

# BASIC TEMPLATE FOR MATHEMATICS REPORTS

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ABSTRACT. This is a latex template that demonstrates how you may type up a mathematics report in Latex. The reports in the summer REU at Indiana University can be found in <https://math.indiana.edu/undergraduate/reu-summer-research-program/past-reu/index.html>.

issue of "for any set  $A$ " in statements, but only as  $A$  gets sufficiently large  
We presume the reader is familiar with basic ... such as ...  
note that all of the contents are known, and I am not passing this off as my own original thoughts

## 1. INTRODUCTION AND MOTIVATION

For any sets  $A, B$  and binary operation  $\cdot$  which acts on elements of  $A$  and  $B$ , we define

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}.$$

Observe the following example which motivates the study of the sum-product problem.

Let  $A = \{1, 2, 3, \dots, n\}$  and  $G = \{2, 2^2, 2^3, \dots, 2^n\}$ .

Notice that  $A$  is given by an arithmetic sequence and  $G$  by a geometric sequence. We are going to calculate  $|A + A|$ ,  $|AA|$ ,  $|G + G|$ , and  $|GG|$ .

Observe that

$$\begin{aligned} |A + A| &= |\{2, 3, \dots, 2n\}| \\ &= 2n - 1 \\ &= 2|A| - 1, \end{aligned}$$

and by the same argument,

$$\begin{aligned} |GG| &= |\{2^2, 2^3, \dots, 2^{2n}\}| \\ &= 2|G| - 1. \end{aligned}$$

We have

$$|G + G| = |\{2^i + 2^j : i, j \in \{1, \dots, n\}\}|$$

For all  $i, j$ ,  $2^i + 2^j$  is a number written in base 2. By the uniqueness of representations in different bases (see appendix), we have that  $2^i + 2^j$  is distinct for every choice of  $\{i, j\}$ .

Therefore,

$$\begin{aligned} |G + G| &\geq |\{\{i, j\} : i, j \in \{1, \dots, n\}\}| \\ &= \binom{n}{2} + n \\ &= \binom{|G| + 1}{2} \end{aligned}$$

Finally, I prove in the appendix that **PROVE IT IN THE APPENDIX**.

Observing the trivial bounds  $2|S| - 1 \leq |S \cdot S| \leq \binom{|S|}{2}$  for any set  $S$  and any commutative operation  $\cdot$ , it is clear that both  $AA$  and  $G + G$  are almost as large as they can be, and  $A + A, GG$  are as small as they can be.

This is the phenomenon which motivates this problem. One questions is: “does there exist a set for which both the sum and product sets are small?” The sum-product conjecture states that such a set does not exist. Another question we study in this report is: “what determines if the sum set is large or the product set is large?” A conjecture which tries to partially answer this question states that the sum set is large when the set itself is convex.

The rest of the report will proceed in the following way : **finish this**

## 2. PRELIMINARIES

As a useful shorthand, for any natural number  $n$ , let

$$[n] = \{1, 2, \dots, n\}.$$

The study of these problems requires the notion of orders of magnitude. For any functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , write

$$f(x) \gg g(x) \text{ as } x \rightarrow \infty$$

if

$$\exists x_0, c \in \mathbb{R}^+ \text{ s.t. } x > x_0 \implies |f(x)| \geq c |g(x)|,$$

write

$$f(x) \ll g(x) \text{ as } x \rightarrow \infty$$

if

$$\exists x_0, c \in \mathbb{R}^+ \text{ s.t. } x > x_0 \implies |f(x)| \leq c |g(x)|,$$

and write

$$f(x) \asymp g(x) \text{ as } x \rightarrow \infty$$

if

$$f(x) \ll g(x) \text{ and } f(x) \gg g(x) \text{ as } x \rightarrow \infty.$$

We write  $\ll_\epsilon, \gg_\epsilon, \asymp_\epsilon$  if the constant depends on  $\epsilon$ . For example,

$$f(x) \gg_\epsilon g(x)^\epsilon$$

if there is some function  $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  so

$$\forall \epsilon > 0, \exists x_0 \in \mathbb{N} \text{ s.t. } x > x_0 \implies |f(x)| \geq c(\epsilon) |g(x)|^\epsilon.$$

This should be interpreted as

$$\forall \epsilon > 0, f(x) \gg g(x)^\epsilon.$$

Throughout this report, the asymptotic parameter (in this case  $x$ ) will always tend to infinity, so it will no longer be mentioned. Oftentimes the parameter will not even be in the expression. For example, if for some set  $A$  we write

$$|A + A| \gg |A|,$$

it is taken to mean that  $A$  is defined implicitly by  $|A|$ , and  $|A|$  is the parameter which tends to  $\infty$ .

For any sets  $A, B$  and any binary operation  $\cdot$  acting on elements of  $A$  and  $B$ , define the representation function  $r_{A \cdot B} : A \cdot B \rightarrow \mathbb{N}$  by

$$r_{A \cdot B}(x) = |\{(a, b) \in A \times B : x = a \cdot b\}|.$$

Throughout this report the shorthand

$$\delta_{A,B}(x) = r_{A-B}(x), \quad \sigma_{A,B}(x) = r_{A+B}(x), \quad \delta_A(x) = \delta_{A,A}(x), \quad \sigma_A(x) = \sigma_{A,A}(x)$$

will be used.

