# Fuzzy Type Theory

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We develop a new type theory with semantics in categories enriched in categories of fuzzy sets, in analogy with Martin-Löf type theory and its interpretation in categories. First, we examine specific categories of fuzzy sets and categories enriched in them, with a particular focus on weighted pullbacks and their role in the (fuzzy) substitution rule. Next, we introduce a generalization of the notion of a display-map category designed for categories enriched in fuzzy sets. Third, we give structural rules for our new type theory and prove soundness.

## 1 Introduction

In this work, we introduce and study the semantics of a fuzzy version of Martin-Löf type theory using the tools of enriched category theory: just as type theory has semantics in categories (or, to be precise, certain structures on categories, see Section 1.1), our type theory has semantics in categories enriched in fuzzy sets.

Recall that Martin-Löf type theory has a logical interpretation (that falls under the umbrella of the Curry-Howard correspondence): types are interpreted as propositions, and terms are interpreted as proofs of these propositions. We can only speak of absolute truth (and, with some work, falsity). In contrast, we construct in this paper a type theory with a *fuzzy* logical [17] interpretation. That is, we want to speak of *degrees* of truth. Thus, this work can be seen as the development of *proof relevant* fuzzy logic.

There are many applications of fuzzy logic for which we hope our type-theoretic version will be relevant. In this paper, we choose one in which to couch our work: opinion analysis [14]. That is,

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instead of a logical interpretation, we give an *opinion* interpretation of our fuzzy type theory: types are interpreted as opinions and terms are interpreted as reasons for those opinions.[1]

| | Logical interpretation of type theory | Opinion interpretation of fuzzy type theory |
|---|---|---|
| types | propositions | opinions |
| terms | proofs | reasons |

Thus, we can revise our characterization of our work as proof relevant fuzzy logic and instead characterize it as *reason* relevant (fuzzy) logic.

To give a toy example, suppose that one holds the opinion “Bees should be protected.” One could have multiple reasons for holding such an opinion. For instance, two reasons could be “They carry pollen between plants,” and “I like honey,” but perhaps these reasons do not hold the same weight. This is where fuzziness comes into play.

## 1.1 Martin-Löf type theory and display-map categories

In this section we briefly recall Martin-Löf type theory (MLTT) [13] and its interpretation in display-map categories [15]. Martin-Löf type theory is an intuitionistic, proof-relevant logic that has found great success in inspiring several computer proof assistants (for instance, Coq, Agda, and Lean) and research programs (for instance, homotopy type theory).

MLTT first consists of the following five judgments

$$\vdash \Gamma\ \texttt{ctx} \qquad \Gamma \vdash A\ \texttt{Type} \qquad \Gamma \vdash A = A'\ \texttt{Type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash a = a' : A$$

which convey, respectively, that $\Gamma$ is a context, that $A$ is a type in context $\Gamma$, that types $A, A'$ in context $\Gamma$ are equal, that $a$ is a term of type $A$ in context $\Gamma$, and that terms $a, a'$ of type $A$ in context $\Gamma$ are equal.

We refer the reader to [6] for a comprehensive introduction to the topic, and to either [6] or Appendix A for the structural rules of MLTT.

There are many notions of categorical models of type theory, such as [1, 9, 2]. Here, we focus on *display-map categories*.

**Definition 1.1.1** (Display-map category [7, 15])**.** A *display-map category* is a pair $(\mathscr{C}, \mathfrak{D})$ of a category $\mathscr{C}$ and a class of morphisms $\mathfrak{D} = \{p_A : \Gamma.A \to \Gamma\}$ of $\mathscr{C}$ called *display maps* such that the following hold.

1. For each display map $p_A : \Gamma.A \to \Gamma$ and morphism $s : \Delta \to \Gamma$ in $\mathscr{C}$, there exists a display map $p_{A[s]}$ which is a pullback of $p_A$ along $s$.

$$\begin{tikzcd} \Delta.A[s] \arrow[r, "\bar{s}"] \arrow[d, "p_{A[s]}"'] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "s"'] & \Gamma \end{tikzcd}$$

2. $\mathfrak{D}$ is closed under pre- and post-composition with isomorphisms.

3. $\mathscr{C}$ has a terminal object 1.

[1] To be clear, in this short paper, we will neither give a Curry-Howard style correspondence between our fuzzy type theory and fuzzy logic (which we leave to future work) or any detailed analysis of opinions (which is best left to experts). Rather, we use this opinion interpretation as motivation for our work and as a way for the reader to make sense of our syntactic rules.

If $p_A$ is a display map, we write $\Gamma$ and $\Gamma.A$ for its codomain and domain, respectively, and this is because we think of objects of $\mathscr{C}$ as contexts and of projections as types: given a type $p_A$ (or, simply, $A$), we can always recover both its context and the one obtained extending by it with $A$ itself.

Pullbacks represent substitution of a type $p_A$ (or, $A$) along a morphism $s$ of contexts ($\text{Subst}_{tm}$ of Appendix A), and we denote the resulting type with $p_{A[s]}$ (or, $A[s]$). The terminal object 1 represents the empty context (C-Emp of Appendix A). In this setting, terms of a given type $p_A$ are represented by sections of $p_A$: such a section gives us, in particular, for each point $a : 1 \to \Gamma$, a point $sa : 1 \to \Gamma.A$ lying over $a$.

### 1.2 Enriched categories

Enriched categories are a widely used and important tool in category theory. Since Martin-Löf type theory can be seen as the "internal language" of a display map category, the goal of this paper is to devise an analogue of display map categories for certain enriched categories together with an internal language for it. For the definition of an enriched category, we refer the reader to [11] or Appendix B.

Here, we consider categories enriched in categories of fuzzy sets, which we term a *fuzzy category*. For that we fix a commutative ordered monoid $\mathbb{M}$ (Definition 2.0.1), perhaps the booleans $\mathbb{B}$ with conjunction or the unit interval $\mathbb{I}$ with multiplication. Then a fuzzy category is a category together with an assignment that takes every morphism of the category to an element of $\mathbb{M}$, which we call its *value* (satisfying some axioms). Thus, since we will identify terms of our internal language with certain morphisms of a fuzzy category, the terms (or reasons) themselves come with values.

We can interpret these values as confidence levels for reasons. When $\mathbb{M}$ is a singleton, we obtain Martin-Löf type theory. When $\mathbb{M} = \mathbb{B}$, the confidence levels are 0 and 1, and we obtain (a very slight variation) on Martin-Löf type theory, where we sometimes have reasons of value 0 (which we interpret as non-reasons) but otherwise have reasons of value 1, which we interpret as proofs. When $\mathbb{M} = \mathbb{I}$, we obtain a type theory in which the confidence level for a reason can be any real number between 0 and 1, inclusive.

### 1.3 Strategy and structure of the paper

Our strategy is to first generalize Definition 1.1.1 to our enriched context. Looking at Definition 1.1.1, we need to generalize the following notions: (1) category, (2) display map, and (3) pullbacks. We generalize (1) in Section 2 and (2) and (3) in Section 3. In Section 4, we give the generalization of Definition 1.1.1 and discuss how terms will be interpreted. In Section 5, we give structural rules for our new type theory together with a theorem that they can be interpreted in any fuzzy display-map category.

## 2 The category of fuzzy sets

We consider categories enriched in categories of fuzzy sets. Here, a fuzzy set will be a set together with a function into a fixed commutative ordered monoid. We require the target to be a monoid so that the category of fuzzy sets is monoidal, which makes it possible to enrich in it. We require the target to be ordered so that we can compare confidences. We require that it is commutative so that the category of fuzzy sets is symmetric monoidal, which allows us to formulate weighted limits (Section 3).

**Definition 2.0.1** (Commutative ordered monoid)**.** We say that $\mathbb{M} = (M, \cdot, 1, \leq)$ is a *commutative ordered monoid* if $(M, \leq)$ is a partial order, $(M, \cdot, 1)$ is a commutative monoid, and for all $m, n, x \in M$,

$$\text{if } m \leq n \text{ then } x \cdot m \leq x \cdot n.$$

We say that $\mathbb{M}$ is

1. *integral* when the unit of the monoid is the top element of the order, and
2. *idempotent* when $x \cdot x = x$, for all $x \in \mathbb{M}$.

**Remark 2.0.2.** A commutative monoid $(M, \cdot, 1)$ is in fact a small, thin, symmetric monoidal category.

**Example 2.0.3** (Commutative ordered monoids)**.**

($*$) The singleton has a trivial integral, idempotent, commutative monoid structure.

($\mathbb{B}$) The booleans, or the *boolean quantale*, $\mathbb{B} = (\{0,1\}, \wedge, 1, \leq)$ is an integral, idempotent, commutative monoid.

($\mathbb{L}$) The *Lawvere quantale* $\mathbb{L} = ([0,\infty], +, 0, \geq)$ (where $+, 0, \geq$ have their usual meaning on the extended real numbers) is an integral, commutative monoid. This is called the *Lawvere quantale* since Lawvere formulated his generalized metric spaces as categories enriched over $\mathbb{L}$ [12].

