

Pricing Swaptions Within an Affine Framework

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Proposed here is a very fast and accurate algorithm for pricing swaptions when the underlying term structure dynamics are affine. The algorithm is efficient because the moments of the underlying asset (e.g., a coupon bond) have simple closed-form solutions. These moments uniquely identify the cumulants of the distribution. The probability distribution of the asset's future price is then estimated using an Edgeworth expansion technique. The approach is fast because no numerical integrations are ever performed; it is accurate because the cumulants decay very quickly.

Using as an example a three-factor Gaussian model, the authors obtain prices of a 2-10 swaption in under 0.05 seconds, with an absolute error of only a few parts in 10^{-6} . An added benefit of the approach is that prices of swaptions across multiple strikes can be estimated at virtually no additional computational cost. Finally, the method provides an intrinsic estimate of the pricing error, and remains feasible even when the number of factors is infinite.

The swaps and LIBOR-based derivatives market has become by far the world's largest fixed-income market. According to the Bank for International Settlements, in 2000 the notional value in the swap market was approximately \$40 trillion, of which the combined cap and swaption market totaled \$9 trillion. As a consequence, the search for efficient algorithms to price caps and swaptions has received considerable attention.¹

From a practitioner's perspective, development of efficient formulas to price caps and swaptions is necessary in order to evaluate and hedge large portfolios of LIBOR-based derivatives. From an academic perspective, the great amount of data on interest rate derivatives provides an important source of information that might provide new insights into the factors that drive term structure dynamics. For an estimation procedure like maximum-likelihood to be feasible, however, it is essential that researchers have access to algorithms that provide fast and accurate estimates of derivative prices.

We propose a very fast and accurate algorithm for pricing swaptions when the underlying term structure dynamics are affine. The affine framework has become the dominant framework because of its tractability and flexibility. Affine models allow analytic solutions for the prices of both bonds and bond options, which greatly facilitates empirical investigation.²

In addition, multiple-factor affine models can be calibrated to provide a reasonably good fit for interest rate dynamics (e.g., Dai and Singleton [2000]), and can be improved further by modeling term structure dynamics within an "essentially affine" framework (Duffee [2002]). Further, under certain parameter conditions, affine models are consistent with the empirical observation that derivative securities cannot be hedged by positions in bonds alone (see Collin-Dufresne and Goldstein [2002]).

Finally, if one models forward rate dynamics as affine within a Heath, Jarrow, and Morton [1992] framework, or, more generally, in a random field framework, one still maintains the tractability inherent in the finite-state variable affine models (Collin-Dufresne and Goldstein [2001]).³

Unfortunately, closed-form solutions for swaptions apparently do not exist for multiple-factor affine models. Intuitively, this is because a swaption can be most readily interpreted as an option on a coupon bond, or, equivalently, an option on a portfolio of bonds. Thus, even in the simplest of models, where it is assumed that future bond prices are lognormally distributed, the future value of such a portfolio of bonds would be described by a probability density composed of a sum of lognormals, which has no known analytic solution. It seems unlikely that exact closed-form solutions will ever be found for swaption prices. Hence, an efficient algorithm for estimating swaption prices appears essential.⁴

Define today as date t , and the exercise date of the swaption as date T_0 . Further, define (T_1, \dots, T_N) as the dates that the coupon payments are made, where by construction $t < T_0 < T_1 < \dots < T_N$. A swaption is effectively an option on a coupon bond with these payment dates, and the date t price of a swaption with strike K is related to the probability that it ends up in the money. Define $CB(T_0)$ as the date T_0 price of this coupon bond. Then, at date t , the value of the swaption depends on the probability that the value of the coupon bond ends up higher than the strike price: $\pi(\overline{CB(T_0)} > K | \mathcal{F}_t)$.

The insight of our approach is to note that, even though the probability density $\pi(\overline{CB(T_0)} | \mathcal{F}_t)$ does not have an analytic solution, within an affine framework, all of the moments

$$E_t \left[(CB(T_0))^m \right] = \int_0^\infty \pi(CB(T_0) = x | \mathcal{F}_t) x^m dx$$

for any finite integer m do have analytic solutions.

We use the first $m \in (1, M)$ moments to approximate the density approx $\pi^{\text{approx}}(\overline{CB(T_0)} | \mathcal{F}_t)$, which in turn provides an estimation of the swaption price. From these first M moments, the first M cumulants of the distribution are uniquely identified. Then $\pi(\overline{CB(T_0)} | \mathcal{F}_t)$ is estimated by performing an Edgeworth expansion. The Edgeworth expansion is particularly advantageous as it permits swaption prices to be written as sums of terms, each of which involves at worst the cumulative normal function. Hence, no numerical integrations are ever performed.

Several other approximation schemes have been proposed in the literature. For example, Singleton and Umantsev [2001] propose an approximation for coupon bond options by approximating the exercise boundary with a linear function of the state variables (i.e., a hyperplane). They show that their technique dominates the speed and accuracy of the stochastic duration approach developed by Wei [1997] and Munk [1999]. They report that it takes approximately 1.4 seconds to estimate the price of a swaption in a two-factor CIR model with an absolute pricing error of $\sim (5 \times 10^{-4})$.

The Singleton-Umantsev approach, however, does not appear to provide an estimate of the magnitude of the pricing error. Further, a separate (and thus computationally costly) approximation needs to be performed for every strike of interest. Finally, the approach becomes infeasible when the number of state variables becomes large.

In comparison, for the case of a three-factor Gaussian model, our algorithm prices a 2-10 swaption in approximately 0.05 seconds, while obtaining a pricing accuracy of a few parts in 10^{-6} . Additionally, the highest-order term in the expansion provides an intrinsic estimate of the magnitude of the pricing error.

Furthermore, our approach also provides swaption prices across multiple strike prices at virtually zero computational cost, which is advantageous in pricing a portfolio of swaptions or dealing with a panel data set in empirical work. Finally, our approach remains efficient for arbitrarily large dimensions. Indeed, Collin-Dufresne and Goldstein [2001] demonstrate that swaption prices can be estimated quickly and accurately even for the infinite-factor, or random field, affine models.

Other approximation schemes for pricing swaptions have been proposed for Gaussian, affine, and so-called market models. For multifactor Gaussian models, Brace and Musiela [1995] obtain a formula in terms of a multidimensional Gaussian integral. For simple affine models, the problem can also be reduced to a multidimensional integral that can be solved by quadrature. For dimensions higher than two, however, the problem often becomes numerically very burdensome, and approximations such as the one-dimensional approximation proposed in Brace and Musiela [1995] become imprecise.⁵

Lacking an efficient and accurate pricing formula for coupon bond options has led to the development of the so-called swap market model (Jamshidian [1997]), which is closely related to the LIBOR market model of Brace, Gatarek, and Musiela [1997]. By choosing a suitable distribution of the forward swap rate underlying the

swaption, it is possible to obtain a closed-form (and arbitrage-free) solution to the swaption price. Indeed, this solution resembles the simple Black formula, and thus can easily be calibrated to market quotes.

Yet it is well known that the assumptions leading to closed-form solutions for swaptions in the swap market model (namely, lognormally distributed forward swap rates) are inconsistent with the assumptions leading to closed-form solutions for caps and floors in the LIBOR market model (namely, lognormally discrete forward LIBOR). Empirical evidence also seems to reject the swap market model in favor of the LIBOR market model (De Jong, Driessen, and Pelsser [2000]).

In response, some approximation schemes have been proposed to estimate swaption prices in a standard LIBOR market model setup (Brace, Gatarek, and Musiela [1997], Andersen and Andreasen [2000]). Unfortunately, these schemes are uncontrolled in that there is no sense in which these approximations converge to the exact formula.

