BASIC TEMPLATE FOR MATHEMATICS REPORTS

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ABSTRACT. This is a latex template that demonstrates how you may type up a mathematics report in Latex. The reports in the summer REU at Indiana University can be found in https://math.indiana.edu/undergraduate/reu-summer-research-program/past-reu/index.html.

There is an issue of not specifying what tends to infinity in some places. see crossing number inequalities and sum product theorems, say something like "when asymptotic parameter is not specified, take it to be..."

for all names add Dr or Mr or Professor or sum

less than less than epsilon notation

issue of "for any set A" in statements, but only as A gets sufficiently large

We presume the reader is familiar with basic ... such as ...

note that all of the contents are known, and I am not passing this off as my own original thoughts

1. Introduction and Motivation

I need to define all things I use like little o and big O.

verify I have n-th moment energy defined

For any sets A, B and binary operation $\cdot$ which acts on elements of A and B, we define

$$A \cdot B = \{a \cdot b : a \in A\ ,\ b \in B\}\,.$$

Observe the following 2 examples which motivate the study of the sum-product problem.

Let A be a finite set of numbers given by an arbitrary arithmetic sequence, and G by an arbitrary geometric sequence. We will determine the size of the sets $A+A, AA, G+G$, and GG.

A is of the form

$$A = \{a, a+d, a+2d, \cdots, a+(n-1)d\}$$

and G of the form

$$G = \left\{b, br, br^2, \cdots, br^{k-1}\right\}.$$

Observe that

$$\begin{aligned}|A+A| &= |\{2a, 2a+d, 2a+2d, \cdots, 2a+2(n-1)d\}| \\ &= 2(n-1)+1 \\ &= 2\,|A| - 1,\end{aligned}$$

and by the same argument,

$$\begin{aligned}|GG| &= \left|\left\{b^2, b^2r, \cdots, b^2r^{2(k-1)}\right\}\right| \\ &= 2\,|G| - 1.\end{aligned}$$

Date: August 7, 2024.

We have

$$\begin{aligned}|G+G| &= |\{g_1+g_2 : g_1, g_2 \in G\}| \\ &= \left|\left\{b(r^i+r^j) : i,j \in \{0,\cdots,k-1\}\right\}\right| \\ &= \left|\left\{r^i+r^j : i,j \in \{0,\cdots,k-1\}\right\}\right|.\end{aligned}$$

For $i \neq j$, $r^i + r^j$ represents a number in base r. By the uniqueness of representations in different bases (see appendix) **PUT IT IN APPENDIX**, we can conclude that there must be at least k^2-k many numbers in this set, that is

$$|G+G| = \left|\left\{r^i+r^j : i,j \in \{0,\cdots,k-1\}\right\}\right| \geq k^2-k = |G|^2 - |G|.$$

I NEED TO FINISH THIS EXAMPLE fix G sum set

Observing the trivial bound $|S \cdot S| \leq |S|^2$ for any set S and any operation $\cdot$, it is clear that both AA and $G+G$ are almost as large as they can be.

The main questions is: "does there exist a set for which both the sum and product sets are small?" The sum-product conjecture states that such a set does not exist. Another good question is: "what determines whether the sum set is large or the product set is large?" A separate conjecture tries to partially answer this question by stating that the sum set is large when the set itself is convex.

ABOVE PARAGRAPH CONVEX PRODUCT SET ARGUMENT

The rest of the report will proceed in the following way : **finish this**

2. Preliminaries

For any natural number n, define

$$[n] = \{1, 2, \cdots, n\}.$$

For any functions $f, g : \mathbb{N} \to \mathbb{R}$, write

$$f(x) \gg g(x)$$

if

$$\exists x_0 \in \mathbb{N} \text{ s.t. } x > x_0 \implies |f(x)| \geq c\,|g(x)|,$$

write

$$f(x) \ll g(x)$$

if

$$\exists x_0 \in \mathbb{N} \text{ s.t. } x > x_0 \implies |f(x)| \leq c\,|g(x)|,$$

and write

$$f(x) \asymp g(x)$$

if

$$f(x) \ll g(x) \text{ and } f(x) \gg g(x).$$

For any sets A, B and any binary operation $\cdot$ acting on elements of A and B, define the representation function $r_{A\cdot B} : A \cdot B \to \mathbb{N}$ by

$$r_{A\cdot B}(x) = |\{(a,b) \in A \times B : x = a \cdot b\}|.$$

For any sets A, B, define the Additive Energy $E(A,B)$ and Multiplicative Energy $M(A,B)$ by

$$E(A,B) = \left|\left\{(a_1,a_2,b_1,b_2) \in A^2 \times B^2 : a_1+b_1 = a_2+b_2\right\}\right|$$

and

$$M(A,B) = \left|\left\{(a_1,a_2,b_1,b_2) \in A^2 \times B^2 : a_1b_1 = a_2b_2\right\}\right|.$$

