Active Calculus - Multivariable

2015 edition

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November 14, 2018

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Chapter 12

Vector Calculus

12.1 Vector Fields

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a vector field?
- How do we draw a vector field?
- What are some familiar contexts in which vector fields arise?

Introduction

Thus far vectors have played a central role in our study of multivariable calculus. We know how to do operations on vectors (addition, scalar multiplication, dot product, etc.), and we have seen how vectors can be used to describe curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. The examples of using vectors to describe curves was our first example of a vector-valued function. That is, a curve $\mathbf{r}(t)$ is really a function that takes as input a real number and produces a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. In this section we will expand our understanding of vector-valued functions to take as input a point (x, y) in $\mathbb{R}^2$ or a point (x, y, z) in $\mathbb{R}^3$ and produce a vector (typically in $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively).

Preview Activity 12.1. It's common for weather forecasters to discuss the wind *speed*, but as any student who has gotten this far in the text will know, this nomenclature is imprecise. It's not terribly helpful to tell someone the wind is blowing at 10 km/h without telling them the direction in which the wind is blowing. If you're trying to make a decision based on what the wind is doing, you need to know about the direction as well. (Perhaps you are taking off in a hot air balloon and need to know which direction the chase team should head to keep track of you.) Because of the swirling nature of wind, it makes sense to give the wind *velocity* at every point in a region (two-dimensional or three-dimensional).

(a) Suppose that given a point (x, y) in the plane, you know that the wind velocity at that point is given by the vector $\mathbf{F}(x, y) = \langle y, x\rangle$. For example, we'd then know that at the point $(1, -1)$, the wind velocity is $\mathbf{F}(1, -1) = \langle -1, 1\rangle$. In the table below, fill in the wind velocity vectors for the given points.

(x, y)	$(2, 1)$	$(0, 0)$	$(-1, 2)$	$(3, -1)$	$(-2, -1)$
$\mathbf{F}(x, y)$					

(b) Suppose that we associate the vector $\mathbf{G}(x, y) = -x\mathbf{j}$ to a point (x, y) in the plane. Complete the table below by giving the vector associated to each of the given points.

(x, y)	$(-2, 0)$	$(-1, 2)$	$(0, -2)$	$(1, 1)$	$(2, 3)$	$(3, 2)$	$(-1, 0)$	$(1, 3)$
$\mathbf{G}(x, y)$								

A table of values of these vector-valued functions is useful, but perhaps even better is a method of visualizing the vectors. In keeping with our wind velocity analogy, if $\mathbf{F}(2, 1) = \langle 1, 2\rangle$, we draw the vector $\langle 1, 2\rangle$ with its tail at the point $(2, 1)$.

(c) Using the first set of axes in Figure 9.1, plot the vectors $\mathbf{F}(x, y)$ for the five points in the table in part (a). The example $\mathbf{F}(1, -1) = \langle -1, 1\rangle$ is drawn for you.

(d) Using the second set of axes in Figure 9.1, plot the vectors $\mathbf{G}(x, y)$ for the eight points in the table in part (b).

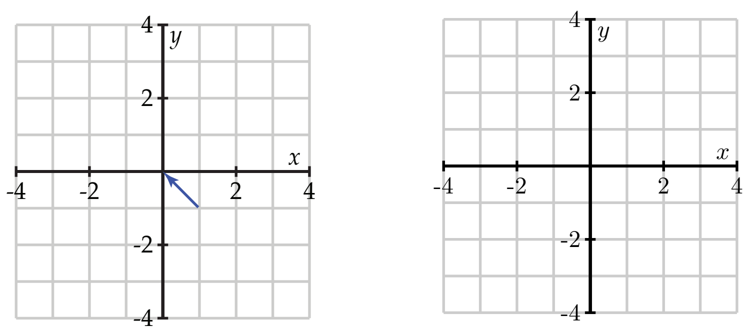

Figure 12.1: Axes for plotting some vectors from $\mathbf{F}(x, y)$ and $\mathbf{G}(x, y)$.

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Examples of Vector Fields

As Preview Activity 9.1 showed you, one example of a time where it makes sense to associate a vector to each point in a region is a *velocity vector field* $\mathbf{F}(x, y)$ or $\mathbf{F}(x, y, z)$, where the vector associated to the point (x, y) or (x, y, z) is the velocity of something at that point. Wind velocity is one example, but another example would be the velocity of a flowing fluid. Figure 9.2 shows such a velocity vector field. Technically, it only shows some of the vectors in the vector field, since the figure would be unintelligible if all of the vectors were shown. This is illustrated by the inset in the upper left corner, which gives a better picture of what we would see if we zoomed in on the red square of the main figure.

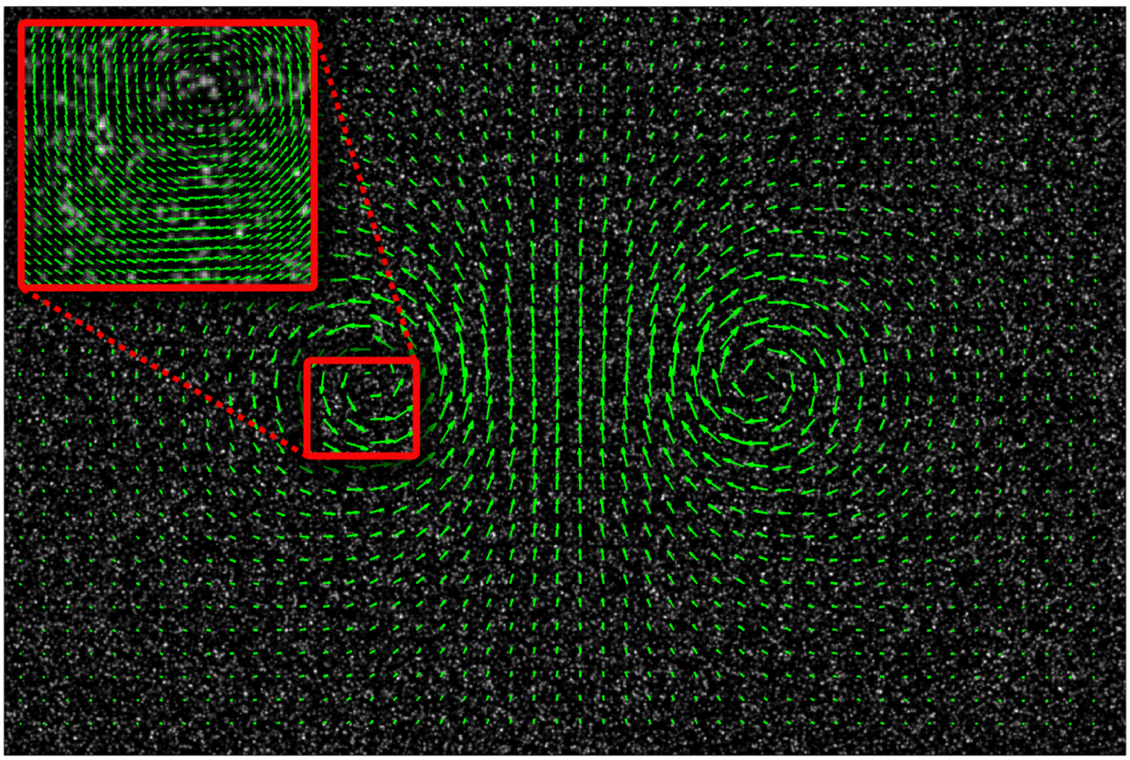

Figure 12.2: An illustration of some of the vectors in a fluid velocity vector field.

Force fields, such as those created by gravity, are also examples of vector fields. For example, the earth exerts a gravitational force on objects. The force is directed from the center of the object to the center of the earth, and its magnitude is determined by the distance between the object and the earth. An illustration of this vector field can be seen in Figure 9.3, where the earth is positioned at the origin, but not shown. Notice that the vectors get shorter as the distance from the origin increases, reflecting the fact that the gravitational force is weaker at that distance.

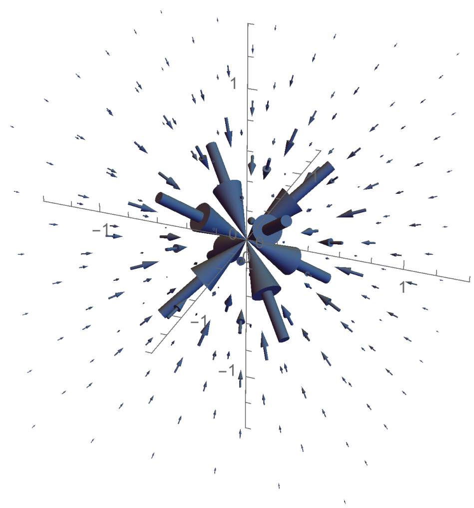

Figure 12.3: Gravitational vector field.

Mathematical Vector Fields

As suggested in the introduction and Preview Activity 9.1, vector fields can be given by formulas.

Definition 12.1. A **vector field** in 2-space function whose value at a point (x, y) is a 2-dimensional vector $\mathbf{F}(x, y)$. In 3-space, a vector field is similarly a function $\mathbf{F}(x, y, z)$ whose value at the point (x, y, z) is a 3-dimensional vector.