For any sets  $A, B$ , define the Additive Energy  $E(A, B)$  and Multiplicative Energy  $M(A, B)$  by

$$E(A, B) = |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : a_1 - b_1 = a_2 - b_2\}|$$

and

$$M(A, B) = \left| \left\{ (a_1, a_2, b_1, b_2) \in A^2 \times B^2 : \frac{a_1}{b_1} = \frac{a_2}{b_2} \right\} \right|.$$

Observe that

$$E(A, B) = \sum_{x \in A-B} \delta_{A,B}(x)^2$$

and

$$M(A, B) = \sum_{x \in \frac{A}{B}} r_{\frac{A}{B}}(x)^2.$$

This definition is symmetric in the sense that a 4-tuple  $(a_1, a_2, b_1, b_2) \in A^2 \times B^2$  is a solution to

$$a_1 - b_1 = a_2 - b_2$$

if and only if it is a solution to

$$a_1 + b_2 = a_2 + b_1,$$

and therefore

$$E(A, B) = \sum_{x \in A+B} \sigma_{A,B}(x)^2 = \sum_{x \in A-B} \delta_{A,B}(x)^2.$$

There is a similar argument for multiplicative energy. Any 4-tuple  $(a_1, a_2, b_1, b_2) \in A^2 \times B^2$  with nonzero entries is a solution to

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

if and only if it is a solution to

$$\frac{a_1}{b_2} = \frac{a_2}{b_1}.$$

There are at most

$$\sum_{i=1}^4 \binom{4}{i} = 15$$

4-tuples with zero entries, so

$$M(A, B) = \sum_{x \in \frac{A}{B}} r_{\frac{A}{B}}(x)^2 \asymp \sum_{x \in AB} r_{AB}(x)^2.$$

We also define higher energies

$$E_n(A, B) = \sum_{x \in A-B} \delta_{A,B}(x)^n,$$

so

$$E(A) = E_2(A),$$

and as a shorthand use

$$E_n(A) = E_n(A, A).$$

Similar definitions and shorthand are used for multiplicative energy.

We can relate the energies to the sizes of the sum and product sets by the Cauchy-Schwarz Inequality.

$$|A| |B| = \sum_{x \in A+B} \sigma_{A,B}(x) \leq |A+B|^{\frac{1}{2}} E(A, B)^{\frac{1}{2}},$$

and

$$|A| |B| = \sum_{x \in AB} r_{AB}(x) \leq |AB|^{\frac{1}{2}} M(A, B)^{\frac{1}{2}}.$$

Similar inequalities can be derived for  $|A - B|$  and  $|\frac{A}{B}|$ .

Let  $I \subset \mathbb{R}$  be an interval. We call a function  $f : I \rightarrow \mathbb{R}$  convex if for any 2 points  $x_1, x_2 \in I$ , and any  $\lambda \in [0, 1]$ ,

$$f((x_1 - x_2)\lambda + x_2) \leq (f(x_1) - f(x_2))\lambda + f(x_2).$$

A finite set  $A \subset \mathbb{R}$  is convex if there is a function  $f : [1, |A|] \rightarrow \mathbb{R}$  such that

$$A = \{f(i) : i \in \{1, \dots, |A|\}\}.$$

A property of convex functions which will come up later, and is worth proving now, is

**Lemma 2.1.** *Let  $I \subset \mathbb{R}$  be an interval. Let  $f : I \rightarrow \mathbb{R}$  be convex. Let  $T \subset \mathbb{R}^2$ .*

*Take  $\ell = \{(x, f(x)) : x \in I\}$  to be the graph of  $f$ . We have that*

$$\forall t_1, t_2 \in T, |(\ell + t_1) \cap (\ell + t_2)| \leq 1.$$

*That is that translations of the graph of a convex function intersect in at most one point.*

*Proof.* **PROVE THIS** □

### 3. STATEMENT OF PROBLEMS

We'll now move on to precise statements of the mentioned problems, and statements of the most modern results.

The idea that there does not exist a set with a small sum and product set is stated precisely as

**Conjecture 3.1** (Sum-Product Conjecture). *For every finite set  $A \subset \mathbb{R}$ ,*

$$\max(|A + A|, |A \cdot A|) \gg_{\epsilon} |A|^{2-\epsilon}$$

The idea that convexity opposes additive structure is stated precisely as

**Conjecture 3.2** (Convex Sumset Conjecture). *For finite and convex set  $A$ ,*

$$|A + A| \gg_{\epsilon} |A|^{2-\epsilon}$$

To date, the best results for both of these conjectures are proven in [RS21].

**Theorem 3.3.** *For every sufficiently large finite set  $A \subset \mathbb{R}$ ,*

$$\max(|A + A|, |A \cdot A|) \gg_{\epsilon} |A|^{\frac{4}{3} + \frac{2}{1167} - \epsilon}$$

and

**Theorem 3.4.** *For every sufficiently large, finite, and convex set  $A$ ,*

$$|A + A| \gg_{\epsilon} |A|^{\frac{30}{19} - \epsilon}$$

These results will not be covered in this report. **best in this report is ...** ,

A final result I'd like to mention, which I found interesting, is a result by Olmezov proven in ref

**Theorem 3.5.** *Let  $n \geq 1$ . Let  $f : [1, n] \rightarrow \mathbb{R}$  be a convex function satisfying*

$$f'(x) > 0, \quad f''(x) > 0, \quad f'''(x) < 0, \quad f^{(IV)}(x) \leq 0,$$

*and let  $A = \{f(i) : i = 1, \dots, n\}$ . Then*

$$|A \pm A| \gg_{\epsilon} |A|^{\frac{5}{3} - \epsilon}.$$

I will not discuss this result further in this paper.

#### 4. GRAPHS AND THE CROSSING NUMBER INEQUALITY

**Make a note about how this is the dumbed down version (no topology)**

A very useful tool in proving sum-product theorems is the Szemerédi-Trotter theorem. The easiest proof of this theorem is as a corollary of the Crossing Number Inequality, which gives an estimate on how close a graph is to being planar. The purpose of this section is to provide a brief introduction to graphs and prove the Crossing Number Inequality.

