Technical Appendix

A Conditional Quantile Estimation

A.1 Univariate Conditional Quantile Estimation

Given a vector of continuous portfolio returns r_t, the value-at-risk associated with a probability level θ satisfies

$$P(r_t < VaR_t(\theta)) = \theta. \tag{A.1}$$

We rewrite $VaR_t(\theta)$ in the quantile form $q_t(\beta;\theta)$, and obtain the coefficient estimates $\hat{\beta}$ by minimizing the objective function

$$\frac{1}{T}\sum_{t=1}^{T}[\theta - I(r_t < q_t(\beta;\theta))][r_t - q_t(\beta;\theta)]. \tag{A.2}$$

The indicator function $I(r_t < q_t(\beta;\theta))$ takes value 1 when the actual return falls below the value-at-risk and 0 otherwise.

We have three candidate models, namely the HYBRID-quantile model, the MIDAS-quantile model, and the CAViaR model. The HYBRID structure and the quantile version of the MIDAS model have been proposed by Chen, Ghysels, and Wang (2015) and Ghysels, Plazzi, and Valkanov (2016), respectively. The CAViaR model is introduced by Engle and Manganelli (2004).

The three models take the following general form

$$HYBRID : q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 \sum_{d=0}^{N-1} \omega(\kappa_\theta) f(r_{t-1+d/N}), \tag{A.3}$$

$$MIDAS : q_t(\beta;\theta) = \beta_1 + \beta_2 \sum_{d=0}^{N-1} \omega(\kappa_\theta) f(r_{t-1+d/N}), \tag{A.4}$$

$$CAViaR : q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 f(r_{t-1}). \tag{A.5}$$

The first two models incorporate a mixed-frequency component as the last term of the specification. The weighting polynomial, $\omega(\kappa_\theta)$, assigns higher weights to more recent daily returns. The term represents a projection of daily returns to a monthly frequency. All three models take into account past returns, which can be seen in the term $f(r_{t-1})$ or $f(r_{t-1+d/N})$. In the CAViaR case, the applicable past return is the previous monthly return.

The first specification we examine is the symmetric absolute value (SAV) form. The three models can be written as follows

$$HYBRID : q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 \sum_{d=1}^{20} \omega(\kappa_\theta)|r_{t-d/20}|, \tag{A.6}$$

$$MIDAS : q_t(\beta;\theta) = \beta_1 + \beta_2 \sum_{d=1}^{20} \omega(\kappa_\theta)|r_{t-d/20}|, \tag{A.7}$$

$$CAViaR : q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3|r_{t-1}|, \tag{A.8}$$

where $q_t(\beta;\theta)$ is the θ-th quantile at time t.

The second specification that we choose to evaluate is the asymmetric slope (AS) form, for which we allow asymmetric responses to positive and negative past returns. Correspondingly, the functional forms are

$$\begin{aligned} HYBRID : q_t(\beta;\theta) = {} & \beta_1 + \beta_2 q_{t-1}(\beta;\theta) \\ & + \beta_3 \sum_{d=1}^{20} \omega(\kappa_{1,\theta}) r^+_{t-d/20} + \beta_4 \sum_{d=1}^{20} \omega(\kappa_{2,\theta}) r^-_{t-d/20}, \end{aligned} \tag{A.9}$$

$$MIDAS : q_t(\beta;\theta) = \beta_1 + \beta_2 \sum_{d=1}^{20} \omega(\kappa_{1,\theta}) r^+_{t-d/20} + \beta_3 \sum_{d=1}^{20} \omega(\kappa_{2,\theta}) r^-_{t-d/20}, \tag{A.10}$$

$$CAViaR : q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 r^+_{t-1} + \beta_4 r^-_{t-1}, \tag{A.11}$$

where $r^+ = \max(r,0)$, $r^- = -\min(r,0)$.

We use the beta weighting polynomial suggested by Ghysels, Sinko, and Valkanov (2006) in the mixed-frequency component

$$B(k;\theta_1,\theta_2) = \frac{f(\frac{k}{K},\theta_1;\theta_2)}{\sum_{k=1}^{K} f(\frac{k}{K},\theta_1;\theta_2)},$$

where

$$f(x,a,b) = \frac{x^{a-1}(1-x)^{b-1}\Gamma(a+b)}{\Gamma(a)\Gamma(b)},$$
$$\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx.$$

For our purposes, we use daily returns as the inputs to estimate monthly return quantiles. We fix $\theta_1 = 1$ in our estimation, and obtain a weighting parameter θ_2 that in general assigns heavier weights to more recent observations.