Since $\mathbf{F}(x, y, z)$ is a vector, it has **i**, **j**, and **k** components. Each of these components is a scalar function of the point (x, y, z), and so we will often write

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}.$$

For example, if $\mathbf{F}(x, y, z) = \langle x^2, xy\sin(z), y^3\rangle$, then $F_1(x, y, z) = x^2$, $F_2(x, y, z) = xy\sin(z)$, and $F_3(x, y, z) = y^3$. Any time we are considering a vector field $\mathbf{F}(x, y, z)$, the definitions of functions F_1, F_2, and F_3 should be assumed in this manner. (For a vector field $\mathbf{F}(x, y)$ in 2-space, we only have the functions F_1 and F_2, defined analogously.)

Plotting Vector Fields

Preview Activity 9.1 gave you a chance to plot some vectors in the vector fields $\mathbf{F}(x, y) = \langle y, x\rangle$ and $\mathbf{G}(x, y) = \langle 0, -x\rangle$. It would be impossible to sketch *all* of the vectors in these vector fields, since there is one for every point in the plane. In fact, even sketching by hand many more of the vectors than you were asked to in the preview activity rapidly becomes tedious. Fortunately, computer algebra systems can do a great job of making such sketches. One thing to keep in mind, however, is that the magnitudes of the vectors are typically scaled in these plots, including plots of vector fields we will encounter later in the text. To illustrate this, consider the two plots of the vector field $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ in Figure 9.4. The left plot shows some of the vectors but accurately

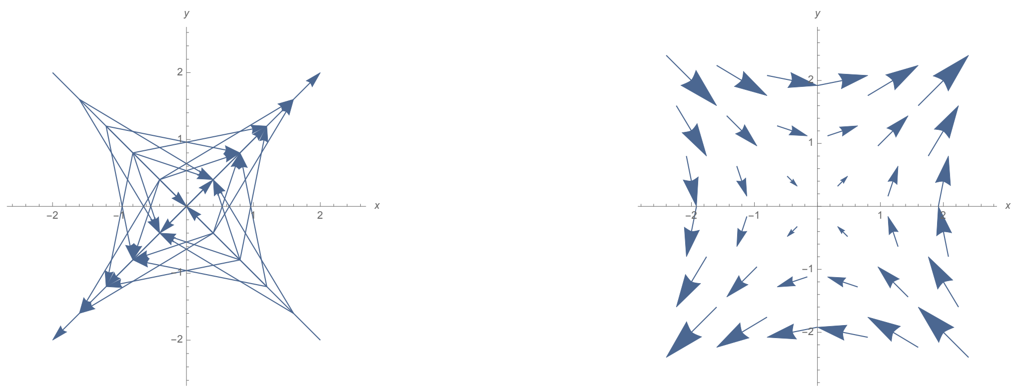

Figure 12.4: Two plots of $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ from a computer algebra system

depicts all of their magnitudes, making the figure very hard to understand, especially along the lines $y = x$ and $y = -x$. The plot on the right, however, uses a uniform scaling to make the figure easier to read. As before, each vector's direction is completely accurate, but now the magnitudes are much smaller. However, the *relative* magnitudes are preserved, helping us to see that vectors farther from the origin have larger magnitude than those closer to the origin.

Activity 12.1.

The plot in Figure 9.5 illustrates the vector field $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$.

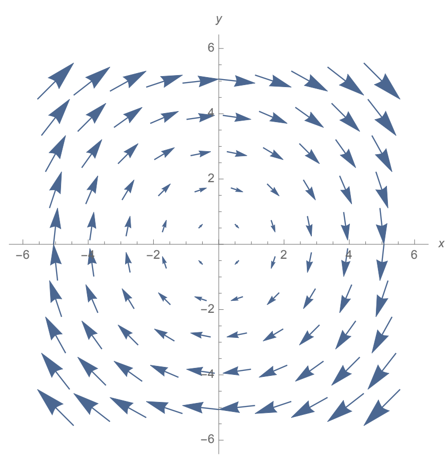

Figure 12.5: The vector field $y\mathbf{i} - x\mathbf{j}$

(a) Starting with one of the vectors near the point $(2, 0)$, sketch a curve that follows the direction of the vector field $\mathbf{F}$. To help visualize what you are doing, it may be useful to think of the vector field as the velocity vector field for some flowing water and that you are imagining tracing the path that a tiny particle inserted into the water would follow as the water moved it around.

(b) Repeat the previous step for at least two other starting points not on the curve you previously sketched.

(c) What shape do the curves you sketched in the previous two steps form?

(d) Verify that $\mathbf{F}(x, y)$ is orthogonal to $\langle x, y\rangle$.

(e) What is the relationship between the function $f(x, y) = x^2 + y^2$ and the vector $x\mathbf{i} + y\mathbf{j}$?

(f) What does this tell you about the relationship between $\mathbf{F}(x, y)$ and circles centered at the origin? What is the relationship between $|\mathbf{F}(x, y)|$ and the radius of the appropriate circle?

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Gradient Vector Fields

Without using the terminology, we've actually already encountered one very important family of vector fields a number of times. Given a function f of two or three variables, the gradient of f is a vector field, since for any point where f has first-order partial derivatives, ∇f assigns a vector to that point.

Activity 12.2.

(a) In Figure 9.6 there are three sets of axes showing level curves for functions f, g, and h, respectively. Sketch at least six vectors in the gradient vector field for each function. In making your sketches, you don't have to worry about getting vector magnitudes precise, but you should ensure that the relative magnitudes (and directions) are correct for each function independently.

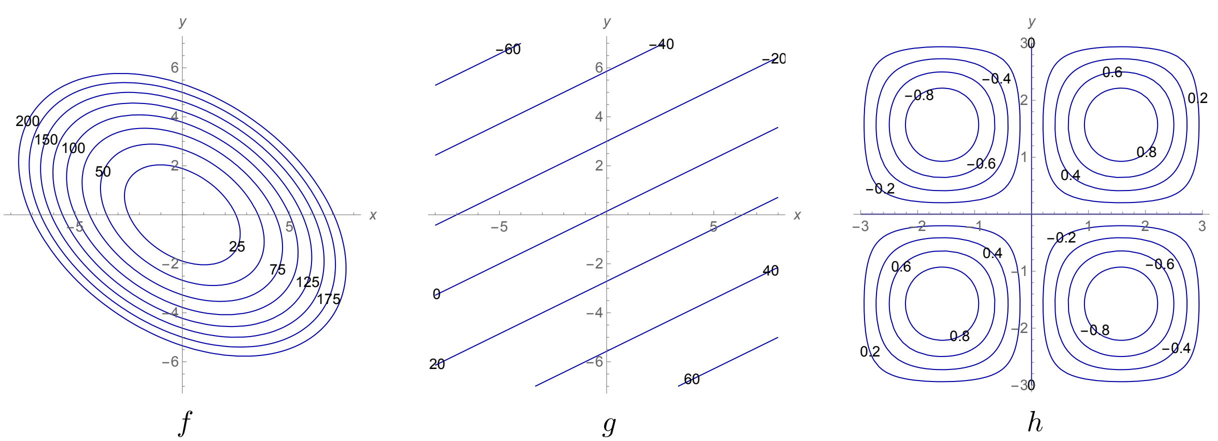

Figure 12.6: Three sets of level curves

(b) Verify that $\mathbf{F}(x,y) = \langle 6xy, 3x^2 + 9\sqrt{y}\rangle$ is a gradient vector field by finding a function f such that $\nabla f(x,y) = \mathbf{F}(x,y)$. For reasons originating in physics, such a function f is called a *potential function* for the vector field $\mathbf{F}$.

(c) Is the function f found in part (b) unique? That is, can you find another function g such that $\nabla g(x,y) = \mathbf{F}(x,y)$ but $f \neq g$?

(d) Is the vector field $\mathbf{F}(x,y) = 6xy\mathbf{i} + (2x + 9\sqrt{y})\mathbf{j}$ a gradient vector field? Why or why not?

$\triangleleft$

Summary

- A 2-dimensional vector field is a function defined on part of $\mathbb{R}^2$ whose value is a 2-dimensional vector. A 3-dimensional vector field is a function defined on part of $\mathbb{R}^3$ whose value is a 3-dimensional vector.
- Vector fields arise in familiar contexts such as wind velocity, fluid velocity, and gravitational force.
- Vector fields are generally plotted in ways that ensure the direction and relative magnitudes of the vectors sketched are correct instead of ensuring that each vector's magnitude is depicted correctly.
- The gradient of a function f of two or three variables is a vector field defined wherever f has partial derivatives.

12.2 The Idea of a Line Integral

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a line integral of a vector-valued function along a curve?
- How can we estimate if a line integral of a vector-valued function along a curve is positive, negative, or zero?

Introduction

As we discussed in section 9.1, vector fields are often useful as representations of forces such as gravity, wind, and flowing water. We learned in section **??** that the dot product of a force vector and a displacement vector tells us how much work the force did on the object as it moved from the tail of its displacement vector to the tip. However, things more complicated when an object's movement is not in a straight line and when the force is not uniform throughout the area in which the object moves. For example, how much work does a wind of 30 mph toward the northwest do on an airplane that's flying 500 miles due north? What if the wind gets weaker the farther north the plane gets? In this section, we begin investigating how integration can be used to calculate the work a force field does in such circumstances.

