Problem 1

Co-benefits and pre-existing policies

You are analyzing a carbon pricing policy. The policy will generate cobenefits in the form of reduced local air pollution. (I.e., a carbon tax will reduce the use of coal in the electricity market, which will reduce particulate matter.)

You know that local air pollutants are also subject to regulation. (I.e., the Clean Air Act limits ambient levels of particulate matter.) But, you think the regulation of the local air pollutants is imperfect and likely does not yield the optimal amount of abatement. The question you wish to investigate is how the cobenefits should figure into the optimal price on carbon, taking into account that there are some policies in place.

Write down a mathematical model that allows you to characterize the optimal carbon tax that takes into account cobenefits and the existence of imperfect policies regulating air pollution. Solve the model and describe how the results answer the question.

There are myriad ways of complicating this problem and a variety of interesting related questions. Your goal is to shed light on the issue in the bold text. Make your modeling choices accordingly.

This question is motivated by one of the critiques that I suspect hampered Knittel and Sandler. In their paper, they calculated the optimal second-best gasoline tax as if it were an instrument acting on local air pollution. There are already regulations impacting the local air pollution per mile of vehicles. Their discussion of welfare did not take those policies into account, and this made it harder to judge the welfare conclusions.

Consider a model of firms with two pollutants as follows[1]:

- J identical firms produce a homogenous good in a perfectly competitive industry, and produce two pollutants (1 and 2) as byproducts.
- Firm j produces the good at level x_j with production costs $C(x_j)$, where $C(\cdot)$ is twice continuously differentiable and $C' > 0$ and $C'' \geq 0$
- In producing x_j, firm j also produces pollutants 1 and 2 as byproducts in amounts $e_{1j}(x_j)$ and $e_{2j}(x_j)$, where pollutant 1 is a greenhouse gas that will be targeted by a carbon tax, and pollutant 2 is particulate matter that will be targeted by an emissions limit.
- Emissions depend directly and linearly on output such that $e_{1j}(x_j) = \delta_1 x_j$ and $e_{2j}(x_j) = \delta_2 x_j$, where δ_1 and δ_2 are the emission intensities of production of the two types of pollutants.
- Profits for firm j are then $\pi_j = p \cdot x_j - C(x_j)$, where p is the output price (taken by the firms as given).
- Assume that the representative consumer's utility is quasilinear of the form $U = u(X) + z - D_1(E_1) - D_2(E_2)$, where $u(X)$ is the sub-utility from consuming $X = \sum_j x_j$ total

[1] This model brorrows heavily from D. Phaneuf and T. Requate's textbook, "*A Course in Environmental Economics.*"

units of the good, z is all other spending, $E_k = \sum_j e_{kj}$ is aggregate emissions of type k, and $D_k(\cdot)$ is the money-metric damages from aggregate emissions of type k.

- The representative consumer maximizes their utility subject to the budget constraint $Y = px + z$, where Y is aggregate income.

The representative consumer maximizes utility (subject to their budget constrain) by choosing amount X to consume of the polluting good. We can assume that the consumer acts as if their choice of X has no impact on the aggregate level of emissions for either pollutant. Therefore, utility maximization results in the following first order condition:

$$p = u'(X)$$

This defines the inverse demand function

$$p(X) = u'(X)$$

Then total consumer benefits from consuming X is

$$\int_0^X p(w)dw = u(X)$$

The aggregate production X and aggregate emissions E_1 and E_2 are

$$X = \sum x_j; \quad E_1 = \sum e_{1j} = \sum \delta_1 x_j; \quad E_2 = \sum e_{2j} = \sum \delta_2 x_j$$

So social welfare from production of the good is

$$\begin{aligned} W(x_1, ..., x_j) &= \int_0^X p(w)dw - \sum C(x_j) - D_1(E_1) - D_2(E_2) \\ &= \int_0^{\sum x_j} p(w)dw - \sum C(x_j) - D_1\left(\sum \delta_1 x_j\right) - D_2\left(\sum \delta_2 x_j\right) \end{aligned}$$