Observe that

$$E(A,B) = \sum_{x \in A+B} r_{A+B}(x)^2$$

and

$$M(A,B) = \sum_{x \in AB} r_{AB}(x)^2.$$

A 4-tuple $(a_1,a_2,b_1,b_2) \in A^2 \times B^2$ is a solution to

$$a_1 + b_1 = a_2 + b_2$$

if and only if it is a solution to

$$a_1 - b_2 = a_2 - b_1,$$

and therefore

$$E(A,B) = \sum_{x \in A+B} r_{A+B}(x)^2 = \sum_{x \in A-B} r_{A-B}(x)^2.$$

FINISH BELOW Any 4-tuple $(a_1,a_2,b_1,b_2) \in A^2 \times B^2$ with nonzero entries is a solution to

$$a_1b_1 = a_2b_2$$

if and only if it is a solution to

$$\frac{a_1}{b_2} = \frac{a_2}{b_1},$$

and therefore

By the Cauchy-Schwarz Inequality

$$|A|\,|B| = \sum_{x \in A+B} r_{A+B}(x) \leq |A+B|^{\frac{1}{2}}\, E(A,B)^{\frac{1}{2}},$$

and

$$|A|\,|B| = \sum_{x \in AB} r_{AB}(x) \leq |AB|^{\frac{1}{2}}\, M(A,B)^{\frac{1}{2}}.$$

Similar inequalities can be derived for $|A-B|$ and $\left|\frac{A}{B}\right|$.

CONVEXITY Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is convex if for any 2 points $x_1, x_2 \in I$, and any $\lambda \in [0,1]$,

$$f((x_1 - x_2)\lambda + x_2) \leq (f(x_1) - f(x_2))\,\lambda + f(x_2).$$

A finite set $A \subset \mathbb{R}$ is convex if there is a function $f : [1, |A|] \to \mathbb{R}$ such that

$$A = \{f(i) : i \in \{1, \cdots, |A|\}\}.$$

3. IDK what to call this

The idea that there does not exist a set with a small sum and product set is stated precisely as

Conjecture 3.1 (Sum-Product Conjecture)**.** *For every sufficiently large finite set $A \subset \mathbb{R}$,*

$$\max\left(|A+A| \ , \ |A \cdot A|\right) \gg_\epsilon |A|^{2-\epsilon}$$

The idea that

Conjecture 3.2. *For every sufficiently large, finite, and convex set A,*

$$|A+A| \gg_\epsilon |A|^{2-\epsilon}$$

To date, the best results for both of these conjectures are proven in **ref**.
proven in this report?
they are

Theorem 3.3. *For every sufficiently large finite set $A \subset \mathbb{R}$,*

$$\max\left(|A+A| \ , \ |A \cdot A|\right) \gg_\epsilon |A|^{\frac{4}{3}+\frac{2}{1167}-\epsilon}$$

and

Theorem 3.4. *For every sufficiently large, finite, and convex set A,*

$$|A+A| \gg_\epsilon |A|^{\frac{30}{19}-\epsilon}$$

if not, best results in this are ... , others are only slight improvements
A final result I'd like to mention, which I found interesting, is a result by Olmezov proven in **ref**

Theorem 3.5. *Let $n \geq 1$. Let $f : [1, n] \to \mathbb{R}$ be a convex function satisfying*

$$f'(x) > 0, \quad f''(x) > 0, \quad f'''(x) < 0, \quad f^{(IV)}(x) \leq 0,$$

and let $A = \{f(i) : i = 1, \ldots, n\}$. Then

$$|A \pm A| \gg_\epsilon |A|^{\frac{5}{3}-\epsilon}.$$

I will not discuss this result further in this paper.

4. Graphs and the Crossing Number Inequality

Make a note about how this is the dumbed down version (no topology)
A very useful tool in proving sum-product theorems is the Szemeredi-Trotter theorem. The easiest proof of this theorem is as a corollary of the Crossing Number Inequality, which gives an estimate on how close a graph is to being planar. The purpose of this section is to provide a brief introduction to graphs and prove the Crossing Number Inequality.
define connected Tao's article reread to see what I missed
Abstractly, a graph G is a pair $G = (V, E)$ where each $e \in E$ is of the form $e \subset V$ with $|e| = 2$. The call the set V the vertices, and the set E the edges. A drawing of a graph is a depiction of a graph with vertices as points in the plane and edges as curves between the vertices they consist of. For example:
There are infinitely many ways to draw any given graph. A crossing in a drawing of a graph is an intersection between 2 curves which represent edges. The crossing number of a

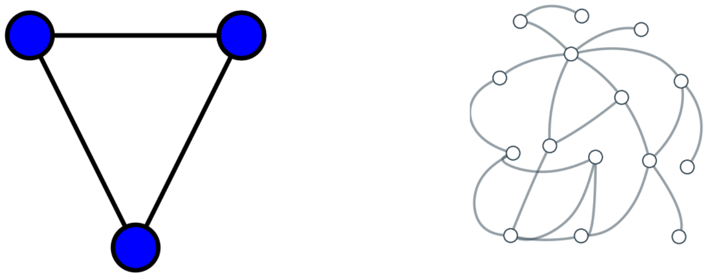

FIGURE 1. Drawings of Graphs

graph is the minimum number of crossings over all drawings of the graph G. Denote this by $\mathrm{cr}(G)$. A graph G is called planar if its crossing number is 0.

