# A Tour of $V$-Gph: Graphs over a Category

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We provide a toolset for handling generalized weighted graphs in the unifying language of category theory. We define the scale of a graph based on the categorification of the algebraic path problem, and identify loopless graphs of scale 0 with the underlying graphs of categories enriched over Bénabou cosmoi with idempotent coproducts. This paper includes a few worked demonstrations of applications of the toolset in interesting settings.

## Introduction

The study and application of magnitude homology has been hampered by the domain of its definition: metric spaces [5] and more generally enriched categories [6]. By contrast, persistence homology has seen widespread adoption in topological data analysis, due in part to its applicability to any real-weighted graph.[13] This paper is part of a larger project to determine the broadest family of real-weighted graphs that admit magnitude homology. Nevertheless, our initial paper aims to be self-contained, and the reader is assumed to be familiar with only the basic principles and terminology of category theory. Any mention of homology is relegated to the Future Goals section.

A general theory of weighted graphs must respect the subtle difference between a graph and a category. Both have objects and edges (morphisms) that connect them. An enriched category has a designated identity edge for each vertex and a strict notion of composition of edges; graphs, on the other hand, simply do not. The subtlety lifts to the study of $V$-**Gph** versus $V$-**Cat**, where the former is the category of weighted graphs and the latter is the category of enriched categories. If we let $V$ be Set, the category of sets and set functions, then Set-**Gph** is the well-known functor category of directed multigraphs, and all limits and colimits are computed pointwise. A general $V$ does not make $V$-**Gph** a functor category, so limits and colimits need to be constructed and verified manually.

An early formalization of the ties between $V$-**Gph** and $V$-**Cat** is found in Wolff [12], for $V$ a symmetric monoidal closed category. More recently, Plessas' dissertation [8] established a categorical recasting of the traditional types of unweighted graphs (and multigraphs). Weber [11] provides an elegant proof that $V$-**Gph** inherits completeness and cocompleteness from $V$. A technical characterization of the underlying graphs of enriched categories was carried out by Allouch [1].

In this paper, we explore $V$-**Gph** in its own right, laying out the tools and properties available for nice choices of $V$. We extend the characterization of $V$-**Gph** by advocating for a specific notion of subgraph, picking a tensor product for graphs, and describing the associated internal hom. The tensor product allows us to construct an action of $V$ on $V$-**Gph**. We define the scale of a $V$-graph in terms of stabilization of the algebraic path problem. We say a graph $G$ is loopless when, for all $x \in \mathrm{obj}(G)$, $G(x, x)$ is isomorphic to the identity of the tensor product in $V$. We show that loopless graphs of scale 0 are exactly the underlying graphs of $V$-categories with idempotent coproducts. We work through three examples in depth, chosen to represent various applications of the toolset.

Preliminary Report. Final version to appear in:
ACT 2023

# Outline

Most of the heavy-lifting is done in Section 1: we choose some conditions on $V$, define $V$-graphs and their morphisms, and run through some fundamental constructions. We also introduce a family of endofunctors $\Gamma_k : V\text{-}\mathbf{Gph} \to V\text{-}\mathbf{Gph}$ and the notion of scale. In Section 2, we restrict our attention to the case where $V$ is a bounded lattice, presenting a characterization of split monomorphisms in $V$-**Gph**. Finally, in Section 3, we focus on real graphs, time varying graphs, and social network graphs. Section 4 lists some work in progress. The fiddliest of our proofs reside in the Appendix.

# 1 Categorical Foundations

## 1.1 Graphs and Subgraphs

Let $V = (V, \otimes, 1_\otimes)$ be a Bénabou cosmos [10], that is, a complete and cocomplete closed symmetric monoidal category, with initial object $v_I$ and terminal object $v_T$. As we will see, many friendly and familiar categories form Bénabou cosmoi.

*Note.* Since $V$ is closed, the tensor product is a left adjoint and thus preserves all colimits. Furthermore, $v_I \otimes k \cong k \otimes v_I \cong v_I$ for any $k \in V$.

As for the definition of a graph in $V$, we will work in as much generality as possible: we work with complete graphs with loops, and recover the notions of loopless graphs and partial graphs by assigning $v_I$ to "missing" edges.

**Definition 1.1.** Let $V$-**Gph** denote the category of complete graphs with edges labelled in $V$. An object $G$ in $V$-**Gph** is a set $\mathrm{obj}(G)$ such that every pair $a, b \in \mathrm{obj}(G)$ is assigned an object in $V$, denoted $G(a, b)$ and often referred to as the value, weight, or label of the edge $(a, b)$. A morphism $f : G \to H$ is a set function $f^\# : \mathrm{obj}(G) \to \mathrm{obj}(H)$, and, for every pair $a, b \in \mathrm{obj}(G)$, a morphism $f_{a,b} : G(a, b) \to H(f^\#(a), f^\#(b))$ in $V$.

For notational clarity, we will usually denote $f^\#$ by $f$ and $f_{a,b}$ by $f_{ab}$.

**Example 1.2.** Consider $V = (\mathbb{R}_+, +, 0)$, the poset $([0, \infty], \geq)$ equipped with addition as a tensor product. We will call the objects of $\mathbb{R}_+$-**Gph** *real graphs*. A real graph is equivalent to the traditional notion of a weighted graph with loops. Since the only morphisms in $\mathbb{R}_+$ are $(\geq)$, the definition of a morphism in $\mathbb{R}_+$-**Gph** corresponds to a nonincreasing map of weighted graphs, i.e., maps $f : G \to H$ such that $G(a, b) \geq H(f(a), f(b))$ for all $a, b \in \mathrm{obj}(G)$.

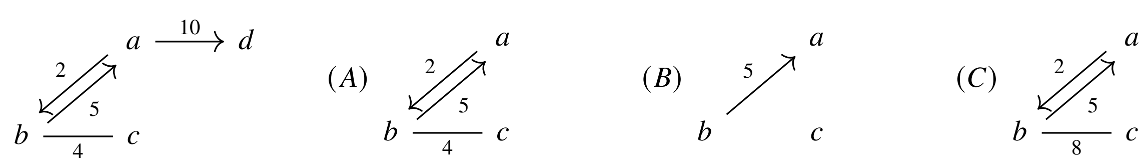
**Example 1.3.** Let $V = \mathbf{Set}$. Graphs in **Set** correspond to the unweighted multigraphs. The definition of a morphism carries only one restriction: edges with weight $\emptyset$ must be sent to an edge with weight $\emptyset$.

