Effect Algebras as Omega-categories

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We show how an effect algebra $\mathscr{X}$ can be regarded as a category, where the morphisms $x \to y$ are the elements f such that $x \leq f \leq y$. This gives an embedding $\mathbf{EA} \to \mathbf{Cat}$. The interval $[x,y]$ proves to be an effect algebra in its own right, so $\mathscr{X}$ is an $\mathbf{EA}$-enriched category. The construction can therefore be repeated, meaning that every effect algebra can be identified with a strict ω-category. We describe explicitly the strict ω-category structure for two classes of operators on a Hilbert space.

Introduction

In the theory of quantum logic, a pervasive structure is that of an effect algebra (or equivalently, that of a D-poset). It can be motivated by quantum theoretical arguments, and arises naturally in the study of positive semidefinite operators on Hilbert spaces. (Higher) Category theory also plays a key role in the study of quantum theory, specifically in quantum field theory (e.g. M. Benini and A. Schenkel's work [1]) and its relation with cobordism (e.g. Freed, Hopkins, Lurie and Teleman's work [4]); literature on the subject is not scarce[1].

In this paper we present a novel link between quantum logic and (strict) higher category theory: every effect algebra gives rise to a strict ω-category. We hence give an explicit description of such a construction, and briefly discuss two distinct ways to see how it arises: one is more formal and employs iterated enrichment/internalization, the second is more direct and requires explicit calculations.

We also briefly sketch a physical interpretation of the n-cells of such an ω-category, in the cases where the starting effect algebra arises from a Hilbert space.

1 D-Posets and Effect Algebras

We review the definitions of D-posets and effect algebras and the equivalence between them.

Definition 1.1. *D-poset*

A *D-poset* (or *difference poset*) $(X, \leq, \top, \ominus)$ is given by a poset $(X, \leq)$ with a top element $\top$ and a partial binary operation

$$\ominus : X \times X \to X$$

satisfying the following axioms:

1. $y \ominus x$ is defined precisely when $x \leq y$
2. whenever $x \leq y$ we also have $y \ominus x \leq y$ and $y \ominus (y \ominus x) = x$
3. if $z \leq y \leq x$ then $x \ominus y \leq x \ominus z$ and $(x \ominus z) \ominus (x \ominus y) = y \ominus z$

[1] As an example among many we direct the reader to [10], a volume curated by H. Sati and U. Schreiber, for a collection of papers related to the matter.

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Points 1, 2 and the first part of point 3 can be rephrased by saying that for any $x \in X$ we can look at $x \ominus -$ as a (total) function

$$(x \ominus -) : X_{\leq x} \to X_{\leq x}$$

which is an order-reversing involution, where $X_{\leq x} = \{t \in X : t \leq x\}$.

Remark 1.1. *Results on D-posets (see [6])*
Given a D-poset $(X, \leq, \top, \ominus)$, the following statements hold:

(a) for any elements $x, y \in X$, we have $x \ominus x = y \ominus y$; moreover, such an element (which we'll denote by $\perp$) is the bottom element of $(X, \leq)$;

(b) for all $x \in X$ we have $x \ominus \perp = x$;

(c) if $y \leq x$ and $x \ominus y = x$ then $y = \perp$;

(d) if $z \leq y \leq x$ then $y \ominus z \leq x \ominus z$ and $(x \ominus z) \ominus (y \ominus z) = x \ominus y$;

(e) if $y \leq x$ and $z \leq x$ then $x \ominus y = z$ if and only if $x \ominus z = y$;

(f) if $y \leq x$ and $z \leq x \ominus y$ then $z \leq x$, $y \leq x \ominus z$ and $(x \ominus y) \ominus z = (x \ominus z) \ominus y$;

(g) for any $x \in X$ we can look at $- \ominus x$ as an order preserving (total) function;

$$(- \ominus x) : X_{x \leq} \to X_{\leq (\top \ominus x)}$$

(h) for any $x \in X$, the antitonic function $x \ominus -$ is an order-isomorphism;

There is a canonical notion of morphism of D-posets and sub-D-posets, which will turn out to be too strict for our purposes:

Definition 1.2. (Generalized) D-monotonic functions and sub-D-posets
Given D-posets $(X, \leq, \top, \ominus)$ and $(Y, \preceq, \uparrow, \boxminus)$ a function $f : X \to Y$ is said to be *D-monotonic* whenever the following axioms hold:

1. $f : (X, \leq) \to (Y, \preceq)$ is a monotonic function;
2. whenever $x \leq y$ we have $f(y \ominus x) = f(y) \boxminus f(x)$;
3. $f(\top) = \uparrow$.

In addition, we'll say that it is a *generalized D-monotonic* function whenever it satisfies axioms 1 and 2 (but not necessarily 3). Notice how by point 2, every D-monotonic function will necessarily preserve the bottom element: $f(\perp) = f(x \ominus x) = f(x) \boxminus f(x) =: \downarrow$.

A subset $U \subseteq X$ of a D-poset is a *sub-D-poset* if the following hold:

(a) $\top \in U$;

(b) whenever $x \leq y$ are elements of U, then so is $y \ominus x$.

Moreover, we'll say that it is a *generalized sub-D-poset* if it has a (possibly different) top element and axiom (b) holds. It is clear that for a (generalized) sub-D-poset, the inclusion is a (generalized) D-monotonic function.

This allows us to define categories **DPos** (of D-posets and D-monotonic functions) and **gDPos** (of D-posets and generalized D-monotonic functions). We also have a forgetful functor **DPos** $\to$ **gDPos**.

We will now define a class of (D-posets isomorphic to) generalized sub-D-posets that will play an important role in the following sections:

Remark 1.2. *Intervals*

Given a D-poset $(X,\leq,\top,\ominus)$ and elements $x\leq y$, we can define the structure of a D-poset on $[x,y] := \{z\in X \mid x\leq z\leq y\}$. Whenever $x=\perp$, this D-poset will canonically be a generalized sub-D-poset of X; in general $[x,y]\simeq[\perp,y\ominus x]$ will only be isomorphic to one.

Proof. See 5.1 in the appendix. □

Before moving on to effect algebras, there's another important definition we need:

Definition 1.3. *Product of D-posets*

Given D-posets $(X,\leq,\top,\ominus)$ and $(Y,\preceq,\uparrow,\boxminus)$, we can endow $X\times Y$ with the structure of a D-poset $(X\times Y,\sqsubseteq,(\top,\uparrow),\ominus\times\boxminus)$ defined as follows:

- $(x_1,y_1)\sqsubseteq(x_2,y_2) \iff x_1\leq x_2$ and $y_1\preceq y_2$;
- $(x_2,y_2)(\ominus\times\boxminus)(x_1,y_1) := (x_2\ominus x_1, y_2\boxminus y_1)$.

It is immediate to see how this is the Cartesian product in both the category of D-posets and D-monotonic functions and the category of D-posets and generalized monotonic functions: as a consequence, this defines functors

$$\times : \mathbf{DPos}\times\mathbf{DPos}\to\mathbf{DPos}$$

$$\times : \mathbf{gDPos}\times\mathbf{gDPos}\to\mathbf{gDPos}$$

It is also clear how this product is associative (up to isomorphism), and how the singleton set $\{*\}$ can be endowed with the structure of a D-poset (which is immediately recognized as the terminal object in both categories) which acts as a unit for the Cartesian product (again, up to isomorphism). This allows us to endow both categories with the structure of a Cartesian monoidal category; we can also notice how the forgetful functor $\mathbf{DPos}\to\mathbf{gDPos}$ is a strictly monoidal functor.