($\mathbb{I}$) The *interval quantale* $\mathbb{I} = ([0,1], \cdot, 1, \leq)$ (where $\cdot, 1, \leq$ have their usual meaning on the real numbers) is an integral, commutative monoid. Note that $\mathbb{L}$ and $\mathbb{I}$ are isomorphic via the map $x \mapsto e^{-x}$.

($\mathbb{T}$) Consider $\mathbb{T} = ([0,1], T, 1, \leq)$ where $1, \leq$ have their usual meaning on the real numbers and where $T$ is a *t-norm* [8]: any monoidal product making $\mathbb{T} = ([0,1], T, 1, \leq)$ into a commutative ordered monoid.

($\mathbb{M}^X$) Consider $\mathbb{M}^X = (M^X, \cdot, const_1, \leq)$ for any commutative monoid $\mathbb{M}$ and set $X$. Here, $M^X$ is the set of all functions $X \to M$, $(f \cdot g)(x) := f(x) \cdot g(x)$ and $const_1(x) := 1$ for all $x \in X$, and and $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. This is a commutative, ordered monoid. It is integral (respectively, idempotent) if $\mathbb{M}$ is.

($\Delta$) Consider the *quantale of distribution functions* $\Delta = (\triangle, \otimes, const_1, \leq)$. Here,

$$\triangle = \left\{ f : [0,\infty] \to [0,1] \,:\, f \text{ is monotone and } f(x) = \bigvee_{y<x} f(y) \right\},$$

$(f \otimes g)(t) = \bigvee_{r+s \leq t} f(r) \cdot g(s)$ and $const_1(t) = 1$ for all $t \in [0,\infty]$, and $f \leq g$ if $f(t) \leq g(t)$ for all $t \in [0,\infty]$. In [5], quantales of distributive functions are used to see probabilistic metric spaces as enriched categories over $\Delta$.

Throughout this paper we use the examples $*, \mathbb{B}$, and $\mathbb{I}$ to illustrate our theory.

**Definition 2.0.4** ($\mathbf{Set}(\mathbb{M})$)**.** Call $\mathbf{Set}(\mathbb{M})$ the category whose

- objects $X$ are pairs $(X^0, |-|_X)$ where $X^0$ is a set and $|-|_X$ is a function $X^0 \to M$, and whose
- morphisms $f : X \to Y$ are functions $f : X^0 \to Y^0$ such that for all $x \in X^0$

$$|x|_X \leq |f(x)|_Y$$

We call the objects of $\mathbf{Set}(\mathbb{M})$ *fuzzy sets*. For a fuzzy set $X = (X^0, |-|_X)$, we will call $X^0$ the *underlying set of* $X$, $|-|_X$ the *valuation of* $X$, and $|x|_X$ the *value* of $x \in X^0$.

**Remark 2.0.5.** Other works have considered other choices for morphisms – hence, categories – of fuzzy sets. See for example [17], [16], [4].

**Example 2.0.6** (Fuzzy sets)**.**

($*$) Objects of $\mathbf{Set}(*)$ are pairs $(X, !)$ where ! is the function $X \to *$. Thus, $\mathbf{Set}(*) \cong \mathbf{Set}$.

($\mathbb{B}$) Objects of $\mathbf{Set}(\mathbb{B})$ are pairs $(X, \chi_A)$ where $X$ is a set and $\chi_A$ is the characteristic function of a subset $A$ of $X$. Thus, if we identify the objects of $\mathbf{Set}(\mathbb{B})$ with subset inclusions $i_A : A \hookrightarrow X$, a morphism $f : i_A \to i_{A'}$ is a function $f : X \to X'$ of the ambient sets which restricts to a function $f : A \to A'$ of the subsets.

($\mathbb{I}$) Objects of $\mathbf{Set}(\mathbb{I})$ are pairs $(X, \mu_A)$. We interpret $\mu_A$ as indicating the degree to which an element $x \in X$ is in $A$:

$$\mu_A(x) = \begin{cases} 0, \text{ if } x \text{ is not a member of } A \\ \mu_A(x) \in (0,1), \text{ if } x \text{ is a fuzzy member of } A \\ 1, \text{ if } x \text{ is a full member of } A. \end{cases}$$

($\mathbb{L}$) Consider $(X, \delta_z)$ where $X$ is a pseudo-metric space and $\delta_z(x) = \delta(x,z)$ is the distance between $x \in X$ and a fixed point $z \in X$: these are objects of $\mathbf{Set}(\mathbb{L})$. Then any non-expansive map $f : X \to Y$ is an example of a morphism $(X, \delta_z) \to (Y, \delta_{fz})$ since $\delta_X(x,z) \geq \delta_Y(f(x), f(z))$.

($\Delta$) Let $S$ denote the set of non-negative random variables: that is, measurable functions $X : \Omega \to [0,\infty]$ where $\Omega$ is the space of possible outcomes. To each $X$ we may associate its cumulative distribution function $f_X : [0,\infty] \to [0,1]$ defined by $f_X(x) = P(X \leq x)$. Note that $f_X$ is an element of $\Delta$ and so we have an example of a fuzzy $\Delta$-set. If $\phi : S \to S'$ is a function, then a morphism from $X$ to $\phi(X)$ is a measuring-preserving map $\Gamma : \Omega \to \Omega'$ such that $\phi(X) \circ \Gamma = X$. Then $P(X \leq x) = P(\phi(X) \leq x)$, from which it follows that we have a morphism in $\mathbf{Set}(\Delta)$.

We describe a few properties of the category $\mathbf{Set}(\mathbb{M})$.

**Proposition 2.0.7** (Properties of $\mathbf{Set}(\mathbb{M})$)**.**

1. If $\mathbb{M}$ is integral, then $\mathbf{Set}(\mathbb{M})$ has a terminal object $1_1 = (\{*\}, * \mapsto 1)$. Actually, the set $\{*\}$ supports different $\mathbb{M}$-set structures: for an $\alpha$ in $\mathbb{M}$ we write $1_\alpha$ for the pair $(\{*\}, * \mapsto \alpha)$.
2. If $\mathbb{M}$ has binary meets, then $\mathbf{Set}(\mathbb{M})$ also has pullbacks. Given objects $X$ and $Y$ in $\mathbf{Set}(\mathbb{M})$, their product is $X \times Y = (X^0 \times Y^0, |-|_{X\times Y})$, where $|(x,y)|_{X\times Y} = |x|_X \wedge |y|_Y$.
3. Suppose again that $\mathbb{M}$ has binary meets. For maps $f \colon X \to Z$ and $g \colon Y \to Z$ the underlying set of the pullback $X \times_Z Y$ is the pullback of underlying sets
$$(X \times_Z Y)^0 = X^0 \times_{Z^0} Y^0 = \{(x,y) \,|\, f(x) = g(y)\}$$
and $|(x,y)|_{X\times_Z Y} = |x|_X \wedge |y|_Y$.

**Remark 2.0.8.** Note that $\mathbf{Set}(\mathbb{M})$ is concrete. A morphism $f$ in $\mathbf{Set}(\mathbb{M})$ is

1. a monomorphism if and only if its underlying function is injective,
2. an epimorphism if and only if its underling function is surjective, and
3. an isomorphism if and only if its underlying function is a bijection and $|x|_X = |f(x)|_Y$ for all $x \in X^0$.

**Proposition 2.0.9.** There is a pair of adjoint functors as below.

$$\begin{tikzcd} \mathbf{Set}(\mathbb{M}) \arrow[r, shift left, "U_1"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, shift left, "F"] \end{tikzcd}$$

*Proof.* The functor $F$ maps a set $S$ to the pair $(S, const_1)$. Conversely, $U_1$ acts as

$$X = (X^0, |-|_X) \quad \mapsto \quad \{x \in X^0 : |x|_X = 1\}. \qquad \square$$

**Construction 2.0.10** ($U_\alpha$)**.** For each $\alpha \in M$ we can also define a functor $U_\alpha : \mathbf{Set}(\mathbb{M}) \to \mathbf{Set}$

$$X = (X^0, |-|_X) \quad \mapsto \quad \{x \in X^0 : |x|_X \geq \alpha\}.$$

If there is a least element 0 of $\mathbb{M}$, then $U_0(X) = X^0$.

**Proposition 2.0.11.** If $\beta \leq \alpha$, there is a natural transformation $\lambda : U_\alpha \Rightarrow U_\beta$.

*Proof.* $\lambda$ is simply defined as set inclusion at each component. $\square$

**Proposition 2.0.12.** Monoidal structures in $\mathbb{M}$ induce monoidal structures in $\mathbf{Set}(\mathbb{M})$. If $\mathbb{M}$ has binary meets, then this induces a cartesian monoidal product $X \times Y = (X^0 \times Y^0, |-|_{X\times Y})$ introduced in Item 2 of Proposition 2.0.7. Similarly, the product in $\mathbb{M}$ induces a monoidal product $X \otimes Y := (X^0 \times Y^0, |-|_{X\otimes Y})$ where $|(x,y)|_{X\otimes Y} = |x|_X \cdot |y|_Y$.

