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Аналогичное представление получаем и для $f(s, h)$. Теорема 2 полностью доказана.

Автор выражает свою благодарность профессорам Ж.-Ж. Лоебу и П. Грачику за интересные обсуждения характеризационных задач на неабелевых группах. Настоящая работа возникла в результате этих дискуссий.

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GEIGER J., KERSTING G.**

THE SURVIVAL PROBABILITY OF A CRITICAL BRANCHING PROCESS IN RANDOM ENVIRONMENT[1)]

В статье описывается асимптотическое поведение вероятности выживания для критического ветвящегося процесса в случайной среде. В частном случае, когда число потомков определяется независимыми одинаково (геометрически) распределенными случайными величинами, и в несколько более общем случае, когда распределение числа потомков имеет дробно-линейную производящую функцию, М. В. Козлов [10] доказал,

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что при $n \to \infty$ вероятность невырождения в n-м поколении пропорциональна $n^{-1/2}$. В статье асимптотика Козлова обобщается на случай любых независимых одинаково распределенных величин.

Ключевые слова и фразы: ветвящиеся процессы, случайные среды, условное случайное блуждание.

1. Introduction and main result. Consider a branching process $(Z_n)_{n\geqslant 0}$ in random environment represented by a sequence of probability generating functions f_n, $n \geqslant 0$. Given the environment, individuals reproduce independently of each other. The offspring of an individual in the n-th generation has generating function f_n. Thus if Z_n denotes the number of individuals in generation n, then

$$\mathbf{E}(s^{Z_n} \mid Z_0, \dots, Z_{n-1}, f_0, \dots, f_{n-1}) = (f_{n-1}(s))^{Z_{n-1}}, \quad 0 \leqslant s \leqslant 1. \tag{1.1}$$

Here we consider the case, where the f_n are independent and identically distributed random variables. For convenience we assume that the initial size of the population Z_0 equals one. A branching process in i.i.d. random environment is called ***subcritical***, ***critical*** or ***supercritical*** according to whether $\mathbf{E}\ln f'(1)$ is less, equal or greater than zero, where $f'(1)$ is the conditional mean number of children per individual. It is well-known that in critical and subcritical cases Z_n eventually becomes extinct,

$$\mathbf{E}\ln f'(1) \leqslant 0,\ \mathbf{P}\{Z_1 = 1\} < 1 \implies \lim_{n\to\infty} \mathbf{P}\{Z_n > 0\} = 0.$$

For details we refer to the papers of Smith, Wilkinson [11] and Athreya, Karlin [3]. The survey article of Vatutin, Zubkov [13] gives further references.

In this paper we study the critical case. Our main result describes the asymptotic decay of the probability of non-extinction at generation n.

Theorem 1.1. *Suppose* $\mathbf{E}\ln f'(1) = 0$, $0 < \mathbf{E}(\ln f'(1))^2 < \infty$, *and assume that*

$$\mathbf{E}\,\frac{f''(1)}{(f'(1)^2)}\Big(1 + \ln^+ f'(1)\Big) < \infty. \tag{1.2}$$

Then, for some $0 < \beta < \infty$,

$$\mathbf{P}\{Z_n > 0\} \sim \beta n^{-1/2} \quad \text{as } n \to \infty. \tag{1.3}$$

Kozlov [10] proved (1.3) for i.i.d. random environments with linear fractional generating functions and derived upper and lower bounds in the general case. Vatutin, Dyakonova [12] determined the asymptotic probability of extinction at but not prior to the nth generation for critical branching processes with linear fractional generating functions. The analog of (1.3) for subcritical processes of this special kind was proved by Afanasyev [1]. For subcritical processes with general offspring distributions only estimates of the survival probability are known (see [1], [5]).

A key ingredient for the proof of Theorem 1.1 is an expression of the survival probability at generation n in terms of a random walk, which generalizes a well-known formula in the case of linear fractional generating functions due to Agresti [2]. This formula is derived in Section 2. In Section 3 we discuss the relevant properties of the random walk. Section 4 contains the proof of Theorem 1.1.

