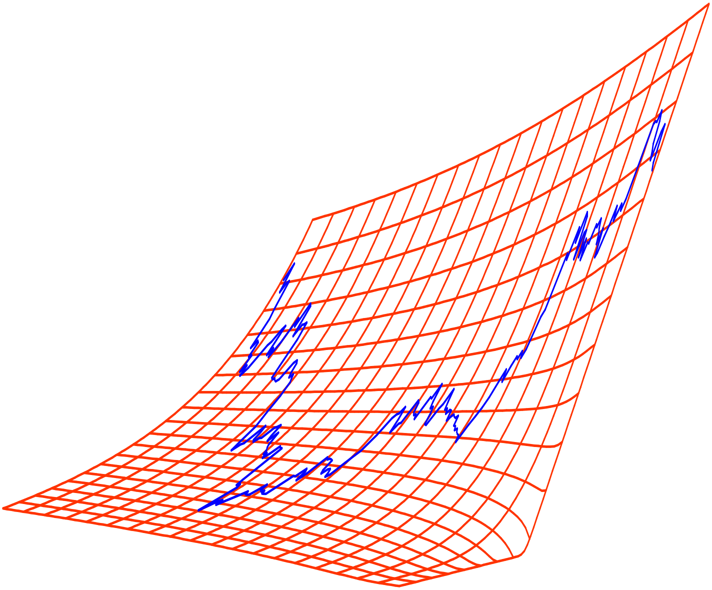
Nicolas Privault

Notes on Stochastic Finance



This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

Preface

This text is an introduction to pricing and hedging in discrete and continuous time financial models without friction (*i.e.* without transaction costs), with an emphasis on the complementarity between analytical and probabilistic methods. Its contents are mostly mathematical, and also aim at making the reader aware of both the power and limitations of mathematical models in finance, by taking into account their conditions of applicability. The book covers a wide range of classical topics including Black-Scholes pricing, exotic and american options, term structure modeling and change of numéraire, as well as models with jumps. It is targeted at the advanced undergraduate and graduate level in applied mathematics, financial engineering, and economics. The point of view adopted is that of mainstream mathematical finance in which the computation of fair prices is based on the absence of arbitrage hypothesis, therefore excluding riskless profit based on arbitrage opportunities and basic (buying low/selling high) trading. Similarly, this document is not concerned with any "prediction" of stock price behaviors that belong other domains such as technical analysis, which should not be confused with the statistical modeling of asset prices. The text also includes 104 figures and simulations, along with about 20 examples based on actual market data.

The descriptions of the asset model, self-financing portfolios, arbitrage and market completeness, are first given in Chapter 1 in a simple two time-step setting. These notions are then reformulated in discrete time in Chapter 2. Here, the impossibility to access future information is formulated using the notion of adapted processes, which will play a central role in the construction of stochastic calculus in continuous time.

In order to trade efficiently it would be useful to have a formula to estimate the "fair price" of a given risky asset, helping for example to determine whether the asset is undervalued or overvalued at a given time. Although such a formula is not available, we can instead derive formulas for the pricing of options that can act as insurance contracts to protect their holders against adverse changes in the prices of risky assets. The pricing and hedging of options in discrete time, particularly in the fundamental example of the

Cox-Ross-Rubinstein model, are considered in Chapter 3, with a description of the passage from discrete to continuous time that prepares the transition to the subsequent chapters.

A simplified presentation of Brownian motion, stochastic integrals and the associated Itô formula, is given in Chapter 4. The Black-Scholes model is presented from the angle of partial differential equation (PDE) methods in Chapter 5, with the derivation of the Black-Scholes formula by transforming the Black-Scholes PDE into the standard heat equation wich is then solved by a heat kernel argument. The martingale approach to pricing and hedging is then presented in Chapter 6, and complements the PDE approach of Chapter 5 by recovering the Black-Scholes formula via a probabilistic argument. An introduction to volatility estimation is given in Chapter 7, including historical, local, and implied volatilities. This chapter also contains a comparison of the prices obtained by the Black-Scholes formula with option price market data.

Exotic options such as barrier, lookback, and Asian options in continuous asset models are treated in Chapters 8, 9 and 10 respectively. Optimal stopping and exercise, with application to the pricing of American options, are considered in Chapter 11. The construction of forward measures by change of numéraire is given in Chapter 12 and is applied to the pricing of interest rate derivatives in Chapter 14, after an introduction to the modeling of forward rates in Chapter 13, based on material from [84].