**define connected Tao's article reread to see what I missed**

Abstractly, a graph  $G$  is a pair  $G = (V, E)$  where each  $e \in E$  is of the form  $e \subset V$  with  $|e| = 2$ . We call the set  $V$  the vertices, and the set  $E$  the edges. A drawing of a graph is a depiction of a graph with vertices as points in the plane and edges as curves between the vertices they consist of. For example:

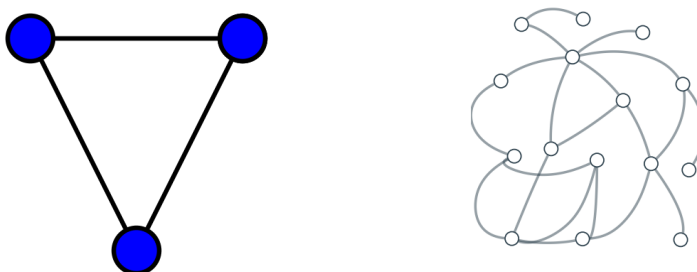


FIGURE 1. Drawings of Graphs

There are infinitely many ways to draw any given graph. A crossing in a drawing of a graph is an intersection between 2 curves which represent edges. The crossing number of a graph is the minimum number of crossings over all drawings of the graph  $G$ . Denote this by  $\text{cr}(G)$ . A graph  $G$  is called planar if its crossing number is 0.

A precise statement of the Crossing Number Inequality is

**Theorem 4.1.** *If  $G = (V, E)$  is a sufficiently large graph, with  $|E| \geq 4|V|$ , then*

$$\text{cr}(G) \gg \frac{|E|^3}{|V|^2}.$$

For a drawing of a planar graph, we call any region of the plane which is bounded by edges a face. We also call the unbounded region of the plane a face. Here is an example of a drawing of a planar graph with labelled faces:

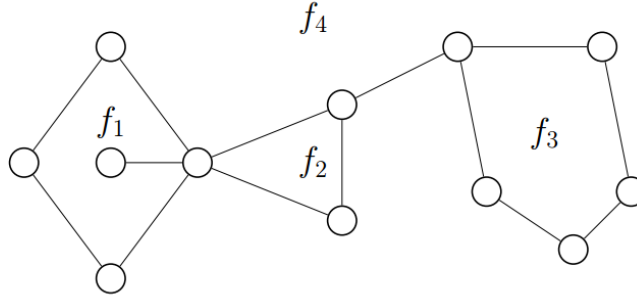


FIGURE 2. Drawing of Planar Graph with Labeled Faces  $f_i$

Observe that any non-planar graph  $G = (V, E)$  can be turned into a planar graph by removing at most  $\text{cr}(G)$  edges from  $E$ . Therefore, an upper bound on the number of edges of a planar graph can be used to find a lower bound on the crossing number of a non-planar graph. An obvious tool to use for a statement about planar graphs is

**Theorem 4.2** (Euler's Formula for Planar Graphs). *Let  $G = (V, E)$  be a connected planar graph, with  $|V| \geq 1$ , and consider some drawing with 0 crossings. Let  $F$  be the set of all faces of this drawing.*

$$|V| - |E| + |F| = 2.$$

*Proof.* We may construct our drawing of  $G$  by first drawing a vertex, and then doing combination of the following steps:

**FINISH PROOF** □

**below paragraph is ugly** The dependence on  $|F|$  in Euler's formula can be removed by bounding it in terms of  $|E|$ . This can be done by double counting the face-edge incidences. Let an edge be incident to a face if the edge is one of the bounding edges which defines the face. Define  $\chi : F \times E \rightarrow \{0, 1\}$  by  $\chi(f, e) = 1$  if  $f$  and  $e$  are incident, and  $\chi(f, e) = 0$  otherwise. The total number of face edge incidences is

$$I = \sum_{f \in F} \sum_{e \in E} \chi(f, e).$$

We may assume  $E \geq 3$ . It follows that every face is incident to at least 3 edges, so

$$I \geq \sum_{f \in F} 3 = 3|F|.$$

Every edge is incident to at most 2 faces, so

$$I \leq \sum_{e \in E} 2 = 2|E|.$$

It follows that

$$3|F| \leq 2|E|$$

or

$$|F| \leq \frac{2}{3}|E|.$$

Applying this to Euler's formula,

$$|V| - |E| + \frac{2}{3}|E| \geq 2$$

or

$$|E| \leq 3|V| - 6.$$

Now suppose that  $G = (V, E)$  is non-planar. As mentioned before,  $G$  may be turned planar  
**I NEED BETTER LOGIC FOR WHY I CAN REMOVE CRG AND MAKE IT PLANAR**

Therefore, if  $|E| \geq 3|V|$ , then  $\text{cr}(G) \geq |E| - 3|V|$ . To further improve this inequality, apply the probabilistic method to the deletion of vertices of  $G$ .

Let each  $v \in V$  be removed with a probability  $1 - p$ ,  $p \in (0, 1)$ . Let the remaining set of vertices be  $V'$ . An edge is removed whenever either of the corresponding vertices are removed. Let the remaining set of edges be  $E'$ . The remaining graph is then  $G' = (V', E')$ . We have

$$\text{cr}(G') \geq |E'| - 3|V'|,$$

and so

$$\mathbb{E}(\text{cr}(G')) \geq \mathbb{E}(|E'| - 3|V'|),$$

or, by the linearity of the expected value,

$$\mathbb{E}(\text{cr}(G')) \geq \mathbb{E}(|E'|) - 3\mathbb{E}(|V'|).$$

Each  $v \in V$  is removed with probability  $1 - p$ , so

$$\mathbb{E}(|V'|) = p|V|.$$

Each edge remains only when both corresponding vertices remain. Each vertex remains independently with a probability  $p$ , so