2. The probability of non-extinction. From (1.1) we see that the conditional probability generating function of Z_n is

$$\mathbf{E}(s^{Z_n} \mid f_0, \dots, f_{n-1}) = f_0(f_1(\cdots f_{n-1}(s)\cdots)), \quad 0 \leqslant s \leqslant 1.$$

In particular, the probability of non-extinction at generation n given the environment $f_0, f_1, \dots, f_{n-1}$ is

$$\mathbf{P}\{Z_n > 0 \mid f_0, \dots, f_{n-1}\} = 1 - f_0(f_1(\cdots f_{n-1}(0)\cdots)) =: q_n, \tag{2.1}$$

so that

$$\mathbf{P}\{Z_n > 0\} = \mathbf{E}\,q_n, \quad n \geqslant 0.$$

In order to rewrite this equation, we introduce some notation. Let

$$\begin{aligned} f_{k,n} &:= f_k \circ f_{k+1} \circ \cdots \circ f_{n-1}, \quad 0 \leqslant k \leqslant n-1; \quad f_{n,n} := \mathrm{id}, \\ a_k &:= (f_0'(1) f_1'(1) \cdots f_{k-1}'(1))^{-1}, \quad k \geqslant 1; \quad a_0 := 1, \\ g_k(s) &:= \frac{1}{1-f_k(s)} - \frac{1}{f_k'(1)(1-s)}, \quad 0 \leqslant s < 1. \end{aligned}$$

Our starting point is the following simple identity

$$\begin{aligned} \frac{1}{1-f_0(f_1(\cdots f_{n-1}(s)\cdots))} &= \frac{a_0}{1-f_{0,n}(s)} = \frac{a_n}{1-f_{n,n}(s)} \\ &\quad + \sum_{k=0}^{n-1} \left(\frac{a_k}{1-f_{k,n}(s)} - \frac{a_{k+1}}{1-f_{k+1,n}(s)} \right) \\ &= \frac{a_n}{1-s} + \sum_{k=0}^{n-1} a_k g_k(f_{k+1,n}(s)), \quad 0 \leqslant s < 1. \end{aligned}$$

Introducing the random walk $S_n := X_1 + \cdots + X_n$, $n \geqslant 1$, $S_0 := 0$, with the increments X_n being defined as $X_n := \ln f_{n-1}'(1)$, we end up with the following formula:

$$q_n = \left(\exp(-S_n) + \sum_{k=0}^{n-1} \eta_{k,n} \exp(-S_k) \right)^{-1}, \tag{2.2}$$

where $\eta_{k,n} := g_k(f_{k+1,n}(0))$. Note that the summands in (2.2) are non-negative. This is due to the fact that f_k is convex and thus g_k is non-negative.

R e m a r k s. 1. Our expression (2.2) for the survival probability generalizes several known formulas. In the case of linear fractional generating functions the formula has been derived by Agresti [2]. In this case $\eta_{k,n}$ depends on f_k, but not on $f_{k+1}, \ldots, f_{n-1}$ nor n. For Galton–Watson branching processes in non-varying environment the formula has been used by Kesten et al. [9].

2. The random variables $\eta_{k,n}$ have a probabilistic interpretation. To this end regard the random family tree associated with the branching process as a rooted planar tree with the distinguishable offspring of each individual ordered from left to right. Given that $Z_n > 0$, let $v_k, 0 \leqslant k \leqslant n$, be the left-most individual in generation k with a descendant in generation n. Then $\eta_{k,n} f_k'(1)$ is the expected number of siblings to the right of v_{k+1}, given the environment $f_0, \ldots, f_{n-1}$ and the event $Z_n > 0$. This fact follows from a backward construction of the conditioned tree due to Geiger (see [7] for the special case of Galton–Watson branching processes).

The following lemma gives an upper bound for the $\eta_{k,n}$.

Lemma 2.1. *Let f be a probability generating function satisfying $(f''(1))^2 < \infty$. Then, for any $s \in [0,1)$,*

$$0 \leqslant g(s) := \frac{1}{1-f(s)} - \frac{1}{f'(1)(1-s)} \leqslant \frac{f''(1)}{(f'(1))^2}. \tag{2.3}$$

P r o o f. Since f is convex, we have

$$\begin{aligned} f'(1)g(s) &\leqslant f'(1)g(s) + \frac{s}{1-s}\left(1 - \frac{f'(s)(1-s)}{f(1)-f(s)}\right) \\ &= \frac{f'(1) - sf'(s)}{1-f(s)} - 1, \quad 0 \leqslant s < 1. \end{aligned} \tag{2.4}$$

Let $(p_k)_{k \geqslant 0}$ be the probability measure represented by f, then

$$\frac{f'(1) - sf'(s)}{1-f(s)} = \sum_{k=1}^{\infty} k r_k(s), \tag{2.5}$$

where $r_k(s) := p_k(1-s^k)/(1-f(s))$, $k \geqslant 1$. Note that

$$\frac{r_{k+1}(s)}{r_k(s)} = \frac{p_{k+1}(1-s^{k+1})}{p_k(1-s^k)} = \frac{p_{k+1}}{p_k}\left(1+\frac{1}{\sum_{j=1}^{k} s^{-j}}\right) \tag{2.6}$$

is increasing in s for any $k \geqslant 1$, and that the $r_k(s)$ sum to one for any $0 \leqslant s < 1$. Hence, the right-hand side of (2.5) increases with s. Consequently, by (2.4),

$$f'(1)\,g(s) \leqslant \lim_{u\uparrow 1} \frac{f'(1)-uf'(u)}{1-f(u)} - 1 = \frac{f''(1)}{f'(1)}, \quad 0 \leqslant s < 1,$$

which is the assertion of the lemma.