A.2 Joint Estimation

Consider two return series, Y_{1t} and Y_{2t}. The information set $\mathcal{F}_{t-1}$ represents all information available at time t. For a certain confidence level $\theta \in (0, 1)$, the conditional quantile q_{it} for Y_{it} at time t is

$$P(Y_{it} \leq q_{it}|\mathcal{F}_{t-1}) = \theta, \quad i = 1, 2,$$

which is analogous to the univariate definition.

We adopt the methodology proposed by White, Kim, and Manganelli (2015) to estimate the conditional quantiles of the market returns jointly. The conditional quantiles q_{1t} and q_{2t} can be linked by a vector autoregressive (VAR) structure:

$$\begin{aligned} q_{1t} &= X_t'\beta_1 + b_{11}q_{1t-1} + b_{12}q_{2t-1}, \\ q_{2t} &= X_t'\beta_2 + b_{21}q_{1t-1} + b_{22}q_{2t-1}. \end{aligned}$$

The predictors X_t belong to $\mathcal{F}_{t-1}$ and typically include lagged returns.

The coefficient $\hat{\beta}_T$ is a quasi-maximum likelihood estimator that solves the optimization problem below:

$$\min_{\beta} \bar{S}_T(\beta) = \frac{1}{T}\sum_{t=1}^{T}\{\sum_{i=1}^{n}\sum_{j=1}^{p}\rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \beta))\}, \tag{A.12}$$

where $\rho_\theta(\cdot)$ is the standard check function used in quantile regressions. We view

$$S_t(\beta) = -\sum_{i=1}^{n}\sum_{j=1}^{p}\rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \beta)) \tag{A.13}$$

as the quasi log-likelihood for the observation at time t.

If $b_{12} = b_{21} = 0$, the structure above is reduced to the univariate CAViaR. In that case, the two conditional quantiles can be estimated independently. The off-diagonal coefficients b_{12} and b_{21} indicate the level of tail codependence of Y_{1t} and Y_{2t}, and can be assessed by testing the null hypothesis $H_0 : b_{12} = b_{21} = 0$.

B Backtests

To validate the conditional quantile predictions and provide a basis for selecting the most effective model, we refer to several backtesting procedures. We list here the dynamic quantile (Engle and Manganelli (2004)) test, the Kupiec (1995) test, and the Christoffersen (1998) test.

B.1 Hit Statistic and DQ test

Following Engle and Manganelli (2004), we calculate the Hit statistic:

$$Hit(\beta) \equiv I_t(\beta) - \theta,$$
$$I_t(\beta) = I(r_t < q_t(\beta)).$$

The function $Hit_t(\beta)$ is equal to $(1-\theta)$ when the return falls below the corresponding quantile and $(-\theta)$ otherwise. The expected value of this indicator is thus 0. Moreover, $Hit_t(\beta)$ must be uncorrelated with its lagged values and with $q_t(\beta)$.

The dynamic quantile test examines whether $T^{-1/2}X'(\hat{\beta})Hit(\hat{\beta})$ is significantly different from 0. The in-sample and out-of-sample dynamic quantile tests are

$$DQ_{IS} \equiv \frac{Hit'(\hat{\beta})X(\hat{\beta})(\hat{M}_T\hat{M}_T')^{-1}X'(\hat{\beta})Hit'(\hat{\beta})}{\theta(1-\theta)} \overset{d}{\sim} \chi^2_q, \quad T \to \infty, \tag{B.14}$$

where

$$\hat{M}_T \equiv X'(\hat{\beta}) - \{(2T\hat{c_T})^{-1}\sum_{t=1}^{T} I(|r_t - q_t(\hat{\beta})| < \hat{c_T}) \times X_t'(\hat{\beta})\nabla q_t(\hat{\beta})\}\hat{D_T^{-1}}\nabla' q(\hat{\beta}),$$

and

$$DQ_{OOS} \equiv N_R^{-1}Hit'(\hat{\beta}_{TR})X(\hat{\beta}_{TR})[X'(\hat{\beta}_{TR})X(\hat{\beta}_{TR})]^{-1} \tag{B.15}$$
$$\times X'(\hat{\beta}_{TR})Hit'(\hat{\beta}_{TR})/(\theta(1-\theta)) \overset{d}{\sim} \chi^2_q, \tag{B.16}$$

where T_R denotes the number of in-sample observations and N_R the number of out-of-sample observations.