$$\mathbb{E}(|E'|) = p^2|E|.$$

Considering a drawing of  $G$  with the minimum number of crossings. Each crossing remains only if both corresponding edges remain. Each edge remains independently with a probability  $p^2$ , so the expected value of the number of crossings remaining in the drawing is  $p^4 \text{cr}(G)$ . There is no guarantee that this drawing is optimal to minimize the crossings of  $G'$ , but we may conclude that

$$\mathbb{E}(\text{cr}(G')) \leq p^4 \text{cr}(G),$$

and therefore that

$$p^4 \operatorname{cr}(G) \geq \mathbb{E}(\operatorname{cr}(G)) \geq p^2 |E| - 3p |V|$$

for any  $p \in (0, 1)$ . Assuming  $|E| \geq 4|V|$ , and taking  $p = \frac{4|V|}{|E|}$ ,

$$\operatorname{cr}(G) \geq \frac{|E|}{\left(\frac{4|V|}{|E|}\right)^2} - \frac{3|V|}{\left(\frac{4|V|}{|E|}\right)^3} = \frac{1}{16} \left( \frac{|E|^3}{|V|^2} - \frac{3|E|^3}{4|V|^2} \right) \gg \frac{|E|^3}{|V|^2}.$$

## 5. THE SZEMEREDI-TROTTER THEOREM

A precise statement of the Szemerédi-Trotter Theorem is

**I think there is a better way to write the set of curves below that is less restrictive on their domain**

**Theorem 5.1** (Szemerédi-Trotter Theorem). *Let  $P \subset \mathbb{R}^2$  be a finite set of points. Let  $\mathcal{L}$  be a finite set of curves in  $\mathbb{R}^2$ .*

*More precisely, every  $l \in \mathcal{L}$  is of the form  $l = \{(x(t), y(t)) : t \in \mathbb{R}\}$  for some  $x, y \in C^0(\mathbb{R})$ .*

*Let  $\chi : P \times \mathcal{L} \rightarrow \{0, 1\}$  be the incidence function between a point and a line, so*

$$\chi(p, l) = \begin{cases} 1 & \text{if } p \in l \\ 0 & \text{otherwise} \end{cases}$$

*If any two  $l \in \mathcal{L}$  intersect in at most one point, then the total number of point-line incidences,*

$$I(P, \mathcal{L}) = \sum_{(p, l) \in P \times \mathcal{L}} \chi(p, l)$$

*satisfies*

$$I(P, \mathcal{L}) \ll |P|^{\frac{2}{3}} |\mathcal{L}|^{\frac{2}{3}} + |P| + |\mathcal{L}|.$$

**First proven in ...**

**I need to prove that the crossing number inequality can be used**

**Proof**

**revise below dialogue**

A particular case of the Szemerédi-Trotter theorem is when all the curves in  $\mathcal{L}$  are lines. Heuristically, the Szemerédi-Trotter theorem is useful in the study of Sum-Product problems because equations of lines are given by addition and multiplication. You can construct a system of lines and points whose incidences relate to the sum or product set you are studying, and apply the Szemerédi-Trotter theorem to bound these incidences.

Recalling the property of convexity, ... **finish this dialogue.**

We'll begin with a theorem which applies the Szemerédi-Trotter theorem to a system whose curves consist of translations of convex functions.

**EMERGENCY : REWIRTE SO S HAS ARBITRARY : , NOT [n], THEN CHANGE BELOW.**

**Theorem 5.2.** *For some  $n \in \mathbb{N}$ , let  $f : [1, n] \rightarrow \mathbb{R}$  be convex.*

*Let  $S = \{(i, f(i)) : i \in [n]\}$  and  $T \subset \mathbb{R}^2$  be finite. We have*

$$|S + T| \gg \max \left( |S|^{\frac{3}{2}} |T|^{\frac{1}{2}}, |S| |T| \right).$$



*Proof.* Let

$$L_t = \{(x, f(x)) + t : x \in [1, n], t \in T\},$$

and let

$$\mathcal{L} = \{L_t : t \in T\}.$$

For every  $x \in [n]$ ,  $(x, f(x)) + t \in S + T$ . Therefore, there are  $|S|$  incidences between  $L_t$  and the point set  $S + T$  for all  $t \in T$ . The set  $\mathcal{L}$  consists of translations of the graph of a convex function, so the Szemerédi-Trotter theorem is satisfied.

$$|S| |T| \ll |S + T|^{\frac{2}{3}} |T|^{\frac{2}{3}} + |S + T| + |T|.$$

Trivially,

$$|S + T| \geq |T|,$$

so

$$|S| |T| \ll \max \left( |S + T|^{\frac{2}{3}} |T|^{\frac{2}{3}}, |S + T| \right),$$

or

$$|S + T| \gg \max \left( |S|^{\frac{3}{2}} |T|^{\frac{1}{2}}, |S| |T| \right).$$

□

**this theorem very useful blah blah blah**

**Theorem 5.3.** *For sufficiently large sets  $A \subset \mathbb{R}$ , which are finite and convex, and any set  $B \subset \mathbb{R}$  with  $|A| = |B|$ ,*

$$|A + B| \gg |A|^{\frac{3}{2}}.$$

*Proof.* Let  $|A| = n$  and  $f : [1, n] \rightarrow \mathbb{R}$  be the convex function such that

$$A = \{f(i) : i \in [n]\}.$$

Take

$$S = \{(i, f(i)) : i \in [n]\}$$

and

$$T = [n] \times B.$$

Observe that  $S + T \subset [2n] \times (A + B)$ , so  $|S + T| \ll |A| |A + B|$ . Applying Theorem 5.2,

$$|A| |A + B| \gg \max \left( |S|^{\frac{3}{2}} |T|^{\frac{1}{2}}, |S| |T| \right) = \max \left( |A|^{\frac{3}{2}} |A|, |A|^3 \right) = |A|^{\frac{5}{2}},$$