We sketch the proof of the Proposition 2.0.12 in the Appendix C. Now we observe that we need a closed monoidal structure on $\mathbf{Set}(\mathbb{M})$, which is given iff $\mathbb{M}$ is a quantale.

**Definition 2.0.13.** A quantale $\mathbb{Q} = (Q, \cdot, 1, \leq)$ is an ordered monoid with all meets and joins that satisfies the following distributive laws: for all $a \in Q$ and $\{b_i\}_{i\in I} \subseteq Q$

$$a \cdot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a \cdot b_i) \quad \text{and} \quad (\bigvee_{i\in I} b_i) \cdot a = \bigvee_{i\in I} (b_i \cdot a)$$

Equivalently,

**Lemma 2.0.14** (Quantale)**.** A *quantale* $\mathbb{Q}$ is an ordered monoid which has all meets and joints and admits an operations $\to_\ell, \to_r$: $\mathbb{Q}^{op} \times \mathbb{Q} \to \mathbb{Q}$ satisfying $a \cdot b \leq c \iff a \leq (b \to_r c) \iff b \leq (a \to_\ell c)$ for all $a, b, c \in \mathbb{Q}$.

**Lemma 2.0.15.** In a quantale, $b \to c = \bigvee\{a \in Q \,:\, a \cdot b \leq c\}$.

This can be directly computed or seen as an instance of the general adjoint functor theorem.

**Remark 2.0.16.** A quantale is in fact a small thin symmetric closed monoidal category.

**Example 2.0.17.** All examples except $\mathbb{T}$ and $\mathbb{M}^X$ of Example 2.0.3 are quantales, as their names suggest. The commutative ordered monoid $\mathbb{T}$ is a quantale when the $t$-norm is left-continuous [10]. The commutative ordered monoid $\mathbb{M}^X$ is a quantale when $\mathbb{M}$ is.

Now we state the following proposition and refer the reader to Appendix C to read the proof.

**Proposition 2.0.18.** Let $\mathbb{M}$ be an integral commutative monoid. Then $\mathbf{Set}(\mathbb{M})$ is monoidal closed if and only if $\mathbb{M}$ is a quantale.

In particular, when $\mathbf{Set}(\mathbb{M})$ is monoidal closed, for fuzzy sets $Y, Z \in \mathbf{Set}(\mathbb{M})$, we have that $((Z^0)^{Y^0}, |-|_{Z^Y})$ is the exponential of $Y$ and $Z$, where $|h|_{Z^Y} = \bigvee_{y\in Y} (|y|_Y \to |h(y)|_Z)$.

**Corollary 2.0.19.** If $\mathbb{Q}$ is an integral quantale and $b \leq c \in Q$, then $b \to c = 1$ (by Lemma 2.0.15). Thus, for a morphism $f : Y \to Z$ in $\mathbf{Set}(\mathbb{Q})$, we have $|f|_{Z^Y} = 1$ since $|y|_Y \leq |f(y)|_Z$ for all $y \in Y$.

For functions $f : Y^0 \to Z^0$ in $\mathbf{Set}(\mathbb{Q})$, we interpret $|f|_{Z^Y}$ as denoting the extent to which $f$ is a morphism in $\mathbf{Set}(\mathbb{Q})$.

Now we consider categories enriched in $\mathbf{Set}(\mathbb{M})$ (see Appendix B for the definition of enriched category).

**Definition 2.0.20.** We call $\mathbf{Set}(\mathbb{M})$-enriched categories *fuzzy categories*.

**Example 2.0.21.** When $\mathbb{M}$ is a quantale, $\mathbf{Set}(\mathbb{M})$ is monoidal closed, and so is enriched in itself. Thus, $\mathbf{Set}(\mathbb{M})$ is a fuzzy category.

# 3 Substitution in the enriched setting

As we saw in Definition 1.1.1, the categorical tool describing substitution is that of *pullback* in a category. Since we are working in an enriched category, it is now appropriate to consider the enriched analogues of limits and pullbacks.

Informally, the universal property of the pullback in a regular set-based category states that, provided a cospan $A \to C \leftarrow B$, there are three maps from the limiting object: one into $A$, one into $B$, and one into $C$ such that the triangles they form commute, and they are universal with the respect to this property. Now that we are in a setting where maps come equipped with certain values, what do we do with them? How do we formally ask that such maps into $A, B, C$ have precise values? Can we? What universal property do they need to have? In order to answer these questions, we use the notion of *weighted pullback*, specialize it to the enriched setting, and – considering the logic we want to interpret – describe what we need.

In this section, we provide a diagrammatic definition of weighted pullbacks for $\mathbf{Set}(\mathbb{M})$-enriched categories and refer the reader to Appendix D to see how these are specialisations of the general theory of weighted limits in enriched categories.

**Construction 3.0.1.** (Weighted pullbacks) Let $\mathscr{C}$ be a $\mathbf{Set}(\mathbb{M})$-enriched category. We are only concerned with studying pullbacks, so we pick $\mathscr{D}$ to be the enriched cospan $0 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 1$: that is, the $\mathbf{Set}(\mathbb{M})$-enriched category with objects $0, 1, 2$ and with morphisms generated by $\underline{\mathsf{hom}}_{\mathscr{D}}(0,2) = (\{*\}, \alpha)$ and $\underline{\mathsf{hom}}_{\mathscr{D}}(1,2) = (\{*\}, \beta)$.

We will see in Section 4 that our display maps will always have value 1. Thus, the only kinds of weighted pullbacks we are interested in are those where one of the maps to $C$ have value 1 and so we make the assumption that $\alpha = 1$. The case when $\alpha$ is not necessarily 1 is dealt with in Appendix D.

Suppose we have the following enriched functors (Definition B.0.2): a diagram $D : \mathscr{D} \to \mathscr{C}$ and weights $W : \mathscr{D} \to \mathbf{Set}(\mathbb{M})$.

$$
\begin{tikzcd}
 & 0 \arrow[d, "\alpha"] & & & A \arrow[d, "|f|=\alpha"] \\
1 \arrow[r, "\beta"] & 2 \arrow[r, mapsto, "D"] \arrow[rrd, mapsto, "W"'] & & B \arrow[r, "|g|=\beta"'] & C \\
 & & & & 1_{w_A} \arrow[d, "\alpha"] \\
 & & & 1_{w_B} \arrow[r, "\beta"] & 1_{w_C}
\end{tikzcd}
$$

Note that the weights must satisfy $w_A \cdot 1 \leq w_C$ and $w_B \cdot \beta \leq w_C$ because the image of $W$ is in $\mathbf{Set}(\mathbb{M})$.

Our aim is to mimic pullbacks in an enriched category, so while $D$ does what it usually does in **Set**-enriched categories, we pick the weights to be singletons with constant value possibly less than 1:

- singletons because to each vertex $A,B,C$ we want to get with *one* arrow,
- with value possibly less than 1 because if it was only 1 we would recover only maps with value 1 and thus lose all the *fuzziness* of substitution.

The weight functor $W$ controls the fuzziness of arrows to each of the objects $A,B$ and $C$. In other words, we are looking for the object $X \in \mathscr{C}$ such that there are maps to $A,B$ and $C$ with fuzziness greater than $w_A, w_B$ and $w_C$ respectively such that the triangles commute appropriately. Further, we want the pullback $X$ to be universal among all other objects that also have the maps to $A,B$ and $C$ with the required commutativity.

**Definition 3.0.2.** (Weighted pullbacks) For a diagram $D$ and weight $W$ as above, the weighted pullback of $D$ is an object $A \times_C B \in \mathscr{C}$ given by the following universal property: For every $H \in \mathscr{C}$,

$$\underline{\mathsf{hom}}_{\mathscr{C}}(H, A \times_C B)_0 \cong \{(a,c,b) \mid f \circ a = c = g \circ b\} \subseteq \underline{\mathsf{hom}}(H,A)^0 \times \underline{\mathsf{hom}}(H,C)^0 \times \underline{\mathsf{hom}}(H,B)^0 \qquad (1)$$

and the valuation is given by $|(a,c,b)| = (w_A \to |a|) \wedge (w_C \to |c|) \wedge (w_B \to |b|)$.

**Definition 3.0.3** (Having weighted pullbacks for substitution)**.** A $\mathbf{Set}(\mathbb{M})$-category *has weighted pullbacks for substitution* if for each cospan $(f,g)$ with valuations $|f| = 1$ and $|g| = \beta$, it has weighted pullbacks with weights given by $w_A = \beta$ and $w_B = 1$.

The relations $w_A \cdot 1 \leq w_C$ and $w_B \cdot \beta \leq w_C$ imply that $w_C \geq \beta$.