B.2 Kupiec test

A standard unconditional coverage test is the Kupiec (1995) test, which focuses on the proportion of VaR violations. The violation count at confidence level $(1-\theta)$ should not differ considerably from $(\theta \times 100\%)$ over any time span.

The test statistic assumes the form

$$LR_{POF} = -2\log[\frac{(1-\theta)^{T-I(\theta)}\theta^{I(\theta)}}{(1-\hat{\theta})^{T-I(\theta)}\hat{\theta}^{I(\theta)}}] \sim \chi^2(1) \tag{B.17}$$
$$\hat{\theta} = \frac{1}{T}I(\theta) = \frac{1}{T}\sum_{t=1}^{T} I_t(\theta)$$

where $I_t(\theta)$ is the number of VaR violations and T is the sample size.

B.3 Time Until First Failure (TUFF) Test

Kupiec (1995) also suggested the time until first failure (TUFF) test. The TUFF-test measures the time it takes for the first VaR violation to occur. The test statistic is

$$LR_{TUFF} = -2\log[\frac{\theta(1-\theta)^{v-1}}{\frac{1}{v}(1-\frac{1}{v})^{v-1}}] \sim \chi^2(1), \tag{B.18}$$

where v denotes the time of first violation.

B.4 Christoffersen test

The Christoffersen (1998) independence test is a conditional coverage test identifying unusally frequent consecutive VaR exceedances. The test examines whether the probability of a VaR violation depends on the outcome of the previous day.

Define n_{ij} as the number of days that condition j occurred subsequent to condition i on the day before. All possible outcomes are displayed in the contingency table below. Following notations in earlier sections, the indicator variable I_t is set to 1 if a violation occurs and 0 under compliance.

Let π_i represent the probability of observing a violation conditional on state i on the previous day

$$\pi_0 = \frac{n_{01}}{n_{00}+n_{01}}, \pi_1 = \frac{n_{11}}{n_{10}+n_{11}}.$$

The unconditional probability of observing state $i = 1$ at time t is

$$\pi = \frac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}} = \frac{n_{01}+n_{11}}{N}.$$

	$I_{t-1} = \mathbf{0}$	$I_{t-1} = \mathbf{1}$	
$I_t = \mathbf{0}$	n_{00}	n_{10}	$n_{00}+n_{10}$
$I_t = \mathbf{1}$	n_{01}	n_{11}	$n_{01}+n_{11}$
	$n_{00}+n_{01}$	$n_{10}+n_{11}$	N

If the model is an accurate characterization of the VaR, an exception occurring today should be independent of the prior state. Namely, the null hypothesis states that $\pi_0 = \pi_1$. The likelihood ratio for this test is

$$LR_{IND} = -2\log[\frac{(1-\pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\pi_0^{n_{01}}(1-\pi_1)^{n_{10}}\pi_1^{n_{11}}}] \sim \chi^2(1). \tag{B.19}$$

We obtain a joint test of unconditional coverage and independence by combining the corresponding likelihood ratios

$$LR_{CC} = LR_{POF} + LR_{IND} \sim \chi^2(2). \tag{B.20}$$

A model passes the test when LR_{CC} is lower than the $\chi^2(2)$ critical value. We acknowledge that it is possible for a model to pass the joint test while failing either the unconditional coverage or the independence test, hence we will present the results for all three tests separately.

C Structural Break Tests

The CUSUM processes contain cumulative sums of standardized residuals (Brown, Durbin, and Evans (1975)):

$$W_n(t) = \frac{1}{\tilde{\sigma}\sqrt{\eta}} \sum_{i=k+1}^{k+\lfloor t\eta \rfloor} \tilde{u}_i. \tag{C.1}$$

Under the null hypothesis, $W_n \Rightarrow W$. Under the alternative, the recursive residuals should be close to 0 up to the structural change point t_0 and leave its mean afterwards.

Instead of analyzing the cumulative sums, an alternative is to detect a structural change through the moving sums of the residuals. The resulting sum is based on a moving time window, whose size is determined by the bandwidth $h \in (0, 1)$.

The recursive MOSUM process is defined as follows

$$\begin{aligned} M_n(t|h) &= \frac{1}{\tilde{\sigma}\sqrt{\eta}} \sum_{i=k+\lfloor N_\eta t \rfloor + 1}^{k+\lfloor N_\eta t \rfloor + \lfloor \eta h \rfloor} \hat{u}_i \\ &= W_n(\frac{\lfloor N_\eta t \rfloor + \lfloor \eta h \rfloor}{\eta}) - W_n(\frac{\lfloor N_\eta t \rfloor}{\eta}), \end{aligned} \tag{C.2}$$

where $N = (\eta - \lfloor \eta h \rfloor)/(1 - h)$.

