Modelling and Numerical Methods

Lecture 2

Stress and Tensors

Outline

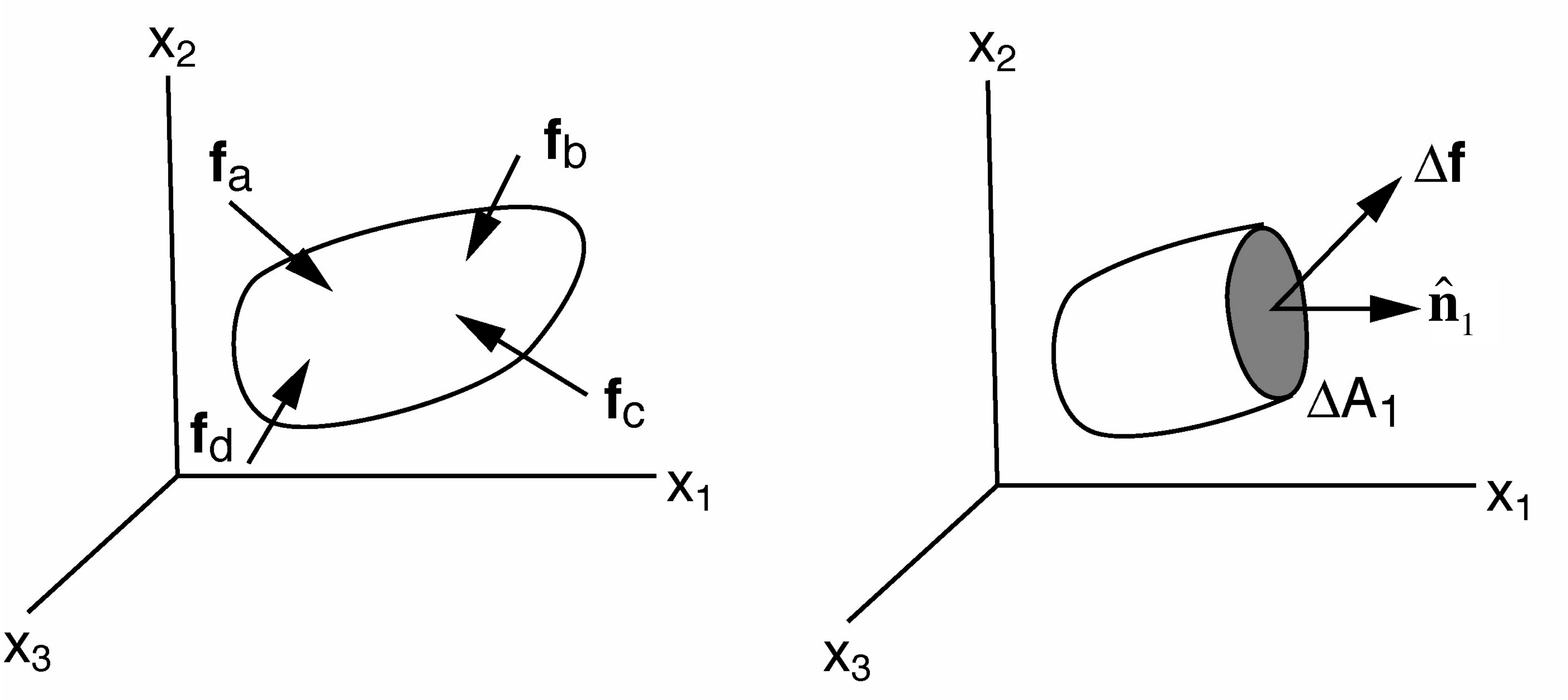
- Cauchy stress tensor recap
- Coordinate transformation (stress) tensors
- (Stress) tensor symmetry
- Tensor invariants
- Diagonalising, eigenvalues, eigenvectors
- Special stress states
- Equation of motion

Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to transform rank 2 tensor to a new basis.
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

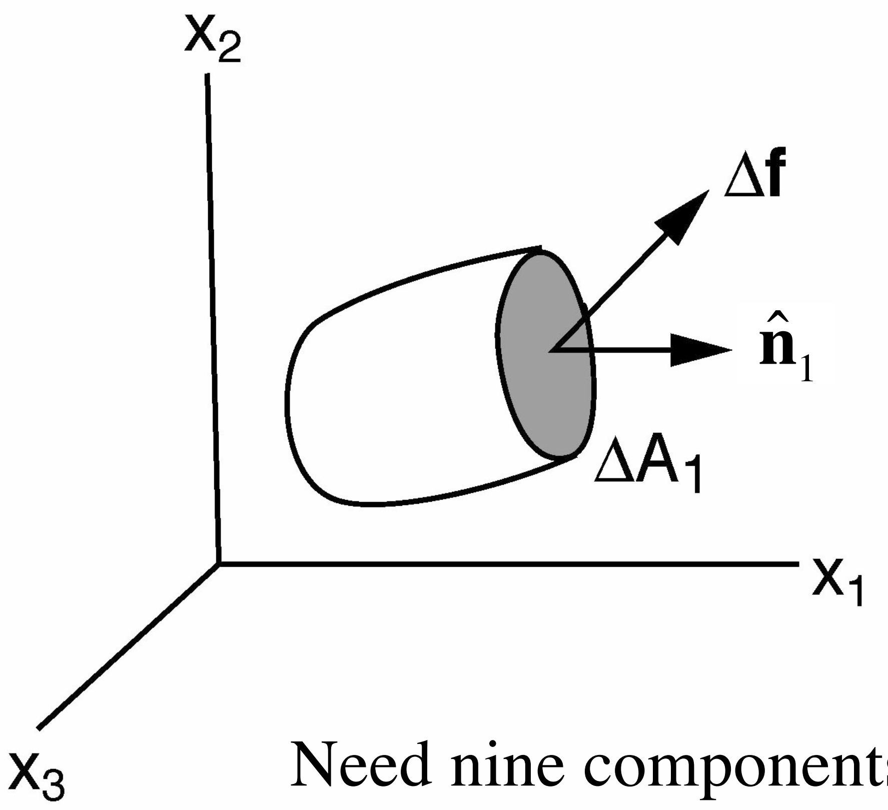
Cauchy Stress

Stress in a point, measured in medium as deformed by the stress experienced.



forces introduce a state of stress in a body

(Other stress measures, e.g., Piola-Kirchhoff tensor, used in Lagrangian formulations)



traction, stress vector

$$\mathbf{t_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta \mathbf{f}/\Delta A_1$$

$$\mathbf{t_1} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1
$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2
$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

