Introduction Tensors

- Tensors, generalisation of vectors to more dimensions
- Use when properties depend on direction in more than one way.
- Stress tensor as example
- Stress is force per area, depends on the direction of the force as well as the chosen cross sectional area (which can be described by its normal) on which the stress is evaluated.

Tensors

Used in

Stress, strain, moment tensors

Electrostatics, electrodynamics, rotation, crystal properties

Tensors describe properties that depend on direction

Tensor rank 0 - scalar - independent of direction

Tensor rank 1 - vector - depends on direction in 1 way

Tensor rank 2 - tensor - depends on direction in 2 ways

Tensor comes from the word tension (= stress)

Notation

- Tensors as **T**
- for second order: $\overline{\overline{T}}$ or $\underline{\underline{T}}$
- Index notation T_{ij}, $i,j=x,y,z$ or $i,j=1,2,3$
- But also higher order T_{ijkl}

An example tensor

Gradient of velocity depends on direction in two ways

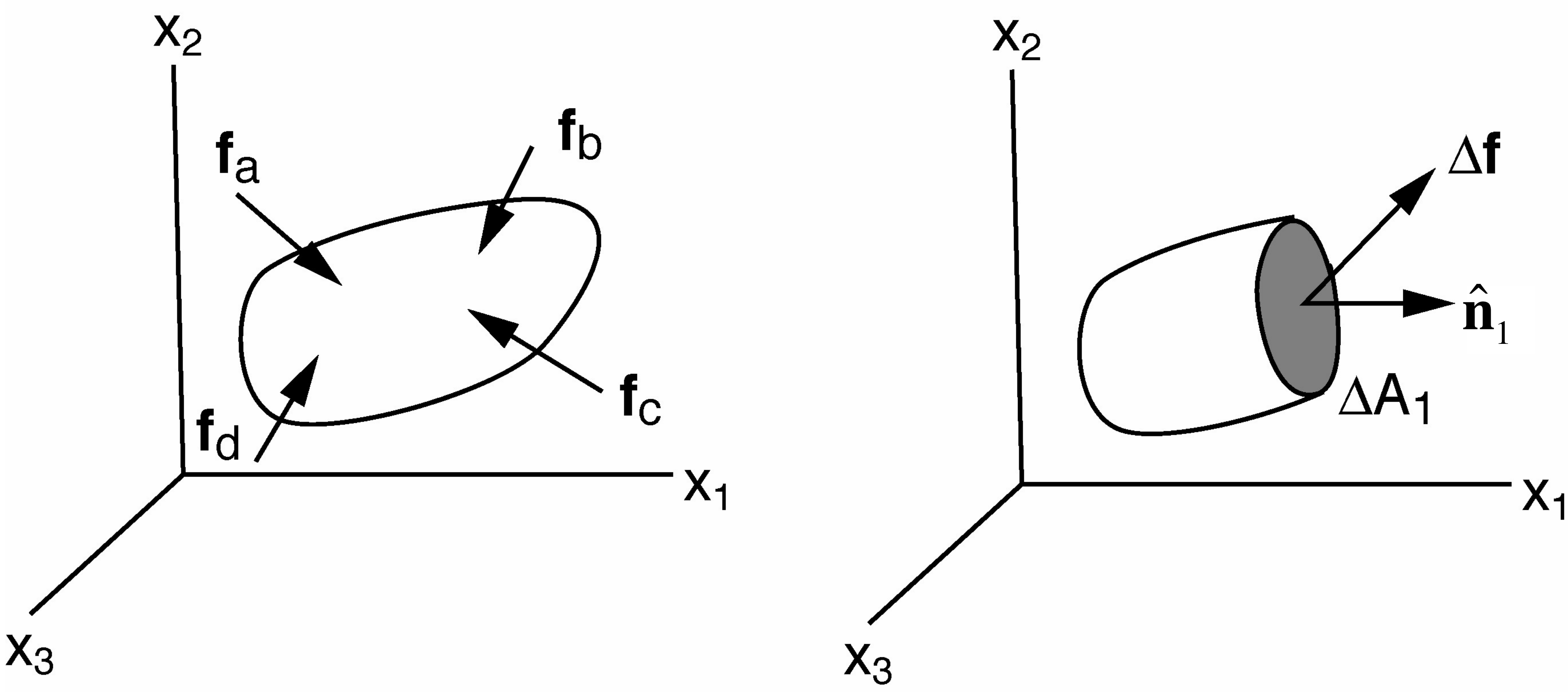
Component of velocity

Coordinate direction

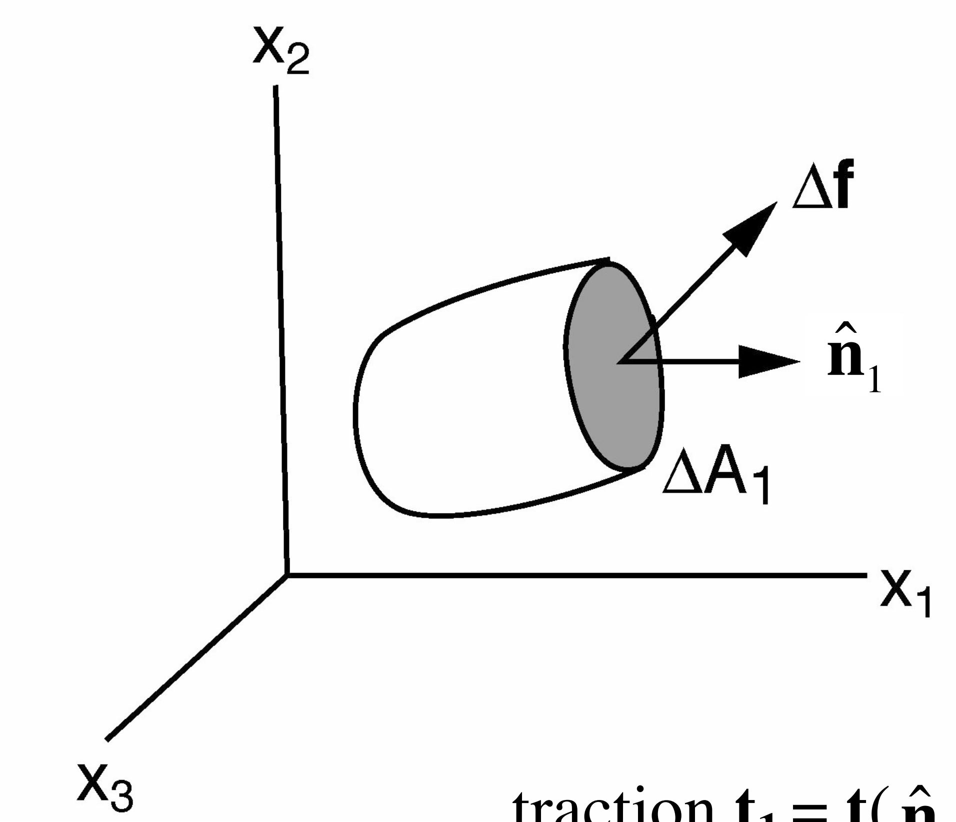
$$\nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Another example: Stress