**Definition 1.4.** A graph $G$ is **loopless** if $G(x, x) \cong 1_\otimes$ for all $x \in \mathrm{obj}(G)$.

**Theorem 1.5.** *A monomorphism in $V$-**Gph** is exactly a morphism $f : G \to H$ such that $f^\#$ is injective and $f_{ab}$ is a monomorphism in $V$ for every $a, b \in \mathrm{obj}(G)$.*

**Definition 1.6.** Given $H \in V$-**Gph**, a **strict subgraph** of $H$ is a graph $G$ and monomorphism $\iota : G \to H$ such that $G(a, b) \cong v_I$ or $G(a, b) \cong H(\iota(a), \iota(b))$ for all $a, b \in \mathrm{obj}(G)$. We write $G \subseteq H$.

Let us take a moment to unpack the definition of a subgraph, using $\mathbb{R}_+$-**Gph** as a motivating example. Below is a graph $G$ followed by three candidates for strict subgraphs of $G$. Edges of weight $v_I$ are omitted.



We argue that (A) and (B) are the natural choices, while (C) should not be counted as a strict subgraph since $C(b,c) = 8 \neq 4 = G(b,c)$. As is standard in category theory, we begin by requiring a subobject $G$ of $H$ to be monomorphism $\iota : G \to H$. Every morphism in $\mathbb{R}_+$ is a monomorphism, so all three candidates are equally valid without further criteria. It is too strict to demand isomorphisms on edge weights. This condition is equivalent to the traditional notion of a full subgraph and rules out (B). Therefore, we ease the requirement by allowing subgraphs to have edges of weight $v_I$ regardless of the corresponding weight in the parent graph.

## 1.2 Limits and Colimits

We now move on to construct common limits and colimits in $V$-**Gph**. While $V$-graphs are not enriched categories, the category of $V$-graphs inherits most of its structure from $V$. Products and pullbacks are computed pointwise, while some subtlety is required to define coproducts and pushouts.

**Proposition 1.7.** *Given two graphs $G$ and $H$, the* ***coproduct*** *of $G$ and $H$ is the graph $G \amalg H$ with* $\mathrm{obj}(G \amalg H) = \mathrm{obj}(G) \amalg \mathrm{obj}(H)$ *and*

$$(G \amalg H)(a,b) = \begin{cases} G(a,b), & a,b \in \mathrm{obj}(G) \\ H(a,b), & a,b \in \mathrm{obj}(H) \\ v_I, & \textit{otherwise.} \end{cases}$$

*Recall that the coproduct in Set is disjoint union. For coproducts of more than two graphs, we will often use summation notation, e.g., $\sum_i X_i$.*

**Proposition 1.8.** *Given two morphisms of graphs $f_1 : G \to H_1$ and $f_2 : G \to H_2$, the* ***pushout*** *of $f_2$ along $f_1$, written $H_1 \amalg_G H_2$, is the graph with* $\mathrm{obj}(H_1 \amalg_G H_2) = \mathrm{obj}(H_1) \amalg_G \mathrm{obj}(H_2)$ *and*

$$(H_1 \amalg_G H_2)([x],[y]) = \sum_{a \in [x], b \in [y]} (H_1 \amalg H_2)(a,b).$$

*Recall that the pushout in Set is given by*

$$\mathrm{obj}(H_1) \amalg_G \mathrm{obj}(H_2) = \mathrm{obj}(H_1) \amalg \mathrm{obj}(H_2)/(\sim)$$

*where $f_1(x) \sim f_2(x)$ for all $x \in$* $\mathrm{obj}(G)$.

It will also be useful to define a tensor product for graphs.

**Definition 1.9.** Given two graphs $G$ and $H$, the **tensor product** of $G$ and $H$ is the graph $G \otimes H$ with $\mathrm{obj}(G \otimes H) = \mathrm{obj}(G) \times \mathrm{obj}(H)$ and $(G \otimes H)((a,b),(a',b')) = G(a,a') \otimes H(b,b')$.

**Theorem 1.10.** *$V$-**Gph** admits an* ***internal hom*** *functor which sends every pair of graphs $(G,H)$ to a graph $[G,H]$ with objects $V$-**Gph** morphisms and*

$$[G,H](f,g) = \prod_{a,b \in \mathrm{obj}(G)} [G(a,b), H(f(a), g(b))],$$

*where $[-,=]$ inside the product denotes the internal hom of $V$.*

Equipped with a tensor product and an internal hom, we can now construct two actions of $V$ on $V$-**Gph**. Take an element $v \in V$ and a $V$-**Gph** $G$. Let $|v|$ be the singleton graph $\{\star\}$ with $|v|(\star,\star) = v$. Define $v \odot G := G \otimes |v|$ and $v \pitchfork G := [|v|, G]$.

**Theorem 1.11.** *$|\cdot| : V \to V$-**Gph** is a functor, and $(v \odot -) \dashv (v \pitchfork -)$.*

*Proof.* Functoriality follows trivially from the definition of $|v|$. The adjunction is given by

$$\mathrm{Hom}(v \odot G, H) = \mathrm{Hom}(G \otimes |v|, H) \cong \mathrm{Hom}(G, [|v|, H]) = \mathrm{Hom}(G, v \pitchfork H).$$

□

**Example 1.12.** The internal hom in $\mathbb{R}_+$ is given by truncated subtraction: $[m, n] = \max\{n - m, 0\}$ for $m, n \in \mathbb{R}_+$. Given $r \in \mathbb{R}_+$ and a real graph $G$, the operation $r \odot G$ increases the weight of each edge in $G$ by $r$, while $r \pitchfork G$ decreases the weight of each edge by $r$ (w.r.t. truncated subtraction).

The initial object in $V$-**Gph** is given by $\emptyset$, and the terminal object is given by $|v_T|$. In certain contexts, it may be beneficial to use pointed graphs, i.e., graphs with designated vertices; the initial object for pointed graphs is $|v_I|$, and the category of pointed graphs has a zero object if and only if $V$ has a zero object.

**Definition 1.13.** Given a graph morphism $f : G \to H$, define the **kernel graph** of $f$, $\ker(f)$, to be the graph given by the pullback of $f$ along itself, equipped with a morphism $\ker(f) \hookrightarrow G$. Define the **quotient graph** of $f$, $f/\ker(f)$, to be the pushout of $\ker(f) \to G$ along itself, equipped with a morphism $G \to G/\ker(f)$.

**Definition 1.14.** Given a graph morphism $f : G \to H$, define the **image graph** of $f$, $\mathrm{im}(f)$, to be the graph with objects $f(\mathrm{obj}(G))$ and weights $\mathrm{im}(f)(f(x), f(y)) := H(f(x), f(y))$, equipped with a morphism $G \twoheadrightarrow \mathrm{im}(f)$. Occasionally, we will write $f(G) := \mathrm{im}(f)$.