Preview Activity 12.2. Recall from Section **??** that the work done by a force $\mathbf{F}$ on an object that moves with displacement vector $\mathbf{v}$ is $\mathbf{F}\cdot\mathbf{v}$. In this Preview Activity, we will consider the work done by a wind blowing due east at 45 miles per hour on an airplane at various stages of its journey.

(a) A pilot flies for an hour and finds that he is 300 miles from where he started at a heading of 20° degrees east of due north. Find the work the wind has done on the airplane during the flight.

(b) An hour later, the pilot determines that he is 275 miles due north of where he previously checked his position. Find the work done by the wind on the airplane during the second hour of the flight.

(c) Find the pilot's displacement from his original position after two hours of flying and use that to find the work done by the wind on the airplane during the first two hours of flight.

(d) How does your answer to the previous part connect to the answers to the first two parts?

(e) Suppose that the pilot then flies 45° west of due north for 200 miles. Find the work done by the wind on the airplane during this part of the journey.

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Orientations of Curves

Given our motivation for calculating the work that a force field does on an object as it moves through the field, it is natural to concern ourselves with *how* the object moves. In particular, in many circumstances it will be different if an object moves from the point $(0, 1)$ to the point $(4, 3)$ by first going up the y-axis to $(0, 3)$ and then moving horizontally to $(4, 3)$ than if the object moves along the line segment from $(0, 1)$ directly to $(4, 3)$. Similarly, given a fixed force field, we would expect the work done to be different (in fact, opposite) if the object moves from $(4, 3)$ to $(0, 1)$ directly along a line segment. We say that a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is *oriented* if we have specified the direction of travel along the curve. When a curve is given parametrically (including as a vector-valued function), our convention will be that the orientation follows from the smallest allowable value of the parameter to the largest.

Activity 12.3.

Find parameterizations of each of the curves described below. Ensure that each curve's orientation matches the one specified.

(a) The line segment in $\mathbb{R}^3$ from $(0, 1, -2)$ to $(3, -1, 2)$.

(b) The line segment in $\mathbb{R}^3$ from $(3, -1, 2)$ to $(0, 1, -2)$.

(c) The circle of radius 3 (in $\mathbb{R}^2$) centered at the origin, beginning at the point $(0, -3)$ and proceeding clockwise around the circle.

(d) The portion of the parabola $y^2 = x$ from the point $(4, 2)$ to the point $(1, -1)$.

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Line Integrals

Just as when we differentiated a vector-valued function $\mathbf{r}(t)$ to find a tangent vector, we begin by dividing a curve C oriented from a point P to a point Q into n small, straight pieces. Each of these pieces is in an area where the vector field $\mathbf{F}$ is nearly constant. In Figure 9.7, we show this situation. Each $\mathbf{r}_i$ is the tip of a vector that traces out the curve, and then the $\Delta\mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$ (shown in blue) approximate the curve C. The green vectors are the vectors in the vector field $\mathbf{F}$ at each of the designated points along the curve.

If we are trying to figure out how much a wind current helps or hinders an aircraft flying along a path determined by the curve, then calculating the dot product $\mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i$ makes sense for the local amount of help/hindrance, as if the vector $\mathbf{r}_i$ along the curve and the force field vector $\mathbf{F}(\mathbf{r}_i)$ point in similar directions, the dot product will be positive.[1] On the other hand, if the angle between them is obtuse, the dot product will be negative and we also would note that the force field is hindering the aircraft's progress. Taking the sum over $i = 0, \ldots, n-1$, we have a Riemann

[1] We are abusing notation here a tiny bit, since technically the domain of $\mathbf{F}$ is points in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $\mathbf{r}_i$ is a vector. By $\mathbf{F}(\mathbf{r})$, we mean $\mathbf{F}(r_1, r_2)$, where $\mathbf{r} = \langle r_1, r_2 \rangle$.

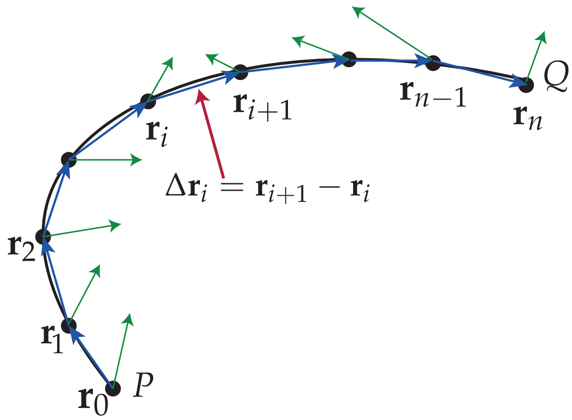

Figure 12.7: A curve C oriented from the point P to the point Q. The tips of the vectors $\mathbf{r}_i$ that trace out the curve are shown as points. The blue vectors are the $\Delta\mathbf{r}_i$, while the green vectors are the vectors associated to each of the points by a vector field $\mathbf{F}$.

sum that approximates the work done by the vector field on the aircraft as it flies along C:

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i.$$

This suggests the following definition.

Definition 12.2. Let C be an oriented curve and $\mathbf{F}$ a vector field defined in a region containing C. The **line integral** of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{|\Delta\mathbf{r}_i| \to 0} \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i,$$

provided the limit exists.

The limit in definition 9.2 exists provided that $\mathbf{F}$ is a continuous vector field, by which we mean that each component function of $\mathbf{F}$ is continuous as a function of 2 or 3 variables, and that C is a piecewise smooth curved traced out from its initial point to its terminal point without retracing any portion of the curve.

Because the dot products in the definition of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ can each be viewed as the work done by $\mathbf{F}$ as an object moves along the (very small) vector $\Delta\mathbf{r}$, the line integral gives the total work done by the vector field on an object that moves along C (in the direction of its orientation).

Activity 12.4.

Shown in Figure 9.8 are two vector fields, $\mathbf{F}$ and $\mathbf{G}$ and four oriented curves, as labeled in the plots. For each of the line integrals below, determine if its value should be positive, negative, or zero.

(a) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ (b) $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ (c) $\int_{C_3} \mathbf{G} \cdot d\mathbf{r}$ (d) $\int_{C_4} \mathbf{G} \cdot d\mathbf{r}$

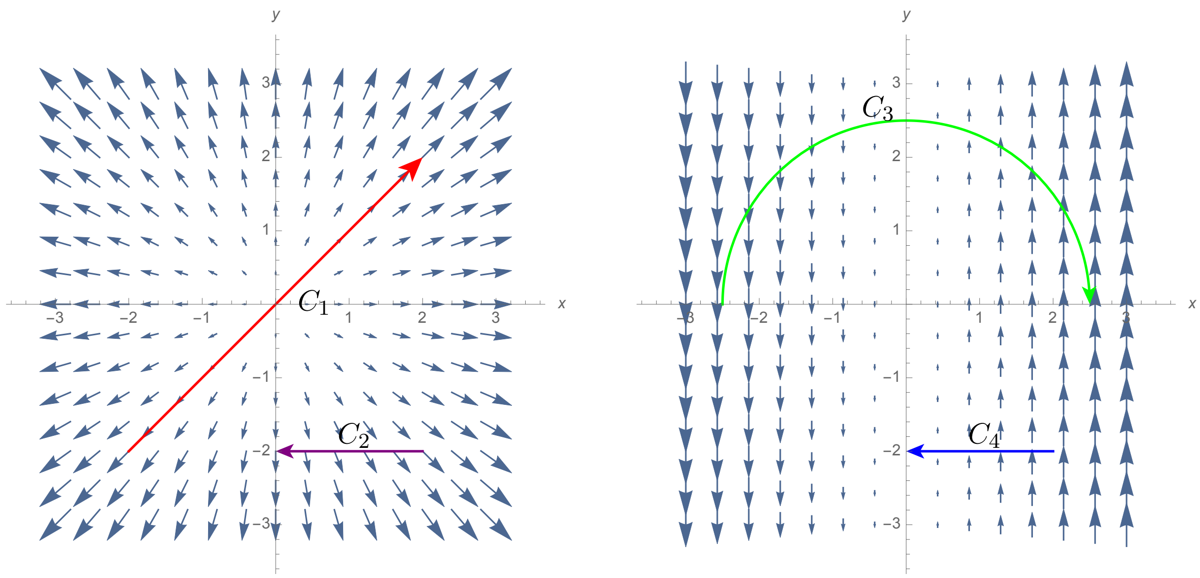

Figure 12.8: Vector fields **F** (left) and **G** (right)

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The next several sections will be devoted to determining ways to calculate line integrals, since the limit in the definition, just like the limit in the definition of every other type of integral we've studied so far, is cumbersome to work with in most cases. However, in the case where the oriented curve C is composed of horizontal and vertical line segments, we can make a rather quick reduction to a single-variable integral, as the following example shows.

