Problem Set 1

1) a) Show $\Gamma^1_{10} = \Gamma^1_{01} = \frac{\dot{a}}{a}$ and $\Gamma^0_{11} = \frac{a\dot{a}}{c^2(1-Kr^2)}$ for $\underbrace{\text{FLRW-metric}}$

$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$

Proof: $\Gamma^j_{ik} = \frac{1}{2} g^{jl} \cdot \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right)$

with $i,j,k,l = t, r, \theta, \phi$

$\Gamma^1_{10} = \Gamma^1_{01}$ because of Symmetry (1.13)

$\Gamma^1_{01} = \frac{1}{2} g^{1l} \cdot \left(\frac{\partial g_{0l}}{\partial x^1} + \frac{\partial g_{1l}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^l}\right)$

with $g^{ab} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \cdot \begin{pmatrix} dt \\ dr \cdot a^2 \\ ds \cdot a^2 \\ d\phi \cdot a^2 \end{pmatrix}$

$= \frac{1}{2} \underbrace{g^{10}}_{0} \cdot \left(\frac{\partial g_{00}}{\partial t} \dots\right) + \frac{1}{2} g^{11} \cdot \left(\frac{\partial g_{01}}{\partial t} \dots\right) + \frac{1}{2} \underbrace{g^{12}}_{0} \cdot \left(\frac{\partial g_{02}}{\partial t} \dots\right) + \frac{1}{2} \underbrace{g^{13}}_{0} \cdot (\dots$

$= \frac{1}{2} \cdot g^{11} \cdot \left(\underbrace{\frac{\partial g_{01}}{\partial r}}_{0} + \frac{\partial g_{11}}{\partial t} - \underbrace{\frac{\partial g_{01}}{\partial r}}_{0}\right)$

$= \frac{1}{2} \cdot \underbrace{g^{11}}_{g^{ab} = \frac{1}{g_{ab}}} \cdot \left(\frac{\partial \frac{a^2(t)}{1-kr^2}}{\partial t}\right)$

weil Diagonale $= 0 \rightarrow \frac{1}{g_{ab}}$ ist inverses

$$= \frac{1}{2} \cdot \frac{(1-kr^2)}{a^2} \cdot \frac{2a \cdot \dot{a}}{(1-kr^2)}$$

$$= \frac{\dot{a}}{a} \quad \text{Slay!}$$

$$\Gamma^0_{11} = \frac{1}{2} g^{0l} \cdot \left(\frac{\partial g_{1l}}{\partial x^1} + \frac{\partial g_{1l}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^l} \right)$$

$$= \frac{1}{2} \cdot \left(g^{00} \dots + \underbrace{g^{01} \dots + g^{02} \dots + g^{03}}_{0} \right)$$

$$= \frac{1}{2} \cdot g^{00} \cdot \left(\underbrace{\frac{\partial g_{10}}{\partial x^1} + \frac{\partial g_{10}}{\partial x^1}}_{0} - \frac{\partial g_{11}}{\partial x^0} \right)$$

$$= \frac{1}{2} g^{00} \cdot \frac{-\partial g_{11}}{\partial t}$$

$$= \frac{1}{2} \cdot \underbrace{g^{00}}_{-\frac{1}{c^2}} \cdot \frac{-2a \cdot \dot{a}}{1-kr^2}$$

$$= \frac{a \cdot \dot{a}}{c^2 \cdot (1-kr^2)} \quad \text{Slay!}$$

b) Show $R_{00} = -3 \frac{\ddot{a}}{a}$

$R_{ij} = R^k_{ikj} \rightarrow$ Trace of Riemann-Tensor R^i_{jkl}

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{mk} \Gamma^m_{jl} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{ml} \Gamma^m_{jk}$$

$$\Gamma^0_{11} = \frac{a\dot{a}}{c^2(1-\kappa r^2)} \quad \Gamma^0_{22} = \frac{1}{c^2}a\dot{a}r^2 \quad \Gamma^0_{33} = \frac{1}{c^2}a\dot{a}r^2\sin^2\theta$$

$$\Gamma^1_{10} = \Gamma^1_{01} = \frac{\dot{a}}{a} \quad \Gamma^1_{11} = \frac{kr}{1-\kappa r^2} \quad \Gamma^1_{22} = -r(1-\kappa r^2) \quad \Gamma^1_{33} = -r(1-\kappa r^2)\sin^2\theta$$

$$\Gamma^2_{20} = \Gamma^2_{02} = \frac{\dot{a}}{a} \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r} \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{a}}{a} \quad \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r} \quad \Gamma^3_{32} = \Gamma^3_{23} = \cot\theta.$$