A precise statement of the Crossing Number Inequality is

Theorem 4.1. *If $G = (V, E)$ is a sufficiently large graph, with $|E| \geq 4|V|$, then*

$$\mathrm{cr}\,(G) \gg \frac{|E|^3}{|V|^2}.$$

For a drawing of a planar graph, we call any region of the plane which is bounded by edges a face. We also call the unbounded region of the plane a face. Here is an example of a drawing of a planar graph with labelled faces:

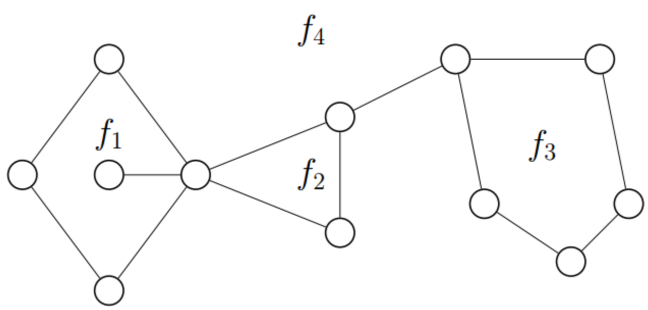

FIGURE 2. Drawing of Planar Graph with Labeled Faces f_i

Observe that any non-planar graph $G = (V, E)$ can be turned into a planar graph by removing at most $\mathrm{cr}\,(G)$ edges from E. Therefore, an upper bound on the number of edges of a planar graph can be used to find a lower bound on the crossing number of a non-planar graph. An obvious tool to use for a statement about planar graphs is

Theorem 4.2 (Euler's Formula for Planar Graphs)**.** *Let $G = (V, E)$ be a connected planar graph, with $|V| \geq 1$, and consider some drawing with 0 crossings. Let F be the set of all faces of this drawing.*

$$|V| - |E| + |F| = 2.$$

Proof. We may construct our drawing of G by first drawing a vertex, and then doing combination of the following steps:

FINISH PROOF □

below paragraph is ugly The dependence on $|F|$ in Euler's formula can be removed by bounding it in terms of $|E|$. This can be done by double counting the face-edge incidences. Let an edge be incident to a face if the edge is one of the bounding edges which defines the face. Define $\chi : F \times E \to \{0,1\}$ by $\chi(f,e) = 1$ if f and e are incident, and $\chi(f,e) = 0$ otherwise. The total number of face edge incidences is

$$I = \sum_{f \in F} \sum_{e \in E} \chi(f,e).$$

We may assume $E \geq 3$. It follows that every face is incident to at least 3 edges, so

$$I \geq \sum_{f \in F} 3 = 3\,|F|\,.$$

Every edge is incident to at most 2 faces, so

$$I \leq \sum_{e \in E} 2 = 2\,|E|\,.$$

It follows that

$$3\,|F| \leq 2\,|E|$$

or

$$|F| \leq \frac{2}{3}\,|E|\,.$$

Applying this to Euler's formula,

$$|V| - |E| + \frac{2}{3}\,|E| \geq 2$$

or

$$|E| \leq 3\,|V| - 6.$$

Now suppose that $G = (V, E)$ is non-planar. As mentioned before, G may be turned planar **I NEED BETTER LOGIC FOR WHY I CAN REMOVE CRG AND MAKE IT PLANAR**

Therefore, if $|E| \geq 3\,|V|$, then $\mathrm{cr}\,(G) \geq |E| - 3\,|V|$. To further improve this inequality, apply the probabilistic method to the deletion of vertices of G.

Let each $v \in V$ be removed with a probability $1 - p$, $p \in (0,1)$. Let the remaining set of vertices be V'. An edge is removed whenever either of the corresponding vertices are removed. Let the remaining set of edges be E'. The remaining graph is then $G' = (V', E')$. We have

$$\mathrm{cr}\,(G') \geq |E'| - 3\,|V'|\,,$$

and so

$$\mathbb{E}\,(\mathrm{cr}\,(G')) \geq \mathbb{E}\,(|E'| - 3\,|V'|)\,,$$

or, by the linearity of the expected value,

$$\mathbb{E}\,(\mathrm{cr}\,(G')) \geq \mathbb{E}\,(|E'|) - 3\mathbb{E}\,(|V'|)\,.$$

Each $v \in V$ is removed with probability $1 - p$, so

$$\mathbb{E}\,(|V'|) = p\,|V|\,.$$

Each edge remains only when both corresponding vertices remain. Each vertex remains independently with a probability p, so

$$\mathbb{E}\,(|E'|) = p^2\,|E|\,.$$

Considering a drawing of G with the minimum number of crossings. Each crossing remains only if both corresponding edges remain. Each edge remains independently with a probability p^2, so the expected value of the number of crossings remaining in the drawing is $p^4 \operatorname{cr}(G)$. There is no guarantee that this drawing is optimal to minimize the crossings of G', but we may conclude that

$$\mathbb{E}(\operatorname{cr}(G')) \leq p^4 \operatorname{cr}(G),$$

and therefore that

$$p^4 \operatorname{cr}(G) \geq \mathbb{E}(\operatorname{cr}(G)) \geq p^2 |E| - 3p|V|$$

for any $p \in (0,1)$. Assuming $|E| \geq 4|V|$, and taking $p = \frac{4|V|}{|E|}$,

$$\operatorname{cr}(G) \geq \frac{|E|}{\left(\frac{4|V|}{|E|}\right)^2} - \frac{3|V|}{\left(\frac{4|V|}{|E|}\right)^3} = \frac{1}{16}\left(\frac{|E|^3}{|V|^2} - \frac{3|E|^3}{4|V|^2}\right) \gg \frac{|E|^3}{|V|^2}.$$

5. The Szemeredi-Trotter Theorem

A precise statement of the Szemeredi-Trotter Theorem is

I think there is a better way to write the set of curves below that is less restrictive on their domain

Theorem 5.1 (Szemeredi-Trotter Theorem)**.** *Let $P \subset \mathbb{R}^2$ be a finite set of points. Let $\mathcal{L}$ be a finite set of curves in $\mathbb{R}^2$.*

More precisely, every $l \in \mathcal{L}$ is of the form $l = \{(x(t), y(t)) : t \in \mathbb{R}\}$ for some $x, y \in C^0(\mathbb{R})$.