The socially optimal allocation of production

Given the social welfare function above, the social planner with control over allocation/production of x would choose to maximize W. The corresponding first order conditions for the optimal $\{x_j\}$ are:

$$\left.\frac{\partial W}{\partial x_j}\right|_{x_j^*} = p\left(\sum x_j^*\right) - C'(x_j^*) - D_1'\left(\sum \delta_1 x_j^*\right)\delta_1 - D_2'\left(\sum \delta_2 x_j^*\right)\delta_2 = 0$$

$$\implies p\left(\sum x_j^*\right) = C'(x_j^*) + \delta_1 D_1'\left(\sum \delta_1 x_j^*\right) + \delta_2 D_2'\left(\sum \delta_2 x_j^*\right) \quad \forall j \tag{1}$$

Intuitively, this means that the social planner would choose all J (identical) firms to produce in equal amounts until the marginal social benefit equals the marginal social costs, where the marginal social benefit is given by the price consumers are willing to pay for one more unit of the good, and the total marginal social costs come from marginal costs of production and marginal damages of each of the pollutants being produced by one more unit of the good.

Because we are now including the second pollutant, the price determined by the optimal allocation is now higher, and the optimal total production will be lower.

The optimal carbon tax absent any constraints on PM emissions

The ultimate goal is to find the optimal emissions tax for e_1, imposed on firms, given some constraint on e_2 emissions. To compare, let's first find the optimal (pigouvian) tax on e_1 given no limitations on either e_1 or e_2.

The regulator wishes to maximize social welfare, given that a per-unit tax t on e_1 is the only policy tool available to the regulator. Under the tax t on e_1, the perfectly competitive firms will maximize profit, taking prices as given, according to

$$\begin{aligned}\max_{x_j} \pi_j &= \max_{x_j}\{p \cdot x_j - C(x_j) - te_{1j}\} \\ &= \max_{x_j}\{p \cdot x_j - C(x_j) - t\delta_1 x_j\}\end{aligned}$$

This leads to the first order conditions for profit maximization under the tax

$$C'(x_j) = p - t\delta_1 \tag{2}$$

If the regulator new where the new equilibrium would be and were to set a naive tax equal to the sum of marginal damages from both emissions at that equilibrium ($t = D_1'(E_1)+D_2'(E_2) = D_1'(\delta_1 X)+D_2'(\delta_2 X)$), then at equilibrium, firm's would choose x_j according to

$$\begin{aligned}C'(x_j) &= p(X) - \delta_1[D_1'(\delta_1 X) + D_2'(\delta_2 X)] \\ &= p(X) - \delta_1 D_1'(\delta_1 X) + \delta_1 D_2'(\delta_2 X)\end{aligned}$$

Compare this to the optimal allocation in Eq. (1). Depending on the relative per-unit amounts of emissions produced (δ_1 and δ_2), the last two terms above may be more or less than the last two terms in (1). To correct for this, we can solve for the correct tax t by setting $t\delta_1$ from (2) equal to the last two terms of (1). This gives us an optimal tax on emissions of e_1, accounting for the co-benefits of reducing e_2, assuming *no constraint* on e_2:

$$t^{nc} = D_1'(E_1) + \frac{\delta_2}{\delta_1} D_2'(E_2) = D_1'(\delta_1 X) + \frac{\delta_2}{\delta_1} D_2'(\delta_2 X) \tag{3}$$

Assume type 1 emissions are carbon emissions (greenhouse gases) and type 2 are local particulate mater emissions. Depending on the relative values of δ_1 and δ_2 (how emissions-intensive production is in carbon emissions compared to particulate matter emissions), the optimal carbon tax may be more or less than the sum of marginal damages of emissions, but is certainly greater than the marginal damages of carbon emissions alone (assuming that type 2 emissions bring damages, not benefits).

The optimal carbon tax with a constraint on PM emissions

Simplifying the complex process of Clean Air Act emissions regulation, let's assume that the result of the CAA National Ambient Air Quality Standards is to provide a hard cap on type 2 emissions at the firm level. Each firm is restricted to produce no more than $\bar{e}_2$.

Firms now maximize profit subject to the constraint on type 2 emissions, which is equivalent to a constraint on output:

$$e_{2j} \leq \bar{e}_2 \quad \implies x_j \leq \frac{\bar{e}_2}{\delta_2} \equiv \bar{x}$$

We can split the possible results into two categories: either $\bar{x} > x_j^*$ where x_j^* is given by Eq. (1); or $\bar{x} \leq x_j^*$. In the first case, imposing the optimal tax t^{nc} calculated previously that ignores the existing constraint will produce equilibrium levels of production below the existing constraint. The optimal carbon tax internalizing the co-benefits of decreasing the second pollutant would effectively supersede the existing constraint in that the firms would choose to emit at or below the limit even without the constraint, given the tax.