Stochastic calculus with jumps is dealt with in Chapter 15 and is restricted to compound Poisson processes which only have a finite number of jumps on any bounded interval. Those processes are used for option pricing and hedging in jump models in Chapter 16, in which we mostly focus on risk minimizing strategies as markets with jumps are generally incomplete. Chapter 17 contains an elementary introduction to finite difference methods for the numerical solution of PDEs and stochastic differential equations, dealing with the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milshtein schemes for SDEs. The text is completed with an appendix containing the needed probabilistic background.

The material in this book has been used for teaching in the Masters of Science in Financial Engineering at City University of Hong Kong and at the Nanyang Technological University in Singapore. The author thanks Ju-Yi Yen (University of Cincinnati) for several corrections and improvements.

The cover graph represents the time evolution of the HSBC stock price from January to September 2009, plotted on the price surface of a European call option on that asset, expiring on October 05, 2009, cf. § 5.5.

This pdf file contains external links, and animated figures and embedded videos in Chapters 8, 9, 10, 11, 13 and 15, that may require using Acrobat Reader for viewing on the complete pdf file. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on the problem number sends the reader to the corresponding solution, however this feature should not be misused.

Contents

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

List of Figures

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

* Animated figures (work in Acrobat reader).

Introduction

Modern mathematical finance and quantitative analysis require a strong background in fields such as stochastic calculus, optimization, partial differential equations (PDEs) and numerical methods, or even infinite dimensional analysis. In addition, the emergence of new complex financial instruments on the markets makes it necessary to rely on increasingly sophisticated mathematical tools. Not all readers of this book will eventually work in quantitative financial analysis, nevertheless they may have to interact with quantitative analysts, and becoming familiar with the tools they employ be an advantage. In addition, despite the availability of ready made financial calculators it still makes sense to be able oneself to understand, design and implement such financial algorithms. This can be particularly useful under different types of conditions, including an eventual lack of trust in financial indicators, possible unreliability of expert advice such as buy/sell recommendations, or other factors such as market manipulation. To some extent we would like to have some form of control on the future behaviour of random (risky) assets, however, since knowledge of the future is not possible, the time evolution of the prices of risky assets will be modelled by random variables and stochastic processes.

Historical Sketch

We start with a description of some of the main steps, ideas and individuals that played an important role in the development of the field over the last century.

Robert Brown, botanist, 1827

Brown observed the movement of pollen particles as described in his paper "A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies."

Phil. Mag. 4, 161-173, 1828.

Philosophical Magazine, first published in 1798, is a journal that "publishes articles in the field of condensed matter describing original results, theories and concepts relating to the structure and properties of crystalline materials, ceramics, polymers, glasses, amorphous films, composites and soft matter."

Louis Bachelier, mathematician, PhD 1900

Bachelier used Brownian motion for the modelling of stock prices in his PhD thesis "Théorie de la spéculation", Annales Scientifiques de l'Ecole Normale Supérieure 3 (17): 21-86, 1900.

Albert Einstein, physicist

Einstein received his 1921 Nobel Prize in part for investigations on the theory of Brownian motion: "... in 1905 Einstein founded a kinetic theory to account for this movement", presentation speech by S. Arrhenius, Chairman of the Nobel Committee, Dec. 10, 1922.

Albert Einstein, "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen", Annalen der Physik 17 (1905) 223.

Norbert Wiener, mathematician, founder of cybernetics

Wiener is credited, among other fundamental contributions, for the mathematical foundation of Brownian motion, published in 1923. In particular he constructed the Wiener space and Wiener measure on $\mathcal{C}_0([0,1])$ (the space of continuous functions from $[0,1]$ to $\mathbb{R}$ vanishing at 0).

Norbert Wiener, "Differential space", Journal of Mathematics and Physics of the Massachusetts Institute of Technology, 2, 131-174, 1923.

Kiyoshi Itô (伊藤清), mathematician, Gauss prize 2006

Itô constructed the Itô integral with respect to Brownian motion, cf. Itô, Kiyoshi, Stochastic integral. Proc. Imp. Acad. Tokyo 20, (1944). 519-524. He also constructed the stochastic calculus with respect to Brownian motion, which laid the foundation for the development of calculus for random processes, cf. Itô, Kiyoshi, "On stochastic differential equations", Mem. Amer. Math. Soc. (1951).

