Part B

$$\frac{\partial T}{\partial t} = \underbrace{-\vec{v}\cdot\nabla T}_{\text{adv.}} + \underbrace{K\nabla^2 T}_{\text{diff}}$$

$$\frac{\partial T}{\partial t} = -\begin{pmatrix} v_x \\ v_y \end{pmatrix}\cdot\begin{pmatrix} \partial T/\partial x \\ \partial T/\partial y \end{pmatrix} + K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t} = -v_x\frac{\partial T}{\partial x} - v_y\frac{\partial T}{\partial y} + K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

RHS:

need FD for $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, $\frac{\partial^2 T}{\partial x^2}$, $\frac{\partial^2 T}{\partial y^2}$

$\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ – upwind schemes

$$\frac{\partial T}{\partial x}^n \approx \begin{cases} \dfrac{T^n_{i+1,j} - T^n_{i,j}}{\Delta x} & v_x < 0 \\ \dfrac{T^n_{i,j} - T^n_{i-1,j}}{\Delta x} & v_x > 0 \end{cases}$$

(this will be v_{ij}; n – current time step)

$$\frac{\partial T}{\partial y}^n \approx \begin{cases} \dfrac{T^n_{j+1} - T^n_{j}}{\Delta y}, & v_y < 0 \\ \dfrac{T^n_{j} - T^n_{j-1}}{\Delta y}, & v_y > 0 \end{cases}$$

(this will be v_{ij})

$\frac{\partial^2 T}{\partial x^2}$, $\frac{\partial^2 T}{\partial y^2}$

$$\frac{\partial^2 T_{ij}}{\partial x^2}^n \approx \left\{ \frac{T^n_{i+1,j} + T^n_{i-1,j} - 2T^n_{i,j}}{\Delta x^2} \right.$$

$$\frac{\partial^2 T_{ij}}{\partial y^2}^n \approx \frac{T^n_{i,j+1} + T^n_{i,j-1} - 2T^n_{i,j}}{\Delta y^2}$$

LHS

$$\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^{n}}{\Delta t}$$

Combined

$$T_{ij}^{n+1} = \left(-V_x \begin{cases} \frac{T_{i+1,j}^{n} - T_{ij}^{n}}{\Delta x} & u_{ij} < 0 \\ \frac{T_{ij}^{n} - T_{i-1,j}^{n}}{\Delta x} & u_{ij} > 0 \end{cases} - V_{ij} \begin{cases} \frac{T_{j+1}^{n} - T_{i,j}^{n}}{\Delta y} & v_{ij} < 0 \\ \frac{T_{ij}^{n} - T_{ij-1}^{n}}{\Delta y} & v_{ij} > 0 \end{cases} + \right.$$

$$\left. + K \left(\frac{T_{i+1,ij}^{n} + T_{i-1,j}^{n} - 2T_{ij}^{n}}{\Delta x^2} + \frac{T_{ij+1}^{n} + T_{i,j-1}^{n} - 2T_{i,j}^{n}}{\Delta y^2} \right) \right) \Delta t \; 4$$

$$+ T_{ij}^{n}$$