I hope there is no easier way to do this, which I am overlooking :(

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{mk}\Gamma^m_{jl} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{ml}\Gamma^m_{jk}$$

$j = l = 0$ for $i = k = \text{range}(0,4)$ for $m = \text{range}(0,4)$

$$= R^0_{000} + R^1_{010} + R^2_{020} + R^3_{030}$$

$$= \frac{\partial \Gamma^i_{00}}{\partial x^k} + \Gamma^i_{mk}\Gamma^m_{00} - \frac{\partial \Gamma^i_{0k}}{\partial x^0} - \Gamma^i_{m0}\Gamma^m_{0k}$$

$$= \underbrace{\frac{\partial \Gamma^0_{00}}{\partial x^0} + \frac{\partial \Gamma^1_{00}}{\partial x^1} + \frac{\partial \Gamma^2_{00}}{\partial x^2} + \frac{\partial \Gamma^3}{\partial x^3}}_{0} + \dots$$

$$= 0 + \Gamma^0_{m0}\Gamma^m_{00} - \dots \quad // \; i = k = 0$$

$$= 0 + \underbrace{\Gamma^0_{00}\Gamma^0_{00} + \Gamma^0_{10}\Gamma^1_{00} + \Gamma^0_{20}\Gamma^2_{00} + \Gamma^0_{30}\Gamma^3_{00}}_{0} - \dots$$

$$= 0 + \Gamma^1_{m1}\Gamma^m_{00} - \dots \quad // \; i = k = 1$$

$$= 0 + \Gamma^1_{01}\Gamma^0_{00} + \dots \quad // \; m = 0$$

$$\frac{\dot a}{a}\cdot 0 \rightarrow \text{Since } \Gamma^m_{00} = 0\text{, this term cancels out!}$$

$$= 0 + 0 - \frac{\partial \Gamma^i_{0k}}{\underbrace{\partial x^0}_{\partial t}} - \ldots$$

$$= 0 + 0 - \left(\frac{\partial \Gamma^1_{01}}{\partial t} + \frac{\partial \Gamma^2_{02}}{\partial t} + \frac{\partial \Gamma^3_{03}}{\partial t}\right) - \ldots \quad // \Gamma^0_{00} = 0$$

$$= 0 + 0 - \underbrace{\frac{\partial}{\partial t}\left(3\frac{\dot a}{a}\right)}_{3\frac{\ddot a \cdot a - \dot a^2}{a^2}} - \Gamma^i_{m0}\Gamma^m_{0k} \quad // \Gamma^0_{m0}\Gamma^m_{00} = \Gamma^1_{00}\Gamma^0_{01} = 0$$

$$= 0 + 0 - 3\cdot\frac{\ddot a \cdot a - \dot a^2}{a^2} - \underbrace{\left(\underbrace{\Gamma^1_{10}\Gamma^1_{01}}_{\frac{\dot a}{a}\cdot\frac{\dot a}{a}} + \Gamma^2_{20}\Gamma^2_{02} + \ldots\right)}_{3\left(\frac{\dot a}{a}\right)^2}$$

$$= 0 + 0 - 3\cdot\left(\frac{\ddot a \cdot a - \dot a^2}{a^2} + \frac{\dot a^2}{a^2}\right)$$

$$= -3\cdot\left(\frac{\ddot a \cdot a - \dot a^2 + \dot a^2}{a^2}\right)$$

$$= -3\cdot\frac{\ddot a}{a}$$

Slay!

c) Show that $G^0_0 = -3\left(\frac{1}{c^2}\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2}\right)$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$$

$$R_{00} = -3\frac{\ddot a}{a},\quad R = 6\left(\frac{1}{c^2}\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2}\right) + \frac{k}{a^2}\right)$$

with $g^{ab} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \cdot \begin{pmatrix} dt \\ dr \cdot a^2 \\ ds \cdot a^2 \\ d\varphi \cdot a^2 \end{pmatrix}$

