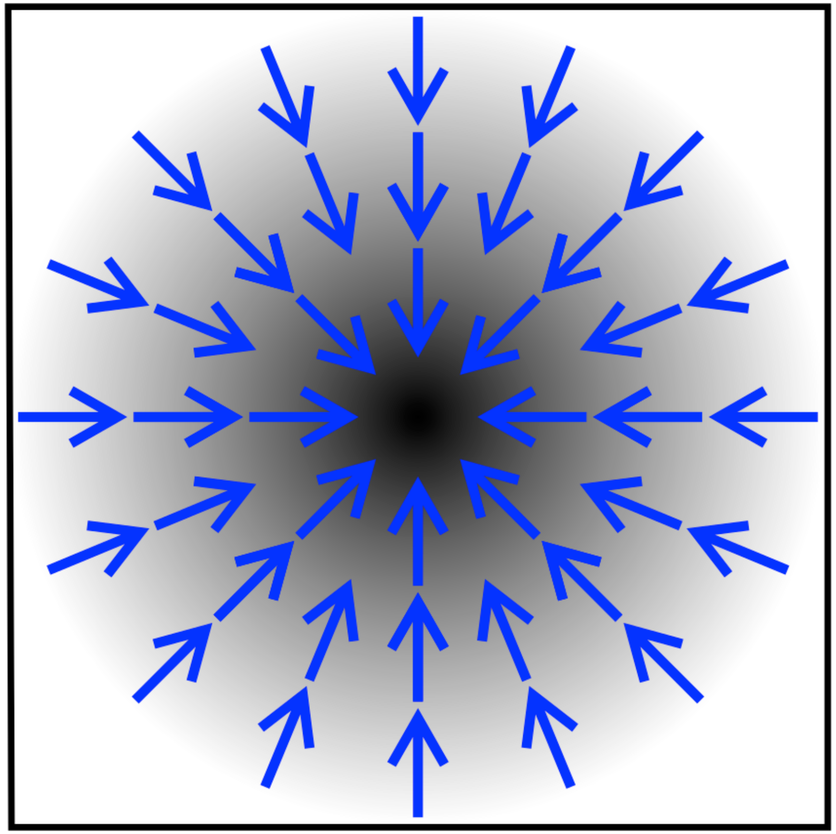


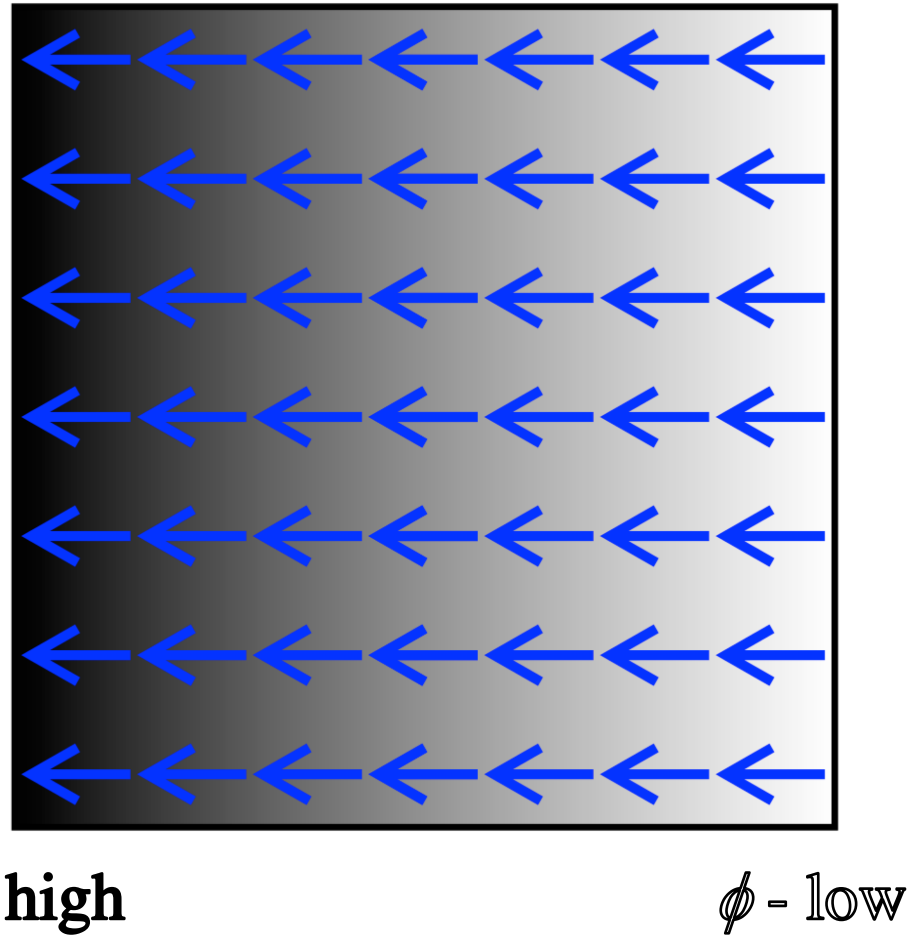
usually has a different direction than **v**

