Contents

1 Abstract

In this report, we derive the analytical Jacobians for particle tracks (1) in the absence of electromagnetic fields and (2) in the presence of a constant B-field in z-direction. Our work is a follow-up to Sec. 5.3.3 and 5.3.4 of Ref. [1]: We (1) simplify the results obtained there by evaluating the Jacobians at the PCA and (2) add the time coordinate to the calculations.

2 General Notions

In the Perigee representation, a track is parametrized at its point of closest approach (PCA) P to the origin R of a reference coordinate system (see Fig. 3.1 and Fig. 4.1 for the definition of the points). The corresponding parameter vector $\mathbf{q}$ reads

$$\mathbf{q} \coloneqq \begin{pmatrix} d_0 \\ z_0 \\ \varphi_P \\ \theta_P \\ (q/p)_P \\ t_P \end{pmatrix}, \tag{2.1}$$

where

- d_0 is the signed distance between P and R in the x-y plane
- $z_0 = z_P - z_R$ is the z-distance between P and R
- $\varphi_P \in [-\pi, \pi)$ is the polar angle of the momentum at P
- $\theta_P \in (0, \pi)$ is the azimuthal angle of the momentum at P
- $(q/p)_P$ is the charge of the particle divided by the absolute value of its momentum at P
- t_P is the track time at P

The sign convention for d_0 requires special care. We have

$$d_0 \begin{cases} > 0 \text{ if } \exists\ n \in \mathbb{Z} \text{ s.t. } \varphi_0 - \varphi_P = \frac{\pi}{2} + 2\pi n \\ < 0 \text{ otherwise} \end{cases},$$

where $\varphi_0 \in [-\pi, \pi)$ is the polar angle of the vector pointing from R to P. Note that for linear tracks (no EM fields) this translates to

$$\operatorname{sgn}(d_0) = \operatorname{sgn}(y_R - y_P), \tag{2.2}$$

and for helical tracks (constant B-field in z-direction, $\mathbf{B} = B\ \hat{\mathbf{e}}_z$) we have

$$\operatorname{sgn}(d_0) = \operatorname{sgn}(B) \operatorname{sgn}(q) \operatorname{sgn}\left(\rho^2 - (\mathbf{r}_R - \mathbf{r}_O)^2\right), \tag{2.3}$$

where ρ is the helix radius.[1]

One can write the six parameters from Eq. 2.1 as a function of a 4D point on the track (point V in Fig. 3.1) and the corresponding momentum, e.g.:

$$d_0 = d_0(x_V, y_V, z_V, t_V, \varphi_V, \theta_V, (q/p)_V),$$

In the following, we will compute the Jacobian of the Perigee parameters in this representation, i.e.:

$$J := \left.\begin{pmatrix} \overbrace{\partial_{x_V} d_0 \quad \partial_{y_V} d_0 \quad \partial_{z_V} d_0 \quad \partial_{t_V} d_0}^{=:\, A} & \overbrace{\partial_{\varphi_V} d_0 \quad \partial_{\theta_V} d_0 \quad \partial_{(q/p)_V} d_0}^{=:\, B} \\ \partial_{x_V} z_0 \quad \ddots & \vdots \\ \partial_{x_V} \varphi & \vdots \\ \partial_{x_V} \theta & \vdots \\ \partial_{x_V} q/p & \ddots \quad \vdots \\ \partial_{x_V} t_P \quad \dots \quad \dots \quad \dots & \dots \quad \dots \quad \partial_{(q/p)_V} t_P \end{pmatrix}\right|_{V=P} \tag{2.4}$$

where we evaluate the Jacobian at the PCA P. We follow the literature convention and split the Jacobian into the submatrices A and B, which we call position and momentum Jacobian. Note that it is often useful to rewrite the derivative with respect to q/p like

$$\begin{aligned} \partial_{q/p} &= \partial_{q/p}\ p\ \partial_p \\ &= q\ \partial_{q/p} \left(\frac{q}{p}\right)^{-1} \partial_p \\ &= -q \left(\frac{q}{p}\right)^{-2} \partial_p \\ &= -\frac{p^2}{q}\partial_p, \end{aligned} \tag{2.5}$$

where we dropped the subscript for readability.

It is important to keep in mind that the Jacobian should only depend on the track parameters at the PCA. However, as we will see in the following, the terms involving time will often depend on the particle speed v, which cannot be extracted directly from $\mathbf{q}$. To obtain v nonetheless, we need to exploit a mass and a charge hypothesis:

$$v = c\beta = \frac{p}{\sqrt{p^2 + (cm_0)^2}},$$

where c is the speed of light and m_0 is the rest mass of the particle, which is fixed by the mass hypothesis. The momentum p can be determined from the track parameters using the charge hypothesis.

[1] $\operatorname{sgn}\left(\rho^2 - (\mathbf{r}_R - \mathbf{r}_O)^2\right)$ is (negative) positive if R is (outside) inside of the helix.