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

"Renowned math wiz Itô, 93, dies." (The Japan Times, Saturday, Nov. 15, 2008).

Kiyoshi Itô, an internationally renowned mathematician and professor emeritus at Kyoto University died Monday of respiratory failure at a Kyoto hospital, the university said Friday. He was 93. Itô was once dubbed "the most famous Japanese in Wall Street" thanks to his contribution to the founding of financial derivatives theory. He is known for his work on stochastic differential equations and the "Itô Formula", which laid the foundation for the Black-Scholes model, a key tool for financial engineering. His theory is also widely used in fields like physics and biology.

Paul Samuelson, economist, Nobel Prize 1970

In 1965, Samuelson rediscovered Bachelier's ideas and proposed geometric Brownian motion as a model for stock prices. In an interview he stated "In the early 1950s I was able to locate by chance this unknown [Bachelier's] book, rotting in the library of the University of Paris, and when I opened it up it was as if a whole new world was laid out before me." We refer to "Rational theory of warrant pricing" by Paul Samuelson, Industrial Management Review, p. 13-32, 1965.

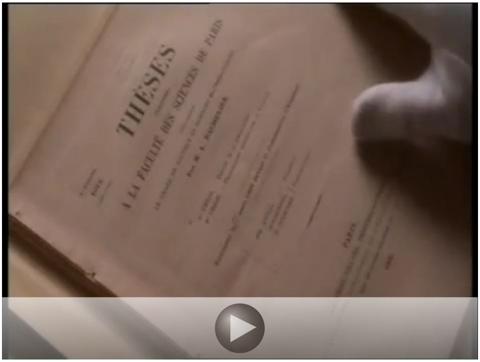


Fig. 0.1: [15] "As if a whole new world was laid out before me."*

In recognition of Bachelier's contribution, the Bachelier Finance Society was started in 1996 and now holds the World Bachelier Finance Congress every

* Click on the figure to play the video (works in Acrobat reader on the entire pdf file).

2 years.

Robert Merton, Myron Scholes, economists

Robert Merton and Myron Scholes shared the 1997 Nobel Prize in economics: "In collaboration with Fisher Black, developed a pioneering formula for the valuation of stock options ... paved the way for economic valuations in many areas ... generated new types of financial instruments and facilitated more efficient risk management in society."*

Black, Fischer; Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities". Journal of Political Economy 81 (3): 637-654.

The development of options pricing tools contributed greatly to the expansion of option markets and led to development several ventures such as the "Long Term Capital Management" (LTCM), founded in 1994. The fund yielded annualized returns of over 40% in its first years, but registered lost US$ 4.6 billion in less than four months in 1998, which resulted into its closure in early 2000.

Oldrich Vasiček, economist, 1977

Interest rates behave differently from stock prices, notably due to the phenomenon of mean reversion, and for this reason they are difficult to model using geometric Brownian motion. Vasiček was the first to suggest a mean-reverting model for stochastic interest rates, based on the Ornstein-Uhlenbeck process, in "An equilibrium characterisation of the term structure", Journal of Financial Economics 5: 177-188.

David Heath, Robert Jarrow, A. Morton

These authors proposed in 1987 a general framework to model the evolution of (forward) interest rates, known as the HJM model, see their joint paper "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", Econometrica, (January 1992), Vol. 60, No. 1, pp 77-105.

Alan Brace, Dariusz Gatarek, Marek Musiela (BGM)

The BGM model is actually based on geometric Brownian motion, and it is specially useful for the pricing of interest rate derivatives such as caps and

* This has to be put in relation with the modern development of risk societies; "societies increasingly preoccupied with the future (and also with safety), which generates the notion of risk".

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

swaptions on the LIBOR market, see "The Market Model of Interest Rate Dynamics". Mathematical Finance Vol. 7, page 127. Blackwell 1997, by Alan Brace, Dariusz Gatarek, Marek Musiela.

European Call and Put Options

We close this introduction with a description of European call and put options, which are at the basis of risk management. As mentioned above, an important concern for the buyer of a stock at time t is whether its price S_T can fall down at some future date T. The buyer of the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t. This contract is called a put option with strike price K and exercise date T.