**Remark 3.0.4.** (Interpretation of weighted pullbacks for substitution) If $\mathscr{C}$ has weighted pullbacks, then for each $f,g$

$$\begin{tikzcd}
H \arrow[dr, dashed, "u"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\
& A \times_C B \arrow[r, "\geq\beta"'] \arrow[d, "1"] & A \arrow[d, "|f|=1"] \\
& B \arrow[r, "|g|=\beta"'] & C
\end{tikzcd}$$

and $a,b$ such that $f \circ a = g \circ b$ there is a unique $u\colon H \to A \times_C B$ with value $|u| = w_A \to |a| \wedge w_C \to |c| \wedge w_B \to |b|$ where $w_A = \beta$ and $w_B = 1$. Since, $w_A \cdot 1 \leq w_C$ and $|c| = |f \circ a| \geq 1 \cdot |a|$, we have that $w_A \to |a| \leq w_C \to |c|$. Similarly, $w_B \to |b| \leq w_C \to |c|$. So, the valuation of $u$ reduces to

$$|u| = w_A \to |a| \wedge w_B \to |b| = \beta \to |a| \wedge 1 \to |b|$$

Note that when we take $H = A \times_C B$, the map $u$ is just the identity map and so has valuation 1. In particular, this means that $1 = \beta \to |a| \wedge 1 \to |b|$ and so $|a| \geq \beta$ and $|b| \geq 1$. which explains the valuations of the right and down arrow in the above diagram. By composition in $\mathbf{Set}(\mathbb{M})$, the middle arrow $A \times_C B \to C$ has valuation $\geq 1 \cdot \beta$. As promised, the valuations of maps from the weighted pullback $A \times_C B$ to $A,B$ and $C$ are greater than $w_A, w_B$ and $w_C$ respectively.

**Example 3.0.5.** (Weighted pullbacks in $\mathbf{Set}(\mathbb{I})$ for substitution) Suppose we have maps $a,b$ as in the diagram given in Remark 3.0.4 with valuations $|a|$ and $|b|$ in $\mathbb{I}$ respectively, then there exists a unique $u$ with valuation

$$|u| = \frac{|a|}{\beta} \wedge \frac{|b|}{1}.$$

**Example 3.0.6.** (Weighted pullbacks in $\mathbf{Set}(\mathbb{B})$ for substitution) Now suppose that we have maps $a, b$ as in the diagram given in Remark 3.0.4 but with valuations $|a|$ and $|b|$ in $\mathbb{B}$. If $|a|$ and $|b|$ are both 1, then the weighted pullback corresponds to the ordinary pullback in non-enriched categories. If either of $|a|$ or $|b|$ is 0 then there is no pullback.

Finally, we show that weighted pullbacks compose appropriately.

**Lemma 3.0.7** (Pullback pasting lemma)**.** Consider a commutative diagram in $\mathbf{Set}(\mathbb{M})$-enriched category $\mathscr{C}$ as follows

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "|h|=\beta"] \\ A \arrow[r, "|f|=\gamma"'] & B \arrow[r, "|g|=\alpha"'] & C \end{tikzcd}$$

such that $|h'| \leq |h''|$. If the right square is a weighted pullback, then:

- the outer rectangle is a weighted pullback if the left square is a weighted pullback;
- the left square is a weighted pullback if the outer rectangle is a weighted pullback.

Refer to Appendix D for a proof of the lemma.

# 4 From syntax to semantics and back

We can now finally describe the necessary categorical structure needed in order to interpret fuzzy dependent types.

## 4.1 Display-map fuzzy categories

Following Definition 1.1.1, we generalize it to the fuzzy-set-enriched case.

**Definition 4.1.1** (Display-map fuzzy category)**.** A *display-map fuzzy category* is a pair $(\mathscr{C}, \mathfrak{D})$ of a $\mathbf{Set}(\mathbb{M})$-category $\mathscr{C}$ and a class of morphisms $\mathfrak{D} = \{p_A : \Gamma.A \to \Gamma\}$ in $\mathscr{C}$ called *display maps* such that:

1. for each display map $p_A : \Gamma.A \to \Gamma$ and $s : \Delta \to \Gamma$ in $\mathscr{C}$, there is a display map $p_{A[s]}$ which is the weighted pullback for substitution (Definition 3.0.3) of $p_A$ along $s$

$$\begin{tikzcd} \Delta.A[s] \arrow[r, "\bar{s}"] \arrow[d, "p_{A[s]}"'] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "s"'] & \Gamma \end{tikzcd}$$

2. $\mathfrak{D}$ is closed under pre and post-composition with isomorphisms;
3. $\mathscr{C}$ has a terminal object 1;
4. for all $A$, $|p_A|_{\underline{\mathsf{hom}}(\Gamma.A,\Gamma)} = 1$.

Again, we think of objects in $\mathscr{C}$ as contexts, projections as types, (weighted) pullback as substitution, 1 as the empty context. Notice that we additionally ask for projections to always have the maximum possible value, so that types themselves are not fuzzy.

**Remark 4.1.2.** Our types are not fuzzy. One could easily change that by removing axiom 4, that makes the value of each type be constant 1. This is a direction we would clearly like to extend this project in, and it is clear that the majority of the work we are presenting does not really rely on this hypothesis.

### 4.2 Sections

As in Section 1.1, we now want to look at sections of projections, and say that they describe terms in our new setting. We need to be able to *fully* exploit our enrichment, and have sections come equipped with a given $\mathbb{M}$-value, which we interpret to be the desired "extent" discussed in Section 1. This is the motivation behind the following definition. Recall Construction 2.0.10 for the subtleties in the notation below.

**Definition 4.2.1.** ($\alpha$-sections) Let $p_A$ a projection. An *$\alpha$-section of $p_A$* is a morphism $a$ in $\underline{\mathrm{hom}}(\Gamma, \Gamma.A)$ such that

- $p_A \circ a = \mathrm{id}$, and
- $|a| \geq \alpha$.

$$\Gamma \xrightarrow[a]{} \Gamma.A \xrightarrow[p_A]{} \Gamma$$

This means that we can finally describe our interpretation to its full extent: we understand objects of $\mathscr{C}$ as contexts, projections as types, terms (of confidence *at least* $\alpha$) as ($\alpha$-)sections. If $a$ is a term of type $A$ in context $\Gamma$ with confidence $\alpha$ we write

$$\Gamma \vdash a :_\alpha A.$$

Classical judgements involving types are written in the usual manner. Notice that variables which are written in contexts only do not possess a confidence of their own.

**Remark 4.2.2** (Confidence is preserved)**.** For all $\beta \leq \alpha$, the following holds.

$$\frac{\Gamma \vdash a :_\alpha A}{\Gamma \vdash a :_\beta A}$$

When $\mathbb{M} = \mathbb{B}$ then of course this adds no information to our system (given that $0 < 1$), but in the case of the unit interval $\mathbb{I}$, for example, this is meant to say that whenever a person holds a belief with a certain degree, they also hold it with all possible degrees below that.

## 5 Structural rules for fuzzy type theory

Now that we have all the basic elements of our theory, we need to express *what* we can do with them. This section of the paper flirts with moving swiftly between the logical side of things and the categorical one, so we better spend some time explaining each, and how the two are related.

Classical type theory as described in [6] is a logical system in which one gives an account of judgements pertaining terms and types in context. Out of these basic blocks, one builds up a system imposing rules that dictate the behaviour of all of these different pieces together: for example, out of a type in context, one can produce a new context "pasting" the type to its context. Each rule has a label that is meant to be descriptive of its meaning, for example the rule we just described is usually denoted (C-Ext) for context extension.

On its categorical side, the rules represent operations one can perform in the category, given the axioms one starts from, so that they are usually built in the original definition of the categorical structure one chooses. In the case of context extension, for example, the action of computing the context obtained by extending a context with a type amounts to computing the domain of the corresponding projection. If this sounds tautological at all, it is because the categorical structure one considers is meant to precisely mimic the logic. When our effort is successful, we say that the structure *verifies* the rules.

**Remark 5.0.1** (Our strategy)**.** Here, we sort of work the other way round: we have defined a structure that is directly built on top of the classical definition, now we repeat the same constructions of the classical case, and *read* into the category the correct formulation of the rules. Nonetheless, Theorem 5.0.2 is expressed in the traditional form.

The four judgements allowed in our theory are of the following form,

$$\Gamma \vdash A \ \mathtt{Type} \qquad \Gamma \vdash A = A' \ \mathtt{Type} \qquad \Gamma \vdash a :_\alpha A \qquad \Gamma \vdash a = a' :_\alpha A$$

and they respectively declare that $A$ is a type in context $\Gamma$, that types $A, A'$ in context $\Gamma$ are definitionally equal, that with confidence at least $\alpha$ $a$ is a term of type $A$ in context $\Gamma$, that with confidence at least $\alpha$ $a, a'$ are definitionally equal terms of type $A$ in context $\Gamma$. We do not dwell further on the treatment of definitional equality, as it is not in itself fuzzy in our context, and a notion of identity is in itself available, though some might say "controversial", in categories. In this sense, all rules prescribing that it is an equivalence relation are trivial in our setting. Of course it is very easy to see how this, too, can be extended from the intuition of the present work: one could simply enrich in a category of fuzzy sets where not only the membership predicate, but the equality, too, takes fuzzy values. We discuss this more in Section 6.

