Economics 202A: Section 2

Dynamic Optimization

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In this section we will review how to analyze dynamic optimization problems. Analyzing these problems generally involves two steps

1. use of the Kuhn-Tucker Theorem or the Maximum Principle to obtain the first-order conditions that are necessary (and often sufficient in economic applications) conditions that must hold at an optimum

2. analyze this dynamic system (i.e. system of difference equations plus boundary conditions) to discover the properties of the solution

This second step is where we will use economic reasoning, graphical analysis and approximations, and we'll discuss these methods later.

David's syllabus suggests multiple texts for learning this material. I'll add one more: Acemoglu's *Introduction to Modern Economic Growth* which provides more formal statements of the results here in addition to proofs.

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

Our outline for handling dynamic optimization problems under *certainty* is as follows:

- Discrete Time (use Lagrangian)
    1. Static Problems (hopefully familiar)
    2. Finite Horizon Problems (actually the same as above)
    3. Infinite Horizon Problems (pretty similar)
- Continuous Time (use Hamiltonian)
    1. Infinite Horizon Problems

The most important stuff is at the very end!

Static Optimization: Constraints Hold With Equality

Many problems in economics take the following form:

$$\max_{\mathbf{x}} \; F(\mathbf{x}) \; s.t. \tag{1}$$
$$G(\mathbf{x}) = 0$$

where $\mathbf{x}$ is a vector and G is a vector-valued function. The problem is to choose the elements of $\mathbf{x}$ so as to maximize the function F while obeying the constraints. For example, F could be a utility function, $\mathbf{x}$ a vector of 2 goods, and G the constraint that purchases equal income.[1]

How does one solve such a problem? Recall from calculus that solving (1) is equivalent to solving the unconstrained maximization problem

$$\max_{\mathbf{x},\lambda} \; F(\mathbf{x}) + \lambda \cdot G(\mathbf{x}) \tag{2}$$

The function to be maximized in (2) is known as the *Lagrangian*, $\mathcal{L}$. When we differentiate with respect to $\mathbf{x}$, we now take into account both the direct effect of adjusting an element of $\mathbf{x}$ on the objective function *and* the indirect effect on the constraint, given an appropriate weight: λ which we can interpret as the *shadow price* of the constraint, or the amount that the objective function would increase if we relaxed the constraint by one unit. For example, in the consumer problem we can interpret λ as the utility value of having one more unit of income to spend.

Note that in general we need to be careful that after we characterize the optimum, we ensure that it also exists! These are conceptually distinct exercises in the approaches presented here.

[1] Although the consumer is free to not spend all of her income, since the marginal utility of goods is always positive it will never be optimal. This fact allows us to replace the inequality constraint with an equality constraint. I'll come back to this later.

Exercise 1 *Consider the lagrangian for the static optimization problem of a consumer maximizing the utility of consuming two goods $U(x_1, x_2)$ subject to the budget constraint $I = P_1X_1 + P_2X_2$:*

$$\mathcal{L} = U(x_1, x_2) + \lambda(I - P_1x_1 - P_2x_2)$$

1. *Write down the first order conditions of the problem*
2. *Use these to show that the utility obtained by the consumer is given by evaluating $\mathcal{L}$ at the optimum choices for consumption, x_1^* and x_2^**
3. *Evaluate $\frac{d\mathcal{L}(x_1^*, x_1^*, \lambda^*, I)}{dI}$ at the optimum to show that λ^* is the marginal utility gain from an increase in I.*

To begin, we maximize

$$\max_{x_1,x_2,\lambda} \mathcal{L} = \max_{x_1,x_2,\lambda} U(x_1, x_2) + \lambda(I - P_1x_1 - P_2x_2)$$

resulting in the following necessary conditions:

$$\begin{aligned} U_1 &= \lambda P_1 \\ U_2 &= \lambda P_2 \\ I &= P_1x_1 + P_2x_2 \end{aligned}$$

Given appropriate assumptions on U (e.g. twice differentiable and concave) solving this system yields unique values of x_1^*, x_2^* and λ^* which are presumably functions of income (and prices.) Since the solutions satisfy all the necessary conditions, plugging them into the Lagrangian yields

$$\mathcal{L}(x_1^*, x_2^*) = U(x_1^*, x_2^*) + \lambda^*(I - P_1x_1^* - P_2x_2^*) = U(x_1^*, x_2^*)$$

Thus the lagrangian, when evaluated at the optimum consumption choices, yields the maximum possible utility the consumer can obtain.