## 1.3 Scale and Enriched Categories

In this section, we develop a categorical approach to the algebraic path problem [2], and relate the approach to the close bond between graphs and enriched categories.

**Definition 1.15.** Given two graphs $G$ and $H$ with a bijection $\alpha : \mathrm{obj}(H) \to \mathrm{obj}(G)$, define the **pointwise sum** graph, $G + H$, with objects those of $H$ and

$$(G + H)(x, y) := G(\alpha(x), \alpha(y)) \amalg H(x, y).$$

**Definition 1.16.** Given two graphs $G$ and $H$ with a bijection $\alpha : \mathrm{obj}(H) \to \mathrm{obj}(G)$, define the **composite** graph, $G \circ H$, with objects those of $H$ and

$$(G \circ H)(x, y) := \sum_{z \in \mathrm{obj}(H)} G(\alpha(x), \alpha(z)) \otimes H(z, y).$$

The unit for composition of graphs on $n$ objects is $I_n := \sum_{i=1}^{n} |1_\otimes|$.

**Theorem 1.17.** *Up to isomorphism, composition of graphs is associative and composition distributes over pointwise addition.*

*Proof.* See Appendix. □

*Observation.* For finite graphs, graph composition is equivalent to matrix multiplication of the adjacency matrices.

**Definition 1.18.** For $k \in \mathbb{N}$, the **optimal k-step graph** of a graph $G$, $\Gamma_{k-1}G$, is given by

$$\Gamma_{k-1}G := \sum_{i=1}^{k} G^i,$$

where $G^k := \underbrace{G \circ \cdots \circ G}_{k}$. Explicitly, $\Gamma_{k-1}G$ has the same object-set as $G$ and, for every $x, y \in \mathrm{obj}(G)$,

$$\Gamma_{k-1}G \cong G(x,y) \amalg \sum_{n<k, \{x_i\}_{i=1}^{n} \subseteq \mathrm{obj}(G)} G(x,x_1) \otimes G(x_1,x_2) \otimes \cdots \otimes G(x_n,y).$$

*Observation.* For every graph $G$ and $v \in V$, $\Gamma_k(v \odot G) \cong \Gamma_k(|v|) \otimes \Gamma_k G$.
If the coproduct in $V$ is idempotent, that is, $x \amalg x \cong x$ for all $x \in V$, then we have the computationally cheaper formula

$$\Gamma_{k-1}G \cong (I+G)^k,$$

for $k > 1$.
**Theorem 1.19.** *For any graph G, there is a sequence in V-**Gph***

$$\Gamma_0 G \to \Gamma_1 G \to \Gamma_2 G \to \cdots .$$

*The sequential colimit $\Gamma G := \mathrm{colim}_k \Gamma_k G$ always exists, and we call $\Gamma G$ the **optimal step completion** of $G$. Furthermore, each $\Gamma_k$ is an endofunctor over $V$-**Gph**, as is $\Gamma$.*
**Definition 1.20.** We say $G$ has scale $n$ if there exists a least integer $n$ for which the sequence $\{\Gamma_k G\}$ stabilizes up to isomorphism, i.e., $n = \min\{k \in \mathbb{N} \mid \Gamma_k G \cong \Gamma_{k+1} G\}$. If the sequence never stabilizes, we say $\mathrm{scale}(G) = \infty$.
*Note.* If $G$ has finite scale $n$, then $\Gamma G \cong \Gamma_n G$.
**Proposition 1.21.** *Consider the adjunction $F \dashv U$, where $F : V$-**Gph** $\to V$-**Cat** is the free enriched category functor and $U : V$-**Cat** $\to V$-**Gph** is the underlying graph functor. Then $\Gamma = UF$.*
See Wolff [12] for a rigourous characterization of the free-underlying adjunction.[1]
We can use scale to break up $V$-**Gph** into a sequence of full subcategories

$$V\text{-}\mathbf{Gph}_0 \subset V\text{-}\mathbf{Gph}_1 \subset \cdots \subset V\text{-}\mathbf{Gph}_\infty = V\text{-}\mathbf{Gph}$$

where $V$-$\mathbf{Gph}_k$ is the category of graphs $G$ such that $\mathrm{scale}(G) \leq k$, for finite $k$. For $k \leq n < \infty$, $\Gamma_k$ can be viewed as a functor from $V$-$\mathbf{Gph}_n$ to $V$-$\mathbf{Gph}_{n-k}$. Likewise, for $n \leq \infty$, $\Gamma$ can be viewed as a functor from $V$-$\mathbf{Gph}_n$ to $V$-$\mathbf{Gph}_0$.
**Definition 1.22.** For $k = 0, 1, \ldots$, let $V$-$\mathbf{Gph}_k^\times$ denote the subcategory of loopless graphs of scale $k$.
**Theorem 1.23.** *For a Bénabou cosmos with idempotent coproduct, the categories $V$-**Cat** and $V$-$\mathbf{Gph}_0^\times$ are equivalent.*

*Proof.* The underlying graph of a $V$-category is a $V$-graph of scale 0; this follows from the definition of scale. $U$ forgets the The counit of the adjunction $\varepsilon : FU \Rightarrow 1$ is a natural isomorphism, so $V$-**Cat** is a reflective subcategory of $V$-**Gph**. Recall $U$ sends every $V$-category to a $V$-graph of scale 0, by forgetting unit and composition maps. Furthermore, for all $x \neq y$ in a graph $G$, $FG(x,x) \cong \Gamma(1_\otimes \amalg G(x,x))$ and $FG(x,y) \cong \Gamma G(x,y)$. the restriction of $F$ to graphs of scale 0 makes no changes to edge weights, only adding in unit and composition maps. Since every graph in $V$-$\mathbf{Gph}_0^\times$ is loopless, $F$ has a canonical choice for the unit maps. Thus, the restriction of $F$ to graphs of scale 0 turns the unit of the adjunction $\eta : 1 \Rightarrow UF$ into a natural isomorphism. Both $\eta$ and $\varepsilon$ are natural isomorphims under this restriction, so $V$-**Cat** is equivalent to the subcategory of $V$-**Gph** of graphs with scale 0. □

[1]The author is grateful to Emily Roff for highlighting this canonical adjunction between $V$-**Gph** and $V$-**Cat**.