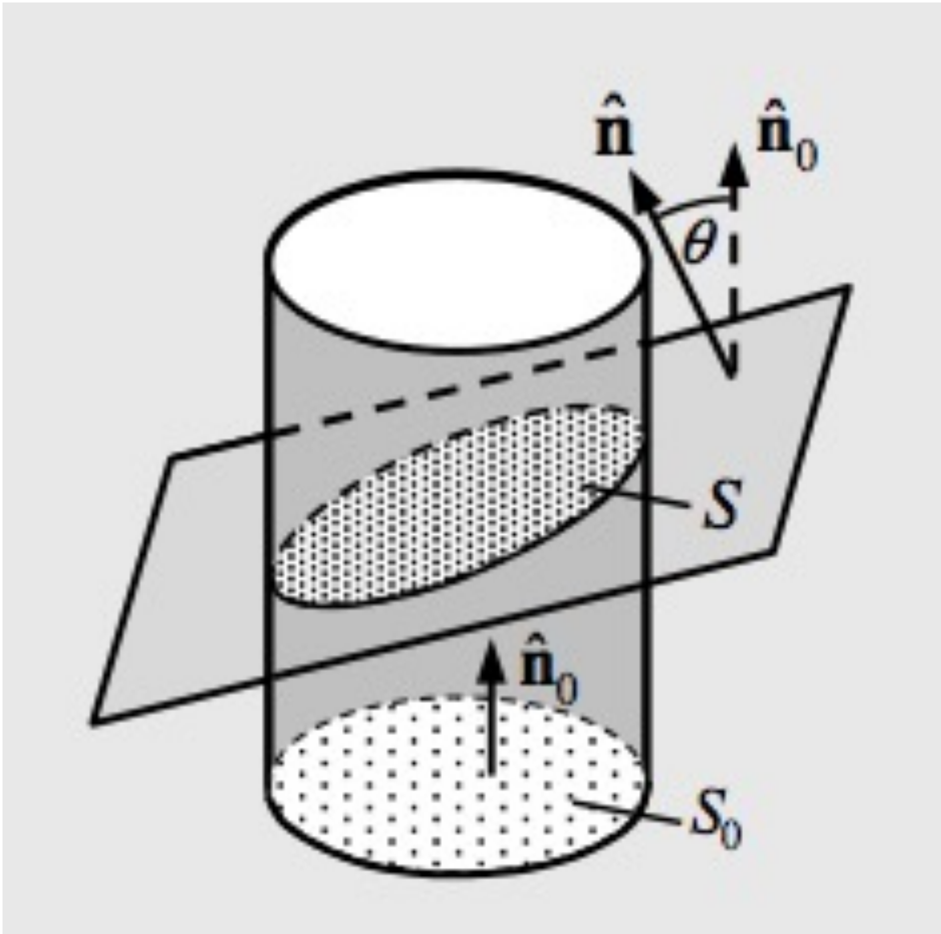
first index = orientation of plane
second index = orientation of force

Plane area as a vector

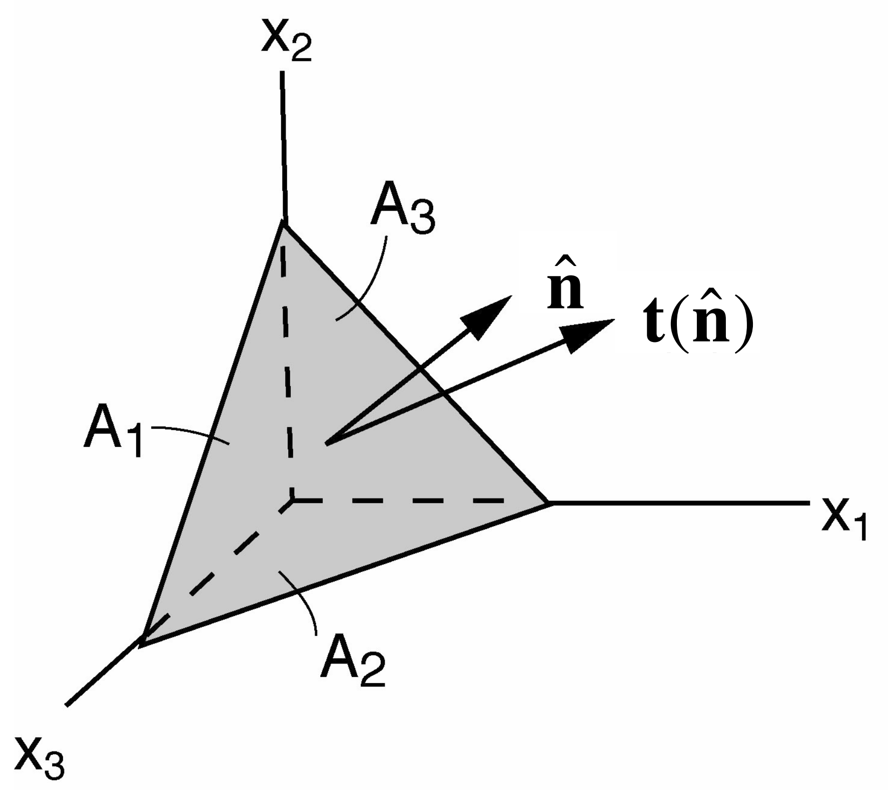
The area of plane S can be defined in terms of vectors assuming S_0 and θ are known.

$$S_0 = \mathbf{S} \cdot \hat{\mathbf{n}}_0 =$$
$$S\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 = S\cos\theta$$

$$\Rightarrow S = S_0 / \cos\theta$$

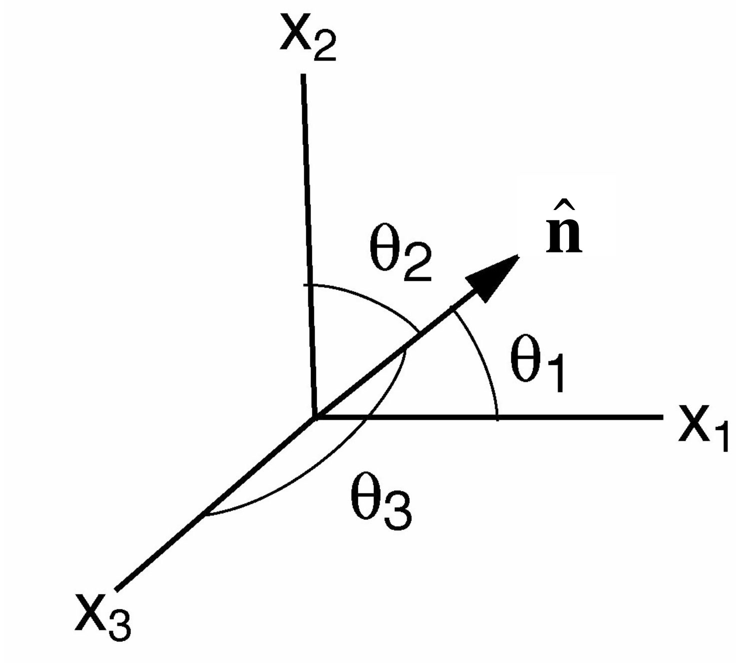


Are nine components sufficient? Demonstrate with equilibrium for a tetrahedron



Given: stress on A_1, A_2, A_3

Find: $\mathbf{t}(\hat{\mathbf{n}})$



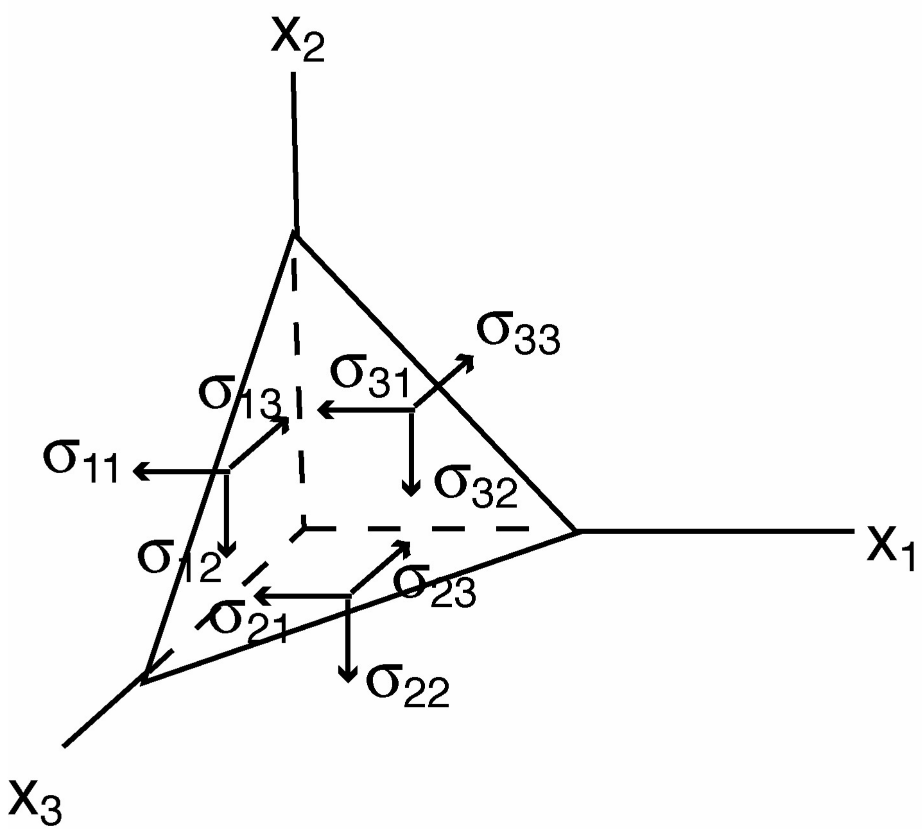
1: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_1$, $\Delta A_1 = \Delta A\cos\theta_1$