An equivalent notion to D-posets is that of effect algebras, a notion that at first glance seems much more algebraic:

Definition 1.4. *Effect Algebra*

An Effect Algebra $(X,1,\oplus,(-)^\dagger)$ is given by a set X, an element $1\in X$, a function $(-)^\dagger : X\to X$ and a partial binary operation $\oplus$ such that (in the following we write $0 := 1^\dagger$ as is customary):

1. Whenever $a\oplus b$ is defined, so is $b\oplus a$ and they are equal;
2. Whenever $b\oplus c$ and $a\oplus(b\oplus c)$ are defined, so are $a\oplus b$ and $(a\oplus b)\oplus c$; moreover
$$a\oplus(b\oplus c) = (a\oplus b)\oplus c$$
3. If $a\oplus 1$ is defined, then $a=0$;
4. $a\oplus a^\dagger$ is always defined and equals 1; moreover whenever $a\oplus b=1$ then $b=a^\dagger$.

As we did for D-posets, we'll state some important results on effect algebras that we'll employ throughout the following section

Remark 1.3. *Results on Effect Algebras (see [7])*

Given an effect algebra $(X,1,\oplus,(-)^\dagger)$ the following hold:

(a) $(-)^\dagger$ is an involution, meaning $(a^\dagger)^\dagger = a$;

(b) $a\oplus 0$ is always defined and equals a;

(c) if $a\oplus b$ is defined and equals 0, then $a=b=0$;

(d) if $a \oplus b = a \oplus c$ then $b = c$;

Effect algebras come with their own notion of homomorphism and subobject: homomorphisms have to preserve 1, $(-)^\dagger$ and $\oplus$, while a subobject has to contain 1 and be closed under $(-)^\dagger$ and $\oplus$. As for D-poset, we need a slightly generalized version for both: a *generalized homomorphism* of effect algebras will only preserve 0 and $\oplus$ (but not necessarily 1 or $(-)^\dagger$), and a *generalized sub-effect-algebra* will only be assumed to contain 0 and be closed under $\oplus$. As before, we have categories **EA** and **gEA** and a forgetful functor $\mathbf{EA} \to \mathbf{gEA}$.

As previously stated, the notions of D-poset and effect algebra are equivalent: this is also a standard result in the theory of quantum logic.

Remark 1.4. *Equivalence of effect algebras and D-posets*

Any effect algebra $(A, 1, \oplus, (-)^\dagger)$ is canonically a D-poset, and vice-versa a D-poset $(X, \leq, \top, \ominus)$ is canonically an effect algebra. More specifically, the D-poset structure on A is given by:

(a) $\top := 1$;

(b) $a \leq b$ iff there is some c such that $a \oplus c = b$ and in such a case $b \ominus a := c$.

In the other direction, the effect algebra structure on X is given by

(c) $1 := \top$;

(d) $x^\dagger := \top \ominus x$;

(e) $x \oplus y$ is defined whenever there is some z such that $z \ominus y = x$ and in such a case $x \oplus y := z$;

We can give an explicit formula for writing $\ominus$ in terms of $\oplus$ and $(-)^\dagger$ and vice-versa:

$$a \ominus b := (a^\dagger \oplus b)^\dagger$$

$$x \oplus y := \top \ominus ((\top \ominus x) \ominus y)$$

Proof. The fact the two notions are equivalent is, once again, a standard result: a reference might be, e.g., [8]. The explicit formulas are easily seen to follow from points b and e. □

We now move on to state some results on how the two (necessarily coexisting) structures interact, which will play a key role in the following section.

Remark 1.5. *Interactions between* $\oplus$ *and* $\ominus$

Given a D-poset $(X, \leq, \top, \ominus)$ with the canonical structure of an effect algebra $\oplus$, $(-)^\dagger$, $0 = \bot = \top^\dagger$, the following hold:

1. $x \oplus y$ is defined iff $y \leq x^\dagger$;
2. given $x \leq y \leq z$ we have $(z \ominus y) \oplus (y \ominus x) = z \ominus x$;
3. given $x \leq z$ and $y \leq z \ominus x$ we have $(z \ominus x) \ominus y = z \ominus (x \oplus y)$;
4. given $x \leq w$ and $z \leq y \leq (w \ominus x)^\dagger$ we have
$$(w \ominus x) \oplus (y \ominus z) = ((w \ominus x) \oplus y) \ominus z$$
5. $\oplus$ is D-monotonic in an appropriate sense. In particular, whenever well-defined, the following equality holds:
$$(w \ominus x) \oplus (y \ominus z) = (w \oplus y) \ominus (x \oplus z)$$

Proof. Point 1. is in [5] (remark 1.14), points 2 and 3 can be found in [3] (prop 1.1.6), and point 4 easily follows. For a proof of point 5, see 5.2 in the appendix. □

We can now look back at remark 1.2 and notice how our definition for $f : [\bot, y \ominus x] \to [x, y]$ is nothing more than $- \oplus x$, and similarly $a \boxminus b := (a \ominus b) \oplus x$.

2 Effect Algebras as (Higher) Categories

It is tradition in category theory to identify a poset with a thin category, that is, a category whose hom-sets have cardinality *at most* 1. Here we will show a *different* way to turn D-posets into categories; in this way, the categories will not be thin. We will then show an appropriate sense in which our construction can be iterated indefinitely, yielding (strict) higher categories and, in the limiting case, (strict) ω-categories.

Definition 2.1. *D-posets as categories*

Given a D-poset $\mathscr{X} = (X, \leq, \top, \ominus)$ with the usual additional structure of an effect algebra $(\oplus, \dots)$. We define a category $\mathbb{B}\mathscr{X}$ as follows:

- Objects are given by elements of the D-poset $\mathbf{Ob}(\mathbb{B}\mathscr{X}) := X$;
- Hom-sets are given by

$$\mathbb{B}\mathscr{X}[x,y] := \begin{cases} \{x\} \times X_{\leq y \ominus x} \times \{y\} & \text{if } x \leq y, \\ \emptyset & \text{otherwise;} \end{cases}$$

- Since $\mathbb{B}\mathscr{X}[x,x] = \{(x,\perp,x)\}$ we can only set $1_x := (x,\perp,x)$;
- Composition is given by $\oplus$ in the following way:

$$\begin{aligned} \circ : \mathbb{B}\mathscr{X}[y,z] \times \mathbb{B}\mathscr{X}[x,y] &\to \mathbb{B}\mathscr{X}[x,z] \\ (y,f,z), \quad (x,g,y) &\mapsto (x, f \oplus g, z) \end{aligned}$$

  which is well-defined because $(z \ominus y) \oplus (y \ominus x) = z \ominus x$ (cf. pt. 2 in 1.5).

Proof. See 5.3 in the appendix. □

We could have also defined the hom-sets in terms of intervals $[x,y]$ instead of $X_{\leq y \ominus x}$; this would have avoided the unpleasantness of defining them by case analysis: if $x \leq y$ doesn't hold, we simply have $[x,y] = \emptyset$. On the other hand, we would have needed a slightly different definition for the composition: $(y,f,z) \circ (x,g,y) := (x,(f \ominus y) \oplus g, z)$ which would have made our proof a bit more complex. But since the two (D-)posets are equivalent, as shown in 1.2, we reserve the right to switch between the two descriptions with little notice[2].

Moreover, generalized D-monotonic functions $f : \mathscr{X} \to \mathscr{Y}$ lift to functors $\mathbb{B}f : \mathbb{B}\mathscr{X} \to \mathbb{B}\mathscr{Y}$, exhibiting the category **gDPos** of D-posets as a sub-category of the category **Cat** of categories and functors:

Definition 2.2. *Generalized D-monotonic functions as functors*

Let $\mathscr{X} = (X, \leq, \top, \ominus)$ and $\mathscr{Y} = (Y, \preceq, \uparrow, \boxminus)$ be D-posets, endowed with the usual effect algebra structure (resp. $\oplus$ and $\boxplus$); let also $f : X \to Y$ be a generalized D-monotonic function between them.We can "upgrade" f to a functor

$$\mathbb{B}f : \mathbb{B}\mathscr{X} \to \mathbb{B}\mathscr{Y}$$

as expected (i.e. $(\mathbb{B}f)(\phi : x \to y) = f(\phi) : f(x) \to f(y)$).

[2] The description in terms of intervals will be extremely convenient later on, when explicitly describing the (higher) categorical structure we'll define in the language of globular sets.