In the second case, the constraint forces all firms to produce below the socially optimal amount x_j^*. We might think the optimal tax in this case would be negative (a subsidy to bring production back up to the socially optimal level). But because, in this simplified model, emissions of both pollutants are linearly tied to production, no amount of subsidy can increase production above $\bar{x}$. Therefore, the reduced production will fall short of demand (at the optimum) and prices will exceed marginal costs, producing profits for the fixed number of perfectly competitive firms. This could happen if there are fixed costs larger than the profits firm get, preventing any new firms from entering. There is no benefit to imposing a tax, except to capture some of the profits from industry, which would only serve as a wealth transfer.

The optimal tax under a constraint on e_2 would then be a piecewise function:

$$t^c = \begin{cases} D_1'\left(\sum \delta_1 x_j^*\right) + \frac{\delta_2}{\delta_1} D_2'\left(\sum \delta_2 x_j^*\right), & \text{if } \dfrac{\bar{e}_2}{\delta_2} > x_j^* \\ 0, & \text{if } \dfrac{\bar{e}_2}{\delta_2} \leq x_j^* \end{cases}$$

If the e_2 emissions threshold is so high that we observe firms producing below the threshold before imposing any tax, we can assume the observed unconstrained and untaxed equilibrium is above the optimal production level. This is simply because marginal damages are positive for the pollutants, thus any unconstrained and untaxed equilibrium must be above the socially optimal production. Thus, we should impose the non-zero tax above.

If we observe firms producing at the level of the constraint before the tax, then we can assume the unconstrained (and untaxed) level of production would be above the constraint. We can again assume that the optimal output is below the unconstrained output, but is it below or above the constraint?

This requires estimating the inverse demand, cost function, and marginal damage functions at least up to the point of the constraint. Once we have those functions estimated from $x_j = 0$ to $x_j = \bar{x}$, and given

δ_1 and δ_2, we can determine if the socially optimal level of production is at or below the constraint. If the $x_j^* < \bar{x}$, then we need to impose the associated tax. If $x_j^* \geq \bar{x}$, then nothing would be gained from the carbon tax and we should abandon all efforts.

This is informative because, given this model setup, we would not need to estimate the functions beyond the range of the existing constraint. If the existing constraint has been in place for a while, it may be hard to get reliable estimates of costs, demand, and damages because of how firms, consumers, and technology has changed.

In summary, we should impose the following tax based on observed production $\{\hat{x}_j\}$:

$$t^c = \begin{cases} D_1'\left(\sum \delta_1 x_j^*\right) + \frac{\delta_2}{\delta_1} D_2'\left(\sum \delta_2 x_j^*\right), & \text{if we observe } \hat{x}_j < \dfrac{\bar{e}_2}{\delta_2} \\ D_1'\left(\sum \delta_1 x_j^*\right) + \frac{\delta_2}{\delta_1} D_2'\left(\sum \delta_2 x_j^*\right), & \text{if we observe } \hat{x}_j = \dfrac{\bar{e}_2}{\delta_2} \text{ and then estimate } x_j^* < \dfrac{\bar{e}_2}{\delta_2} \\ 0, & \text{if we observe } \hat{x}_j = \dfrac{\bar{e}_2}{\delta_2} \text{ and then estimate } x_j^* \geq \dfrac{\bar{e}_2}{\delta_2} \end{cases}$$

This model is highly simplified and would likely have a non-zero carbon tax if the pollutant relationship to output were modeled more flexibly. Two possible modifications for future versions: (1) modeling emissions as an input that can be substituted away from for increased costs; or (2) emissions as a separate production function with abatement inputs.

Problem 2

Externalities with monopoly

In "External Diseconomies, Corrective Taxation, and Market Structure" James Buchanan critiqued Pigouvian taxation in the presence of market power, which he argued constituted the bulk of relevant congestion and pollution problems. He presents a graphical argument of a monopolist whose production produces a negative externality on a second competitive industry, summarized by this quote: "The monopolist simultaneously imposes two external diseconomies... He 'pollutes' and hence increases costs of firms in the damaged industry. Also, however, he holds down output and hence increases costs of his product to buyers. So long as the second diseconomy is more highly valued than the former [at the margin], any levy of a per unit tax on the monopolist's output will decrease total welfare." That is, he concluded that the use of any corrective tax would cause a net decrease in social welfare.

