# Problem 1

**The elasticity of substitution with constant-relative-risk-aversion utility: Romer 2.2**

Consider an individual who lives for two periods and whose utility is given by equation (2.43). Let $P_1$ and $P_2$ denote the prices of consumption in the two periods, and let $W$ denote the value of the individual's lifetime income; thus the budget constraint is $P_1C_1 + P_2C_2 = W$.

(a) What are the individual's utility-maximizing choices of $C_1$ and $C_2$, given $P_1$, $P_2$, and $W$?

The Lagrangian for this problem is

$$\begin{aligned}\mathcal{L} &= U(t) + \lambda(W - P_1C_1 - P_2C_2) \\ &= \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2(t+1)}^{1-\theta}}{1-\theta} + \lambda(W - P_1C_1 - P_2C_2)\end{aligned}$$

Supressing the $t$ subscript, we the FOCs are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_1} &= 0 \implies \frac{1}{\lambda} = P_1C_1^\theta \\ \frac{\partial \mathcal{L}}{\partial C_2} &= 0 \implies \frac{1}{\lambda} = P_2C_2^\theta(1+\rho)\end{aligned}$$

So

$$\begin{aligned}P_1C_1^\theta &= P_2C_2^\theta(1+\rho) \\ C_1 &= C_2\left[\frac{P_2}{P_1}(1+\rho)\right]^{1/\theta}\end{aligned}$$

Plugging this into the budget constraint...

$$\begin{aligned}W &= P_1C_1 + P_2C_2 \\ &= P_1C_2\left[\frac{P_2}{P_1}(1+\rho)\right]^{1/\theta} + P_2C_2 \\ &= P_2C_2\left\{\left[\left(\frac{P_1}{P_2}\right)^{\theta-1}(1+\rho)\right]^{1/\theta} + 1\right\}\end{aligned}$$

So the optimal period 2 consumption is

$$C_2^* = \frac{W}{P_2 \left\{ \left[ \left( \frac{P_1}{P_2} \right)^{\theta - 1} (1+\rho) \right]^{1/\theta} + 1 \right\}}$$

And the optimal period 1 consumption is

$$C_1^* = C_2^* \left[ \frac{P_2}{P_1} (1+\rho) \right]^{1/\theta}$$

$$= \frac{W \left[ \frac{P_2}{P_1} (1+\rho) \right]^{1/\theta}}{P_2 \left\{ \left[ \left( \frac{P_1}{P_2} \right)^{\theta - 1} (1+\rho) \right]^{1/\theta} + 1 \right\}}$$

Which probably simplifies...

(b) The elasticity of substitution between consumption in the two periods is $-[(P_1/P_2)/(C_1/C_2)][\partial(C_1/C_2)/\partial(P_1/P_2)]$, or $-\partial\ln(C_1/C_2)/\partial\ln(P_1/P_2)$. Show that with the utility function (2.43), the elasticity of substitution between $C_1$ and $C_2$ is $1/\theta$.

Since we know that

$$\frac{C_1^*}{C_2^*} = \left[\frac{P_2}{P_1}(1+\rho)\right]^{1/\theta}$$

Then

$$\ln\frac{C_1^*}{C_2^*} = \frac{1}{\theta}\ln\left[\frac{P_2}{P_1}(1+\rho)\right]$$

$$= \frac{1}{\theta}\left[\ln(1+\rho) - \ln\frac{P_1}{P_2}\right]$$

So

$$\frac{\partial\ln\frac{C_1^*}{C_2^*}}{\partial\ln\frac{P_1}{P_2}} = -\frac{1}{\theta}$$

# Problem 2

**Find the utility-maximizing path of $C$: Romer 2.5**

Consider a household with utility given by (2.2) (2.3). Assume that the real interest rate is constant, and let $W$ denote the household's initial wealth plus the present value of its lifetime labor income (the right-hand side of [2.7]). Find the utility-maximizing path of $C$, given $r$, $W$, and the parameters of the utility function.

First note that since $r(t) = r$ is a constant,

$$R(t) = \int_0^t r(s)dt = \int_0^t r dt = rt$$

so the path of consumption satisfies

$$\underset{C(t)}{\text{argmax}} \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt$$

subject to the conditions

$$\int_0^\infty e^{-rt} C(t) \frac{L(t)}{H} dt \leq W$$

$$\dot{k} = f(k) - C - (n+g)k$$

$$f'(k) = r(t) = r$$

So the Hamiltonian for this problem is

$$H = e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} + \lambda(t) \left[ f(k(t)) - C(t) - (n+g)k(t) \right]$$

And the first FONC is

$$H_C = 0 \implies \lambda(t) = e^{-\rho t} C(t)^{-\theta} \frac{L(t)}{H}$$

taking logs and then the time derivative gives

$$\implies \ln \lambda(t) = -\rho t - \theta \ln C(t) + \ln L - \ln H$$

$$\implies \frac{\dot{\lambda}}{\lambda} = -\rho - \theta \frac{\dot{C}}{C} + \frac{\dot{L}}{L}$$