Differentiating the Lagrangian evaluated at this optimum w.r.t. I:

$$\begin{aligned}
\frac{d\mathcal{L}(x_1^*, x_2^*, \lambda^*, I)}{dI} &= \frac{\partial \mathcal{L}}{\partial x_1}\frac{\partial x_1^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial x_2}\frac{\partial x_2^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda_1^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial I}\frac{\partial I}{\partial I} \\
\frac{d\mathcal{L}(x_1^*, x_2^*, \lambda^*, I)}{dI} &= 0 + 0 + 0 + \frac{\partial \mathcal{L}}{\partial I} \\
\frac{d\mathcal{L}(x_1^*, x_2^*, \lambda^*, I)}{dI} &= \lambda^*
\end{aligned}$$

Note that all the partial derivatives of $\mathcal{L}$ are all evaluated at the optimum (I have not written this to keep the expression clean) which is why the first three terms are zero. Thus, we can see now why the costate variable λ is often called the "shadow price" of relaxing the constraint! It is literally the marginal value (in utils) of an extra unit of income.

Static Optimization with Inequality Constraints

What about inequality constraints? The Kuhn-Tucker theorem gives a similar set of necessary conditions and sufficient conditions for a maximum. Start with a problem

$$
\begin{aligned}
\max_{\mathbf{x}} \ & F(\mathbf{x}) \ s.t. \\
& G(\mathbf{x}) = 0 \\
& H(\mathbf{x}) \geq 0
\end{aligned} \tag{3}
$$

The Kuhn-Tucker Theorem says that solving this problem is equivalent to solving the following problem

$$
\begin{aligned}
& \max_{\mathbf{x}\lambda} F(\mathbf{x}) + \lambda \cdot G(\mathbf{x}) + \mu \cdot H(\mathbf{x}) \ s.t. \\
& \quad H(\mathbf{x}) \geq 0 \\
& \quad \mu H = 0 \\
& \quad \mu \geq 0
\end{aligned} \tag{4}
$$

Note the following:

- This is close to an unconstrained problem, except for the condition that *either* $H(\mathbf{x}) = 0$ *or* $\mu = 0$. This is called the *complementary slackness* condition and can be interpreted as requiring that either the constraint is binding and the shadow price is positive, or the constraint is not binding and the shadow price is zero.
- This makes intuitive sense: if the constraint is not binding then relaxing the constraint should not increase the objective function.
- The mechanics of solving a problem with inequality constraints that sometimes bind and sometimes do not can be kind of messy.

We can often use economic logic to conclude that our constraints always bind (or never bind). This will be our approach in this part of 202A: likely, in the second part, you will learn to deal with "occasionally binding" constraints.

Dynamic Optimization in Finite, Discrete Time

Dynamic optimization with a finite horizon is just a special case of static optimization.

Exercise 2 *Consider a consumer whose instantaneous utility is given by:*

$$u(c_t) = \log c_t \tag{5}$$

The consumer lives for T periods, $t = 0$ up to $t = T - 1$. She discounts future utility by $\beta < 1$. Thus her objective function is

$$\max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t \log c_t \tag{6}$$

Each period she receives perfectly forseeable income $\bar{y} > 0$. She can accumulate interest-bearing assets a_t by saving. Assume that $\beta(1 + r) = 1$. Her budget constraint each period is

$$a_{t+1} = \bar{y} - c_t + (1 + r)a_t \ \forall \ t = 1, ..., T \tag{7}$$

with her initial assets given by $a_0 = 0$. Note that consumption cannot be negative so

$$c_t \geq 0 \ \forall \ t = 0, ..., T \tag{8}$$

Finally, the consumer cannot leave any debt behind so

$$\left(\frac{1}{1+r}\right)^{T-1} a_T \geq 0, \tag{9}$$

which is also known as the No Ponzi scheme condition.