so

$$|A + B| \gg |A|^{\frac{3}{2}}.$$

□

**better dialogue** Another variation of this theorem is

**Theorem 5.4.** *For finite sets  $A \subset \mathbb{R}$  which are convex and sufficiently large, and any set  $B \subset \mathbb{R}$ ,*

$$|A + B| \gg |A| |B|^{\frac{1}{2}}.$$

*Proof.* **PROOF**

□

In particular, both of these theorems give the result

**Theorem 5.5.** *For sufficiently large sets  $A \subset \mathbb{R}$  which are finite and convex,*

$$|A + A| \gg |A|^{\frac{3}{2}}.$$

**Theorem 5.6.** *For sufficiently large sets  $A \subset \mathbb{R}$ ,*

$$\max(|A + A|, |A \cdot A|) \gg |A|^{\frac{5}{4}}.$$

*Proof.* Take  $P = (A + A) \times (A \cdot A)$ .

For any  $a, b \in A$ , let  $\ell_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $\ell_{a,b}(x) = (x - a) \cdot b$ . Let  $L_{a,b} = \{(x, \ell_{a,b}(x)) : x \in \mathbb{R}\}$  be the graph of  $\ell_{a,b}$ .

Take  $\mathcal{L} = \{L_{a,b} : a, b \in A\}$ . There are  $|A|$  many numbers in  $A + A$  of the form  $x + a$  where  $x \in A$ . For all of these numbers,  $\ell_{a,b}(x + a) = xb \in A \cdot A$ .

We have shown that for each choice of  $a, b \in A$ , there are  $|A|$  many numbers  $z \in A + A$  such that  $\ell_{a,b}(z) \in A \cdot A$ , or that there are  $|A|$  many incidences between  $P$  and  $L_{a,b}$ . It follows that there are  $|A|^3$  total incidences between  $\mathcal{L}$  and  $P$ . Because  $\mathcal{L}$  is a set of lines, the Szemerédi-Trotter theorem holds.

$$|A|^3 \ll |A + A|^{\frac{2}{3}} |A \cdot A|^{\frac{2}{3}} |A|^{\frac{4}{3}} + |A + A| |A \cdot A| + |A|^2$$

or

$$|A|^3 \ll \max\left(|A + A|^{\frac{2}{3}} |A \cdot A|^{\frac{2}{3}} |A|^{\frac{4}{3}}, |A + A| |A \cdot A|, |A|^2\right).$$

Applying trivial inequalities,

$$|A|^2 \leq |A + A| |A \cdot A| \leq |A + A| |A \cdot A| \left( \frac{|A|^{\frac{4}{3}}}{|A + A|^{\frac{1}{3}} |A \cdot A|^{\frac{1}{3}}} \right),$$

so

$$\begin{aligned} |A|^3 \ll |A + A|^{\frac{2}{3}} |A \cdot A|^{\frac{2}{3}} |A|^{\frac{4}{3}} &\implies \max(|A + A|, |A \cdot A|)^{\frac{4}{3}} \gg |A|^{\frac{5}{3}} \\ &\implies \max(|A + A|, |A \cdot A|) \gg |A|^{\frac{5}{4}}. \end{aligned}$$

□

We also have results like

**Theorem 5.7.** *Let  $A \subset \mathbb{R}$  be a sufficiently large convex and finite set, then for every finite set  $B \subset \mathbb{R}$  we have*

$$|\{x \in A - B : \delta_{A,B}(x) \geq \tau\}| \ll \frac{|A| |B|^2}{\tau^3}.$$

*Proof.* **PROOF**

□

An immediate and useful corollary of this is

**REPLACE WITH A,B VERSION**

**Corollary 5.8.** *Let  $A \subset \mathbb{R}$  be a sufficiently large, convex, finite set. Order elements  $a_i \in A$  such that*

$$\delta_A(s_1) \geq \delta_A(s_2) \geq \cdots \geq \delta_A(s_{|A+A|}).$$

*For every  $1 \leq r \leq |A + A|$  we have*

$$\delta_A(s_r) \ll \frac{|A|}{r^{\frac{1}{3}}}.$$

*Proof.*

$$r = |\{x \in A - A : \delta_A(x) \geq \delta_A(s_r)\}| \ll \frac{|A|^3}{\delta_A(s_r)^3} \implies \delta_A(s_r) \ll \frac{|A|}{r^{\frac{1}{3}}}.$$

□

any more results here?

## 6. ADDITIVE AND MULTIPLICATIVE ENERGY RESULTS **EW BAD TITLE**

### Dyadic partitioning introduction

#### Lemma 6.1. *dyadic partitioning*

Recall that

$$|A + A| \geq \frac{|A|^4}{E(A)}$$

and

$$|AA| \geq \frac{|A|^4}{M(A)}.$$

Observe that finding an upper bound on  $E(A)$  or  $M(A)$  in terms of  $|A + A|$ ,  $|AA|$ , and  $|A|$  yields a sum product theorem. This section consists of results which employ this general idea.

The simplest of which is a result in [Sol09], which gives a stronger result on the sum-product conjecture than the Szemerédi-Trotter theorem.

**Theorem 6.2.** *Let  $A \subset \mathbb{R}^+$  be finite and sufficiently large.*

$$\max(|A + A|, |AA|) \gg |A|^{\frac{4}{3}-o(1)}.$$

*Proof.* **FIX THIS BEGINNING PART**

#### THE WHOLE THING NEEDS REVISION

We begin with a construction. Consider the set  $A^2$ , along with the smallest set of lines through the origin which cover  $A^2$ .