And noting that the growth rate of labor is $n$

$$\implies \frac{\dot{\lambda}}{\lambda} = -\rho - \theta \frac{\dot{C}}{C} + n$$

Then the second FONC is

$$H_k = -\dot{\lambda} \implies -\frac{\dot{\lambda}}{\lambda} = f'(k) - (n+g)$$

And noting that $f'(k) = r(t) = r$,

$$\implies -\frac{\dot{\lambda}}{\lambda} = r - n - g$$

Putting these first two FONCs together, we have

$$r - n - g = \rho + \theta\frac{\dot{C}}{C} - n$$

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - g - \rho)$$

Since this is a constant, this implies that

$$C(t) = Ae^{-\frac{1}{\theta}(g+\rho-r)}$$

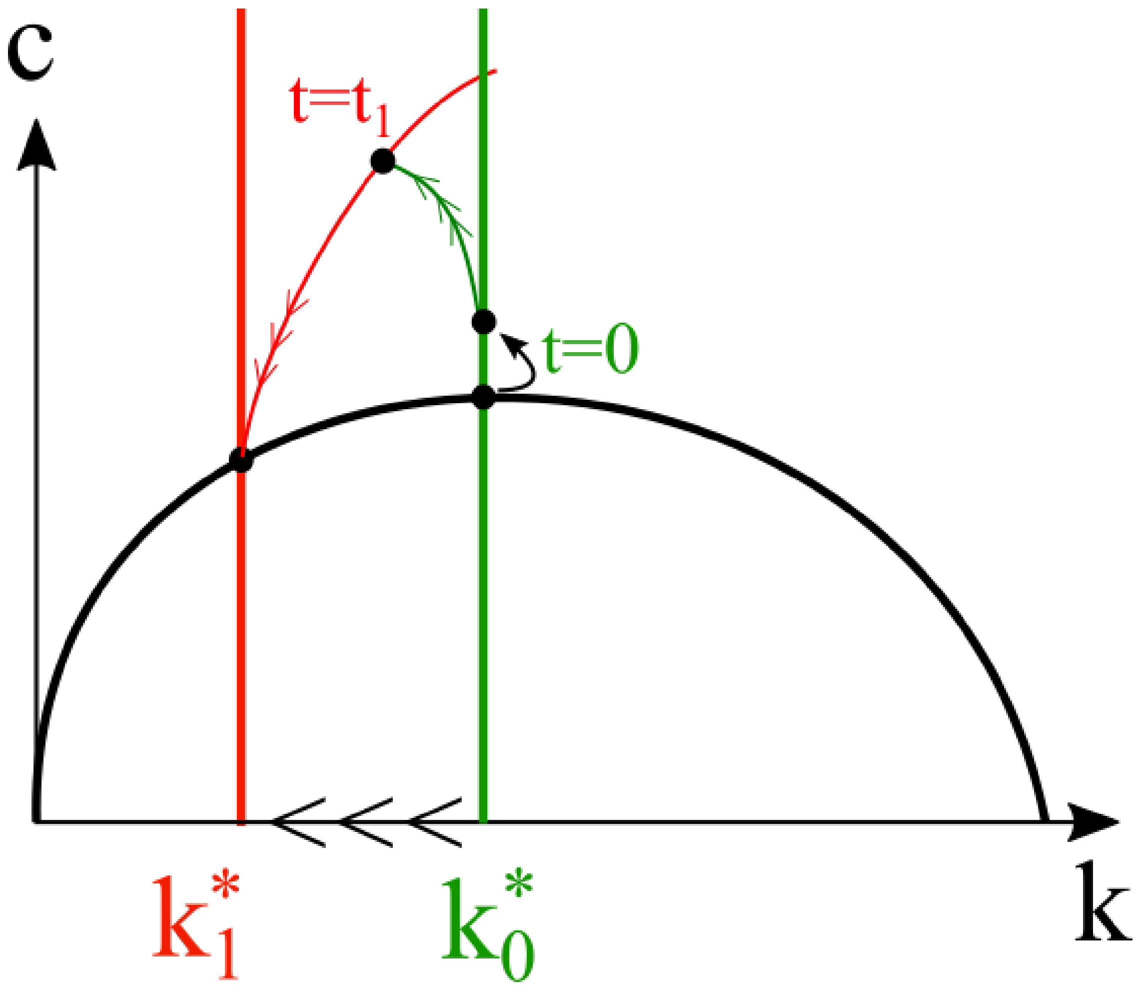
For some constant $A$ determined by the initial capital stock $k_0$ and the law of motion $\dot{k} = f(k) - C(t) - (n+g)k$.