Write down the Lagrangian and find the first order conditions for maximization. Determine which constraints are binding and which are not, then rewrite the first order conditions to reflect which constraints bind. Solve for the optimal consumption profile $\{c_t\}_{t=0}^{T-1}$ using the Lagrange FOCs for a maximum and argue that this indeed constitutes a maximum.

If we multiply each costate variable / constraint by a constant β^t, The Lagrangian is given by

$$\max_{\{c_t\}_{t=0}^{T-1},\{a_t\}_{t=1}^{T},\{\lambda_t\}_{t=0}^{T-1}} \mathcal{L} = \sum_{t=0}^{T-1} \beta^t \left[\log c_t + \lambda_t(-a_{t+1} + \bar{y} - c_t + (1+r)a_t) + \theta_t c_t\right] + \\ + \mu \left(\frac{1}{1+r}\right)^{T-1} a_T$$

so the FOC are

$$\begin{aligned}
c_t : &\ \beta^t(c_t^{-1} - \lambda_t + \theta_t) = 0 \\
\lambda_t : &\ \beta^t(-a_{t+1} + \bar{y} - c_t + (1+r)a_t) = 0 \\
a_{t+1} : &\ \beta^t(-\lambda_t + (1+r)\beta\lambda_{t+1}) = 0 \\
a_T : &\ -\beta^{T-1}\lambda_{T-1} + \left(\frac{1}{1+r}\right)^{T-1} \mu = 0 \\
Constraints : &\ c_t \geq 0 \\
Constraints : &\ \left(\frac{1}{1+r}\right)^{T-1} a_T \geq 0 \\
CS : &\ \theta_t c_t = 0, \quad \mu \cdot \left(\frac{1}{1+r}\right)^{T-1} a_T = 0
\end{aligned}$$

We have two inequality constraints to rule out: since $log(c)$ satisfies the Inada condition at 0, consumption is always positive and the constraint does not bind (why?) For the No Ponzi Scheme, it's intuitive that $a_T = 0$ because if it's positive we are leaving consumption on the table, and the constraint binds.

Formally, note we can solve for $\mu = (\beta(1+r))^{T-1}\lambda_{T-1}$ using the FOC for a_T, then use the FOC for c_t to substitute for λ_{T-1}. Substituting this expression for μ into the original complementary slackness constraint, we get

$$\beta^{T-1}\frac{1}{c_{T-1}}a_T = 0 \tag{10}$$

This is known as the *Transversality Condition* for the finite horizon problem. Since $c_{T-1} <$

∞ it must be that $a_T = 0$. This transversality condition plays an important role in infinite horizon problems, which we will mention shortly.

Armed with our two simplifications from classifying our constraints as binding or not, we can rewrite the FOCs as

$$\begin{aligned}
c_t :&\ \beta^t (c_t^{-1} - \lambda_t) = 0 \\
\lambda_t :&\ \beta^t (-a_{t+1} + \bar{y} - c_t + (1+r)a_t) = 0 \\
a_{t+1} :&\ \beta^t (-\lambda_t + (1+r)\beta\lambda_{t+1}) = 0 \\
a_T :&\ -\beta^{T-1}\lambda_{T-1} + \left(\frac{1}{1+r}\right)^{T-1} \mu = 0 \\
\mu :&\ \mu \left(\frac{1}{1+r}\right)^{T-1} a_T = 0
\end{aligned}$$

The first step to solving for the optimal consumption profile is to eliminate the multiplier λ from the system by solving the FOC for c_t to get $\lambda_t = c_t^{-1}$. We can then substitute this result into the FOC for a_{t+1} to derive (using the assumption that $\beta(1+r) = 1$)

$$c_t = c_{t+1} = \bar{c}\ \forall t \tag{11}$$

This is known as the Euler Equation relating optimal consumption in successive periods, although it takes a particularly simple form here.[2] It remains to show what the constant level of consumption is equal to. We can recursively substitute for a_t in the budget constraints to get

$$a_T = \sum_{t=0}^{T-1} (1+r)^{T-1-t}(\bar{y} - \bar{c}) + (1+r)^T a_0 \tag{12}$$

and, using the facts that $a_T = a_0 = 0$, we get $\bar{c} = \bar{y}$.

