Chaotic Midpoints

Mauricio Sevilla

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We have done several examples regarding chaotic dynamical systems, namely, standard map, baker map, forced harmonic oscillator, duffing oscillator and so on. We learned to measure *chaoticity* using the first Lyapunov exponents. In this problem we are about to calculate the Lyapunov exponents for a particular dynamical system constructed not by evolving single points on a physical phase space but two and considering the midpoint.

1 Problem

Consider two non interacting particles ψ_+ and ψ_- under the presence of an external force. These particles will evolve following the Hamilton equations for the external force but independently between them.

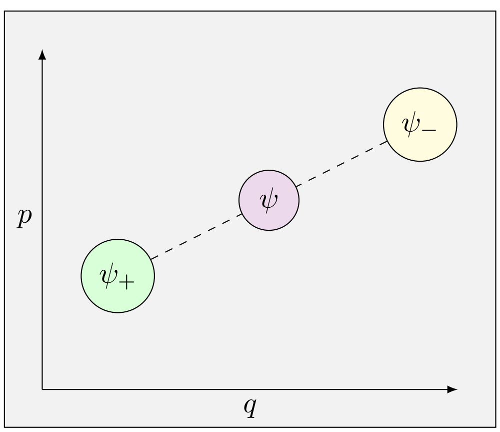

Figure 1: Midpoint visualization (on phase space).

We are interested on the dynamics of the midpoint between ψ_+ and ψ_-, but the midpoint on phase space instead of real space, as it is shown in the figure 1. This has no implications on the way Hamilton equations have to be solved, but just we just have to know that we always are going to need two particles and of course that the coordinates on phase space for the midpoint are constructed as follows,

$$q(\psi) = \frac{1}{2}(q(\psi_+) + q(\psi_-)), \tag{1a}$$

and

$$p(\psi) = \frac{1}{2}(p(\psi_+) + p(\psi_-)). \tag{1b}$$

Then, we want to evaluate how sensitive is the midpoint *trajectory* with respect to the initial conditions by measuring the Lyapunov exponent.

2 System

With no particular reason, we are going to use a one dimensional Morse Oscillator potential [Morse, 1929],

$$V(q) = D\left(1 - \exp\left(-a\left(q - q_0\right)\right)\right)^2, \tag{2}$$

with $D = 15$ and $a = 0.18$.

To be sure we are taking points on the bounded zone, consider as an initial configurations points close to $\mathbf{r} = (q_0 = 4, p_0 = 0)$. Consider for this case $q_0 = 0$ and a mass $m = 1.$.

3 Evolution Method

We need to solve Hamilton's equations for the two points ψ_+ and ψ_- numerically, so we need to construct a function that receives any combination of points $(q(t), p(t))$ and returns $((q(t+\Delta t), p(t+\Delta t)))$. Hence, if we want to enhance the integration afterwards, we have only to change a single function. The time evolution of p and q go as,

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{3a}$$

and

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q} = F(q, p) \tag{3b}$$

where F is the force.

Let us consider a very simple but powerful method based on the common Euler method, the Euler-Cromer Method [Cromer, 1981] which is a simple modification the normal Euler method but it yields to solutions that are stable for oscillatory systems. The Euler-Cromer algorithm is constructed with the following difference equations

$$p_{n+1} = p_n - \frac{\partial H(q_n, p_n)}{\partial q}\Delta t = p_n + F(q_n, p_n)\Delta t, \tag{4a}$$

$$q_{n+1} = q_n + \frac{\partial H(q_n, p_{n+1})}{\partial p}\Delta t = q_n + \frac{p_{n+1}}{m}\Delta t, \tag{4b}$$

This simple modification of the algorithm, makes the method *symplectic* which means that conserves energy, unlike Euler method which creates energy artificially when the dynamics occur.

4 Final remarks

The straightforward path to solve this problem is constructing a class `pair` in which two positions and two momenta are defined as variables, then a single method can perform the evolution for both points, and of course, some simple methods that return the midpoint. This class creator may receive the initial conditions as arguments and also the size of the timestep.

4.1 Class `pair`

- Functions
  - Initializer
  - Hamilton Equation for p
  - Hamilton Equation for q
  - Euler Cromer Method for a single timestep
  - Dynamics that calls the Euler Cromer function for both trajectories
  - Midpoint q and p
- Variables
  - $q(\psi_+)$, $p(\psi_+)$
  - $q(\psi_-)$, $p(\psi_-)$
  - Δt

Then, the class is used to compare the differences of two close initial midpoints with time and therefore, calculate the Lyapunov exponent.

References

[Cromer, 1981] Cromer, A. (1981). Stable solutions using the euler approximation. *American Journal of Physics*, 49(5):455–459.

[Morse, 1929] Morse, P. M. (1929). Diatomic molecules according to the wave mechanics. ii. vibrational levels. *Phys. Rev.*, 34:57–64.