Finally, Monte Carlo techniques following Boyle [1977] have been successfully applied to pricing swaptions. Standard variance reduction techniques and control variates can improve the speed of convergence (Clewlow, Pang, and Strickland [1996]). Even though these techniques have the potential to achieve arbitrary accuracy, they still lack the computational efficiency of closed-form approximations.

The Edgeworth expansion has been used previously in the finance literature as an approximation scheme for pricing stock and Asian and basket options (Jarrow and Rudd [1982], Turnbull and Wakeman [1991]).⁶ Unfortunately, the pricing accuracy of the Edgeworth expansion is rather limited for these cases (see, e.g., Ju [2001]).

This occurs because the Edgeworth expansion is basically an expansion about the normal distribution, while the underlying distributions for these three cases are not well approximated by normal distributions, but rather lognormal distributions. In contrast, the relatively low volatility associated with interest rates ($\sigma \approx 0.01$) compared to stocks ($\sigma \approx 0.3$) generates probability distributions for coupon bonds that are close enough to normally distributed that the Edgeworth expansion provides an excellent approximation scheme for pricing swaptions.

Fortran programs for selected examples can be found at www.andrew.cmu.edu/user/dufresne/.

I. CUMULANT EXPANSION APPROXIMATION

A European swaption at date t gives its holder the right to enter a swap at some future date T_0 . A swaption is most readily interpreted as an option on a coupon bond, where the strike is equal to the nominal of the contract, and the coupon rate is equal to the swap rate strike of the swaption.⁷

We propose a very accurate and computationally efficient algorithm for pricing swaptions in a general affine framework. Following Duffie and Kan (DK [1996]), and Duffie, Pan, and Singleton (DPS [2000]), we characterize a general J -factor affine model of the term structure by a vector of Markov processes $\{X_j\}_{j=1,\dots,J}$ whose risk-neutral dynamics are such that the instantaneous drifts and covariances are linear in the state variables. Further, the instantaneous short rate is defined as a linear combination of the state variables: $r_t = \delta_0 + \sum_{j=1}^J \delta_1 X_j(t)$.⁸

Within an affine framework, DK demonstrate that bond prices have an exponentially affine form:

$$P^T(t) = E_t^Q \left[e^{-\int_t^T r_s ds} \right] = e^{B_0(T-t) + \sum_{j=1}^J B_j(T-t) X_j(t)} \quad (1)$$

where the deterministic functions $B_0(\tau)$ and $\{B_j(\tau)\}$ satisfy a system of ordinary differential equations known as Riccati equations. Furthermore, since the characteristic function of log bond prices is exponentially affine, DPS demonstrate that bond options also have analytic solutions. Unfortunately, swaption prices for multivariate models apparently do not have closed-form solutions.

In searching for an efficient algorithm to price a swaption, it is convenient to define $CB(T_0)$ as the date T_0 price of the underlying coupon bond that the option is written on:

$$CB(T_0) = \sum_{i=1}^N C_i P^{T_i}(T_0) \quad (2)$$

The date t price of a swaption with exercise date- T_0 and with payments C_i on dates T_i $i = 1, \dots, N$ and strike price K is given by the expected discounted cash flows, where the expectation is under the so-called risk-neutral measure:⁹

$$\begin{aligned}
\text{Swn}(t, \{X_j(t)\}) &= \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \max(C B(T_0) - K, 0) \right] \\
&= \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \left(C B(T_0) \mathbf{1}_{(CB(T_0) > K)} - K \mathbf{1}_{(CB(T_0) > K)} \right) \right] \\
&= \sum_{i=1}^N C_i \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \mathbf{1}_{(CB(T_0) > K)} P^{T_i}(T_0) \right] - K \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \mathbf{1}_{(CB(T_0) > K)} \right] \\
&= \sum_{i=1}^N C_i \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \mathbf{1}_{(CB(T_0) > K)} \mathbb{E}_{T_0}^Q \left[e^{-\int_{T_0}^{T_i} r_s ds} \right] \right] - K \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \mathbf{1}_{(CB(T_0) > K)} \right] \\
&= \sum_{i=1}^N C_i \mathbb{E}_t^Q \left[e^{-\int_t^{T_i} r_s ds} \mathbf{1}_{(CB(T_0) > K)} \right] - K \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r_s ds} \mathbf{1}_{(CB(T_0) > K)} \right] \tag{3}
\end{aligned}$$

where the last line follows from the law of iterated expectations.

We sometimes use Equation (3) to estimate swaption prices. In addition, however, it is sometimes more convenient to price swaptions by calculating expectations under the so-called forward measures rather than the risk-neutral measure, as first demonstrated by El Karoui and Rochet [1989] and Jamshidian [1989].

We do this by rewriting Equation (3) as:

$$\begin{aligned}
\text{Swn}(t, \{X_j(t)\}) &= \\
&= \sum_{i=1}^N C_i P^{T_i}(t) \mathbb{E}_t^Q \left[\frac{e^{-\int_t^{T_i} r_s ds}}{P^{T_i}(t)} \mathbf{1}_{(CB(T_0) > K)} \right] - K P^{T_0}(t) \mathbb{E}_t^Q \left[\frac{e^{-\int_t^{T_0} r_s ds}}{P^{T_0}(t)} \mathbf{1}_{(CB(T_0) > K)} \right] \\
&= \sum_{i=1}^N C_i P^{T_i}(t) \mathbb{E}_t^{T_i} \left[\mathbf{1}_{(CB(T_0) > K)} \right] - K P^{T_0}(t) \mathbb{E}_t^{T_0} \left[\mathbf{1}_{(CB(T_0) > K)} \right] \\
&= \sum_{i=1}^N C_i P^{T_i}(t) \pi^{T_i} \left(\widetilde{CB(T_0)} > K \middle| \mathcal{F}_t \right) - K P^{T_0}(t) \pi^{T_0} \left(\widetilde{CB(T_0)} > K \middle| \mathcal{F}_t \right) \tag{4}
\end{aligned}$$

where the first line on the right-hand-side comes from multiplying and dividing by $P^{T_i}(t)$, which is an observable number at date t , so it can be placed inside or outside the expectation, and the second line follows from the definition of the forward measures.

Equation (4) can be interpreted as stating that the price of a swaption is related to a series of probabilities $\pi^{T_i}(\widetilde{CB(T_0)} | \mathcal{F}_t)$ that the underlying coupon bond will end up in the money. As emphasized by the superscript π^{T_i} , these probabilities are to be determined for *each* of the $(N + 1)$ relevant forward measures (T_0, \dots, T_N) .¹⁰

As we have noted, the probability densities $\pi^{T_i}(\widetilde{CB(T_0)} | \mathcal{F}_t)$ do not have analytic solutions. To approximate these densities, we determine the first M moments of the distribution, each of which does have an analytic solution. That is, for each of the $i = 0, 1, \dots, N$ forward measures, we determine the first $m = 1, 2, \dots, M$ moments of $CB(T_0)$: $\mathbb{E}_t^{T_i}[(CB(T_0))^m]$.

Note that for any m , $(CB(T_0))^m$ can be written as a sum of terms, each involving a product of m bond prices:

$$\begin{aligned}
(CB(T_0))^m &= (C_1 P^{T_1}(T_0) + C_2 P^{T_2}(T_0) + \dots + C_N P^{T_N}(T_0))^m \\
&= \sum_{i_1, i_2, \dots, i_m=1}^N \left(C_{i_1} \dots C_{i_m} \right) \times \left(P^{T_{i_1}}(T_0) \dots P^{T_{i_m}}(T_0) \right) \tag{5}
\end{aligned}$$

Since all bond prices have an exponential affine structure as in Equation (1), it follows that products of bond prices also have an exponential affine form. Hence, Equation (5) can be written as a sum of terms, each written in an exponential form:

$$(C B T_0)^m = \sum_{i_1, i_2, \dots, i_m=1}^N (C_{i_1} \cdots C_{i_m}) \times (e^{F_0 + \sum_{j=1}^J X_j(T_0) F_j}) \quad (6)$$

where the coefficients F_0 and F_j are sums of the $B_0(T_i - T_0)$ and $B_j(T_i - T_0)$ functions defined above.

