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The Yoneda Embedding Expresses Whether, What, How, Why

Eugenia Cheng to Emily Riehl about the Yoneda Lemma: "...I love what you said about how you don't ever really understand it..." [1]

Eugenia Cheng about category theory: "... deep down it's about what is really making something work ..." [1]

I applaud the intention of deep understanding, and yet, as an outsider, I would say that category theory doesn't understand whether, what, how, why category theory works or fails. At its best, I would think of it as Metaphor theory, where the object of study is metaphors, which is to say, adjunctions, which synchronize windows upon Worlds, which is what I would call categories, so as to emphasize the worlds of relationships and not their individual objects. I would understand the Yoneda Lemma as its Fundamental Theorem in a holistic way that fits together our hard-won intuitions like the parts of an elephant.

I claim this elephant has four parts, which I refer to as Whether, What, How, Why.

- Whether. "The Yoneda-y principle" "the one thing we know we have in any world of homsets is the identity, and all the Yoneda functors and natural transformations are acting by composition on one side or the other." [2]
- What. "A functor is a picture of one category in another category." [3]
- How. "A good way to think of a functor is as a kind of construction." [3]
- Why. "The Yoneda perspective." "Mathematical objects are completely determined by their relationships to other objects." [4]

Specifically, let us focus on the Yoneda embedding $\mathrm{Hom}(\mathrm{Hom}(A,\_),\mathrm{Hom}(B,\_)) \cong \mathrm{Hom}(B,A)$ and the bijection $\theta^{B\overset{u}{\to}A} \Leftrightarrow B \overset{u}{\to} A$ that it defines.

$$\begin{array}{cc}
\text{Whether is } B \overset{u}{\to} A & \theta_A^{B\overset{u}{\to}A}(A \overset{\mathrm{id}_A}{\to} A) = B \overset{u}{\to} A \overset{\mathrm{id}_A}{\to} A \\
\text{What is } B \overset{u}{\to} A & B \overset{u}{\to} A \in \mathrm{Hom}(B,A) \\
\text{How is } B \overset{u}{\to} A & \theta^{B\overset{u}{\to}A} \in \mathrm{Hom}(\mathrm{Hom}(A,_),\mathrm{Hom}(B,_)) \\
\text{Why is } B \overset{u}{\to} A & B \overset{u}{\to} A \overset{f}{\to} X = \theta_X^{B\overset{u}{\to}A}(A \overset{f}{\to} X)
\end{array}$$

What is $B \overset{u}{\to} A$? It is an arrow from B to A, an element in the output of the functor $\mathrm{Hom}(B,\_)$ applied to A. How is $B \overset{u}{\to} A$? It is the action $\theta^{B\overset{u}{\to}A}$ which prepends $B \overset{u}{\to} A$ as a puzzle piece that was missing. What and How refer to possibilities within sets in the familiar world of Sets.

Whether is $B \overset{u}{\to} A$? This refers to the way that $\theta^{B\overset{u}{\to}A}$ defines $B \overset{u}{\to} A$, which is by acting on the do-nothing action $A \overset{\mathrm{id}_A}{\to} A$ and thus prepending $B \overset{u}{\to} A$ onto nothing. Why is $B \overset{u}{\to} A$? This refers to the way that $B \overset{u}{\to} A$ grounds $\theta^{B\overset{u}{\to}A}$, whereby extending $B \overset{u}{\to} A$ variously with $A \overset{f}{\to} X$ defines $\theta_X^{B\overset{u}{\to}A}(A \overset{f}{\to} X)$, the unifying, comprehensive view upon all of the relationships.

We see that we have a system of four parts that seems complete. How is the natural transformation on the left hand side. What is the arrow on the right hand side. Why takes us from the right hand side to the left hand side. Whether takes us from the left hand side to the right hand side. Is there room for a fifth intuition? Probably not anything elementary. Thus I claim this framework lets us fully understand the Yoneda lemma, tentatively yet pragmatically, comprehensively and absolutely, with more to learn from here.

Given a cup, children distinguish What it appears to be to their senses, and How it is used, affected and created. They can contemplate Whether the cup is still there when nobody sees it. Knowing Why there is a cup means knowing how it relates to absolutely everything. The Yoneda Lemma begs us to consider how it organizes knowledge in terms of actions that do nothing or do something or do anything or do everything.

Our human experience informs us that our minds move easily from How to What, but not the other way around, which is why architects say, "Form follows function". The missing piece gets filled and never goes missing again. Subsequently, our minds move easily from Why to Whether, but not the other way around.[5] We can think of the Yoneda Lemma as relating two minds, two hemispheres, two genders, the Unconscious mind that knows the answer What, and the Conscious mind that does not know and thus asks questions How. A third mind, Consciousness, waits for them to balance, considering Why. It keeps us Conscious until they are balanced and then lets us be Unconscious, at which time we lose Consciousness and what was softwired, Why, becomes hardwired, Whether. [6] In modeling this experience, we can interpret the mathematics of the Yoneda Embedding, but human experience constrains us more than the mathematics suggests of itself.

History shows that category theory may have the ambition for such holistic insights, yet lacks the requisite language of frameworks for absolute and comprehensive understanding. Metaphysics is this language of cognitive frameworks, of wisdom, of absolute truth, which comes from ever testing the limits of our imagination. Philosophers, in their private languages, speak of this same framework, which expresses the four vantage points requisite for knowledge. Plato's Republic speaks of wisdom (why), true opinion (how), false opinion (what), ignorance (whether). Aristotle distinguishes four explanations (four causes): material (why), efficient (how), formal (what), final (whether). Peirce's three kinds of signs appeal to these levels of knowledge: symbols (why), indexes (how), icons (what), and we can add the thing (whether) they signify. Few now believe, as they believed, that there could be a unifying language of frameworks. [7] Believers include Jesus, whose Great Commandment is structured like the two sides of the Yoneda embedding. Love God, the whole at once. Alternatively, love your neighbor as yourself, extending outwards, arrow by arrow. [8]

Mathematics can serve as a field for cultivating, clarifying and confirming a language of wisdom, which is not axiomatic, not explicit, not logical, but prior to all that, anthropological and quite possibly universal, underlying all systems, enabling them to unfold from a state of contradiction. We can study, for example, how the Yoneda Embedding expresses the heart of propositional logic: $\forall X(P(X) \leq Q(X)) \Leftrightarrow P \leq Q$. How does that relate to adjunctions with existential and universal quantifiers? [9] In computability theory, the four levels of knowledge are suggested by Yates's Index Set Theorem. [10] They can also be compared with the four mappings of a short exact sequence or chain complex, where 0 is understood as $A \stackrel{\mathrm{id}_A}{\to} A$, which is mapped to $B \stackrel{u}{\to} A \stackrel{\mathrm{id}_A}{\to} A$ and then $B \stackrel{u}{\to} A$ and then $\theta^{B \stackrel{u}{\to} A}$ and perhaps back to $A \stackrel{\mathrm{id}_A}{\to} A$. As an outsider, I admit that category theory takes care of the drudgery that doesn't interest us, but I seek a language of wisdom which teaches us to recognize the structures and relationships that are key, such as the Yoneda embedding.

References

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