Chapter 2

Fundamental concepts

2.1 Norms

In this course we will consider numerical methods that produce approximations of the solution(s) of various problems. Typically, we will be considering some problem (e.g. a system of equations) with an unknown true solution $\boldsymbol{x}$ belonging to a certain vector space[1]. We will typically be interested in a numerical method that produces a sequence of approximations $\boldsymbol{x}_n$, $n \in \mathbb{N}$, belonging to the same vector space. The index n is usually related to the amount of computational effort (e.g. number of calculations) invested in calculating $\boldsymbol{x}_n$. Our goal will be to prove precise statements about the performance of the method, for instance by proving the convergence of $\boldsymbol{x}_n$ to $\boldsymbol{x}$ as $n \to \infty$.

To measure the size of elements of a vector space, for instance to quantify the size of the error in the approximation, we use a norm.

Definition 2.1.1 (Norms and normed vector spaces)**.** *Let V be a vector space over $\mathbb{R}$. A function $\|\cdot\| : V \to \mathbb{R}$ is a* norm *on V if*

1. *(i) $\|\boldsymbol{v}\| \geq 0$, $\forall \boldsymbol{v} \in V$ and (ii) $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$;*
2. *$\|\alpha \boldsymbol{v}\| = |\alpha| \|\boldsymbol{v}\|$ $\forall \alpha \in \mathbb{R}$ and $\forall \boldsymbol{v} \in V$;*
3. *$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$, $\forall \boldsymbol{v}, \boldsymbol{w} \in V$.*

We then say that $(V, \|\cdot\|)$ is a normed vector space.

The standard norm on $\mathbb{R}$ is the absolute value function, i.e. $\|x\| = |x|$ for $x \in \mathbb{R}$.

[1] Common examples include finite-dimensional vector spaces such as $\mathbb{R}$ or $\mathbb{R}^n$ ($n \geq 2$), and infinite-dimensional function spaces such as $C([a,b])$, the space of all continuous functions $f : [a,b] \to \mathbb{R}$. For simplicity we consider only real vector spaces in this course. But the concepts we consider generalise easily to complex vector spaces, with only minor modifications.

Examples of norms on $\mathbb{R}^n$ are given by the p-norm

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \text{ for } 1 \leq p < \infty, \tag{2.1}$$

where x_i denotes the ith component of the vector $\boldsymbol{x} \in \mathbb{R}^n$. Taking $p = 2$ in (2.1) leads to the classical Euclidean norm

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{(\boldsymbol{x}, \boldsymbol{x})},$$

where[2] the inner product $(\boldsymbol{x}, \boldsymbol{y})$ is defined by $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$. In this case the Cauchy-Schwarz inequality holds:

$$|(\boldsymbol{x}, \boldsymbol{y})| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2.$$

Observe that $p = \infty$ is excluded in the definition of the p-norm above. The *infinity-norm* or *max-norm* is defined separately by

$$\|\boldsymbol{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Each of the above norms measures the "length" of a vector $\boldsymbol{x} \in \mathbb{R}^n$ in a different way. (Exercise: sketch the unit ball $\{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\}$ for the cases $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$.)

One can define norms on the infinite-dimensional space $C([a, b])$ in an analogous way. For instance, the *infinity-norm* or *max-norm* is defined for $f \in C([a, b])$ by

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|.$$

Here the maximum is well-defined because a continuous function on a bounded interval is bounded and attains its bounds.

Definition 2.1.2 (Norm equivalence)**.** *Two norms* $\|\cdot\|$ *and* $\|\cdot\|'$ *on a vector space* V *are said to be* equivalent *if there exist constants* $0 < c < C$ *such that*

$$c\|\boldsymbol{v}\| \leq \|\boldsymbol{v}\|' \leq C\|\boldsymbol{v}\| \qquad \forall \boldsymbol{v} \in V. \tag{2.2}$$

On a finite-dimensional vector space (such as $\mathbb{R}^n$) all norms are equivalent. But this result does not in general extend to infinite-dimensional spaces such as $C([a, b])$.

[2] It is a general result that whenever one has an inner product $(\cdot, \cdot)$ defined on vector space V, one can generate a norm on V by the formula $\|\boldsymbol{v}\| = \sqrt{(\boldsymbol{v}, \boldsymbol{v})}$, and that the Cauchy-Schwarz inequality holds.

2.2 Errors and convergence

If $\tilde{\boldsymbol{x}} \in V$ is an approximation to $\boldsymbol{x} \in V$ we can consider the *absolute error*

$$E_{\text{abs}} = \|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|$$

and the *relative error* (provided that $\boldsymbol{x} \neq 0$)

$$E_{\text{rel}} = \frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|}.$$

Note that if $V = \mathbb{R}$ then $-\log_{10}(E_{\text{rel}})$ indicates to how many decimal digits the approximate and exact solutions agree.

Convergence of sequences and series in a normed vector space is defined in the obvious way. For instance, we say a sequence $\{\boldsymbol{x}_n\}_{n=1}^{\infty} \subset V$ converges to $\boldsymbol{x} \in V$ with respect to the norm $\|\cdot\|$ if $\|\boldsymbol{x}_n - \boldsymbol{x}\| \to 0$ as $n \to \infty$ (as a sequence of real numbers)[3]. Note that if a sequence converges with respect to a given norm, then it also converges with respect to any equivalent norm.

To determine whether a numerical method is likely to be useful in a practical application, it is important to know how fast $\boldsymbol{x}_n$ converges to $\boldsymbol{x}$ as $n \to \infty$.