## 2 Graphs Over Bounded Lattices

A poset $(P, \leq)$ is a category with objects $P$ and, for every pair $x, y \in P$, at most one morphism $\leq_{x,y} \in \mathrm{Hom}(x, y)$, which we will write $\leq$ with implicit reliance on context. A bounded lattice is a poset $(\mathscr{L}, \leq)$ such that every finite subset $L \subseteq \mathscr{L}$ has a product, written $\bigwedge L$ and pronounced "the join of $L$," and a coproduct, written $\bigvee L$ and pronounced "the meet of $L$." Categorically speaking, a bounded lattice is a finitely complete and cocomplete poset with an initial object and a terminal object. Let $\mathscr{L}$ be a bounded lattice $(\mathscr{L}, \leq, 0, \infty)$ with a symmetric tensor product functor $\otimes : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$. The coherence axioms for the tensor product force it to be order-preserving.

We would like $\mathscr{L}$ to be a closed monoidal category, so let us take a moment to discuss a robust technique for finding the internal hom. We want a right adjoint to the tensor product that takes values in $\mathscr{L}$. We could certainly try a guess-and-check method, but, as we will see in the next section, the internal hom is not always as obvious as it is for $\mathbb{R}_+$. It would be preferable to generate the right adjoint instead. To this end, we note that the right adjoint to a functor $F : C \to D$, if it exists, is generated by the left Kan extension of the identity functor along $F$, that is, $U \cong \mathrm{Lan}_F\, \mathrm{id}_C$. Kan extensions may be computed rather directly using coend calculus:

$$(\mathrm{Lan}_F\, G)(d) \cong \int^{c \in C} \mathrm{Hom}(Fc, d) \times Gc.$$

The analogy between coends and traditional integration is quite powerful in the context of posets. In this setting, we can faithfully interpret the formula above to mean "take the potentially infinite coproduct of $G$ applied to every $c$ such that $Fc \leq d$." To learn everything there is to know about our (co)friends, see [7]. Section 2.2 contains a illuminating discussion of the ties between coends and traditional integration. Also, see [9] for an indepth discussion on discovering adjoints and computing Kan extensions via slice categories. Returning to the tensor product, we now have the formula

$$[x, y] \cong (\mathrm{Lan}_{-\otimes x}\, \mathrm{id}_{\mathscr{L}})(y) \cong \int^{a \in \mathscr{L}} \mathrm{Hom}(a \otimes x, y) \times a \cong \bigvee_{a \otimes x \leq y} a$$

whenever the internal hom exists. As an aside, there is also a formula for a potential left adjoint to the tensor product via a right Kan extension; the reader is encouraged to work it out and to check its existence or nonexistence for the examples provided in the next section.

**Example 2.1.** Consider the poset of extended positive integers with partial order ($\vdots$) defined by $a \vdots b$ whenever $b|a$, i.e., $b$ divides $a$. We will denote this poset $\mathbb{Z}_+$. Note that every pair $x, y \in \mathbb{Z}_+$ has a product, $x \wedge y = \mathrm{lcm}(x, y)$, and a coproduct, $x \vee y = \gcd(x, y)$, so $\mathbb{Z}_+$ forms a lattice. Letting $\infty \vdots x$ for all $x \in \mathbb{Z}_+$, the lattice is bounded with initial object $\infty$ and terminal object 1. Furthermore, integer multiplication forms a symmetric tensor product on $\mathbb{Z}_+$. Applying our hom-hunting method, we have

$$[x, y] = \bigvee_{x \otimes a \leq y} a = \gcd\{a : ax \vdots y\} = \frac{y}{\gcd(x, y)} = \frac{\mathrm{lcm}(x, y)}{x}.$$

Here is an example of using the internal hom to prove results obout $\mathscr{L}$.

**Proposition 2.2.** *Suppose $\mathscr{L}$ is semicartesian. For all $x, y \in \mathscr{L}$, $x \otimes y \leq x$ and $x \otimes y \leq y$. By the universal property of the coproduct, $x \otimes y \leq x \vee y$.*

*Proof.* Consider $x, y \in \mathscr{L}$.

$$\begin{aligned}\mathrm{Hom}(x \otimes y, x \vee y) &\cong \mathrm{Hom}(x \otimes y, x) \times \mathrm{Hom}(x \otimes y, y)\\ &\cong \mathrm{Hom}(y \otimes x, x) \times \mathrm{Hom}(x \otimes y, y)\\ &\cong \mathrm{Hom}(y, [x, x]) \times \mathrm{Hom}(x, [y, y])\\ &\cong \mathrm{Hom}(y, 1_\otimes) \times \mathrm{Hom}(x, 1_\otimes)\end{aligned}$$

Since $1_\otimes = v_T$ is the terminal object in $\mathscr{L}$, the last line is satisfied by $y \leq 1_\otimes$ and $x \leq 1_\otimes$. The last isomorphism comes from the fact that $1_\otimes = v_T$ is an object in the set $\{a : a \otimes x \leq x\}$ and $v_T \vee a \cong v_T$ for any $a \in \mathscr{L}$, so $[x, x] \cong \bigvee\{a : a \otimes x \leq x\} \cong 1_\otimes$. Thus, $x \otimes y \leq x \vee y$ by the chain of natural isomorphisms. □

Once we have constructed a bounded symmetric monoidal lattice $\mathscr{L}$, it is natural to seek bounds for the scale of a $\mathscr{L}$-graph. The following theorem is a result of Baras [2] that sets a worst-case bound for a particular family of graphs.

Call a map $\gamma : \{0, \ldots, k\} \to G$ a **path in $G$ in $k$ steps**, and define the length of $\gamma$ by

$$L(\gamma) := \bigotimes_{i=0}^{k-1} G(\gamma(i), \gamma(i+1)).$$

If $\gamma(0) = \gamma(k) = x \in G$, then $\gamma$ is referred to as a **circuit in $G$ based at $x$**.

**Theorem 2.3.** *Let $G$ be an $\mathscr{L}$-graph on $n$ objects such that $L(\gamma) \leq 1_\otimes$ for every circuit in $G$. Then* $\mathrm{scale}(G) < n$.