Example 12.1. Consider the constant vector field $\mathbf{F}(x, y) = \langle 2, 1\rangle$. Let C be the curve that follows the horizontal line segment from $(1, 1)$ to $(4, 1)$ and then continues down the vertical line segment to $(4, -2)$. Figure 9.9 shows $\mathbf{F}$ and C, including the orientation.

To calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, we'll start by working with the horizontal line segment. Along that part of C, notice that $d\mathbf{r} \approx \Delta\mathbf{r} = \Delta x\mathbf{i}$. Thus, the Riemann sum that calculates the line integral along this portion of C consists of terms of the form $\langle 2, 1\rangle \cdot (\Delta x\mathbf{i}) = 2\Delta x$. Along this part of C, x ranges from 1 to 4, and thus we can turn the Riemann sum here into the definite integral $\int_1^4 2\,dx = 6$. Since the vectors are generally pointing in a direction that agrees with the orientation of C, we are not surprised to have a positive value here.

Now we turn our attention to the vertical portion of C. Here $d\mathbf{r} \approx \Delta\mathbf{r} = \Delta y\mathbf{j}$, which means that $\mathbf{F} \cdot d\mathbf{r} \approx 1\Delta y$. Hence, our Riemann sum can be calculated by the definite integral $\int_1^{-2} 1\,dy = -3$. Notice that the limits of integration here were set up to match the orientation of C. Also, the

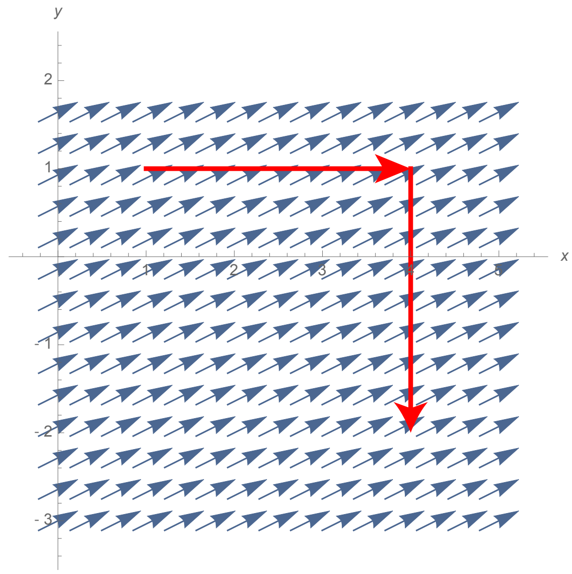

Figure 12.9: An oriented curve from $(1, 1)$ to $(4, -2)$ in a vector field $\mathbf{F}$.

negative value should not be unexpected, since C is oriented in a direction for which the vectors of $\mathbf{F}$ point in a direction that would hinder motion along C.

Combining these two pieces of work, we find that $\int_C \mathbf{F} \cdot d\mathbf{r} = 6 - 3 = 3$.

Properties of Line Integrals

In Example 9.1, we implicitly made use of the idea that if C can be broken up into two curves C_1 and C_2 such that the endpoint of C_1 is the start point of C_2, then the line integral of $\mathbf{F}$ along C is the sum of the line integrals of $\mathbf{F}$ along C_1 and along C_2. This probably seems natural, based on the property for definite integrals which tells us that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The table below summarizes some other properties of line integrals, each of which has a familiar analogue amongst the properties of definite integrals. If C_1 and C_2 are oriented curves, with C_1 from a point P to a point Q and C_2 from Q to a point R, we denote by $C_1 + C_2$ the oriented curve from P to R that follows C_1 to Q and then continues along C_2 to R. Also, if C is an oriented curve,

$-C$ denotes the same curve but with the opposite orientation.

For a constant scalar k, vector fields $\mathbf{F}$ and $\mathbf{G}$, and oriented curves C, C_1, and C_2, the following properties hold:

1. $\displaystyle\int_C (k\mathbf{F}) \cdot d\mathbf{r} = k \int_C \mathbf{F} \cdot d\mathbf{r}$
2. $\displaystyle\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$
3. $\displaystyle\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$
4. $\displaystyle\int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Activity 12.5.

Figure 9.10 shows a vector field $\mathbf{F}$ as well as six oriented curves, as labeled in the plot.

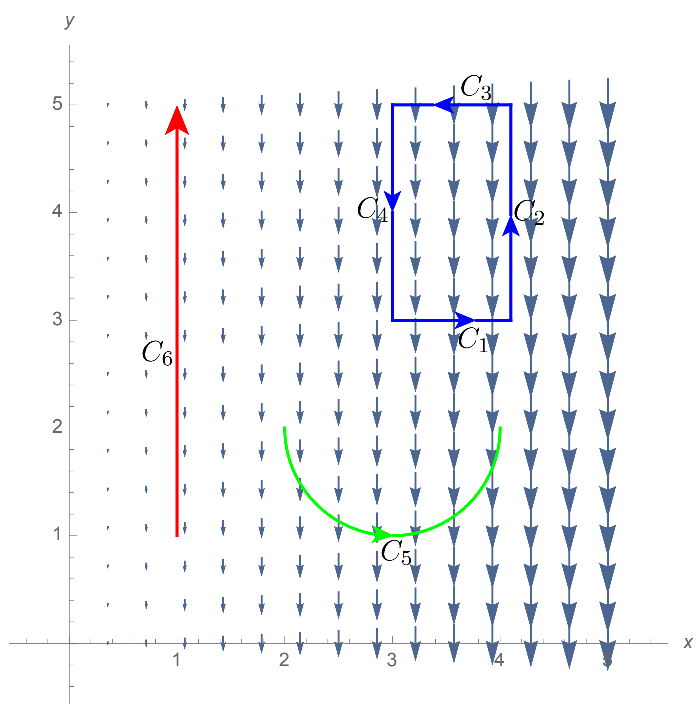

Figure 12.10: A vector field $\mathbf{F}$ and six oriented curves.

(a) Is $\int_{C_5} \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain.

(b) Let $C = C_1 + C_2 + C_3 + C_4$. Determine if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

(c) Order the line integrals below from smallest to largest.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \qquad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \qquad \int_{C_3} \mathbf{F} \cdot d\mathbf{r} \qquad \int_{C_4} \mathbf{F} \cdot d\mathbf{r} \qquad \int_{C_5} \mathbf{F} \cdot d\mathbf{r} \qquad \int_{C_6} \mathbf{F} \cdot d\mathbf{r}$$

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The Circulation of a Vector Field

If an oriented curve C ends at the same point where it started, we say that C is **closed**. The line integral of a vector field $\mathbf{F}$ along a closed curve C is called the **circulation** of $\mathbf{F}$ around C. To emphasize the fact that C is closed, we sometimes write $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\int_C \mathbf{F} \cdot d\mathbf{r}$. Circulation servies as a measure of a vector field's tendency to rotate in a manner consistent with the orientation of the curve.

Activity 12.6.

Determine if the circulation of the vector field around each of the closed curves shown in Figure 9.11 below is positive, negative, or zero.

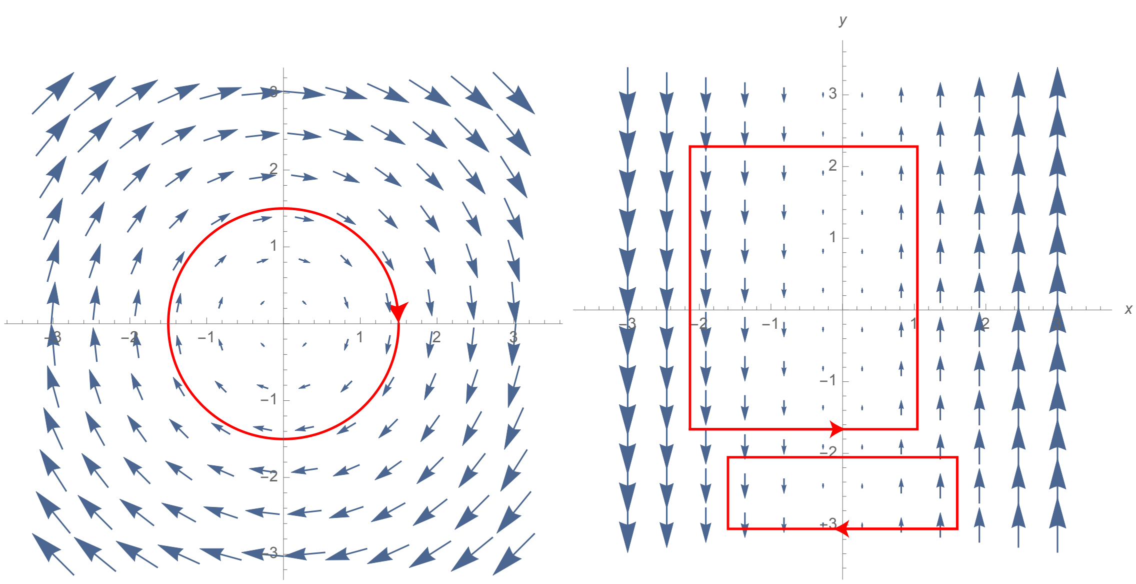

Figure 12.11: Vector fields and closed curves

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Summary

- A line integral measures of a vector field along an oriented curve measures the extent to which the vector field points in a direction consistent with the orientation of the curve.