Note that $(CB[T_0])^m$ depends only on the state variables $X_j(T_0)$ in an exponentially affine manner. This implies that the date t expectation of $(CB[T_0])^m$ also has an exponentially affine solution:¹¹

$$E_t^{T_i} [(C B T_0)^m] = \sum_{i_1, i_2, \dots, i_m=1}^N (C_{i_1} \cdots C_{i_m}) \times e^{H_0(T_0 - t) + \sum_{j=1}^J X_j(t) H_j(T_0 - t)} \quad (7)$$

where the deterministic functions $H_0(\cdot)$ and $H_j(\cdot)$ satisfy a set of Ricatti equations. Hence, Equation (7) demonstrates that all moments of coupon bond prices have analytic solutions within an affine framework.

After determining the exact first M moments of $CB(T_0)$ under each forward measure of interest, we estimate $\pi^{T_i}(CB[T_0]) > K$ for each of the T_i forward measures of interest by performing a cumulant expansion on $\pi^{T_i}(CB[T_0])$. The cumulants of a distribution are no more mysterious than the underlying moments of a distribution. Indeed, there is a one-to-one relationship between moments and cumulants. For example, the first two cumulants of a distribution are its mean and its variance. More generally, cumulants are defined as the coefficients of a Taylor series expansion of the logarithm of the characteristic function. In other words, define:

$$G(k, t) \equiv \int_{-\infty}^{\infty} e^{iky} \pi(\widetilde{CB(T_0)} = y | \mathcal{F}_t) dy \quad (8)$$

as the characteristic function of the random variable $CB(T_0)$. Then the cumulants $\{c_j\}$ are defined via:

$$\log [G(k, t)] = \sum_{j=1}^{\infty} c_j \frac{(ik)^j}{j!} \quad (9)$$

The n -th order cumulant is uniquely defined by the first n moments of the distribution (see, for example, Gardiner [1983]). As a reference, the first seven cumulants are provided in Appendix A.

Armed with an explicit expression for the cumulants, we can obtain the probability density $\pi(\cdot)$ of $CB(T_0)$ by inverse Fourier transform:

$$\pi(C B T_0) = y | \mathcal{F}_t = \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{-iky} G(k, t) dk \quad (10)$$

We can then make use of our cumulant expansion of the characteristic function to obtain:

$$\begin{aligned} \pi(C B T_0) = y | \mathcal{F}_t &= \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{-iky} e^{\log(G(k, t))} dk \\ &= \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{-iky} e^{\sum_{j=1}^{\infty} \frac{(ik)^j}{j!} c_j} dk \\ &= \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{-iky} e^{ikc_1 - \frac{k^2}{2} c_2} e^{\Lambda} dk \end{aligned} \quad (11)$$

where $\Lambda \equiv \sum_{j=3}^{\infty} [(ik)^j / j!] c_j$.

Up to this point, the solution is exact. The approximation occurs when one truncates the Taylor series expansion $e^{\Lambda} \approx \sum_{n=0}^{[M/3]} [\Lambda^n / n!]$, where $[M/3]$ is the largest integer less than or equal to $M/3$. To expand to order $M = 7$, it is sufficient to approximate:¹²

$$\begin{aligned}
e^\Lambda &\approx 1 + \Lambda + \frac{\Lambda^2}{2} \\
&= 1 + \left(\frac{(ik)^3}{3!} c_3 + \frac{(ik)^4}{4!} c_4 + \frac{(ik)^5}{5!} c_5 + \frac{(ik)^6}{6!} c_6 + \frac{(ik)^7}{7!} c_7 \right) + \frac{1}{2} \left(\frac{(ik)^3}{3!} c_3 + \frac{(ik)^4}{4!} c_4 \right)^2 \\
&\approx 1 - ik^3 \left(\frac{c_3}{3!} \right) + k^4 \left(\frac{c_4}{4!} \right) + ik^5 \left(\frac{c_5}{5!} \right) - k^6 \left(\frac{c_6}{6!} + \frac{1}{2} \left(\frac{c_3}{3!} \right)^2 \right) - ik^7 \left(\frac{c_7}{7!} + \frac{c_3 c_4}{3! 4!} \right) \\
&\equiv 1 + \sum_{n=3}^7 \alpha_n k^n \tag{12}
\end{aligned}$$

$$\equiv \sum_{n=0}^7 \alpha_n k^n \tag{13}$$

Equation (13) is equivalent to Equation (12) (with $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 0$). We choose $M = 7$ because it offers an excellent balance between speed and accuracy.

For parameters of interest, however, we find that:

$$\frac{c_6}{6!} \ll \frac{1}{2} \left(\frac{c_3}{3!} \right)^2, \quad \frac{c_7}{7!} \ll \frac{c_3 c_4}{3! 4!}$$

Because it is computationally expensive to determine the higher-order cumulants c_6 and c_7 , we find it convenient to set these both to zero. This is not equivalent to choosing $M = 5$; rather, it is simply making the two approximations $\alpha_6 \approx -1/2(c_3/3!)^2$ and $\alpha_7 \approx -i[c_3 c_4/3! 4!]$ within the $M = 7$ framework.¹³

Hence, using Equations (11) and (12), to order k^7 , we find:

$$\begin{aligned}
\pi^7(CBT_o) = y|\mathcal{F}_t &= \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{-\left(\frac{k^2}{2}\right) c_2 - ik(y-c_1)} \times \left(1 + \sum_{n=3}^7 \alpha_n k^n \right) dk \\
&= \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(y-c_1)^2}{2c_2}} + \sum_{n=3}^7 \alpha_n \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} k^n e^{-\left(\frac{k^2}{2}\right) c_2 - i(y-c_1)k} dk \tag{14}
\end{aligned}$$

Note that the first term in Equation (14) approximates the transition density of the future coupon bond price as distributed normally about the actual mean and variance of the coupon bond. Hence, as claimed previously, we can see that the cumulant expansion generates an expansion about a normal distribution. The remaining terms in Equation (14) improve upon this approximation.

It is more convenient, however, to rewrite Equation (14) using Equations (11) and (13):

$$\pi^7(CBT_o) = y|\mathcal{F}_t = \sum_{n=0}^7 \alpha_n \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} k^n e^{-\left(\frac{k^2}{2}\right) c_2 - i(y-c_1)k} dk \tag{15}$$

This expansion results in a sum of simple integrals, which can easily be solved by noting that:

$$\begin{aligned}
& \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} k^n e^{\left(-\frac{k^2}{2} c_2 - i(y-c_1)k \right)} dk \\
&= \frac{\partial^n}{\partial \beta^n} \left[\left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{-\frac{k^2}{2} c_2 + \beta k} \right] \Big|_{\beta = -i(y-c_1)} dk \\
&= \left(\frac{1}{\sqrt{2\pi c_2}} \right) \frac{\partial^n}{\partial \beta^n} \left[e^{\frac{\beta^2}{2c_2}} \right] \Big|_{\beta = -i(y-c_1)} \\
&\equiv \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(y-c_1)^2}{2c_2}} \left[\sum_{j=0}^7 a_j^n (y-c_1)^j \right] \quad (16)
\end{aligned}$$

where the last line defines the coefficients a_j^n .

Combining Equations (14) and (16), we find that the probability density can be written

$$\pi^7(CB(T_0) = y | \mathcal{F}_t) = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(y-c_1)^2}{2c_2}} \left(\sum_{j=0}^7 \gamma_j (y-c_1)^j \right) \quad (17)$$

where

$$\gamma_j = \left(\sum_{n=0}^7 \alpha_n a_j^n \right) \quad (18)$$

The coefficients γ_j are provided in Appendix B.