Also, just to capture the flavor of the discourse, the article begins thusly: "This note is presented as a contribution to the continuing dismantling of the Pigouvian tradition in applied economics, defined here as the emphasis on internalizing externalities through the imposition of corrective taxes and subsidies."

The optimal policy solution when there are two market failures is to have two corrective policies. If one fixes the competition failure, then the Pigouvian prescription will apply unaltered. (This point is made in the summary article by Cropper and Oates, as well as the Baumol and Oates text.) But, let us suppose that we must take the competition failure as given. The point of this exercise is to get you to consider how the Pigouvian tax is (or is not) altered by the existence of market power.

For ease, let us consider a monopolist who produces a good in quantity q. Production of q generates a constantly increasing externality, with net effect on welfare of ϕq. Write the inverse demand curve of consumers as $p(q)$. The production cost to the monopolist is $c(q)$. The planner's only available policy tool is a per unit of output tax, t.

Derive the optimal (second-best) tax rate t, and thoroughly interpret its economic meaning.

The following might be useful:

- The planner's optimization problem can be written as $\max_t \int_0^q p(q)dw - C(q) - \phi q$
- The monopolist's optimization problem can be written $\max_q \pi = p(q)q - C(q) - tq$

Please note that the analysis here for a monopolist need not extend directly to alternative models of imperfect competition. Such models are highly varied and are capable of delivering different results about the implications for externalities.

Also note that in this model, the only way to abate is to reduce quantity. The lessons are different when there is an abatement action that can be taken independent of quantity. You can see a more general version in Barnett (AER 1980).

The Social Planner's Optimal Allocation

Given inverse demand $p(q)$ and costs of production $C(q)$, the social planner chooses the quantity to produce q to maximize the social welfare:

$$\max_q SWF = \max_q \int_0^q p(q)dw - C(q) - \phi q$$

which result in the the first order condition

$$\frac{\partial SWF}{\partial q} = 0 \qquad \Longrightarrow \qquad p(q) = C'(q) + \phi$$

So the social planner would choose to produce upto the point where the marginal benefit to society (the price consumers are willing to pay) equals the marginal costs to society (the marginal costs of production and the marginal damages of production).

The Monopolists Problem given a Tax

Given inverse demand $p(q)$ and a unit-tax on production, the monopolist maximizes profit by choosing the quantity to produce:

$$\max_q \pi = \max_q \{p(q)\ q - C(q) - t\ q\}$$

which implies the first order condition

$$\frac{\partial \pi}{\partial q} = 0 \qquad \Longrightarrow \qquad p'(q^*)\ q^* + p(q^*) = C'(q^*) + t \tag{4}$$

where the left side of the condition is the marginal revenue of the monopolist, and the right side is the marginal cost to the monopolist. Given the demand function, this defines an implicit, profit-maximizing quantity response of the monopolist based on the tax t

$$q^* = q(t)$$

We can also find the comparative static of the profit maximizing quantity with respect to a change in the tax:

$$\begin{aligned} d[p'\ q + p] &= d[C' + t] \\ p''qdq + p'dq + p'dq &= C''dq + dt \\ \Longrightarrow \frac{dq}{dt} &= [p''q + 2p' - C'']^{-1} \end{aligned} \tag{5}$$

The Regulators Problem

The regulator only has the option to tax output from the firms. So the regulator's problem is to maximize social welfare by using the tax, taking the demand function and monopolists profit-maximizing quantity response as given:

$$\max_t SWF\big|_{q=q(t)} = \max_t \left\{ \int_0^{q(t)} p(w)dw - C(q(t)) - \phi\, q(t) \right\}$$

With a quick application of the fundamental theorem of calculus and the chain rule, we have the following first order condition:

$$\frac{\partial SWF}{\partial t} = 0 \qquad \Longrightarrow \qquad p(q(t))\; q'(t) = C'(q(t))\; q'(t) + \phi\; q'(t) \tag{6}$$

Assuming that $q'(t)$ is nonzero in the range of the tax being considered, we can divide all terms by $q'(t)$. From comparative static (5) above, this assumption seems reasonable in that it would require one or more of p'', p', or C'' to go to $\pm\infty$ for $dq/dt = 0$. After dividing out $q'(t)$, we have

$$p(q(t)) = C'(q(t)) + \phi \tag{7}$$

This is simply the same condition of the social planner, but instead of being able to choose the quantity directly, we can induce the optimal quantity produced by applying the tax. In order to calculate the optimal tax, I next assume function forms for the demand and cost curves.