$\hookrightarrow g_{00} = -\frac{1}{c^2}$

$\rightarrow$ Index heben mit Metrik

$$G^j{}_v = g^{ij} G_{iv}$$

$$G_{uv} = R_{uv} - \frac{1}{2} R g_{uv}$$

$$G^j{}_v = g^{00} R_{00} - \frac{1}{2} R g^{00} g_{00}$$

$$= \frac{-1}{c^2}\left(\frac{-3\ddot{a}}{a}\right) - \frac{1}{2} \cdot \left(6 \cdot \left(\frac{1}{c^2} \cdot \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{k}{a^2}\right) \cdot \frac{-1}{c^2} \cdot -c^2\right.$$

$$= 3\frac{\ddot{a}}{c^2 a} - 3 \cdot \left(\frac{\ddot{a}}{c^2 a} + \frac{\dot{a}^2}{c^2 a^2} + \frac{k}{a^2}\right)$$

$$= -3 \cdot \left(\frac{1}{c^2} \cdot \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)$$ Slay!

d) Show H^2 follows from $G^0{}_0, T^v{}_v$

$$G_{\mu v} + \Lambda g_{uv} = \frac{8\pi G}{c^4} T_{uv}$$ // 1.44, only equation I found to contain everything

$$\hookrightarrow G_{00} + \Lambda g_{00} = \frac{8\pi G}{c^4} T_{00}$$

$$= R_{00} - \frac{1}{2} R g_{00} + \Lambda g_{00} = \frac{8\pi G}{c^4} \cdot \rho$$

$$= -3\frac{\ddot{a}}{a} - \frac{1}{2} \cdot \left(6\left(\frac{1}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{k}{a^2}\right)\right) \cdot -\frac{1}{c^2} - \frac{\Lambda}{c^2} = \frac{8\pi G}{c^4} \rho$$

$$= -3\frac{\ddot{a} \cdot c^2}{a} - \left(3\frac{\ddot{a}}{c^2 a} + 3\frac{\dot{a}^2}{c^2 a^2} + 3\frac{k}{a^2}\right) - \Lambda = \frac{8\pi G}{c^4} \rho \cdot c^2$$

$$= -3\left(\frac{\ddot{a}}{a}c^2 + \frac{\dot{a}}{c^2 a} + \frac{\dot{a}^2}{c^2 a^2} + \frac{k}{a^2}\right) = \frac{8\pi G}{c^2}\rho + \Lambda \qquad // \cdot c^2 \,,\; :3 \,,\; -\frac{k}{a^2}$$

$$\underbrace{-\frac{\ddot{a}c^4}{a} + \frac{\ddot{a}}{a}}_{\text{Cancels out for } c=1\,?} + \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

2) starts on the next page :)

2) $H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda$

where $\Omega_k = 1 - \Omega_r - \Omega_m - \Omega_\Lambda$

a) Find equalities

$H_0 = 67{,}74$
$\Omega_m = 0.3089 \pm 0.0062$
$\Omega_\Lambda = 0.6911 \pm 0.062$
$\Omega_k = 0$:)
$\Omega_r = 8{,}4 \cdot 10^{-5}$

(1.72)

Equality Λ, m: $\frac{\Omega_m}{a^3 \Omega_\Lambda} \overset{!}{=} 1$

$\hookrightarrow a^3 = \frac{\Omega_m}{\Omega_\Lambda}$

$a = \sqrt[3]{\frac{0.3089}{0.6911}}$

Unsicherheiten sind für Experimentalphysiker

$a(t) = 0.7645$

$\rightarrow t \approx 13{,}8 \cdot 0{,}7645 = 10.5501$ Gj

Equality m, r: $a = \frac{\Omega_r}{\Omega_m} = \frac{8.4 \cdot 10^{-5}}{0.3089} = 0.00027$

$\rightarrow t \approx 0{,}0037$ Gj

Discussion: For a rather short time, the universe was dominated by radiation. For the next ~ 10,5 billion years, the universe is in the matter dominated era..