- Line integrals are defined using Riemann sums by breaking the curve into many small segments along which the vector field is essentially constant.
- Line integrals have many properties that are analogous to those of definite integrals of functions of a single variable.

12.3 Using Parameterizations to Calculate Line Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we use a parameterization of an oriented curve C to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$?
- If a parameterization is used to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, does the parameterization chosen alter the value of the line integral?

Introduction

We begin this section by taking a look at how we might go about calculating a line integral of a vector field along some line segments and will use this as inspiration to see how treating oriented curves as vector-valued functions can allow us to quickly turn a line integral of a vector field into an integral of a single variable.

Preview Activity 12.3. Let $\mathbf{F} = \langle xy, y^2 \rangle$, let C_1 be the line segment from from $(1,1)$ to $(4,1)$, let C_2 be the line segment from $(4,1)$ to $(4,3)$, and let C_3 be the line segment from $(1,1)$ to $(4,3)$. Also let $C = C_1 + C_2$. This vector field and the curves are shown in Figure 9.12.

(a) Every point along C_1 has $y = 1$. Therefore, along C_1, the vector field $\mathbf{F}$ can be viewed purely as a function of x. In particular, along C_1, we have $\mathbf{F}(x,1) = \langle x, 1 \rangle$. Since every point along C_2 has the same x-value, what (in terms of y only) is $\mathbf{F}$ along C_2?

(b) Recall that $d\mathbf{r} \approx \Delta\mathbf{r}$, and along C_1, we have that $\Delta\mathbf{r} = \Delta x\mathbf{i} \approx dx\mathbf{i}$. Thus, $d\mathbf{r} = \langle dx, 0 \rangle$. We know that along C_1, $\mathbf{F} = \langle x, 1 \rangle$. What does this mean $\mathbf{F} \cdot d\mathbf{r}$ is along C_1? What interval of x-values describes C_1? Use these facts fact to write $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ as an integral of the form $\int_a^b f(x)\,dx$ and evaluate the integral.

(c) Use an analogous approach to write $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ as a limit of the form $\int_\alpha^\beta g(y)\,dy$ and evaluate the integral.

(d) Use the previous parts and a property of line integrals to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ without having to evaluate any additional integrals.

(e) Evaluating $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ takes more work at this stage, so let's break the process into smaller pieces.

i. Since C_3 is a line segment, find the slope-intercept ($y = mx + b$) form of the equation of this line.

ii. Just as we noticed that along C_1 we always had $y = 1$, we now know how to express y in terms of x for all points along C_3. Use this to to express $\mathbf{F}(x, y) = \mathbf{F}(x, mx + b)$ as a vector purely in terms of x for points along C_3.

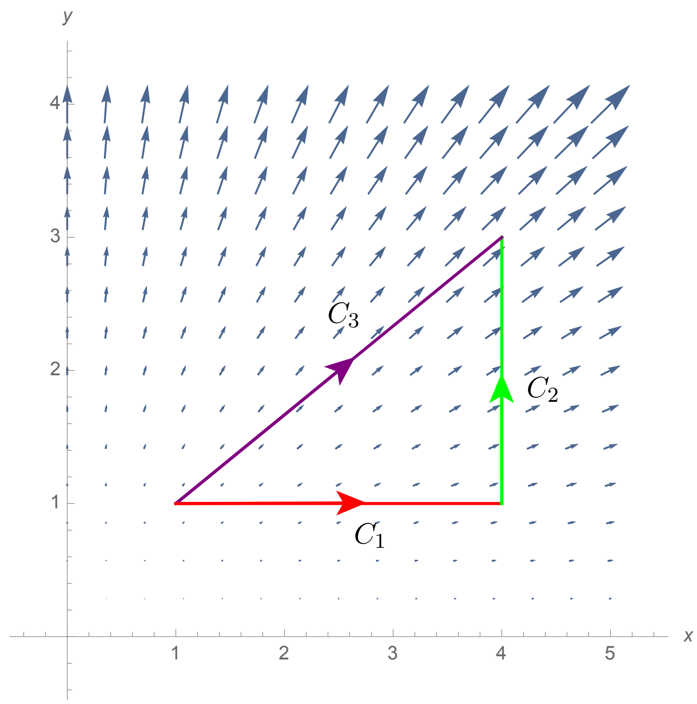

Figure 12.12: A vector field $\mathbf{F}$ and three oriented curves.

iii. We often think of the slope of a line as being $\Delta y/\Delta x$. Use this fact and the slope of the line containing C_3 to express Δy as a multiple of Δx.

iv. We may view $\Delta \mathbf{r}$ as $\langle \Delta x, \Delta y \rangle$. Since $\Delta x \approx dx$ and $\Delta y \approx dy$, write $d\mathbf{r}$ as a vector in terms of dx.

v. Use the range of x-values covered by the line segment C_3 to write $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ as a single-variable integral of the form $\int_a^b f(x)\,dx$ and evaluate the integral.

(f) Notice that C and C_3 both start at $(1, 1)$ and end at $(4, 3)$. How do the values of $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ compare?

(g) Is $\mathbf{F}$ a gradient vector field? Why or why not?*Hint*: If $\mathbf{F}$ were a gradient vector field, then there would be a function f such that $\mathbf{F} = \nabla f$. What would Clairaut's theorem say in this case?

$\bowtie$

Parameterizations in the Definition of $\int_C \mathbf{F} \cdot d\mathbf{r}$

Preview Activity 9.3 has shown us that it is possible to evaluate line integrals without needing to resort to working with Riemann sums directly. However, the approaches taken there seem rather cumbersome to use for oriented curves that are not line segments. Fortunately, parameterizing the oriented curve along which a line integral is calculated provides a powerful tool for evaluating line integrals.

Suppose that C is an oriented curve traced out by the vector-valued function $\mathbf{r}(t)$ for $a \le t \le b$, and let $\mathbf{F}$ be a continuous vector field defined near C. We can divide the interval $[a, b]$ into n subintervals, each of length $\Delta t = (b-a)/n$, by letting $t_i = a + i\Delta t$ for $i = 0, 1, \ldots, n$. This subdivision of $[a, b]$ then can be used to break C up into n pieces by letting $\Delta\mathbf{r}_i = \mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)$ for $i = 0, 1, \ldots, n-1$. Now notice that

$$\Delta\mathbf{r}_i = \mathbf{r}(t_{i+1}) - \mathbf{r}(t_i) = \mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i) = \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t}\Delta t.$$

Since the ratio in the last expression for $\Delta\mathbf{r}_i$ is approximately $\mathbf{r}'(t_i)$, we can say that $\Delta\mathbf{r}_i \approx \mathbf{r}'(t_i)\Delta t$. Substituting this into the Reimann sum in the definition of the line integral, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{|\Delta\mathbf{r}_i| \to 0} \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)\Delta t.$$

This final Riemann sum is $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$, allowing us to reduce the evaluation of a line integral of a vector-valued function along an oriented curve to an ordinary integral of a function of one variable, since after evaluating the dot product, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ is (scalar) function of t. We restate this result below for easy reference.

> Let C be a smooth, oriented curve traced out by the vector-valued function $\mathbf{r}(t)$ for $a \le t \le b$ and let $\mathbf{F}$ be a continuous vector field defined near C. Then
>
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Example 12.2. Let $\mathbf{F}(x, y) = x\mathbf{i} + y^2\mathbf{j}$ and let C be the quarter of the circle of radius 3 from $(0, 3)$ to $(3, 0)$. This vector field and curve are shown in Figure 9.13. By properties of line integrals, we know that $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}$, and we will use this property since $-C$ is the usual clockwise orientation of a circle, meaning we can parameterize $-C$ by $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le \pi/2$.

To evaluate $\int_{-C} \mathbf{F} \cdot d\mathbf{r}$ using this parameterization, we need to note that

$$\mathbf{F}(\mathbf{r}(t)) = \langle 3\cos(t), 9\sin^2(t)\rangle \qquad \text{and} \mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t)\rangle.$$

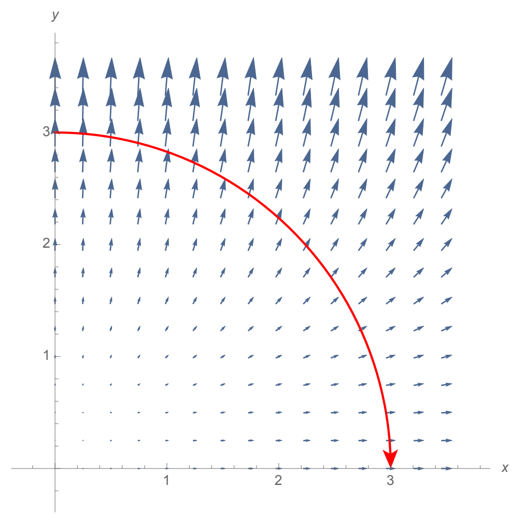

Figure 12.13: The vector field $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j}$ and an oriented curve C

Thus, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r} &= -\int_0^{\pi/2} \langle 3\cos(t), 9\sin^2(t)\rangle \cdot \langle -3\sin(t), 3\cos(t)\rangle \, dt \\
&= -\int_0^{\pi/2} \left(-9\sin(t)\cos(t) + 27\sin^2(t)\cos(t)\right) \, dt \\
&= -\int_0^1 (-9u + 27u^2) \, du = -\left[-\frac{9}{2}u^2 + 9u^3\right]_0^1 = -\left(-\frac{9}{2} + 9\right) = -\frac{9}{2}.
\end{aligned}$$

Activity 12.7.