3 Track Linearization in the Absence of EM Fields

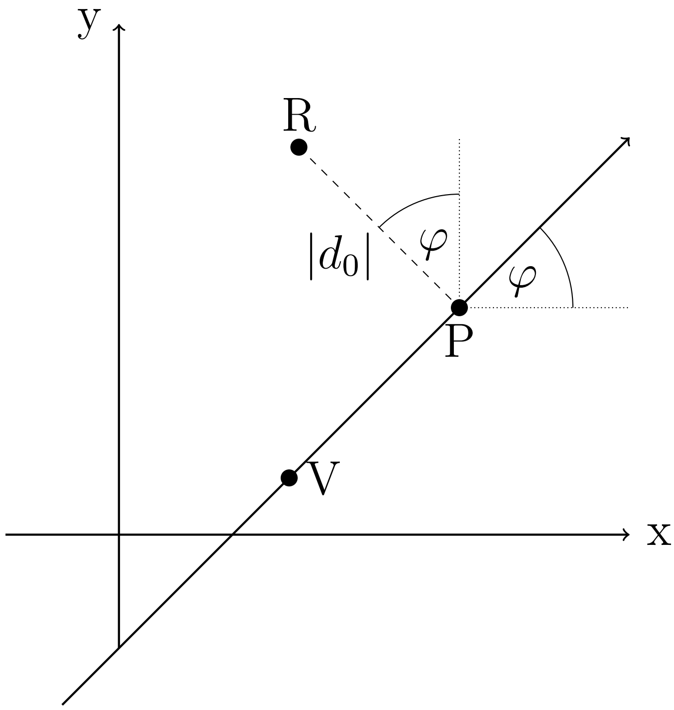

Figure 3.1: Projection of a track on the x-y plane in the absence of a magnetic field. The Perigee parametrization is given with respect to a coordinate system with origin in point R, whose axes are parallel to the global coordinate axes. d_0 is the x-y-distance between the reference point R and the PCA P of the trajectory to it. V denotes a general point on the trajectory. Note that we have $d_0 < 0$, $\varphi > 0$, and $\varphi_0 < 0$ in this plot.

If no electromagnetic field is present, the particle is not accelerated ($\ddot{\mathbf{r}} = 0$) and it thus moves on a straight trajectory, see Fig. 3.1. Therefore, φ, θ, and q/p are constant along the track, and we have

$$\begin{aligned} \varphi_V &= \varphi_P =: \varphi \\ \theta_V &= \theta_P =: \theta \\ (q/p)_V &= (q/p)_P =: q/p \end{aligned}$$

in the following.

Note that we perform all calculations for the situation shown in Fig. 3.1. One can (and should!) verify that we obtain the same results for different parameter signs and reference positions (e.g., when the particle moving in the opposite direction or when the reference R is below the track).

Let us start by expressing the coordinates of the PCA to the reference point R (i.e., the point P) with respect to the coordinates of the point V:

$$\mathbf{r}_P = \mathbf{r}_V + v \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} (t_P - t_V),$$

where v denotes the speed of the particle. Using the definition from Eq. 2.2 and keeping the sign of φ in mind, we can find another equation for $\mathbf{r}_P$:

$$\mathbf{r}_P = \mathbf{r}_R + \begin{pmatrix} d_0\sin\varphi \\ -d_0\cos\varphi \\ z_0 \end{pmatrix},$$

as one can easily verify from Fig. 3.1. Equating the two expressions for $\mathbf{r}_P$, we obtain:

$$\mathbf{r}_V + v \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} (t_P - t_V) = \mathbf{r}_R + \begin{pmatrix} d_0\sin\varphi \\ -d_0\cos\varphi \\ z_0 \end{pmatrix}. \tag{3.1}$$

Note that, the equation above contains only the Perigee parameters and the space-time coordinates of V.[1]

3.1 Derivatives of t_p

Before calculating the Jacobian, we must derive explicit functions for the Perigee parameters from Eq. 3.1. To obtain an expression for the time coordinate t, we rearrange the equations in the first two dimensions of Eq. 3.1:

$$\begin{aligned} d_0\sin\varphi &= x_V - x_R + v\sin\theta\cos\varphi\Delta t \\ -d_0\cos\varphi &= y_V - y_R + v\sin\theta\sin\varphi\Delta t, \end{aligned}$$

where we introduced $\Delta t \coloneqq t_P - t_V$. Division of the above equations furnishes:

$$\begin{aligned} -\tan\varphi &= \frac{x_V - x_R + v\sin\theta\cos\varphi\Delta t}{y_V - y_R + v\sin\theta\sin\varphi\Delta t} \\ -\tan\varphi\left(y_V - y_R + v\sin\theta\sin\varphi\Delta t\right) &= x_V - x_R + v\sin\theta\cos\varphi\Delta t \\ -\sin\varphi\left(y_V - y_R\right) - v\sin\theta\sin^2\varphi\Delta t &= \cos\varphi\left(x_V - x_R\right) + v\sin\theta\cos^2\varphi\Delta t, \end{aligned}$$

[1] The momentum at V coincides with the Perigee momentum due to the absence of a magnetic field.

where we multiplied by $\cos\varphi$ and used $\tan\varphi = \frac{\sin\varphi}{\cos\varphi}$ in the last step. We can simplify this expression by recalling that $\sin^2\varphi + \cos^2\varphi = 1$:

$$v \sin\theta \Delta t = -\cos\varphi\,(x_V - x_R) - \sin\varphi\,(y_V - y_R)\,.$$

Finally:

$$\begin{aligned} \Delta t &= -\frac{1}{v\sin\theta}\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right) \\ \implies t_P &= t_V - \frac{1}{v\sin\theta}\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right). \end{aligned} \tag{3.2}$$