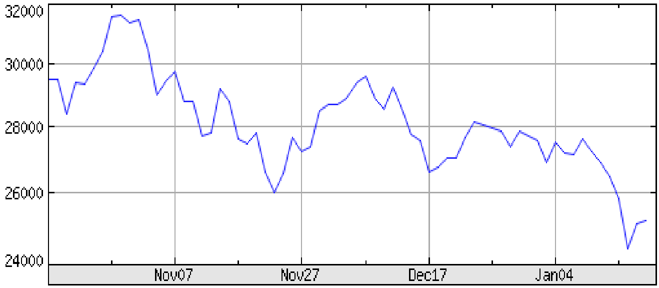


Fig. 0.2: Graph of the Hang Seng index - holding a put option might be useful here.

Definition 0.1. *A (European)* put *option is a contract that gives its holder the right (but not the obligation) to* sell *a quantity of assets at a predefined price K called the strike price (or exercise price) and at a predefined date T called the maturity.*

In case the price S_T falls down below the level K, exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option and sell the asset at the market price S_T. In turn, the issuer of the option will register a loss also equal to $K - S_T$ (in the absence of transaction costs and other fees).

If S_T is above K then the holder of the option will not exercise the option as he may choose to sell at the price S_T. In this case the profit derived from the option is 0.

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

In general, the payoff of a (so called European) put option will be of the form

$$\phi(S_T) = (K - S_T)^+ = \begin{cases} K - S_T, & S_T \leq K, \\ 0, & S_T \geq K. \end{cases}$$

Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 0.3.

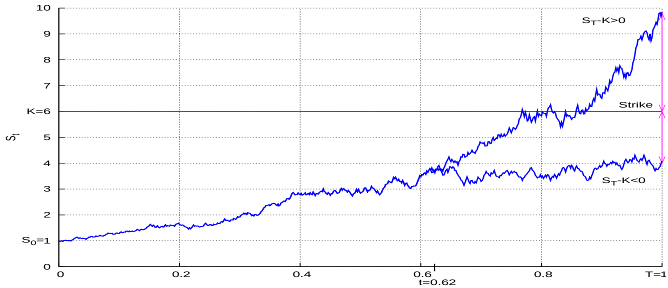


Fig. 0.3: Sample price processes simulated by a geometric Brownian motion.

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t.

Here, in the event that S_T goes above K, the buyer of the option will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Definition 0.2. *A (European)* call *option is a contract that gives its holder the right (but not the obligation) to* buy *a quantity of assets at a predefined price K called the strike and at a predefined date T called the maturity.*

In general, a (European) call option is an option with payoff function

$$\phi(S_T) = (S_T - K)^+ = \begin{cases} S_T - K, \; S_T \geq K, \\ 0, \qquad\qquad S_T \leq K. \end{cases}$$

In market practice, options are often divided into a certain number n of *warrants*, the (possibly fractional) quantity n being called the *entitlement ratio*.

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

In order for an option contract to be fair, the buyer of the option should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, which is known as option *pricing*.

The second important issue is that of *hedging*, *i.e.* how to manage a given portfolio in such a way that it contains the required random payoff $(K-S_T)^+$ (for a put option) or $(S_T-K)^+$ (for a call option) at the maturity date T.

The next figure illustrates a sharp increase and sharp drop in asset price, making it valuable to hold a call option during the first half of the graph, whereas holding a put option would be recommended during the second half.

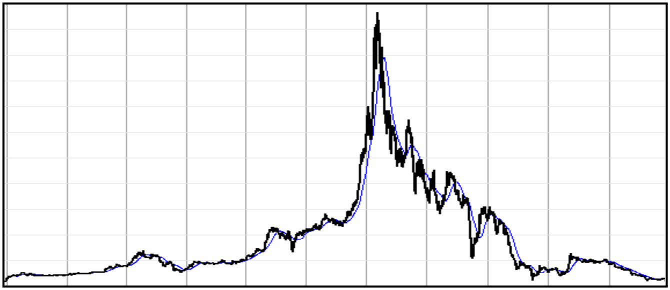


Fig. 0.4: "Infogrames" stock price curve.

An illustration - pricing and hedging in a binary model

We close this introduction with a simplified illustration of the pricing and hedging technique in a binary model. Consider a risky stock price S valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values

$$S_1 = \begin{cases} \$5 \\ \\ \$2 \end{cases}$$

at time $t = 1$. In addition, consider an option that yields a payoff P whose values are contingent to the data of S:

$$P = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases}$$

At time $t = 0$ we choose to invest α units in the risky asset S, while keeping $\$\beta$ on our bank account, meaning that we invest a total amount

$$\alpha S_0 + \$\beta \qquad \text{at } t = 0.$$

The following issues can be addressed:

a) Hedging: how to choose the portfolio allocation $\{\alpha, \$\beta\}$ so that the value

$$\alpha S_1 + \$\beta$$

of the portfolio matches the future payoff P at time $t = 1$?

b) Pricing: how to determine the amount $\alpha S_0 + \$\beta$ to be invested in such a portfolio at time $t = 0$?