Let $\chi : P \times \mathcal{L} \to \{0,1\}$ be the incidence function between a point and a line, so

$$\chi(p,l) = \begin{cases} 1 & \text{if } p \in l \\ 0 & \text{otherwise} \end{cases}$$

If any two $l \in \mathcal{L}$ intersect in at most one point, then the total number of point-line incidences,

$$I(P,\mathcal{L}) = \sum_{(p,l) \in P \times \mathcal{L}} \chi(p,l)$$

satisfies

$$I(P,\mathcal{L}) \ll |P|^{\frac{2}{3}} |\mathcal{L}|^{\frac{2}{3}} + |P| + |\mathcal{L}|.$$

First proven in ...

I need to prove that the crossing number inequality can be used

Proof

revise below dialogue

A particular case of the Szemeredi-Trotter theorem is when all the curves in $\mathcal{L}$ are lines. Heuristically, the Szemeredi-Trotter theorem is useful in the study of Sum-Product problems because equations of lines are given by addition and multiplication. You can construct a system of lines and points whose incidences relate to the sum or product set you are studying, and apply the Szemeredi-Trotter theorem to bound these incidences.

Recalling the property of convexity, **... finish this dialogue**.

We'll begin with a theorem which applies the Szemeredi-Trotter theorem to a system whose curves consist of translations of convex functions.

EMERGENCY : REWIRTE SO S HAS ARBITRARY : , NOT [n], THEN CHANGE BELOW.

Theorem 5.2. *For some $n \in \mathbb{N}$, let $f : [1, n] \to \mathbb{R}$ be convex.*
Let $S = \{(i, f(i)) : i \in [n]\}$ and $T \subset \mathbb{R}^2$ be finite. We have

$$|S + T| \gg \max\left(|S|^{\frac{3}{2}} |T|^{\frac{1}{2}}, |S| |T|\right).$$

Proof. Let

$$L_t = \{(x, f(x)) + t : x \in [1, n], t \in T\},$$

and let

$$\mathcal{L} = \{L_t : t \in T\}.$$

For every $x \in [n]$, $(x, f(x)) + t \in S + T$. Therefore, there are $|S|$ incidences between L_t and the point set $S + T$ for all $t \in T$. The set $\mathcal{L}$ consists of translations of the graph of a convex function, so the Szemeredi-Trotter theorem is satisfied.

$$|S| |T| \ll |S + T|^{\frac{2}{3}} |T|^{\frac{2}{3}} + |S + T| + |T|.$$

Trivially,

$$|S + T| \geq |T|,$$

so

$$|S| |T| \ll \max\left(|S + T|^{\frac{2}{3}} |T|^{\frac{2}{3}}, |S + T|\right),$$

or

$$|S + T| \gg \max\left(|S|^{\frac{3}{2}} |T|^{\frac{1}{2}}, |S| |T|\right).$$

$\square$

this theorem very useful blah blah blah

Theorem 5.3. *For sufficiently large sets $A \subset \mathbb{R}$, which are finite and convex, and any set $B \subset \mathbb{R}$ with $|A| = |B|$,*

$$|A + B| \gg |A|^{\frac{3}{2}}.$$

Proof. Let $|A| = n$ and $f : [1, n] \to \mathbb{R}$ be the convex function such that

$$A = \{f(i) : i \in [n]\}.$$

Take

$$S = \{(i, f(i)) : i \in [n]\}$$

and

$$T = [n] \times B.$$

Observe that $S + T \subset [2n] \times (A + B)$, so $|S + T| \ll |A| |A + B|$. Applying Theorem 5.2,

$$|A| |A + B| \gg \max\left(|S|^{\frac{3}{2}} |T|^{\frac{1}{2}}, |S| |T|\right) = \max\left(|A|^{\frac{3}{2}} |A|, |A|^3\right) = |A|^{\frac{5}{2}},$$

so

$$|A + B| \gg |A|^{\frac{3}{2}}.$$

$\square$

better dialogue Another variation of this theorem is

Theorem 5.4. *For finite sets $A \subset \mathbb{R}$ which are convex and sufficiently large, and any set $B \subset \mathbb{R}$,*

$$|A + B| \gg |A| |B|^{\frac{1}{2}}.$$

Proof. **PROOF** ☐

In particular, both of these theorems give the result

Theorem 5.5. *For sufficiently large sets $A \subset \mathbb{R}$ which are finite and convex,*

$$|A+A| \gg |A|^{\frac{3}{2}}.$$

Theorem 5.6. *For sufficiently large sets $A \subset \mathbb{R}$,*

$$\max\left(|A+A|, |A\cdot A|\right) \gg |A|^{\frac{5}{4}}.$$

Proof. Take $P = (A+A) \times (A \cdot A)$.

For any $a, b \in A$, let $\ell_{a,b} : \mathbb{R} \to \mathbb{R}$ be defined by $\ell_{a,b}(x) = (x-a)\cdot b$. Let $L_{a,b} = \{(x, \ell_{a,b}(x)) : x \in \mathbb{R}\}$ be the graph of $\ell_{a,b}$.