**Theorem 5.0.2.** (Soundness) Let $(\mathscr{C}, \mathfrak{D})$ a display-map fuzzy category. Then it verifies the following rules for fuzzy type theory.

$$\frac{}{\vdash \diamond \ \mathtt{ctx}}(\text{C-Emp}) \qquad \frac{\Gamma \vdash A \ \mathtt{Type}}{\vdash \Gamma, x : A \ \mathtt{ctx}}(\text{C-Ext}) \qquad \frac{\vdash \Gamma, x : A, \Delta \ \mathtt{ctx}}{\Gamma, x : A, \Delta \vdash x :_1 A}(\text{Var})$$

$$\frac{\Gamma, \Delta \vdash B \ \mathtt{Type} \quad \Gamma \vdash A \ \mathtt{Type}}{\Gamma, x : A, \Delta \vdash B \ \mathtt{Type}}(\text{Weak}_{ty}) \qquad \frac{\Gamma, \Delta \vdash b : B \quad \Gamma \vdash A \ \mathtt{Type}}{\Gamma, x : A, \Delta \vdash b : B}(\text{Weak}_{tm})$$

$$\frac{\Gamma, x : A, \Delta \vdash B \ \mathtt{Type} \quad \Gamma \vdash a :_\alpha A}{\Gamma, \Delta[a/x] \vdash B[a/x] \ \mathtt{Type}}(\text{Subst}_{ty}) \qquad \frac{\Gamma, x : A, \Delta \vdash b :_\beta B \quad \Gamma \vdash a :_\alpha A}{\Gamma, \Delta[a/x] \vdash b[a/x] :_\beta B[a/x]}(\text{Subst}_{tm})$$

We leave the detailed proof in Appendix E. All rules not involving terms do not contain more information than in the classical case: (C-Emp), (C-Ext), (Weak$_{ty}$) are the same. The variable rule (Var) says that, given a well-formed context (of types, hence of confidence 1), one can always extract a variable in (a type in) it with confidence 1. Weakening for terms (Weak$_{tm}$) says that when weakening a context, confidence in a term is not increased, nor lost. The most curious looking rules are those involving substitution: (Subst$_{ty}$) and (Subst$_{tm}$) seem to forget the confidence of $a$. However, this is made clear by looking at the corresponding weighted pullback used to define these substitutions, as discussed in Appendix E.

The key idea is to note that $B[a/x]$ already takes into account the confidence of $a$, and so in some sense, $B[a/x]$ can be viewed as a kind of fuzzy type. So we could define $B[a/x]$ to be an example of a fuzzified version of $B$ with confidence $\alpha$. However, more work needs to be done in determine how exactly this kind of fuzziness arises, as although this is a consequence of our currently chosen framework, it could be that there are other ways of implementing fuzzy types in the first place that encapsulates this behavior, which would simplify the type theory. At minimum though, we can use these perhaps surprising results to guide the design of future fuzzy type theories.

# 6 Conclusions and future work

As we mentioned above, fuzzifying terms produces an apparently unexpected formula for the last substitution rule. This is the result of two choices that can easily be avoided, namely that we ask each projection to be of value 1, and that the two triangles insisting on the same arrow should maximally coincide (see Appendix D). Nevertheless, we have decided to present this work in a simpler form, in order to better be introduced, but with a mathematical structure solid enough to accommodate different uses that we plan to explore in the future. Clearly, these go beyond tweaking the two choices above, and we are interested in extending our strategy even further, to attach fuzziness to other relations, starting from definitional equality.

Another immediate topic to explore is the behavior of $\mathbf{Set}(\mathbb{M})$ for other quantales $\mathbb{M}$, calculate pullbacks in such context and interpret its type theory. Perhaps, different $\mathbb{M}$ will provide a fuzzy type theory that describes other fuzzy real-life situations besides peoples' beliefs.

In a more applied perspective, we motivated the introduction of a fuzzy type theory by arguing it would provide a good model for opinions. Ghrist and Hansen recently applied categorical methods to study how opinions evolve over time, but modeled opinions using vector spaces [3]. We want to import some of their categorical ideas and mix it with our fuzzy type theory in order to provide a new, dynamic, representation of opinions.

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# A Structural rules of MLTT

The analogous rules of Theorem 5.0.2 for non-fuzzy type theory are

$$\frac{}{\vdash \diamond\ \mathtt{ctx}}(\text{C-Emp}) \qquad \frac{\Gamma \vdash A\ \mathtt{Type}}{\vdash \Gamma, x : A\ \mathtt{ctx}}(\text{C-Ext}) \qquad \frac{\vdash \Gamma, x : A, \Delta\ \mathtt{ctx}}{\Gamma, x : A, \Delta \vdash x : A}(\text{Var})$$

$$\frac{\Gamma, \Delta \vdash B\ \mathtt{Type} \quad \Gamma \vdash A\ \mathtt{Type}}{\Gamma, x : A, \Delta \vdash B\ \mathtt{Type}}(\text{Weak}_{ty}) \qquad \frac{\Gamma, \Delta \vdash b : B \quad \Gamma \vdash A\ \mathtt{Type}}{\Gamma, x : A, \Delta \vdash b : B}(\text{Weak}_{tm})$$

$$\frac{\Gamma, x : A, \Delta \vdash B\ \mathtt{Type} \quad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash B[a/x]\ \mathtt{Type}}(\text{Subst}_{ty}) \qquad \frac{\Gamma, x : A, \Delta \vdash b : B \quad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash b[a/x] : B[a/x]}(\text{Subst}_{tm})$$

Observe that we expressed the weakening and the substitution rules separately for types and for terms. Since in non-fuzzy type theory neither (terms or types) of both are fuzzy, this separation is not necessary, so in [6], for instance, the weakening and the substitution rules are presented as follows

$$\frac{\Gamma, \Delta \vdash J \quad \Gamma \vdash A\ \mathtt{Type}}{\Gamma, x : A, \Delta \vdash J}(\text{Weak}) \qquad \frac{\Gamma, x : A, \Delta \vdash J \quad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash J[a/x]}(\text{Subst})$$

where $J$ ranges over $B$ $\mathtt{Type}$ and $a : A$ (and also $a = b$, and $A = B$ $\mathtt{Type}$, but for the moment we stick with a trivial, identity-of-objects in categories, notion of definitional equality).

# B Definition of enriched category

**Definition B.0.1.** Let $(\mathscr{V}, \otimes, I)$ be a monoidal category. A *$\mathscr{V}$-enriched category $\mathscr{C}$* consists of

1. a set $\mathrm{Ob}(\mathscr{C})$ of *objects of $\mathscr{C}$*,
2. for each pair $x, y$ of objects of $\mathscr{C}$, an object $\underline{\mathsf{hom}}_{\mathscr{C}}(x, y)$ of $\mathscr{V}$,
3. for each object $x$ of $\mathscr{C}$, a point $\mathsf{id}_x : I \to \underline{\mathsf{hom}}_{\mathscr{C}}(x, x)$, and
4. for each triple $x, y, z$ of objects of $\mathscr{C}$, a morphism in $\mathscr{V}$

$$\circ : \underline{\mathsf{hom}}_{\mathscr{C}}(x, y) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y, z) \to \underline{\mathsf{hom}}_{\mathscr{C}}(x, z)$$

such that the following diagrams commute for all objects $w,x,y,z$ of $\mathscr{C}$ (and where the unlabelled isomorphisms come from the monoidal structure of $\mathscr{V}$).

$$
\begin{tikzcd}
\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \arrow[r, "\cong"] \arrow[d, equal] & I \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \arrow[d, "\mathsf{id}_x \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(x,y)"] \\
\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) & \underline{\mathsf{hom}}_{\mathscr{C}}(x,x) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \arrow[l, "\circ"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \arrow[r, "\cong"] \arrow[d, equal] & \underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \otimes I \arrow[d, "\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \otimes \mathsf{id}_y"] \\
\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) & \underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y,y) \arrow[l, "\circ"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\underline{\mathsf{hom}}_{\mathscr{C}}(w,x) \otimes (\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y,z)) \arrow[rr, "\underline{\mathsf{hom}}_{\mathscr{C}}(w,x) \otimes \circ"] \arrow[d, "\cong"] & & \underline{\mathsf{hom}}_{\mathscr{C}}(w,x) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(x,z) \arrow[d, "\circ"] \\
(\underline{\mathsf{hom}}_{\mathscr{C}}(w,x) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(x,y)) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y,z) \arrow[dr, "\circ \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y,z)"'] & & \underline{\mathsf{hom}}_{\mathscr{C}}(w,z) \\
& \underline{\mathsf{hom}}_{\mathscr{C}}(w,y) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y,z) \arrow[ur, "\circ"] &
\end{tikzcd}
$$

**Definition B.0.2.** Let $(\mathscr{V},\otimes,I)$ be a monoidal category, and let $\mathscr{C}$ and $\mathscr{D}$ be $\mathscr{V}$-enriched categories. A *$\mathscr{V}$-functor $F : \mathscr{C} \to \mathscr{D}$* consists of