? Aren't the constants dependent on each other ?

b) Find the solutions for $\Omega_i = \Omega_i, \Omega_j = 0$

Radiation: $H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \Omega_K a^{-4}$

$\hookrightarrow$

$$\dot{a}^2 = \Omega_K a^{-2}$$

Trennen der Variablen:

$$\left(\frac{da}{dt}\right)^2 = \Omega_K a^{-2}$$

$$\frac{da}{dt} \cdot a = \sqrt{\Omega_K}$$

$$a\,da = \sqrt{\Omega_K}\,dt$$

$$\frac{a^2}{2} \quad \sqrt{\Omega_K}\,t + C$$

$$a(t) = (2(t\sqrt{\Omega_K} + C)^{\frac{1}{2}}$$

(for whatever value Ω_K would have for $\Omega_j = 0$)

Matter: $\left(\frac{da}{dt}\right)^2 = \Omega_m a^{-1}$

$$\frac{da}{dt}\sqrt{a} = \sqrt{\Omega_m}$$

$$\frac{2}{3} a^{3/2} = \sqrt{\Omega_m}\,t + C$$

$$a(t) = \left(\frac{3}{2} \cdot (\sqrt{\Omega_m}\,t + C\right)^{\frac{2}{3}}$$

(for whatever value Ω_m would have for $\Omega_j = 0$)

Λ: $\left(\frac{\dot{a}}{a}\right)^2 = \Omega_\Lambda$

$$\left(\frac{da}{dt}\right)^2 = \Omega_\Lambda a^2$$

$$\frac{da}{dt} = \sqrt{\Omega_\Lambda} \cdot a$$

$$\frac{da}{a} = \sqrt{\Omega_\Lambda}\,dt$$

A certified Hood-classic:

$$a = e^{\sqrt{\Omega_\Lambda}\,t + C}$$

Curvature: $\dot{a}^2 = \Omega_k$

$$da = \sqrt{\Omega_k}\, dt$$

$$a = \sqrt{\Omega_k}\, t + C$$

c) Provide these expressions in the limiting cases when $\Omega_m = 0.3, 1, 1.3$ and plot them, using e.g. Python

So I can use something "different" ☺

Verify that $a(\eta)$ for $t(\eta)$ works for $(H)^2$

$$a(\eta) = \frac{\Omega_m}{2\cdot(1-\Omega_m)} \cdot \cosh(\sqrt{1-\Omega_m}\,\eta) - 1$$

$$t(\eta) = \frac{\Omega_m}{2\cdot(1-\Omega_m)} \cdot \frac{\sinh(\sqrt{1-\Omega_m}\,\eta)}{\sqrt{1-\Omega_m}} - \eta$$

This took me way too long:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2}$$

$$\rightarrow \dot{a}^2 = \frac{\Omega_m}{a} + \Omega_k$$

$$= \left(\frac{da(\eta)}{d(t(\eta))}\right)^2 = \left(\frac{da}{dt}\,\frac{dt}{d\eta}\right)^2 = \left(\frac{da}{dt}\cdot\left(\frac{d\eta}{dt}\right)^{-1}\right)^2$$

$$= \frac{\cancel{\frac{\Omega_m}{2\cdot(1-\Omega_m)}}}{\cancel{\frac{\Omega_m}{2\cdot(1-\Omega_m)}}} \cdot \frac{\left(\sqrt{1-\Omega_m}\cdot \sinh(\sqrt{1-\Omega_m}\,\eta)\cdot 1\right)^2}{\left((\cosh(\sqrt{1-\Omega_m}\,\eta)\cdot 1)-1\right)^2}$$

$$= \frac{(1-\Omega_m)\cdot\sinh^2(\sqrt{1-\Omega_m}\,\eta)}{\cosh^2(\sqrt{1-\Omega_m}\,\eta) - 2\cdot\cosh(\sqrt{1-\Omega_m}\,\eta) + 1} \qquad // \cosh^2(x) - \sinh^2(x) = 1$$

$$= \frac{(1-\Omega_m)\cdot\sinh^2(\sqrt{1-\Omega_m}\,\eta)}{\sinh^2(\sqrt{1-\Omega_m}\,\eta) - 2\cdot\cosh(\sqrt{1-\Omega_m}\,\eta)}$$

From equation (1), we can tell that

$$a = \frac{\Omega_m}{2\cdot(1-\Omega_m)}\cdot\left(\cosh(\sqrt{1-\Omega_m}\,\eta) - 1\right)$$