I assumed quadratic production costs and I tried several functional forms for the demand curve. The results depend dramatically on the demand curve: when applying a Cobb-Douglas utility function for a representative consumer and deriving the inverse demand curve (which looks something like $p(q) = b\, q^{-1}$ for some positive constant b), the tax required turns out to be unambiguously negative (a subsidy). I present below the results when assuming linear demand.

The Optimal Tax under Linear Demand and Quadratic Production Costs

Assume quadratic production costs of the form

$$C(q) = \frac{1}{2}cq^2$$

where $c \in \mathbb{R}_+$, then

$$C'(q) = cq \quad \text{and} \quad C''(q) = c$$

Also assume that the inverse demand function is linear of the form

$$p(q) = b - aq$$

where $b, a \in \mathbb{R}_+$; then

$$p'(q) = -a \quad \text{and} \quad p''(q) = 0$$

Then returning to the monopolist's FOC from (4), we have

$$\begin{aligned} p'(q^*)\ q^* + p(q^*) &= C'(q^*) + t \\ -aq^* + (b - aq^*) &= cq^* + t \\ \implies q^* = q(t) &= \frac{b-t}{2a+c} \quad \text{and} \quad q'(t) = -\frac{1}{2a+c} \end{aligned}$$

To reiterate, we see $q'(t)$ in our regulator's FOC in (6). Since $a, c \in \mathbb{R}_+$, $q'(t) < 0$ for all t, so we can move straight to the simplified FOC in (7) to find the optimal tax.

$$\begin{aligned} p(q(t)) &= C'(q(t)) + \phi \\ b - aq(t) &= cq(t) + \phi \\ b - \phi &= (a+c)q(t) \\ b - \phi &= (a+c)\frac{b-t}{2a+c} \end{aligned}$$

After some algebra, we get

$$t = \phi + \frac{a}{a+c}(\phi - b)$$

b is the price-intercept of the inverse demand curve, so we can think of it as the maximum willingness to pay for a unit of the good. We can then think of $(\phi - b)$ as the difference between the marginal damage to society and the maximum marginal benefit. $(\phi - b)$ will be greater than zero if the marginal damages outweigh the maximum marginal benefit, meaning that we would tax higher than the naive Pigouvian tax. If $(\phi - b)$ is less than zero, and at least someone in society gets a benefit from the good larger than the marginal social damage, then the optimal tax is lower than the Pigouvian prescription – essentially offering a subsidy from the naive Pigouvian tax because monopoly power is keeping the production below the optimal quantity.

a and c represent the slopes of the inverse demand and marginal cost curves. So $\dfrac{a}{a+c}$ will be closer to one if marginal cost is relatively steep compared to demand, and will be closer to zero if supply is relatively flat. So our modification to the Pigouvian prescription is scaled by how relatively steep the monopoly's cost curve is compared to consumers' demand – holding demand fixed, this means a larger deviation from the Pigouvian prescription for lower marginal costs of the monopoly.

Problem 3

Optimal subsidy for two goods

This problem asks you to consider results related to a second-best problem in which a planner wants to subsidize two products, but faces a revenue constraint. When revenue is costly, the optimal subsidies will depend on the number of inframarginal agents because of a marginal cost of funds consideration. If you just want to kick start a market for a new good with a limited budget, the logic of the relative subsidies here should be a useful guide. (Note, however, that this objective is not particularly well grounded in theory, but it does seem of practical relevance.) This is a version of the problem that we considered in Workshop 1, and it relates to the theory in DeShazo, Sheldon and Carson's article.

Consider a setup with two goods, 1 and 2, consumed in quantities x_1 and x_2. Each is produced at constant marginal cost, denoted c_1 and c_2, by a competitive industry. The utility of a representative consumer is denoted $u(x_1) + v(x_2) + n$, where u and v are increasing and concave. n is a numeraire good. Consumer income Y is exogenous.