2: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_2$, $\Delta A_2 = \Delta A\cos\theta_2$

3: $\hat{\mathbf{n}} = -\hat{\mathbf{x}}_3$, $\Delta A_3 = \Delta A\cos\theta_3$

4: $\hat{\mathbf{n}} = \left(n_1, n_2, n_3\right)$, $n_i = \cos\theta_i$, $\Delta A_4 = \Delta A$

$$\Sigma f_1 = t_1\Delta A - \sigma_{11}\Delta A\cos\theta_1 - \sigma_{21}\Delta A\cos\theta_2 - \sigma_{31}\Delta A\cos\theta_3 = 0$$



this gives:

$$t_1 = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3$$

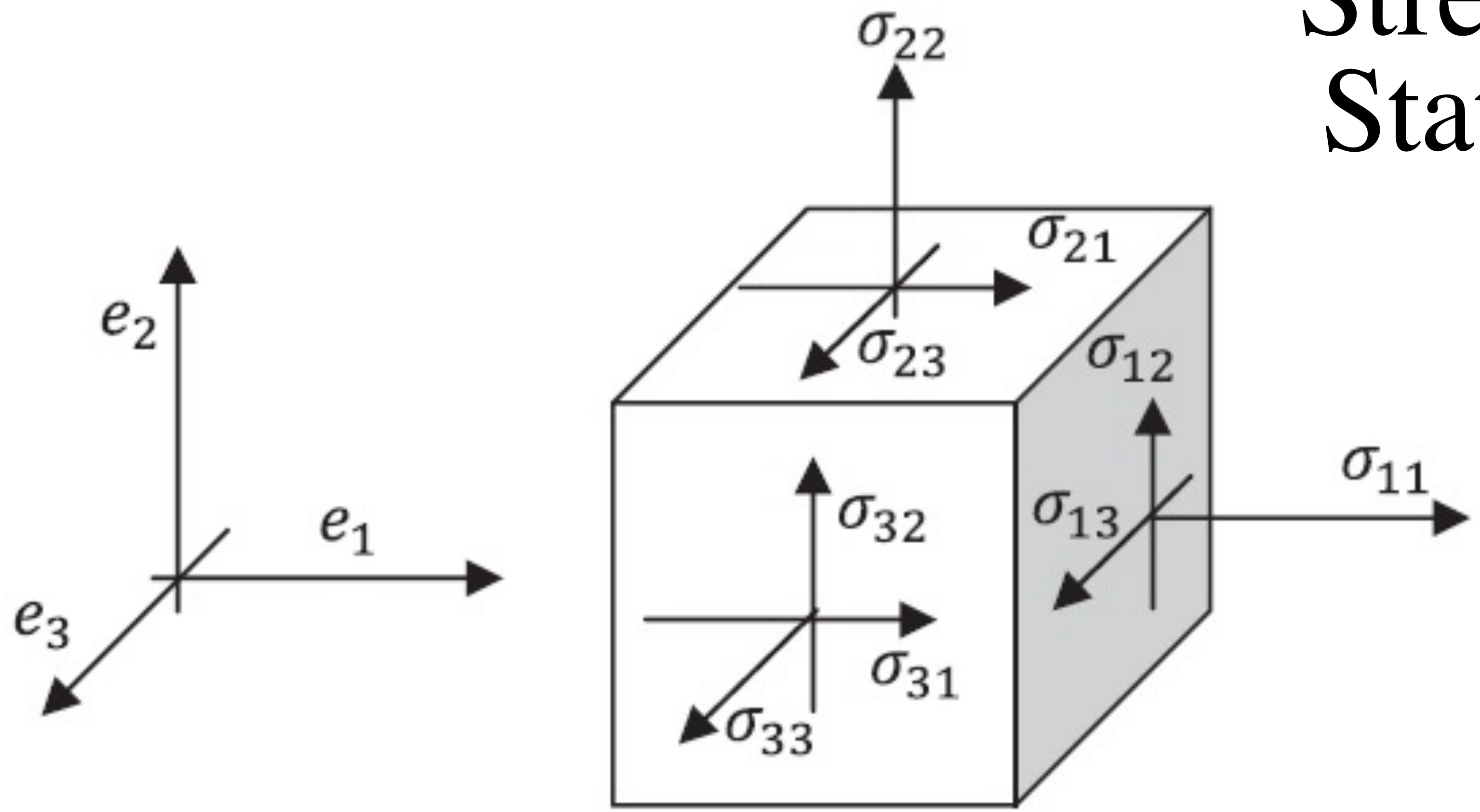
similarly:

$$t_2 = \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3$$
$$t_3 = \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3$$

$\Rightarrow$ $\boxed{t_i = \sigma_{ji}n_j}$ (Einstein convention)

How many stress components required in 2D?

3-D Stress State



first index = orientation of plane
second index = orientation of force

Positive if force in direction of normal (as shown)

$t_i = \sigma_{ji} n_j$ $\qquad$ $\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$ $\qquad$ Transpose: $\sigma_{ji} = \sigma^T_{ij}$

Note: unusual index order

in matrix notation: $\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$

$\mathbf{t}$ and $\hat{\mathbf{n}}$ - tensors of rank 1 (vectors) in 3-D

$\underline{\boldsymbol{\sigma}}$ - tensor of rank 2 in 3-D

compression - negative

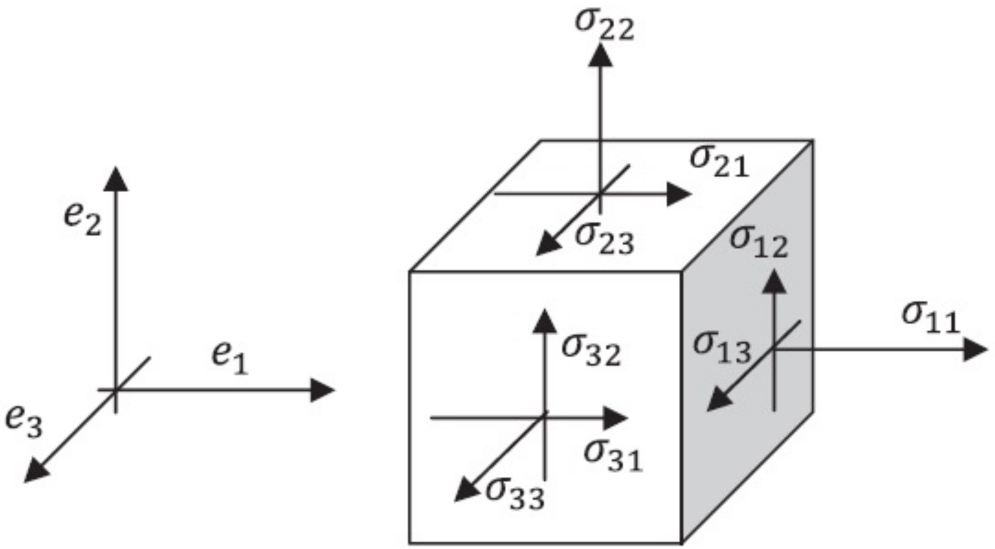
tension - positive

σ_{ji} where i=j - normal stresses

σ_{ji} where i≠j - shear stresses

2nd order tensors can be written as square matrices and have algebraic properties similar to some of those of matrices.

