GLOSSARY OF TERMS - SURVIVAL MODELS

Prepared by David Forfar, MA, FFA with the assistance of David Raymont, Librarian of Institute of Actuaries, London.

Deterministic Approach (where you know exactly how many persons will die at each age)

Life table functions

l_x :- the number of persons alive aged x in a life table. For example, we might have $l_0=100$, $l_1=90$, $l_2=82$, $l_3=75$, etc. which would mean that out of 100 births, 90 persons are alive at exact age 1, 82 alive at exact age 2, 75 alive at exact age 3 etc.

d_x :- the number of persons who die aged x last birthday. In the above example, $d_0=10$, $d_1=8$, $d_2=7$ etc.

p_x :- the probability that a person aged exactly x survives one year to exactly age (x+1) i.e. the probability of surviving one year. Example, $p_0=90/100=.9$, $p_1=82/90=.9111$, $p_2=75/82=.9146$, etc. Hence $p_x= l_{x+1}/l_x$.

q_x :-the probability that a person aged exactly x dies before exact age (x+1), $q_x = (1-p_x)$ i.e. the probability of dying in the next year. Example, $q_0=10/100=.1$, $q_1= 8/90=.0889$, $q_3=7/82=.0854$. Hence $q_x= d_x/l_x$.

$_tp_x$:-the probability that a person aged exactly x lives for another t years to exact age (x+t). Hence
$_tp_x =l_{(x+t)}/l_x$.

$_tq_x$:- the probability that a person aged exactly x dies before exact age (x+t).
Hence $_tq_x = 1- {}_tp_x = (l_x-l_{(x+t)})/l_x$.

$_{m|n}q_x$:- the probability that a person aged exactly x dies between exact ages (x+n) and (x+m+n). Hence $_{m|n}q_x = (l_{(x+n)}-l_{(x+m+n)})/l_x$. The period m is sometimes referred to as the deferred period.

μ_x , **force of mortality** :- $\mu_x = -\dfrac{1}{l_x}\dfrac{dl_x}{dx}$. Hence μ_x represents the instantaneous rate at which people are dying. If x is fixed and t is the variable then $\mu_{x+t} = -\dfrac{1}{l_{x+t}}\dfrac{dl_{x+t}}{dt}$. By integration of this equation it can

be seen that $l_{x+t} = l_x\, e^{-\int_{s=0}^{s=t}\mu_{x+s}ds}$ and therefore that $_tp_x = e^{-\int_{s=0}^{s=t}\mu_{x+s}ds}$

Assumption of uniform of distribution of deaths between exact ages x and (x+1) :- The d_x deaths between ages x and (x+1) are assumed uniformly spread across the year of age. Hence $l_{(x+t)} = l_x - t.d_x$ for $0 \le t \le 1$.

Assumption of constant force of mortality between exact ages x and (x+1) :- The force of mortality between ages x and (x+1) is assumed constant (μ, say). Hence, $l_{(x+t)} = l_x e^{-\mu t}$ for $0 \le t \le 1$.

T_x **(N.B. this a different from the T_x defined below in the statistical approach although the symbol is the same apart from the italics):-** Total years of life to be lived by those aged exactly x (not the random variable $\mathbf{T_x}$) until they all die, $T_x = \int_{t=0}^{t=\infty} l_{x+t} dt$.

L_x **:-** number of years of life lived between ages x and (x+1) of those currently aged x. $L_x = \int_{t=0}^{t=\infty} l_{x+t} dt$.

$\overset{o}{e_x}$ **, complete expectation of life :-** expectation of life allowing for the whole of the lifetime of the individual (years and days). $\overset{o}{e_x} = \dfrac{T_x}{l_x}$

e_x **, curtate expectation of life : -** expectation of life where only complete years of life count.

$$e_x = \frac{1}{l_x} \sum_{t=1}^{t=\infty} l_{x+t}$$

m_x **, the central death rate :-** defined as $m_x = d_x / L_x$.