Chu, Hornik, and Kuan (1995a) show that the limiting process for the empirical MOSUM processes is the increments of a Brownian motion. The Rec-MOSUM path will have a strong shift around the potential structural break point t_0.

As an extension to Chow (1960), we can also calculate the F statistics for all potential changing points. Andrews (1993) and Andrews and Ploberger (1994) suggested three

Assume that we obtain a vector of portfolio returns, $\{r_t\}_{t=1}^T$. As is typical in the literature, all of the returns referred to are log returns to allow temporal aggregation.

The n-period log return is defined as

$$r_{t,n} = \sum_{j=0}^{n-1} r_{t+j}. \tag{2.1}$$

Denote the probability associated with a target VaR as θ, i.e.:

$$P[r_{t,n} < q_{t,n}(\beta;\theta)] = \theta, \tag{2.2}$$

where β is a vector of parameters that need to be estimated.

The estimates $\hat{\beta}$ are set up to solve

$$\min_{\beta} \frac{1}{T} \sum_{t=1}^{T} [\theta - I(r_{t,n} < q_{t,n}(\beta;\theta))][r_{t,n} - q_{t,n}(\beta;\theta)], \tag{2.3}$$

where the indicator function $I(r_{t,n} < q_{t,n}(\beta;\theta)) = 1$ if $r_{t,n}$ is indeed below $q_{t,n}(\beta;\theta)$. For the ease of notation, we will omit the subscript n and proceed with the term $q_t(\beta;\theta)$.

2.1 CAViaR Model

The first model we cite is the CAViaR model proposed by Engle and Manganelli (2004). For portfolio returns $\{r_t\}_{t=1}^T$ and a vector of time t observable variables x_t, the CAViaR model can be written as follows

$$q_t(\beta;\theta) = \beta_0 + \sum_{i=1}^{q} \beta_i q_{t-i}(\beta;\theta) + \sum_{j=1}^{r} \beta_j l(x_{t-j}), \tag{2.4}$$

where $q_t(\beta;\theta)$ is the θ-quantile of the portfolio returns at time t. The notation signifies that each quantile level has a different set of coefficient estimates.

We expect the value-at-risk to increase as the returns from the previous period become higher, and to decrease otherwise. Therefore, a natural step to proceed is to choose the lagged returns as x_{t-1}.

We consider multiple functional forms of (1)

- **Symmetric absolute value**

  $q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 |r_{t-1}|$,

  where VaR depends symmetrically on the previous period return.

- **Asymmetric slope**

  $q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 r^+_{t-1} + \beta_4 r^-_{t-1}$,

  where $r^+ = \max(r, 0)$, $r^- = -\min(r, 0)$. This specification allows the conditional quantile to respond differently to positive and negative past returns.

- **Indirect GARCH**

  $q_t(\beta;\theta) = (\beta_1 + \beta_2 q_{t-1}(\beta;\theta)^2 + \beta_3 r^2_{t-1})^{1/2}$,

- **Adaptive**

  $q_t(\beta;\theta) = q_{t-1}(\beta;\theta) + \beta_1\{[1 + \exp(G(r_{t-1} - q_{t-1}(\beta;\theta)))]^{-1} - \theta\}$,

  where G is a finite, positive constant. This model corresponds to a strategy where the VaR should be increased immediately when exceeded, and decreased slightly otherwise.

All three terms, $q_t(\beta;\theta)$, $q_{t-1}(\beta;\theta)$, and r_{t-1}, are of weekly frequency. We intend to include daily observations in the forecast, and will address this issue further in later sections.

2.2 MIDAS Model

The second model that we choose is the MIDAS quantile forecasting model put forth by Ghysels, Plazzi, and Valkanov (2016). The conditional quantiles pertain to multiple horizon returns, and the regressors are lagged daily returns.

A weighting scheme of these daily returns is adopted, and we investigate the following equation

$$q_{\theta,t}(r_{t,n};\delta_{\theta,n}) = \alpha_{\theta,n} + \beta_{\theta,n}\sum_{d=1}^{D}\omega(\kappa_{\theta,n})x_{t-d/D}, \tag{2.5}$$

where $\delta_{\theta,n} = (\alpha_{\theta,n}, \beta_{\theta,n}, \kappa_{\theta,n})$ are the unknown parameters that need to be estimated. The vector $Z_{\theta,t-1}$ takes the form of $Z_{t-1}(\kappa_{\theta,n})$, and consists of daily returns with a lag of d days. It is an affine function of state variables.