The bound (2.3) is sharp up to a factor 2, since $\lim_{s\uparrow 1} g(s) = f''(1)/2(f'(1))^2$.

R e m a r k. The stochastic monotonicity expressed in (2.6) says that given the event $Z_n > 0$, the number of v_k's children is dominated by the ***size-biased*** distribution with generating function $\hat{f}_k(s) = sf_k'(s)/f_k'(1)$ (compare the second remark above).

3. Conditioned environments. In this section we discuss relevant results from fluctuation theory for random walks. Suppose that survival of the population at generation n is observed, what can one infer about the environment? Apparently, environments $f_0, \ldots, f_{n-1}$, which give rise to a high non-extinction probability q_n, will be favored. In particular, one might expect that the successive minima of $S_0, \ldots, S_n$ remain bounded from below with increasing n (compare formula (2.2)). Thus we are led to the study of random walks conditioned not to drop below some given level. To this end let us introduce the function h,

$$h(x) := \sum_{j=0}^{\infty} \mathbf{P}\{S_{\tau_j} \geqslant -x\}, \quad x \in \mathbf{R},$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are the descending strict ladder epochs of the random walk (S_n) being defined as

$$\tau_{j+1} := \min\{n > \tau_j \colon S_n < S_{\tau_j}\}, \quad j \geqslant 0.$$

In fluctuation theory, h is known as the renewal function of the ladder height process (Ch. 12 in Feller's monograph [6] is the standard reference; compare also the paper by Bertoin, Doney [4]). Note that $h(0) = 1$ and $h(x) = 0$ for $x < 0$. Assuming $\mathbf{E}X_1 = 0$ and $0 < \mathbf{E}X_1^2 < \infty$, the elementary renewal theorem implies $h(x) \sim cx$ as $x \to \infty$ for some $c > 0$. The function h satisfies

$$h(x) = \mathbf{E}\,h(x+X_1), \quad x \geqslant 0, \tag{3.1}$$

which is a version of the celebrated Wiener–Hopf equation and may be derived from XII, (3.7b) in [6] by an application of Fubini's theorem. In other words, h is harmonic on $\mathbf{R}_0^+$ for the random walk (S_n). This allows the construction of transition kernels Q_x, $x \geqslant 0$, where

$$Q_x(y;B) := (h(x+y))^{-1}\,\mathbf{E}(h(x+y+X_1);\, X_1 \in B-y), \tag{3.2}$$

for any Borel set $B \subset [-x,\infty)$ and $y \geqslant -x$. For each $x \geqslant 0$, Q_x gives rise to a Markov chain on $[-x,\infty)$ starting at 0. It is convenient to denote this Markov chain by (S_n) and its increments by X_n, again, and rather view this construction as a change of measures on the path space from $\mathbf{P}$ to $\widehat{\mathbf{P}}_x$, say. The measure $\widehat{\mathbf{P}}_x$ is characterized by the property that

$$\widehat{\mathbf{E}}_x\psi(S_0,\ldots,S_n) = (h(x))^{-1}\,\mathbf{E}(\psi(S_0,\ldots,S_n)\,h(x+S_n);\, M_n \geqslant -x), \tag{3.3}$$

for any positive functional ψ that depends only on the n first steps of the Markov chain or random walk (S_n). Here, M_n, $n \geqslant 0$, are the successive minima of the chain,

$$M_n := \min(S_0, S_1, \ldots, S_n).$$

It is well-known that (S_n) with respect to $\widehat{\mathbf{P}}_x$ can be interpreted as a random walk conditioned not to hit $(-\infty,-x)$. This intuition can be made precise by means of the following result (Theorem A in [10]). For $x \geqslant 0$ write

$$m_n(x) := \mathbf{P}\{M_n \geqslant -x\}.$$

Proposition 3.1. *Suppose that* $\mathbf{E}X_1 = 0$ *and* $0 < \mathbf{E}X_1^2 < \infty$. *Then there are finite constants* $c_1, c_2 > 0$ *such that, as* $n \to \infty$,

$$m_n(x) \sim c_1 h(x) n^{-1/2}, \quad x \geqslant 0, \tag{3.4}$$

$$m_n(x) \leqslant c_2 h(x) n^{-1/2}, \quad x \geqslant 0,\ n \geqslant 1. \tag{3.5}$$