Hedging means that at time $t = 1$ the portfolio value matches the future payoff P, *i.e.*

$$\alpha S_1 + \$\beta = P.$$

This condition can be rewritten as

$$P = \begin{cases} \$3 = \alpha \times \$5 + \$\beta & \text{if } S_1 = \$5, \\ \\ \$0 = \alpha \times \$2 + \$\beta & \text{if } S_1 = \$2, \end{cases}$$

i.e.

$$\begin{cases} 5\alpha + \beta = 3 \\ \\ 2\alpha + \beta = 0, \end{cases} \qquad \text{which yields} \qquad \begin{cases} \alpha = 1 \\ \\ \$\beta = -\$2. \end{cases}$$

In other words, we buy 1 unit of the stock S at the price $S_0 = \$4$, and we borrow \$2 from the bank. The price of the option contract is given by the portfolio value

$$\alpha S_0 + \$\beta = 1 \times \$4 - \$2 = \$2.$$

at time $t = 0$.

Conclusion: in order to deliver the random payoff $P = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases}$

at time $t = 1$, one has to:

1. receive \$2 (the option price) at time $t = 0$,

2. borrow $-\$\beta = \$2$ from the bank,

3. invest those $\$2 + \$2 = \$4$ into the purchase of $\alpha = 1$ unit of stock valued at $S_0 = \$4$ at time $t = 0$,

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html

4. wait until time $t = 1$ to find that the portfolio value evolved into

$$P = \begin{cases} \alpha \times \$5 + \$\beta = 1 \times \$5 - \$2 = \$3 & \text{if } S_1 = \$5, \\ \\ \alpha \times \$2 + \$\beta = 1 \times \$2 - \$2 = 0 & \text{if } S_1 = \$2. \end{cases}$$

so that the option contract is fulfilled whatever the evolution of S.

We note that the initial amount of \$2 can be turned to $P = \$3$ (%50 profit) ... or into $P = \$0$ (total ruin).

Thinking further

1) The expected gain of our portfolio is

$$\begin{aligned} \mathbb{E}[P] &= \$3 \times \mathbb{P}(P = \$3) + \$0 \times \mathbb{P}(P = \$0) \\ &= \$3 \times \mathbb{P}(S_1 = \$5) \\ &= \$3 \times \mathbb{P}(S_1 = \$5). \end{aligned}$$

In absence of arbitrage opportunities (“fair market”) this expected gain $\mathbb{E}[P]$ should equal the initial amount \$2 invested in the option. In that case we should have

$$\begin{cases} \mathbb{E}[P] = \$3 \times \mathbb{P}(S_1 = \$5) = \$2 \\ \\ \mathbb{P}(S_1 = \$5) + \mathbb{P}(S_1 = \$2) = 1. \end{cases}$$

from which we can *infer* the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \dfrac{2}{3} \\ \\ \mathbb{P}(S_1 = \$2) = \dfrac{1}{3}. \end{cases} \tag{0.1}$$

We see that the stock S has twice more chances to go up than to go down in a “fair” market.

2) Based on the probabilities (0.1) we can also compute the expected value $\mathbb{E}[S_1]$ of the stock at time $t = 1$. We find

$$\begin{aligned} \mathbb{E}[S_1] &= \$5 \times \mathbb{P}(S_1 = \$5) + \$2 \times \mathbb{P}(S_1 = \$2) \\ &= \$5 \times \frac{2}{3} + \$2 \times \frac{1}{3} \\ &= \$4 \\ &= S_0. \end{aligned}$$

Here this means that, on average, no profit can be made from an investment on the risky stock. In a more realistic model we can assume that the riskles bank account yields an interest rate equal to r, in which case the above analysis is modified by letting \$$\beta$ become \$$(1+r)\beta$ at time $t = 1$, nevertheless the main conclusions remain unchanged.

This version: June 23, 2016
http://www.ntu.edu.sg/home/nprivault/indext.html