Endogenizing fuel economy and quality

Note: in order to endogenize fuel economy and quality into the West, Hoekstra, Meer and Puller (WHMP) model, this generates maximizer miles traveled (m), fuel consumption rate (g), and car quality (x) that are all functions of the car price function p(x, g) – a smooth function of fuel consumption rate and car quality. I diverge from the WHMP derivation of elasticity by taking a derivative with respect to a changing car price function, instead of w.r.t. the fuel economy (or fuel consumption rate). This change seems necessary to me because fuel economy has now been endogenized, and I no longer have a maxmizer miles traveled m as a function of fuel economy, and cannot take a derivative with respect to fuel economy. I do not understand how to avoid this issue, but have written down all that I understand of the problem below.

We can begin by assuming a utility-maximizing consumer i with consumer demographics W_i is in the market to purchase a new car. In the purchase, they simultaneously choose their predicted total miles to drive m, the quality of the vehicle to purchase x (size, comfort, etc.), the gas consumption rate g (gallons consumed per mile), and all other non-vehicle consumption y (the numeraire good, having a price normalized to 1), subject to their budget constraint. This car buyer faces car prices that are a smooth function of quality and fuel consumption $p(x, g; W_i)$ that increase as the quality of the car increases ($\partial p/\partial x > 0$) and decrease in the fuel consumption rate ($\partial p/\partial g < 0$); they also face gas prices $p_{\text{gas}}(W_i)$ (assuming W_i contains the consumer's location). Then, with budget I_i, the maximization problem for the vehicle-buyer is

$$\max_{m,x,g,y} U = \phi(m; W_i)\ \theta(x; W_i) + y \qquad \text{s.t.} \qquad I_i = y + p_{\text{gas}}(W_i)\ m\ g + p(x, g; W_i) \tag{1}$$

where $\phi(m; W_i)$ is an increasing function describing the consumer's change in utility from miles driven and $\theta(x; W_i)$ is an increasing function describing the change in utility from the car's quality; both of these internal utility functions are shared among consumers with the same demographics W_i. Suppressing demographics, the resulting first order conditions are then

$$\phi(m)\ \theta'(x) = p_x(x, g) \tag{2}$$

$$\phi'(m)\ \theta(x) = p_{\text{gas}}\ g \tag{3}$$

$$p_{\text{gas}}\ m = -p_g(x, g) \tag{4}$$

where $p_k(x, g) = \partial p/\partial k$. Condition (2) says that the consumer increases the quality of the car to purchase until the marginal utility of quality equals the marginal cost of quality (increased sticker price). Since marginal utility of driving miles is assumed to be positive, condition (3) can be interpreted as the consumer increasing the total miles to drive until the marginal utility of driving another mile equals the marginal cost of driving another mile. Condition (4) shows the consumer decreasing the fuel consumption rate of the car (making it more efficient) until the marginal total gas cost of fuel consumption equals the marginal cost of a more efficient car (from the increased sticker price).

Assume the price of gas is exogenous to the consumer's choice and fixed over time.[1] Then the first order conditions (2), (3), and (4) can be used to solve to consumer i's demand for miles-to-drive, car quality,

[1] Previous research has provided good reason to think that consumers, on average, make decisions as if future gas prices will be fixed at current prices (Anderson et al. 2013). Price of gas here is still a function of the consumer's location.

and fuel consumption:

$$\begin{aligned} m_i^* &= m(p_{\text{gas}}; p(\cdot,\cdot), W_i) \\ x_i^* &= x(p_{\text{gas}}; p(\cdot,\cdot), W_i) \\ g_i^* &= g(p_{\text{gas}}; p(\cdot,\cdot), W_i) \end{aligned} \tag{5}$$

As will be explained in section 3, the Cash for Clunkers (CfC) program served as a quasi-random source of variation in the car prices that a new car buyer will face.[2] CfC offered rebates to car owners with clunkers that matched eligibility criteria. These rebates are a function of the gas consumption rate of the car buyer's old car (call this g_0) and the gas consumption rate of the new car (g). So the new prices that the car buyers face can be written as the old price minus the rebate:

$$\tilde{p}(x, g; W_i) = p(x, g_0; W_i) - R(g_0, g) \tag{6}$$

where $R(g_0, g)$ is the rebate offered to a consumer depending on the difference between the old and new gas consumption rates. Under CfC, (6) changes the demands from (5) to[3]

$$\begin{aligned} m_i^* &= m(p_{\text{gas}}; \tilde{p}(\cdot,\cdot), W_i) \\ x_i^* &= x(p_{\text{gas}}; \tilde{p}(\cdot,\cdot), W_i) \\ g_i^* &= g(p_{\text{gas}}; \tilde{p}(\cdot,\cdot), W_i) \end{aligned} \tag{7}$$

Given this setup, we can find how the total amount of gasoline consumption changes when there is a (exogenous) change in car prices. A household's total gasoline consumption is

$$gal_i = m(p_{\text{gas}}; \tilde{p}(\cdot,\cdot), W_i)\ g(p_{\text{gas}}; \tilde{p}(\cdot,\cdot), W_i)$$

Taking logs and differentiating with respect to car price yields the elasticity of gasoline consumption with respect to car prices (ε_{gal-p}):

$$\frac{d\log(gal_i)}{d\log(p)} = \frac{d\log(m)}{d\log(p)} + \frac{d\log(g)}{d\log(p)}$$

This elasticity tells us the percentage reduction in gallons of gasoline consumed that is achieved with a given percent shift in the car price function.

[2] Section 3 would need to be rewritten slightly to change the emphasis from exogenous variation in fuel economy to exogenous variation in prices.

[3] This added rebate is actually discontinuous and increasing in g, which creates a new price function $\tilde{p}$ that is no longer a smooth function of the fuel consumption rate. This violates convexity assumptions made during derivation of the maximizers (m_i^*, x_i^*, g_i^*) and may result in some corner solutions where people choose to purchase cars right above the eligibility threshold but are not matching their marginal utility from the gas consumption rate to the marginal price increase.

The problem left to solve is how to take a derivative of a piecewise-changing function. I have attempted to derive $\frac{dm}{dp}$ and $\frac{dg}{dp}$, but both allude me because the first order conditions already contain the derivatives of p and I cannot find a way to massage dp from the total derivatives. Here is what I have:

Taking the total derivatives of the first order conditions (2), (3), (4) gives

$$\begin{aligned}
d\left[\phi(m)\ \theta'(x) = p_x(x,g)\right] &\implies \theta' d\phi + \phi d\theta' = dp_x(x,g) \\
&\implies \theta'\phi' dm + \phi\theta'' dx = p_{xx}dx + p_{xg}dg \\
d\left[\phi'(m)\ \theta(x) = p_{\text{gas}}\ g\right] &\implies \theta d\phi' + \phi' d\theta = p_{\text{gas}}dg \\
&\implies \theta\phi'' dm + \phi'\theta' dx = p_{\text{gas}}dg \\
d\left[p_{\text{gas}}\ m = -p_g(x,g)\right] &\implies -p_{\text{gas}}dm = dp_g(x,g) \\
&\implies -p_{\text{gas}}dm = p_{xg}dx + p_{gg}dg
\end{aligned}$$

We also note that the total derivative of p is

$$dp = p_x dx + p_g dg$$

We can substitute p_x and p_g in from the first order conditions, which are assumed to hold during a utility-maximizing purchase, to get

$$dp = \phi\theta' dx - p_{\text{gas}} m dg$$

However, I do not see a way to get either of the above forms of dp from the combination of the total derivatives at the top of the page.