Consumption of the sum of x_1 and x_2 has a marginal external benefit ϕ. That is, a positive externality is equal to $\phi(x_1 + x_2)$. You could think of x_1 and x_2 as two versions of a new technology, where the externality is a function of the total market demand for either product; the externality coming from network externalities that aid diffusion.

A planner can subsidize the two goods with marginal price subsidies s_1 and s_2.

1. Derive expressions for the optimal subsidies s_1^* and s_2^*, assuming that revenue from the tax is simply recycled lump-sum to consumers.

The Social Planner's Optimal Allocation

The social planner's problem is to maximize social welfare

$$\max_{x_1,x_2} SWF = \max_{x_1,x_2} \{u(x_1) + v(x_2) + n - c_1x_1 - c_2x_2 + \phi\ (x_1 + x_2)\}$$

leading to the first order conditions for the optimal allocation:

$$\begin{aligned} u'(x_1^*) + \phi &= c_1 \\ v'(x_2^*) + \phi &= c_2 \end{aligned} \tag{8}$$

where on the left hand side of the conditions is the marginal benefit to society from one more unit of each good, and on the right hand side is the marginal cost.

The consumer's problem and the optimal subsidies

I will use a representative consumer model. In this model, because the consumer assumes they do not affect the total amount produced, they also assume their lump-sum tax burden is exogenous. Then

$$T = s_1 x_1 + s_2 x_2$$

is the tax burden the representative consumer faces, and the derivatives of which do not appear in the first order utility maximization conditions. Then the consumer's problem is

$$\max_{x_1,x_2,n} U = \max_{x_1,x_2,n} \{u(x_1) + v(x_2) + n\} \quad \text{s.t.} \quad Y \geq T + (p_1 - s_1)x_1 + (p_2 - s_2)x_2 + n$$

Since the utility is quasi-linear in the numeraire good, the resulting Lagrange multiplier from the budget constraint (call it μ) is equal to one. Meaning the consumer's marginal utility of income is equal to one at the utility maximizing quantity. This leads to the consumer's first order conditions for utility maximization:

$$\begin{aligned} u'(x_1) + s_1 &= p_1 \\ v'(x_2) + s_2 &= p_2 \end{aligned}$$

Because the producing industry is perfectly competitive, the prices will be equal to the marginal costs. So our first order conditions for utility maximization can be interpreted as first order conditions for the equilibrium by swapping in marginal costs c_1 and c_2 for the prices:

$$\begin{aligned} u'(x_1) + s_1 &= c_1 \\ v'(x_2) + s_2 &= c_2 \end{aligned} \tag{9}$$

Comparing these to the social planner's FOCs in (8), we can see the optimal allocation can be achieved by setting both subsidies equal to the marginal externality evaluated at the equilibrium:

$$s_1^* = s_2^* = \phi \tag{10}$$

Problem 3 continued...

2. The setup assumes additive separability of the utility from x_1 and x_2. Suppose instead that utility is described by $w(x_1, x_2) + n$, with $w_1', w_2' > 0$, and $w_1'', w_2'' < 0$ and, importantly, $w_{12}'' < 0$ – that is, x_1 and x_2 are substitutes. Would this modification change your answer to part (a)? Explain how, or explain why not.

Replacing $u(x_1) + v(x_2)$ with $w(x_1, x_2)$ would not change my answer to the previous part. The reason is that the optimality conditions change in both the social planner's optimal allocation and in the market response to the subsidies. The updated social planner's first order conditions would be

$$w_1(x_1^*, x_2^*) + \phi = c_1$$
$$w_2(x_1^*, x_2^*) + \phi = c_2$$

and the equilibrium conditions under the subsidies would be

$$w_1(x_1^*, x_2^*) + s_1 = c_1$$
$$w_2(x_1^*, x_2^*) + s_2 = c_2$$

Therefore the subsidies that decentralize the optimal allocation would still be

$$s_1^* = s_2^* = \phi$$

Problem 3 continued...

3. Assume separability again (utility is $u(x_1) + v(x_2) + n$). Suppose that the government faces a binding revenue constraint R that limits the total amount that can be spent on subsidies: $s_1x_1 + s_2x_2 \leq R$. Derive expressions for the optimal subsidies, $\tilde{s}_1$ and $\tilde{s}_2$, that illustrates the additivity property of Pigouvian taxation, where the optimal subsidy is marginal damages plus some other term.