Two pairs  $(a_1, a_2), (b_1, b_2) \in A^2$  give the same representation as a quotient if and only if

$$\frac{a_2}{a_1} = \frac{b_2}{b_1}.$$

Observe that this is the slope of a line through the origin and the points  $(a_1, a_2), (b_1, b_2)$ .

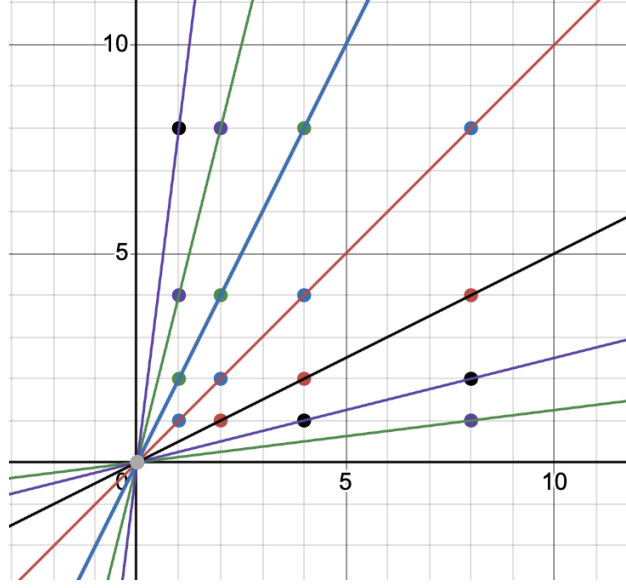
It follows that  $\left|\frac{A}{A}\right|$  is the number of lines through the origin necessary to cover the point set  $A^2$ . The slope of the line is the value the line represents in  $\frac{A}{A}$ , and the number of points on the line is the number of representations of that number in  $\frac{A}{A}$ .

Consider 2 consecutive lines and the set of all vector sums between a point on each line. If our points are  $(a_1, a_2)$  and  $(b_1, b_2)$ , with

$$\frac{a_2}{a_1} > \frac{b_2}{b_1},$$

then the slope of their sum is

$$\frac{a_2 + b_2}{a_1 + b_1}$$

FIGURE 3. Example with  $A = \{1, 2, 4, 8\}$ .

which satisfies

$$\frac{b_2}{b_1} < \frac{a_2 + b_2}{a_1 + b_1} < \frac{a_2}{a_1}.$$

That is, the vector sum must “lie between” the two lines which the original vectors are on. More precisely, for any pairs of consecutive lines, the vector sums of all points along the lines are disjoint.

We also have that the sums of any 2 points on each line are distinct. This is because a solution to

$$\lambda_1 v + \lambda_2 w = \lambda_3 v + \lambda_4 w \iff (\lambda_1 - \lambda_3) v + (\lambda_2 - \lambda_4) w$$

where  $\lambda_1 \neq \lambda_3$  or  $\lambda_2 \neq \lambda_4$  exists only if  $v$  and  $w$  are linearly dependent.

By dyadic partitioning on  $M(A)$ , we have

$$M(A) = \sum_{x \in \frac{A}{A}} r_{\frac{A}{A}}(x)^2 \leq \log \left( \left| \frac{A}{A} \right| \right) \tau^2 |S|$$

for some  $\tau$ , where  $S = \left\{ x \in \frac{A}{A} : r_{\frac{A}{A}}(x) \asymp \tau \right\}$ .

Consider the reduced system of points and lines, consisting only of the  $|S|$  many lines which have  $\asymp \tau$  many points on them. Consider the set of all vector sums between points on consecutive lines. Because all pairs of lines give disjoint sets, each with  $\asymp \tau^2$  many sums, there are  $\tau^2 |S|$  many vector sums. Because this is a subset of  $(A + A)^2$ ,

$$\tau^2 |S| \leq |A + A|^2,$$

so

$$\frac{|A|^4}{|AA|} \leq M(A) \ll \log \left( \left| \frac{A}{A} \right| \right) |A + A|^2 \implies \max(|A + A|, |AA|) \gg |A|^{\frac{4}{3} - o(1)}$$

□

This is nearly the best known result, all others are only slight improvements,  
... Only result purely obtaining upper bound on energy

Another instance of this type of result is

**Theorem 6.3.** *For convex sets  $A \subset \mathbb{R}$  which are finite and sufficiently large,*

$$E(A) \ll |A|^{\frac{32}{13}-o(1)},$$

which immediately leads to the result

**Corollary 6.4.** *For convex sets  $A \subset \mathbb{R}$  which are finite and sufficiently large,*

$$|A + A| \gg |A|^{\frac{3}{2} + \frac{1}{26} - o(1)}.$$

**FIX BELOW PARAGRAPH** This is proven in [Shk12]. I did not read this paper because it includes a lot of unique ideas and complex notation. I plan to read it in the future.

**REWRITE BELOW** Many arguments involving additive energy are not as straightforward as finding an upper bound. We often explore quantities such as  $E(A, S)$ ,  $E(D)$ ,  $E(S)$ ,  $E_3(A)$  etc. The following are examples of those style of arguments.

I'll first prove a handful of theorems which will be useful. The first of which I will restate from before, still leaving out proof.