Finally, this is a maximum because the objective function is concave and the constraint function is convex.

[2] Indeed, the fact that optimal consumption is a constant is a special "knife edge" result that follows from the assumption of log utility and $\beta(1+r) = 1$; you should note that this also holds for any exogenous income stream y_t, not just $y_t = \bar{y}$!

There are three important lessons to be learned from this problem:

1. **We need an initial condition.** If we hadn't specified $a_0 = 0$, then the problem is ill-defined. The consumer will not be able to solve for the optimal consumption path since she does not know her initial wealth.

2. **We need a terminal condition.** Suppose we hadn't imposed $\left(\frac{1}{1+r}\right)^{T-1} a_T \geq 0$. Then the consumer would borrow an infinite amount each period.

3. **We needed special assumptions to get a simple closed-form solution!** Most of the time these aren't available.

Infinite Time Horizon

Most of our tools still apply to problems with an infinite time horizon, but there are two complications:

1. the objective function is now an infinite sum and could potentially diverge to ∞. This would be bad because we cannot discriminate between two solutions that both deliver infinite utility. We can usually deal with this by assuming sufficient discounting, i.e. $\beta < 1$

2. A more subtle problem is that we need to impose the appropriate "terminal" conditions on the long run level of assets (or the capital stock in the Ramsey model) in order to find the optimum, but we don't have a "terminus" in an infinite horizon problem.

We can impose a limiting No Ponzi Scheme condition that is a natural analogue of the finite time condition:

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T a_{T+1} \geq 0 \tag{13}$$

However, this does not pin down the long-run level of assets uniquely because there are infinitely many positive levels of assets that will satisfy this condition. Recall that in the finite time horizon problem we derived the transversality condition from the complementary slackness constraint AND the FOC for a_T, which allowed us to conclude that $a_T = 0$. In an infinite-horizon problem we will still have a complementary slackness constraint on the limiting No Ponzi constraint, but *we no longer have an FOC for terminal assets* because there is no terminal period. Thus the Kuhn-Tucker conditions as we've described them do not suffice to pin down the long-run asset position, and hence we cannot determine the optimal consumption profile. This does NOT mean that all paths that satisfy the Kuhn-Tucker conditions maximize utility; there is in fact a unique maximum under the usual concavity conditions. Rather, it means that we need an additional condition to pin down the maximizing path for c_t.

This condition turns out to be the limit of the finite time transversality condition:[3]

$$\lim_{T\to\infty} \beta^T \lambda_T a_{T+1} = 0 \tag{14}$$

- This condition does NOT require that assets go to zero in the limit; instead, it requires that they not grow faster than the rate at which discounted marginal utility shrinks. This rules out paths where the *present value* of assets explodes.

- Intuition is similar to the finite horizon case: it cannot be optimal to postpone consumption forever. In contrast to the finite horizon case where the transversality condition was derived from the Kuhn-Tucker conditions, for the infinite horizon *this is an additional restriction that we impose*

It turns out that the Kuhn-Tucker conditions PLUS the transversality condition (which replaces the terminal condition we had before in our finite horizon problem) are both necessary (all discounted problems) and sufficient (discounted problems with concave objective and convex constraints) for a wide range of problems of practical relevance in economics.

[3] Technically, to link this to David's lecture, the "don't do globally stupid things" constraint is $\lim_{T\to\infty} \beta^T \lambda_T a_{T+1} \leq 0$ but we can strengthen this to an equality given the no-ponzi scheme assumption as he discussed in lecture – I've skipped this step here as well.