   (Hint: for the parts below, it is most useful to write the answer in terms of price derivatives and quantities, x and $\partial x/\partial s$, rather than to substitute in using the definition of the elasticity. Also, if you have the correct answer, your answer in part (a) should be a special case that is easy to see.)

Though the revenue source for the subsidies has changed, because the consumer in both this part and part 1 does not consider their actions to change the overall revenue needed, the same first order conditions apply as in Eq. (9). From these FOCs, I define implicit equilibrium quantities x_1 and x_2 based on subsidies imposed:

$$\begin{aligned} u'(x_1) + s_1 = c_1 \quad &\implies \quad x_1(c_1 - s_1) \\ v'(x_2) + s_2 = c_2 \quad &\implies \quad x_2(c_2 - s_2) \end{aligned}$$

Because u and v are increasing and concave, the law of demand holds and the own-price derivative is negative:

$$\frac{\partial x_k}{\partial p_k} = \frac{\partial x_k}{\partial c_k} < 0 \quad \forall k \in \{1, 2\}$$

Let x_k' be the own-price derivative. By the chain rule, the own-subsidy derivative is just the opposite sign:

$$x_k' \equiv \frac{\partial x_k}{\partial p_k} = -\frac{\partial x_k}{\partial s_k} \quad \forall k \in \{1, 2\}$$

Under the revenue constraint, the regulator's problem becomes

$$\max_{s_1,s_2} SWF = \max_{s_1,s_2} \{u(x_1(c_1 - s_1)) + v(x_2(c_2 - s_2)) + n - c_1x_1 - c_2x_2 + \phi\ (x_1(c_1 - s_1) + x_2(c_2 - s_2))\}$$

$$\text{s.t.} \quad R \geq s_1x_1 + s_2x_2$$

which lead to the following first order conditions for the optimal subsidies $\tilde{s}_1$ and $\tilde{s}_2$:

$$\begin{aligned} 0 &= -x_1' \left[u' + \phi - c_1 - \lambda\tilde{s}_1\right] - \lambda x_1 \\ 0 &= -x_2' \left[v' + \phi - c_2 - \lambda\tilde{s}_2\right] - \lambda x_2 \end{aligned}$$

where λ is the shadow price of the revenue constraint. From part 1 of this question, the Lagrange multiplier μ is equal to one, which is also the consumer's marginal utility of income. So λ here is also λ/μ, which is the Marginal Cost of Public Funds . We can divide by them an simplify to

$$\begin{aligned} 0 &= u' + \phi - c_1 - \lambda\tilde{s}_1 + \lambda\frac{x_1}{x_1'} \\ 0 &= v' + \phi - c_2 - \lambda\tilde{s}_2 + \lambda\frac{x_2}{x_2'} \end{aligned}$$

We can substitute in the utility-maximizing response from consumers from (9) to get

$$
\begin{aligned}
0 &= (c_1 - \tilde{s}_1) + \phi - c_1 - \lambda \tilde{s}_1 + \lambda \frac{x_1}{x_1'} \\
0 &= (c_2 - \tilde{s}_2) + \phi - c_2 - \lambda \tilde{s}_2 + \lambda \frac{x_2}{x_2'}
\end{aligned}
$$

Partially solving for $\tilde{s}_1$ and $\tilde{s}_2$, and doing some algebra, we get

$$
\begin{aligned}
\tilde{s}_1 &= \left(1 - \frac{\lambda}{1+\lambda}\right)\phi + \frac{\lambda}{1+\lambda}\frac{x_1}{x_1'} \\
\tilde{s}_2 &= \left(1 - \frac{\lambda}{1+\lambda}\right)\phi + \frac{\lambda}{1+\lambda}\frac{x_2}{x_2'} \qquad (11)
\end{aligned}
$$

Comparing the optimal subsidies under no constraint in part 1 (where they both equal ϕ), we can see that is a special case of this problem that arises when $\lambda = 0$ (when the revenue constraint does not bind).

When the regulator's revenue constraint does bind and $\lambda > 0$, then the optimal subsidy will scale down the Pigouvian element and scale up the second term, which is based on consumption and price-responsiveness.

Note that this is not a full solution to the question of optimal subsidies. While the above relationships do hold and the subsides on the left would equal the quantities on the right, both x_k and x_k' are implicit functions of s_k. If we wanted to impose specific functional forms on the utility (and thus the derived demand curve), we could solve for the subsidy fully in terms of parameters of the model and functional form of the utility.