remember: $\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$

$$\frac{d\mathbf{v}}{dt} = \frac{dv_1}{dt}\hat{\mathbf{e}}_1 + \frac{dv_2}{dt}\hat{\mathbf{e}}_2 + \frac{dv_3}{dt}\hat{\mathbf{e}}_3 + \underbrace{v_1\frac{d\hat{\mathbf{e}}_1}{dt} + v_2\frac{d\hat{\mathbf{e}}_2}{dt} + v_3\frac{d\hat{\mathbf{e}}_3}{dt}}_{\text{for Cartesian systems} = 0}$$

Vector derivatives
directional derivative: space

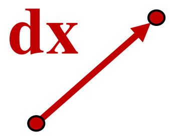




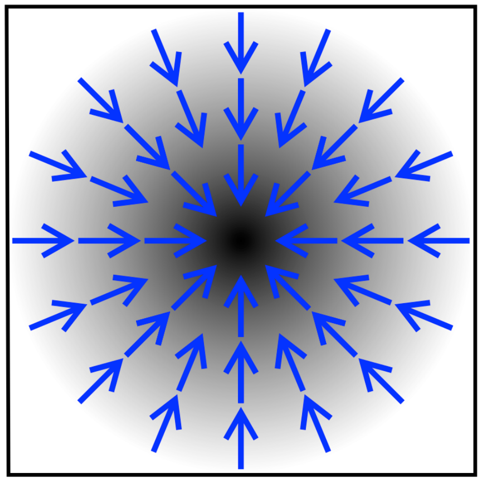
Vector derivatives
directional derivative: space

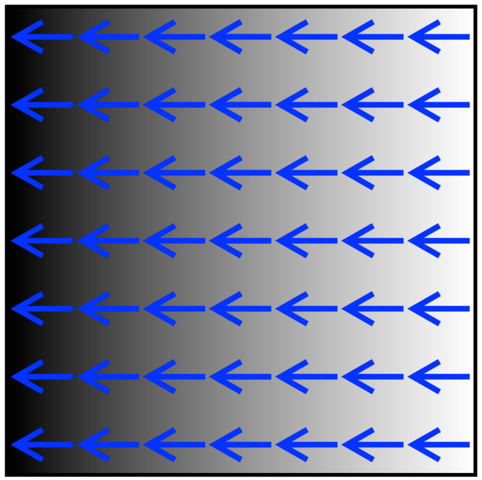
∂ - partial derivative

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3$$



$$d\phi = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \nabla \phi \cdot \mathbf{dx}$$





Gradient $\nabla \phi$: vector that is a measure of change in scalar field with direction

Del operator

$$\nabla = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix}$$

Has some properties of a vector, but not all

$$\mathbf{v} \cdot \nabla \phi \neq \left(\nabla \cdot \mathbf{v} \right) \phi$$

Vector products with derivatives
divergence, curl

- Divergence of a vector: $$\nabla \cdot \mathbf{v} = \sum_{i=1,3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

- Curl of a vector: $$\nabla \times \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\[2ex] \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\[2ex] \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix}$$

Useful calculus theorems

Gauss or divergence theorem: $\int_V \nabla \cdot \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$

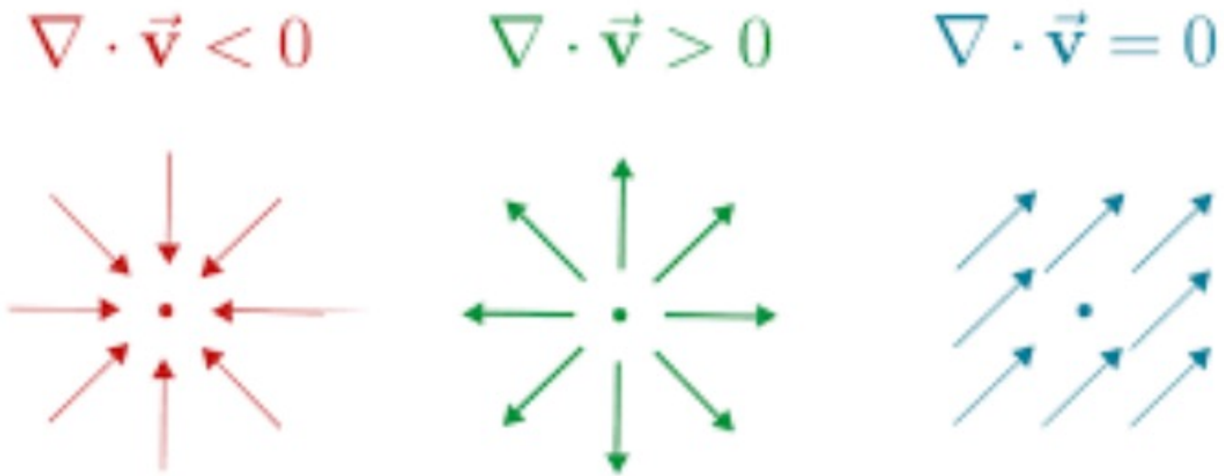
Stokes or curl theorem: $\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$