Proof. We will always assume the hom-sets are non-empty (otherwise the claim is vacuously true). We'll also identify a morphism (x,ϕ,y) with the middle element ϕ whenever there's no ambiguity.

We know by assumption (definition 1.2) that f preserves bottom elements and differences. This implies that $\mathbb{B}f$ sends morphisms in the correct hom-sets and preserves identities. To see it preserves compositions, what we need to show is that $f(\phi\oplus\psi)=f(\phi)\oplus f(\psi)$: this can be shown directly with almost no effort, and a reference be found in section 1.3 of [3] □

This endows $\mathbb{B}$ with the structure of a functor $\mathbb{B}:\mathbf{gDPos}\to\mathbf{Cat}$. We are soon going to venture into higher categorical structures; before doing so, it is convenient to notice that such a functor preserves products "on the nose":

Remark 2.1. *$\mathbb{B}$ is a strict monoidal functor*

Given D-posets $\mathscr{X}_i$ and (generalized) D-monotonic functions $f_k:\mathscr{X}_k\to\mathscr{X}_{k+2}$ (for any $i\in\{1,2,3,4\}$ and $k\in\{1,2\}$), we have

$$\mathbb{B}(\mathscr{X}_i\times\mathscr{X}_j)\simeq(\mathbb{B}\mathscr{X}_i)\times(\mathbb{B}\mathscr{X}_j)$$

and

$$\mathbb{B}(f_1\times f_2)\simeq(\mathbb{B}f_1)\times(\mathbb{B}f_2):\mathbb{B}(\mathscr{X}_1\times\mathscr{X}_2)\to\mathbb{B}(\mathscr{X}_3\times\mathscr{X}_4)$$

Proof. See 5.4 in the appendix. □

As it turns out, for any given D-poset $\mathscr{X}=(X,\le,\top,\ominus)$, its non-empty hom-sets $\mathbb{B}\mathscr{X}[x,y]$ can also be seen as D-posets; in fact, they can be endowed with the structure of a generalized sub-D-poset of the original poset: given $x\le y$, we can define

$$\mathscr{X}_{x,y}=(\{x\}\times X_{\le y\ominus x}\times\{y\},\preceq,\uparrow,\downarrow,\boxminus)$$

as follows: first, we define $(x,\phi,y)\preceq(x,\psi,y)\iff\phi\le\psi$; then we can set $\uparrow:=(x,y\ominus x,y)$ and $\downarrow:=(x,\perp,y)$. As for the difference, we stipulate

$$(x,\psi,y)\boxminus(x,\phi,y):=(x,\psi\ominus\phi,y)$$

This allows us to notice that whenever $x\le y\le z$, we can see composition as a D-monotonic function

$$\circ:\mathscr{X}_{y,z}\times\mathscr{X}_{x,y}\to\mathscr{X}_{x,z}$$

In order to see why this is true, first notice how

$$y\ominus x=(z\ominus x)\ominus(z\ominus y)\le\top\ominus(z\ominus y)=(z\ominus y)^*$$

hence $\mathscr{X}_{x,y}$ is a generalized sub-D-poset of $X_{\le(z\ominus y)^*}$. This implies that we have a generalized D-monotonic inclusion[3]

$$\mathscr{X}_{y,z}\times\mathscr{X}_{x,y}\hookrightarrow(X_{\le z\ominus y},\le,(z\ominus y),\perp,\ominus)\times(X_{\le(z\ominus y)^*},\le,(z\ominus y)^*,\perp,\ominus)$$

into the D-poset we mentioned in the proof of point 5 of 1.5. Recall that there we proved how

$$\oplus:(X_{\le z\ominus y},\le,(z\ominus y),\perp,\ominus)\times(X_{\le(z\ominus y)^*},\le,(z\ominus y)^*,\perp,\ominus)\to\mathscr{X}$$

[3] This isn't actually an inclusion, but it is isomorphic to one by projecting to the middle element (recall the carrier is a set of triples); in particular, it is a monomorphism in the category **gDPos**.

is D-monotonic; hence the composition with the aforementioned inclusion is also (generalized) D-monotonic. Thus, we only need to show it factors through the inclusion $\mathscr{X}_{x,z} \hookrightarrow \mathscr{X}$; this is immediate: indeed, if $\perp \leq \phi \leq z \ominus y$ and $\perp \leq \psi \leq y \ominus x$ then

$$\perp = \perp \oplus \perp \leq \phi \oplus \psi \leq (z \ominus y) \oplus (y \ominus x) = z \ominus x$$

This suggests applications of the functor $\mathbb{B}$ can be "iterated", yielding the structure of a higher category.

Theorem 2.1. *D-posets as (higher) categories*
Given a D-poset $\mathscr{X} = (X, \leq, \top, \perp, \ominus)$, we can define a strict 2-category $\mathbb{B}^2\mathscr{X}$ as follows:

- The set of 0-cells $(\mathbb{B}^2\mathscr{X})_0 := X$ is the carrier itself;
- The hom-categories are given by $\mathbb{B}^2\mathscr{X}[x,y] := \mathbb{B}\mathscr{X}_{x,y}$

where we stipulate a convention according to which $y \nleq x \implies \mathbb{B}\mathscr{X}_{x,y} := \emptyset$ the empty category (with no object or morphisms).

Proof. Both horizontal and vertical composition are given by $\oplus$, hence interchange follows from commutativity. All other axioms for a strict 2-category hold because definition 2.1 makes sense. □

Once again, given $x \leq y$ and $\phi \leq \psi \leq y \ominus x$, it's easy to see how $\mathbb{B}^2\mathscr{X}[x,y][\phi,\psi]$ is again a D-poset: this construction can be iterated once more, producing a strict 3-category $\mathbb{B}^3\mathscr{X}$.

We however prefer to cut the chase and prove the limiting result:

Theorem 2.2. *Effect Algebras are ω-categories*
We have a strict ω-category $\mathbb{B}^\infty\mathscr{X}$, obtained by transfinitely applying the construction of definition 2.1 to the effect algebra $\mathscr{X}$.

If every hom-set of $\mathbb{B}\mathscr{X}$ was an effect algebra, we would only have needed to invoke remark 2.1. This is not the case: they can be empty as well. But the empty set also has a canonical structure of a category (the empty one), so we have the following proof:

Proof.

- Let $\mathbf{gEA}^*$ be the category whose objects are either an effect algebra or $\emptyset$; and whose morphisms are generalized effect algebra homomorphisms *or* the empty function;
- Define a functor $\mathbf{gEA}^* \to \mathbf{Cat}_{\mathbf{gEA}^*}$ that sends every effect algebra $\mathscr{X}$ to the $\mathbf{gEA}^*$-enriched category $\mathbb{B}\mathscr{X}$;
- This functor preserves the empty object, in the sense that it maps the empty set to the empty category.

This clearly implies that to each effect algebra corresponds a strict omega category, obtained by using $\mathbb{B}$ as the "change of enriching category" functor (a definition of what this is can be found in any introductory text on enriched category theory, e.g. lemma 3.4.3 in [9]) inductively. □

In the following section we'll describe more explicitly this ω-category for a fixed (though arbitrary) effect algebra, in terms of globular sets. For ease of treatment, from now on, we'll use $[x,y]$ instead of $[\perp, y \ominus x]$ in the definition of n-cells for all $n > 0$.

3 Effect Algebras as ω-categories

In this section we'll make heavy use of the simplex category Δ of non-empty finite ordinals and monotonic non-decreasing functions between them (as well as its face and degeneracy maps δ_i^n, σ_i^n and the simplicial identities relating them); we'll later unwind the definition and show how to explicitly describe our notion.