(a) Find the work done by the vector field $\mathbf{F}(x, y, z) = 6x^2z\mathbf{i} + 3y^2\mathbf{j} + x\mathbf{k}$ on a particle that moves from the point $(3, 0, 0)$ to the point $(3, 0, 6\pi)$ along the helix given by $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$.

(b) Let $\mathbf{F}(x, y) = \langle 0, x\rangle$. Let C be the closed curve consisting of the top half of the circle of radius 2 centered at the origin and the portion of the x-axis from $(2, 0)$ to $(-2, 0)$, oriented clockwise. Find the circulation of $\mathbf{F}$ around C.

◁

Activity 12.8.

Let $\mathbf{F}(x, y) = \langle y^2, 2xy + 3\rangle$.

(a) Let C_1 be the portion of the graph of $y = x^3 - x$ from $(1, 0)$ to $(-1, 0)$. Calculate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

(b) Let C_2 be the line segment from $(1, 0)$ to $(-1, 0)$. Calculate $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

(c) Let C_3 be the circle of radius 3 centered at the origin, oriented counterclockwise. Calculate $\oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$.

◁

Alternative Notation for Line Integrals

In contexts where the fact that the quantity we are measuring via a line integral is best measured via a dot product (such as calculating work), the notation we have used thus far for line integrals is fairly common. However, sometimes the vector field is such that the units on x, y, and z are not distances. In this case, a dot product may not have a physical meaning, and an alternative notation using differentials can be common. Specifically, if $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz.$$

(If $\mathbf{F}$ is a vector field in $\mathbb{R}^2$, the $F_3\,dz$ term is omitted.) As a concrete example, if $\mathbf{F}(x,y,z) = \langle x^2y, z^3, x\cos(z)\rangle$ and C is some oriented curve in $\mathbb{R}^3$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x^2y\,dx + z^3\,dy + x\cos(z)\,dz.$$

It is important to recognize that the integral on the right-hand side is still a line integral and must be evaluated using techniques for evaluating line integrals. We cannot simply try to treat it as if it were a definite integral of a function of one variable.

Independence of Parameterization

Up to this point, we've just been choosing whatever parameterization of an oriented curve C came to mind, and our argument for how we can use parameterizations to calculate line integrals did not depend on the specific choice of parameterization. However, it is not immediately obvious that different parameterizations don't result in different values of the line integral. Our next example explores this question.

Example 12.3. Consider the vector field $\mathbf{F} = x\mathbf{i}$. Let us consider two different oriented curves from $(0,1)$ to $(3,3)$. The first oriented curve C travels horizontally to $(3,1)$ and then proceeds vertically to $(3,3)$. The second oriented curve C_3 is the line segment from $(0,1)$ to $(3,3)$. Notice that, as depicted in Figure 9.14, we can break C up into two oriented curves C_1 (the horizontal portion) and C_2 (the vertical portion) so that $C = C_1 + C_2$.

We first note that since $\mathbf{F}$ is orthogonal to C_2, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$; threfore $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$. We can parameterize C_1 as just $x\mathbf{i} + \mathbf{j}$ for $0 \le x \le 3$, which leads to

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \langle x, 0\rangle \cdot \langle 1, 0\rangle\,dx = \int_0^3 x\,dx = \frac{9}{2}.$$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 9/2$.

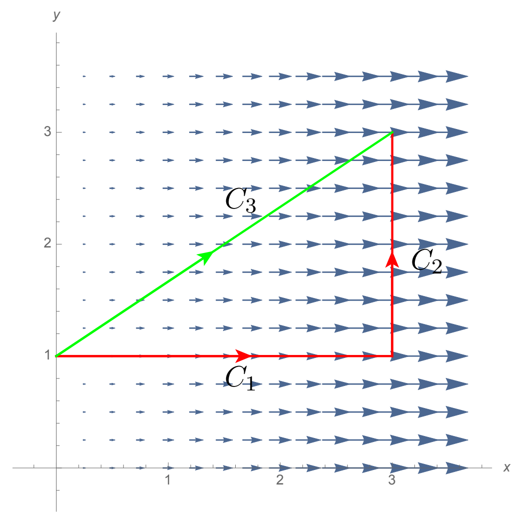

Figure 12.14: The vector field $\mathbf{F} = x\mathbf{i}$ and some oriented curves.

Now we look at $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$, but we parameterize C_3 in a bit of a nonstandard way by letting $\mathbf{r}(t) = \langle 3\sin(t), 1 + 2\sin(t)\rangle$ for $0 \leq t \leq \frac{\pi}{2}$. Then $\mathbf{r}'(t) = \langle 3\cos(t), 2\cos(t)\rangle$, and

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \langle 3\sin(t), 0\rangle \cdot \langle 3\cos(t), 2\cos(t)\rangle \, dt = \int_0^{\pi/2} 9\sin(t)\cos(t)\, dt = \frac{9}{2}.$$

In the next activity, you are asked to consider the more typical parameterization of C_3 and verify that using it gives the same value for the line integral.

It's also worth observing here that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$, so at least two (very different) paths from $(0, 1)$ to $(3, 3)$ give the same value of the line integral here. The next section will further investigate this phenomenon and when it happens.

Activity 12.9.

(a) The typical parameterization of the line segment from $(0, 1)$ to $(3, 3)$ (the oriented curve C_3 in Example 9.3) is $\mathbf{r}(t) = \langle 3t, 1 + 2t\rangle$. Use this parameterization to calculate $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = x\mathbf{i}$.

(b) Calculate $\int_C (3xy + e^z)\, dx + x^2\, dy + (4z + xe^z)\, dz$ where C is the oriented curve consisting of the line segment from the origin to $(1, 1, 1)$ followed by the line segment from $(1, 1, 1)$ to $(0, 0, 2)$.

(c) Calculate $\int_{C'} (3xy + e^z)\, dx + x^2\, dy + (4z + xe^z)\, dz$ where C' is the line segment from $(0, 0, 0)$ to $(0, 0, 2)$.

◁

Summary

- If C_1 and C_2 are different paths from P to Q, it is possible for $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ to have a different value to $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

- Line integrals of vector fields along oriented curves can be evaluated by parameterizing the curve in terms of t and then calculating the integral of $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ on the interval $[a, b]$.
- The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ can also be written as $\int_C F_1\,dx + F_2\,dy$ if $\mathbf{F} = \langle F_1, F_2 \rangle$ is a vector field in $\mathbb{R}^2$ or $\int_C F_1\,dx + F_2\,dy + F_3\,dz$ if $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field in $\mathbb{R}^3$.

12.4 Path-Independent Vector Fields and the Fundamental Theorem of Calculus for Line Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we characterize the vector fields $\mathbf{F}$ for which $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for every oriented path from a point P to a point Q?
- What special properties do gradient vector fields have?

Introduction

In some of the activities and examples in this chapter we have encountered situations where C_1 and C_2 are different oriented curves from a point P to a point Q and $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. In this section, we explore vector fields which have the property that for all points P and Q, if C_1 and C_2 are oriented paths from P to Q, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Preview Activity 12.4. In Activity 9.8 we considered the vector field $\mathbf{F}(x, y) = \langle y^2, 2xy + 3\rangle$ and two different oriented curves from $(1, 0)$ to $(-1, 0)$. We found that the value of the line integral of $\mathbf{F}$ was the same along those two oriented curves.

(a) Verify that $\mathbf{F}(x, y) = \langle y^2, 2xy + 3\rangle$ is a gradient vector field by showing that $\mathbf{F} = \nabla f$ for the function $f(x, y) = xy^2 + 3y$.

(b) Calculate $f(-1, 0) - f(1, 0)$. How does this value compare to the value of the line integral $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ you found in Activity 9.8?

(c) Let C_3 be the line segment from $(1, 1)$ to $(3, 4)$. Calculate $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ as well as $f(3, 4) - f(1, 1)$. What do you notice?

We've used Clairaut's Theorem to argue that a vector field in $\mathbb{R}^2$ is not a gradient vector field when $\partial F_1/\partial y \neq \partial F_2/\partial x$, and earlier in this preview activity, you verified that a given vector field was the gradient of a particular function of two variables. Clairaut's Theorem holds for functions of three variables. However, in that case there are six mixed partials to calculate, and thus it can be rather tedious. The remaining parts of this preview activity suggest a process for determining if a vector field in $\mathbb{R}^3$ is conservative as well as finding a potential function for the vector field.

Let $\mathbf{G}(x, y, z) = \langle 3e^{y^2} + z\sin(x), 6xye^{y^2} - z, 3z^2 - y - \cos(x)\rangle$ and $\mathbf{H}(x, y, z) = \langle 3x^2y, x^3 + 2yz^3, xz + 3y^2z^2\rangle$.