More specifically, our approach is to use daily return data in the forecast of weekly return quantiles. The corresponding functional forms are

- **Symmetric absolute value**

  $q_t(\beta;\theta) = \beta_1 + \beta_2 \sum_{d=1}^{5} \omega(\kappa_\theta)|r_{t-d/5}|,$

- **Asymmetric slope**

  $q_t(\beta;\theta) = \beta_1 + \beta_2 \sum_{d=1}^{5} \omega(\kappa_{1,\theta})r^+_{t-d/5} + \beta_3 \sum_{d=1}^{5} \omega(\kappa_{2,\theta})r^-_{t-d/5},$

  where $r^+ = \max(r,0)$, $r^- = -\min(r,0)$.

- **Indirect GARCH**

  $q_t(\beta;\theta) = (\beta_1 + \beta_2 \sum_{d=1}^{5} \omega(\kappa_\theta)r^2_{t-d/5})^{1/2},$

- **Adaptive**

  $q_t(\beta;\theta) = \beta_1 + \beta_2\{[1+\exp(G(\sum_{d=1}^{5} \omega(\kappa_\theta)|r_{t-d/5}| - \beta_1))]^{-1} - \theta\},$

  where G is a finite, positive constant.

In this model, information from the higher frequency data is taken into consideration. Another piece of information that merits some attention is the autoregressive component from the CAViaR model in the previous section. It is reasonable that higher return quantiles should be followed by high return quantiles in the next time period, and vice versa. This also represents a component in the HYBRID structure model that we would like to introduce in the next section.

2.3 HYBRID Model

2.3.1 Multi-Period Returns and the HYBRID structure

We have already stated that the forecasting problem we address involves multiple time periods. We have also alluded to the existence of both weekly and daily observations. We detail the assumptions made about the return series in this section.

Since returns tend to have time-varying conditional second moments, we state the following condition

$$r_{t,n} = \mu + \sigma_{t,n}\epsilon_{t,n}. \tag{2.6}$$

The notion of HYBRID models is put forward by Chen, Ghysels, and Wang (2015). The advantage of this class of models is that they can consist of data sampled at any frequency. In the context of a generic HYBRID-GARCH model

$$V_{t+1|t} = \alpha + \beta V_{t|t-1} + \gamma H_t, \tag{2.7}$$

where $V_{t+1|t}$ is the conditional volatility. Depicted from a weekly basis, H_t can assume the form of a simple weekly squared return, a weighted sum of five daily squared returns, or a more convoluted structure.

2.3.2 Model Specification

As a natural extension to the two models discussed in the previous sections, we introduce a new model with a mixed frequency term and impose a HYBRID structure on the return series.

We would like to acknowledge the autoregressive characteristic of the return quantiles. This follows the literature on dynamic quantiles (Gourieroux and Jasiak (2008), Koenker and Xiao (2009)), and can be shown in our notation through the state variables. We will elaborate on the assumptions regarding the return series in another section.

We present the model as follows

- **Symmetric absolute value**

  $$q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 \sum_{d=1}^{5} \omega(\kappa_\theta)|r_{t-d/5}|,$$

- **Asymmetric slope**

  $$q_t(\beta;\theta) = \beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 \sum_{d=1}^{5} \omega(\kappa_{1,\theta})r^{+}_{t-d/5} + \beta_4 \sum_{d=1}^{5} \omega(\kappa_{2,\theta})r^{-}_{t-d/5},$$

- **Indirect GARCH**

  $$q_t(\beta;\theta) = (\beta_1 + \beta_2 q_{t-1}(\beta;\theta) + \beta_3 \sum_{d=1}^{5} \omega(\kappa_\theta)r^{2}_{t-d/5})^{1/2},$$

- **Adaptive**

  $$q_t(\beta;\theta) = q_{t-1}(\beta;\theta) + \beta_1\{[1+\exp(G(\sum_{d=1}^{5} \omega(\kappa_\theta)|r_{t-d/5}| - q_{t-1}(\beta;\theta)))]^{-1} - \theta\},$$

  where G is a finite, positive constant.

As previously stated, $q_t(\beta;\theta)$ and $q_{t-1}(\beta;\theta)$ represent current weekly return quantile and the return quantile from the previous week. We choose a time horizon with fractions for the weighted aggregation of past returns. This conveys the notion of utilizing all the daily return information available to us. Compared to the CAViaR model, we will be