Theorem 3.1. *The structure of* $\mathbb{B}^\infty\mathscr{X}$

Given a D-poset $\mathscr{X} = (X, \leq, \top, \bot, \ominus)$ with the usual definitions for $\oplus$ and $(-)^*$, we claim that the n-cells in $\mathbb{B}^\infty\mathscr{X}$, as well as sources, targets, identities, and compositions can be described as follows:

1. The set of n-cells as monotonic functions (i.e. morphisms in the category of posets):
$$(\mathbb{B}^\infty\mathscr{X})_n := \mathbf{Pos}[[2n], X]$$
2. Source maps $s_n : (\mathbb{B}^\infty\mathscr{X})_{n+1} \to (\mathbb{B}^\infty\mathscr{X})_n$ as the following composition
$$s_n(f) : [2n] \xrightarrow{\delta_{n+1}^{2n+2}} [2n+1] \xrightarrow{\delta_{n+2}^{2n+3}} [2n+2] \xrightarrow{f} X$$
3. Target maps $t_n : (\mathbb{B}^\infty\mathscr{X})_{n+1} \to (\mathbb{B}^\infty\mathscr{X})_n$, similarly, as
$$t_n(f) : [2n] \xrightarrow{\delta_{n}^{2n+2}} [2n+1] \xrightarrow{\delta_{n+2}^{2n+3}} [2n+2] \xrightarrow{f} X$$
4. Identity maps $Id_n : (\mathbb{B}^\infty\mathscr{X})_n \to (\mathbb{B}^\infty\mathscr{X})_{n+1}$ as
$$Id_n(f) : [2n+2] \xrightarrow{\sigma_{n+1}^{2n+2}} [2n+1] \xrightarrow{\sigma_{n}^{2n+1}} [2n] \xrightarrow{f} X$$
5. Compositions
$$c_n : (\mathbb{B}^\infty\mathscr{X})_{n+1} \ {}_{s_n}\times_{t_n} (\mathbb{B}^\infty\mathscr{X})_{n+1} \to (\mathbb{B}^\infty\mathscr{X})_{n+1}$$
are instead defined by an explicit formula:
$$c_n(f,g)(i) := \begin{cases} g(i) & i \leq n \\ (f(n+1) \ominus f(n)) \oplus g(n+1) & i = n+1 \\ f(i) & i \geq n+2 \end{cases}$$

Here δ, σ denote face and degeneracy maps, respectively. The domain of the composition maps is a pullback of the following diagram:

$$(\mathbb{B}^\infty\mathscr{X})_{n+1} \xrightarrow{s_n} (\mathbb{B}^\infty\mathscr{X})_n \xleftarrow{t_n} (\mathbb{B}^\infty\mathscr{X})_{n+1}$$

which from now on will simply be denoted by $(\mathbb{B}^\infty\mathscr{X})_{n+1} \times_n (\mathbb{B}^\infty\mathscr{X})_{n+1}$ with no explicit mention of the source and target maps.

Proof. See 5.5 in the appendix. □

Our goal for this section is to prove that the structure we just defined is indeed a *strict* ω *category*. In order to do so we'll follow the following plan:

- First, we'll show it defines a globular set,
- Second, we'll upgrade it to a reflexive globular set,
- Third, we'll show that it defines a category for any $i < j$,
- Fourth, we'll show that it defines a strict 2-category for any $i < j < k$.

From these points, it is well-known in the literature (e.g. see chapter 1 of [2]; there the name ∞-category is used instead) that what we just described is indeed a strict ω-category.

Before going any further, it will be useful to describe face and degeneracy maps more explicitly; to this end, recall how the simplex category is a full subcategory of the *augmented* simplex category of (possibly empty) finite ordinals and order-preserving maps.

$$\Delta \hookrightarrow \Delta_\alpha$$

In Δ_α the empty set $[-1]$ is the initial object, while the singleton $[0]$ is terminal; moreover, we can define a bifunctor called *ordinal sum*

$$\boxtimes : \Delta_\alpha \times \Delta_\alpha \to \Delta_\alpha$$

defined by

$$[n] \boxtimes [m] := [n+m+1]$$

and (given $f : [n] \to [n\prime]$, $g : [m] \to [m\prime]$)

$$(f \boxtimes g)(i) := \begin{cases} f(i) & 0 \le i \le n \\ g(i-n-1)+n\prime+1 & i > n \end{cases}$$

endowing Δ_α with the structure of a strict monoidal category $(\Delta_\alpha, [-1], \boxtimes)$. We also have canonical maps

$$[-1] \xrightarrow{u} [0] \xleftarrow{\mu} [1]$$

endowing $[0]$ with the structure of a monoid object (verifying the axioms is immediate); the diagonal of the diagram encoding associativity will be useful later on, so we'll write it as

$$\mu^2 := \mu \circ (\mu \boxtimes 1_{[0]}) = \mu \circ (1_{[0]} \boxtimes \mu)$$

Even more is true: every object of Δ_α can be written as the ordinal sum of copies of $[0]$, and every face and degeneracy map as the ordinal sum of identities, u and μ:

$$[n] := [0]^{\boxtimes n+1}$$

$$\delta_i^n := [n-2] \xrightarrow{1_{[i-1]} \boxtimes u \boxtimes 1_{[n-i-2]}} [n-1]$$

$$\sigma_i^n := [n] \xrightarrow{1_{[i-1]} \boxtimes \mu \boxtimes 1_{[n-i-2]}} [n-1]$$

and hence (since any morphism in Δ_α is a finite composition of faces and degeneracies), morphisms are finite compositions of ordinal sums of identities, u and μ. This justifies calling Δ_α the "free monoidal category with a monoid object", or the "walking monoid". As a final note, the simplicial identities can be derived using the monoid axioms on $([0], u, \mu)$ and the bifunctoriality of $\boxtimes$ (that is, interchange).

This means, in particular, that we can provide explicit formulas for s_n, t_n, Id_n defined in 3.1 (just before) in terms of this monoidal structure. We can write

$$s^n : [2n] \xrightarrow{1_{[n]} \boxtimes u \boxtimes u \boxtimes 1_{[n-1]}} [2n+2]$$

$$t^n : [2n] \xrightarrow{1_{[n-1]} \boxtimes u \boxtimes u \boxtimes 1_{[n]}} [2n+2]$$

$$i^n : [2n+2] \xrightarrow{1_{[n-1]} \boxtimes \mu^2 \boxtimes 1_{[n-1]}} [2n]$$

and define

$$s_n(f) = f \circ s^n, \;\; t_n(f) = f \circ t^n, \;\; Id_n(f) = f \circ i^n.$$

We are now ready to start tackling the main goal:

Theorem 3.2. $(\mathbb{B}^\infty \mathscr{X})_\bullet$ *is a globular set.*

Explicitly, this means

$$s_n \circ s_{n+1} = s_n \circ t_{n+1}$$

$$t_n \circ s_{n+1} = t_n \circ t_{n+1}$$

Proof. See 5.6 in the appendix. □

As a consequence, any (finite) composition of source and target morphisms is completely determined by the first term: in the following we'll abuse notation and write

$$s_n, t_n : (\mathbb{B}^\infty \mathscr{X})_{n+k} \to (\mathbb{B}^\infty \mathscr{X})_n$$

meaning $s_n \circ \dots$ and $t_n \circ \dots$ respectively, for any $k > 0$.

The second claim is also within reach:

Theorem 3.3. $(\mathbb{B}^\infty \mathscr{X})_\bullet$ *is a reflexive globular set.*

Since we already proved it is an (ordinary) globular set, we only need to prove

$$s_n \circ Id_n = 1_{(\mathbb{B}^\infty \mathscr{X})_n} = t_n \circ Id_n$$

Proof. See 5.7 in the appendix. □

As for theorem 3.2, this justifies abusing notation even more: we'll write

$$Id_n : (\mathbb{B}^\infty \mathscr{X})_n \to (\mathbb{B}^\infty \mathscr{X})_{n+k+1}$$

for the appropriate composition of identity maps, for all $k \geq 0$.

