A compositional game to fairly divide homogeneous cake

Abel Jansma

Based on work under review at *Compositionality* (`https://arxiv.org/abs/2301.02281`)

Introduction

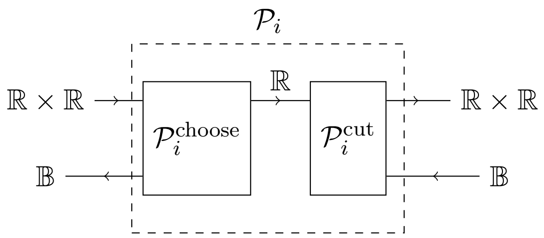
The central question in the game theory of cake-cutting is how to distribute a finite resource among players in a 'fair' way, where fairness can be defined in different ways. The prototypical example of a fair cake-cutting game among two people is referred to as 'I cut you choose': One player cuts the cake, the other player chooses a piece, and the first player get the leftover piece. Why do we call this fair? Even if the two players have private valuation functions, the equilibrium guarantees both players at least $\frac{1}{2}$ of the cake's value. However, the rule is not *meta-envy-free* [5], as it is sometimes more profitable to be the chooser than the cutter. Furthermore, there is no nice way to extend this game to more than two players. While there are fair cake-cutting mechanisms for more than two players (*e.g.* the *moving-knife* procedures [2, 9, 1, 6] and the *last-diminisher* method [8]), these require continuous and universally agreed upon time, perfect and direct communication between all players, an external referee, or an unbounded number of cuts that leave each player with many disconnected pieces of cake. In fact, a finite algorithm (*i.e.* involving a finite number of steps) with a proportional equilibrium distribution of contiguous pieces does not exist:

Theorem 1 (Impossibility of contiguous pieces [7]) *No finite algorithm can guarantee each of n players at least $\frac{1}{n}$ of the cake using only $n-1$ cuts when $n \geq 3$.*

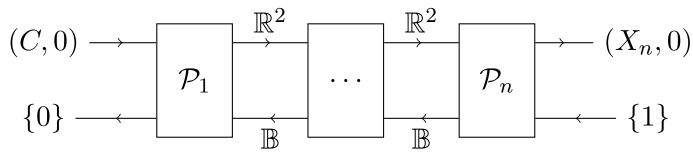
Most cake-cutting literature focuses on the general case in which the cake is heterogeneous, so that the players do not know how the others value each piece. However, when the cake is homogeneous—representing a homogeneous resource or pool of fungible items—then there is a trivially fair mechanism, namely a benevolent dictator that simply cuts the cake for everyone. However, not only is this a very centralised and contrived game with uninteresting dynamics, it also requires identifying a player that can exchange information and cakes with all other players. Furthermore, it is not meta-envy-free. If the partition results in pieces that are not exactly $\frac{1}{n}$, the other players cannot judge if this is the result of luck or malice, so they might still be envious of the cutting player. Such imperfect partitions might seem artificial, but naturally arise whenever $C \bmod n \neq 0$ where C is the size of the cake, or when perfect division is simply not possible (as is the case with real cake-cutting). To get rid of the strong centralisation and restrictions on the communication channels, one could define the game starting from its compositional structure. In this paper, the structure of iterated 'I cut you choose' is investigated with compositional game theory [3], which leads to a new game for cutting a homogeneous cake with n players that has a fair equilibrium of contiguous pieces.

Cake-cutting as lenses

Consider an iterated version of 'I cut you choose', where each player chooses a piece presented to them by the previous player, and subsequently cuts their chosen piece to offer some cake to the next player. For a single player $\mathcal{P}_i$, this open decision-making process can be summarised as follows:



where $\mathbb{R}$ represents a piece of cake, and $\mathbb{B}$ represents the binary choice. Multiple players can then be composed (and put in a context of a cake of size C) as



Iterated cake-cutting is thus modelled as lens composition, where the pieces of cake are propagated along the 'forward' direction, and the decisions are propagated in the 'backward' direction. It is easy to see that this composed game has an exponentially unfair equilibrium distribution. However, changing the composition rule fixes this, through something called the BigPlayer rule.

The BigPlayer rule

Consider the following game, which is just an iterated version of 'I cut you choose', composed with the *BigPlayer* rule:

Definition 1 (The BigPlayer Rule) *Let $\{P_1, P_2, \ldots, P_n\}$ be the players in the n-player game of dividing a cake of size C. The BigPlayer rule then says the following:*

1. *P_1 cuts the cake in two, resulting in two pieces of sizes $(\alpha C, (1-\alpha)C)$, where $\alpha \in [0, 1]$.*
2. *P_2 chooses one of the two pieces.*
3. *If there are any players left who did not play yet: Let the last cutter and the chooser be (P_a, P_b), respectively, and the size of their pieces (a, b), respectively. Then, let $P_{BP} = P_a$ if $a \geq b$, and $P_{BP} = P_b$ otherwise. P_{BP} then has to cut their piece in two.*
4. *A player that did not play yet chooses one of P_{BP}'s pieces.*
5. *Move to 3 if there are any players that have not played yet.*

This is just iterated 'I cut you choose', but after each round, the player who ends up with the biggest piece has to be the cutter in the next round. In the paper, it is shown that sharing a homogeneous cake among any number of players with the BigPlayer rule has a fair equilibrium where each player ends up with a contiguous piece:

Theorem 2 *Cutting a homogeneous cake of size C with the BigPlayer rule has a Nash equilibrium at a proportional distribution with $n-1$ cuts where each of n players gets a contiguous piece of size C/n.*

By starting from the compositional structure of iterated cake-cutting, we have thus found a way to completely eliminate the price of anarchy, which implies decentralisation at no extra welfare cost.

Cake-cutting in the Open Game Engine

This shows the power of compositional game theory as a framework for analysing games—the BigPlayer rule achieves fairness through the compositional structure of iterated games, not their individual implementation. Furthermore, the fact that the compositional structure was central to the definition made the implementation in the Open Game Engine [4] natural, and analysis straightforward. Both versions of iterated 'I cut you choose' are implemented and analysed in the engine, and the appropriate equilibria and deviations are found.

References

[1] AK Austin. Sharing a cake. *The Mathematical Gazette*, 66(437):212–215, 1982.

[2] Lester E Dubins and Edwin H Spanier. How to cut a cake fairly. *The American Mathematical Monthly*, 68(1P1):1–17, 1961.

[3] Neil Ghani, Jules Hedges, Viktor Winschel, and Philipp Zahn. Compositional game theory. In *Proceedings of the 33rd annual ACM/IEEE symposium on logic in computer science*, pages 472–481, 2018.

[4] Jules Hedges, Philipp Zahn, Andre Videla, and Sjoerd Visscher. open-game-engine. `https://github.com/CyberCat-Institute/open-game-engine`, 2022.

[5] Yoshifumi Manabe and Tatsuaki Okamoto. Meta-envy-free cake-cutting protocols. In *International Symposium on Mathematical Foundations of Computer Science*, pages 501–512. Springer, 2010.

[6] Elisha Peterson and Francis Edward Su. Four-person envy-free chore division. *Mathematics Magazine*, 75(2):117–122, 2002.

[7] Jack Robertson and William Webb. *Cake-cutting algorithms: Be fair if you can.* AK Peters/CRC Press, 1998.

[8] Hugo Steinhaus. The problem of fair division. *Econometrica*, 16:101–104, 1948.

[9] Walter Stromquist. How to cut a cake fairly. *The American Mathematical Monthly*, 87(8):640–644, 1980.