Problem 3 continued...

4. Suppose that x_1 and x_2 both have the same constant own-price demand derivative $\partial x_1/\partial s_1 = \partial x_2/\partial s_2$. If good one is sold in higher quantities (at the second-best policy; $\tilde{x}_1 > \tilde{x}_2$), what does this imply about the relative subsidy for good one compared to good two?

   Answer the question in two parts. First, apply intuition from class lectures. Second, use your result from part (c), in particular take the difference between $\tilde{s}_1 - \tilde{s}_2$ and examine the expression. Use the theoretical intuition to make sure that your algebra signs are correct. Briefly state the intuition and algebra separately in your write up.

Assuming that the own-price derivatives are equal, I cannot think of an intuitive reason that the subsidy should be different. If we know the regulator is optimizing for maximum social welfare under the assumption that both types of goods produce the same amount of constant marginal externality, then I would guess the subsidies should be the same when they have equal own-price derivatives. One could argue in a more general sense that, if the regulator is interested in maximum production, they might want to subsidize the good that is not selling as much more than than the good that is selling more. An argument like, "if we are going to provide help to a producer, we should provide it to the producer that needs more help."

But, honestly, this is a bit of made up reasoning to justify the math.

Let's take a look at what the equations tell us. Subtracting the second subsidy from the first from (11), we have

$$\begin{aligned}
\tilde{s}_1 - \tilde{s}_2 &= \frac{\lambda}{1+\lambda}\left[\frac{x_1}{x_1'} - \frac{x_2}{x_2'}\right] && (x_1' = x_2') \\
&= \frac{\lambda}{(1+\lambda)x_1'}\left[x_1 - x_2\right] && (x_1 > x_2 \text{ and } x_1' < 0) \\
&< 0
\end{aligned}$$

So $\tilde{s}_1 < \tilde{s}_2$ if $\tilde{x}_1 > \tilde{x}_2$, meaning the good that sells more must have a *smaller* optimal subsidy applied.

Problem 3 continued...

5. Suppose instead that x_1 and x_2 have the same quantities sold (at the second-best policy $\tilde{x}_1 = \tilde{x}_2$), but now x_1 is more price responsive, so that $|\partial x_1/\partial s_1| > |\partial x_2/\partial s_2|$. What does this imply about the relative subsidy for good one compared to good two?

   Answer the question in two parts. First, apply intuition from class lectures. Second, use your result from part (c), in particular take the difference between $\tilde{s}_1 - \tilde{s}_2$ and examine the expression. Use the theoretical intuition to make sure that your algebra signs are correct. Briefly state the intuition and algebra separately in your write up.

Ramsey taxation intuition tells us that to be the least distortionary, we want to put higher taxes on goods that are less price-responsive. We might think the following modification could be true: to be the most distortionary and increase consumption of the good, we should put the larger price distortion (larger subsidy) on the good that is more price-responsive. So if good one is more price responsive, I would expect the subsidy should be larger.

Subtracting the second subsidy from the first from (11), we have

$$
\begin{aligned}
\tilde{s}_1 - \tilde{s}_2 &= \frac{\lambda}{1+\lambda}\left[\frac{x_1}{x_1'} - \frac{x_2}{x_2'}\right] && (x_1 = x_2) \\
&= \frac{\lambda x_1}{(1+\lambda)}\left[\frac{1}{x_1'} - \frac{1}{x_2'}\right] && (x_1', x_2' < 0) \\
&= -\frac{\lambda x_1}{(1+\lambda)}\left[\frac{1}{|x_1'|} - \frac{1}{|x_2'|}\right] && \left(|x_1'| > |x_2'| \implies \frac{1}{|x_1'|} < \frac{1}{|x_2'|}\right) \\
&= \frac{\lambda x_1}{(1+\lambda)}\left[\frac{1}{|x_2'|} - \frac{1}{|x_1'|}\right] && \left(\frac{1}{|x_1'|} < \frac{1}{|x_2'|} \implies \frac{1}{|x_2'|} - \frac{1}{|x_1'|} > 0 \text{ and } x_1 > 0\right) \\
&> 0
\end{aligned}
$$

So $\tilde{s}_1 > \tilde{s}_2$ if $\tilde{x}_1' > \tilde{x}_2'$, meaning that there is a *larger* subsidy on the good that consumers are more price responsive to.