Now assume $k \leqslant n$. Then

$$\begin{aligned} &\mathbf{E}\Big(\psi(S_0,\dots,S_k);\ M_n \geqslant -x\Big) \\ &= \mathbf{E}\Big(\psi(S_0,\dots,S_k);\ M_k \geqslant -x,\ \min(S_k, S_{k+1},\dots,S_n) \geqslant -x\Big) \\ &= \mathbf{E}\Big(\psi(S_0,\dots,S_k)\, m_{n-k}(x+S_k);\ M_k \geqslant -x\Big). \end{aligned} \tag{3.6}$$

In view of Proposition 3.1, the dominated convergence theorem and (3.3), we obtain for any bounded ψ,

$$\begin{aligned} \lim_{n\to\infty} \mathbf{E}(\psi(S_0,\dots,S_k) \mid M_n \geqslant -x) &= (h(x))^{-1}\mathbf{E}(\psi(S_0,\dots,S_k)h(x+S_k);\ M_k \geqslant -x) \\ &= \widehat{\mathbf{E}}_x\, \psi(S_0,\dots,S_k); \end{aligned} \tag{3.7}$$

this relation clarifying the interpretation of $\widehat{\mathbf{P}}_x$.

For later reference we record the following properties of the transition kernels Q_x, $x \geqslant 0$, both being easy consequences of (3.2) and the monotonicity of h:

$$Q_x(z; z+B) = Q_{x+z}(0; B), \quad z \geqslant -x,\ B \subset [-x-z, \infty), \tag{3.8}$$

$$Q_x(0;\cdot) \geqslant \mathbf{P}\{X_1 \in \cdot\} \quad \text{stochastically},\ x \geqslant 0. \tag{3.9}$$

Note that the Q_x need not retain the monotonicity of the random walk kernel.

The following integrability result will be useful in the proof of Theorem 1.1.

Lemma 3.1. *Suppose that* $\mathbf{E}X_1 = 0$ *and* $0 < \mathbf{E}X_1^2 < \infty$. *Then, for any* $x \geqslant 0$,

$$\widehat{\mathbf{E}}_x \sum_{k=0}^{\infty} \exp(-S_k) < \infty. \tag{3.10}$$

Proof. We first show that it is sufficient to establish (3.10) for $x = 0$. Let D be the set of all $x \geqslant 0$ which satisfy (3.10). From (3.3) and the monotonicity of h we see that the estimate

$$\widehat{\mathbf{E}}_x \sum_{k=0}^{\infty} \exp(-S_k) \leqslant \frac{h(y)}{h(x)} \widehat{\mathbf{E}}_y \sum_{k=0}^{\infty} \exp(-S_k)$$

holds for any $0 \leqslant x \leqslant y$. Therefore, D is a (possibly empty) interval of the form $[0, u)$ or $[0, u]$. Furthermore, by (3.8),

$$\begin{aligned} \widehat{\mathbf{E}}_x \sum_{k=0}^{\infty} \exp(-S_k) &= 1 + \widehat{\mathbf{E}}_x\left(\widehat{\mathbf{E}}_x\left(\sum_{k=1}^{\infty} \exp(-S_k) \mid S_1\right)\right) \\ &= 1 + \int_{-x}^{\infty} Q_x(0; dz) \exp(-z)\, \widehat{\mathbf{E}}_{x+z} \sum_{k=0}^{\infty} \exp(-S_k). \end{aligned} \tag{3.11}$$

Relation (3.11) combined with (3.9) shows that there is a $\delta > 0$, such that $x \in D$ implies $x + \delta \in D$. Therefore, $D = \varnothing$ or $\mathbf{R}_0^+$ and it suffices to prove $0 \in D$. Following Kozlov [10] let

$$U_k(x) := \mathbf{P}\{M_k \geqslant 0,\ S_k \leqslant x\} \quad \text{and} \quad U(x) := \sum_{k=0}^{\infty} U_k(x), \quad x \geqslant 0.$$

From (3.3) we see that

$$\widehat{\mathbf{E}}_0 \sum_{k=0}^{\infty} \exp(-S_k) = \int_0^{\infty} \exp(-x) h(x)\, dU(x). \tag{3.12}$$

By the duality lemma (see, e.g., Ch. 12 in [6]), $U(x)$ and the expected number of ascending weak ladder heights in the interval $[0, x]$ agree. The elementary renewal theorem implies $U(x) \sim cx$ as $x \to \infty$ for some $c > 0$. Hence, the integral on the right-hand side of (3.12) is finite, which completes the proof of the lemma.