- *Body forces* - depend on volume, e.g., gravity
- *Surface forces* - depend on surface area, e.g., friction



forces introduce a state of stress in a body

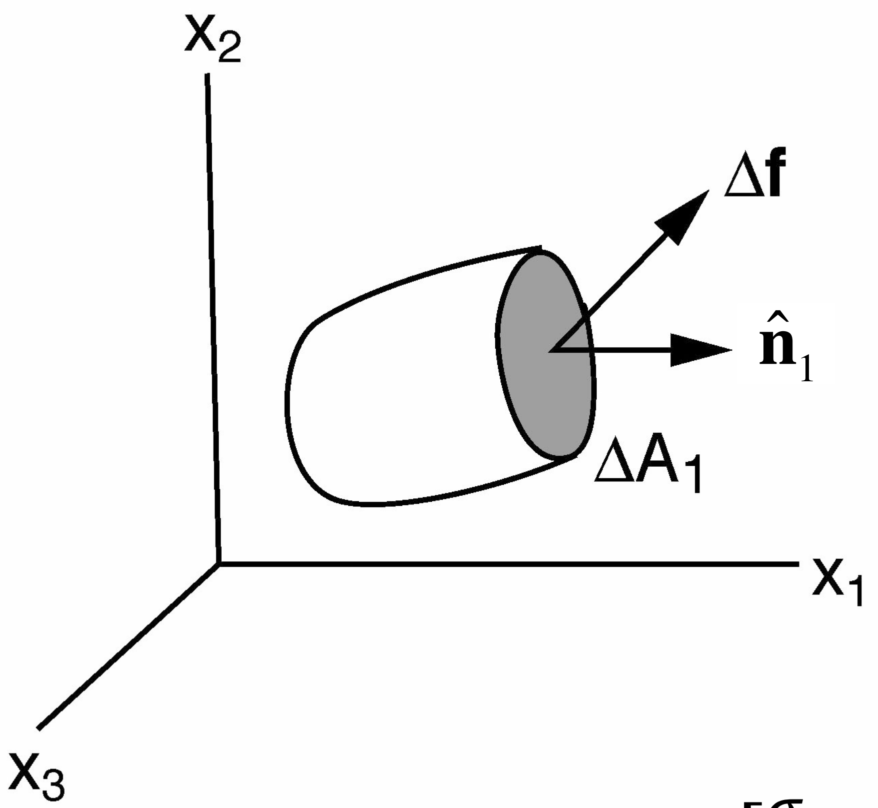


- $\Delta\mathbf{f}$ necessary to maintain equilibrium depends on orientation of the plane, $\hat{\mathbf{n}}_1$

$$\text{traction } \mathbf{t_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta\mathbf{f}/\Delta A_1$$

$$\mathbf{t_1} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

$$\sigma_{11} = \lim_{\Delta A1 \to 0} \Delta\mathbf{f}_1/\Delta A_1$$
$$\sigma_{12} = \lim_{\Delta A1 \to 0} \Delta\mathbf{f}_2/\Delta A_1$$
$$\sigma_{13} = \lim_{\Delta A1 \to 0} \Delta\mathbf{f}_3/\Delta A_1$$



Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1
$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2
$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

first index = orientation of plane
second index = orientation of force

Difference between a tensor and its matrix

Tensor – physical quantity that is independent of coordinate system used

Matrix of a tensor – contains components of that tensor in a particular coordinate frame

Could test that indeed tensor addition and multiplication satisfy transformation laws

Summation (Einstein) convention

When an index in a single term is a duplicate, dummy index, summation implied without writing summation symbol

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \sum_{i=1}^{3} a_i v_i = a_i v_i$$

$$\begin{aligned}\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i y_j = a_{ij} x_i y_j &= a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{13} x_1 y_3 \\ &+ a_{21} x_2 y_1 + a_{22} x_2 y_2 + a_{23} x_2 y_3 \\ &+ a_{31} x_3 y_1 + a_{32} x_3 y_2 + a_{33} x_3 y_3\end{aligned}$$

Invalid, indices repeated more than twice

$$\sum_{i=1}^{3} a_i b_i v_i \neq a_i b_i v_i$$

Notation conventions

index notation

$$\alpha_{ij} x_i y_j =$$

matrix-vector notation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y} =$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

other versions index notation

$$\alpha_{ij} x_i y_j = x_i \alpha_{ij} y_j =$$

$$\alpha_{ij} y_j x_i$$

Dummy vs free index

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \sum_{i=1}^{3} a_i v_i = \sum_{k=1}^{3} a_k v_k$$