Exercise 3 *Consider the exact same setup as before, but with an infinite horizon. Write down the Lagrangian and the first order conditions for maximization. Find the optimal consumption path.*

Ignoring the non-negativity constraints on consumption (which still do not bind), the Lagrangian is now

$$\max_{\{c_t\}_{t=0}^{\infty},\{a_t\}_{t=1}^{\infty},\{\lambda_t\}_{t=0}^{\infty}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \lambda_t(-a_{t+1} + \bar{y} - c_t + (1+r)a_t) + \theta_t c_t\right] + \\ + \mu \cdot \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T-1} a_T$$

In this problem, we still have that $c_t > 0$ and so $\theta_t = 0$, and so the remaining FOCs are

$$\begin{aligned} c_t &: \ \beta^t(c_t^{-1} - \lambda_t) = 0 \\ \lambda_t &: \ \beta^t(-a_{t+1} + \bar{y} - c_t + (1+r)a_t) = 0 \\ a_{t+1} &: \ \beta^t(-\lambda_t + (1+r)\beta\lambda_{t+1}) = 0 \\ \textit{Constraint} &: \ \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T-1} a_T \geq 0 \\ CS &: \ \mu \cdot \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T-1} a_T = 0 \end{aligned}$$

In addition, we have the transversality condition

$$\lim_{T\to\infty} \beta^T \lambda_T a_{T+1} = 0 \tag{15}$$

Following the same steps as before, we can derive that $c_t = c_{t+1} = \bar{c}$ and use iterative substitution for a_t in the budget constraint as well as the assumption that $a_0 = 0$ to get

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T} a_{T+1} = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^{t} (\bar{y} - \bar{c}) \tag{16}$$

which gives us

$$\bar{c} = \bar{y} - \frac{r}{1+r} \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T} a_{T+1} \tag{17}$$

- We do not know the value of the second term on the RHS.
- The No Ponzi Scheme condition tells us that it must be weakly positive, but there are infinitely many values that satisfy this condition.
- That's why we need the transversality condition, which implies that that the second term is 0.

To see this, note that since $\beta(1+r) = 1$ and $\lambda_T = 1/c_T = 1/\bar{c}$ we can write the transversality condition,

$$\lim_{T\to\infty} \beta^T \lambda_T a_{T+1} = 0, \tag{18}$$

as

$$\begin{aligned} & \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T \frac{a_{T+1}}{\bar{c}} = 0 \\ \Rightarrow \quad & \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T a_{T+1} = 0 \end{aligned}$$

which implies that $\bar{c} = \bar{y}$.

This is really all there is to deriving the FOCs using the Kuhn-Tucker conditions. More complex problems will have more complex FOCs, but the beauty of the theorem is that you just need to know how to differentiate and do algebra in order to derive the FOCs.

[Let's take a 5m Break at this point]

Dynamic Optimization in Continuous Time

In continuous time models, economists make use of variations on the Kuhn-Tucker approaches above that can seem a little intimidating at first. We will focus on one particular tool in this class, but you will study other techniques later in the course.

This tool is called the *Hamiltonian.* One way to think about this tool is as the limiting form of the discrete time optimization methods reviewed above. Some terminology:

Definition 1 *The* ***control*** *variables are variables chosen directly by the agent or planner. In most economic problems, control variables appear as things like consumption, investment, etc.*

Definition 2 *The* ***state*** *variables are function of past choices by the agent or planner. Often these state variables play a key part in dynamic constraints. In a savings / consumption problem, for example, the agent's stock of wealth is influenced by past consumption choices and affects how much the agent is able to consume today.*

(Heuristic) Derivation of the Hamiltonian Approach

A problem is given by:

$$\max_{c_t} V(0) = \int_0^{\infty} v(k_t, c_t, t) \cdot dt \text{ subject to} \tag{19}$$

$$\dot{k_t} = g(k_t, c_t, t) \tag{20}$$

$$k_0 > 0 \tag{21}$$

$$lim_{t \to \infty} k_t \cdot e^{-R_t} \geq 0 \tag{22}$$

- Here $V(0)$ is total value of the objective function at time 0 (the value of the optimal "plan"); $v(k_t, c_t, t)$ is some function translating the instantaneous value of each variable into contributions to the objective function (i.e., a utility or profit function); c_t is a control variable. We let $R_t = \int_0^t r_s ds$ represent the cumulative discount rate.
- The first constraint says that k_t behaves dynamically according to $g(\cdot)$. This is commonly called a *law of motion*
- The second constraint specifies an initial condition for the state variable.
- The final constraint is what is known a no-ponzi constraint. Intuitively, it states that present value of the state variable must be positive in the limit. In a savings problem, this means that agents cannot rack-up infinite debt and thus consume an infinite amount (solutions would be boring if we allowed this).