*Proof.* By Theorem 1.19, we know $\Gamma_{n-1}G(x, y) \leq \Gamma_n G(x, y)$ for all $x, y$, so it is sufficient to verify $\Gamma_n G(x, y) \leq \Gamma_{n-1}(x, y)$ for all $x, y$, given the hypothesis of the theorem. Let $\gamma$ be a path in $n$ steps. Note that $\gamma$ is indexed over $n + 1$ objects while $G$ has only $n$ objects to traverse, so $\gamma$ cannot be injective. Then $\gamma$ contains a circuit $\rho$ based at some object. Removing the circuit produces a path $\widetilde{\gamma}$ in $k$ steps for some $k < n$. By the hypothesis, the length of the removed circuit is less than or equal to the identity of the tensor product. By construction,

$$\Gamma_n G(x, y) \leq L(\gamma) = L(\widetilde{\gamma}) \otimes L(\rho) \leq L(\widetilde{\gamma}) \otimes 1_\otimes = L(\widetilde{\gamma}) \leq \Gamma_k G(x, y) \leq \Gamma_{n-1} G(x, y).$$

Since $x$, $y$, and $\gamma$ were chosen arbitrarily, $\Gamma_n G(x, y) \leq \Gamma_{n-1} G(x, y)$ for all $x, y$. Thus, $\Gamma_{n-1} G = \Gamma_n G$ and $\mathrm{scale}(G) \leq n - 1 < n$. □

The hypothesis for Theorem 2.3 selects graphs in which optimal paths cannot "stand still." Checking this condition is computationally expensive in practice, but we are fortunate to be interested in lattices in which the condition comes for free.

**Corollary 2.4.** *If $\mathscr{L}$ is semicartesian, i.e., $1_\otimes = v_T$, then* $\mathrm{scale}(G) < |\,\mathrm{obj}(G)|$ *for any finite graph $G$.*

**Example 2.5.** The above corollary to Theorem 2.3 holds in our favourite setting, $\mathbb{R}_+$, since $1_\otimes = v_T = 0$.

**Example 2.6.** $\mathbb{Z}_+$ is semicartesian, $1_\otimes = 1 = v_T$, so all $\mathbb{Z}_+$-graphs on $n$ objects have scale less than $n$.

A monomorphism (mono) is a morphism $\iota : A \to B$ that satisfies the left cancellation property: for every pair of morphisms $g, h : X \to A$, $\iota \circ g = \iota \circ h$ implies $g = h$. Dually, an epimorphism (epi) satisfies the right cancellation property. Monos and epis are often introduced as the generalization of the notions of injective and surjective functions, respectively. Indeed, the definitions coincide in the category of sets and set functions. However, monos and epis are much broader classifications in general.

A split monomorphism is a monomorphism such that there exists a morphism $s : B \longrightarrow A$ such that $s \circ \iota = \mathrm{id}$. Implicitly, the left inverse $s$ must be a split epimorphism. Split monos act like embeddings of an object $A$ into an object $B$, that is, maps that preserve the structure of $A$ which may be retrieved by applying the split epi. Instead of working with the usual definition of a split mono, we will opt to use a characterization particular to graphs over bounded lattices.

**Theorem 2.7.** *A split monomorphism $i : X \hookrightarrow G$ is equivalent to a decomposition of* $\mathrm{obj}(G)$ *into a disjoint union of subsets $S_j$, and, for each $j$, a designated point $a_j \in S_j$ such that $G(a_j, a_k) = \bigvee_{x \in S_j, y \in S_k} G(x, y)$ for all $j, k$.*

This characterization allows us to begin with a parent graph $G$ and view split monos into $G$ as special subgraphs.

# 3 Examples and Applications

### Cheatsheet for Bénabou Cosmoi

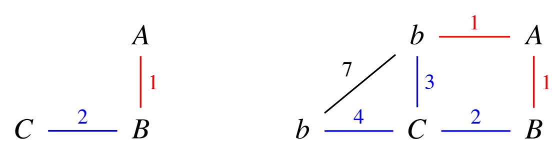
| | initial object | terminal object | product | coproduct | tensor product | tensor unit |
|---|---|---|---|---|---|---|
| $V$ | $v_I$ | $v_T$ | $\Pi$ | $\amalg$ | $\otimes$ | $1_\otimes$ |
| $(\mathscr{L}, \leq)$ | $\bigwedge \mathscr{L}$ | $\bigvee \mathscr{L}$ | $\wedge$ | $\vee$ | $\otimes$ | $1_\otimes$ |
| $([0, \infty], \geq)$ | $\infty$ | $0$ | max | min | $(+)$ | $0$ |
| $(\mathbb{Z}_+, \vdots)$ | $\infty$ | $1$ | lcm | gcd | $(\cdot)$ | $1$ |
| $(E, \ll)$ | $0$ | $1$ | $\min^{\ll}$ | $\max^{\ll}$ | $(\cdot)$ | $1$ |

## 3.1 Real Graphs

Recall that $\mathbb{R}_+$ forms a Bénabou cosmos with addition as the tensor product, and internal hom given by truncated subtraction. Since the coproduct is idempotent, real graphs of scale 0 correspond to the underlying graphs of Lawvere metric spaces, a categorical notion of metric space initially described by Lawvere in 1973. [4]

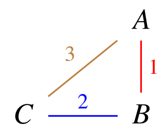
*Observation.* Let $G$ and $H$ be loopless graphs of scale 0. Then $[G, H]$ is the graph of nonincreasing functions from $G$ to $H$, and the weight between functions $f$ and $g$ corresponds to the infinity norm $\|f - g\|_\infty := \max_{a \in \mathrm{obj}(G)} H(f(a), g(a))$.

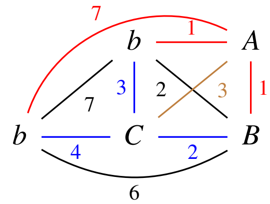
We now illustrate the characterization of split monomorphisms from the previous section, as well as the effect of $\Gamma$ on real graphs. There is a split monomorphism embedding the left graph $X$ into the right graph $G$.




All self-loops have weight 0 and edges of weight 0 or $\infty$ are omitted. The vertices of $G$ have been partitioned into classes $[A] = [a]$, $[B] = [b]$, and $[C] = [c]$, and labelled accordingly, with uppercase marking the designated vertex in each.

Since all functors preserve split monos, the functor $\Gamma_k$ does also, for every $k$. This particular graph $G$ has scale 2.




