Probability and Statistics course

Introduction

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Week 9

Likelihood function

Definition (Likelihood function)

$$L(\vartheta; \mathbf{x}) = f_\vartheta(\mathbf{x}) = \prod_{i=1}^{n} f_\vartheta(x_i)$$

, if the distribution is absolutely continuous

$$L(\vartheta; \mathbf{x}) = P_\vartheta(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^{n} P_\vartheta(X_i = x_i)$$

, if the distribution is discrete.

Definition (Log-likelihood function)

$l(\vartheta; \mathbf{x}) = \ln(L(\vartheta; \mathbf{x}))$

Parameter estimation - Maximum Likelihood

We search for the parameter value, for which the likelihood function is maximal (the reasoning is that the parameter value is chosen for which the observed sample has the maximal probability): $\max_{\vartheta} L(\vartheta; \mathbf{x})$. This obviously coincides with the parameter value, where the log-likelihood function is maximal: $\max_{\vartheta} l(\vartheta; \mathbf{x})$.

If the likelihood function is differentiable with respect of ϑ the maximum can be found by finding the root of its derivative. However, it is much easier to perform the maximization using the log-likelihood function.

If ϑ is 1 dimensional, then $\partial_{\vartheta} l(\vartheta, \mathbf{x}) = 0$, while if $\vartheta = (\vartheta_1, ..., \vartheta_p)$ is p dimensional, then solving $\partial_{\vartheta_i} l(\vartheta, \mathbf{x}) = 0$ gives the estimator. (It has to be checked that we indeed found the maximum.)

Theorem (Invariant property of the ML-estimator)

If the ML estimator of ϑ is $\hat{\vartheta}$, then for any function g the estimator of $g(\vartheta)$ is $g(\hat{\vartheta})$.

Example for ML method

Examples

Let the frequency table below be a sample of size 20 from the following discrete distribution: $P(X_i = -1) = c, P(X_i = 1) = 3c, P(X_i = 2) = 1 - 4c$ $\left(i = 1, \ldots 20 \text{ and } c \text{ is the unknown parameter, } 0 < c < \frac{1}{4}\right)$.

value	-1	1	2
frequency	4	10	6

Determine the Maximum Likelihood (ML) estimator for c!

Parameter estimation - Moment method

1. The kth moment of the distribution, if ϑ is the unknown parameter: $\mu_{k,\vartheta} = \mathbb{E}_\vartheta(X_1^k)$.
2. Let $\hat{\mu}_k = \frac{1}{n}\sum_{j=1}^n X_j^k$ be the kth empirical moment.
3. Let us consider the equations below for the smallest k for which there is a unique solution for ϑ (if there exists such an integer k):

$$\mathbb{E}_\vartheta(X_1) = \frac{1}{n}\sum_{j=1}^n X_j;$$
$$\mathbb{E}_\vartheta(X_1^2) = \frac{1}{n}\sum_{j=1}^n X_j^2;$$
$$\dots$$
$$\mathbb{E}_\vartheta(X_1^k) = \frac{1}{n}\sum_{j=1}^n X_j^k.$$

4. The moment method estimate of ϑ is the solution $\hat{\vartheta}$ of the above system of equations, if this is uniquely determined.

Example for Moment method

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Determine the Moment method estimator for c!

Examples

1. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from the following distributions. Calculate the Maximum Likelihood (ML) estimator for the unknown parameter!
   a a) $Bin(m, p)$ binomial distribution, where $m \in \mathbb{N}$ is given and p is the parameter
   b $Exp(\lambda)$ exponential distribution
   c $N(\mu, \sigma^2)$ normal distribution, where $\sigma \in \mathbb{N}$ is given and μ is the parameter
2. Determine the Maximum Likelihood (ML) estimator for the unknown parameter, if the sample is from an $U[a, 1]$ distribution!
3. Let $X_1, X_2, \ldots, X_n$ be independent, identically $U[a, b]$ distributed random variables. Calculate the estimators for the unknown parameters using the Method of Moments!