So far we have only discussed changes in the distribution of $(S_n)_{n \geqslant 0}$. Clearly, this entails a change of the whole process. The following 3-step scheme covers all of the various branching models:

1. Run the random walk or Markov chain $(S_n)_{n \geqslant 0}$ starting at $S_0 = 0$.
2. Generate independent probability distributions with generating functions $f_0, f_1, f_2, \ldots$, where the law of f_n conditioned on $\exp(S_{n+1} - S_n) = x$ is the same as the law of f given that $f'(1) = x$.
3. Construct the branching process $(Z_n)_{n \geqslant 0}$ in the environment $f_0, f_1, f_2, \ldots$.

4. Proof of the theorem. Before we begin with the proof of Theorem 1.1 we formalize the 3-step construction at the end of Section 3. For each $x \geqslant 0$, we construct a measure on $\sigma(\{f_n, Z_n : n \geqslant 0\})$ defined on the same probability space as the measure $\mathbf{P}$. These measures will be denoted $\widehat{\mathbf{P}}_x$ again, which should cause no confusion. Retaining the notation $X_n = \ln f'_{n-1}(1)$, $S_n = X_1 + \cdots + X_n$, etc. from the previous sections, the measure $\widehat{\mathbf{P}}_x$ is characterized by properties (4.1)–(4.3) below.

The marginal distribution of $\widehat{\mathbf{P}}_x$ on $\sigma(\{S_n : n \geqslant 0\})$ is the measure denoted by $\widehat{\mathbf{P}}_x$ in Section 3, i.e.,

$$\widehat{\mathbf{E}}_x \psi(S_0, \ldots, S_n) = (h(x))^{-1}\, \mathbf{E}(\psi(S_0, \ldots, S_n)\, h(x + S_n);\ M_n \geqslant -x) \tag{4.1}$$

for any positive functional ψ that depends only on $S_0, \ldots, S_n$. The conditional distribution of $(f_n)_{n \geqslant 0}$ under $\widehat{\mathbf{P}}_x$ given $0 = S_0, S_1, \ldots$ is

$$\widehat{\mathbf{P}}_x\{f_k \in A_k,\ 0 \leqslant k \leqslant n \mid S_i = s_i,\ i \geqslant 1\} = \mathbf{P}\{f_k \in A_k,\ 0 \leqslant k \leqslant n \mid S_i = s_i,\ i \geqslant 1\} \tag{4.2}$$

for any measurable sets A_k and $(s_i) \in [-x, \infty)^{\mathbf{N}}$. The conditional distribution of $(Z_n)_{n \geqslant 0}$ under $\widehat{\mathbf{P}}_x$ given $f_0, f_1, \ldots$ is the same as under $\mathbf{P}$,

$$\widehat{\mathbf{E}}_x(s^{Z_n} \mid Z_0, \ldots, Z_{n-1}, f_0, \ldots, f_{n-1}) = (f_{n-1}(s))^{Z_{n-1}}, \quad 0 \leqslant s \leqslant 1. \tag{4.3}$$

Note that equations as (2.1) remain valid if $\mathbf{P}$ is replaced with $\widehat{\mathbf{P}}_x$.

Our proof of Theorem 1.1 is based on the observation that contributions to $\mathbf{P}\{Z_n > 0\} = \mathbf{E}\, q_n$ other than those on the event $\{M_n \geqslant -x\}$ are asymptotically negligible if only x is large enough (a similar idea was used by Hirano [8] in a somewhat different context, whereas Kozlov [10] considered the event that only few ladder epochs occur until time n). First, we study the quantity q_n on the event $\{M_{\rho n} \geqslant -x\}$ for some fixed $\rho > 1$ and $x \geqslant 0$, writing $M_{\rho n}$ instead of $M_{\lfloor \rho n \rfloor}$ for convenience. Using first (3.7) and then (4.2) we have for any fixed m,

$$\begin{aligned} \lim_{n \to \infty} \mathbf{P}\{Z_m > 0 \mid M_{\rho n} \geqslant -x\} &= \lim_{n \to \infty} \mathbf{E}(\mathbf{E}(q_m \mid S_0, \ldots, S_m) \mid M_{\rho n} \geqslant -x) \\ &= \widehat{\mathbf{E}}_x(\mathbf{E}(q_m \mid S_0, \ldots, S_m)) = \widehat{\mathbf{E}}_x q_m = \widehat{\mathbf{P}}_x\{Z_m > 0\}. \end{aligned} \tag{4.4}$$