To price a swaption with strike K , we need to compute the date 0 probability that $CB(T_0)$ will fall above the strike price. That is, we need to compute the integral:

$$\int_K^{\infty} \pi^7(CB(T_0) = y | \mathcal{F}_t) dy = \sum_{j=0}^7 \gamma_j \lambda_j \quad (19)$$

where

$$\lambda_j = \frac{1}{\sqrt{2\pi c_2}} \int_K^{\infty} (y-c_1)^j e^{-\frac{(y-c_1)^2}{2c_2}} dy \quad (20)$$

$$= \frac{1}{\sqrt{2\pi}} (c_2)^{\frac{j}{2}} \int_{\frac{K-c_1}{\sqrt{c_2}}}^{\infty} z^j e^{-\frac{1}{2}z^2} dz \quad (21)$$

Note that all λ_j can be solved in closed form and involve, at worst, the one-dimensional cumulative normal distribution function, for which there are standard numerical routines that do not require any numerical integration. We have thus obtained a very simple expression for the probability that the coupon bond price will be in the money. It involves only simple summations. In Appendix B we present the expressions for the coefficients γ_j, λ_j for $j = 0, \dots, 7$.

The swaption can then be written as

$$\begin{aligned}
\text{Sw}_{\text{N}}^{T_0, \{T_i\}}(t) &= \sum_{i=1}^N C_i P^{T_i}(t) \left(\sum_{j=0}^7 \gamma_j^{T_i} \lambda_j^{T_i} \right) - \\
&KP^{T_0}(t) \left(\sum_{j=0}^7 \gamma_j^{T_0} \lambda_j^{T_0} \right) \quad (22)
\end{aligned}$$

where $\gamma_j^{T_i}$ and $\lambda_j^{T_i}$ are the various coefficients computed under each T_i forward-neutral measure.

II. NUMERICAL RESULTS

We consider two models: a three-factor Gaussian model and a two-factor CIR model. Since the approach is model-independent, a single program can be written for all models, needing only a call to a subroutine for each specific model. We choose $M = 7$ for the order of expansion, since it appears to offer an excellent compromise between speed and accuracy. For both cases, we compute prices of swaptions for various strikes and compare them to Monte Carlo simulated prices for accuracy. Note that the normalized highest-order cumulant provides a good estimate of the attained accuracy.

Three-Factor Gaussian Model

We consider a three-dimensional Gaussian model with state variable dynamics as follows:

$$dx_i = -\kappa_i x_i dt + \sigma_i dz_i^Q \quad (23)$$

where $dz_i^Q dz_j^Q = \rho_{ij} dt$, and $r = \delta + \sum_{i=1}^3 x_i$.¹⁴

The bond prices take the form (see Langetieg [1980]):

$$P^T(s, x_s) = e^{B_0(T-s) - \sum_{i=1}^3 B_{\kappa_i}(T-s) x_i(s)} \quad (24)$$

where

$$B_{\kappa_i}(\tau) = \frac{1 - e^{-\kappa_i \tau}}{\kappa_i} \quad (25)$$

$$B_0(\tau) = -\delta\tau + \sum_{i,j} \frac{\sigma_i \sigma_j \rho_{ij}}{\kappa_i \kappa_j} \left[\tau - B_{\kappa_i}(\tau) - B_{\kappa_j}(\tau) + B_{\kappa_i + \kappa_j}(\tau) \right] \quad (26)$$

where we define $\rho_{ii} = 1$.

Under the W forward measure, the state variables have the dynamics

$$dx_i(t) = \left(-\kappa_i x_i - \sum_{j=1}^3 \sigma_i \sigma_j \rho_{ij} B_{\kappa_j}(W-t) \right) dt + \sigma_i dz_i^W \quad (27)$$

The expectation of products of bond prices at some future date can be computed using the expression for the Laplace transform of the state variable under the forward-neutral measure:

$$\begin{aligned} L(t, T, W) &\equiv E_t^W \left[e^{-\sum_{i=1}^3 F_i x_i(T)} \right] \\ &= e^{M(T-t) - \sum_{i=1}^3 N_i(T-t) x_i(t)} \end{aligned} \quad (28)$$

where M and N_i are given by:

$$N_i(\tau) = F_i e^{-\kappa_i \tau} \quad (29)$$

$$\begin{aligned} M(\tau) &= \sum_{i,j} \frac{\sigma_i \sigma_j \rho_{ij}}{\kappa_j} F_i \left[B_{\kappa_i}(\tau) - e^{-\kappa_j(W-T)} B_{\kappa_i + \kappa_j}(\tau) \right] + \\ &\quad \sum_{i \geq j} \sigma_i \sigma_j \rho_{ij} F_i F_j B_{\kappa_i + \kappa_j}(\tau) \end{aligned} \quad (30)$$

These formulas allow us to compute all the moments of the coupon bond price at the maturity date T_0 . We can thus compute the relevant cumulants (see Appendix A) and the parameters $\gamma_j^{T_i}, \lambda_j^{T_i}$ to be used in Equation (22).

The parameter values for the numerical illustration are given in Exhibit 1. Exhibits 2 and 3 show, respectively, the absolute and relative deviations of our approximation compared to a Monte Carlo solution.

The Monte Carlo prices are obtained using the exact (Gaussian) distribution of the state variable at maturity to avoid any time discretization bias. The number of simulations is set to obtain standard errors of order 10^{-7} (2 million random draws with standard variance reduction techniques).

As the graphs show, the approximation is excellent. The absolute error relative to the true solution is less than a few parts in 10^{-6} . The relative error is very small, less than a few parts in 10^{-3} , with the biggest errors for highly out-of-the-money options, which have negligible values, thus making this type of metric somewhat misleading. The approximation takes less than 0.05 seconds to compute all 50 swaption prices (corresponding to different strikes).

Another advantage of the Edgeworth expansion approach is that the order of magnitude of the error term can be predicted by looking at the scaled cumulants $(c_k/k!c_2^{k/2})$.¹⁵ In Exhibit 4, we present the mean, variance, and the third through fifth scaled cumulants for each of the $(N+1) = 21$ measures. Two notable features are apparent.

First, the scaled cumulants decay quickly, which provides an indication of the appropriateness of the Edgeworth expansion approach. Further, it also provides an estimate of the truncation error. Indeed, at the rate at which the scaled cumulants are decaying, one can guess that the sixth scaled cumulant, and hence the error, is indeed of the order of 10^{-6} .

Second, the fifth scaled cumulants are nearly identical across measures. Hence, for time efficiency, one needs to calculate only the fifth scaled cumulant for a single measure.