We can now calculate the last row of the Jacobian from Eq. 2.4 using Eq. 3.2:

$$\begin{aligned} \partial_{x_V} t_P\Big|_{V=P} &= -\frac{\cos\varphi}{v\sin\theta} \\ \partial_{y_V} t_P\Big|_{V=P} &= -\frac{\sin\varphi}{v\sin\theta} \\ \partial_{z_V} t_P\Big|_{V=P} &= 0 \\ \partial_{t_V} t_P\Big|_{V=P} &= 1 \\ \partial_{\varphi} t_P\Big|_{V=P} &= \frac{1}{v\sin\theta}\left(\sin\varphi\,(x_V - x_R) - \cos\varphi\,(y_V - y_R)\right)\Big|_{V=P} \\ &= -\frac{d_0}{v\sin\theta} \\ \partial_{\theta} t_P\Big|_{V=P} &= -\left(\partial_\theta \frac{1}{v\sin\theta}\right)\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right)\Big|_{V=P} \\ &= 0 \\ \partial_{q/p} t_P\Big|_{V=P} &= -\left(\partial_{q/p} \frac{1}{v\sin\theta}\right)\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right)\Big|_{V=P} \\ &= 0, \end{aligned} \tag{3.3}$$

where we used that

$$\begin{aligned} (x_V - x_R)\Big|_{V=P} &= (x_P - x_R) \\ &= -\sin\varphi\; d_0 \\ (y_V - y_R)\Big|_{V=P} &= (y_P - y_R) \\ &= \quad \cos\varphi\; d_0. \end{aligned} \tag{3.4}$$

3.2 Derivatives of q/p

The fifth row of the Jacobian is obtained by noting that q/p is constant in the absence of an electric field and thus

$$\partial_{(q/p)_V} (q/p)_P\Big|_{V=P} = \partial_{q/p}\; q/p\Big|_{V=P} = 1,$$

while all other derivatives vanish.

3.3 Derivatives of θ

Again, θ is constant along the track in the absence of an electric field, and we have

$$\partial_{(\theta)_V}(\theta)_P\Big|_{V=P} = \partial_\theta\ \theta\Big|_{V=P} = 1,$$

while all other derivatives in the fourth row of the Jacobian vanish.

3.4 Derivatives of φ

φ is constant along the track in the absence of electric *and* magnetic field. Therefore, we find as before:

$$\partial_{(\varphi)_V}(\varphi)_P\Big|_{V=P} = \partial_\varphi\ \varphi\Big|_{V=P} = 1,$$

while all other derivatives in the third row of the Jacobian vanish.

3.5 Derivatives of z_0

To obtain an expression for z_0, we consider the third dimension of Eq. 3.1, i.e.:

$$z_V + v\cos\theta\Delta t = z_R + z_0.$$

Then,

$$z_0 = z_V - z_R + v\cos\theta\Delta t,$$

and we can find the derivatives of z_0 by exploiting

$$\partial_{q_i}\Delta t = \partial_{q_i}t_P - \delta_{q_i t_V},$$

in combination with the derivatives from Eq. 3.3. We find:

$$\begin{aligned}
\partial_{x_V} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{x_V}\Delta t\Big|_{V=P} \\
&= -v\cos\theta\frac{\cos\varphi}{v\sin\theta} \\
&= -\cot\theta\cos\varphi \\
\partial_{y_V} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{y_V}\Delta t\Big|_{V=P} \\
&= -v\cos\theta\frac{\sin\varphi}{v\sin\theta} \\
&= -\cot\theta\sin\varphi \\
\partial_{z_V} z_0\Big|_{V=P} &= 1 + v\cos\theta\ \partial_{z_V}\Delta t\Big|_{V=P} \\
&= 1 \\
\partial_{t_V} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{t_V}\Delta t \\
&= 0 \\
\partial_{\varphi} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{\varphi}\Delta t\Big|_{V=P} \\
&= -v\cos\theta\frac{d_0}{v\sin\theta} \\
&= -d_0\cot\theta \\
\partial_{\theta} z_0\Big|_{V=P} &= -v\sin\theta\Delta t\Big|_{V=P} + v\cos\theta\ \partial_{\theta}\Delta t\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{q/p}\Delta t\Big|_{V=P} \\
&= 0,
\end{aligned}$$

where we used that

$$\Delta t\Big|_{V=P} = (t_P - t_V)\Big|_{V=P} = 0. \tag{3.5}$$

3.6 Derivatives of d_0

An expression for d_0 can be found by rearranging the first two dimensions of Eq. 3.1 like:

$$\begin{aligned}
d_0\sin\varphi - v\sin\theta\cos\varphi\Delta t &= x_V - x_R \\
-d_0\cos\varphi - v\sin\theta\sin\varphi\Delta t &= y_V - y_R.
\end{aligned}$$

Squaring and adding these equations furnishes

$$d_0^2 + v^2\sin^2\theta(\Delta t)^2 = (x_V - x_R)^2 + (y_V - y_R)^2,$$

which could have been deduced geometrically by noting that the speed in the x-y-plane is given by $v_T = v \sin\theta$ and by applying the Pythagorean theorem in Fig. 3.1. Solving for d_0 furnishes

$$|d_0| = \sqrt{(x_V - x_R)^2 + (y_V - y_R)^2 - v^2 \sin^2\theta(\Delta t)^2},$$

and, by using Eq. 2.2,

$$d_0 = \operatorname{sgn}(y_R - y_P)\sqrt{(x_V - x_R)^2 + (y_V - y_R)^2 - v^2 \sin^2\theta(\Delta t)^2}.$$

The derivatives of d_0 read

$$\begin{aligned}
\partial_{x_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(x_V - x_R - v^2\sin^2\theta\Delta t\ \partial_{x_V}\Delta t)\Big|_{V=P} \\
&= \frac{x_P - x_R}{d_0} \\
&= \frac{-\sin\varphi d_0}{d_0} \\
&= -\sin\varphi \\
\partial_{y_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(y_V - y_R - v^2\sin^2\theta\Delta t\ \partial_{y_V}\Delta t)\Big|_{V=P} \\
&= \frac{y_P - y_R}{d_0} \\
&= \frac{\cos\varphi d_0}{d_0} \\
&= \cos\varphi \\
\partial_{z_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2\sin^2\theta\Delta t\ \partial_{z_V}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{t_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2\sin^2\theta\Delta t\ \partial_{t_V}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{\varphi} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2\sin^2\theta\Delta t\ \partial_{\varphi}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{\theta} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2\sin^2\theta\Delta t\ \partial_{\theta}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2\sin^2\theta\Delta t\ \partial_{q/p}\Delta t)\Big|_{V=P} \\
&= 0,
\end{aligned}$$

where we used Eqs. 3.4 and 3.5.