We would like to replace Z_m by Z_n on the left-hand side of (4.4). First, however, we describe the asymptotic behavior of the right-hand side of (4.4) as $m \to \infty$. Note that the fact that $f_{k,m}(0)$ is increasing in m and g_k is continuous implies

$$\eta_{k,m} \to \eta_{k,\infty} \quad \text{as } m \to \infty. \tag{4.5}$$

Lemma 4.1. *Let the conditions of Theorem* 1.1 *be satisfied. Then, for any* $x \geqslant 0$,

$$\lim_{m \to \infty} \widehat{\mathbf{P}}_x\{Z_m > 0\} = \widehat{\mathbf{E}}_x \left(\sum_{k=0}^{\infty} \eta_{k,\infty} \exp(-S_k) \right)^{-1}. \tag{4.6}$$

P r o o f. We will show that

$$\lim_{m \to \infty} \widehat{\mathbf{E}}_x |q_m^{-1} - q_\infty^{-1}| = 0, \tag{4.7}$$

where $q_\infty^{-1} := \sum_{k=0}^\infty \eta_{k,\infty} \exp(-S_k)$. Clearly, (4.7) and the fact that $q_m \leqslant 1$ imply $q_\infty \leqslant 1$. Hence, $|q_m - q_\infty| = q_m q_\infty |q_m^{-1} - q_\infty^{-1}| \leqslant |q_m^{-1} - q_\infty^{-1}|$, and using (4.7) again we have

$$\lim_{m\to\infty} \widehat{\mathbf{E}}_x |q_m - q_\infty| = 0.$$

In particular, $\lim_{m\to\infty} \widehat{\mathbf{P}}_x\{Z_m > 0\} = \lim_{m\to\infty} \widehat{\mathbf{E}}_x q_m = \widehat{\mathbf{E}}_x\, q_\infty$, which is assertion (4.6).

To show (4.7) let $\ell \leqslant m$ and apply Lemma 2.1 to deduce

$$|q_m^{-1} - q_\infty^{-1}| \leqslant \exp(-S_m) + \sum_{k=0}^{\ell-1} |\eta_{k,m} - \eta_{k,\infty}| \exp(-S_k) + \sum_{k=\ell}^{\infty} \eta_k \exp(-S_k),$$

where $\eta_k := f_k''(1)/(f_k'(1))^2$. Taking expectations with respect to $\widehat{\mathbf{P}}_x$ we obtain

$$\begin{aligned}\widehat{\mathbf{E}}_x|q_m^{-1} - q_\infty^{-1}| \;\leqslant\; & \widehat{\mathbf{E}}_x \exp(-S_m) + \sum_{k=0}^{\ell-1} \widehat{\mathbf{E}}_x|\eta_{k,m} - \eta_{k,\infty}| \exp(-S_k) \\ & + \widehat{\mathbf{E}}_x \sum_{k=\ell}^{\infty} \eta_k \exp(-S_k). \end{aligned} \tag{4.8}$$

Now $\widehat{\mathbf{E}}_x \exp(-S_m) \to 0$ as $m \to \infty$ by Lemma 3.1. So to prove (4.7) it suffices to show

$$\widehat{\mathbf{E}}_x \sum_{k=0}^{\infty} \eta_k \exp(-S_k) < \infty. \tag{4.9}$$

(Indeed, in view of (4.5) we may then apply the dominated convergence theorem to the second term on the right-hand side of (4.8). Also, we see from (4.9) that the last term on the right-hand side of (4.8) can be made arbitrarily small by choosing ℓ sufficiently large.)

To verify (4.9) first note that (4.2) and the independence of the f_n under $\mathbf{P}$ imply

$$\begin{aligned}\widehat{\mathbf{E}}_x\eta_k \exp(-S_k) &= \widehat{\mathbf{E}}_x(\exp(-S_k)\,\widehat{\mathbf{E}}_x(\eta_k \mid S_0, \ldots, S_{k+1})) \\ &= \widehat{\mathbf{E}}_x(\exp(-S_k)\,\mathbf{E}(\eta_k \mid S_0, \ldots, S_{k+1})) = \widehat{\mathbf{E}}_x(\exp(-S_k)\,\mathbf{E}(\eta_k \mid X_{k+1})).\end{aligned}$$