Step 1/2

Notice that the law of motion for the state variable implies there are a *continuum* of constraints: at every instant, equation 20 applies with equality. You might be tempted therefore write down a Lagrangian that looks like:

$$\mathcal{L} = \int_0^\infty v(k_t, c_t, t) \cdot dt + \int_0^\infty \mu_t \left[g(k_t, c_t, t) - \dot{k}_t \right] dt + \lambda \cdot \left[lim_{t\to\infty} k_t \cdot e^{-R_t} \right] \tag{23}$$

Where we've replaced sums with integrals. This introduces another important concept:

Definition 3 *The* ***costate*** *variables, here μ_t are features of the optimization approach that capture the "shadow" value of the state variables in terms of present value utility.*

We can't just start solving first order conditions here, because the derivative of $\dot{k}$ is undefined. To begin getting rid of $\dot{k}$, note we can write the following:

$$\begin{aligned} \frac{\partial \mu k}{\partial t} &= \dot{\mu} k + \mu \dot{k} && (24) \\ \mu \dot{k} &= \frac{\partial \mu k}{\partial t} - \dot{\mu} k && (25) \\ \int_0^\infty \mu_t \dot{k}_t dt &= \lim_{s\to\infty} \mu_s k_s - \mu_0 k_0 - \int_0^\infty \dot{\mu}_t k_t dt && (26) \end{aligned}$$

This means we can write our Lagrangian attempt in equation 23 as:

$$\begin{aligned} \mathcal{L} = \int_0^\infty \left[v(k_t, c_t, t) + \mu_t \cdot g(k_t, c_t, t) \right] dt + \int_0^\infty \dot{\mu}_t k_t dt + \mu_0 k_0 - \lim_{s\to\infty} \mu_s k_s + \lambda \cdot \left[lim_{t\to\infty} k_t \cdot e^{-R_t} \right] && (27) \\ \mathcal{L} = \int_0^\infty \left[H(k_t, c_t, t) + \dot{\mu}_t k_t \right] dt + \mu_0 k_0 - \lim_{s\to\infty} \mu_s k_s + \lambda \cdot \left[lim_{t\to\infty} k_t \cdot e^{-R_t} \right] && (28) \end{aligned}$$

The Hamiltonian function, $H(k_t, c_t, t)$ makes some intuitive sense. We have transformed the optimization problem so that the choice of consumption influences the objective function directly, through v, and through the state variable appropriately priced using the costate variable.

Step 2/2

Now suppose that path of the control variable we've chosen is optimal. Appealing to Envelope theorem-type reasoning, any small change to the path of the state variable or control variable should not affect the value of our Lagrangian, since its derivative with respect to these variables equals zero on the optimal path. Take derivatives of this Lagrangian with respect to c_t and k_t. This gives us the necessary FOC for Hamiltonians:

$$\frac{\partial H}{\partial c_t} = 0 \tag{29}$$

$$\frac{\partial H}{\partial k_t} + \dot{\mu}_t = 0 \tag{30}$$

What about the TVC? In general, some sort of TVC will hold in an optimal solution whenever there is time discounting and bounded utility. This is why we have two proofs in the handout, for both a general and special case. Note that when working with the special case of time discounting (as we usually do in this course) an additional confusion is the distinction between *present value* and *current value* Hamiltonians. In the examples above, utility was discounted exponentially. Implicitly, the costate variable is as well, so that $\mu(t)$ gives the shadow price of the state variable in terms of utility today (at time 0). The Hamiltonian in this formulation is therefore known as a *present value* Hamiltonian.