Stress components



traction on a plane

$$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

what is (1) $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$?

what is (2) $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_1$? what is (3) $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_2$?

Tensor symmetry

A tensor can be symmetric in one or more indices

For rank 2:

$S_{ij}= S_{ji}$ => $\mathbf{S}=\mathbf{S}^T$ symmetric

$S_{ij}= -S_{ji}$ => $\mathbf{S}=-\mathbf{S}^T$ antisymmetric

Higher rank:

e.g., $S_{ijk}= S_{jik}$ for all i,j,k => symmetric in i,j

antisymmetric **T** of rank 2

=> $T_{ij}=0$ for i=j, trace(**T**)=0

has n(n-1)/2 independent components

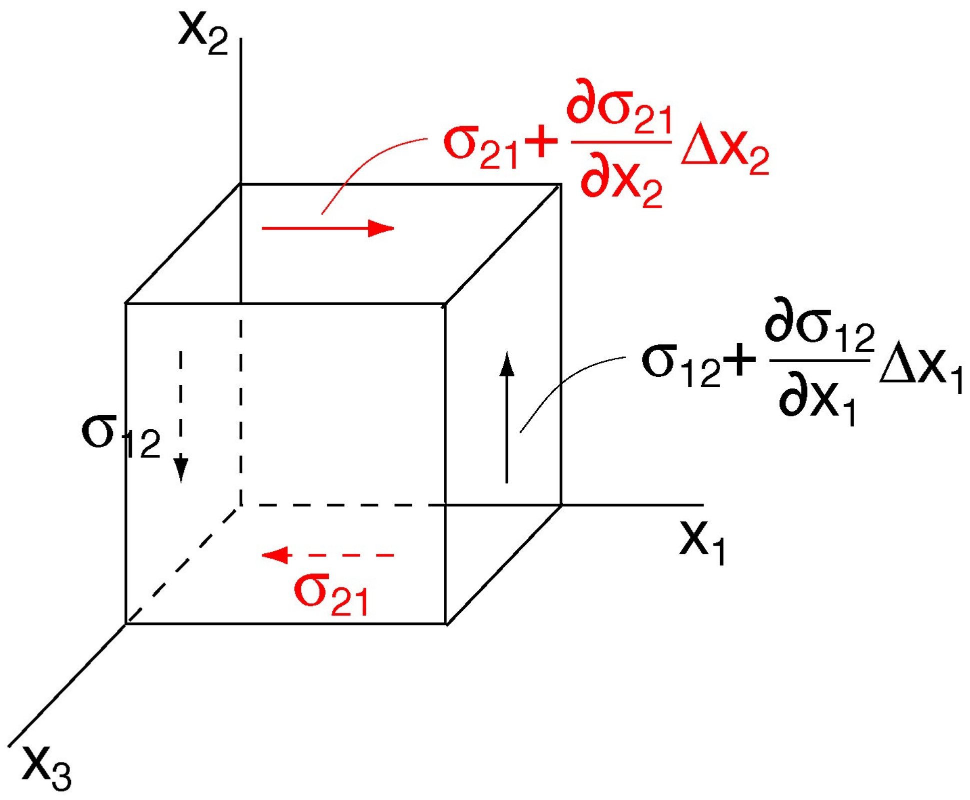
symmetric **T** of rank 2

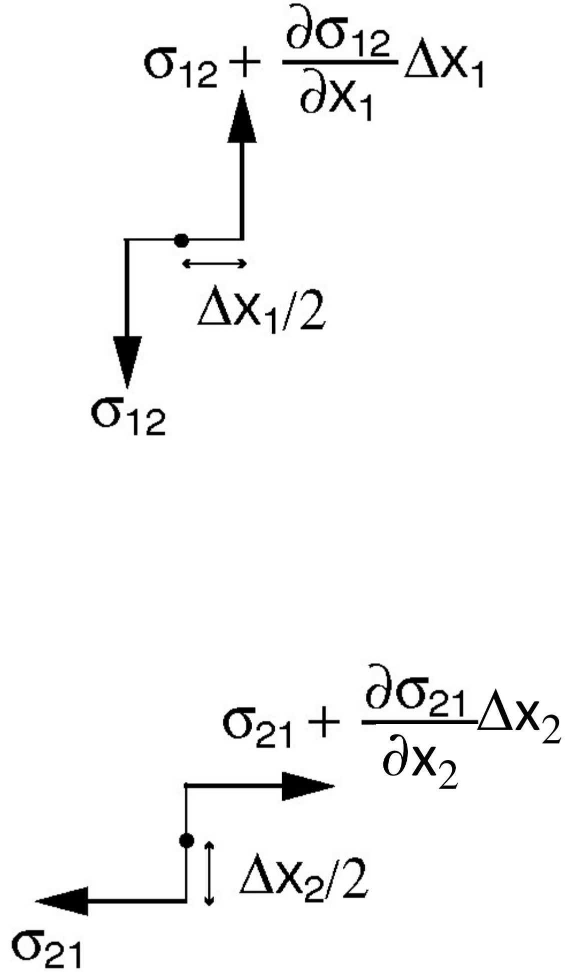
has n(n+1)/2 independent components

Any **T** of rank 2 can be decomposed in symm. and antisymm. part:

$$T_{ij} = (T_{ij}+T_{ji})/2 + (T_{ij}-T_{ji})/2$$

Symmetry of the stress tensor





Try writing out the balance of moments in x_3 direction,

assuming static equilibrium

A balance of moments in x_3 direction:

$$m_3 = [\qquad\qquad\qquad\qquad\qquad] \quad \cdot \Delta x_1 / 2$$

$$-[\qquad\qquad\qquad\qquad\qquad] \quad \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \to 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

Note: if body force induced rotation:

$$I_{33} \frac{\partial \omega}{\partial t} = O(\Delta x^2)$$

Balancing m_1 and m_2: $\boxed{\sigma_{23} = \sigma_{32}}$ *and* $\boxed{\sigma_{13} = \sigma_{31}}$

*thus, the stress tensor is **symmetric***

$$\mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \hat{\mathbf{n}} \Rightarrow \mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

A balance of moments in x_3 direction:

$$m_3 = [\sigma_{12} + (\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})]\Delta x_2 \Delta x_3 \cdot \Delta x_1 / 2$$

$$-[\sigma_{21} + (\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})]\Delta x_1 \Delta x_3 \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \to 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

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Balancing m_1 and m_2: $\boxed{\sigma_{23} = \sigma_{32}}$ *and* $\boxed{\sigma_{13} = \sigma_{31}}$

thus, the stress tensor is ***symmetric***

$$\mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \hat{\mathbf{n}} \Rightarrow \mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