- i,k – dummy index – appears in duplicates and can be substituted without changing equation

$$F_j = A_j \sum_{i=1}^{3} B_i C_i \quad \Rightarrow \quad \begin{aligned} F_1 &= A_1 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right) \\ F_2 &= A_2 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right) \\ F_3 &= A_3 \left(B_1 C_1 + B_2 C_2 + B_3 C_3 \right) \end{aligned}$$

- j – free index, appears once in each term of the equation

Addition and subtraction of tensors

$\mathbf{W} = a\mathbf{T}+b\mathbf{S}$

add each component: $W_{ijkl} = aT_{ijkl} + bS_{ijkl}$

T and **S** must have same rank, dimension and units

W has same rank, dimension and units as **T** and **S**

T and **S** are tensors => **W** is a tensor

commutative, associative

This is same as how vectors and matrices are added.

Multiplication of tensors

Inner product = dot product

$\mathbf{W} = \mathbf{T}\cdot\mathbf{S}$

involves contraction over 1 index: $W_{ik} = T_{ij}S_{jk}$

As normal matrix and matrix-vector multiplication

T and **S** can have different rank, but same dimension

rank**W** = rank**T**+rank**S**-2, dimension as **T** and **S**,

units as product of units **T** and **S**

T and **S** are tensors => **W** is a tensor

Examples: $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$

$\boldsymbol{\sigma} = \mathbf{C}:\boldsymbol{\varepsilon}$ or $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ (Hooke's law)

Multiplication of tensors

Tensor product=outer product = dyadic product
≠ cross product

$\mathbf{W} = \mathbf{TS}$ often written as $\mathbf{W} = \mathbf{T} \otimes \mathbf{S}$
no contraction: $W_{ijkl} = T_{ij}S_{kl}$

T and **S** can have different rank, but same dimension
rank$\mathbf{W}$ = rank$\mathbf{T}$+rank$\mathbf{S}$, dimension as **T** and **S**,
units as product of units **T** and **S**

T and **S** are tensors => **W** is a tensor

Examples: $\nabla\mathbf{v}$ (gradient of a vector) ≠ $\nabla\cdot\mathbf{v}$ (divergence)

remember gradient is a vector $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}\right)$

Multiplication of tensors

For both multiplications

Distributive: $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$

Associative: $\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$

Not commutative: $\mathbf{TS} \neq \mathbf{ST}$, $\mathbf{T{\cdot}S} \neq \mathbf{S{\cdot}T}$

but: $\mathbf{T{\cdot}S}=\mathbf{S}^{\mathrm{T}}\cdot\mathbf{T}^{\mathrm{T}}$

and: $\mathbf{ab}=(\mathbf{ba})^{\mathbf{T}}$ but only for rank 2

Remember transpose: $\mathbf{a{\cdot}T{\cdot}b}=\mathbf{b}\cdot\mathbf{T}^{\mathrm{T}}\cdot\mathbf{a} \Rightarrow T_{ji}=T^{T}_{ij}$

Special tensor:
Kronecker delta δ_{ij}

$$\delta_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$$

$\delta_{ij} = 1$ for i=j, $\delta_{ij} = 0$ for $i \neq j$

In 3-D:

$$\boldsymbol{\delta} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Isotropic tensors, invariant upon coordinate transformation
- Scalars
- **0** vector
- δ_{ij}

$\mathbf{T} \cdot \boldsymbol{\delta} = \mathbf{T} \cdot \mathbf{I} = \mathbf{T}$ or $T_{ij}\delta_{jk} = T_{ik}$

δ is isotropic: $\delta_{ij} = \delta'_{ij}$ upon coordinate transformation
can be used to write dot product: $T_{ij}S_{jl} = T_{ij}S_{kl}\delta_{jk}$
can be used to write trace: $A_{ii} = A_{ij}\delta_{ij}$
orthonormal transformation: $\alpha_{ij}\alpha^T{}_{jk} = \delta_{ik}$