Commutation columns

D_x :- defined as $D_x = v^x l_x$ and $\overline{D}_x = (1+i)^{\frac{1}{2}} D_x$

N_x:- defined as $N_x = \sum_{t=0}^{t=\infty} D_{x+t}$ and $\overline{N}_x = (1+i)^{\frac{1}{2}} N_x$

C_x:- defined as $C_x = v^{(x+1)} d_x$ and $\overline{C}_x = (1+i)^{\frac{1}{2}} C_x$

M_x:-defined as $M_x = \sum_{t=0}^{t=\infty} C_{x+t}$ and $\overline{M}_x = (1+i)^{\frac{1}{2}} M_x$

Assurances and Annuities

Whole of life assurance A_x **:-** a payment of £1 is made at the end of the year of death of a person aged x. The symbol represents the present value of the payments. The present value of the payments is $\sum_{t=0}^{t=\infty} v^{t+1} \dfrac{d_{x+t}}{l_x} = \dfrac{M_x}{D_x}$

Immediate whole of life assurance $\bar{A}_x$ **:-** a payment of £1 is made immediately on death of a person aged x. The symbol represents the present value of the payments and therefore the single premium for a whole of life assurance. The present value of the payments is $\sum_{t=0}^{\infty} v^{t+\frac{1}{2}} \frac{d_{x+t}}{l_x} = (1+i)^{\frac{1}{2}} \frac{M_x}{D_x} = \frac{\bar{M}_x}{D_x}$

Term (temporary) assurance $A_{x:\overline{n}|}^{1}$ **:-** a payment of £1 is made at the end of the year of death of a person aged x provided the person dies within n years. The symbol represents the present value of the payments and therefore the single premium for a term (temporary) assurance. The present value of the payments is $\sum_{t=0}^{t=n-1} v^{t+1} \frac{d_{x+t}}{l_x} = \sum_{t=0}^{t=n-1} \frac{C_{x+t}}{D_x} = \frac{M_x - M_{x+n}}{D_x}$

Pure endowment, $A_{x:\overline{n}|}^{1}$ **:-** a payment of £1 is made after n years provided the person aged x is alive at age (x+n). The symbol represents the present value of the payments and therefore the single premium for a pure endowment. The present value is $v^n \frac{l_{x+n}}{l_x} = v^n {}_np_x = \frac{D_{x+n}}{D_x}$

Endowment assurance, $A_{x:\overline{n}|}$ **:-** a payment of £1 is made to a person aged x at the end of the year of death provided the person dies within n years OR if they survive n years, a payment of £1 is made at the end of n years. The symbol represents the present value of the payments and therefore the single premium for an endowment assurance. $A_{x:\overline{n}|} = A_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{1}$. Hence present value of payments is $\frac{M_x - M_{x+n} + D_{x+n}}{D_x}$

Immediate annuity in arrear a_x **:-** payments of £1 are made at the end of each year which the person aged x survives. The symbol represents the present value of the payments and therefore the single premium for an immediate annuity in arrear. The present value of the payments is given by Halley's formula (Sir Edmund Halley, 1656-1752, of comet fame) $a_x = \sum_{t=1}^{\infty} v^t \frac{l_{x+t}}{l_x} = \sum_{t=1}^{\infty} v^t {}_tp_x = \frac{N_{x+1}}{D_x}$. Note that Halley also produced the first life table with numbers alive at each age, the so-called Breslau table and devised the above formula to evaluate the cost of an annuity. Halley was Savilian Professor of Geometry at Oxford as was, somewhat later, Professor J. J. Sylvester, one of the founding members of the Institute of Actuaries and actuary to the Equity and Law Insurance Company for 11 years. It is interesting to note that a recent holder of the Savilian Chair is Sir Michael Atiyah who is an Honorary Fellow of the Faculty of Actuaries.

Immediate annuity in advance $\ddot{a}_x$ **:-** payments of £1 are made immediately and at the beginning of each subsequent year which the person aged x survives. The symbol represents the present value of the

payments and therefore the single premium for an immediate annuity in advance. The present value

of the payments is $\sum_{t=0}^{\infty} v^t \frac{l_{x+t}}{l_x} = \sum_{t=0}^{\infty} v^t {}_t p_x = \frac{N_x}{D_x}$. Hence $\ddot{a}_x = 1 + a_x$.