Therefore:

$$H^2 = \frac{(1-\Omega_m)\cdot\sinh^2(\sqrt{1-\Omega_m}\,\eta)}{\sinh^2(\sqrt{1-\Omega_m}\,\eta) - 2\cdot\cosh(\sqrt{1-\Omega_m}\,\eta)} = \frac{\Omega_m}{\left(\frac{\Omega_m}{2(1-\Omega_m)}\cdot\left(\cosh(\sqrt{1-\Omega_m}\,\eta) - 1\right)\right)} + \Omega_k$$

$$= \frac{2(1-\Omega_m)}{\cosh(\sqrt{1-\Omega_m}\,\eta) - 1} + (1-\Omega_m)$$

Divide by $(1-\Omega_m)$:

$$\rightarrow \frac{\sinh^2(\sqrt{1-\Omega_m}\,\eta)}{\sinh^2(\sqrt{1-\Omega_m}\,\eta) - 2\cdot\cosh(\sqrt{1-\Omega_m}\,\eta)} = \frac{2}{\cosh(\sqrt{1-\Omega_m} - 1)} + 1$$

$$= \frac{\sinh^2(\sqrt{1-\Omega_m}\,\eta)}{\left(\cosh(\sqrt{1-\Omega_m}\,\eta - 1)\right)^2} = \frac{2}{\cosh(\sqrt{1-\Omega_m} - 1)} + 1$$

$$\rightarrow \sinh^2\left(\sqrt{1-\Omega_m}\,\eta\right) = 2\cdot\cosh\left(\sqrt{1-\Omega_m}\,\eta\right)-1) + \left(\cosh\left(\sqrt{1-\Omega_m}\,\eta\right)-1\right)^2$$

$$\rightarrow \sinh^2\left(\sqrt{1-\Omega_m}\,\eta\right) = \cosh^2\left(\sqrt{1-\Omega_m}\,\eta\right)-1$$

$$= \sinh^2(x) = \cosh^2(x)-1$$

Slay!

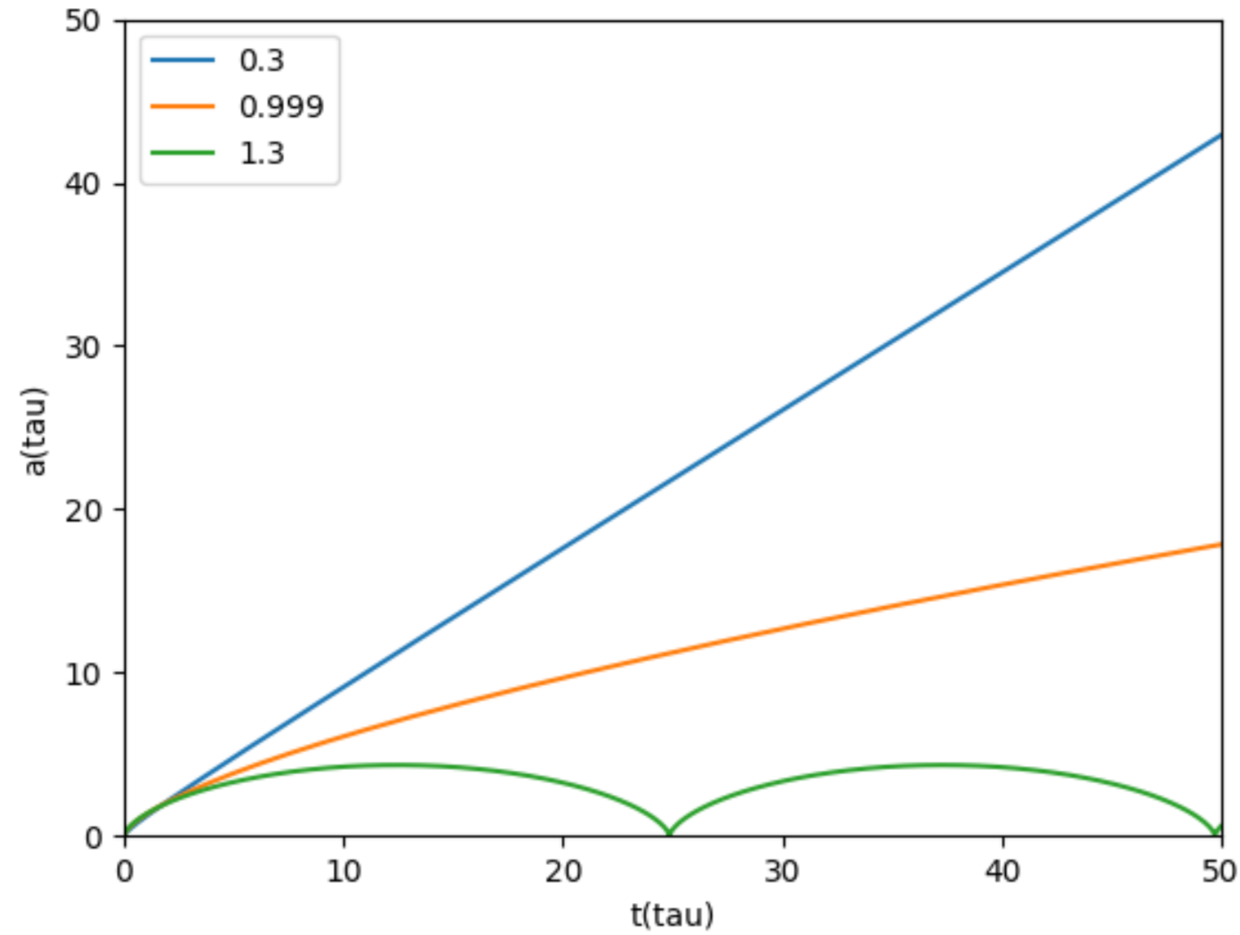
Originally, I wanted to use my own modified python-version:

https://github.com/actopozipc/German-Python-Interpreter

```
a = lambda omega_m, eta: (omega_m/(2*(1-omega_m)))*((np.cosh(np.emath.sqrt(1-omega_m)*eta)-1)) #emath needed for omega_m=1.3 >:(
t = lambda omega_m, eta: (omega_m/(2*(1-omega_m)))*((np.sinh(np.emath.sqrt(1-omega_m)*eta)/(np.emath.sqrt(1-omega_m))-eta))
a_values = []
t_values = []
für omega_m in [0.3,0.999,1.3]:
    liste(karte(lambda t_value: t_values.erweitere(karte(lambda a_value: np.real(t(a_value, t_value)), [omega_m])), np.arange(0,10,0.1)))
    liste(karte(lambda t_value: a_values.erweitere(karte(lambda a_value: np.real(a(a_value, t_value)), [omega_m])), np.arange(0,10,0.1)))
    plt.plot(t_values, a_values, label=f"{omega_m}")
    a_values.aufräumen()
    t_values.aufräumen()
```

Sadly, there seems to be a bug in my lambda implementation, so I will do it with default python:

```
a = lambda omega_m, eta: (omega_m/(2*(1-omega_m)))*((np.cosh(np.emath.sqrt(1-omega_m)*eta)-1)) #emath needed for omega_m=1.3 >:(
t = lambda omega_m, eta: (omega_m/(2*(1-omega_m)))*((np.sinh(np.emath.sqrt(1-omega_m)*eta)/(np.emath.sqrt(1-omega_m))-eta))
a_values = []
t_values = []
for omega_m in [0.3,0.999,1.3]:
    list(map(lambda t_value: a_values.extend(map(lambda a_value: np.real(a(a_value, t_value)), [omega_m])), np.arange(0,10,0.1)))
    list(map(lambda t_value: t_values.extend(map(lambda a_value: np.real(t(a_value, t_value)), [omega_m])), np.arange(0,10,0.1)))
    plt.plot(t_values, a_values, label=f"{omega_m}")
    a_values.clear()
    t_values.clear()
plt.xlabel("t(tau)")
plt.ylabel("a(tau)")
plt.show()
```



So, when does the big crunch happen?