Before carrying on with the plan sketched above, we need to define how the composition will work; for now, we only have

$$c_n : (\mathbb{B}^\infty \mathscr{X})_{n+1} \times_n (\mathbb{B}^\infty \mathscr{X})_{n+1} \to (\mathbb{B}^\infty \mathscr{X})_{n+1}$$

while we need

$$c_{n,k} : (\mathbb{B}^\infty \mathscr{X})_{n+k} \times_n (\mathbb{B}^\infty \mathscr{X})_{n+k} \to (\mathbb{B}^\infty \mathscr{X})_{n+k}$$

where here $\times_n$ is to be intended as a pullback over the new diagram (note how here s_n, t_n arbitrary compositions as justified by theorem 3.2):

$$(\mathbb{B}^\infty \mathscr{X})_{n+k} \xrightarrow{s_n} (\mathbb{B}^\infty \mathscr{X})_n \xleftarrow{t_n} (\mathbb{B}^\infty \mathscr{X})_{n+k}$$

Trying to keep our abused notation slightly coherent, we'll continue writing c_n instead of $c_{n,k}$. Defining such maps is surprisingly easy:

$$c_n(f,g)(i) := \begin{cases} g(i) & \text{if } 0 \leq i \leq n \\ (f(i) \ominus f(n)) \oplus g(i) & \text{if } n < i < n+2k \\ f(i) & \text{if } n+2k \leq i \leq 2n+2k \end{cases}$$

Remark 3.1. *The definitions for c_n make sense*

As previously noted, we need to show that whenever $s_n(f) = t_n(g)$ then $c_n(f,g)$ is defined and it is a monotonic function. Since the above definitions for c_n reduce to the ones in 3.1 in the case $k = 1$, proving monotonicity for a general $k > 0$ suffices.

Proof. See 5.8 in the appendix. □

We now have a definition for all of the structure required to conclude the initial plan, which we now carry on.

Theorem 3.4. *For any $i < j$ we have a category.*

The category is defined by the following diagram:

$$\begin{tikzcd} (\mathbb{B}^\infty \mathscr{X})_j \times_i (\mathbb{B}^\infty \mathscr{X})_j \arrow[d, "c_i"] & \\ (\mathbb{B}^\infty \mathscr{X})_j \arrow[r, bend left, "s_n"] \arrow[r, bend right, "t_n"'] & (\mathbb{B}^\infty \mathscr{X})_i \arrow[l, "Id_n"] \end{tikzcd}$$

In the proof, since there's no ambiguity, we'll drop all subscripts for source, target, identity, and composition.

Proof. See 5.9 in the appendix. □

In order to finish proving our definition 3.1 really does define a strict ω-category, the only piece left is the following

Theorem 3.5. *For any $i < j < k$ we have a strict 2-category.*

The strict 2-category is defined by the following diagram:

$$\begin{tikzcd} (\mathbb{B}^\infty \mathscr{X})_k \times_j (\mathbb{B}^\infty \mathscr{X})_k \arrow[d, "c_j"] & (\mathbb{B}^\infty \mathscr{X})_j \times_i (\mathbb{B}^\infty \mathscr{X})_j \arrow[d, "c_i"] & \\ (\mathbb{B}^\infty \mathscr{X})_k \arrow[r, bend left, "s_j"] \arrow[r, bend right, "t_j"'] & (\mathbb{B}^\infty \mathscr{X})_j \arrow[l, "Id_j"] \arrow[r, bend left, "s_i"] \arrow[r, bend right, "t_i"'] & (\mathbb{B}^\infty \mathscr{X})_i \arrow[l, "Id_i"] \end{tikzcd}$$

More precisely: there is a (strict) 2-category in which:

1. (0/1/2)-cells are elements of $(\mathbb{B}^\infty \mathscr{X})_i$, $(\mathbb{B}^\infty \mathscr{X})_j$ and $(\mathbb{B}^\infty \mathscr{X})_k$ (respectively);
2. Identity 1-cells, sources and targets for 1-cells are given by Id_i, s_i and t_i;
3. Identity 2-cells, sources and targets for 2-cells are given by Id_j, s_j and t_j;
4. Horizontal and vertical compositions are given by c_i and c_j (with c_i being functorial with respect to the category structure induced by j and k).

Proof. See 5.10 in the appendix. □

4 Applications to Quantum Theory

In this section we'll discuss how the construction $\mathbb{B}^\infty\mathscr{X}$ relates to the effect algebras arising from quantum theory. We'll be concerned with mostly 2 examples:

Example 4.1. *Effect algebras arising from a Hilbert space*

Given a Hilbert space $\mathbb{H}$, we can produce two distinct effect algebras out of the algebra $\mathscr{L}(\mathbb{H})$ of bounded operators over it.

- The effect algebra $\mathbf{Proj}(\mathbb{H})$ is made of projection operators $P : \mathbb{H} \to \mathbb{H}$. Addition $P \oplus Q := P + Q$ is defined whenever they project on orthogonal subspaces $\mathbf{Ker}(P)^\perp \cap \mathbf{Ker}(Q)^\perp = \mathbf{0}$ (equivalently, when the result is still a projection operator); the involution is then the difference with the identity operator $P^\dagger := \mathbf{Id} - P$.
- The effect algebra $\mathbf{Bound}(\mathbb{H})$ is made of positive semidefinite operators whose norm is bounded by 1. Addition $P \oplus Q := P + Q$ is defined whenever the resulting operator's norm is still within the bound; the involution is the same as before. It seems to also be possible, with a small amount of care, to bound the operators with an arbitrary (positive semidefinite) operator T instead of the identity.

It is of relevance to observe that the first example is also an orthoalgebra: $P \oplus P$ is defined only when $P = 0$. This does not hold in the second example.

In these cases, there's a quantum mechanical meaning to the n-cells in $\mathbb{B}^\infty\mathscr{X}$: they are a sequence of non-decreasing[4] operators

$$P_1 \leq P_2 \leq \cdots \leq P_{2n} \leq P_{2n+1}$$

We can make physical sense of this: this corresponds to a sequence of measurements (for $\mathbf{Proj}(\mathbb{H})$) or observables (for $\mathbf{Bound}(\mathbb{H})$); in both cases, the requirement that the sequence be increasing has meaning as well:

- For $\mathbf{Proj}(\mathbb{H})$ the operators correspond to (projective) measurements, and the fact they are non-decreasing corresponds to them encompassing more and more eigenvectors of the corresponding observable;
- For $\mathbf{Bound}(\mathbb{H})$ the operators correspond to observables, and the fact they are non-decreasing corresponds to the eigenvalues being non-decreasing (for any fixed eigenspace).

Moreover, the composition of n-cells satisfies a universal property if we restrict ourselves to an *orthoalgebra* (instead of a more general effect algebra):

Lemma 4.1. Let $\mathscr{X}$ be an orthoalgebra and $a \in \mathscr{X}$. Then $[0,a]$ is a (generalized) suborthoalgebra of $\mathscr{X}$.

Theorem 4.1. Let $\mathscr{X}$ be an orthoalgebra. Let $x \leq f \leq y \leq g \leq z$. Then the following is a pullback in thin category corresponding to the poset structure on $[f,g]$:

$$\begin{tikzcd} f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & y \arrow[d] \\ g \circ f \arrow[r] & g \end{tikzcd}$$

A lot of questions remain open; among those: what is the physical meaning of this (if any), what happens in the case of specific effect algebras (e.g. Boolean algebras), how this relates to other constructions (e.g. nerve constructions, the Kalmbach monad). These are all possible directions for future research.

[4]Following the prescriptions in remark 1.4, we have $P \leq Q$ if $Q - P$ is a projection operator (resp. positive semidefinite).

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5 Appendix: proofs

Proof 5.1. *Remark 1.2*

Clearly, $[x,y]$ is partially ordered by the restriction of $\leq$, and x (y) is the top (bottom) element. It is similarly clear that whenever $z \leq y \leq x$ (that is, $z \leq y$ are elements of $[\bot,x]$), $y \ominus z \leq y \leq x$ and hence $y \ominus z \in [\bot,x]$. Hence, by definition 1.2, it is a generalized sub-D-poset.