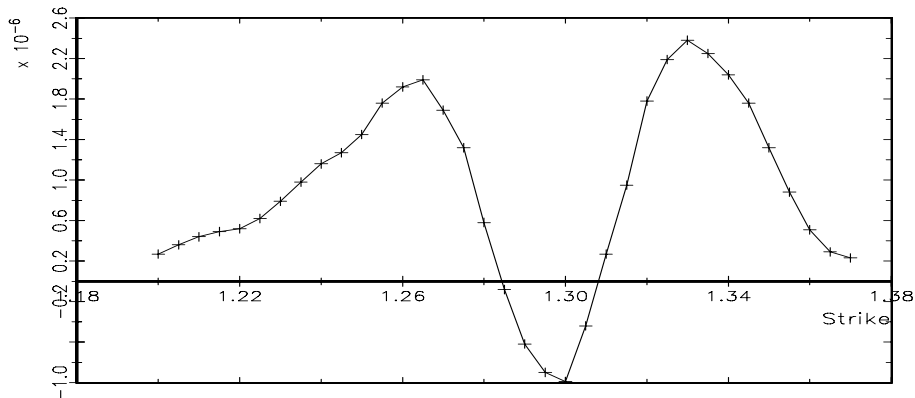
EXHIBIT 1

Parameters for Gaussian Three-Factor Model

$x_1(0)$	$x_2(0)$	$x_3(0)$	δ	κ_1	κ_2	κ_3	σ_1	σ_2	σ_3	ρ_{12}	ρ_{13}	ρ_{23}
0.01	0.005	-0.02	0.06	1.0	0.2	0.5	0.01	0.005	0.002	-0.2	-0.1	0.3

EXHIBIT 2

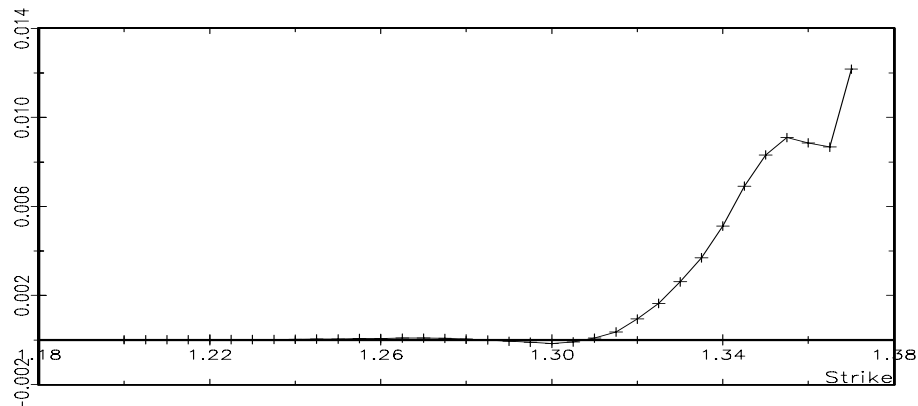
Difference Between Cumulant Approximation and Monte Carlo Swaption Prices for Various Strike Prices



Parameters as in Exhibit 1. Monte Carlo run using the exact (Gaussian) distribution of the state variable at maturity to avoid a time discretization bias. Standard error of Monte Carlo prices less than 5×10^{-7} .

EXHIBIT 3

Relative Difference Between Cumulant Approximation and Monte Carlo Swaption Prices for Various Strike Prices



Parameters as in Exhibit 1. Monte Carlo run using the exact (Gaussian) distribution of the state variable at maturity to avoid a time discretization bias. Standard error of Monte Carlo prices less than 5×10^{-7} .

Two-Factor CIR Model

To investigate whether these results are specific to the Gaussian case, we apply the same approach to a second example where the state variables do not follow a Gaussian process. We choose a standard two-factor CIR model of the term structure. The spot rate is defined as $r = \delta + x_1 + x_2$, where the two state variables follow independent square root processes:

$$dx_i = \kappa_i(\theta_i - x_i) dt + \sigma_i \sqrt{x_i} dz_i^Q \quad (31)$$

where the Brownian motions are independent. Bond prices are a simple extension of the original CIR bond pricing formula:

$$P^T(s, x_1(s), x_2(s)) = e^{B_0(T-s) - x_1(s)B_1(T-s) - x_2(s)B_2(T-s)} \quad (32)$$

EXHIBIT 4

Mean, Variance, and Scaled Cumulants for Forward Measures and Risk-Neutral Measure for Three-Factor Gaussian Model

Measure	Mean	Variance	$\frac{c_3}{3! c_2^{3/2}}$	$\frac{c_4}{4! c_2^{4/2}}$	$\frac{c_5}{5! c_2^{5/2}}$
0	1.291480	0.00073789	0.01166506	0.00036705	0.0000136
1	1.291569	0.00073801	0.01166509	0.00036705	0.0000136
2	1.291647	0.00073810	0.01166513	0.00036706	0.0000136
3	1.291714	0.00073819	0.01166515	0.00036706	0.0000136
4	1.291773	0.00073826	0.01166518	0.00036706	0.0000136
5	1.291826	0.00073833	0.01166521	0.00036706	0.0000136
6	1.291872	0.00073839	0.01166523	0.00036706	0.0000135
7	1.291914	0.00073844	0.01166525	0.00036706	0.0000136
8	1.291951	0.00073849	0.01166527	0.00036706	0.0000137
9	1.291985	0.00073853	0.01166528	0.00036707	0.0000136
10	1.292015	0.00073857	0.01166530	0.00036707	0.0000136
11	1.292042	0.00073861	0.01166531	0.00036707	0.0000136
12	1.292066	0.00073864	0.01166532	0.00036707	0.0000135
13	1.292088	0.00073867	0.01166533	0.00036707	0.0000136
14	1.292108	0.00073869	0.01166534	0.00036707	0.0000135
15	1.292126	0.00073871	0.01166535	0.00036707	0.0000136
16	1.292142	0.00073873	0.01166536	0.00036707	0.0000136
17	1.292157	0.00073875	0.01166537	0.00036707	0.0000137
18	1.292170	0.00073877	0.01166537	0.00036707	0.0000136
19	1.292182	0.00073879	0.01166538	0.00036707	0.0000136
20	1.292193	0.00073880	0.01166538	0.00036707	0.0000136

where

$$\gamma_i \equiv \sqrt{\kappa_i^2 + 2\sigma_i^2}$$

$$B_0(\tau) = -\delta\tau + \sum_{i=1}^2 \times \left(\frac{2\kappa_i\theta_i}{\gamma_i - \kappa_i} \tau - \frac{2\kappa_i\theta_i}{\sigma_i^2} \log \left[\frac{(\kappa_i + \gamma_i)(e^{\gamma_i\tau} - 1) + 2\gamma_i}{2\gamma_i} \right] \right) \quad (33)$$

$$B_i(\tau) = \frac{2(e^{\gamma_i\tau} - 1)}{(\kappa_i + \gamma_i)(e^{\gamma_i\tau} - 1) + 2\gamma_i} \quad (34)$$

and where we have defined

From Equation (32), we note that products of bond prices (with differing maturities) will take the form:

$$P^{T_i}(T_0) P^{T_j}(T_0) \dots P^{T_n}(T_0) = e^{F_0 - x_1(T_0) F_1 - x_2(T_0) F_2} \quad (35)$$

As in the Gaussian case, we can compute (for all relevant measures) the moments of the distribution of a coupon bond by noting

$$L(t, T_0, W) \equiv E_t^W \left[e^{F_0 - x_1(T_0) F_1 - x_2(T_0) F_2} \right] \quad (36)$$

$$= \frac{1}{PW(t)} E_t^Q \left[e^{-\int_t^T dv r_v} e^{F_0 - x_1(T_0) F_1 - x_2(T_0) F_2} \right] \quad (37)$$

$$= \frac{1}{PW(t)} E_t^Q \left[e^{-\int_t^T dv r_v} e^{F_0^* - x_1(T_0) F_1^* - x_2(T_0) F_2^*} \right] \quad (38)$$

where $F_i^* = F_i + B_i(W - T_0)i = 0, 1, 2$.

It is well known that the solution to this expectation takes the form:

$$L(t, T_0, W) = \frac{1}{PW(t)} e^{M(T-t) - x_1(t) N_1(T-t) - x_2(t) N_2(T-t)} \quad (39)$$

where the functions M , N_1 , and N_2 satisfy the Riccati equations:

$$-N_i' = -1 + \kappa_i N_i + \frac{\sigma_i^2}{2} N_i^2 \quad (40)$$

$$M' = -\delta - \sum_{i=1}^2 \kappa_i \theta_i N_i \quad (41)$$

with initial conditions $N_i(0) = F_i^*$, $M(0) = F_0^*$.