Special tensor:
Permutation symbol ε_{ijk}

$$\varepsilon_{ijk} = \left(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j\right) \cdot \hat{\mathbf{e}}_k$$

$\varepsilon_{ijk} = 1$ if i,j,k an even permutation of 1,2,3

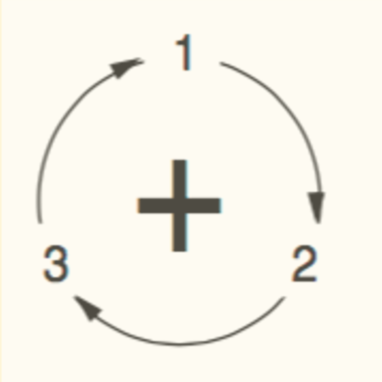
$\varepsilon_{ijk} = -1$ if i,j,k an odd permutation of 1,2,3

$\varepsilon_{ijk} = 0$ for all other i,j,k

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{222} = \ldots = 0$



Note that $\varepsilon_{ijk} a_i b_j \hat{e}_k$ where $\hat{e}_k$ is the unit vector in k direction is index notation for cross product **a** x **b**

Exercise: useful identity $\varepsilon_{ijm}\, \varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$

Vector derivatives - curl

Curl of a vector: $$\nabla \times \mathbf{v} = \varepsilon_{ijk} \frac{\partial}{\partial x_i} v_j \hat{\mathbf{e}}_k = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\[2ex] \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\[2ex] \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix}$$

In index notation, using special tensor

Some tensor calculus

Gradient of a vector is a tensor:
$$\nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Such that the change **dv** in field **v** in direction **dx** is: $\mathbf{dv} = \nabla \mathbf{v} \cdot \mathbf{dx}$

Divergence of a vector: $\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$

$$\nabla \cdot \mathbf{v} = tr\left(\nabla \mathbf{v}\right)$$

Trace of a tensor is the sum of diagonal elements

Some tensor calculus

Divergence of a tensor:

$$\nabla \cdot T = \frac{\partial T_{ij}}{\partial x_j} = \begin{pmatrix} \dfrac{\partial T_{1j}}{\partial x_j} \\ \dfrac{\partial T_{2j}}{\partial x_j} \\ \dfrac{\partial T_{3j}}{\partial x_j} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{12}}{\partial x_2} + \dfrac{\partial T_{13}}{\partial x_3} \\ \dfrac{\partial T_{21}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{23}}{\partial x_3} \\ \dfrac{\partial T_{31}}{\partial x_1} + \dfrac{\partial T_{32}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} \end{pmatrix}$$

Laplacian = div(grad *f*), where *f* is a scalar function

$$\nabla \cdot \nabla f = \nabla^2 f = \Delta\ f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial f}{\partial x_1^2} + \frac{\partial f}{\partial x_2^2} + \frac{\partial f}{\partial x_3^2}$$

Objectives

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention

Summary

- **Vectors**
  - Addition, linear independence
  - Orthonormal Cartesian bases, transformation
  - Multiplication
  - Derivatives, del, div, curl
- **Tensors**
  - Tensors, rank, stress tensor
  - Index notation, summation convention
  - Addition, multiplication
  - Special tensors, δ_{ij} and ε_{ijk}
  - Tensor calculus: gradient, divergence, curl, ..

*Further reading/studying e.g: **Reddy** (2013) 2.2.1-2.2.3, 2.2.5, 2.2.6, 2.4.1, 2.4.4, 2.4.5, 2.4.6, 2.4.8 (not co/contravariant), **Lai, Rubin, Kremple** (2010): 2.1-2.13, 2.16, 2.17, 2.27-2.32, 4.1-4.3, **Khan Academy** – linear algebra, multivariate calculus*