Immediate annuity payable continuously $\overline{a}_x$ **:-** payments of £1p.a. are made continuously so long as the

person aged x survives. Present value is $\dfrac{\overline{N}_x}{D_x}$

Temporary annuity in arrear $a_{x:\overline{n}|}$:- payments of £1 are made at the end of each year which the person

aged x survives for a maximum of n years (i.e. n payments). The symbol represents the present
value of the payments and therefore the single premium for a temporary immediate annuity in

arrear. Present value of payments is $\sum_{t=1}^{t=n} v^t \frac{l_{x+t}}{l_x} = \sum_{t=1}^{t=n} v^t {}_t p_x = \frac{N_{x+1} - N_{x+n+1}}{D_x}$

Temporary annuity in advance $\ddot{a}_{x:\overline{n}|}$ **:-** payments of £1 are made immediately and at the beginning of

each subsequent year which the person aged x survives for a maximum of n years (i.e. n payments).
The symbol represents the present value of the payments and therefore the single premium for a
temporary immediate annuity in advance. Present value of payments is

$$\sum_{t=0}^{t=n-1} v^t \frac{l_{x+t}}{l_x} = \sum_{t=0}^{t=n-1} v^t {}_t p_x = \frac{N_x - N_{x+n}}{D_x}.$$

Temporary annuity payable continuously $\overline{a}_{x:\overline{n}|}$ **:-** payments of £1p.a. are made continuously so long as

the person aged x survives for a maximum of n years. Present value is $\dfrac{\overline{N}_x - \overline{N}_{x+n}}{D_x}$

Deferred annuities ${}_m|a_x$, ${}_m|\ddot{a}_x$, ${}_m|\overline{a}_x$, ${}_m|a_{x:\overline{n}|}$, ${}_m|\ddot{a}_{x:\overline{n}|}$, ${}_m|\overline{a}_{x:\overline{n}|}$:- represent the discounted value of the

payments defined by the symbols defined to the right of the bar but when the payments are deferred

m years. For example ${}_m|a_x = \dfrac{D_{x+m}}{D_x} a_{x+m} = \dfrac{N_{x+m+1}}{D_x}$,

and ${}_m|a_{x:\overline{n}|} = \dfrac{D_{x+m}}{D_x} a_{x+m:\overline{n}|} = \dfrac{N_{x+m+1} - N_{x+m+n+1}}{D_x}$.

Annual premium for a pure endowment assurance $P_{x:\overline{n}|}^{1}$: the annual premium for a pure endowment

policy with a sum assured of £1 and is $\dfrac{A_{x:\overline{n}|}^{1}}{\ddot{a}_{x:\overline{n}|}}$.

Annual Premium for a term (temporary) assurance $P_{x:\overline{n}|}^{1}$ **:-** is the annual premium for a term

(temporary) assurance policy with a sum assured of £1 and is $\dfrac{A_{x:\overline{n}|}^{1}}{\ddot{a}_{x:\overline{n}|}}$.

Annual premium for an endowment policy $P_{x:\overline{n|}}$ **:-** is the annual premium for an endowment assurance

with a sum assured of £1 and is $\dfrac{A_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$.

Useful relations:- $A_x = 1 - d\ddot{a}_x$, $A_{x:\overline{n|}} = 1 - d\ddot{a}_{x:\overline{n|}}$, $\overline{A}_x = 1 - \delta\overline{a}_x$, $\overline{A}_{x:\overline{n|}} = 1 - \delta\overline{a}_{x:\overline{n|}}$ which can be proved using d=(1-v), or by general reasoning.

Statistical Approach (where the number of persons that die at each age is a random variable)

Note:- Random Variables are shown in bold.

$\mathbf{T_x}$:- a random variable representing the length of the future lifetime of a person ages exactly x. It is a continuous random variable. The (cumulative) distribution function of $\mathbf{T_x}$ is $F_x(t)=\text{Probability}(\mathbf{T_x} \le t) = 1 - {}_tp_x = {}_tq_x$ and the density function of $\mathbf{T_x}$ is $f_x(t)$ which is equal to ${}_tp_x\mu_{x+t}$ which is derived by differentiation of the (cumulative) distribution function $F_x(t)$. We have:-

$$f_x(t) = \frac{dF_x(t)}{dt} = \frac{d(1-{}_tp_x)}{dt} = -\frac{d_tp_x}{dt} = -\frac{1}{l_x}\cdot\frac{dl_{x+t}}{dt} = -\frac{l_{x+t}}{l_x}\cdot\frac{1}{l_{x+t}}\cdot\frac{dl_{x+t}}{dt} = {}_tp_x\mu_{x+t}$$