We find

$$M(\tau) = F_0^* - \delta \tau + \sum_{i=1}^2 \left(\frac{2\kappa_i \theta_i}{\gamma_i - \kappa_i} \tau - \frac{2\kappa_i \theta_i}{\sigma_i^2} \log \left[\frac{(F_i^* - \lambda_{i,-}) e^{\gamma_i \tau} - (F_i^* - \lambda_{i,+})}{\frac{2\gamma_i}{\sigma_i^2}} \right] \right) \quad (42)$$

$$N_i(\tau) = \frac{F_i^* (\lambda_{i,+} e^{\gamma_i \tau} - \lambda_{i,-}) + \frac{2}{\sigma_i^2} (e^{\gamma_i \tau} - 1)}{F_i^* (e^{\gamma_i \tau} - 1) - (\lambda_{i,-} e^{\gamma_i \tau} - \lambda_{i,+})} \quad (43)$$

where we have defined

$$\gamma_i \equiv \sqrt{\kappa_i^2 + 2\sigma_i^2}$$

$$\lambda_{i,\pm} \equiv \frac{-\kappa_i \pm \gamma_i}{\sigma_i^2}$$

We can thus determine the relevant cumulants and parameter inputs $\{\gamma_j^i, \lambda_j^i\}$ that are needed to price the swaption using Equation (22). The parameter values are

provided in Exhibit 5. Exhibits 6 and 7 show, respectively, the absolute and relative deviations of our approximation compared to a Monte Carlo solution.

The Monte Carlo prices are obtained using a standard Euler discretization scheme of the stochastic differential equation. To reduce the time discretization bias, we choose a very small time step: $dt = 3 \times 10^{-5}$. The number of simulations is set to obtain standard errors of order less than 10^{-6} (e.g., 5 million paths with standard variance reduction techniques).¹⁶

As the graphs show, the approximation is still excellent, although slightly less accurate than the three-factor Gaussian case. The absolute error relative to the true solution is less than a few parts in 10^{-5} , and the relative error is very small, less than a few parts in 10^{-2} . The approximation takes less than 0.2 seconds to compute all 50 swaption prices (corresponding to different strikes).

Exhibit 8 presents the mean, variance, and the third through fifth scaled cumulants for each of the $(N + 1) = 21$ measures. Note that the third cumulant is now negative. This can be understood as follows. Under the square root process, higher interest rates lead to higher volatility, in turn leading to an upward skew in interest rates, which produces a downward skew for (coupon) bond prices. Also note that the cumulants do not decay as quickly as in the Gaussian case, leading to a slightly larger error for this case.¹⁷

Finally, note that the fifth scaled cumulants are not as similar as they were in the Gaussian case. Thus, for numerical efficiency one can choose to compute only two of them, corresponding to the shortest and longest forward measure maturities, and then estimate the others via interpolation as a function of forward measure maturity.

III. CONCLUSION

We have presented a new approach based on a cumulant expansion to price coupon bond options and hence swaptions in affine frameworks. Our approximation performs very well for both Gaussian and square root affine models. For example, for the three-factor Gaussian model, we obtain prices in fewer than 0.05 seconds and accurate to a few parts in 10^{-6} .

Given the size of fixed-income markets for swaps and swaps derivatives, this approach should attract widespread interest. Practitioners need fast and accurate formulas to mark to market and hedge their books of derivatives. Academics need fast and accurate solutions to estimate likelihood functions with multiple parameters.

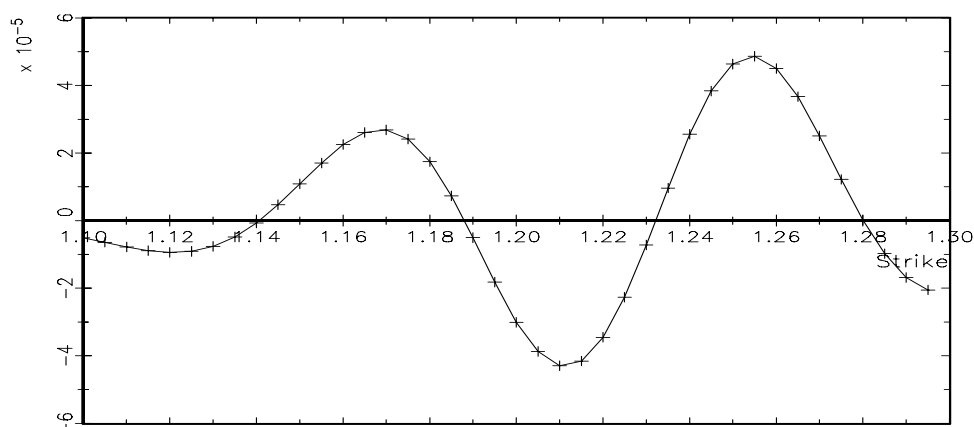
EXHIBIT 5

Parameters for Two-Factor CIR Model

$x_1(0)$	$x_2(0)$	δ	κ_1	κ_2	θ_1	θ_2	σ_1	σ_2
0.04	0.02	0.02	0.02	0.02	0.03	0.01	0.04	0.02

EXHIBIT 6

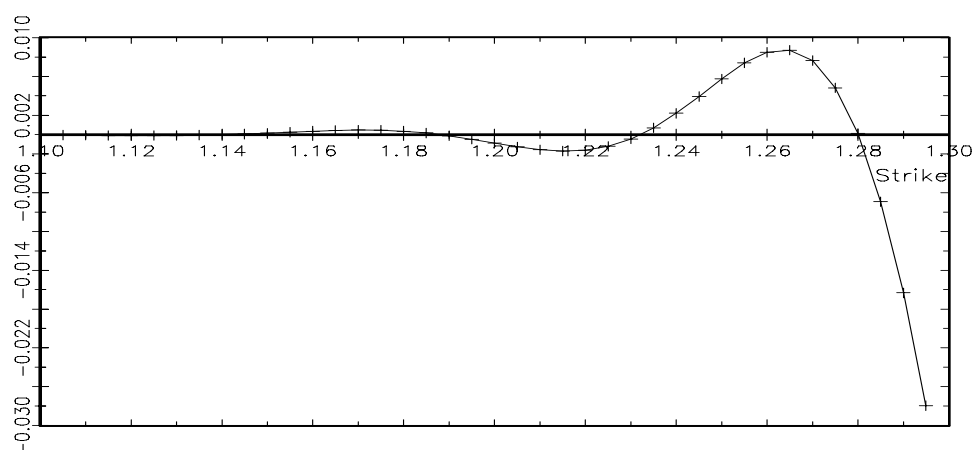
Difference Between Cumulant Approximation and Monte Carlo Swaption Prices for Various Strike Prices



Parameters as in Exhibit 5. Monte Carlo run using 5 million paths and setting $dt = 3 \times 10^{-5}$. Standard error of Monte Carlo prices less than 5×10^{-6} .

EXHIBIT 7

Relative Difference Between Cumulant Approximation and Monte Carlo Swaption Prices for Various Strike Prices



Parameters as in Exhibit 5. Monte Carlo run using 5 million paths and setting $dt = 3 \times 10^{-5}$. Standard error of Monte Carlo prices less than 5×10^{-6} .