(d) If $\mathbf{G}$ and $\mathbf{H}$ are to be gradient vector fields, then there are functions g and h for which $\mathbf{G} = \nabla g$ and $\mathbf{H} = \nabla h$. What would this tell us about the partial derivatives g_x, g_y, g_z, h_x, h_y, and h_z?

(e) Find a function g so that $\partial g/\partial x = 3e^{y^2} + z\sin(x)$. Find a function h so that $\partial h/\partial x = 3x^2y$.

(f) When finding the most general antiderivative for a function of one variable, we add a constant of integration (usually denoted by C) to capture the fact that any constant will vanish through differentiation. When taking the partial derivative with respect to x of a function of x, y, and z, what variables can appear in terms that vanish because they are treated as constants? What does this tell you should be added to g and h in the previous part to make them the most general possible functions with the desired partial derivatives with respect to x?

(g) Now calculate $\partial g/\partial y$ and $\partial h/\partial y$. Explain why this tells you that we must have

$$g(x, y, z) = 3xe^{y^2} - z\cos(x) - yz + m_1(z)$$

and

$$h(x, y, z) = x^3y + y^2z^3 + m_2(z)$$

for some functions m_1 and m_2 depending only on z.

(h) Calculate $\partial g/\partial z$ and $\partial h/\partial z$ for the functions in the part above. Notice that m_1 and m_2 are functions of z alone, so taking a partial derivative with respect to z is the same as taking an ordinary derivative, and thus you may use the notation $m_1'(z)$ and $m_2'(z)$.

(i) Explain why $\mathbf{G}$ is a gradient vector field but $\mathbf{H}$ is not a gradient vector field. Find a potential function for $\mathbf{G}$.

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Path-Independent Vector Fields

We say that a vector field $\mathbf{F}$ defined on a domain D is **path-independent** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ whenever C_1 and C_2 are oriented paths in D having the same initial point and same terminal point.

We've already encountered some situations where we had evidence that a vector field was path-independent, but since the definition requires that for every pair of points in the domain and every possible pair of paths from oine point to the other we must get the same value, it doesn't appear that verifying a vector field is path-independent is an easy task. Fortunately, one familiar class of vector fields can be shown to be path-independent. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a function for which ∇f is continuous on a domain D. Suppose that P and Q are points in D and let C be a smooth oriented path from P to Q. Let's take a look at $\int_C \nabla f \cdot d\mathbf{r}$ by fixing an arbitrary parameterization $\mathbf{r}(t)$ of C, $a \le t \le b$. Since $\nabla f(\mathbf{r}(t)) = \langle f_x(\mathbf{r}(t)), f_y(\mathbf{r}(t)), f_z(\mathbf{r}(t))\rangle$, we know that

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \langle f_x(\mathbf{r}(t)), f_y(\mathbf{r}(t)), f_z(\mathbf{r}(t))\rangle \cdot \mathbf{r}'(t)\, dt.$$

If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then the integrand above is

$$\langle f_x(\mathbf{r}(t)), f_y(\mathbf{r}(t)), f_z(\mathbf{r}(t))\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle = \\ f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t).$$

Notice that this is exactly what the chain rule tells us $\frac{d}{dt}f(\mathbf{r}(t))$ is equal to. Therefore,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \frac{d}{dt}f(\mathbf{r}(t))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(Q) - f(P).$$

In other words, gradient vector fields are path-independent vector fields, and we can evaluate line integrals of gradient vector fields by using a potential function. (Technically the argument above assumed that C was smooth, but we can replace C by a piecewise smooth curve by splitting the line integral up into the sum of finitely many line integrals along smooth curves.)

This result is so important that it is frequently called the Fundamental Theorem of Calculus for Line integrals, because of its similarity to the Fundamental Theorem of Calculus, which can be written as

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Fundamental Theorem of Calculus for Line Integrals Let f be a function for which ∇f is continuous on a domain D. If P and Q are points in D and C is a piece-wise smooth oriented path from P to Q in D, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(Q) - f(P).$$

Activity 12.10.

Calculate each of the following line integrals.

(a) $\int_C \nabla f \cdot d\mathbf{r}$ if $f(x, y) = 3xy^2 - \sin(x) + e^y$ and C is the top half of the unit circle oriented from $(-1, 0)$ to $(1, 0)$.

(b) $\int_C \nabla g \cdot d\mathbf{r}$ if $g(x, y, z) = xz^2 - 5y^3\cos(z) + 6$ and C is the portion of the helix $\mathbf{r}(t) = \langle 5\cos(t), 5\sin(t), 3t\rangle$ from $(5, 0, 0)$ to $(0, 5, 9\pi/2)$.

(c) $\int_C \nabla h \cdot d\mathbf{r}$ if $h(x, y, z) = 3y^2e^{y^3} - 5x\sin(x^3z) + z^2$ and C is the curve consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 1)$, followed by the line segment from $(1, 1, 1)$ to $(-1, 3, -2)$, followed by the line segment from $(-1, 3, -2)$ to $(0, 0, 10)$.

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By following the methodology laid out in Preview Activity 9.4 to show that a given vector field is a gradient vector field, the Fundamental Theorem of Calculus for Line Integrals makes evaluating a large number of line integrals simpler now.

Activity 12.11.

Calculate each of the following line integrals.

(a) $\int_C \nabla f \cdot d\mathbf{r}$ if $f(x,y) = 3xy^2 - \sin(x) + e^y$ and C is the top half of the unit circle oriented from $(-1,0)$ to $(1,0)$.

(b) $\int_C \nabla g \cdot d\mathbf{r}$ if $g(x,y,z) = xz^2 - 5y^3\cos(z) + 6$ and C is the portion of the helix $\mathbf{r}(t) = \langle 5\cos(t), 5\sin(t), 3t\rangle$ from $(5,0,0)$ to $(0,5,9\pi/2)$.

(c) $\int_C \nabla h \cdot d\mathbf{r}$ if $h(x,y,z) = 3y^2e^{y^3} - 5x\sin(x^3z) + z^2$ and C is the curve consisting of the line segment from $(0,0,0)$ to $(1,1,1)$, followed by the line segment from $(1,1,1)$ to $(-1,3,-2)$, followed by the line segment from $(-1,3,-2)$ to $(0,0,10)$.

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Activity 12.12.

Calculate each of the following line integrals.

(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F}(x,y) = \langle 2x, 2y\rangle$ and C is the line segment from $(1,2)$ to $(-1,0)$.

(b) $\int_C \mathbf{G} \cdot d\mathbf{r}$ if $\mathbf{G}(x,y) = \langle 4x^3 - 12y\cos(xy), 9y^2 - 12x\cos(xy)\rangle$ and C is the portion of the unit circle from $(0,-1)$ to $(0,1)$.

(c) $\int_C \mathbf{H} \cdot d\mathbf{r}$ if $\mathbf{H}(x,y,z) = \langle H_1, H_2, H_3\rangle$ with

$$\begin{aligned}
H_1(x,y,z) &= e^{z^2} + 2xy^3z + \cos(x) - y^3\sin(x)\\
H_2(x,y,z) &= 2ye^{y^2} + 3x^2y^2z + 3y^2z^2 + 3y^2\cos(x)\\
H_3(x,y,z) &= x^2y^3 + 2xze^{z^2} + 2y^3z - 4z^3
\end{aligned}$$

and C is the curve consisting of the line segment from $(1,1,1)$ to $(3,0,3)$, followed by the line segment from $(3,0,3)$ to $(1,5,-1)$, followed by the line segment from $(1,5,-1)$ to $(0,0,0)$.

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Line Integrals Along Closed Curves

Recall that we call an oriented curve C **closed** if it has the same initial and terminal point. A typical example of a closed curve would be a circle (with orientation), but we could also consider something like the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$, oriented clockwise (or counterclockwise).

Activity 12.13.

Suppose that $\mathbf{F}$ is a continuous path-independent vector field (in $\mathbb{R}^2$ or $\mathbb{R}^3$) on some domain D.

(a) Let P and Q be points in D and let C_1 and C_2 be oriented curves from P to Q. What can you say about $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

(b) Let $C = C_1 - C_2$. Explain why C is a closed curve.

(c) Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$. (Recall that we sometimes use the symbol $\oint$ for a line integral when the curve is closed.)

(d) What does the previous part show must be the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for any closed curve C and continuous path-independent vector field $\mathbf{F}$?

From the first part of this activity, you now now that the line integral around any closed curve in a path-independent vector field is 0. What can we say about the converse? That is, suppose that $\mathbf{F}$ is a continuous vector field on a domain D for which $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C.

(e) Pick two points P and Q in D. Let C_1 and C_2 be oriented curves from P to Q. What type of curve is $C = C_1 - C_2$?

(f) What is $\oint_C \mathbf{F} \cdot d\mathbf{r}$? Why?

(g) What does that tell you about the relationship between $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

(h) Explain why this shows that $\mathbf{F}$ is path-independent.

◁

From Activity 9.13, we now know that $\mathbf{F}$ is path-independent if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C in the domain of $\mathbf{F}$. Although this is not a terribly useful way to show that a vector field is path-independent, it can be a useful way to show that a vector field is *not* path-independent: find a closed curve around which the circulation is not zero.