1. a function $F_{\mathsf{Ob}} : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ and
2. for each pair of objects $x,y$ of $\mathscr{C}$, a morphism $F_{x,y} : \underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \to \underline{\mathsf{hom}}_{\mathscr{D}}(Fx,Fy)$ in $\mathscr{V}$.

such that the following diagrams commute.

$$
\begin{tikzcd}
I \arrow[r, "\mathsf{id}_x"] \arrow[dr, "\mathsf{id}_{fx}"'] & \underline{\mathsf{hom}}_{\mathscr{C}}(x,x) \arrow[d, "F_{x,x}"] \\
& \underline{\mathsf{hom}}_{\mathscr{D}}(Fx,Fx)
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathsf{hom}}_{\mathscr{C}}(x,y) \otimes \underline{\mathsf{hom}}_{\mathscr{C}}(y,z) \arrow[r, "\circ"] \arrow[d, "F_{x,y} \otimes F_{y,z}"] & \underline{\mathsf{hom}}_{\mathscr{C}}(x,z) \arrow[d, "F_{x,z}"] \\
\underline{\mathsf{hom}}_{\mathscr{D}}(Fx,Fy) \otimes \underline{\mathsf{hom}}_{\mathscr{D}}(Fy,Fz) \arrow[r, "\circ"] & \underline{\mathsf{hom}}_{\mathscr{D}}(Fx,Fz)
\end{tikzcd}
$$

# C The category of fuzzy sets

Here we provide technical details omitted in Section 2.

*Proof of Proposition 2.0.18.* Since the meet is a particular (idempotent) case of the product we will prove the statement only for the product.

Given $X$ and $Y$ objects in $\mathbf{Set}(\mathbb{M})$, their monoidal product is $X \otimes Y = (X^0 \otimes Y^0, |-|_{X\otimes Y})$, where $|(x,y)|_{X\otimes Y} = |x|_X \cdot |y|_Y$.

The tensor unit is given by $I = (I^0, |-|_I)$ where $|i|_I = 1_{\mathbb{M}}$.

There are associators because

$$
\begin{aligned}
|((x,y),z)|_{(X\otimes Y)\otimes Z} &= |(x,y)|_{X\otimes Y} \cdot |z|_Z \\
&= |x|_X \cdot |y|_Y \cdot |z|_Z \\
&= |x|_X \cdot |(y,z)|_{Y\otimes Z} \\
&= |((x,(y,z))|_{X\otimes(Y\otimes Z)}
\end{aligned}
$$

There are left unitors since

$$|(i,x)|_{I\otimes X} = |i|_I \cdot |x|_X = 1 \cdot |x|_X = |x|_X$$

The existence of right unitors follows analogously.

The triangle identity holds since

$$\begin{aligned} |((x,i),z)|_{(X\otimes I)\otimes Z} &= |(x,(i,z))|_{X\otimes(I\otimes Z)} \\ &= |x|_X \cdot |(i,z)|_{I\otimes Z} \\ &= |x|_X \cdot 1 \cdot |z|_Z \\ &= |x|_X \cdot |z|_Z = |(x,z)|_{X\otimes Z} \end{aligned}$$

The pentagon identity follows by the same reasoning. □

**Remark C.0.1.** Defining $X\otimes Y = (X^0 \times Y^0, |-|_{X\times Y})$, where $|(x,y)|_{X\times Y} = |x|_X \cdot |y|_Y$ it is not monoidal cartesian. Remind that the cartesian product of two sets is equipped with projections $\pi_1$ and $\pi_2$ such that for any pair of morphism $f : Z \to X$ and $g : Z \to Y$ there is a unique morphism $h : Z \to X \times Y$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Moreover, know that $h(z) = (f(z), g(z))$, for any $z \in Z$. However, such $h$ may not be a morphism in $\mathbf{Set}(\mathbb{M})$. Take $\mathbb{M} = ([0,1], \cdot, 1, \leq)$, then $|h(z)|_{X\times Y} = |(f(z),g(z))|_{X\times Y} = |f(z)|_X \cdot |g(z)|_Y \geq |z|_Z \cdot |z|_Z \leq |z|_Z$.

Of course, when $\cdot = \wedge$, then $|(f(z),g(z))|_{X\times Y} = |f(z)|_X \wedge |g(z)|_Y \geq |z|_Z \wedge |z|_Z = |z|_Z$. This arguments works because $\wedge$ is an idempotent operation.

It is known that $\mathbf{Set}(\mathbb{M})$ is cartesian closed category if and only if $\mathbb{M}$ is a complete Heyting Algebra, see [16, Proposition 71.4]. Using an analogous argument we now prove that $\mathbf{Set}(\mathbb{M})$ is monoidal closed if and only if $\mathbb{M}$ is a quantale.

*Proof of Proposition 2.0.12.* Given $\mathbb{M}$-fuzzy sets $Y$ and $Z$, define a $\mathbb{M}$-fuzzy set $Z^Y = (\{h : Y \to Z\}, |-|_{Z^Y})$ where $|h|_{Z^Y} = \bigwedge_{y\in Y} (|y|_Y \to |h(y)|_Z)$.

It is clear that $Hom_{Set}(X, Z^Y) \cong Hom_{Set}(X\times Y, Z)$, since *Set* with the cartesian product is closed, but we want to show that $Hom_{\mathbf{Set}(\mathbb{M})}(X, Z^Y) \cong Hom_{\mathbf{Set}(\mathbb{M})}(X\times Y, Z)$. A morphism in $Hom_{\mathbf{Set}(\mathbb{M})}(X\times Y, Z)$ is a function $f : X\times Y \to Z$ satisfying

$$|x|_X \cdot |y|_Y \leq |f(x,y)|_Z$$

for all $(x,y) \in X \times Y$.

Since $\mathbb{M}$ is a quantale, this is equivalent to

$$|x|_X \leq |y|_Y \to |f(x,y)|_Z,$$

which happens if and only if

$$|x|_X \leq |y|_Y \to |\tilde{f}(x)(y)|_Z,$$

where $\tilde{f}$ is the exponentially adjoint to $f$ in *Set*.

Observe that $|\tilde{f}(x)|_{Y^Z} = \bigwedge_{y\in Y} (|y|_Y \to |\tilde{f}(x)(y)|_Z)$. Thus, by definition of infimum, we have a last equivalence:

$$|x|_X \leq |\tilde{f}(x)|_{Y^Z}$$

Therefore, $\tilde{f}(x) : X \to Z^Y$ is the morphisms in $\mathbf{Set}(\mathbb{M})$ that testifies the desired isomorphism.

Conversely, suppose that $\mathbf{Set}(\mathbb{M})$ is monoidal closed. Consider the $\mathbb{M}$-fuzzy set $(\{*\}, m)$ where $\{*\}$ denotes the singleton set and $m$ denotes the constant function $|*|_{\{*\}} = m$, for each $m \in \mathbb{M}$. Since $\mathbb{M}$ is integral, we have $m_i \cdot m_j \leq m_i$ for all $m_i, m_j \in \mathbb{M}$. So we have morphisms $id_* : (\{*\}, m_i \cdot m_j) \to (\{*\}, m_i)$, for all $i \in I$. A diagram of such morphisms has a colimit cone of morphisms $id_{\{*\}} : (\{*\}, m_i) \to (\{*\}, \bigvee_{i \in I} m_i)$.

$\mathbf{Set}(\mathbb{M})$ is monoidal closed, so the tensor product is a left adjoint functor and thus it has to preserve colimits. Take $\mathbb{M}$-fuzzy set $(\{*\}, n)$. Observe that $(\{*\}, n) \otimes (-)$ preserves the above colimit iff $n \cdot (\bigvee\limits_{i \in I} m_i) = \bigvee\limits_{i \in I} (n \cdot m_i)$. Therefore, $\mathbb{M}$ is a quantale. □

## D Substitution in enriched categories

In this section, we spell out the general construction of weighted pullbacks in enriched categories and show that Definition 3.0.2 is a special case of this construction.

**Definition D.0.1.** (Weighted limits) Let $\mathscr{V}$ be a closed symmetric monoidal category and consider two $\mathscr{V}$-enriched categories $\mathscr{C}$ and $\mathscr{D}$. Then the weighted limit of the diagram $D : \mathscr{D} \to \mathscr{C}$ with weights $W : \mathscr{D} \to \mathscr{V}$ is the object $\lim^W D$ in $\mathscr{C}$ given by the following universal property:

$$\underline{\mathsf{hom}}_{\mathscr{C}}(X, \lim^W D) \cong \int_{\mathscr{D}} [W-, \underline{\mathsf{hom}}(X, D-)]$$

for all objects $X \in \mathscr{C}$. Here $[\cdot,\cdot]$ represents the internal hom in $\mathscr{V}$.

The weight functor can be thought about as prescribing the weight of an arrow to each of the objects in the diagram.