3.7 Results

Summing up the results from the previous sections, the position Jacobian reads:

$$A = \begin{pmatrix} -\sin\varphi & \cos\varphi & 0 & 0 \\ -\cot\theta\cos\varphi & -\cot\theta\sin\varphi & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\cos\varphi}{v_T} & -\frac{\sin\varphi}{v_T} & 0 & 1 \end{pmatrix},$$

and the momentum Jacobian reads:

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -d_0\cot\theta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{d_0}{v_T} & 0 & 0 \end{pmatrix},$$

where $v_T \equiv v\sin\theta$ is the speed in the x-y-plane. When comparing to Eq 5.40 from Ref. [1], we note that several terms in the momentum Jacobian differ from our results. This is because we evaluate the Jacobian at the PCA P while Ref. [1] evaluates the Jacobian at a general point on the trajectory V.[2]

[2]Note that, for all practical applications, we perform the linearization at the PCA.

4 Track Linearization in a Constant Magnetic Field

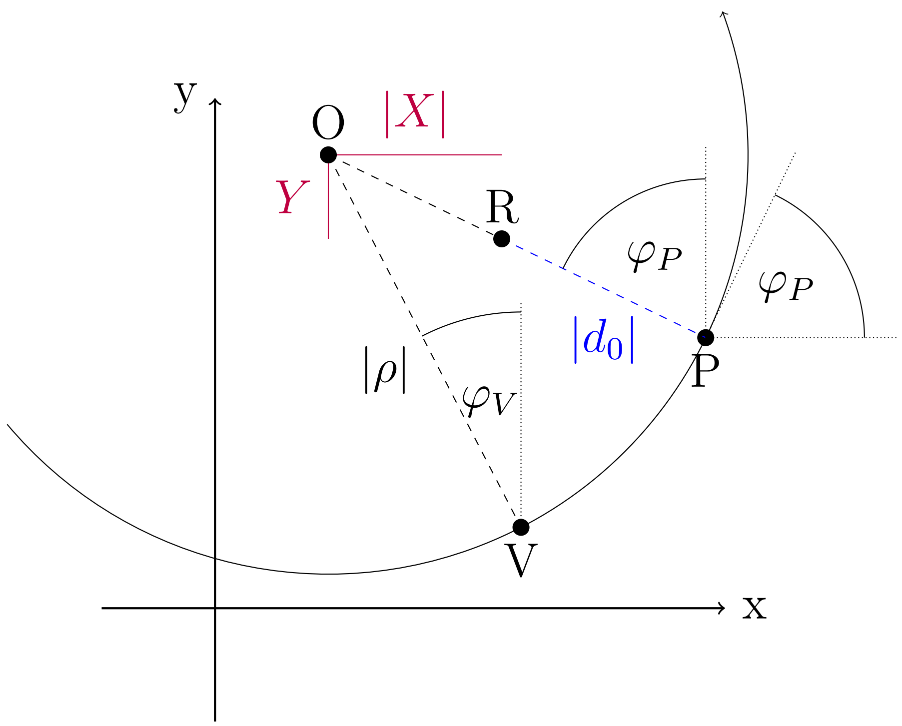

Figure 4.1: Projection of a track on the x-y plane in a constant magnetic field in z-direction. The particle moves counterclockwise on a helix with radius $|\rho|$ (i.e., a negative (positive) particle is moving in a B-field in positive (negative) z-direction). The Perigee parametrization is given with respect to a coordinate system with origin in point R, whose axes are parallel to the global coordinate axes. d_0 is the x-y-distance between the reference point R and the PCA P of the trajectory to it. V denotes a general point on the trajectory. Note that we have $d_0 < 0$, $\rho < 0$, $\varphi_P > 0$, $\varphi_V > 0$, $X < 0$, and $Y > 0$ in this plot. For the angle φ_0 between the x-axis and the vector from R to P, we have $\varphi_0 < 0$.

For a constant B-field in z-direction, the differential equations governing the particle movement read

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = \begin{pmatrix} \dot{y}B \\ -\dot{x}B \\ 0 \end{pmatrix} . \tag{4.1}$$

Note that the acceleration in the transverse plane (i.e., the x-y-plane) is always perpendicular to the velocity in said plane. Therefore, the speed in the transverse plane $v_T \equiv \sqrt{\dot{x}^2 + \dot{y}^2}$ is constant. Furthermore, there is no acceleration in z-direction and thus the speed in said direction $v_z \equiv \dot{z}$ is constant. Consequently, the total speed $v = \sqrt{v_T^2 + v_z^2}$ is also constant. This allows us to conclude that $\theta = \arcsin(v_T/v)$ and $q/p = q/(mv)$ are constant as well, and we can write

$$\begin{aligned} \theta_V = \theta_P &=: \theta \\ (q/p)_V = (q/p)_P &=: q/p \end{aligned}$$

in the following calculations.