## Problem 3

**Using the phase diagram to analyze the impact of an anticipated change: Romer 2.11**

Consider the policy described in Problem 2.10, but suppose that instead of announcing and implementing the tax at time 0, the government announces at time 0 that at some later time, time $t_1$, investment income will begin to be taxed at rate $\tau$.

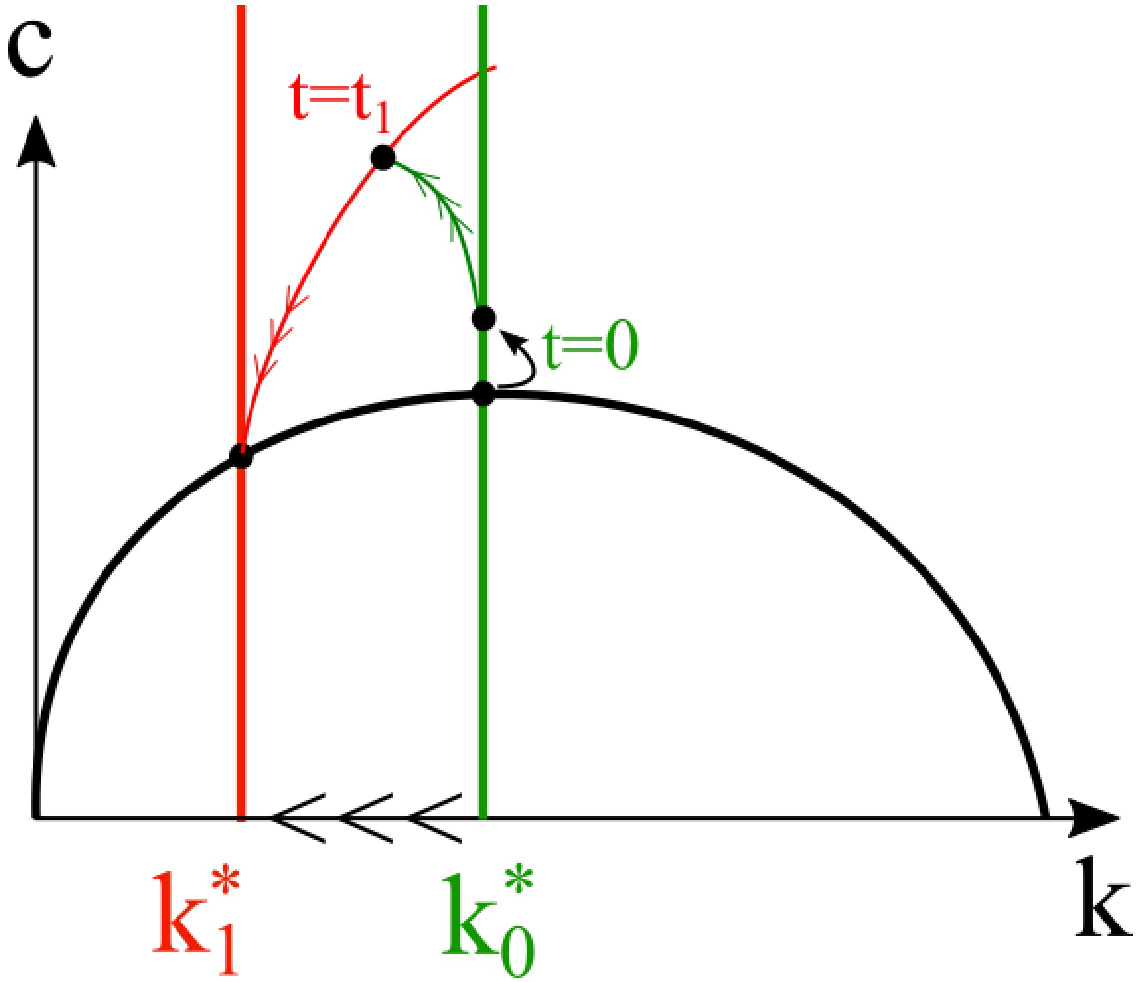
(a) Draw the phase diagram showing the dynamics of $c$ and $k$ after time $t_1$.



(b) Can $c$ change discontinuously at time $t_1$? Why or why not?

No, if $c$ changed discontinuously at $t_1$, that would indicate that the consumers were not consumption smoothing before $t_1$ even though they had know about the change in tax rate since $t = 0$. So, right before $t_1$, $c$ must be approaching the new value it will have at the time of the tax change $t_1$.

(c) Draw the phase diagram showing the dynamics of $c$ and $k$ before $t_1$.



(d) In light of your answers to parts (a), (b), and (c), what must $c$ do at time 0?

It must change discontinuously in order to jump on a path that leads to the saddle path that it will be on at $t_1$.

(e) Summarize your results by sketching the paths of $c$ and $k$ as functions of time.