## 3.2 Time Varying Graphs

Consider the powerset of a set $\Omega$, denoted $\mathscr{P}(\Omega)$, which forms a bounded lattice under set inclusion ($\subseteq$), with initial object $\emptyset$ and terminal object $\Omega$ itself. Products are given by intersection, and coproducts are union. $\mathscr{P}(\Omega)$ forms a cartesian monoidal category, with intersection also playing the role of a symmetric tensor product. The internal hom of $A$ and $B$ is a subset $[A, B] \in \mathscr{P}(\Omega)$ determined by

$$[A, B] = \bigvee_{A \otimes X \leq B} X = \bigcup_{A \cap X \subseteq B} X = A^c \cup B,$$

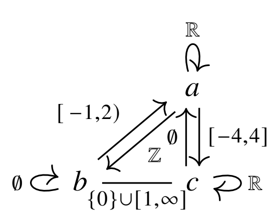
where $A^c$ is the complement of $A$ in $\Omega$, that is, $A^c := \{x : x \in \Omega, x \notin A\}$. We can double check this using the tensor-hom adjunction, which we translate to the notation specific to $\mathscr{P}(\Omega)$, then expand using the set-theoretic definition of set inclusion:

$$\begin{aligned} \mathrm{Hom}(X \otimes A, B) &\cong \mathrm{Hom}(X, [A, B]) \\ X \cap A \subseteq B &\iff X \subseteq A^c \cup B \\ \forall y \in X \cap A, x \in B &\iff \forall x \in X, x \in A^c \cup B \end{aligned}$$

Interestingly, $[A, B] = \Omega$ exactly when $A \subseteq B$.

$$\mathrm{Hom}([-2, 1) \cup (1, 3]\,, (-1, 2)) \cong (-\infty, -2) \cup \{1\} \cup (3, \infty) \cup (-1, 2) = (-\infty, -2) \cup (-1, 2) \cup (3, \infty)$$

Taking $\Omega = \mathbb{R}$, graphs over $\mathscr{P}(\mathbb{R})$ can be thought of as unweighted graphs in a timeseries: for $x$ and $y$ in a powerset graph $G$, the set $G(x, y)$ describes the times when the edge from $x$ to $y$ is available to be travelled along. Graphs over $\mathscr{P}(\mathbb{R})$ are often referred to as **time varying graphs** or TVGs.
Under this setup, $\Gamma G(x, y)$ is the largest time period for which there exists a path in $G$ from $x$ to $y$.
Consider the following graph over $\mathscr{P}(\mathbb{R})$:




We will compute the optimal step completion of $G$, $\Gamma G$. Before we begin, note that $\mathscr{P}(\mathbb{R})$ is cartesian, so $\mathscr{P}(\mathbb{R})$ is also semicartesian. Then Corollary 2.4 guarantees $\mathrm{scale}(G) < |\,\mathrm{obj}(G)| = 3$, so we have at most to compute $G + G \circ G$ and $G + G \circ G \circ G$. First we represent the graph by a matrix to facilitate computations.

$$\Gamma_0 G = G = \begin{pmatrix} \mathbb{R} & \mathbb{Z} & [-4,4] \\ [-1,2) & \emptyset & \{0\} \cup [1,\infty] \\ \emptyset & \{0\} \cup [1,\infty] & \mathbb{R} \end{pmatrix}$$

Next, we compute $\Gamma_1 G = G + G \circ G$.

$$G + G \circ G = \begin{pmatrix} \mathbb{R} & \mathbb{Z} \cup [1,4] & [-4,4] \cup \{5,6,\ldots\} \\ [-1,2) & \{-1,0\} \cup [1,\infty] & [-1,\infty] \\ \{0\} \cup [1,2) & \{0\} \cup [1,\infty] & \mathbb{R} \end{pmatrix}$$

Finally, we compute $\Gamma_2 G = G + G \circ G \circ G$.

$$G + G \circ G \circ G = \begin{pmatrix} \mathbb{R} & \mathbb{Z} \cup [1,4] & [-4,4] \cup \{5,6,\ldots\} \\ [-1,2) & \{-1,0\} \cup [1,\infty] & [-1,\infty] \\ \{0\} \cup [1,2) & \{0\} \cup [1,\infty] & \mathbb{R} \end{pmatrix}$$

We note $\Gamma_1 G = \Gamma_2 G$, so $G$ has scale 1. The optimal step completion of $G$ is presented below.

$$\begin{tikzcd}
 & a \arrow[loop above, "\mathbb{R}"] \arrow[dl, bend right, "{\mathbb{Z}\cup[1,4]}"] \arrow[dr, bend left, "{[-4,4]\cup\{5,6,\ldots\}}"] & \\
b \arrow[loop left, "{\{-1,0\}\cup[1,\infty]}"] \arrow[ur, bend left, "{[-1,2)}"] \arrow[rr, dash, "{\{0\}\cup[1,\infty]}"'] & & c \arrow[loop right, "\mathbb{R}"] \arrow[ul, bend right, "{\{0\}\cup[1,2)}"']
\end{tikzcd}$$

A more sophisticated version of this device using rigs (i.e., rings without negation) is currently being used to algebraically describe satellite communication systems.[3]

### 3.3 Social Networks

Let us test our toolset against a slightly unusual $V$. Let $E$ be the set $[-1,1]$ equipped with the partial order

$$x \ll y \iff |x| < |y| \text{ or } |x| = |y| \text{ and } x \leq y.$$

In words, this is the partial order of dominant absolute value with nonnegatives winning ties. $E$ has 0 and 1 as initial and terminal objects, respectively. $E$ admits a symmetric tensor product given by multiplication, with unit 1. $E$ also admits an internal hom: for $x, y \in [-1,1]$, define $k = \left|\frac{y}{x}\right|$; then

$$[x,y] = \begin{cases} +1, & x = 0 \\ 0, & x \neq y \text{ and } y = 0 \\ -k, & y < 0 < x \text{ and } k \leq 1 \\ \min(1,k), & \text{otherwise.} \end{cases}$$

In an attempt to clarify the computation, we have included some visual aids in Figure 1. The product and coproduct in $E$ are given by $x \sqcap y = \min(x,y)$ and $x \sqcup y = \max(x,y)$, respectively, where min and max are computed with respect to the order $(\ll)$.

*Note.* We can build a family $\{E_n\}$ where each $E_n$ is the set $[-n, n]$ equipped with $(\ll)$, with initial object 0 and terminal object $+n$, with tensor product given by multiplication, and internal hom given by

$$[x, y] = \begin{cases} +n, & x = 0 \\ 0, & x \neq y = 0 \\ -k, & k \leq n \text{ and } y < 0 < x \\ \min(n, k), & \text{otherwise,} \end{cases}$$

for $k = \left|\frac{y}{x}\right|$. The diagonals in the corresponding hom-object visualization tighten around the $y$-axis as $n \to \infty$. In fact, we can construct the limit case $E_\infty$; the reader is encouraged to draw the correct visualization for this case.[2]

(a) $x \ll y$ (b) $[x, y]$

Figure 1: Here are two visualizations of $E$. (a) The blue region (vertically hatched) contains all points $(x, y) \in [-1, 1] \times [-1, 1]$ such that $x \ll y$. The red region (horizontally hatched) forms the complement of the blue. (b) Here the blue region (vertically hatched) denotes points for which $[x, y] = 1$, the black lines denote points where $[x, y] = 0$, the green region (horizontally hatched) denotes points where $[x, y] = +\left|\frac{y}{x}\right|$, and the red region (cross-hatched) denotes points where $[x, y] = -\left|\frac{y}{x}\right|$.