Take a break

Then try Exercises 1 & 2 in the notebook

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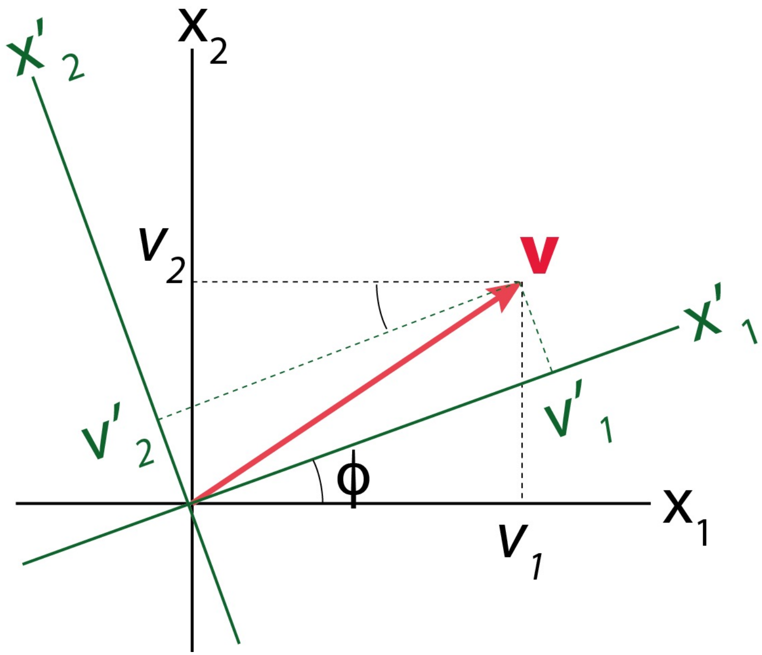
physical parameters should not depend on coordinate frame
$\Rightarrow$ **tensors follow linear transformation laws**

for vectors on orthonormal basis:



$$\mathbf{v}' = \mathbf{A}\mathbf{v}$$

$$\Rightarrow \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v} = \begin{bmatrix} \cos\phi & \cos(90-\phi) \\ \cos(90+\phi) & \cos\phi \end{bmatrix} \mathbf{v}$$

$$\boxed{\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j}$$

Inverse transform: $v_j = \alpha_{ji}\, v'_i$ $\qquad \alpha_{ji} = \hat{e}_j \cdot \hat{e}'_i$

In a new coordinate system:

Traction $\mathbf{t}' = \mathbf{A}\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{A}^T\mathbf{t}'$

normal $\mathbf{n}' = \mathbf{A}\mathbf{n} \Rightarrow \mathbf{n} = \mathbf{A}^T\mathbf{n}'$

$$\mathbf{t} = \boldsymbol{\sigma}^T\mathbf{n}$$

$$\mathbf{t}' = \boldsymbol{\sigma}'^T\mathbf{n}'$$

Relation $\boldsymbol{\sigma}'$ to $\boldsymbol{\sigma}$?

$\Rightarrow$ *transformation for stress tensor*

$$\mathbf{t}' = \mathbf{A}\boldsymbol{\sigma}^T\mathbf{n}$$

$$\mathbf{t}' = \mathbf{A}\boldsymbol{\sigma}^T\mathbf{A}^T\mathbf{n}'$$

$$\mathbf{t}' = \boldsymbol{\sigma}'^T\mathbf{n}'$$

$$\Rightarrow \quad \boldsymbol{\sigma}'^T = \mathbf{A}\boldsymbol{\sigma}^T\mathbf{A}^T$$

- transformation matrices are orthogonal
  $\alpha_{ij}{}^{-1} = \alpha_{ji} \quad (\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}})$
- *remember* $\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$
  $\alpha_{ij}^{-1} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}'_j = \alpha_{ji} = \alpha_{ij}^T$

$\Rightarrow$ each dependence on direction transforms as a vector, requiring two transformations

An *n-dimensional* tensor of *rank r* consists of n^r components

This tensor $T_{i1,i2,\ldots,in}$ is defined relative to a basis of the real, linear n-dimensional space S_n

and under a coordinate transformation **T** transforms as:

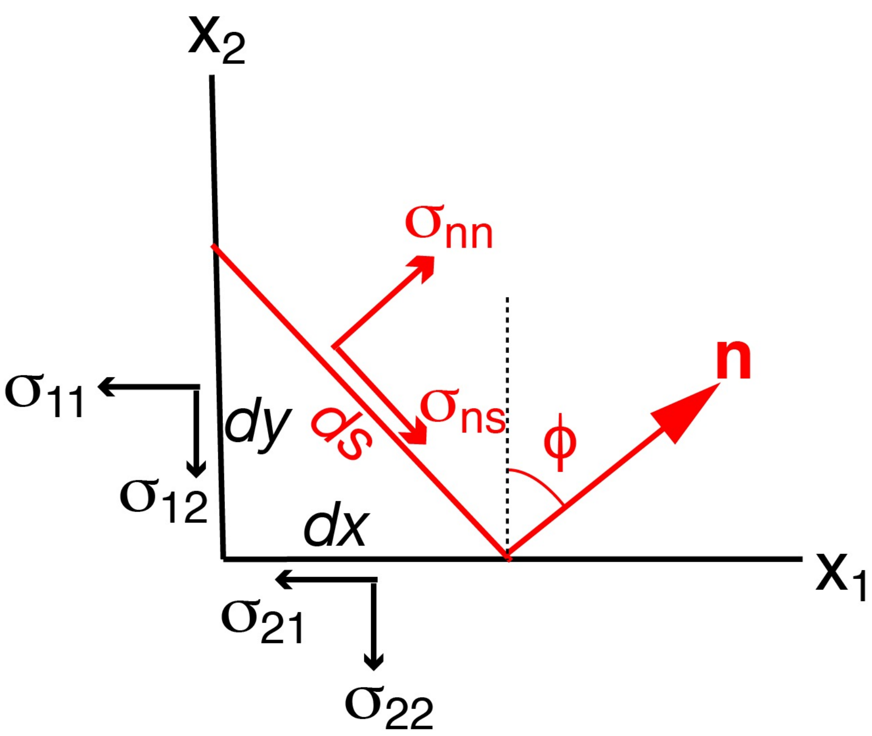
$$T'_{ij\ldots n} = \alpha_{ip}\alpha_{jq}\ldots\alpha_{nt}\, T_{pq\ldots t}$$

For *orthonormal* bases the matrices α_{ik} are *orthogonal* transformations, i.e. $\alpha_{ik}^{-1} = \alpha_{ki}$. (columns and rows are orthogonal and have length =1, i.e., perpendicular unit vectors are transformed to perpendicular unit vectors)

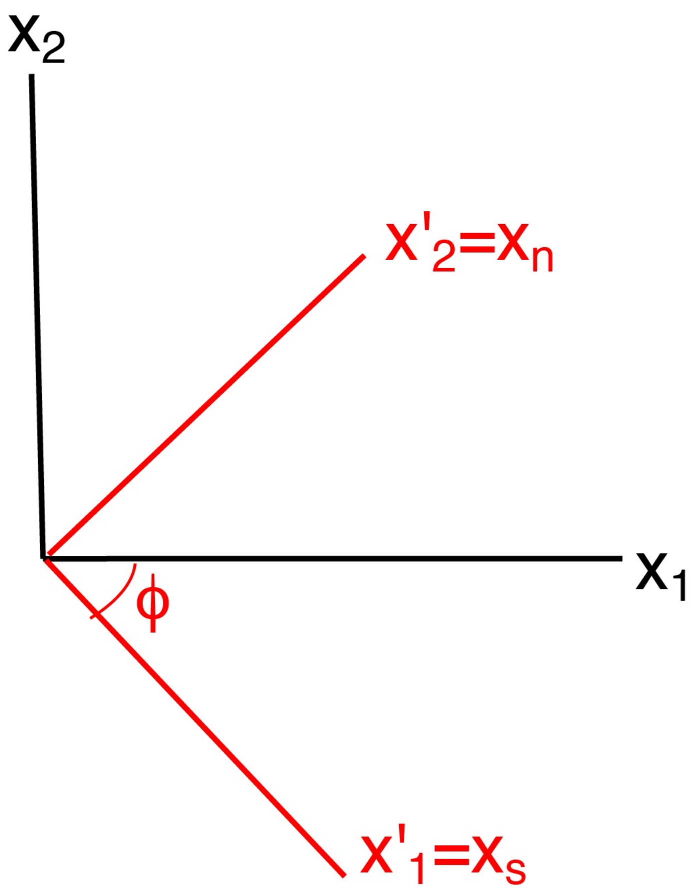
If the basis is *Cartesian*, α_{ik} are *real*.