$\mathbf{K_x}$:- a random variable representing the curtate future lifetime of a person aged x (curtate means rounded to the lower integer e.g. curtate 6.25 is 6) . It is a discrete random variable (i.e. only takes values 0,1,2 etc.). The distribution function of K_x is $\text{Probability}(\mathbf{K_x}=k) = {}_kp_x.q_{x+k} = {}_k|q_x$

$\overset{o}{e}_x$, **complete expectation of life :-** defined by the expected value of $\mathbf{T_x}$ i.e. (where t=∞ denotes the end of the life table):-

$$\overset{o}{e}_x = E(\mathbf{T_x}) = \int_{t=0}^{t=\omega} t\cdot{}_tp_x\cdot\mu_{x+t} = \int_{t=0}^{t=\infty} -t\cdot\frac{d_tp_x}{dt} = \left[-t\cdot{}_tp_x\right]_{t=0}^{t=\infty} + \int_{t=0}^{t=\infty} {}_tp_x.dt = \int_{t=0}^{t=\infty} {}_tp_x.dt = \frac{1}{l_x}\int_{t=0}^{t=\infty} l_{x+t} = \frac{T_x}{l_x}$$ (where T_x

is as defined in the deterministic approach and this formula for $\overset{o}{e}_x$ agrees with the deterministic approach).

e_x, **curtate expectation of life :** - defined by the expected value of $\mathbf{K_x}$ i.e.

$$e_x = E(\mathbf{K_x}) = \sum_{k=1}^{k=\infty} k\cdot{}_kp_x\cdot q_{x+k} = \sum_{k=1}^{k=\infty} k\cdot\frac{d_{x+k}}{l_x} = \sum_{k=1}^{k=\infty} k\cdot\frac{(l_{x+k} - l_{x+k+1})}{l_x} = \sum_{k=1}^{k=\infty} \frac{l_{x+k}}{l_x} = \frac{1}{l_x}\sum_{k=1}^{k=\infty} l_{x+k}$$ (as in deterministic approach) .

Standard deviation of $\mathbf{T_x}$:- is the square root of the variance of $\mathbf{T_x}$.

Variance is $E(\mathbf{T_x}^2)-E(\mathbf{T_x})^2 = \int_{t=0}^{t=\infty} t^2\cdot{}_tp_x\cdot\mu_{x+t}.dt - (\overset{o}{e}_x)^2 = 2\int_{t=0}^{t=\infty} t\cdot{}_tp_x.dt - (\overset{o}{e}_x)^2$

Standard deviation of $\mathbf{K_x}$:- is the square root of the variance of K_x .

Variance is $E(\mathbf{K_x}^2) - E(\mathbf{K_x})^2 = \sum_{k=0}^{k=\infty} k^2\cdot\frac{d_{x+k}}{l_x} - (e_x)^2$

Relations :- d_x and q_x are related to integrals involving the density function of $\mathbf{T_x}$ in the following way

$$q_x = \int_{t=0}^{t=1} {}_tp_x.q_{x+t}dt \text{ and } d_x = \int_{t=0}^{t=1} l_{x+t}.q_{x+t}dt$$

Whole of life assurance (benefits defined as in deterministic approach) :- its value is the random variable v^{K_x+1}. The expected value is $E(v^{K_x+1}) = \sum_{t=0}^{t=\infty} v^{k+1} . P(K_x = k) = \sum_{t=0}^{t=\infty} v^{k+1} . \frac{d_{x+k}}{l_x} = \frac{M_x}{D_x} = A_x$ as expected

but we now are able to evaluate the variance $Var(v^{K_x+1}) = E(v^{2(K_x+1)}) - E(v^{K_x+1})^2 =$

$\sum_{t=0}^{t=\infty} v^{2(k+1)} . P(K_x = k) - A_x^2 = \sum_{t=0}^{t=\infty} v^{2(k+1)} . \frac{d_{x+k}}{l_x} = A_x^j - A_x^2$ (where j is a rate of interest such that $(1+j)=(1+i)^2$).