**Example D.0.2.** Let $\mathscr{V} = \mathbf{Set}$ and $\mathscr{D}$ be the cospan $0 \to 2 \leftarrow 1$. In this case the weight functor $W : \mathscr{D} \to \mathbf{Set}$ represents the set of arrows to each of the objects in the diagram $D$. The object $\lim^W D$ is given by the universal property

$$\mathrm{hom}_{\mathscr{C}}(X, \lim^W D) \cong \int_{\mathscr{D}} [W-, \underline{\mathsf{hom}}(X, D-)] \cong \int_{\mathscr{D}} \mathrm{hom}(W-, \mathrm{hom}(X, D-))$$

for all $X \in \mathscr{C}$. The end above corresponds to the limit of the following cospan $\mathrm{hom}(W0, \mathrm{hom}(X, D0)) \to \mathrm{hom}(W2, \mathrm{hom}(X, D2)) \leftarrow \mathrm{hom}(W1, \mathrm{hom}(X, D1))$ in $\mathbf{Set}$. The limit is just the ordinary pullback

$$\mathrm{hom}_{\mathscr{C}}(X, \lim^W D) \cong \mathrm{hom}(W0, \mathrm{hom}(X, D0)) \times_{\mathrm{hom}(W2, \mathrm{hom}(X, D2))} \mathrm{hom}(W1, \mathrm{hom}(X, D1)).$$

We are interested in the case where $\mathscr{V}$ is the category of fuzzy sets, and $D$ is a cospan diagram, but in order to compute the functor $\underline{\mathsf{hom}}(X, D-)$ we first need to make a technical observation.

**Remark D.0.3** (Postcomposition in an enriched category)**.** Computing the wedge described in Definition D.0.1 requires us to be very careful: in the classical, set-based context, for each map $f : A \to B$ in a category $\mathscr{C}$ one can define for each object $H$ in $\mathscr{C}$ a *function*

$$\mathrm{hom}(H, A) \to \mathrm{hom}(H, B), \quad g \mapsto f \circ g \tag{2}$$

between the hom-sets. This in turn allows us to define for any $X$ in $\mathscr{C}$ the map

$$[X, \mathrm{hom}(H, A)] \to [X, \mathrm{hom}(H, B)],$$

which is the one that is required by the definition of the wedge in Definition D.0.1. This is not always the case in a $\mathscr{V}$-enriched category, because the map described in Eq. (2), while being a morphism in **Set**, might not be a morphism in $\mathscr{V}$ - in fact $\mathscr{V}$ might not even be concrete. What we can do is unfold the behavior underlying Eq. (2) using the monoidal structure, because that is the structure producing composition: in fact, what one can do is compute the following

$$\underline{\mathsf{hom}}(H,A)\otimes I \to \underline{\mathsf{hom}}(H,A)\otimes \underline{\mathsf{hom}}(A,B) \to \underline{\mathsf{hom}}(H,B),$$

in which the first maps picks out $f$ in $\underline{\mathsf{hom}}(A,B)$ and the second computes composition. We only consider the case we are interested in, meaning when $\mathscr{V} = \mathbf{Set}(\mathbb{M})$. In order for this to work, though, we cannot have $I = 1_1$, the singleton with constant value 1, but $1_{|f|}$, so that

$$|g|_{\underline{\mathsf{hom}}(H,A)} \cdot |*|_{1_{|f|}} \leq |g|_{\underline{\mathsf{hom}}(H,A)} \cdot |f|_{\underline{\mathsf{hom}}(A,B)} \leq |f\circ g|_{\underline{\mathsf{hom}}(H,B)}.$$

This will change the nodes in the wedge we compute for the weighted pullback.

**Construction D.0.4** (Weighted pullbacks in categories enriched in fuzzy sets)**.** We work in the setting of Construction 3.0.1 with the exception that $\alpha$ need not be 1. Suppose we have the following enriched functors: diagram $D : \mathscr{D} \to \mathscr{C}$ and weights $W : \mathscr{D} \to \mathbf{Set}(\mathbb{M})$.

$$\begin{tikzcd}
 & 0 \arrow[d, "\alpha"] & & & A \arrow[d, "|f|=\alpha"] \\
1 \arrow[r, "\beta"] & 2 \arrow[rr, mapsto, "D"] \arrow[drrr, mapsto, "W"'] & & B \arrow[r, "|g|=\beta"'] & C \\
 & & & & 1_{w_A} \arrow[d, "\alpha"] \\
 & & & 1_{w_B} \arrow[r, "\beta"] & 1_{w_C}
\end{tikzcd}$$

By the definition of weighted limits (Definition D.0.1), for any $H$ in $\mathscr{C}$ we have that $\underline{\mathsf{hom}}(H,\lim^W D)$ is isomorphic to the limit below.

$$\begin{tikzcd}[column sep=small]
 & & \int_{\mathfrak{D}}[W-,\underline{\mathsf{hom}}(H,D-)] \arrow[dll, dashed] \arrow[d, dashed] \arrow[drr, dashed] & & \\
{[1_{w_A},\underline{\mathsf{hom}}(H,A)]} \arrow[dr] & & {[1_{w_C},\underline{\mathsf{hom}}(H,C)]} \arrow[dl] \arrow[dr] & & {[1_{w_B},\underline{\mathsf{hom}}(H,B)]} \arrow[dl] \\
 & {[1_{w_A\cdot\alpha},\underline{\mathsf{hom}}(H,C)]} & & {[1_{w_B\cdot\beta},\underline{\mathsf{hom}}(H,C)]} &
\end{tikzcd} \tag{3}$$

Let us first of all comment on the mixed term $[1_{w_A\cdot\alpha},\underline{\mathsf{hom}}(H,C)]$: this is precisely an instance of Remark D.0.3 because $1_{w_A\cdot\alpha} \cong 1_{w_A}\otimes 1_\alpha$. For any $w \in \mathbb{M}$, an $\mathbb{M}$-set of the form

$$[1_w,\underline{\mathsf{hom}}(H,X)]$$

is the set of underlying functions $x\colon (1_w)^0 = \{*\} \to \underline{\mathsf{hom}}(H,X)^0$ with valuation

$$|x|_{[1_w,\underline{\mathsf{hom}}(H,X)]} = \bigwedge_{y\in(1_w)^0} |y|_{1_w} \to |x(y)|_{\underline{\mathsf{hom}}(H,X)} = w \to |x(*)|_{\underline{\mathsf{hom}}(H,X)},$$

so that it basically picks up $\mathbb{M}$-set morphisms $x\colon H \to X$ and it assigns them a new value depending on both their original one and on $w$.

Let us show that both

1. $[1_{w_A}, \underline{\mathsf{hom}}(H,A)] \to [1_{w_A\cdot\alpha}, \underline{\mathsf{hom}}(H,C)]$, and
2. $[1_{w_C}, \underline{\mathsf{hom}}(H,C)] \to [1_{w_A\cdot\alpha}, \underline{\mathsf{hom}}(H,C)]$

are well defined. Clearly Item 2 acts as a sort of inclusion, because provided a map $c$ in $\underline{\mathsf{hom}}(H,C)_0$ we have

$$w_C \to |c| \leq w_A\cdot\alpha \to |c|$$

if $w_C \geq w_A\cdot\alpha$, but this is true because $w_C \geq w_A\cdot\alpha$. As for Item 1, we want to perform postcomposition: to a map $a \in \underline{\mathsf{hom}}(H,A)^0$ we want to assign $f\circ a$, which produces a morphism in $\mathbf{Set}(\mathbb{M})$ if

$$w_A \to |a| \leq w_A\cdot\alpha \to |f\circ a|$$

but by definition of composition in $\mathscr{C}$ we have

$$|f|\cdot|a| \leq |f\circ a| \quad \text{or, equivalently,} \quad |a| \leq |f| \to |f\circ a| = \alpha \to |f\circ a|$$

hence

$$w_A \to |a| \leq w_A \to (\alpha \to |f\circ a|) = w_A\cdot\alpha \to |f\circ a|$$

by currying, yielding the desired result.

Now, one can show that the desired limit can be computed by (two) iterated pullbacks in $\mathbf{Set}(\mathbb{M})$, which we know how to compute from Proposition 2.0.7 and Item 3. It follows that $\int_{\mathscr{D}}[W-, \underline{\mathsf{hom}}(H,D-)]$ has underlying set

$$\{(a,c,b)\,|\,f\circ a = c = g\circ b\} \subseteq \underline{\mathsf{hom}}(H,A)^0 \times \underline{\mathsf{hom}}(H,C)^0 \times \underline{\mathsf{hom}}(H,B)^0$$

and valuation $|(a,c,b)| = w_A \to |a| \wedge w_C \to |c| \wedge w_B \to |b|$.

In particular, we can use this to describe the universal maps by checking what happens to the image of $\mathsf{id}_{\lim^W D}$ through

$$\hom(\lim^W D, \lim^W D) \cong \int_{\mathscr{D}} [W-, \underline{\mathsf{hom}}(\lim^W D, D-)]\,.$$

Remember that the identity has value 1, hence maps $a,c,b$ from $\lim^W D$ to, respectively, $A,C,B$ must satisfy

$$1 \leq |(a,c,b)| = w_A \to |a| \wedge w_C \to |c| \wedge w_B \to |b|$$

so that $w_A \to |a| = 1$, hence $w_A \leq |a|$, and so on for each term.