Take $\mathcal{L} = \{L_{a,b} : a, b \in A\}$. There are $|A|$ many numbers in $A+A$ of the form $x+a$ where $x \in A$. For all of these numbers, $\ell_{a,b}(x+a) = xb \in A \cdot A$.

We have shown that for each choice of $a, b \in A$, there are $|A|$ many numbers $z \in A+A$ such that $\ell_{a,b}(z) \in A \cdot A$, or that there are $|A|$ many incidences between P and $L_{a,b}$. It follows that there are $|A|^3$ total incidences between $\mathcal{L}$ and P. Because $\mathcal{L}$ is a set of lines, the Szemeredi-Trotter theorem holds.

$$|A|^3 \ll |A+A|^{\frac{2}{3}} |A\cdot A|^{\frac{2}{3}} |A|^{\frac{4}{3}} + |A+A|\,|A\cdot A| + |A|^2$$

or

$$|A|^3 \ll \max\left(|A+A|^{\frac{2}{3}} |A\cdot A|^{\frac{2}{3}} |A|^{\frac{4}{3}}, |A+A|\,|A\cdot A|, |A|^2\right).$$

Applying trivial inequalities,

$$|A|^2 \le |A+A|\,|A\cdot A| \le |A+A|\,|A\cdot A| \left(\frac{|A|^{\frac{4}{3}}}{|A+A|^{\frac{1}{3}}\,|A\cdot A|^{\frac{1}{3}}}\right),$$

so

$$\begin{aligned} |A|^3 \ll |A+A|^{\frac{2}{3}} |A\cdot A|^{\frac{2}{3}} |A|^{\frac{4}{3}} &\implies \max\left(|A+A|, |A\cdot A|\right)^{\frac{4}{3}} \gg |A|^{\frac{5}{3}} \\ &\implies \max\left(|A+A|, |A\cdot A|\right) \gg |A|^{\frac{5}{4}}. \end{aligned}$$

☐

We also have results like

Theorem 5.7. *Let $A \subset \mathbb{R}$ be a sufficiently large convex and finite set, then for every finite set $B \subset \mathbb{R}$ we have*

$$|\{x \in A - B : \delta_{A,B}(x) \ge \tau\}| \ll \frac{|A|\,|B|^2}{\tau^3}.$$

Proof. **PROOF** ☐

An immediate and useful corollary of this is
REPLACE WITH A,B VERSION

Corollary 5.8. *Let $A \subset \mathbb{R}$ be a sufficiently large, convex, finite set. Order elements $a_i \in A$ such that*

$$\delta_A(s_1) \geq \delta_A(s_2) \geq \cdots \geq \delta_A(s_{|A+A|}).$$

For every $1 \geq r \leq |A+A|$ we have

$$\delta_A(s_r) \ll \frac{|A|}{r^{\frac{1}{3}}}.$$

Proof.

$$r = |\{x \in A - A : \delta_A(x) \geq \delta_A(s_r)\}| \ll \frac{|A|^3}{\delta_A(s_r)^3} \implies \delta_A(s_r) \ll \frac{|A|}{r^{\frac{1}{3}}}.$$

□

any more results here?

6. Additive and Multiplicative Energy Results **ew bad title**

Dyadic partitioning introduction

Lemma 6.1. ***dyadic partitioning***

Recall that

$$|A+A| \geq \frac{|A|^4}{E(A)}$$

and

$$|AA| \geq \frac{|A|^4}{M(A)}.$$

Observe that finding an upper bound on $E(A)$ or $M(A)$ in terms of $|A+A|, |AA|$, and $|A|$ yields a sum product theorem. This section consists of results which employ this general idea.

The simplest of which is a result in [Sol09], which gives a stronger result on the sum-product conjecture than the Szemeredi-Trotter theorem.

Theorem 6.2. *Let $A \subset \mathbb{R}^+$ be finite and sufficiently large.*

$$\max(|A+A|, |AA|) \gg |A|^{\frac{4}{3}-o(1)}.$$

Proof. **FIX THIS BEGINNING PART**

THE WHOLE THING NEEDS REVISION

We begin with a construction. Consider the set A^2, along with the smallest set of lines through the origin which cover A^2.

Two pairs $(a_1, a_2), (b_1, b_2) \in A^2$ give the same representation as a quotient if and only if

$$\frac{a_2}{a_1} = \frac{b_2}{b_1}.$$

Observe that this is the slope of a line through the origin and the points $(a_1, a_2), (b_1, b_2)$.

It follows that $\left|\frac{A}{A}\right|$ is the number of lines through the origin necessary to cover the point set A^2. The slope of the line is the value the line represents in $\frac{A}{A}$, and the number of points on the line is the number of representations of that number in $\frac{A}{A}$.

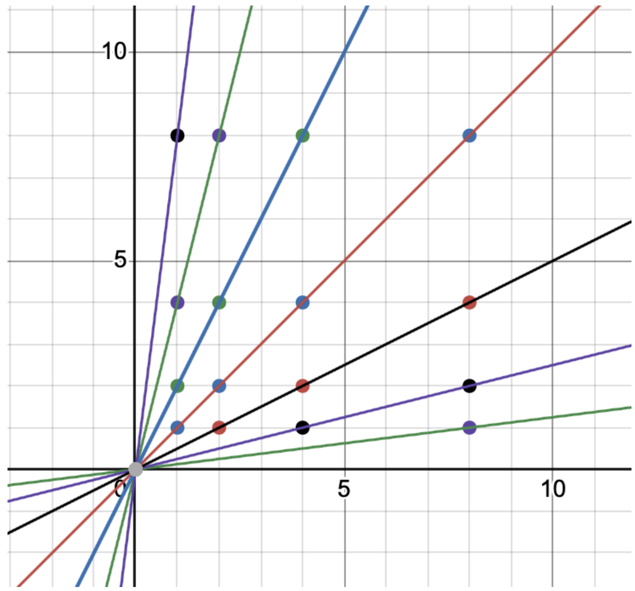

FIGURE 3. Example with $A = \{1, 2, 4, 8\}$.