Now observe that $h(x) - h(y) \leqslant h(x-y) \leqslant c\,(1 + (x-y)^+)$ for any $x, y \geqslant 0$ and some $c \geqslant 1$, where the first inequality follows from the fact that the $\mathbf{P}$-renewal process $(S_{\tau_j})_{j\geqslant 0}$ is zero delayed. Repeatedly using (3.3) and the independence of the f_n under $\mathbf{P}$ we thus have

$$\begin{aligned}&\widehat{\mathbf{E}}_x\eta_k \exp(-S_k) = h(x)^{-1}\mathbf{E}(\exp(-S_k)\,\mathbf{E}(\eta_k \mid X_{k+1})\,h(x+S_{k+1});\ M_{k+1} \geqslant -x) \\ &\leqslant (h(x))^{-1}\mathbf{E}(\exp(-S_k)\,\mathbf{E}(\eta_k \mid X_{k+1})\,h(x+S_k)\,(1 + c\,(1+X_{k+1}^+));\ M_k \geqslant -x) \\ &\leqslant 2c\,\mathbf{E}(\mathbf{E}(\eta_k(1+X_{k+1}^+) \mid X_{k+1}))\,(h(x))^{-1}\mathbf{E}(\exp(-S_k)\,h(x+S_k);\ M_k \geqslant -x) \\ &= 2c\,\mathbf{E}\eta_0(1+X_1^+)\,\widehat{\mathbf{E}}_x \exp(-S_k).\end{aligned}$$

The first expectation above being finite by assumption (1.2), assertion (4.9) now follows from (3.10) which completes our proof of Lemma 4.1.

The following lemma says that we may indeed replace Z_m by Z_n on the left-hand side of (4.4).

Lemma 4.2. *Let the conditions of Theorem* 1.1 *be satisfied. Then, for any* $\rho > 1$ *and* $x \geqslant 0$,

$$\limsup_{m\to\infty}\ \limsup_{n\to\infty}\ \mathbf{P}\{Z_m > 0,\ Z_n = 0 \mid M_{\rho n} \geqslant -x\} = 0. \tag{4.10}$$

Proof. Assume $m \leqslant n$. Using first (3.6) and Proposition 3.1 and then (3.3) and (4.2), we find that

$$\begin{aligned}&\mathbf{P}\{Z_m > 0,\ Z_n = 0 \mid M_{\rho n} \geqslant -x\} = \mathbf{E}(q_m - q_n \mid M_{\rho n} \geqslant -x) \\ &= \mathbf{E}\frac{(q_m - q_n)m_{(\rho-1)n}(x+S_n);\ M_n \geqslant -x}{\mathbf{P}\{M_{\rho n} \geqslant -x\}}\end{aligned}$$

$$\leqslant c\left(\frac{\rho}{\rho-1}\right)^{1/2}(h(x))^{-1}\,\mathbf{E}(\mathbf{E}(q_m-q_n\mid S_0,\dots,S_n)\,h(x+S_n);\ M_n\geqslant -x)$$
$$=c\left(\frac{\rho}{\rho-1}\right)^{1/2}\widehat{\mathbf{E}}_x(\mathbf{E}(q_m-q_n\mid S_0,\dots,S_n))=c\left(\frac{\rho}{\rho-1}\right)^{1/2}\widehat{\mathbf{E}}_x(q_m-q_n)$$
$$=c\left(\frac{\rho}{\rho-1}\right)^{1/2}(\widehat{\mathbf{P}}_x\{Z_m>0\}-\widehat{\mathbf{P}}_x\{Z_n>0\})\quad\text{for some } c>0.$$

Now let first n and then m go to ∞. Applying Lemma 4.1 yields (4.10).

(4.4) combined with Lemmas 4.1 and 4.2 shows that for any $x\geqslant 0$,

$$\lim_{n\to\infty}\mathbf{P}\{Z_n>0\mid M_{\rho n}\geqslant -x\}=\widehat{\mathbf{E}}_x q_\infty=:v(x),\tag{4.11}$$

and, in view of Proposition 3.1,

$$\lim_{n\to\infty}n^{1/2}\,\mathbf{P}\{Z_n>0,M_{\rho n}\geqslant -x\}=c_1\rho^{-1/2}h(x)\,v(x).\tag{4.12}$$