We give a proof of Lemma 3.0.7 below.

*Proof.* (Pullback pasting lemma) Suppose the left square is a pullback. Let $H$ in $\mathscr{C}$ with maps $a$ and $d$ such that $g \circ f \circ a = h \circ d$. Then we have $f \circ a$ and $d$ insisting on the cospan $(g,h)$. Since th right square is a pullback we have a unique $u \colon H \to E$

$$\begin{tikzcd}
H \arrow[dr, dashed, "v"'] \arrow[drr, dashed, "u"] \arrow[drrr, bend left, "d"] \arrow[ddr, bend right, "a"'] & & & \\
& F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "|h|=\beta"] \\
& A \arrow[r, "|f|=\gamma"'] & B \arrow[r, "|g|=\alpha"'] & C
\end{tikzcd}$$

which in turn induces a unique $v \colon H \to F$ with value

$$\begin{aligned}
|v| &= \gamma \to |u| \wedge |h'| \to |a| \\
&= \gamma \to (\alpha \to |d| \wedge \beta \to |a|) \wedge |h'| \to |a| \\
&\geq \gamma \to (\alpha \to |d|) \wedge \gamma \to (\beta \to |a|) \wedge |h'| \to |a| \\
&= \gamma \cdot \alpha \to |d| \wedge \gamma \cdot \beta \to |a| \wedge |h'| \to |a| \\
&\geq |gf| \to |d| \wedge \beta \to |a|
\end{aligned}$$

because $|gf| \geq \gamma \cdot \alpha$, $|h'| = \beta \geq \gamma \cdot \beta$.

Conversely, we wish to show that if the outer rectangle is weighted pullback, then the left square is, too. Consider a pair $a, e$ as below,

$$\begin{tikzcd}
H \arrow[dr, dashed, "v"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "a"'] & & & \\
& F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "|h|=\beta"] \\
& A \arrow[r, "|f|=\gamma"'] & B \arrow[r, "|g|=\alpha"'] & C
\end{tikzcd}$$

then by the universal propery of the rectangle we have a unique $v \colon H \to F$ with value

$$|v| = |gf| \to |g'e| \wedge \beta \to |a|,$$

which we wish to show to be an appropriate universal map for the left square: commutativity is trivial, so we only need to show that it has the appropriate value. Given that the right square is a pullback, we trivially have

$$|e| = \alpha \to |g'e| \wedge \beta \to |h'e|,$$

therefore

$$\begin{aligned}
\gamma \to |e| \wedge \beta \to |a| &= \gamma \to (\alpha \to |g'e| \wedge \beta \to |h'e|) \wedge \beta \to |a| \\
&\geq \gamma \to (\alpha \to |g'e|) \wedge \gamma \to (\beta \to |h'e|) \wedge \beta \to |a| \\
&= \gamma \cdot \alpha \to |g'e| \wedge \gamma \cdot \beta \to |h'e| \wedge \beta \to |a| \\
&\geq |fg| \to |g'e| \wedge \gamma \cdot \beta \to |h'e| \wedge \beta \to |a| \\
&= |fg| \to |g'e| \wedge \beta \to |a|
\end{aligned}$$

because $\beta \leq |h''|$ by hypothesis, so that

$$\gamma \cdot \beta \to |fa| \geq \gamma \cdot |h''| \to |fa| \geq |fh''| \to |fa| = |h'f'| \to |h'e|.$$

$\square$

# E Proof of soundness

*Proof of Theorem 5.0.2.* We interpret contexts to objects in $\mathscr{C}$, types to projections. To compute the context of a type one needs to read the codomain of the associated projection. Terms with confidence $\alpha$ are $\alpha$-sections.

The terminal object in $\mathscr{C}$ provides the empty context. Given a type $A$ in context $\Gamma$, hence a projection

$$p_A : \bullet \to \Gamma$$

we define the extended context of $\Gamma$ with $A$ as $\mathsf{dom}(p_A)$. (In fact we have been writing this as "$\Gamma.A$" this entire time.)

We only prove the variable rule in the case that $\Delta = y : B$ a single type, as it will be clear that the general case can be proved in an entirely similar way. Since $\Gamma, x : A, y : B$ is a context we have that:

- $\Gamma$ is a context;
- $A$ is a type in context $\Gamma$, hence there is a projection $p_A : \Gamma.A \to \Gamma$;
- $B$ is a type in context $\Gamma, x : A$, hence there is a projection $p_B : (\Gamma.A).B \to \Gamma.A$.

Therefore we can consider the following weighted pullback.

$$\begin{tikzcd}
(\Gamma.A).B \arrow[dr, dashed, "x"] \arrow[drr, bend left, "p_B"] \arrow[ddr, bend right, "\mathsf{id}"'] & & \\
& ((\Gamma.A).B).A[p_A \circ p_B] \arrow[r] \arrow[d] & \Gamma.A \arrow[d, "p_A"] \\
& (\Gamma.A).B \arrow[r, "p_A \circ p_B"'] & \Gamma
\end{tikzcd}$$

By hypothesis Item 1 on display-map fuzzy categories, there is a functorial choice of a type $A[p_A \circ p_B]$ making the square a weighted pullback, with all weights 1. Moreover, there is a unique $x'$ and its value is $(1 \Rightarrow 1) \wedge (1 \Rightarrow 1) = 1$. This concludes our proof, see Remark E.0.1 for a discussion on why is that so.

We begin with Substitution: unwinding it for types and terms relies heavily on (Item 1) in Definition 4.1.1. The type $B$ in context $\Gamma, A, \Delta$ amounts to a display map $p_B : \Gamma.A.\Delta.B \to \Gamma.A.\Delta$ built up iteratively, while $a$ is a section of $p_A$. We compute the iterated pullback below, and obtain first $p_{\Delta[a]}$, then $p_{B[a]}$. One can see that it all type-checks. As for terms we additionally have a $b$ section of the aftermentioned $p_B$. The universal property of weighted pullbacks guarantees that we have a section of

$p_B[a]$. We call this $b[a]$.

$$\begin{tikzcd}
\Gamma.\Delta[a] \arrow[r, dashed, "b[a]"] \arrow[rr, bend left, "\zeta\geq\gamma_2\cdot\beta"] \arrow[dr, "|\mathsf{id}|=1"'] & \Gamma.\Delta[a].B[a] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.\Delta.B \arrow[d] \\
& \Gamma.\Delta[a] \arrow[r, "\gamma_2\geq\gamma_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.\Delta \arrow[d] \arrow[u, bend right, "|b|\geq\beta"'] \\
& \Gamma \arrow[r, "\gamma_1\geq\alpha", "a"'] & \Gamma.A \arrow[d] \\
& & \Gamma \arrow[u, bend right, "|a|\geq\alpha"']
\end{tikzcd}$$

We can easily estimate a lower bound for the confidence of $b[a]$:

$$|\,b[a]\,| = \min\{1, \zeta/\gamma_2, 1/1\} = \zeta/\gamma_2 \geq \gamma_2 \cdot \beta/\gamma_2 = \beta,$$

which concludes our proof.

Weakening works in a similar way, but in some sense in the opposite direction. We leave the diagram below to be interpreted by the reader.

$$\begin{tikzcd}
\Gamma.A.\Delta \arrow[r, dashed, "b"] \arrow[rr, bend left, "\zeta\geq 1\cdot\beta"] \arrow[dr, "|\mathsf{id}|=1"'] & \Gamma.A.\Delta.B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.\Delta.B \arrow[d] \\
& \Gamma.A.\Delta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.\Delta \arrow[d] \arrow[u, bend right, "|b|=\beta"'] \\
& \Gamma.A \arrow[r] & \Gamma
\end{tikzcd}$$

Recall the discussion in Remark E.0.1 for how we might denote types and their correspondent in an extended context. □

**Remark E.0.1** (Is $A[p_A \circ p_B]$ equal to $A$?)**.** [6], which we follow for classical rules of type theory, and many others write the variable rule as in Theorem 5.0.2. The main problem with doing so arises in our proof above and is somewhat philosophical: the same type *cannot* have two different contexts, as it seems to be the case in (Var). In fact, it seems that $A$ is supposed to be both in context $\Gamma$ *and* in context $\Gamma, x : A, y : B$. Same goes for $x$. What actually happens is that we substitute $A$ with its correspondent $A[p_A \circ p_B]$ in the extended context, without adding any essentially new information *but* the new context, and, similarly, if we can pick out a variable $x$ of type $A$, there is a way for us to pick out a term in this new type $A[p_A \circ p_B]$, that is $x'$, which is that adding precisely no information. This is also the cause of the confusion of $x$ having no confidence itself and suddenly having it be 1.

Mind that this is in no part due to our working with fuzzy terms, and is in fact a feature of much of the literature on types and their categorical models, we just point it out because we need to be particularly careful when dealing with this extra level of complexity. Still, so that we are not *too* pedantic, we try to stick to the classical notation as much as possible, and write $A$ for $A$[composition of $p$'s].