We will now show that $[x,y]$ is order-isomorphic to $[\bot, y \ominus x]$; this will allow us to transfer the D-poset structure to $[x,y]$. Clearly, we have a monotonic function $(- \ominus x) : [x,y] \to [\bot, y \ominus x]$. To show it is an order-isomorphism, it is enough to exhibit a 2-sided monotonic inverse: we claim that the following definition works.

$$[\bot, y \ominus x] \ni z \mapsto f(z) := \top \ominus ((\top \ominus x) \ominus z) \in [x,y]$$

To see why, first notice how both $\top \ominus -$ and $(\top \ominus x) \ominus -$ are antitonic, hence their composition (if defined) is monotonic. Since $\top$ is the top element, $\top \ominus -$ is defined on all X; moreover, since $z \leq y \ominus x \leq \top \ominus x$, the domain is correct. The codomain is also correct: monotonicity $\bot \leq z \leq y \ominus x$ implies

$$\top \ominus ((\top \ominus x) \ominus \bot) \leq \top \ominus ((\top \ominus x) \ominus z) \leq \top \ominus ((\top \ominus x) \ominus (y \ominus x))$$

Clearly the first term equals x

$$\top \ominus ((\top \ominus x) \ominus \bot) = \top \ominus (\top \ominus x) = x$$

and the last one equals y

$$\top \ominus ((\top \ominus x) \ominus (y \ominus x)) = \top \ominus (\top \ominus y) = y$$

hence the function above is well-defined. We now want to show it is a 2-sided inverse of $-\ominus x$; that is, for all $z \in [\bot, y\ominus x]$ we have $f(z)\ominus x = z$ and for all $w \in [x,y]$ we have $f(w\ominus x) = w$. For the first claim, we simply compute

$$(\top\ominus((\top\ominus x)\ominus z))\ominus x = (\top\ominus((\top\ominus x)\ominus z))\ominus(\top\ominus(\top\ominus x)) = (\top\ominus x)\ominus((\top\ominus x)\ominus z) = z$$

and similarly for the second

$$\top\ominus((\top\ominus x)\ominus(w\ominus x)) = \top\ominus(\top\ominus w) = w$$

which concludes the proof. Clearly then, the difference $\boxminus$ on $[x,y]$ is defined by

$$b\boxminus a := \top\ominus((\top\ominus x)\ominus((b\ominus x)\ominus(a\ominus x))) = \top\ominus((\top\ominus x)\ominus(b\ominus a))$$

As we'll see later on, there's an easier description for all this that employs the canonical structure of an effect algebra every D-poset inherits.

Proof 5.2. *Remark 1.5*

Recall our definition of a generalized sub-D-poset, and notice how for any $x \in X$, there is a generalized sub-D-poset structure on the subset $X_{\leq x}$. Given an $x \in X$, then, we can consider the D-poset

$$(X_{\leq x}, \leq, x, \bot, \ominus)\times(X_{\leq x^\dagger}, \leq, x^\dagger, \bot, \ominus)$$

A more precise statement is then: for any $x \in X$, the restriction of $\oplus$ to such a product of sub-D-posets is D-monotonic[5].

To prove this, first observe that given $y \in X_{\leq x}$ and $z \in X_{\leq x^\dagger}$, we clearly have $z \leq x^\dagger \leq y^\dagger$: hence $y\oplus z$ is defined. Now, fix $y' \leq y \in X_{\leq x}$ and $z' \leq z \in X_{\leq x^\dagger}$; this means there are $v, w \in X$ such that $y'\oplus v = y$ and $z'\oplus w = z$. Hence

$$y\oplus z = (y'\oplus v)\oplus(z'\oplus w) = (y'\oplus z')\oplus(v\oplus w)$$

which means $y'\oplus z' \leq y\oplus z$: it is monotonic. Now, we want to show that it preserves the bottom element, as well as differences; the former is obvious: $\bot\oplus\bot = \bot$. For the latter, consider $(y\ominus y')\oplus(z\ominus z')$; by our assumptions, this equals $v\oplus w$. On the other hand, notice how both $y\oplus z$ and $y'\oplus z'$ are defined:

$$z' \leq z \leq x^\dagger \leq y^\dagger \leq (y')^\dagger$$

moreover, as we already noticed, $y\oplus z = (y'\oplus z')\oplus(v\oplus w)$. As a consequence

$$(y\oplus z)\ominus(y'\oplus z') = ((y'\oplus z')\oplus(v\oplus w))\ominus(y'\oplus z') = v\oplus w$$

and hence we can conclude as promised

$$(y\ominus y')\oplus(z\ominus z') = (y\oplus z)\ominus(y'\oplus z')$$

Proof 5.3. *Definition 2.1*

To show $\mathbb{B}\mathscr{X}$ defines a category, we need to prove composition is unital and associative. More precisely, given objects $w, x, y, z \in X$ and morphisms $\phi : y \to z$, $\psi : x \to y$ and $\theta : w \to x$ we want to prove

$$1_z \circ \phi = \phi = \phi \circ 1_y$$

[5]Notice how such restrictions are *no longer partial*: they become total functions.

$$(\phi \circ \psi) \circ \theta = \phi \circ (\psi \circ \theta)$$

We can clearly assume $w \leq x \leq y \leq z$ (otherwise the claims are vacuously true). The first claim is trivial: $\perp \oplus f = f = f \oplus \perp$ implies

$$(y, \perp, y) \circ (x, f, y) = (x, f, y) = (x, f, y) \circ (x, \perp, x)$$

The second follows from the axioms of a D-poset:

$$\begin{aligned}(\phi \circ \psi) \circ \theta &= ((y, f, z) \circ (x, g, y)) \circ (w, h, x) \\ &= (w, (f \oplus g) \oplus h, z) \\ &= (w, f \oplus (g \oplus h), z) \\ &= (y, f, z) \circ ((x, g, y) \circ (w, h, x)) \\ &= \phi \circ (\psi \circ \theta)\end{aligned}$$

Proof 5.4. *Remark 2.1* Write $\mathscr{X}_i = (X_i, \leq_i, \top_i, \ominus_i)$ with the usual effect algebra structure $(\oplus_i, \dots)$, and recall def. 1.3

$$\mathscr{X}_i \times \mathscr{X}_j = (X_i \times X_j, \preceq_{i,j}, (\top_i, \top_j), \boxminus_{i,j})$$

First we want to define an equivalence of categories; the objects of $\mathbb{B}(\mathscr{X}_i \times \mathscr{X}_j)$ are pairs (x, y) with $x \in X_i$ and $y \in X_j$, and the objects of $(\mathbb{B}\mathscr{X}_i) \times (\mathbb{B}\mathscr{X}_j)$ are pairs (a, b) with $a : \mathbb{B}\mathscr{X}_i$ and $b : \mathbb{B}\mathscr{X}_j$ (meaning $a \in X_i$ and $b \in X_j$): we can define the mapping on objects to be the identity function on the set $X_i \times X_j$. As for morphisms, fixed $a, b \in X_i \times X_j$ such that $a \preceq b$ (meaning $a = (x_1, y_1)$ and $b = (x_2, y_2)$ for some $x_1, x_2 \in X_i$ and $y_1, y_2 \in X_j$, such that $x_1 \leq_i x_2$ and $y_1 \leq_j y_2$), we have

$$\mathbb{B}(\mathscr{X}_i \times \mathscr{X}_j)[a, b] = \{((x_1, y_1), (f, g), (x_2, y_2)) \mid x_1 \leq_i f \leq_i x_2 \ \wedge \ y_1 \leq_j g \leq_j y_2\}$$

$$(\mathbb{B}\mathscr{X}_i) \times (\mathbb{B}\mathscr{X}_j)[a, b] = \{((x_1, f, x_2), (y_1, g, y_2)) \mid x_1 \leq_i f \leq_i x_2 \ \wedge \ y_1 \leq_j g \leq_j y_2\}$$

which are clearly bijective sets. Hence we have established not only an equivalence, but an *isomorphism* of categories (the two functor just described compose to the appropriate identity functors, not to something "just" naturally isomorphic to them).