From (4.9) and the fact that $\eta_{k,\infty}\leqslant\eta_k$, we see that for any $x\geqslant 0$,

$$v(x)>0.\tag{4.13}$$

Next we get rid of ρ. Proposition 3.1 implies

$$\begin{aligned}&\mathbf{P}\{Z_n>0,\ M_n\geqslant -x\}-\mathbf{P}\{Z_n>0,\ M_{\rho n}\geqslant -x\}\\&\quad\leqslant\mathbf{P}\{M_n\geqslant -x\}-\mathbf{P}\{M_{\rho n}\geqslant -x\}\sim c_1h(x)n^{-1/2}(1-\rho^{-1/2})\end{aligned}\tag{4.14}$$

for any $\rho>1$. The last factor on the right-hand side of (4.14) can be made arbitrarily small by choosing ρ sufficiently close to 1. Hence,

$$\lim_{n\to\infty}n^{1/2}\mathbf{P}\{Z_n>0,\ M_n\geqslant -x\}=c_1h(x)\,v(x),\quad x\geqslant 0.\tag{4.15}$$

In the final step we let x go to ∞. Note that the first moment estimate,

$$q_k=\mathbf{P}\{Z_k>0\mid f_0,\dots,f_{k-1}\}\leqslant\mathbf{E}(Z_k\mid f_0,\dots,f_{k-1})=\exp(S_k),$$

implies $q_n=\min_{0\leqslant k\leqslant n}q_k\leqslant\exp(M_n)$. Applying Proposition 3.1 we deduce

$$\begin{aligned}\mathbf{P}\{Z_n>0,\ M_n<-x\}&\leqslant\ \mathbf{E}(\exp(M_n);\ M_n<-x)\\&\leqslant\sum_{k\geqslant\lfloor x\rfloor}\exp(-k)\,\mathbf{P}\{-k-1\leqslant M_n<-k\}\\&\leqslant\ c_2n^{-1/2}\sum_{k\geqslant\lfloor x\rfloor}h(k+1)\exp(-k),\quad x\geqslant 0.\end{aligned}\tag{4.16}$$

Since h grows only linearly in x, the sum on the right-hand side of (4.16) becomes arbitrarily small for sufficiently large x. From (4.16) and (4.15) we see that for any $\varepsilon>0$,

$$\begin{aligned}c_1h(x)\,v(x)-\varepsilon&\leqslant\liminf_{n\to\infty}n^{1/2}\,\mathbf{P}\{Z_n>0\}\leqslant\limsup_{n\to\infty}n^{1/2}\,\mathbf{P}\{Z_n>0\}\\&\leqslant c_1h(x)\,v(x)+\varepsilon,\end{aligned}\tag{4.17}$$

if only x is chosen large enough. From (4.17) the claim of Theorem 1.1 follows with

$$\beta:=c_1\lim_{x\to\infty}h(x)\,\widehat{\mathbf{E}}_x\left(\sum_{k=0}^{\infty}\eta_{k,\infty}\exp(-S_k)\right)^{-1}.$$

By (4.15), $h(x)\,v(x)$ is increasing in x. Hence, (4.13) implies $\beta>0$. Finiteness of β follows from (4.15) and (4.16).

Remark. Our proof of Theorem 1.1 shows that, conditioned on $Z_n>0$, the minimum M_n of the random walk (S_n) has a weak limit. More precisely, (4.15) and (1.3) imply

$$\lim_{n\to\infty}\mathbf{P}\{M_n\geqslant -x\mid Z_n>0\}=\beta^{-1}c_1h(x)\,v(x),\quad x\geqslant 0.$$

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 *LEVENTAL S.**

A MAXIMAL INEQUALITY FOR REAL NUMBERS WITH APPLICATION TO EXCHANGEABLE RANDOM VARIABLES

Пусть $x = (x_1, \ldots, x_n)$ — последовательность вещественных чисел такая, что $\sum_{i=1}^n x_i = 0$, и пусть $\Theta = \{\theta = (\theta_1, \ldots, \theta_n)\colon \theta_i = \pm 1\}$. Мы доказываем, что для любых $\theta \in \Theta$ и $t \geqslant 0$ справедливы неравенства

$$\frac{1}{2}\,\mathbf{P}\{|x_\pi| \geqslant 38\,t\} \leqslant \mathbf{P}\{|\theta \cdot x_\pi| \geqslant t\} \leqslant \mathbf{P}\Big\{|x_\pi| \geqslant \frac{t}{2}\Big\},$$

где $\mathbf{P}$ — равномерное распределение на группе $\{\pi\}$ всех перестановок чисел $\{1, \ldots, n\}$, $x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$, $\theta \cdot x_\pi = (\theta_1 x_{\pi(1)}, \ldots, \theta_n x_{\pi(n)})$ и $|y| = \max_{1 \leqslant k \leqslant n}\{|\sum_{i=1}^k y_i|\}$ для любого $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$.

Наше доказательство элементарно и автономно. В качестве следствия мы доказываем для случая вещественных чисел недавний результат

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