Choosing the initial conditions

$$\begin{aligned} x(0) = x_0 \quad y(0) = y_0 \quad z(0) = z_0 \\ \dot{x}(0) = v \sin\theta \quad \dot{y}(0) = 0 \quad \dot{z}(0) = v\cos\theta, \end{aligned}$$

we find

$$\begin{aligned} x(t) &= x_0 + \rho \sin(\omega_0 t) \\ y(t) &= y_0 + \rho (\cos(\omega_0 t) - 1) \\ z(t) &= z_0 + v \cos\theta \; t \end{aligned} \tag{4.2}$$

as solution for Eq. 4.1. The particle thus follows a helix with radius

$$\begin{aligned} \rho &= \frac{mv \sin\theta}{qB} \\ &= \frac{p \sin\theta}{qB} \end{aligned}$$

and angular frequency

$$\omega_0 = \frac{qB}{m}.$$

Note that the sign of the radius depends on the direction of the B-field and on the charge of the particle. For example, if the B-field is oriented in positive z-direction, (counter)clockwise rotation corresponds to (negative) positive charge and consequently to (negative) positive ρ. Following the literature convention, we define:

$$h := \operatorname{sgn}(\rho) = \operatorname{sgn}(q) \operatorname{sgn}(B) . \tag{4.3}$$

Comparing to Eq. 2.3, we then obtain

$$\operatorname{sgn}(d_0) = h \operatorname{sgn}\left(\rho^2 - (\mathbf{r}_R - \mathbf{r}_O)^2\right).$$

One can relate the radius and the angular momentum like

$$\frac{1}{\omega_0} = \frac{\rho}{v \sin\theta}, \tag{4.4}$$

which will become useful later on. The particle velocity can be retrieved by differentiating Eqs. 4.2:

$$\begin{aligned} \dot{x}(t) &= \rho\omega_0 \cos(\omega_0 t) \\ \dot{y}(t) &= -\rho\omega_0 \sin(\omega_0 t) \\ \dot{z}(t) &= v\cos\theta. \end{aligned} \tag{4.5}$$

Like in Ch. 3, we want to express the Perigee parameters as a function of the free parameters at V. Note that we perform all calculations for the situation shown in Fig. 4.1. One can (and should!) verify that we obtain the same results for different parameter signs and reference positions (e.g., when the particle moving clockwise or when the reference R is in a different quadrant or outside of the helix).

4.1 Derivatives of φ_P

We start by finding an expression for φ_P, which is a convenient choice as we will see a little further down the road. From Fig. 4.1 we find

$$\begin{aligned} x_V &= x_R + |d_0|\sin|\varphi_P| - |\rho|\sin|\varphi_P| + |\rho|\sin|\varphi_V| \\ y_V &= y_R - |d_0|\cos|\varphi_P| + |\rho|\cos|\varphi_P| - |\rho|\cos|\varphi_V|, \end{aligned}$$

and, using the correct signs for the parameters,

$$\begin{aligned} x_V &= x_R - d_0\sin\varphi_P + \rho\sin\varphi_P - \rho\sin\varphi_V \\ y_V &= y_R + d_0\cos\varphi_P - \rho\cos\varphi_P + \rho\cos\varphi_V. \end{aligned} \tag{4.6}$$

Rearranging furnishes

$$\begin{aligned} -\sin\varphi_P(d_0 - \rho) &= x_V - x_R + \rho\sin\varphi_V \\ \cos\varphi_P(d_0 - \rho) &= y_V - y_R - \rho\cos\varphi_V, \end{aligned}$$

and, by dividing the equations,

$$\begin{aligned} -\tan\varphi_P &= \frac{x_V - x_R + \rho\sin\varphi_V}{y_V - y_R - \rho\cos\varphi_V} \\ &\equiv \frac{X}{Y}, \end{aligned} \tag{4.7}$$

where we defined

$$\begin{aligned} X &:= x_V - x_R + \rho \sin \varphi_V \\ Y &:= y_V - y_R - \rho \cos \varphi_V . \end{aligned} \tag{4.8}$$

Using the relation

$$-\tan x = \frac{1}{\tan (x + \pi/2)},$$

we conclude

$$\varphi_P = \arctan \left(\frac{Y}{X} \right) - \frac{\pi}{2}. \tag{4.9}$$

Note that X and Y are the x- and y-coordinate of the helix center O in the reference coordinate system with origin in R.[1] Consequently, X and Y are independent of where we place the point V on the track, and we can write

$$\begin{aligned} X_V &\equiv X \\ Y_V &\equiv Y, \end{aligned}$$

as the choice of notation in Eq. 4.8 already hinted. It is convenient to define the distance S between O and R:

$$S := \sqrt{X^2 + Y^2}.$$

We can then express X and Y via S:

$$\begin{aligned} X &= \quad hS \sin \varphi_P \\ Y &= -hS \cos \varphi_P , \end{aligned} \tag{4.10}$$

where h is the sign of the helix radius as defined in Eq. 4.3.