Transforming the 2-D stress tensor

(determining normal and shear stress on a plane)



$$dx = \cos\phi \, ds$$
$$dy = \sin\phi \, ds$$



Try writing force balance in x_1 direction

Force balance

in x_1 direction: (1)

$$\sigma_{11}dy + \sigma_{21}dx = \sigma_{nn}\sin\phi ds + \sigma_{ns}\cos\phi ds$$

$$\sigma_{11}\sin\phi + \sigma_{21}\cos\phi = \sigma_{nn}\sin\phi + \sigma_{ns}\cos\phi$$

in x_2 direction: (2)

$$\sigma_{12}dy + \sigma_{22}dx = \sigma_{nn}\cos\phi ds - \sigma_{ns}\sin\phi ds$$

$$\sigma_{12}\sin\phi + \sigma_{22}\cos\phi = \sigma_{nn}\cos\phi - \sigma_{ns}\sin\phi$$

(1)·sinϕ + (2)·cosϕ: *verify yourself*

$$\sigma_{nn} = \sigma_{11}\sin^2\phi + \sigma_{21}\cos\phi\sin\phi + \sigma_{12}\cos\phi\sin\phi + \sigma_{22}\cos^2\phi$$

(1)·cosϕ – (2)·sinϕ:

$$\sigma_{ns} = \sigma_{11}\cos\phi\sin\phi + \sigma_{21}\cos^2\phi - \sigma_{12}\sin^2\phi - \sigma_{22}\cos\phi\sin\phi$$

This is equivalent to the **tensor transformation** $\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$

$\sigma'_{nn}=\alpha_{ni}\alpha_{nj}\sigma_{ji}$

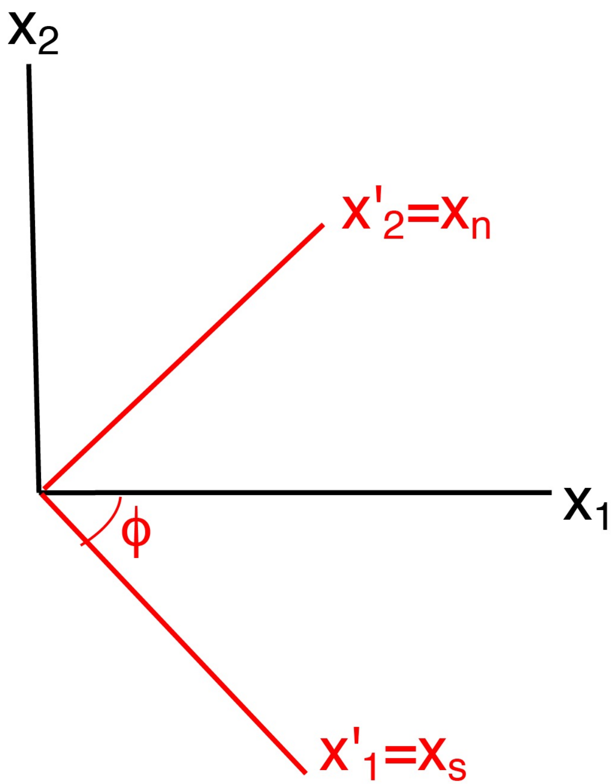
$\sigma'_{ns}=\alpha_{si}\alpha_{nj}\sigma_{ji}$

With $\alpha_{n1} = \sin\phi$, $\alpha_{n2} = \cos\phi$, $\alpha_{s1} = \cos\phi$, $\alpha_{s2} = -\sin\phi$

$x_1' = x_s$
$x_2' = x_n$

Write out transformation

$\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$



$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$

$$\alpha_{s1} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_1 = \cos\phi$$

$$\alpha_{s2} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_2 = -\sin\phi$$

$$\alpha_{n2} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2 = \cos\phi$$

$$\alpha_{n1} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1 = \sin\phi$$

In tensor notation:

$\boldsymbol{\sigma}'^{T} = \mathbf{A}\cdot\boldsymbol{\sigma}^{T}\cdot\mathbf{A}^{T}$

In matrix notation:

$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \alpha_{s1} & \alpha_{s2} \\ \alpha_{n1} & \alpha_{n2} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \alpha_{s1} & \alpha_{n1} \\ \alpha_{s2} & \alpha_{n2} \end{bmatrix}$$

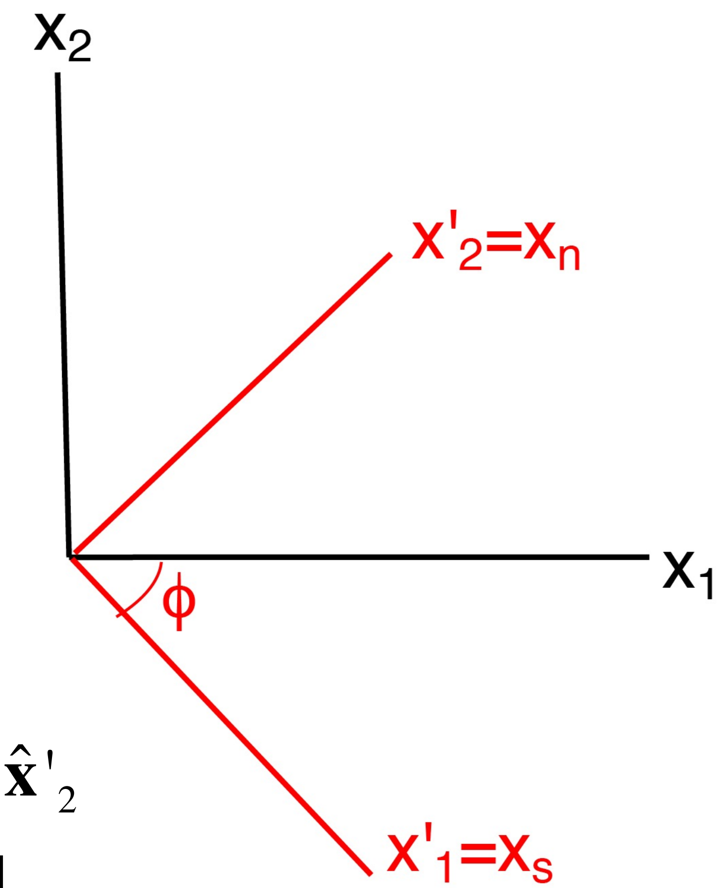
$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

For $\hat{\mathbf{x}}_1=(1,0)$, $\hat{\mathbf{x}}_2=(0,1)$,
first row of $\mathbf{A}$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\hat{\mathbf{x}}'_1 = (\cos\phi, -\sin\phi)$$

$$\hat{\mathbf{x}}'_2 = (\sin\phi, \cos\phi)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}'_1\cdot\mathbf{x}_1 & \mathbf{x}'_1\cdot\mathbf{x}_2 \\ \mathbf{x}'_2\cdot\mathbf{x}_1 & \mathbf{x}'_2\cdot\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$



You may recognise $\mathbf{A}$ *as a matrix that describes a rigid-body rotation over and angle* $-\phi$

A^T *describes a rotation over angle* ϕ

First column of $\mathbf{A}^{\mathrm{T}}$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \mathbf{x}_1\cdot\mathbf{x}'_1 & \mathbf{x}_1\cdot\mathbf{x}'_2 \\ \mathbf{x}_2\cdot\mathbf{x}'_1 & \mathbf{x}_2\cdot\mathbf{x}'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to transform rank 2 tensor to a new basis.
- Be able to find principal stresses and stress invariants and know what they represent
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Take a break

Then try Exercise 5 in the notebook

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Diagonalizing

Real-valued, symmetric rank 2 tensors (square, symmetric matrices) can be diagonalized, i.e. a coordinate frame can be found, such that only the diagonal elements (normal stresses) remain.