Activity 12.14.

Explain why neither of the vector fields in Figure 9.15 is path-independent.

◁

What other vector fields are path-independent?

Recall that in single variable calculus, the Second Fundamental Theorem of Calculus tells us that given a constant c and a continuous function f, there is a unique function $A(x)$ for which $A(c) = 0$ and $A'(x) = f(x)$. In particular, $A(x) = \int_c^x f(t)\,dt$ is this function. We are about to investigate an analog of this result for path-independent vector fields, but first we require two additional definitions.

If D is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, we say that D is **open** provided that for every point in D, there is a disc (in $\mathbb{R}^2$) or ball (in $\mathbb{R}^3$) centered at that point such that every point of the disc/ball is contained in D. For example, the set of points (x, y) in $\mathbb{R}^2$ for which $x^2 + y^2 < 1$ is open, since we can always surround any point in this set by a tiny disc contained in the set. However, if we change the inequality to $x^2 + y^2 \leq 1$, then the set is not open, as any point on the circle $x^2 + y^2 = 1$ cannot be surrounded by a disc contained in the set; any disc surrounding a point on that circle will contain

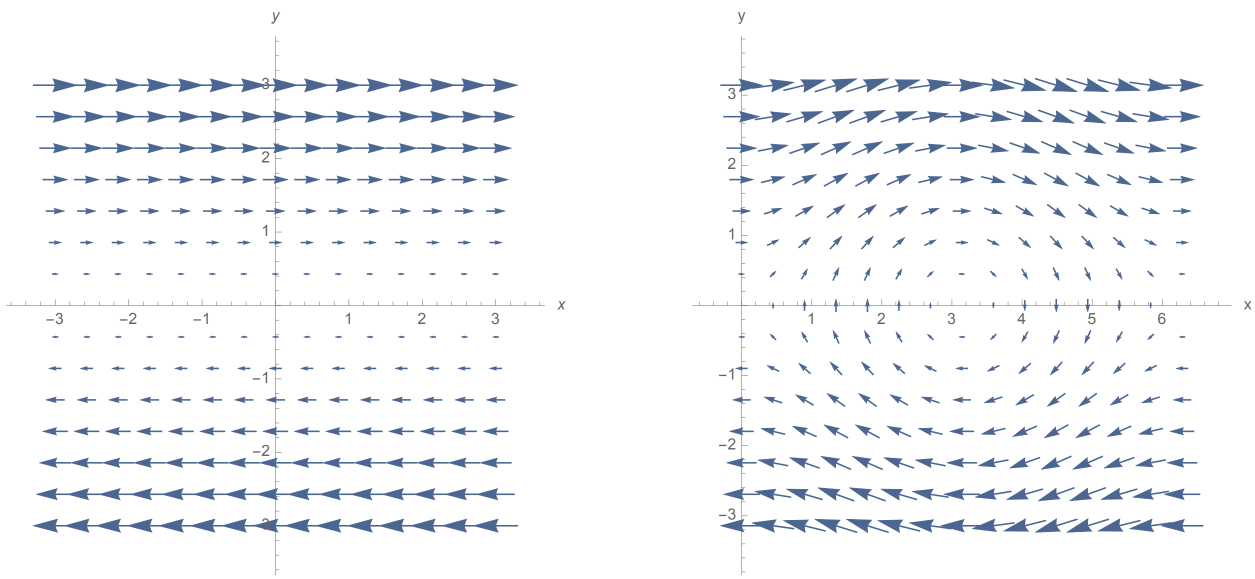

Figure 12.15: Two vector fields that are not path-independent.

points outside the set, that is with $x^2 + y^2 > 1$. We will also say that a region D is **connected** provided that for every pair of points in D, there is a path from one to the other contained in D.

Activity 12.15.

Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a continuous, path-independent vector field on an open, connected region D. We will assume that D is in $\mathbb{R}^2$ and $\mathbf{F}$ is a two-dimensional vector field, but the ideas below generalize completely to $\mathbb{R}^3$. We want to define a function f on D by using the vector field $\mathbf{F}$ and line integrals, much like the Second Fundamental Theorem of Calculus allows us to define an antiderivative of a continuous function using a definite integral. To that end, we assign $f(x_0, y_0)$ an arbitrary value. (Setting $f(x_0, y_0) = 0$ is probably convenient, but we won't explicitly tie our hands. Just assume that $f(x_0, y_0)$ is defined to be some number.) Now for any other point (x, y) in D, define

$$f(x, y) = f(x_0, y_0) + \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any oriented path from (x_0, y_0) to (x, y). Since D is connected, such an oriented path must exist. Since $\mathbf{F}$ is path-independent, f is well-defined. If different paths from (x_0, y_0) to (x, y) gave different values for the line integral, then we'd not be sure what $f(x, y)$ really is.

To better understand this mysterious function f we've now defined, let's start looking at its partial derivatives.

(a) Since D is open, there is a disc (perhaps very small) surrounding (x, y) that is contained in D, so fix a point (a, b) in that disc. Since D is connected, there is a path C_1 from (x_0, y_0) to (a, b). Let C_y be the line segment from (a, b) to (a, y) and let C_x be the line

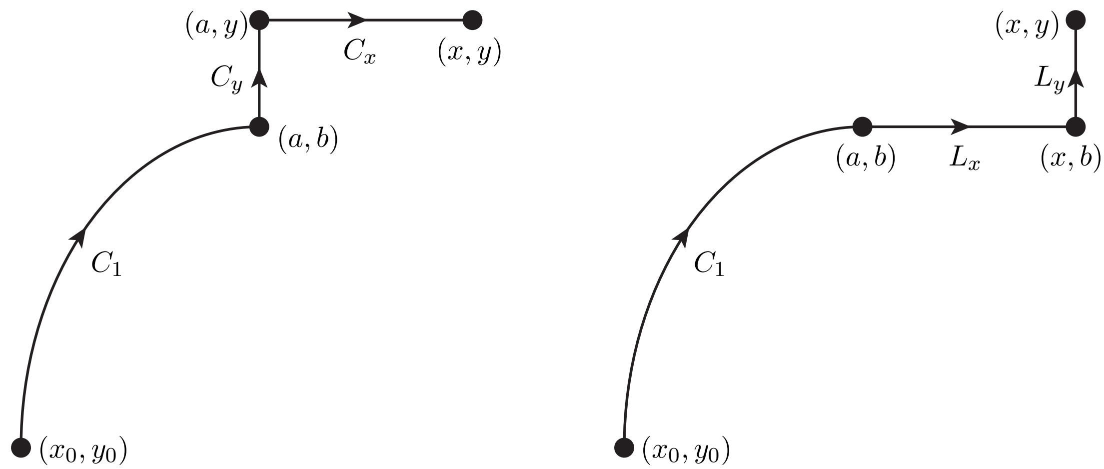

Figure 12.16: Two piecewise smooth oriented curves from (x_0, y_0) to (x, y).

segment from (a, y) to (x, y). (See Figure 9.16.) Rewrite $f(x, y)$ as a sum of $f(x_0, y_0)$ and line integrals along C_1, C_y, and C_x.

(b) Notice that we can parameterize C_y by $\langle a, t\rangle$ for $b \le t \le y$. Find a similar parameterization for C_x.

(c) Use the parameterizations from above to write $\int_{C_y} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_x} \mathbf{F} \cdot d\mathbf{r}$ as single variable integrals in the manner of Section 9.3. (Recall that $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$, enabling you to express your integrals in terms of F_1 and F_2 without any dot products.)

(d) Rewrite your expression for $f(x, y)$ using the single variable integrals above (and a line integral along C_1).

(e) Use your expression for $f(x, y)$ and the Second Fundamental Theorem of Calculus to calculate $f_x(x, y)$.

(f) To calculate $f_y(x, y)$, we continue to consider a path C_1 from (x_0, y_0) to (a, b), but now let L_x be the line segment from (a, b) to (x, b) and let L_y be the line segment from (x, b) to (y, b). Modify the process you used to find $f_x(x, y)$ to find $f_y(x, y)$.

(g) What can you conclude about the relationship between ∇f and $\mathbf{F}$?

◁

We summarize the result of Activity 9.15 below.

Path-Independent Vector Fields If $\mathbf{F}$ is a path-independent vector field on an open, connected domain D, then $\mathbf{F}$ is conservative (or a gradient vector field) on D. Furthermore, if P is a point in D and $f(P)$ is fixed, then the function

$$f(Q) = f(P) + \int_C \mathbf{F} \cdot d\mathbf{r}$$

(where Q is a point in D and C is an oriented curve from P to Q in D) is a potential function for $\mathbf{F}$.

Summary

- Gradient vector fields are path-independent, and if C is an oriented curve from (x_1, y_1) to (x_2, y_2), then $\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$, with the analogous result holding if f is a function of three variables.
- A vector field is path-independent if and only if the ciruclation around every closed curve in its domain is 0.
- If a vector field $\mathbf{F}$ is path-independent, then there exists a function f such that $\nabla f = \mathbf{F}$. That is, $\mathbf{F}$ is a conservative or gradient vector field.

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