Consider 2 consecutive lines and the set of all vector sums between a point on each line. If our points are (a_1, a_2) and (b_1, b_2), with

$$\frac{a_2}{a_1} > \frac{b_2}{b_1},$$

then the slope of their sum is

$$\frac{a_2 + b_2}{a_1 + b_1}$$

which satisfies

$$\frac{b_2}{b_1} < \frac{a_2 + b_2}{a_1 + b_1} < \frac{a_2}{a_1}.$$

That is, the vector sum must "lie between" the two lines which the original vectors are on. More precisely, for any pairs of consecutive lines, the vector sums of all points along the lines are disjoint.

We also have that the sums of any 2 points on each line are distinct. This is because a solution to

$$\lambda_1 v + \lambda_2 w = \lambda_3 v + \lambda_4 w \iff (\lambda_1 - \lambda_3)\, v + (\lambda_2 - \lambda_4)\, w$$

where $\lambda_1 \neq \lambda_3$ or $\lambda_2 \neq \lambda_4$ exists only if v and w are linearly dependent.

By dyadic partitioning on $M(A)$, we have

$$M(A) = \sum_{x \in \frac{A}{A}} r_{\frac{A}{A}}(x)^2 \leq \log\left(\left|\frac{A}{A}\right|\right) \tau^2 \,|S|$$

for some τ, where $S = \left\{x \in \frac{A}{A} : r_{\frac{A}{A}}(x) \asymp \tau\right\}$.

Consider the reduced system of points and lines, consisting only of the $|S|$ many lines which have $\asymp \tau$ many points on them. Consider the set of all vector sums between points on consecutive lines. Because all pairs of lines give disjoint sets, each with $\asymp \tau^2$ many sums, there are $\tau^2 |S|$ many vector sums. Because this is a subset of $(A+A)^2$,

$$\tau^2 |S| \leq |A+A|^2 ,$$

so

$$\frac{|A|^4}{|AA|} \leq M(A) \ll \log\left(\left|\frac{A}{A}\right|\right) |A+A|^2 \implies \max\left(|A+A|, |AA|\right) \gg |A|^{\frac{4}{3}-o(1)}$$

□

This is nearly the best known result, all others are only slight improvements, ... Only result purely obtaining upper bound on energy

Another instance of this type of result is

Theorem 6.3. *For convex sets $A \subset \mathbb{R}$ which are finite and sufficiently large,*

$$E(A) \ll |A|^{\frac{32}{13}-o(1)} ,$$

which immediately leads to the result

Corollary 6.4. *For convex sets $A \subset \mathbb{R}$ which are finite and sufficiently large,*

$$|A+A| \gg |A|^{\frac{3}{2}+\frac{1}{26}-o(1)} .$$

FIX BELOW PARAGRAPH This is proven in [Shk12]. I did not read this paper because it includes a lot of unique ideas and complex notation. I plan to read it in the future.

REWRITE BELOW Many arguments involving additive energy are not as straightforward as finding an upper bound. We often explore quantites such as $E(A,S), E(D), E(S), E_3(A)$ etc. The following are examples of those style of arguments.

I'll first prove a handful of theorems which will be useful. The first of which I will restate from before, still leaving out proof.

Theorem 6.5. *For convex sets $A \subset \mathbb{R}$ which are finite and sufficiently large,*

$$E(A) \ll |A|^{\frac{32}{13}-o(1)} .$$

Theorem 6.6. *For convex sets $A \subset \mathbb{R}$ which are finite and sufficiently large,*

$$E_3(A) \ll |A|^{3-o(1)} .$$

Proof. Recall that upon ordering a_i such that $\delta_A(a_1) \geq \delta_A(a_2) \geq \cdots \geq \delta_A(a_{|A+A|})$, we have that

$$\delta_A(a_r) \ll \frac{|A|}{r^{\frac{1}{3}}}.$$

With this,

$$\begin{aligned} E_3(A) &= \sum_{x\in A-A} \delta_A(x)^3 \\ &\ll |A|^3 \sum_{r=1}^{|A+A|} \frac{1}{r} \\ &\asymp |A|^3 \int_1^{|A+A|} \frac{1}{r}\,\mathrm{d}r \\ &= |A|^{3-o(1)} \end{aligned}$$

□

Theorem 6.7. *For convex sets $A \subset \mathbb{R}$ which are finite and sufficiently large, and any finite set $B \subset \mathbb{R}$*

$$E(A,B) \ll |A|\,|B|^{\frac{3}{2}}.$$

I NEED TO REDO PROOF WITH MORE LOGIC BEHIND WHY I CHOSE B 1/2

Proof. Denote the elements of $A-B$ by s_i where $\delta_{A,B}(s_1) \geq \cdots \geq \delta_{A,B}(s_{|A-B|})$