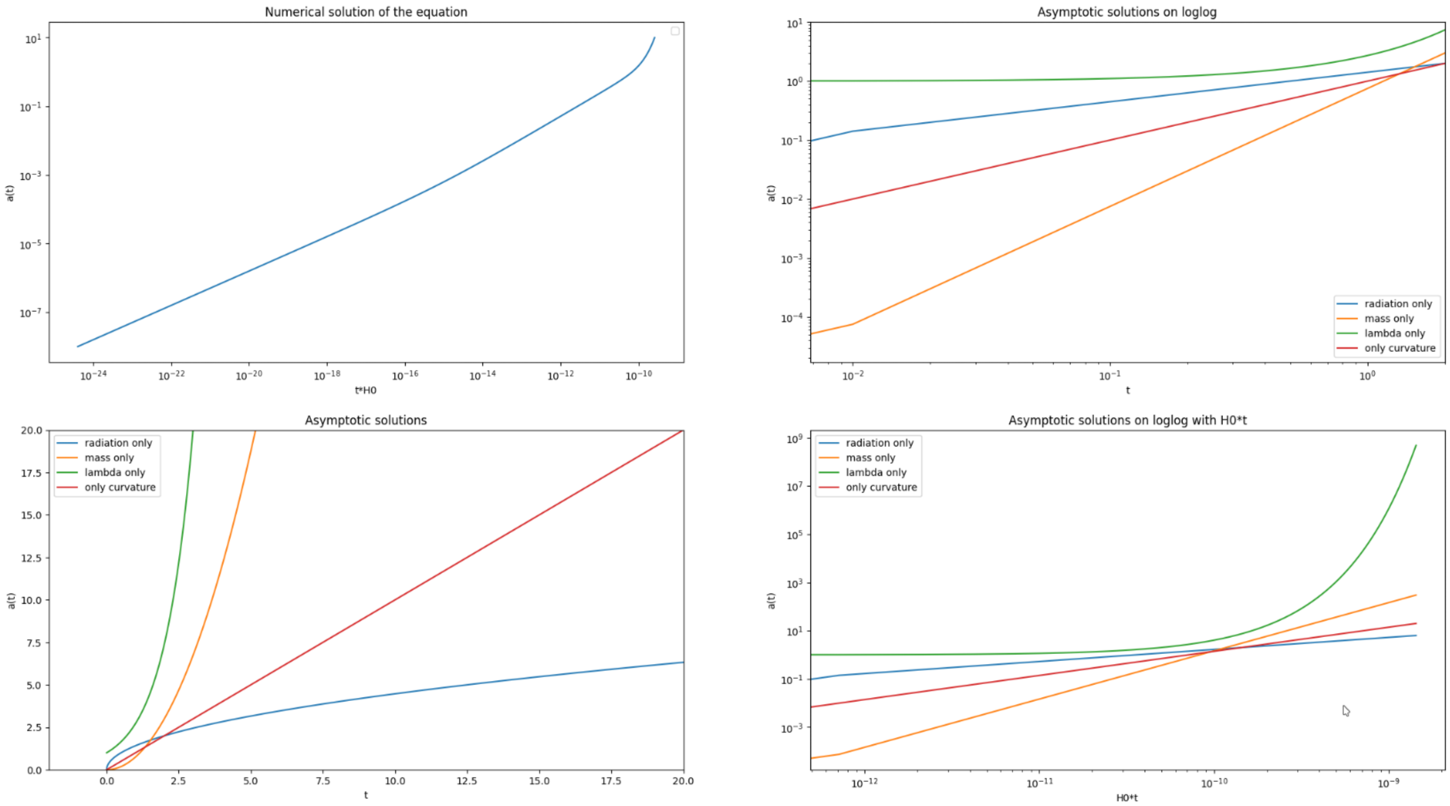
We can find out numerically:

```
big_crunch = lambda l1,l2: [l2[i] for i in range(len(l1)) if l1[i]<0.001]
```

And, as expected from the plot, it doesn't happen for $\Omega_m = 0.3, 1$ but is a returning event for $\Omega_m = 1.3$:

```
[0.0, 2.5008751458474787e-05] 0.3
[0.0] 1
[-0.0, 24.854863755913854, 49.709713900414435 1.3
```

d) Numerical solutions for given parameters



The first window shows the numerical solution.

I wasn't sure about the x-axis for the asymptotic solutions, so I made 3: t, $\log(t)$ and $\log(H0 \cdot t)$

```
names = ["radiation only", "mass only", "lambda only", "only curvature"]
for name_index, f in  enumerate([lambda t: np.sqrt(2*t), lambda t: (3/2 * t)**2/3, lambda t: np.exp(t), lambda t: t]):
    axs[1,0].plot(a,f(a), label=names[name_index])
    axs[0,1].loglog(a,f(a), label=names[name_index])
    axs[1,1].loglog(H0*a,f(a), label=names[name_index])
```

(I will upload the full code)