**Theorem 6.5.** *For convex sets  $A \subset \mathbb{R}$  which are finite and sufficiently large,*

$$E(A) \ll |A|^{\frac{32}{13}-o(1)}.$$

**Theorem 6.6.** *For convex sets  $A \subset \mathbb{R}$  which are finite and sufficiently large,*

$$E_3(A) \ll |A|^{3-o(1)}.$$

*Proof.* Recall that upon ordering  $a_i$  such that  $\delta_A(a_1) \geq \delta_A(a_2) \geq \dots \geq \delta_A(a_{|A+A|})$ , we have that

$$\delta_A(a_r) \ll \frac{|A|}{r^{\frac{1}{3}}}.$$

With this,

$$\begin{aligned} E_3(A) &= \sum_{x \in A+A} \delta_A(x)^3 \\ &\ll |A|^3 \sum_{r=1}^{|A+A|} \frac{1}{r} \\ &\asymp |A|^3 \int_1^{|A+A|} \frac{1}{r} dr \\ &= |A|^{3-o(1)} \end{aligned}$$

□

**Theorem 6.7.** *For convex sets  $A \subset \mathbb{R}$  which are finite and sufficiently large, and any finite set  $B \subset \mathbb{R}$*

$$E(A, B) \ll |A| |B|^{\frac{3}{2}}.$$

**I NEED TO REDO PROOF WITH MORE LOGIC BEHIND WHY I CHOSE**  
**B 1/2**

*Proof.* Denote the elements of  $A - B$  by  $s_i$  where  $\delta_{A,B}(s_1) \geq \cdots \geq \delta_{A,B}(s_{|A-B|})$

Let  $P = \left\{x \in A - B : \delta_{A,B}(x) \geq |B|^{\frac{1}{2}}\right\}$ , and let  $P^* = (A - B) \setminus P$ .

$$\begin{aligned} \sum_{x \in P} \delta_{A,B}(x)^2 &= \sum_{i=1}^{|P|} \delta_{A,B}(s_i)^2 \\ &\ll |A|^{\frac{2}{3}} |B|^{\frac{4}{3}} \sum_{i=1}^{|P|} \frac{1}{r^{\frac{2}{3}}} \\ &\asymp |A|^{\frac{2}{3}} |B|^{\frac{4}{3}} |P|^{\frac{1}{3}} \\ &\ll |A|^{\frac{2}{3}} |B|^{\frac{4}{3}} \left( \frac{|A| |B|^2}{|B|^{\frac{1}{2}}} \right)^{\frac{1}{3}} \\ &= |A| |B|^{\frac{3}{2}}, \end{aligned}$$

and

$$\sum_{x \in P^*} \delta_{A,B}(x)^2 < |B|^{\frac{1}{2}} \sum_{x \in P^*} \delta_{A,B}(x) = |A| |B|^{\frac{3}{2}}.$$

Therefore,

$$E(A, B) = \sum_{x \in P} \delta_{A,B}(x)^2 + \sum_{x \in P^*} \delta_{A,B}(x)^2 \ll |A| |B|^{\frac{3}{2}}.$$

□

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## APPENDIX

### UNIQUENESS OF REPRESENTATIONS OF DIFFERENT BASES

I need to define all things I use like little  $o$  and big  $O$ .

**Theorem** (Erdős multiplication table theorem).

$$|[n] \cdot [n]| = o(n^2).$$

This is known as the “multiplication table theorem” because the quantity

$$|[n] \cdot [n]|$$

is the number of distinct numbers in an  $n \times n$  multiplication table:

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

FIGURE 4.  $10 \times 10$  multiplication table with distinct numbers highlighted in red.

The theorem I am proving here is just an “upper bound” on the asymptotic behavior of this quantity. In [For08], Kevin Ford proved that the exact order of this quantity is

$$|[n] \cdot [n]| \asymp \frac{n^2}{\log(n^2)^\delta (\log \log(n^2))^{\frac{3}{2}}},$$

where  $\delta = 1 - \frac{1+\log \log(2)}{2}$  is the Erdős-Tenenbaum-Ford constant.

A tool that will prove useful in evaluating sums is the Abel summation formula.

**Lemma.** Let  $(a_n)_{n=1}^\infty$  be a sequence of real numbers. Let  $A : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$A(t) = \sum_{n \leq t} a_n.$$

For  $x, y \in \mathbb{R}$ , with  $x < y$ , and any differentiable function  $\phi : [x, y] \rightarrow \mathbb{R}$ ,

$$\sum_{x < n \leq y} a_n \phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(t)\phi'(t) \, dt$$

## FINISH THIS LATER

*Proof.*

$$\begin{aligned} \sum_{x < n \leq y} a_n \phi(n) &= a_{\lceil x \rceil} \phi(\lceil x \rceil) + \cdots + a_{\lfloor y \rfloor} \phi(\lfloor y \rfloor) \\ &= (A(x+1) - A(x)) \phi(\lceil x \rceil) + \cdots + (A(y) - A(y-1)) \phi(\lfloor y \rfloor) \\ &= \\ &= A(y) \phi(\lfloor y \rfloor) - A(x) \phi(\lceil x \rceil) + \sum_{i=1}^{\lfloor y \rfloor - \lceil x \rceil} (\phi(\lceil x \rceil + i) - \phi(\lceil x \rceil + 1 + i)) A(x+i) \end{aligned}$$

□

We will also need to use the Chebyshev psi function

**Lemma 6.8.** *Let  $\psi : \mathbb{R}^+ \rightarrow \mathbb{N}$  be defined by*

$$\psi(t) = \sum_{\substack{p^\alpha \leq t \\ p \text{ prime} \\ \alpha \in \mathbb{N}}} \log(p).$$

*We have*

$$\psi(t) \ll t.$$

*Proof.* First we'll introduce the function  $\theta : \mathbb{R}^+ \rightarrow \mathbb{N}$  defined by

$$\theta(t) = \sum_{\substack{p \leq t \\ p \text{ prime}}} \log(p).$$

The functions  $\theta$  and  $\psi$  are clearly related by

$$\psi(t) = \prod_{\substack{p \leq N^{\frac{1}{\alpha}} \\ p \text{ prime} \\ \alpha \in \mathbb{N}}} \log(p) = \sum_{\alpha \in \mathbb{N}} \theta\left(t^{\frac{1}{\alpha}}\right) = \theta(t) + \sum_{\alpha \geq 2} \theta\left(t^{\frac{1}{\alpha}}\right).$$