EXHIBIT 8

Mean, Variance, and Scaled Cumulants for Forward Measures and Risk-Neutral Measure for Two-Factor CIR Model

Measure	Mean	Variance	$\frac{c_3}{3! c_2^{3/2}}$	$\frac{c_4}{4! c_2^{4/2}}$	$\frac{c_5}{5! c_2^{5/2}}$
0	1.23511950	0.00160541	-0.04087912	0.00205713	0.00012643
1	1.23530353	0.00160333	-0.04089828	0.00206201	0.00012582
2	1.23546977	0.00160145	-0.04091559	0.00206642	0.00012526
3	1.23561989	0.00159976	-0.04093121	0.00207041	0.00012476
4	1.23575544	0.00159823	-0.04094532	0.00207401	0.00012429
5	1.23587782	0.00159684	-0.04095805	0.00207725	0.00012388
6	1.23598829	0.00159560	-0.04096954	0.00208019	0.00012351
7	1.23608800	0.00159447	-0.04097991	0.00208284	0.00012317
8	1.23617799	0.00159346	-0.04098927	0.00208523	0.00012285
9	1.23625919	0.00159254	-0.04099772	0.00208739	0.00012259
10	1.23633246	0.00159172	-0.04100534	0.00208933	0.00012234
11	1.23639857	0.00159097	-0.04101222	0.00209109	0.00012210
12	1.23645821	0.00159030	-0.04101842	0.00209268	0.00012190
13	1.23651202	0.00158969	-0.04102402	0.00209411	0.00012172
14	1.23656055	0.00158915	-0.04102907	0.00209540	0.00012155
15	1.23660433	0.00158865	-0.04103363	0.00209656	0.00012140
16	1.23664382	0.00158821	-0.04103774	0.00209761	0.00012128
17	1.23667944	0.00158781	-0.04104144	0.00209856	0.00012116
18	1.23671156	0.00158745	-0.04104479	0.00209941	0.00012103
19	1.23674053	0.00158712	-0.04104780	0.00210018	0.00012096
20	1.23676666	0.00158683	-0.04105052	0.00210088	0.00012085

The cumulant expansion technique presented may prove useful in other applications in financial economics. First, it can be applied to so-called extended affine models that perfectly fit the initial term structure. These models basically relax the time homogeneity assumption for the state vector by making some parameters time-dependent. The latter are picked to fit the initially observed term structure (Hull and White [1990], Dybvig [1997]).¹⁸

Further, this approach should generalize to jumps within the affine structure (Duffie, Pan, and Singleton [2000]), quadratic models (Longstaff [1989], Beaglehole and Tenney [1991], and Constantinides [1992]), or to Heath, Jarrow, and Morton [1992] models or even random field models with a generalized affine structure; see Collin-Dufresne and Goldstein [2001]. Finally, the approach can be used to approximate the transition density of the state vector, which is useful to perform maximum-likelihood estimation of the parameters.

APPENDIX A

Relation Between Cumulants and Moments

For reference, here we provide the first seven cumulants $\{c_i\}$, in terms of the moments $\{\mu_i\}$. A formula that relates cumulants and moments can be found in Gardiner [1983].

$$c_1 = \mu_1 \quad (\text{A-1})$$

$$c_2 = \mu_2 - \mu_1^2 \quad (\text{A-2})$$

$$c_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \quad (\text{A-3})$$

$$c_4 = \mu_4 - 4\mu_1\mu_3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4 \quad (\text{A-4})$$

$$c_5 = \mu_5 - 5\mu_1\mu_4 - 10\mu_2\mu_3 + 20\mu_1^2\mu_3 + 30\mu_1\mu_2^2 - 60\mu_1^3\mu_2 + 24\mu_1^5 \quad (\text{A-5})$$

$$c_6 = \mu_6 - 6\mu_1\mu_5 - 15\mu_2\mu_4 + 30\mu_1^2\mu_4 - 10\mu_3^2 + 120\mu_1\mu_2\mu_3 - 120\mu_1^3\mu_3 \\ + 30\mu_2^3 - 270\mu_1^2\mu_2^2 + 360\mu_1^4\mu_2 - 120\mu_1^6 \quad (\text{A-6})$$

$$c_7 = \mu_7 - 7\mu_1\mu_6 - 21\mu_2\mu_5 - 35\mu_3\mu_4 + 140\mu_1\mu_3^2 - 630\mu_1\mu_2^3 + 210\mu_1\mu_2\mu_4 - 1260\mu_1^2\mu_2\mu_3 \\ + 42\mu_1^2\mu_5 + 2520\mu_1^3\mu_2^2 - 210\mu_1^3\mu_4 + 210\mu_2^2\mu_3 + 840\mu_1^4\mu_3 - 2520\mu_1^5\mu_2 + 720\mu_1^7 \quad (\text{A-7})$$

APPENDIX B

Coefficients in Approximation of Order M = 7

$$\pi^7(CB(T_o)) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} e^{-\left(\frac{k^2}{2}\right)c_2 - ik(CB(T_o) - c_1)} \times \\ \left[1 + \left(-\frac{ic_3}{3!}k^3 + \frac{c_4}{4!}k^4 + \frac{ic_5}{5!}k^5 - \frac{c_6}{6!}k^6 - \frac{ic_7}{7!}k^7\right) + \frac{1}{2}\left(-\frac{c_3^2}{(3!)^2}k^6 - \frac{2ic_3c_4}{3!4!}k^7\right)\right] dk \quad (\text{B-1})$$

Define $Z \equiv \left(\frac{CB(T_o) - c_1}{\sqrt{c_2}}\right) \Rightarrow \pi(Z) dz = \pi(CB(T_o)) dy$. Then, the probability density can be written as:

$$\pi^7(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} \left(1 + \sum_{i=3}^7 \pi(i)\right) \quad (\text{B-2})$$

where

$$\pi(3) = \left(\frac{c_3}{3! c_2^{3/2}}\right) [Z^3 - 3Z] \quad (\text{B-3})$$

$$\pi(4) = \left(\frac{c_4}{4! c_2^{4/2}}\right) [Z^4 - 6Z^2 + 3] \quad (\text{B-4})$$

$$\pi(5) = \left(\frac{c_5}{5! c_2^{5/2}} \right) \left[Z^5 - 10Z^3 + 15Z \right] \quad (\text{B-5})$$

$$\pi(6) = \left(\frac{c_6}{6! c_2^{6/2}} + \frac{1}{2} \left(\frac{c_3}{3! c_2^{3/2}} \right) \left(\frac{c_3}{3! c_2^{3/2}} \right) \right) \left[Z^6 - 15Z^4 + 45Z^2 - 15 \right] \quad (\text{B-6})$$

$$\pi(7) = \left(\frac{c_7}{7! c_2^{7/2}} + \left(\frac{c_3}{3! c_2^{3/2}} \right) \left(\frac{c_4}{4! c_2^{4/2}} \right) \right) \left[Z^7 - 21Z^5 + 105Z^3 - 105Z \right] \quad (\text{B-7})$$

For pricing options, we eventually want to integrate this density above some strike price K . Defining $\gamma \equiv (CB(T_0) - c_1)$, we have:

$$\int_{K-c_1}^{\infty} \pi(y) dy = \sum_{j=0}^7 \gamma_j \int_{K-c_1}^{\infty} y^j \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{y^2}{2c_2}} dy \quad (\text{B-8})$$

$$= \sum_{j=0}^7 \gamma_j \lambda_j \quad (\text{B-9})$$

All these terms can be written, at worst, in terms of the cumulative normal function, for which there are excellent approximations without the need of numerical integration. The first seven are:

$$\lambda_0 = N \left[\frac{c_1 - K}{\sqrt{c_2}} \right] \quad (\text{B-10})$$

$$\lambda_1 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2] \quad (\text{B-11})$$

$$\lambda_2 = c_2 N \left[\frac{c_1 - K}{\sqrt{c_2}} \right] + \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2 (K - c_1)] \quad (\text{B-12})$$

$$\lambda_3 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2 (K - c_1)^2 + 2c_2^2] \quad (\text{B-13})$$

$$\lambda_4 = 3c_2^2 N \left[\frac{c_1 - K}{\sqrt{c_2}} \right] + \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2 (K - c_1)^3 + 3c_2^2 (K - c_1)] \quad (\text{B-14})$$

$$\lambda_5 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2 (K - c_1)^4 + 4c_2^2 (K - c_1)^2 + 8c_2^3] \quad (\text{B-15})$$

$$\lambda_6 = 15c_2^3 N \left[\frac{K - c_1}{\sqrt{c_2}} \right] + \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2 (K - c_1)^5 + 5c_2^2 (K - c_1)^3 + 15c_2^3 (K - c_1)] \quad (\text{B-16})$$

$$\lambda_7 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} [c_2 (K - c_1)^6 + 6c_2^2 (K - c_1)^4 + 24c_2^3 (K - c_1)^2 + 48c_2^4] \quad (\text{B-17})$$