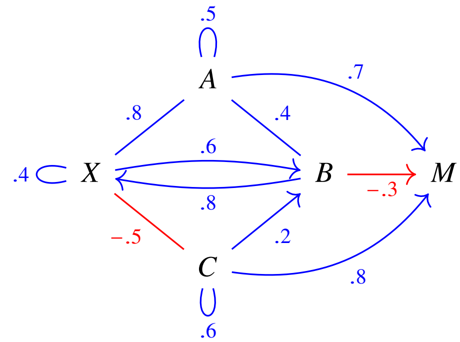
To explore the types of data E-**Gph** can encode, consider the following social scenario: Suppose X is looking for a new movie to watch and asks their good friends A and B for a recommendation. Person A highly recommends movie M while B feels pretty negative towards it. C, a thorn in X's side, overhears the conversation and jumps in to praise movie M for its use of cameras, its employment of actors, and its script of words. What is the strongest opinion X may form of M?

Let us view an $E$-graph as a set of individuals and their opinions of each other, i.e., a social network. A negative edge weight represents a degree of animosity, a positive edge weight a degree of amicability, and an edge weight of 0 neutrality. Then the optimal step completion of an $E$-graph will compute the strongest opinion[3] each individual may form of another individual by taking into account all the opinions in the social network. To answer the question posed in the scenario above, we construct a social network

[2]For $n > 1$, $E_n$ is no longer semicartesian, but retains the other nice properties.
[3]with positive emotions winning in a tie

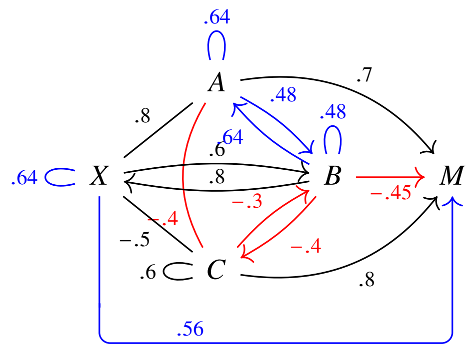
$G$:

$$G = \begin{Bmatrix} 0.4 & 0.8 & 0.6 & -0.5 & 0 \\ 0.8 & 0.5 & 0.4 & 0 & 0.7 \\ 0.8 & 0.4 & 0 & 0 & -0.3 \\ -0.5 & 0 & 0.2 & 0.6 & 0.8 \\ 0 & 0 & 0 & 0 & 0 \end{Bmatrix}$$




All neutral opinions are omitted from the diagram on the right, all symmetric edges are arrowless, and we have chosen to colour positive opinions in blue and negative opinions in red. Running our optimal step completion machine, we discover $G$ has scale

$$\Gamma G = \Gamma_2 G = \begin{Bmatrix} 0.64 & 0.8 & 0.6 & -0.5 & 0.56 \\ 0.8 & 0.64 & 0.48 & -0.4 & 0.7 \\ 0.8 & 0.64 & 0.48 & -0.4 & 0.448 \\ -0.5 & -0.4 & -0.3 & 0.6 & 0.8 \\ 0 & 0 & 0 & 0 & 0 \end{Bmatrix}$$




This time, we have only coloured edges that have changed weight. The strongest opinion X may form of M in this model is +0.56. More interestingly, there is a telling break of symmetry between A and B. Friends A and B have the same opinion of each other in $G$, and both improve their opinions of each other in $\Gamma G$. However, B's opinion of A grows by over 50% while A's opinion of B grows by only 20%. The asymmetry is explained by A and B's respective relationships with X: X slightly prefers A over B. Both A and B value X's opinions equally, so B's opinion of A is boosted by X's preference for A. Similarly, X and C's mutual animosity creates rifts between C and the others.

## 4 Future Goals

- The relationship between split monic subgraphs and magnitude homology needs proper investigating. Likewise, for loopless graphs and graphs.
- We will construct a modified $\Gamma$ functor, $\Theta$, that only changes edges that have weight $v_I$. In $\mathbb{R}_+$-**Gph**, this functor allows us to describe partial real graphs, i.e., graphs in which a subset of edge weights are fixed while the rest are determined by shortest path length. Then we define cyclic graphs as the image under $\Theta$ of a particular class of partial graphs.
- We prove that all real cyclic graphs admit magnitude homology.
- We prove that wedge products preserve the property of admitting magnitude homology, with few conditions.

# 5 Appendix

## 5.1 Proof of Theorem 1.10

We will show that $\mathrm{Hom}(R \otimes P, Q) \cong \mathrm{Hom}(R, [P, Q])$.
Take a morphism $\varphi : R \to [P, Q]$ that takes $r$ to $\varphi(r) : P \to Q$. Define $\widetilde{\varphi} : R \otimes P \to Q$ by $\widetilde{\varphi}(r \otimes \alpha) := \varphi(r)(\alpha)$. We must show that $\widetilde{\varphi}$ is a valid graph morphism, that is, there exists a morphism in $V$ from $(R \otimes P)(r \otimes p, r' \otimes p')$ to $Q(\varphi(r)(p), \varphi(r')(p'))$ for every $r, r' \in R$ and $p, p' \in P$. Fix $r, r', p, p'$. Since $\varphi$ is a morphism of partial graphs, $\varphi_{rr'}$ is $V$-morphism from $R(r, r')$ to

$$[P, Q](\varphi(r), \varphi(r')) = \prod_{a,b \in \mathrm{obj}(P)} [P(a, b), Q(\varphi(r)(a), \varphi(r')(b))].$$

We compose with the appropriate projection out of the product, call it $\Pi_{pp'}$, in $V$ to construct a $V$-morphism

$$\Pi_{pp'} \circ \varphi_{rr'} : R(r, r') \mapsto [P(p, p'), Q(\varphi(r)(p), \varphi(r')(p'))].$$