Take volume V enclosed by a closed surface S
within a vector field **v** with continuous partial derivatives

Flow perpendicular to the boundary: $\mathbf{v} \cdot \hat{\mathbf{n}}$
Flow parallel to the boundary: $\mathbf{v} \cdot \hat{\mathbf{t}}$

These can be used to simplify integration over volumes or closed surfaces as well as to gain understanding of the meaning of div and curl

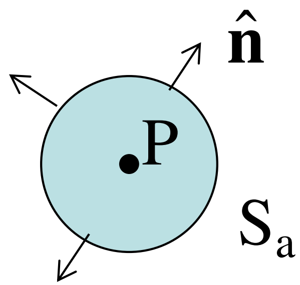
Divergence of a vector field



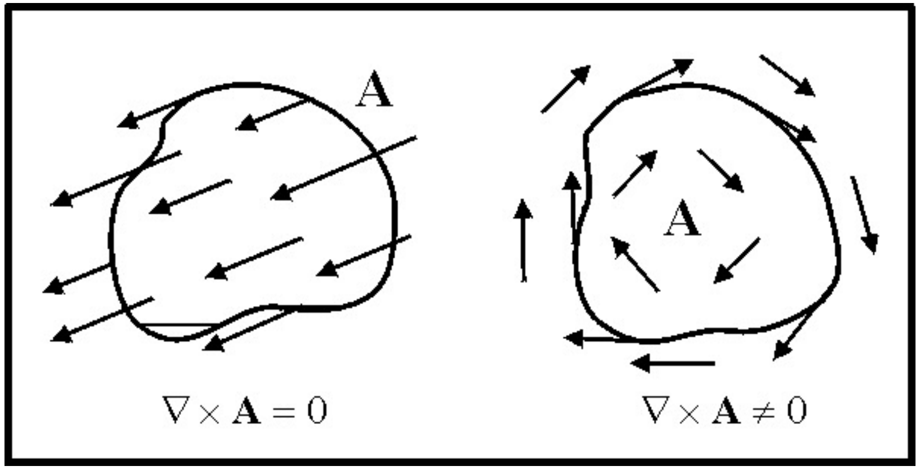
$$\int_V \nabla \cdot \mathbf{v}\, d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$$

Imagine a very small sphere, radius a, with boundary S_a around a point P

$$(\nabla \cdot \mathbf{v})_P \frac{4}{3}\pi a^3 = \oint_{S_a} \mathbf{v} \cdot \hat{\mathbf{n}}\, ds$$



Divergence of a vector field represents the net outward flux per unit volume, i.e. measure of source/sink of flow



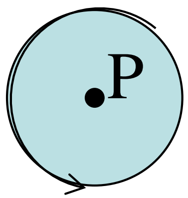
Curl of a vector field

Amount of turn or spin, vorticity, in a vector field

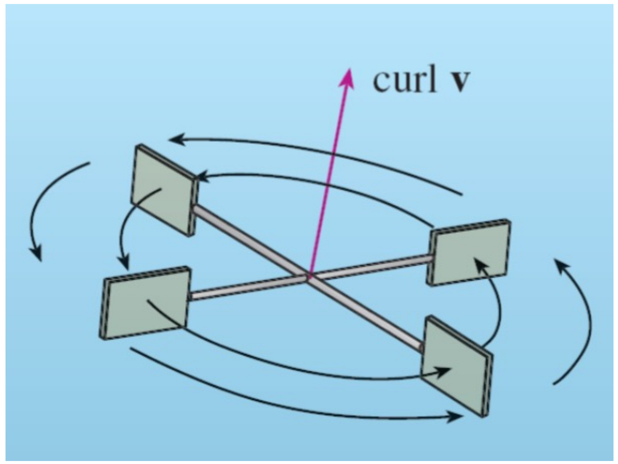
Curl theorem: $\int_V \nabla \times \mathbf{v}\, d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$

Right-hand side larger if velocities more parallel to the boundary, spinning in consistent direction, circulation around the boundary

Imagine a very small disk, radius a, with boundary C_a around a point P



C_a



$$(\nabla \times \mathbf{v})_P\, \pi a^2 = \oint_{C_a} \mathbf{v} \cdot \hat{\mathbf{t}}\, ds$$

Covered so far

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention
- Familiarity with the special tensors δ_{ij} and ε_{ijk}

Try yourself

- Please try Exercises 5 & 6 in the notebook