Let $P = \left\{x \in A-B : \delta_{A,B}(x) \geq |B|^{\frac{1}{2}}\right\}$, and let $P^* = (A-B)\setminus P$.

$$\begin{aligned} \sum_{x\in P} \delta_{A,B}(x)^2 &= \sum_{i=1}^{|P|} \delta_{A,B}(s_r)^2 \\ &\ll |A|^{\frac{2}{3}}\,|B|^{\frac{4}{3}} \sum_{i=1}^{|P|} \frac{1}{r^{\frac{2}{3}}} \\ &\asymp |A|^{\frac{2}{3}}\,|B|^{\frac{4}{3}}\,|P|^{\frac{1}{3}} \\ &\ll |A|^{\frac{2}{3}}\,|B|^{\frac{4}{3}} \left(\frac{|A|\,|B|^2}{|B|^{\frac{1}{2}}}\right)^{\frac{1}{3}} \\ &= |A|\,|B|^{\frac{3}{2}}, \end{aligned}$$

and

$$\sum_{x\in P^*} \delta_{A,B}(x)^2 < |B|^{\frac{1}{2}} \sum_{x\in P^*} \delta_{A,B}(x) = |A|\,|B|^{\frac{3}{2}}.$$

Therefore,

$$E(A,B) = \sum_{x\in P} \delta_{A,B}(x)^2 + \sum_{x\in P^*} \delta_{A,B}(x)^2 \ll |A|\,|B|^{\frac{3}{2}}.$$

□

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Appendix

Theorem (Erdős multiplication table theorem)**.**

$$|[n] \cdot [n]| = o(n^2).$$

This is known as the "multiplication table theorem" because the quantity

$$|[n] \cdot [n]|$$

is the number of distinct numbers in an $n \times n$ multiplication table:

×	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	2	4	6	8	10	**12**	**14**	**16**	**18**	**20**
3	3	6	9	12	**15**	18	**21**	**24**	**27**	**30**
4	4	8	12	16	20	24	**28**	**32**	**36**	**40**
5	5	10	15	20	**25**	30	**35**	40	**45**	**50**
6	6	12	18	24	30	36	**42**	**48**	**54**	**60**
7	7	14	21	28	35	42	**49**	**56**	**63**	**70**
8	8	16	24	32	40	48	56	**64**	**72**	**80**
9	9	18	27	36	45	54	63	72	**81**	**90**
10	10	20	30	40	50	60	70	80	90	**100**

Figure 4. 10×10 multiplication table with distinct numbers highlighted in red.

The theorem I am proving here is just an "upper bound" on the asymptotic behavior of this quantity. In [For08], Kevin Ford proved that the exact order of this quantity is

$$|[n] \cdot [n]| \asymp \frac{n^2}{\log{(n^2)}^{\delta} \left(\log\log{(n^2)}\right)^{\frac{3}{2}}},$$

where $\delta = 1 - \frac{1+\log\log(2)}{2}$ is the Erdős-Tenenbaum-Ford constant.

A tool that will prove useful in evaluating sums is the Abel summation formula.

Lemma. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Let $A : \mathbb{R} \to \mathbb{R}$ be defined by*

$$A(t) = \sum_{n \le t} a_n.$$

For $x, y \in \mathbb{R}$, with $x < y$, and any differentiable function $\phi : [x, y] \to \mathbb{R}$,

$$\sum_{x<n\le y} a_n \phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(t)\phi'(t)\ \mathrm{d}t$$

FINISH THIS LATER

Proof.

$$\begin{aligned}\sum_{x<n\leq y} a_n\phi(n) &= a_{\lceil x\rceil}\phi(\lceil x\rceil)+\cdots+a_{\lfloor y\rfloor}\phi(\lfloor y\rfloor)\\ &= \left(A(x+1)-A(x)\right)\phi(\lceil x\rceil)+\cdots+\left(A(y)-A(y-1)\right)\phi(\lfloor y\rfloor)\\ &=\\ &= A(y)\phi\left(\lfloor y\rfloor\right)-A(x)\phi\left(\lceil x\rceil\right)+\sum_{i=1}\left(\phi\left(\lceil x\rceil+i\right)-\phi\left(\lceil x\rceil+1+i\right)\right)A(x+i)\end{aligned}$$

∎

We will also need to use the Chebyshev psi function

Lemma 6.8. *Let $\psi:\mathbb{R}^+\to\mathbb{N}$ be defined by*

$$\psi(t)=\sum_{\substack{p^\alpha\leq t\\ p\ prime\\ \alpha\in\mathbb{N}}}\log\left(p\right).$$

We have

$$\psi(t)\ll t.$$

Proof. First we'll introduce the function $\theta:\mathbb{R}^+\to\mathbb{N}$ defined by

$$\theta(t)=\sum_{\substack{p\leq t\\ p\text{ prime}}}\log\left(p\right).$$

The functions θ and ψ are clearly related by

$$\psi(t)=\prod_{\substack{p\leq N^{\frac{1}{\alpha}}\\ p\text{ prime}\\ \alpha\in\mathbb{N}}}\log\left(p\right)=\sum_{\alpha\in\mathbb{N}}\theta\left(t^{\frac{1}{\alpha}}\right)=\theta(t)+\sum_{\alpha\geq 2}\theta\left(t^{\frac{1}{\alpha}}\right).$$