Note that the sum over  $\alpha$  has only finitely many terms. The sum terminates when

$$2 \geq t^{\frac{1}{\alpha}} \implies \alpha \leq \log_2(t).$$

A trivial upper bound on  $\theta(t)$  is

$$\theta(t) = \sum_{\substack{p \leq t \\ p \text{ prime}}} \log(p) \leq t \log(t),$$

so

$$\begin{aligned} \psi(t) &= \theta(t) + \sum_{2 \leq \alpha \leq \log_2(t)} \theta(t^{\frac{1}{\alpha}}) \\ &\leq \theta(t) + \log_2(t) \theta(t^{\frac{1}{2}}) \\ &\leq \theta(t) + \frac{t^{\frac{1}{2}} \log(t)^2}{\log(2)} \end{aligned}$$

or

$$\psi(t) \ll \max\left(\theta(t), t^{\frac{1}{2}} \log(t)^2\right).$$

Therefore, it suffices to show  $\theta(t) \ll t$ .

We have

$$\theta(t) = \sum_{\substack{p \leq t \\ p \text{ prime}}} \log(p) = \log\left(\prod_{\substack{p \leq t \\ p \text{ prime}}} p\right),$$

so it is sufficient to show that

$$\prod_{\substack{p \leq t \\ p \text{ prime}}} p \ll e^t.$$

It is also sufficient to prove it for  $t \in \mathbb{N}$  because  $\theta(t) = \theta(\lfloor t \rfloor)$ .



For some natural number  $t$ , and a prime  $p$ ,

$$t+1 < p \leq 2t+1 \implies p \mid \binom{2t+1}{t} = \frac{(2t+1)!}{t!(t+1)!}.$$

Therefore, for any  $t$ ,

$$\prod_{\substack{t+1 < p \leq 2t+1 \\ p \text{ prime}}} p \mid \binom{2t+1}{t} \implies \prod_{\substack{t+1 < p \leq 2t+1 \\ p \text{ prime}}} p \leq \binom{2t+1}{t},$$

which gives us

$$2 \prod_{\substack{t+1 < p \leq 2t+1 \\ p \text{ prime}}} p \leq 2 \binom{2t+1}{t} \leq (1+1)^{2t+1} \implies \prod_{\substack{t+1 < p \leq 2t+1 \\ p \text{ prime}}} p \leq 4^t.$$

The rest follows by induction on  $t$ . Because we are proving a statement about order of magnitude, the base case is trivial. Now suppose that for some  $t \in \mathbb{N}$ ,

$$\prod_{\substack{p \leq m \\ p \text{ prime}}} p \ll e^t.$$

If  $t$  is odd, the induction follows trivially. If  $t$  is even, let  $t = 2m$ , so

$$\prod_{\substack{p \leq 2m \\ p \text{ prime}}} p \ll e^{2m}.$$

We have

$$\begin{aligned} \prod_{\substack{p \leq 2m+1 \\ p \text{ prime}}} p &= \prod_{\substack{p \leq m+1 \\ p \text{ prime}}} p \prod_{\substack{m+1 < p \leq 2m+1 \\ p \text{ prime}}} p \\ &\ll e^{m+1} 4^m \\ &\ll e^{m+1} e^m = e^{2m+1}. \end{aligned}$$

□

More accurately,  $\psi(t) \asymp \theta(t) \asymp t$ , but this is not necessary to get the desired result, so I am omitting it from this report.

The final lemma needed is

**Lemma.**

$$\sum_{\substack{p \leq n \\ p \text{ prime}}} \frac{1}{p} = \log \log(n) + O(1).$$

*Proof of Lemma.* Observe that for some number  $N \in \mathbb{N}$ , the prime factorization of  $N!$  is of the form

$$N! = \prod_{\substack{p \leq N \\ p \text{ prime}}} p^{\alpha(N,p)},$$

where

$$\alpha(N,p) = \sum_{i \in \mathbb{N}} \left\lfloor \frac{N}{p^i} \right\rfloor = \sum_{i \leq \log_p(N)} \left\lfloor \frac{N}{p^i} \right\rfloor = \sum_{i \leq \log_2(N)} \left\lfloor \frac{N}{p^i} \right\rfloor.$$

It follows that

$$\begin{aligned}
\log(N!) &= \sum_{\substack{p \leq N \\ p \text{ prime}}} \alpha(N, p) \log(p) \\
&= \sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_p(N)}} \left\lfloor \frac{N}{p^i} \right\rfloor \log(p) \\
&= \sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_p(N)}} \left( \frac{N}{p^i} - \delta(p) \right) \log(p) \\
&= N \sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_p(N)}} \frac{\log(p)}{p^i} - \sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_p(N)}} \delta(p) \log(p).
\end{aligned}$$

We have that

$$i \leq \log_p(N) \iff p^i \leq N,$$

so

$$\sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_p(N)}} \log(p) = \psi(N),$$

and therefore

$$\sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_2(N)}} \frac{\log(p)}{p^i} \leq \frac{\log(N!)}{N} + \frac{\psi(N)}{N}.$$

We also have, via a Riemann sum,

$$\begin{aligned}
\log(N!) &= \sum_{i=1}^N \log(i) \\
&= \int_2^N \log(i) \, di + O(1) \\
&= [i \log(i) - i]_{i=1}^N + O(1) \\
&= N \log(N) - N + O(1).
\end{aligned}$$

This leads to

$$\sum_{\substack{p \leq N \\ p \text{ prime} \\ i \leq \log_2(N)}} \frac{\log(p)}{p^i} = \frac{N \log(N) - N + O(1)}{N} + \frac{O(N)}{N} = \log(N) + O(1).$$

□