Let us compute some derivatives of these quantities. We have

$$\begin{aligned} \partial_\theta \rho &= \frac{mv \cos \theta}{qB} \\ &= \rho \cot \theta, \\ \partial_{q/p} \rho &= -\frac{p^2}{q} \partial_p \rho \\ &= -\frac{p^2}{q} \frac{\rho}{p} \\ &= -\frac{\rho}{q/p} \end{aligned}$$

[1] Applying this knowledge to Fig. 4.1 confirms Eq. 4.7 geometrically.

while all other derivatives of ρ vanish. Therefore, from Eq. 4.8,

$$\begin{aligned}
\partial_{x_V} X &= 1 \\
\partial_{\varphi_V} X &= \rho \cos \varphi_V \\
\partial_\theta X &= \rho \cot \theta \sin \varphi_V, \\
\partial_{q/p} X &= -\frac{\rho}{q/p} \sin \varphi_V,
\end{aligned}$$

and

$$\begin{aligned}
\partial_{y_V} Y &= 1 \\
\partial_{\varphi_V} Y &= \rho \sin \varphi_V \\
\partial_\theta Y &= -\rho \cot \theta \cos \varphi_V, \\
\partial_{q/p} Y &= \frac{\rho}{q/p} \cos \varphi_V
\end{aligned}$$

while all other derivatives of X and Y vanish. Keeping in mind that

$$\partial_x \arctan x = \frac{1}{1+x^2},$$

we can derive φ_P with respect to X and Y:

$$\begin{aligned}
\partial_X \varphi_P &= \frac{1}{1+\left(\frac{Y}{X}\right)^2} \left(-\frac{Y}{X^2}\right) \\
&= -\frac{Y}{S^2} \\
\partial_Y \varphi_P &= \frac{1}{1+\left(\frac{Y}{X}\right)^2} \frac{1}{X} \\
&= \frac{X}{S^2}
\end{aligned}$$

Finally, we put all pieces together to compute the third row of the Jacobian:

$$\begin{aligned}
\partial_{x_V} \varphi_P \Big|_{V=P} &= \partial_{x_V} X \partial_X \varphi_P \Big|_{V=P} \\
&= -\frac{Y}{S^2} \\
\partial_{y_V} \varphi_P \Big|_{V=P} &= \partial_{y_V} Y \partial_Y \varphi_P \Big|_{V=P} \\
&= \frac{X}{S^2} \\
\partial_{z_V} \varphi_P \Big|_{V=P} &= 0 \\
\partial_{t_V} \varphi_P \Big|_{V=P} &= 0
\end{aligned}$$

$$
\begin{aligned}
\partial_{\varphi_V}\varphi_P\Big|_{V=P} &= \left(\partial_{\varphi_V}X\partial_X\varphi_P + \partial_{\varphi_V}Y\partial_Y\varphi_P\right)\Big|_{V=P} \\
&= \rho\left(\frac{-Y\cos\varphi_V + X\sin\varphi_V}{S^2}\right)\Bigg|_{V=P} \\
&= \rho\left(\frac{h\cos\varphi_P\cos\varphi_V + h\sin\varphi_P\sin\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= h\frac{\rho}{S} \\
&= \frac{|\rho|}{S} \\
\partial_\theta\varphi_P\Big|_{V=P} &= \left(\partial_\theta X\partial_X\varphi_P + \partial_\theta Y\partial_Y\varphi_P\right)\Big|_{V=P} \\
&= \rho\cot\theta\left(\frac{-Y\sin\varphi_V - X\cos\varphi_V}{S^2}\right)\Bigg|_{V=P} \\
&= \rho\cot\theta\left(\frac{h\cos\varphi_P\sin\varphi_V - h\sin\varphi_P\cos\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= 0, \\
\partial_{q/p}\varphi_P\Big|_{V=P} &= \left(\partial_{q/p}X\partial_X\varphi_P + \partial_{q/p}Y\partial_Y\varphi_P\right)\Big|_{V=P} \\
&= \frac{\rho}{q/p}\left(\frac{Y\sin\varphi_V + X\cos\varphi_V}{S^2}\right)\Bigg|_{V=P} \\
&= \frac{\rho}{q/p}\left(\frac{-h\cos\varphi_P\sin\varphi_V + h\sin\varphi_P\cos\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= 0,
\end{aligned}
$$

where we used the chain rule and Eq. 4.10.

4.2 Derivatives of t_P

Let us continue by computing the last row of the Jacobian. From Fig. 4.1 we find geometrically

$$\tan\varphi_V = \frac{\dot{y}(t_V)}{\dot{x}(t_V)}.$$

Using the expressions from Eq. 4.5 allows us to obtain a relation between the time and the polar angle φ:

$$
\begin{aligned}
\tan\varphi_V &= -\tan\left(\omega_0 t_V\right) \\
\implies \varphi_V + 2\pi n_V &= -\omega_0 t_V,\ \ n_V \in \mathbb{N}.
\end{aligned}
$$

Note that

$$n_V \to n_V + 1 \text{ iff } \varphi_V = -\pi,$$

and thus

$$\partial_{\varphi_V} n_V = \delta\left(\varphi_V + \pi\right).$$

Then

$$\begin{aligned}
\Delta t &\equiv t_P - t_V \\
&= -\frac{1}{\omega_0}(\varphi_P - \varphi_V + 2\pi(n_P - n_V)) \\
\implies t_P &= t_V - \frac{\rho}{v \sin\theta}(\varphi_P - \varphi_V + 2\pi(n_P - n_V)),
\end{aligned} \tag{4.11}$$

where we used Eq. 4.4 to replace the angular frequency by the helix radius. The derivatives of t_P follow directly from the calculations for φ_P from Sec. 4.1:

$$\begin{aligned}
\partial_{x_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{x_V}\varphi_P\Big|_{V=P} \\
&= \frac{\rho}{v\sin\theta}\frac{Y}{S^2} \\
\partial_{y_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{y_V}\varphi_P\Big|_{V=P} \\
&= -\frac{\rho}{v\sin\theta}\frac{X}{S^2} \\
\partial_{z_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{z_V}\varphi_P\Big|_{V=P} \\
&= 0 \\
\partial_{t_V} t_P\Big|_{V=P} &= 1 - \frac{\rho}{v\sin\theta}\partial_{t_V}\varphi_P\Big|_{V=P} \\
&= 1 \\
\partial_{\varphi_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\left(\partial_{\varphi_V}\varphi_P - 1 + 2\pi(\partial_{\varphi_V}\varphi_P\ \delta\left(\varphi_P + \pi\right) - \delta\left(\varphi_V + \pi\right))\right)\Big|_{V=P} \\
&= \frac{\rho}{v\sin\theta}\left(1 - \frac{|\rho|}{S}\right)\left(1 + 2\pi\delta\left(\varphi_P + \pi\right)\right) \\
\partial_\theta t_P\Big|_{V=P} &= -\left(\partial_\theta \frac{\rho}{v\sin\theta}\right)(\varphi_P - \varphi_V + 2\pi(n_P - n_V))\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{q/p}\varphi_P\Big|_{V=P} \\
&= 0,
\end{aligned}$$

where we used that

$$\begin{aligned}
(\varphi_P - \varphi_V + 2\pi(n_P - n_V))\Big|_{V=P} &= (\varphi_P - \varphi_P + 2\pi(n_P - n_P)) \\
&= 0.
\end{aligned}$$

4.3 Derivatives of q/p

As in Ch. 3, the fifth row of the Jacobian is obtained by noting that q/p is constant in the absence of an electric field and thus

$$\partial_{(q/p)_V}(q/p)_P\Big|_{V=P} = \partial_{q/p}\ q/p\Big|_{V=P} = 1,$$

while all other derivatives vanish.

4.4 Derivatives of θ

θ is constant along the track in the absence of an electric field, and we have

$$\partial_{(\theta)_V}(\theta)_P\Big|_{V=P} = \partial_{\theta}\ \theta\Big|_{V=P} = 1,$$

while all other derivatives in the fourth row of the Jacobian vanish.

4.5 Derivatives of z_0

From the third equation of Eq. 4.2, we have:

$$\begin{aligned} z_V &= z_P - v\cos\theta(t_P - t_V) \\ &= z_R + z_0 - v\cos\theta(t_P - t_V) \\ \Longrightarrow\ z_0 &= z_V - z_R - \rho\cot\theta\ (\varphi_P - \varphi_V + 2\pi(n_P - n_V)) \end{aligned}$$

where we plugged in the definition of z_0 in the second step and used Eq. 4.11 in the third step. The derivatives of z_0 are then obtained from the derivatives of φ_P from Sec. 4.1:

$$\begin{aligned} \partial_{x_V} z_0\Big|_{V=P} &= -\rho\cot\theta\ \partial_{x_V}\varphi_P\Big|_{V=P} \\ &= -\rho\cot\theta\left(-\frac{Y}{S^2}\right) \\ &= \rho\cot\theta\frac{Y}{S^2} \\ \partial_{y_V} z_0\Big|_{V=P} &= -\rho\cot\theta\ \partial_{y_V}\varphi_P\Big|_{V=P} \\ &= -\rho\cot\theta\frac{X}{S^2} \\ \partial_{z_V} z_0\Big|_{V=P} &= 1 \\ \partial_{t_V} z_0\Big|_{V=P} &= 0 \\ \partial_{\varphi_V} z_0\Big|_{V=P} &= -\rho\cot\theta\left(\partial_{\varphi_V}\varphi_P - 1 + 2\pi(\partial_{\varphi_V}\varphi_P\ \delta\left(\varphi_P + \pi\right) - \delta\left(\varphi_V + \pi\right))\right)\Big|_{V=P} \end{aligned}$$

$$
\begin{aligned}
&= \rho\cot\theta\left(1 - \frac{|\rho|}{S}\right)\left(1 + 2\pi\delta\left(\varphi_P + \pi\right)\right) \\
\partial_\theta z_0\Big|_{V=P} &= -\left(\partial_\theta(\rho\cot\theta)\right)\left(\varphi_P - \varphi_V + 2\pi(n_P - n_V)\right)\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} z_0\Big|_{V=P} &= -\left(\partial_{q/p}\rho\right)\cot\theta(\varphi_P - \varphi_V + 2\pi(n_P - n_V))\Big|_{V=P} \\
&= 0.
\end{aligned}
$$

4.6 Derivatives of d_0

To find an expression for d_0, we can rearrange Eqs. 4.6 like

$$
\begin{aligned}
\sin\varphi_P(\rho - d_0) &= x_V - x_R + \rho\sin\varphi_V \\
&\equiv X \\
-\cos\varphi_P(\rho - d_0) &= y_V - y_R - \rho\cos\varphi_V \\
&\equiv Y.
\end{aligned}
$$

Squaring and adding the two equations leads to

$$
\begin{aligned}
(\rho - d_0)^2 &= X^2 + Y^2, \\
&\equiv S^2,
\end{aligned}
$$

which is what one would expect from geometrical considerations. Taking the square root furnishes

$$
\begin{aligned}
d_0 &= \rho - \operatorname{sgn}(\rho - d_0)\, S \\
&= \rho - \operatorname{sgn}(\rho)\, S \\
&\equiv \rho - hS
\end{aligned}
\tag{4.12}
$$

Let us proof the second equality.