This morphism belongs to

$$V(R(r, r'), [P(p, p'), Q(\varphi(r)(p), \varphi(r')(p'))]),$$

so by the tensor-hom adjunction in $V$, there exists a corresponding morphism

$$\widetilde{\varphi}_{r \otimes p, r' \otimes p'} : R(r, r') \otimes P(p, p') \mapsto Q(\varphi(r)(p), \varphi(r')(p')),$$

as needed.
Take a morphism $\rho : R \otimes P \to Q$, and define $\widehat{\rho} : R \to [P, Q]$ by $\widehat{\rho}(r)(p) := \rho(r \otimes p)$. Each $\rho(r)$ is a well-defined function of sets, so it remains to show that there exists a $V$-morphism from $R(r, r') \to [P, Q](\rho(r), \rho(r'))$ for every pair $r, r' \in R$. Fix a pair $r, r' \in R$. Since $\rho$ is a morphism of graphs, $\rho_{r \otimes p, r' \otimes p'}$ is a $V$-morphism from

$$R(r, r') \otimes P(p, p') = (R \otimes P)(r \otimes p, r' \otimes p')$$

to

$$Q(\rho(r \otimes p), \rho(r' \otimes p')) = Q(\rho(r)(p), \rho(r')(p'))$$

for all $p, p' \in P$. By the tensor-hom adjunction in $V$, we have corresponding $V$-morphisms

$$\theta_{rr'}^{pp'} : R(r, r') \mapsto [P(p, p'), Q(\rho(r)(p), \rho(r')(p'))]$$

for every $p, p' \in P$. This family of morphisms factors through the unique morphism

$$m_{rr'} : R(r, r') \to \prod_{p,p' \in P} [P(p, p'), Q(\rho(r)(p), \rho(r')(p'))]$$

by the universal property of the product in $V$. Note that

$$[P, Q](\rho(r), \rho(r')) := \prod_{p,p' \in P} [P(p, p'), Q(\rho(r)(p), \rho(r')(p'))]$$

so we now have a $V$-morphism $m_{rr'} : R(r, r') \to [P, Q](\rho(r), \rho(r'))$ for every pair $r, r' \in \mathrm{obj}(R)$, as needed.
Checking $\widetilde{\widehat{\rho}} = \rho$ and $\widehat{\widetilde{\varphi}} = \varphi$ follows the same argument as the usual proof of the tensor-hom adjunction. Checking the naturality of the adjunction in each coordinate is equally unoriginal.

### 5.2 Proof of Theorem 1.17

In the following proofs, we will write $Z^X$ for $\mathrm{Hom}(X,Z)$. This choice allows us two write isomorphisms of hom-sets as rules reminiscent of exponentials:

$$(X \sqcap Y)^Z \cong X^Z \times Y^Z, \quad X^{Y \sqcup Z} \cong X^Y \times X^Z, \quad \text{and} \quad X^{Y \otimes Z} \cong \left(X^Z\right)^Y.$$

In this notation, the Yoneda Lemma tells us that $X^Y \cong X^Z$ implies $Y \cong Z$ and $X^Z \cong Y^Z$ implies $X \cong Y$.

Let $A,B,C,D$ be graphs that have the same objects, and $V \in V$. First, we prove composition of graphs distributes over pointwise addition.

$$\begin{aligned}
V^{(A\circ(B+C))(z,y)} &= V^{\sum_z A(x,z)\otimes(B+D)(z,y)} \\
&\cong \prod_z V^{A(x,z)\otimes(B+C)(z,y)} \\
&\cong \prod_z \left(V^{(B+C)(z,y)}\right)^{A(x,z)} \\
&\cong \prod_z \left(V^{B(z,y)} \times V^{C(z,y)}\right)^{A(x,z)} \\
&\cong \prod_z \left(V^{B(z,y)}\right)^{A(x,z)} \times \left(V^{C(z,y)}\right)^{A(x,z)} \\
&\cong \prod_z V^{A(x,z)\otimes B(z,y)} \times V^{A(x,z)\otimes C(z,y)} \\
&\cong V^{\sum_z A(x,z)\otimes B(z,y)} \times V^{\sum_z A(x,z)\otimes C(z,y)} \\
&= V^{(A\circ B)(x,y)} \times V^{(A\circ C)(x,y)} \\
&\cong V^{((A\circ B)+(A\circ C))(x,y)}
\end{aligned}$$

By the Yoneda Lemma, $A \circ (B+C) \cong (A \circ B) + (A \circ C)$. Next, we demonstrate a FOIL-style rule for graphs.

$$\begin{aligned}
V^{((A+B)\circ(C+D))(x,y)} &= V^{\sum_z (A+B)(x,z)\otimes(C+D)(z,y)} \\
&\cong \prod_z V^{(A+B)(x,z)\otimes(C+D)(z,y)} \\
&\cong \prod_z \left(V^{(C+D)(z,y)}\right)^{(A+B)(x,z)} \\
&\cong \prod_z \left(V^{(C+D)(z,y)}\right)^{A(x,z)\amalg B(x,z)} \\
&\cong \prod_z \left(V^{(C+D)(z,y)}\right)^{A(x,z)} \times \left(V^{(C+D)(z,y)}\right)^{B(x,z)} \\
&\cong \prod_z V^{A(x,z)\otimes(C+D)(z,y)} \times V^{B(x,z)\otimes(C+D)(z,y)} \\
&\cong V^{\sum_z A(x,z)\otimes(C+D)(z,y)} \times V^{\sum_z B(x,z)\otimes(C+D)(z,y)} \\
&= V^{(A\circ(C+D))(x,y)} \times V^{(B\circ(C+D))(x,y)} \\
&\cong V^{((A\circ C)+(A\circ D))(x,y)} \times V^{((B\circ C)+(B\circ D))(x,y)} \\
&\cong V^{((A\circ C)+(A\circ D)+(B\circ C)+(B\circ D))(x,y)}
\end{aligned}$$

The second to last isomorphism is an application of the previously proven identity $A \circ (B + C) \cong (A \circ B) + (A \circ C)$. By the Yoneda Lemma, $(A + B) \circ (C + D) \cong (A \circ C) + (A \circ D) + (B \circ C) + (B \circ D)$. Finally, we verify composition is associative.

$$\begin{aligned}
V^{(A\circ(B\circ C))(x,y)} &\cong \prod_z V^{A(x,z)\otimes(B\circ C)(z,y)} \\
&\cong \prod_z \left(V^{(B\circ C)(z,y)}\right)^{A(x,z)} \\
&\cong \prod_z \left(\prod_w V^{B(z,w)\otimes C(w,y)}\right)^{A(x,z)} \\
&\cong \prod_{z,w} \left(V^{B(z,w)\otimes C(w,y)}\right)^{A(x,z)} \\
&\cong \prod_{z,w} \left(\left(V^{C(w,y)}\right)^{B(z,w)}\right)^{A(x,z)} \\
&\cong \prod_{z,w} \left(V^{C(w,y)}\right)^{A(x,z)\otimes B(z,w)} \\
&\cong \prod_w \left(V^{C(w,y)}\right)^{(A\circ B)(x,w)} \\
&\cong \prod_w V^{(A\circ B)(x,w)\otimes C(w,y)} \\
&\cong V^{((A\circ B)\circ C)(x,y)}
\end{aligned}$$

By the Yoneda Lemma, $A \circ (B \circ C) \cong (A \circ B) \circ C$.

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