For stress tensor, these elements, σ_1, σ_2, σ_3 are called the principal stresses

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Such a transformation can be cast as:

$\mathbf{T} \cdot \mathbf{x} = \lambda \mathbf{x}$

where $\mathbf{x}_i$ are eigenvectors or characteristic vectors

and λ_i are the eigenvalues, characteristic or principal values

$\Rightarrow (\mathbf{T} - \lambda \delta) \cdot \mathbf{x} = \mathbf{0}$

Non-trivial solution only if $\det(\mathbf{T} - \lambda \delta) = 0$

Determinant

For 2-dimensional rank 2 tensor

det(**T**)≠0
columns of **T** are linearly independent, and $\mathbf{T}^{-1}$ exists

$$\det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = T_{11}T_{22} - T_{12}T_{21}$$

$$\det(\mathbf{a},\mathbf{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = \mathbf{a} \times \mathbf{b}$$

signed area

For 3-dimensional rank 2 tensor **T**={**a**,**b**,**c**}

$$\mathbf{T} \cdot \hat{\mathbf{e}}_1 = \mathbf{a}$$
$$\mathbf{T} \cdot \hat{\mathbf{e}}_2 = \mathbf{b}$$
$$\mathbf{T} \cdot \hat{\mathbf{e}}_3 = \mathbf{c}$$

$$\det(\mathrm{T}) = \det(\mathbf{a},\mathbf{b},\mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

$$= \varepsilon_{ijk} a_i b_j c_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

signed volume

Determinant and cross product

Can write cross product as a determinant

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\hat{\mathbf{e}}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\mathbf{e}}_2 \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Diagonalizing

Real-valued, symmetric rank 2 tensors (square, symmetric matrices) can be diagonalized, i.e. a coordinate frame can be found, such that only the diagonal elements (normal stresses) remain.

For stress tensor, these elements, σ_1, σ_2, σ_3 are called the principal stresses

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Such a transformation can be cast as:

$\mathbf{T}\cdot\mathbf{x} = \lambda\mathbf{x}$

where $\mathbf{x}_i$ are eigenvectors or characteristic vectors

and λ_i are the eigenvalues, characteristic or principal values

$\Rightarrow(\mathbf{T}-\lambda\delta)\cdot\mathbf{x} = \mathbf{0}$

Non-trivial solution only if $\det(\mathbf{T}-\lambda\delta) = 0$

Eigenvalues, eigenvectors

For real-valued, symmetric rank 2 order *n* tensors

- All eigenvalues are real
- If **T** is positive definite, then eigenvalues are positive
- Eigenvectors for two distinct λ are orthogonal.
- There are *n* linearly independent eigenvectors

$$\mathbf{T}\cdot\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\mathbf{T}\cdot\mathbf{x}_2 = \lambda_2\mathbf{x}_2$$

where $\lambda_1 \neq \lambda_2$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1 = \lambda_1\mathbf{x}_2\cdot\mathbf{x}_1 \qquad \mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2 = \lambda_2\mathbf{x}_1\cdot\mathbf{x}_2 = \lambda_2\mathbf{x}_2\cdot\mathbf{x}_1$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1 = \mathbf{x}_1\cdot\mathbf{T}^{\mathrm{T}}\cdot\mathbf{x}_2 \text{ with symmetry } = \mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1 - \mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2 = (\lambda_1 - \lambda_2)\mathbf{x}_2\cdot\mathbf{x}_1 = 0$$

$$\Rightarrow \mathbf{x}_2\cdot\mathbf{x}_1 = 0$$

Eigenvectors

- If x is an eigenvector with eigenvalue λ, then any multiple $\alpha\mathbf{x}$ is also an eigenvector: $\mathbf{T}\cdot\alpha\mathbf{x} = \alpha\lambda\mathbf{x}$

  $\Rightarrow$ Eigenvectors often scaled to unit vectors
- For repeated λ, infinite range of possible $\mathbf{x}$, usually set of orthonormal vectors chosen

Example: $\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Characteristic equation: $(2-\lambda)^2(3-\lambda)=0$

$\Rightarrow \lambda=2$ (twice), $\lambda=3$

Easy to verify that: $\mathbf{T}\cdot\hat{\mathbf{e}}_1=2\hat{\mathbf{e}}_1$, $\mathbf{T}\cdot\hat{\mathbf{e}}_2=2\hat{\mathbf{e}}_2$, $\mathbf{T}\cdot\hat{\mathbf{e}}_3=3\hat{\mathbf{e}}_3$

$\Rightarrow$ $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ eigenvectors, but so are any $a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2$

Invariants

$$I_1 = \mathrm{tr}(\mathbf{T}) \qquad = T_{11} + T_{22} + T_{33}$$

$$I_2 = \mathrm{minor}(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{32} \\ T_{32} & T_{33} \end{vmatrix}$$

$$= T_{11}T_{22} + T_{22}T_{33} + T_{11}T_{33} - T_{21}^2 - T_{32}^2 - T_{31}^2$$

$$I_3 = \det(\mathbf{T}) \qquad = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = T_{11}T_{22}T_{33} + 2T_{21}T_{32}T_{31}$$

$$- T_{11}T_{32}^2 - T_{22}T_{31}^2 - T_{33}T_{21}^2$$

In terms of eigenvalues, invariants simplify to:

$$I_1 = \mathrm{tr}(\mathbf{T}) \qquad = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \mathrm{minor}(\mathbf{T}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$I_3 = \det(\mathbf{T}) \qquad = \lambda_1\lambda_2\lambda_3$$

Check yourself

Hydrostatic and Deviatoric stress

Diagonalizing
=> principal stress coordinate frame

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

(σ_1 to σ_3 usually ordered from largest to smallest)

$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$

$tr(\boldsymbol{\sigma})$ = sum of normal stresses
$tr(\boldsymbol{\sigma})/3$ = - pressure p = average normal stress = ***hydrostatic stress***
$\Rightarrow$ volume change

$\boldsymbol{\sigma}'_{ij}$ is ***deviatoric stress*** $= \sigma_{ij} + p\delta_{ij}$
$\Rightarrow$ shape change

$tr(\boldsymbol{\sigma}')$ = ?

Second invariant deviatoric stress

σ'_{ij} is deviatoric stress = $\sigma_{ij}+p\delta_{ij}$

$$\text{minor}(\boldsymbol{\sigma}') = \sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{11}\sigma'_{33} - \sigma'^2_{21} - \sigma'^2_{32} - \sigma'^2_{31} \quad (1)$$

$$\begin{aligned} = & - \sigma'^2_{11} - \sigma'^2_{22} - \sigma'^2_{33} \\ & - \sigma'_{11}\sigma'_{33} - \sigma'_{11}\sigma'_{22} - \sigma'_{22}\sigma'_{33} \qquad (2) \\ & - \sigma'^2_{21} - \sigma'^2_{32} - \sigma'^2_{31} \end{aligned}$$

Using that:
$\mathrm{tr}(\sigma')=\sigma'_{11}+\sigma'_{22}+\sigma'_{33}$
$=0$

$$= \tfrac{1}{2}\,[(1)+(2)]$$

$$= -\tfrac{1}{2}\,[\sigma'^2_{11} + \sigma'^2_{22} + \sigma'^2_{33} + \sigma'^2_{21} + \sigma'^2_{32} + \sigma'^2_{31}]$$

$\text{minor}(\boldsymbol{\sigma})=\tfrac{1}{2}[\mathrm{tr}(\boldsymbol{\sigma}^2)-(\mathrm{tr}\boldsymbol{\sigma})^2]$, $\text{minor}(\boldsymbol{\sigma}')=\tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}'^2)$

measure of stress magnitude, important in flow and plastic yielding

Maximum shear stress

Principal stresses include largest and smallest normal stresses in given stress system (*see proof in Lai et al*.)