Note that, for a portfolio of N independent such policies payable by single premium, the expected value of the portfolio increases in proportion to N but the standard deviation only in proportion to $\sqrt{N}$. The ratio of the standard deviation to the expected claims therefore tends to zero.

If the benefit is payable immediately **(immediate whole of life assurance),** the random variable is v^{T_x} with expected value $\overline{A}_x$ and variance $(\overline{A}_x^j - \overline{A}_x^2)$

Term assurance (benefits as in deterministic approach) :- its value is the random variable v^{K_x+1} if $K_x<n$ and zero otherwise.

Its expected value is $\sum_{t=0}^{t=n-1} v^{t+1} \frac{d_{x+t}}{l_x} = \sum_{t=0}^{t=n-1} \frac{C_{x+t}}{D_x} = \frac{M_x - M_{x+n}}{D_x} = A^1_{x:n\rceil}$ and its variance is easily shown to be

$A^{j\,1}_{\ x:n\rceil} - (A^1_{x:n\rceil})^2$.(where j is a rate of interest such that $(1+j)=(1+i)^2$).

Where the benefits are payable immediately the expressions are the same except that there are bars over the As.

Pure endowment (benefits as in deterministic approach) :- its value is the random variable whose value is v^n if $T_x > n$ and zero if $T_x \leq n$. The expected value of is $v^n {}_np_x$ or $\frac{D_{x+n}}{D_x}$ or $A_{x:n\rceil}^{\ \ 1}$ and whose variance is $v^{2n} {}_np_x - (v^n {}_np_x)^2$.

Endowment assurance (benefits as in deterministic approach) : its value is that of the random variable $v^{min(K_x+1,n)}$. The expected value is easily shown to be

$\sum_{k=0}^{k=\infty} v^{min(k+1,n)} . P(K_x = k) = \sum_{k=0}^{k=n-1} v^{k+1} . \frac{d_{x+k}}{l_x} + {}_n p_x v^n = A^1_{x:n\rceil} + A_{x:n\rceil}^{\ \ 1} = A_{x:n\rceil}$ and the variance $A^j_{x:n\rceil} - A^2_{x:n\rceil}$ (where j is a rate of interest such that $(1+j)=(1+i)^2$). Where the benefit is payable immediately the expected value is easily shown to be $\overline{A}_{x:n\rceil}$ and the variance $\overline{A}^j_{x:n\rceil} - \overline{A}^2_{x:n\rceil}$.

Annuity yearly in advance (benefits as in deterministic approach):- its value is the random variable $\ddot{a}_{\overline{K_x+1\rceil}} = (1 - v^{K_x+1})/d$.

So the expected value is $(1 - A_x)/d = \ddot{a}_x$ and variance $(1/d^2) (A_x^j - A_x^2)$ using the results above.

Annuity payable continuously(benefits as in deterministic approach):- the random variable is $\overline{a}_{\overline{T_{\mathbf{x}}}|} =$ (1- $\mathbf{v^{Tx}}$)/δ .

So the expected value is (1- $\overline{A}_x$)/δ $=\overline{a}_x$ and variance (1/δ^2) ($\overline{A}_x^j - \overline{A}_x^2$**).**

Profit :- The profit under an annual premium whole of life assurance (sum assured S) with annual premiums (P) is the random variable P $\ddot{a}_{\overline{\mathbf{K_x}+1}|}$ - S $\mathbf{v^{Kx+1}}$ = (P/d) – (S+P/d) $\mathbf{v^{Kx+1}}$ and its expected value

P $\ddot{a}_x$ - SA$_x$ and variance (S+P/d)^2($A_x^j - A_x^2$).

Note that, for a portfolio of N independent such policies, the expected value of the profit increases in proportion to N but the standard deviation only in proportion to $\sqrt{N}$. The ratio of the standard deviation of the profit on the portfolio to the expected profit therefore tends to zero.

It should be noted that the profit is zero when P=S.A$_x$/$\ddot{a}_x$ as expected. Similar expressions can be obtained for other types of assurance.