DISTRIBUIÇÃO BETA

FDP: $f(x) = \dfrac{1}{\text{BETA}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x)$

SENDO $\alpha > 0$, $\beta > 0$

E $\text{BETA} = \displaystyle\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

NOTAÇÃO: $X \sim \text{BETA}(\alpha,\beta)$

$E(x) = \dfrac{\alpha}{\alpha+\beta}$ $\qquad$ $\text{VAR}(X) = \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

A DISTR. BETA NÃO POSSUI FUNÇÃO GERADORA DE MOMENTOS ESCRITA DE FORMA SIMPLES.

OS MOMENTOS SÃO OBTIDOS PELA EXPRESSÃO:

$$E(X^k) = \frac{1}{\text{BETA}(\alpha,\beta)} \int_0^1 x^{k+\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(k+\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(k+\alpha+\beta)}$$

$$E(X) = \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}$$

PROVAS

$$E(X) = \int_0^1 x \, f(x) \, dx$$

$$= \int_0^1 x \cdot \frac{1}{\text{BETA}(a,B)} \cdot x^{a-1} (1-x)^{B-1} \, dx$$

$$= \frac{1}{\text{BETA}(a,B)} \int_0^1 x \cdot x^{a-1} (1-x)^{B-1} \, dx$$

$$= \frac{1}{\text{BETA}(a,B)} \int_0^1 x^{1+a-1} (1-x)^{B-1} \, dx$$

SE $\text{BETA}(a,B) = \int_0^1 x^{a-1}(1-x)^{B-1} \, dx$

ENTÃO $\text{BETA}(1+a,B) = \int_0^1 x^{1+a-1}(1-x)^{B-1} \, dx$

$$E(X) = \frac{1}{\text{BETA}(a,B)} \cdot \text{BETA}(1+a,B)$$

$$E(X) = \frac{\Gamma(a+B)}{\Gamma(a)\cancel{\Gamma(B)}} \cdot \frac{\Gamma(1+a)\cancel{\Gamma(B)}}{\Gamma(1+a+B)}$$

$$E(X) = \frac{\Gamma(a+B)\,\Gamma(1+a)}{\Gamma(a)\,\Gamma(1+a+B)}$$

$$\boxed{\begin{array}{c}\Gamma(a+1) = a\,\Gamma(a) \\ \text{(IDENTITY SIMILARITY)}\end{array}}$$

$$E(X) = \frac{\cancel{\Gamma(a+\beta)}\, a\, \cancel{\Gamma(a)}}{\cancel{\Gamma(a)}(a+\beta)\cancel{\Gamma(a+\beta)}} = \underline{\underline{\frac{a}{a+\beta}}}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \frac{1}{\mathrm{BETA}(a,\beta)} \int_0^1 x^2 \cdot x^{a-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{\mathrm{BETA}(a,\beta)} \int_0^1 x^{2+a-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{\mathrm{BETA}(a,\beta)} \cdot \mathrm{BETA}(2+a,\beta)$$

$$= \frac{\Gamma(a+\beta)}{\Gamma(a)\cancel{\Gamma(\beta)}} \cdot \frac{\Gamma(2+a)\cancel{\Gamma(\beta)}}{\Gamma(2+a+\beta)}$$

$$E(X^2) = \frac{\Gamma(a+\beta)\Gamma(2+a)}{\Gamma(a)\Gamma(2+a+\beta)}$$

$$E(X^2) = \frac{\Gamma(a+\beta)\Gamma(1+1+a)}{\Gamma(a)\Gamma(1+1+a+\beta)}$$

$$E(x^2) = \frac{\Gamma(\alpha+\beta)(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}$$

$$E(x^2) = \frac{\Gamma(\cancel{\alpha+\beta})(\alpha+1)\,\alpha\,\Gamma(\cancel{\alpha})}{\Gamma(\cancel{\alpha})(\alpha+\beta+1)(\alpha+\beta)\Gamma(\cancel{\alpha+\beta})}$$

$$E(x^2) = \frac{(\alpha+1)\,\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$\mathrm{Var}(X) = \frac{(\alpha+1)\,\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$\mathrm{Var}(X) = \frac{(\alpha+\beta)(\alpha+1)\,\alpha - (\alpha+\beta+1)\,\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\mathrm{Var}(X) = \frac{(\alpha^2+\alpha+\alpha\beta+\beta)\,\alpha - (\alpha^3+\alpha^2\beta+\alpha^2)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\mathrm{Var}(X) = \frac{\cancel{\alpha^3}+\cancel{\alpha^2}+\cancel{\alpha^2\beta}+\alpha\beta-\cancel{\alpha^3}-\cancel{\alpha^2\beta}-\cancel{\alpha^2}}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$