Now for the functors $\mathbb{B}(f_1 \times f_2)$ and $(\mathbb{B}f_1) \times (\mathbb{B}f_2)$; the first acts on objects as the (D-monotonic) function $f_1 \times f_2$, that is

$$\mathbb{B}(f_1 \times f_2)(x, y) = (f_1(x), f_2(y))$$

while the second acts on objects as

$$(\mathbb{B}f_1 \times \mathbb{B}f_2)(x, y) = ((\mathbb{B}f_1)(x), (\mathbb{B}f_2)(y)) = (f_1(x), f_2(y))$$

so, on objects, they coincide on the nose. As for morphisms, the first one acts as

$$(\mathbb{B}(f_1 \times f_2))(\phi, \psi) = (f_1 \times f_2)(\phi, \psi) = (f_1(\phi), f_2(\psi))$$

while the second as

$$((\mathbb{B}f_1) \times (\mathbb{B}f_2))(\phi, \psi) = (\mathbb{B}f_1(\phi), \mathbb{B}f_2(\psi)) = (f_1(\phi), f_2(\psi))$$

hence they also coincide on morphisms; they are the same functor.

Proof 5.5. *Definition 3.1*

We should prove all these maps are monotonic, but s_n, t_n, Id_n clearly are (being compositions of monotonic functions). This means the only ones that are not immediately seen as monotonic are c_n. We'll not provide a proof here since we'll prove a more general result later on (remark 3.1), from which this will naturally follow.

Still, we feel it is useful to make a remark on the definition of c_n: whenever $c_n(f,g)$ is defined, we have $s(f) = t(g)$. This means in our definition, we made two arbitrary (though irrelevant) choices:

1. for $i < n$ and $i > n+2$, since $f(i) = g(i)$, we could have used the two interchangeably;
2. for $i = n+1$, since $f(n) = g(n+2)$, we could have equivalently defined

$$c_n(f,g)(n+1) = (f(n+1) \ominus g(n+2)) \oplus g(n+1)$$

The choices we made are motivated by brevity: regarding point 1, we wanted the least amount of cases in our definition; regarding point 2, our choice produces a (slightly) shorter formula.

In order to show that this is the strict ω-category we defined in 2.2, we proceed by induction: clearly a 1-cell is a non-decreasing sequence of 3 elements of $\mathscr{X}$. Assuming an n-cell is then a non-decreasing sequence of $2n+1$ elements in $\mathscr{X}$, it is clear that the change-of-enriching-category functor we described at the end of the previous section implies an $n+1$-cell is then a triple (a,b,c) of n-cells such that $a \leq b \leq c$ (to be intended as pointwise inequalities). But to make sense of the fact that such an $n+1$-cell is between the n-cells $b : a \to c$, we also want all elements except the middle one to actually be the same; we obtain a non-decreasing sequence of $2n+3 = 2(n+1)+1$ elements of $\mathscr{X}$ as prescribed. This inspection also justifies our definition for sources, targets, identities, and compositions.

Proof 5.6. *Theorem 3.2*

We will show $s^{n+1} \circ s^n = t^{n+1} \circ s^n$ and $s^{n+1} \circ t^n = t^{n+1} \circ t^n$; by the above discussion, this is enough. For the first claim, by unitality and associativity of ordinal sum, we can write

$$s^n = 1_{[n]} \boxtimes u \boxtimes 1_{[-1]} \boxtimes 1_{[-1]} \boxtimes u \boxtimes 1_{[n-1]}$$

$$s^{n+1} = 1_{[n]} \boxtimes 1_{[0]} \boxtimes u \boxtimes u \boxtimes 1_{[0]} \boxtimes 1_{[n-1]}$$

and a trivial computation shows

$$s^{n+1} \circ s^n = 1_{[n]} \boxtimes u \boxtimes u \boxtimes u \boxtimes u \boxtimes 1_{[n-1]}$$

On the other hand we can also write

$$s^n = 1_{[n]} \boxtimes 1_{[-1]} \boxtimes 1_{[-1]} \boxtimes u \boxtimes u \boxtimes 1_{[n-1]}$$

$$t^{n+1} = 1_{[n]} \boxtimes u \boxtimes u \boxtimes 1_{[0]} \boxtimes 1_{[0]} \boxtimes 1_{[n-1]}$$

Now a trivial computation shows

$$t^{n+1} \circ s^n = 1_{[n]} \boxtimes u \boxtimes u \boxtimes u \boxtimes u \boxtimes 1_{[n-1]}$$

and the first claim follows. The proof for the second claim is almost identical, so we omit an explicit calculation.

Proof 5.7. *Theorem 3.3* We'll follow a similar strategy as we did for theorem 3.2: by associativity and functoriality of ordinal sum, we can write

$$s^n = 1_{[n-1]} \boxtimes ((1_{[0]} \boxtimes u \boxtimes 1_{[0]}) \circ (1_{[0]} \boxtimes u)) \boxtimes 1_{[n-1]}$$

$$i^n = 1_{[n-1]} \boxtimes (\mu \circ (1_{[0]} \boxtimes \mu)) \boxtimes 1_{[n-1]}$$

and ignore the outermost $1_{[n-1]}$; our computation then reduces to applying both of the unitality axioms for the monoid structure on $[0]$:

$$\mu \circ (1_{[0]} \boxtimes \mu) \circ (1_{[0]} \boxtimes u \boxtimes 1_{[0]}) \circ (1_{[0]} \boxtimes u) = \mu \circ (1_{[0]} \boxtimes u) = 1_{[0]}$$

and the first part of the claim follows. Similarly, by writing

$$t^n = 1_{[n-1]} \boxtimes ((1_{[0]} \boxtimes u \boxtimes 1_{[0]}) \circ (u \boxtimes 1_{[0]})) \boxtimes 1_{[n-1]}$$

$$i^n = 1_{[n-1]} \boxtimes (\mu \circ (\mu \boxtimes 1_{[0]})) \boxtimes 1_{[n-1]}$$

we can compute

$$\mu \circ (\mu \boxtimes 1_{[0]}) \circ (1_{[0]} \boxtimes u \boxtimes 1_{[0]}) \circ (u \boxtimes 1_{[0]}) = \mu \circ (u \boxtimes 1_{[0]}) = 1_{[0]}$$

We have hence proven $i^n \circ s^n = 1_{[2n]} = i^n \circ t^n$ from which the claim clearly follows.

Proof 5.8. *Remark 3.1* We first need to show the formula for the case $n < i < n+2k$ actually makes sense. Since f is monotonic, $f(n) \leq f(i)$, hence $f(i) \ominus f(n)$ is defined. Also, $g(i) \leq g(n+2k) = f(n)$ and hence $f(i) \ominus f(n) \leq \top \ominus g(i)$. Hence the formula is well-defined.

We now turn to monotonicity: we want to show that

$$i \leq j \implies c_n(f,g)(i) \leq c_n(f,g)(j)$$

We'll tackle the following cases differently:

1. $i \leq j \leq n$
2. $i \leq n < j < n+2k$
3. $n < i \leq j < n+2k$
4. $n < i < n+2k \leq j$
5. $n+2k \leq i \leq j$

Cases 1 and 5 are trivial: f and g are assumed monotonic. Case 3 is similarly immediate: we have already shown that $\ominus$ is monotonic in the first argument and $\oplus$ is monotonic in both; the claim then follows from the fact that the composite of monotonic functions is monotonic.

Without loss of generality, then, we can reduce the remaining points to showing

$$n < i < n+2k \implies c_n(f,g)(n) \leq c_n(f,g)(i) \leq c_n(f,g)(n+2k)$$

Since $c_n(f,g)(n) = g(n)$ and $c_n(f,g)(n+2k) = f(n+2k)$, the claim is equivalent to the following two:

- $g(n) \leq (f(i) \ominus f(n)) \oplus g(i)$
- $(f(i) \ominus f(n)) \oplus g(i) \leq f(n+2k)$

For the first, notice how $g(n) = (f(n) \ominus f(n)) \oplus g(n)$; since $\ominus$ is monotonic in the first argument and $\oplus$ is monotonic in both, the claim follows from the monotonicity of f and g.