Note that the sum over α has only finitely many terms. The sum terminates when

$$2\geq t^{\frac{1}{\alpha}}\implies\alpha\leq\log_2\left(t\right).$$

A trivial upper bound on $\theta(t)$ is

$$\theta(t)=\sum_{\substack{p\leq t\\ p\text{ prime}}}\log\left(p\right)\leq t\log\left(t\right),$$

so

$$\begin{aligned}\psi(t)&=\theta(t)+\sum_{2\leq\alpha\leq\log_2(t)}\theta(t^{\frac{1}{\alpha}})\\ &\leq\theta(t)+\log_2\left(t\right)\theta(t^{\frac{1}{2}})\\ &\leq\theta(t)+\frac{t^{\frac{1}{2}}\log\left(t\right)^2}{\log\left(2\right)}\end{aligned}$$

or

$$\psi(t)\ll\max\left(\theta(t),t^{\frac{1}{2}}\log\left(t\right)^2\right).$$

Therefore, it suffices to show $\theta(t) \ll t$.

We have

$$\theta(t) = \sum_{\substack{p \le t \\ p \text{ prime}}} \log(p) = \log\left(\prod_{\substack{p \le t \\ p \text{ prime}}} p\right),$$

so it is sufficient to show that

$$\prod_{\substack{p \le t \\ p \text{ prime}}} p \ll e^t.$$

It is also sufficient to prove it for $t \in \mathbb{N}$ because $\theta(t) = \theta(\lfloor t \rfloor)$.

For some natural number t, and a prime p,

$$t+1 < p \le 2t+1 \implies p \mid \binom{2t+1}{t} = \frac{(2t+1)!}{t!(t+1)!}.$$

Therefore, for any t,

$$\prod_{\substack{t+1 < p \le 2t+1 \\ p \text{ prime}}} p \mid \binom{2t+1}{t} \implies \prod_{\substack{t+1 < p \le 2t+1 \\ p \text{ prime}}} p \le \binom{2t+1}{t},$$

which gives us

$$2 \prod_{\substack{t+1 < p \le 2t+1 \\ p \text{ prime}}} p \le 2\binom{2t+1}{t} \le (1+1)^{2t+1} \implies \prod_{\substack{t+1 < p \le 2t+1 \\ p \text{ prime}}} p \le 4^t.$$

The rest follows by induction on t. Because we are proving a statement about order of magnitude, the base case is trivial. Now suppose that for some $t \in \mathbb{N}$,

$$\prod_{\substack{p \le m \\ p \text{ prime}}} p \ll e^t.$$

If t is odd, the induction follows trivially. If t is even, let $t = 2m$, so

$$\prod_{\substack{p \le 2m \\ p \text{ prime}}} p \ll e^{2m}.$$

We have

$$\begin{aligned} \prod_{\substack{p \le 2m+1 \\ p \text{ prime}}} p &= \prod_{\substack{p \le m+1 \\ p \text{ prime}}} p \prod_{\substack{m+1 < p \le 2m+1 \\ p \text{ prime}}} p \\ &\ll e^{m+1} 4^m \\ &\ll e^{m+1} e^m = e^{2m+1}. \end{aligned}$$

□

More accurately, $\psi(t) \asymp \theta(t) \asymp t$, but this is not necessary to get the desired result, so I am omitting it from this report.

The final lemma needed is

Lemma.

$$\sum_{\substack{p \le n \\ p\ prime}} \frac{1}{p} = \log\log(n) + O(1).$$

Proof of Lemma. Observe that for some number $N \in \mathbb{N}$, the prime factorization of $N!$ is of the form

$$N! = \prod_{\substack{p \le N \\ p \text{ prime}}} p^{\alpha(N,p)},$$

where

$$\alpha(N,p) = \sum_{i \in \mathbb{N}} \left\lfloor \frac{N}{p^i} \right\rfloor = \sum_{i \le \log_p(N)} \left\lfloor \frac{N}{p^i} \right\rfloor = \sum_{i \le \log_2(N)} \left\lfloor \frac{N}{p^i} \right\rfloor.$$

It follows that

$$\begin{aligned}
\log(N!) &= \sum_{\substack{p \le N \\ p \text{ prime}}} \alpha(N,p)\log(p) \\
&= \sum_{\substack{p \le N \\ p \text{ prime} \\ i \le \log_p(N)}} \left\lfloor \frac{N}{p^i} \right\rfloor \log(p) \\
&= \sum_{\substack{p \le N \\ p \text{ prime} \\ i \le \log_p(N)}} \left(\frac{N}{p^i} - \delta(p) \right) \log(p) \\
&= N \sum_{\substack{p \le N \\ p \text{ prime} \\ i \le \log_p(N)}} \frac{\log(p)}{p^i} - \sum_{\substack{p \le N \\ p \text{ prime} \\ i \le \log_p(N)}} \delta(p)\log(p).
\end{aligned}$$

We have that

$$i \le \log_p(N) \iff p^i \le N,$$

so

$$\sum_{\substack{p \le N \\ p \text{ prime} \\ i \le \log_p(N)}} \log(p) = \psi(N),$$

and therefore

$$\sum_{\substack{p \le N \\ p \text{ prime} \\ i \le \log_2(N)}} \frac{\log(p)}{p} \le \frac{\log(N!)}{N} + \frac{\psi(N)}{N}.$$

Finally, via a Riemann sum,

$$\log(N!) = \sum_{i=1}^{N} \log(i) = \int_2^N \, \mathrm{d}x$$

$$\int_{-\infty}^{\infty} \, \mathrm{d}x$$

$\square$

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