The relevant coefficients γ_m for the λ_m are obtained by collecting terms of the same powers in Equations (B-3)-(B-7). They are

$$\gamma_0 = 1 + \frac{3}{c_2^2} \left(\frac{c_4}{4!} \right) - \frac{15}{c_2^3} \left(\frac{c_6}{6!} + \frac{1}{2} \frac{c_3^2}{(3!)^2} \right) \quad (\text{B-18})$$

$$\gamma_1 = -\frac{3}{c_2^2} \left(\frac{c_3}{3!} \right) + \frac{15}{c_2^3} \frac{c_5}{5!} - \frac{105}{c_2^4} \left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)} \right) \quad (\text{B-19})$$

$$\gamma_2 = -\frac{6}{c_2^3} \left(\frac{c_4}{4!} \right) + \frac{45}{c_2^4} \left(\frac{c_6}{6!} + \frac{1}{2} \frac{c_3^2}{(3!)^2} \right) \quad (\text{B-20})$$

$$\gamma_3 = \frac{1}{c_2^3} \left(\frac{c_3}{3!} \right) - \frac{10}{c_2^4} \frac{c_5}{5!} + \frac{105}{c_2^5} \left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)} \right) \quad (\text{B-21})$$

$$\gamma_4 = \frac{1}{c_2^4} \left(\frac{c_4}{4!} \right) - \frac{15}{c_2^5} \left(\frac{c_6}{6!} + \frac{1}{2} \frac{c_3^2}{(3!)^2} \right) \quad (\text{B-22})$$

$$\gamma_5 = \frac{1}{c_2^5} \frac{c_5}{5!} - \frac{21}{c_2^6} \left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)} \right) \quad (\text{B-23})$$

$$\gamma_6 = \frac{1}{c_2^6} \left(\frac{c_6}{6!} + \frac{1}{2} \frac{c_3^2}{(3!)^2} \right) \quad (\text{B-24})$$

$$\gamma_7 = \frac{1}{c_2^7} \left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)} \right) \quad (\text{B-25})$$

ENDNOTES

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¹See Brace and Musiela [1995], Clewlow, Pang, and Strickland [1996], Brace, Gatarek, and Musiela [1997], Wei [1997], Andersen [1999], Andersen and Andreasen [2000, 2001], Munk [1999], De Jong, Driessen, and Pelsser [2000], Driessen, Klaasen, and Melenberg [2000], Duffie, Pan, and Singleton [2000], and Singleton and Umantsev [2001].

²The proposed methodology also extends to the quadratic term structure models of Longstaff [1989, Beaglehole and Tenney [1992], and Constantinides [1992].

Cox, Ingersoll, and Ross [1985] and Jamshidian [1989] demonstrate that closed-form solutions for options on (zero-coupon) bonds are obtained for one-factor square root and Gaussian models, respectively. Longstaff and Schwartz [1992] extend the result to a two-factor CIR model. More generally, Duffie, Pan, and Singleton [2000] demonstrate that by using inverse Fourier transform methods the entire affine class of models has closed-form solutions for zero-coupon bond options (see Heston [1993]).

³See Kennedy [1994, 1997], Goldstein [2000], and Santa-Clara and Sornette [2001].

⁴Jamshidian [1989] shows that simple solutions for options on coupon bonds can be obtained for *one-factor* models, since in this case the optimal exercise decision at maturity is a one-dimensional boundary. Thus, once the threshold interest rate

r^* is determined, a coupon bond option can be written as a portfolio of zero-coupon bond options. Unfortunately, such a procedure cannot be extended to models with multiple state variables, as the implicit exercise boundary becomes a non-linear function of the state variables.

⁵For example, Jagannathan, Kaplin, and Sun [2000] are unable to compute swaption prices in a three-factor CIR model “due to numerical difficulties.”

⁶One-dimensional expansions have also been recently used to approximate implied risk-neutral distributions. See Jondeau and Rockinger [2000, 2001].

⁷Alternatively, a swaption can also be interpreted as a sum of options on the swap rate that must be exercised at the same date (e.g., Musiela and Rutkowski [1997]).

⁸Duffie, Pan, and Singleton [2000] provide the precise technical regularity conditions on the parameters for the SDE to be well-defined. Dai and Singleton [2000] classify all N -factor affine term structure models into $N + 1$ families depending on how many state variables enter into the conditional variance of the state vector. Our approach is valid for each of these families of models.

⁹Here we price a call option on a coupon bond that is identical to a receiver swaption (e.g., an option to enter a receive fixed, pay floating, swap) when the strike is set to par and the coupon to the strike (rate) of the swaption. Similarly, a payer swaption could be priced as a put option on a coupon bond (or by put-call parity).

¹⁰Comparing Equations (3) and (4), note that $\pi^{T_i}(CB(T_0) > K) \mathcal{F}_t = E_t^Q[\xi \mathbf{1}_{CB(T_0) > K}]$, where we have defined $\xi = e^{-\int_t^{T_i} r_{rs} ds} / P^T(t)$. In general, the expectation of the product of two random variables is not the product of the expectations. Indeed, $E_t^Q[\xi \mathbf{1}_{CB(T_0) > K}] = E_t^Q[\xi \mathbf{1}_{CB(T_0) > K}] + \text{cov}^Q[\xi, \mathbf{1}_{CB(T_0) > K}] = \pi^Q(CB(T_0) > K) \mathcal{F}_t + \text{cov}^Q[\xi, \mathbf{1}_{CB(T_0) > K}]$ since $E_t^Q[\xi] = 1$. Thus we see that if the covariance term is zero the forward-neutral measure is identical to the risk-neutral measure. In general, however, the covariance is not zero, and the change of measure basically modifies the probability of the path of interest rates so that the expectation of the product can be computed as the product of the expectations, but under the new measure. Economically, going to a forward measure amounts to a change of numeraire, namely, using a zero-coupon bond with a specific maturity instead of the continually rolled-over money market fund as numeraire. For a more precise discussion, see Jamshidian [1989] and El Karoui and Rochet [1989].

¹¹See Duffie, Pan, and Singleton [2000] for a general exposition of properties of affine models.

¹²Note that the lowest-order term in Λ^3 is k^9 . Since our expansion goes up only to $M = 7$, it is appropriate to truncate e^Λ at the second order.

¹³It is straightforward to extend this approach to higher-order approximation $M > 7$.

¹⁴It can be shown that this $A_0(3)$ model is maximal, in the sense of Dai and Singleton [2000].

¹⁵That the scaled cumulants are the appropriate measures for estimating the error can be seen from Appendix B. See Equations (B-3)–(B-7).

¹⁶We also used a third pricing approach, a standard numerical integration technique, with similar results (not reported).

¹⁷Note that it may be appropriate to go to the $M = 9$ level, even if we still set $c_6 - c_9$ to zero. Indeed, one can expect a contribution of the order of the third scaled cumulant to the third power, divided by $3!$, which is of the order of 10^{-5} . Indeed, going to higher orders of M is computationally very inexpensive—it is determining the higher-order moments that is computationally costly and grows exponentially in the order.

¹⁸For example, one could simply make δ a deterministic function of time picked to fit the initial term structure, without affecting the approach to price swaptions.

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