If σ_1 is largest and σ_3 smallest principal stress, then maximum shear stress

$$\left|\sigma_s^{\max}\right| = \frac{\sigma_1 - \sigma_3}{2}$$

See Exercise 4

- Show this using case of 2-D stress in σ_1, σ_3 coordinate frame,
- Determine the orientation of the corresponding direction relative to the σ_1, σ_3 coordinate frame

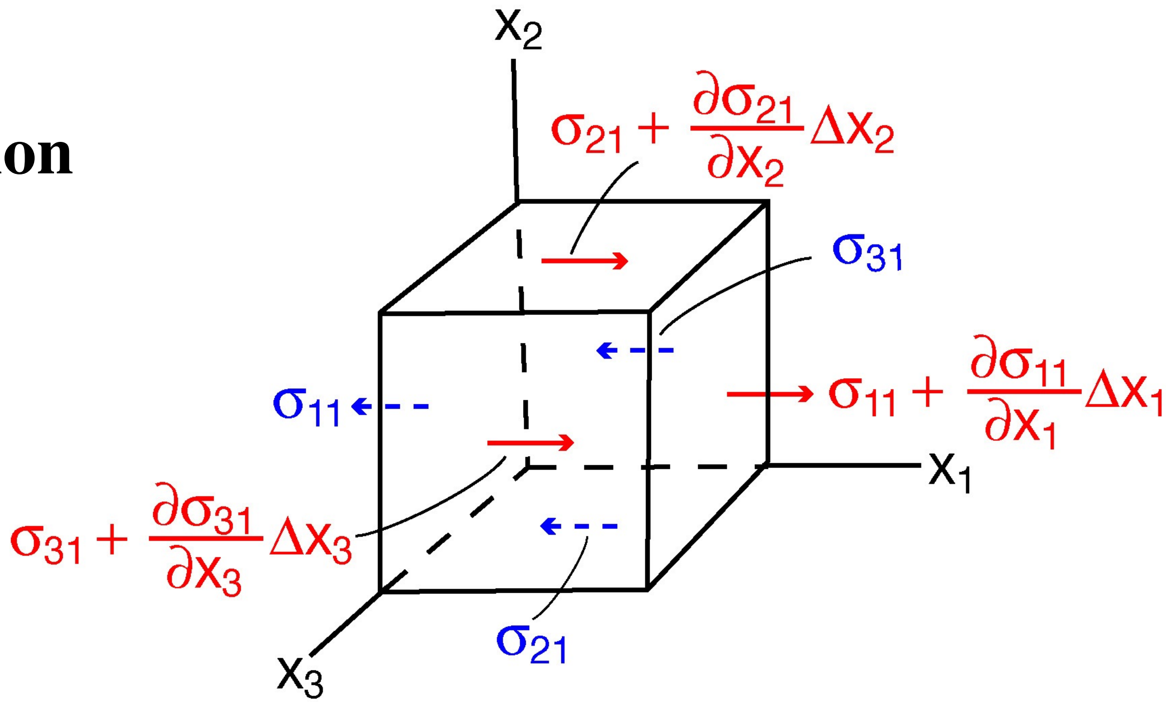
Maximum shear stress important for yield criteria

Equation of motion

Force balance:

$\mathbf{F}_{body}+\mathbf{F}_{stress}=m\mathbf{a}$

In x_1- direction:



$+$

$+$

$+$

$= \rho \Delta x_1 \Delta x_2 \Delta x_3 \partial^2 u_1/\partial t^2$

$\Rightarrow \; f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$

$\Rightarrow \; f_i + \partial\sigma_{ji}/\partial x_j \;= \rho\partial^2 u_i/\partial t^2$

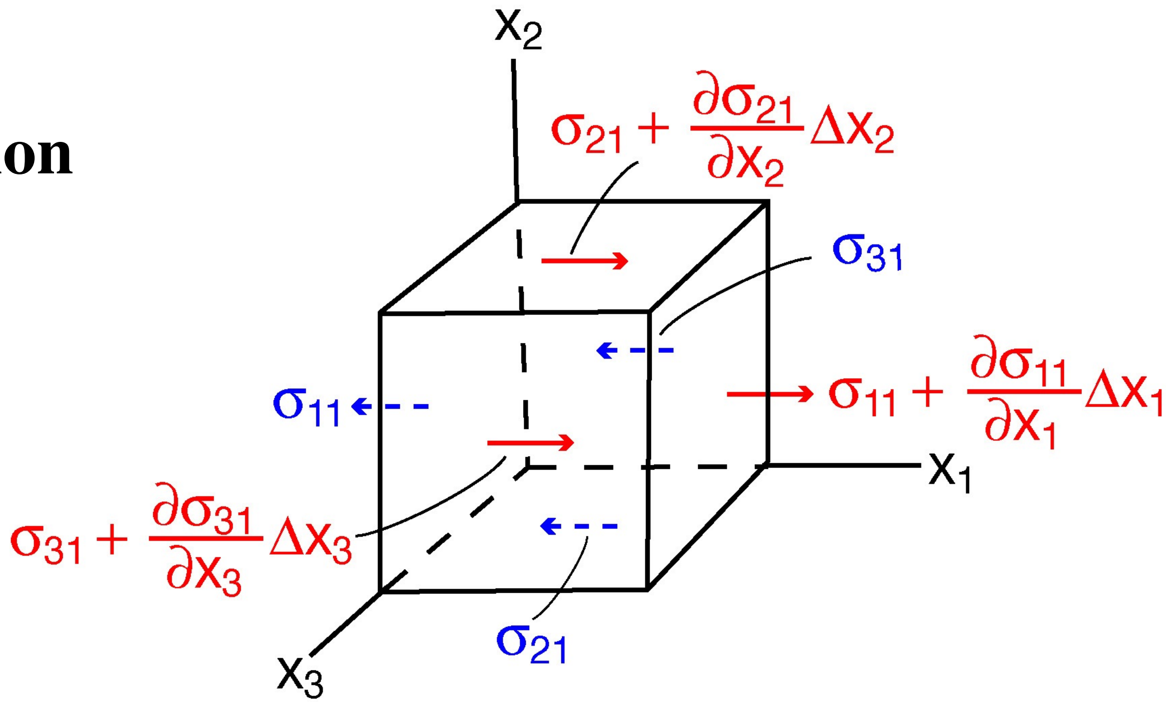
$\Rightarrow \; \mathbf{f} \;+ \nabla\cdot\underline{\boldsymbol{\sigma}} = \rho\partial^2 \mathbf{u}/\partial t^2$

Equation of motion

Force balance:

$\mathbf{F}_{body}+\mathbf{F}_{stress}=m\mathbf{a}$

In x_1- direction:



$$f_1\Delta x_1\Delta x_2\Delta x_3+$$
$$(\sigma_{11}+\Delta x_1\partial\sigma_{11}/\partial x_1 - \sigma_{11})\Delta x_2\Delta x_3 +$$
$$(\sigma_{21}+\Delta x_2\partial\sigma_{21}/\partial x_2 - \sigma_{21})\Delta x_1\Delta x_3 +$$
$$(\sigma_{31}+\Delta x_3\partial\sigma_{31}/\partial x_3 - \sigma_{31})\Delta x_1\Delta x_2 = \rho\Delta x_1\Delta x_2\Delta x_3\partial^2 u_1/\partial t^2$$

$$\Rightarrow\ f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$$

$$\Rightarrow\ f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$$
$$\Rightarrow\ \mathbf{f} + \nabla\cdot\underline{\boldsymbol{\sigma}} = \rho\partial^2\mathbf{u}/\partial t^2$$

Take a break

Then try Exercise 6 & 8 in the notebook

Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to transform rank 2 tensor to a new basis.
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

Summary
Stress Tensors

- Cauchy stress tensor
- Tensor coordinate transformation
- (Stress) tensor symmetry
- Tensor invariants
- Diagonalizing, eigenvalues, eigenvectors
- Special stress states
- Equation of motion

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 2.18 through 2.25, 4.4 through 4.7