Definition 2.2.1. *For a sequence $(\boldsymbol{x}_n)$ that converges to $\boldsymbol{x}$ as $n \to \infty$, we say that the convergence is* linear *if there exists a constant $0 < C < 1$ such that, for n sufficiently large,*

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}\| \leq C\|\boldsymbol{x}_n - \boldsymbol{x}\|.$$

We say that the convergence is quadratic *if there exists a constant $C > 0$ such that, for n sufficiently large,*

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}\| \leq C\|\boldsymbol{x}_n - \boldsymbol{x}\|^2.$$

In general, we say the convergence is of order $p > 1$ *(not necessarily an integer) if there exists a constant $C > 0$ such that, for n sufficiently large,*

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}\| \leq C\|\boldsymbol{x}_n - \boldsymbol{x}\|^p.$$

(Exercise: Explain why the condition $C < 1$ is only needed for the case $p = 1$.)

If a positive quantity $E(n)$ depending on a parameter $n \in \mathbb{N}$ (for example the error $\|\boldsymbol{x}_n - \boldsymbol{x}\|$ in a numerical approximation) behaves like $E(n) \approx Cn^{\alpha}$ for some $\alpha \in \mathbb{R}$, then

$$\log E(n) \approx \log C + \alpha \log n,$$

[3]Recall that a sequence $(c_n)_{n=1}^{\infty} \subset \mathbb{R}$ converges to $c \in \mathbb{R}$ as $n \to \infty$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies $|c_n - c| < \epsilon$.

so plotting $\log E(n)$ against $\log n$ (e.g. using Matlab's `loglog` command) will produce a straight line with slope α. If $E(n)$ behaves like $E(n) \approx Ca^n$ for some $a > 0$, then

$$\log E(n) \approx \log C + n \log a,$$

so plotting $\log E(n)$ against n (e.g. using Matlab's `semilogy` command) will produce a straight line with slope $\log a$.

2.3 Asymptotic notation

We will often need to compare the growth or decay of different functions as their arguments tend to a certain limit. The following notation is very useful in this regard.

Definition 2.3.1 ("Big O" notation)**.** *Let f and g be two functions defined on $\mathbb{R}$, and let $x_0 \in \mathbb{R}$. We write*

$$f(x) = O(g(x)) \text{ as } x \to x_0$$

if there exist constants $\delta > 0$ and $C > 0$ such that

$$|f(x)| \leq C|g(x)| \quad \text{for } |x - x_0| < \delta.$$

Definition 2.3.2 ("Little o" notation)**.** *Let f and g be two functions defined on $\mathbb{R}$, and let $x_0 \in \mathbb{R}$. We write*

$$f(x) = o(g(x)) \text{ as } x \to x_0$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x)| \leq \varepsilon|g(x)| \quad \text{for } |x - x_0| < \delta.$$

Notice that in the big O notation, we require that there is at least one constant C for which the statement holds, whereas for the little o notation, we require that the statement holds for all positive constants ε. Therefore $f(x) = o(g(x))$ as $x \to x_0$ implies $f(x) = O(g(x))$ as $x \to x_0$.

If $g(x) \neq 0$ for all x in a neighbourhood of x_0, then the condition in the little o notation is equivalent to

$$\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = 0.$$

Note that the above definitions can be generalised in an obvious way to the case where $x \to \pm\infty$.

2.4 The mean value theorem and Taylor expansions

A fundamental tool in our analysis of numerical methods will be Taylor expansion. The starting point is the mean value theorem for differentiable functions[4].

Theorem 2.4.1 (Mean value theorem)**.** *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Then, given $x, x_0 \in (a, b)$ with $x \neq x_0$,*

$$f(x) = f(x_0) + f'(\xi)(x - x_0), \tag{2.3}$$

for some ξ between x_0 and x.

This result generalises to higher orders of differentiability as part of Taylor's Theorem.[5]

Theorem 2.4.2 (Taylor's theorem)**.** *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be k-times differentiable on (a, b), for some $k \in \mathbb{N}$. Then, given $x, x_0 \in (a, b)$ with $x \neq x_0$,*

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + R_k(x; x_0), \tag{2.4}$$

where the remainder satisfies

$$R_k(x; x_0) = o(|x - x_0|^k) \quad \text{as } x \to x_0.$$

If f is $k+1$-times differentiable on (a, b) then the remainder can be expressed as

$$R_k(x; x_0) = \frac{f^{k+1}(\xi)}{(k+1)!}(x - x_0)^{k+1}, \tag{2.5}$$

for some ξ between x_0 and x.

Notice that if a function f is $k+1$ times differentiable, and if $|f^{k+1}(\xi)| \leq M$ for some uniform constant M for all arguments ξ in some neighbourhood of x_0, then (2.5) implies that

$$R_k(x; x_0) = O\left(|x - x_0|^{k+1}\right) \quad \text{as } x \to x_0.$$

[4]Recall that $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x_* \in (a, b)$ if the limit $\lim_{h\to 0}(f(x_* + h) - f(x_*))/h$ exists and is finite; this limit is then defined to be the derivative $f'(x_*)$. Differentiability on (a, b) means differentiability at every point $x_* \in (a, b)$. The standard example of a non-differentiable function is the absolute value $|x|$, which is not differentiable at $x_* = 0$ (we get a different limit for positive and negative h).

[5]A function $f : (a, b) \to \mathbb{R}$ is twice differentiable if f is differentiable (in the sense of the previous footnote) and the derivative f' is also differentiable. We then define the second derivative $f'' := (f')'$. Analogously, f is k-times differentiable for some $k \in \mathbb{N}$ if the derivatives f', $f'' := (f')'$, $f''' := (f'')'$, $\ldots f^{(k)} := (f^{(k-1)})'$ all exist.