Proof. *We need to consider four cases:*

- *R is in the helix center ($R = O$)*
  $\Longrightarrow$ *$S = 0$ and the equality holds.*

- *R is inside the helix but not in the helix center*
  $\Longrightarrow$ $sgn(\rho) = sgn(d_0)\,,\ |\rho| > |d_0|$
  $\Longrightarrow$ $sgn(\rho - d_0) = sgn(sgn(\rho)(|\rho| - |d_0|)) = sgn(\rho)$

- *R is on the helix*
  $\Longrightarrow$ $d_0 = 0$
  $\Longrightarrow$ $sgn(\rho - d_0) = sgn(\rho)$

- *R is outside the helix*
  $\implies sgn\,(\rho) = -sgn\,(d_0)$
  $\implies sgn\,(\rho - d_0) = sgn\,(sgn\,(\rho)\,(|\rho| + |d_0|)) = sgn\,(\rho)$

To compute the derivatives of Eq. 4.12, it is useful to note that

$$\partial_X S = \frac{X}{S}$$
$$\partial_Y S = \frac{Y}{S}.$$

Furthermore, thanks to Eq. 4.3, all other derivatives of h vanish.

Then, by virtue of the chain rule,

$$\begin{aligned}
\partial_{x_v} d_0\Big|_{V=P} &= \partial_{x_V} X \partial_X d_0\Big|_{V=P} \\
&= -h\ \partial_X S\Big|_{V=P} \\
&= -h\frac{X}{S} \\
\partial_{y_v} d_0\Big|_{V=P} &= \partial_{y_V} Y \partial_Y d_0\Big|_{V=P} \\
&= -h\ \partial_Y S\Big|_{V=P} \\
&= -h\frac{Y}{S} \\
\partial_{z_v} d_0\Big|_{V=P} &= 0 \\
\partial_{t_V} d_0\Big|_{V=P} &= 0 \\
\partial_{\varphi_V} d_0\Big|_{V=P} &= \left(\partial_{\varphi_V} X \partial_X d_0 + \partial_{\varphi_V} Y \partial_Y d_0\right)\Big|_{V=P} \\
&= \left(\rho\cos\varphi_V\left(-h\frac{X}{S}\right) + \rho\sin\varphi_V\left(-h\frac{Y}{S}\right)\right)\Big|_{V=P} \\
&= -h\rho\left(\frac{X\cos\varphi_V + Y\sin\varphi_V}{S}\right)\Big|_{V=P} \\
&= -h\rho\left(h\sin\varphi_P\cos\varphi_V - h\cos\varphi_P\sin\varphi_V\right)\Big|_{V=P} \\
&= 0 \\
\partial_\theta d_0\Big|_{V=P} &= \left(\partial_\theta\rho - h\left(\partial_\theta X\partial_X S + \partial_\theta Y\partial_Y S\right)\right)\Big|_{V=P} \\
&= \rho\cot\theta - h\rho\cot\theta\left(\sin\varphi_V\frac{X}{S} - \cos\varphi_V\frac{Y}{S}\right)\Big|_{V=P} \\
&= \rho\cot\theta - h\rho\cot\theta\left(\sin\varphi_V(h\sin\varphi_P) - \cos\varphi_V(-h\cos\varphi_P)\right)\Big|_{V=P}
\end{aligned}$$

$$
\begin{aligned}
&= \rho \cot\theta - h^2 \rho \cot\theta \\
&= 0 \\
\partial_{q/p} d_0 \Big|_{V=P} &= \left(\partial_{q/p}\rho - h\left(\partial_{q/p}X \partial_X S + \partial_{q/p}Y \partial_Y S\right)\right) \Big|_{V=P} \\
&= -\frac{\rho}{q/p} - h\frac{\rho}{q/p}\left(-\sin\varphi_V \frac{X}{S} + \cos\varphi_V \frac{Y}{S}\right)\Big|_{V=P} \\
&= -\frac{\rho}{q/p} - h\frac{\rho}{q/p}\left(-\sin\varphi_V(h\sin\varphi_P) + \cos\varphi_V(-h\cos\varphi_P)\right)\Big|_{V=P} \\
&= -\frac{\rho}{q/p} + h^2\frac{\rho}{q/p} \\
&= 0,
\end{aligned}
$$

where we used Eq. 4.10.

4.7 Results

Neglecting the terms containing Kronecker deltas, the position Jacobian for helical tracks reads

$$
A = \begin{pmatrix}
-h\frac{X}{S} & -h\frac{Y}{S} & 0 & 0 \\
\rho\cot\theta\frac{Y}{S^2} & -\rho\cot\theta\frac{X}{S^2} & 1 & 0 \\
-\frac{Y}{S^2} & \frac{X}{S^2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{\rho}{v_T}\frac{Y}{S^2} & -\frac{\rho}{v_T}\frac{X}{S^2} & 0 & 1
\end{pmatrix},
$$

and the momentum Jacobian reads:

$$
B = \begin{pmatrix}
0 & 0 & 0 \\
\rho\cot\theta\left(1 - \frac{|\rho|}{S}\right) & 0 & 0 \\
\frac{|\rho|}{S} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{\rho}{v_T}\left(1 - \frac{|\rho|}{S}\right) & 0 & 0
\end{pmatrix},
$$

where $v_T \equiv v \sin\theta$ is the speed in the x-y-plane. When comparing to Eq 5.36 from Ref. [1], we note that several terms in the momentum Jacobian differ from our results. This is because we evaluate the Jacobian at the PCA P while Ref. [1] evaluates the Jacobian at a general point on the trajectory V.[2]

[2] Note that, for all practical applications, we perform the linearization at the PCA.

List of Figures

Bibliography

[1] G. Piacquadio. *Identification of b-jets and investigation of the discovery potential of a Higgs boson in the* $WH \to l\nu b\bar{b}$ *channel with the ATLAS experiment.* Ph.D. thesis, Freiburg U. (2010).