For the second, notice how $f(n+2k) = (f(n+2k) \ominus f(n)) \oplus f(n)$; the claim follows from the same reasoning as above (using, in addition, the fact that $f(n) = g(n+2k)$).

Proof 5.9. *Theorem 3.4* Since we already know that $(\mathbb{B}^\infty \mathscr{X})_\bullet$ is a reflexive globular set, we only need to show the following claims:

a) $c(f, Id(s(f))) = f$ for all f

b) $c(Id(t(f)), f) = f$ for all f

c) $s(c(f,g)) = s(g)$ for all composable f, g

d) $t(c(f,g)) = t(f)$ for all composable f, g

e) $c(f, c(g,h)) = c(c(f,g)h)$ for all composable f, g, h

The calculation in the following proof will require a lot of case analysis; for this reason we'll preface it with an explicit description of source, target, and identity maps by case splitting. Define $n := j - i$; we have

$$s(f)(k) = \begin{cases} f(k) & 0 \le k \le i \\ f(k+2n) & i < k \le 2i \end{cases}$$

$$t(f)(k) = \begin{cases} f(k) & 0 \le k < i \\ f(k+2n) & i \le k \le 2i \end{cases}$$

$$Id(a)(k) = \begin{cases} a(k) & 0 \le k < i \\ a(i) & i \le k \le i+2n \\ a(k-2n) & i+2n < k \le 2i+2n \end{cases}$$

where for Id we notice how the case $k = i$ (resp. $k = i+2n$) could have been instead included in the first (resp. last) case: the formulas agree, so our choice of case-splitting is arbitrary.

Let us turn to the proof at hand; we start with point a). We can easily compute

$$\begin{aligned} c(f, Id(s(f)))(k) &= \begin{cases} Id(s(f))(k) & 0 \le k \le i \\ (f(k) \ominus f(i)) \oplus Id(s(f))(k) & i < k < i+2n \\ f(k) & i+2n \le k \le 2i+2k \end{cases} \\ &= \begin{cases} s(f)(k) & 0 \le k \le i \\ (f(k) \ominus f(i)) \oplus s(f)(i) & i < k < i+2n \\ f(k) & i+2n \le k \le 2i+2n \end{cases} \\ &= \begin{cases} f(k) & 0 \le k \le i \\ (f(k) \ominus f(i)) \oplus f(i) & i < k < i+2n \\ f(k) & i+2n \le k \le 2i+2n \end{cases} \end{aligned}$$

and since $(f(k) \ominus f(i)) \oplus f(i) = f(k)$, we have $c(f, Id(s(f))) = f$. Similarly, for point b), we can compute

$$c(Id(t(f)), f)(k) = \begin{cases} f(k) & 0 \leq k \leq i \\ (Id(t(f))(k) \ominus Id(t(f))(i)) \oplus f(k) & i < k < i+2n \\ Id(t(f))(k) & i+2n \leq k \leq 2i+2n \end{cases}$$

$$= \begin{cases} f(k) & 0 \leq k \leq i \\ (t(f)(i) \ominus t(f)(i)) \oplus f(k) & i < k < i+2n \\ t(f)(k-2n) & i+2n \leq k \leq 2i+2n \end{cases}$$

$$= \begin{cases} f(k) & 0 \leq k \leq i \\ f(k) & i < k < i+2n \\ f(k) & i+2n \leq k \leq 2i+2n \end{cases}$$

hence we have $c(Id(t(f)), f) = f$. For point c), the computation goes as follows:

$$s(c(f,g))(k) = \begin{cases} c(f,g)(k) & 0 \leq k \leq i \\ c(f,g)(k+2) & i < k \leq 2i \end{cases}$$

$$= \begin{cases} g(k) & 0 \leq k \leq i \\ f(k+2n) & i < k \leq 2i \end{cases}$$

and we conclude by noticing that if $i+2n < k \leq 2i+2n$ then $f(k) = g(k)$. Similarly, for point d):

$$t(c(f,g))(k) = \begin{cases} c(f,g)(k) & 0 \leq k < i \\ c(f,g)(k+2) & i \leq k \leq 2i \end{cases}$$

$$= \begin{cases} g(k) & 0 \leq k < i \\ f(k+2) & i \leq k < 2i \end{cases}$$

and we conclude by noticing that (in analogy of what happened with point c) if $0 \leq k < i$ then $f(k) = g(k)$. The only point left is e). For that, we first compute $c(f, c(g,h))$ and then do the same for $c(c(f,g),h)$:

$$c(f, c(g,h))(k) = \begin{cases} c(g,h)(k) & 0 \leq k \leq i \\ (f(k) \ominus f(i)) \oplus c(g,h)(k) & i < k < i+2n \\ f(k) & i+2n \leq k \leq 2i+2n \end{cases}$$

$$= \begin{cases} h(k) & 0 \leq k \leq i \\ (f(k) \ominus f(i)) \oplus (g(k) \ominus g(i)) \oplus h(k) & i < k < i+2n \\ f(k) & i+2n \leq k \leq 2i+2n \end{cases}$$

$$c(c(f,g),h)(k) = \begin{cases} h(k) & 0 \le k \le i \\ (c(f,g)(k) \ominus c(f,g)(i)) \oplus h(k) & i < k < i+2n \\ c(f,g)(k) & i+2n \le k \le 2i+2n \end{cases}$$

$$= \begin{cases} h(k) & 0 \le k \le i \\ (((f(k) \ominus f(i)) \oplus g(k)) \ominus g(i)) \oplus h(k) & i < k < i+2n \\ f(k) & i+2n \le k \le 2i+2n \end{cases}$$

Since the cases for $0 \le k \le i$ and $i+2n \le k \le 2i+2n$ are clearly equal, it would suffice to show

$$(f(k) \ominus f(i)) \oplus (g(k) \ominus g(i)) = (((f(k) \ominus f(i)) \oplus g(k)) \ominus g(i))$$

which is exactly point 4 in remark 1.5.

Proof 5.10. *Theorem 3.5* Since a strict 2-category is a category enriched in the category **Cat** of categories, our claim can be rephrased by saying that:

a) for any 0-cells a, b there is a category $\mathbf{Hom}[a,b]$

b) with identities in each $\mathbf{Hom}[a,a]$

c) and composition functors $c : \mathbf{Hom}[b,c] \times \mathbf{Hom}[a,b] \to \mathbf{Hom}[a,c]$

such that composition is associative and unital *on the nose*. Such hom-categories can be explicitly defined as

$$\mathbf{Hom}[a,b] := \{f \in (\mathbb{B}^\infty \mathscr{X})_j \mid s(f) = a \wedge t(f) = b\}$$

identities are easily seen to be $Id_i(a) : \mathbf{Hom}[a,a]$ by virtue of $(\mathbb{B}^\infty \mathscr{X})_\bullet$ being a reflexive globular set (cf. the statement of theorem 3.3). Similarly, the composition is easily seen to be c_i: the fact that it respects sources and targets is a consequence of theorem 3.4, from which we can also recover the category structure on $\mathbf{Hom}[a,b]$: just apply the theorem to the appropriate interval[6] in $\mathscr{X}$ (with the usual structure of a D-poset, as described in definition 1.2 and revisited after remark 1.5).

[6]That is, the interval $[a(i), b(i)]$. To see why this is true, consider f such that $s(f) = a$ and $t(g) = b$. In order for this to be true, we also need $s(a) = s(b)$ and $t(a) = t(b)$ (since we are in a globular set), so the only argument on which they can disagree is i. On the other hand, for f to exist in the first place, we need $a(i) = f(i) \le f(i+1) \le f(i+2) = b(i)$, hence $a(i) \le b(i)$, which implies the interval is non-empty.