Also, keep in mind that these are necessary conditions for an optimum, but in general not sufficient. It turns out, however, that just like with Kuhn-Tucker optimization, when the objective function is concave and the restrictions form a convex set, this dynamic approach is also sufficient.[4]

Using either form is a matter of preference. The optimal choices for state and control variables is always the same.

[4]See Acemoglu Chapter 7 for details.

Exercise 4 *Consider the continuous time version of the infinite-horizon savings problem described above, which can be stated as:*

$$\max_{c(t)} \int_0^\infty e^{-\rho t} \log(c(t))\ dt \tag{31}$$

$$s.t.\ \dot{a} = r \cdot a(t) + \bar{y} - c(t) \tag{32}$$

$$\lim_{T\to\infty} e^{-rT} a(T) \geq 0 \tag{33}$$

$$a(0) = 0 \tag{34}$$

Form the Hamiltonian, find the first order conditions for maximization and solve for the optimal consumption profile. As before, assume that $\rho = r$.

The Hamiltonian is:

$$H = e^{-\rho t} log(c(t)) + \lambda(t)\left[r \cdot a(t) + \bar{y} - c(t)\right] \tag{35}$$

The FOCs are:

$$e^{-\rho t}\frac{1}{c(t)} - \lambda(t) = 0 \tag{36}$$

$$\dot{\lambda}(t) = -\lambda(t) r \tag{37}$$

along with the asset accumulation equation and the transversality condition

$$\lim_{T\to\infty} \lambda(T) a(T) = 0 \tag{38}$$

Equation 36 tells us that $\lambda(t) = e^{-\rho t}\frac{1}{c(t)}$. Take the total derivative gives us that:

$$\dot{\lambda(t)} = -\rho e^{-\rho t} c(t)^{-1} - e^{-\rho t} c(t)^{-2} \dot{c(t)} \tag{39}$$

But since we know also from equation 37 that $\dot{\lambda}(t) = \lambda(t) r$, the assumption that $\rho = r$

instantly gives that $\dot{c} = 0$, so consumption is constant on the optimal path.[5]

If consumption is constant, then assets evolve according to:

$$\dot{a}(t) = r \cdot a(t) + \bar{y} - \bar{c} \tag{40}$$

This is a linear differential equation that can be solved explicitly:[6]

$$a(t) = \int_{i=0}^{t} e^{r(t-i)}(\bar{y} - \bar{c})di + e^{rt}a(0) \tag{41}$$

Now multiply each side by e^{-rt}, take limits as $t \to \infty$ and recall that $a(0) = 0$ to get the infinite horizon budget constraint

$$\lim_{T\to\infty} e^{-rT}a_T = \int_{i=0}^{\infty} e^{-ri}(\bar{y} - \bar{c})di \tag{42}$$

which upon rearranging gives us:

$$\bar{c} = \bar{y} - r \lim_{T\to\infty} e^{-rT}a_T \tag{43}$$

Finally, impose the transversality condition to get $\bar{c} = \bar{y}$.

[5]This should have been obvious given log utility, which turns out to be very special.

[6]If you don't know how to solve single first order linear differential equations with constant coefficients, you should invest some time in learning to do so.

Exercise 5 *Consider a similar setup, but in the context of a planner choosing the optimal path of consumption and capital accumulation for an economy with Cobb-Douglas production.*

$$\max_{c(t)} \int_0^{\infty} e^{-\rho t} \log(c(t))\ dt \tag{44}$$

$$s.t.\ \dot{k(t)} = k(t)^a - c(t) - \delta k(t) \tag{45}$$

$$k(0) = 1 \tag{46}$$

$$\lim_{t\to\infty} e^{-R(t)} k(t) \geq 0 \tag{47}$$

Form the Hamiltonian, find the first order conditions for maximization and solve for the optimal consumption profile. Assume that the instantaneous interest rate equals the marginal product of capital.

The Hamiltonian and its FOC are:

$$H = e^{-\rho t} log(c(t)) + \mu(t)\left[k(t)^a - c(t) - \delta k(t)\right] \tag{48}$$

$$e^{-\rho t}\tfrac{1}{c(t)} - \mu(t) = 0 \tag{49}$$

$$\dot{\mu}(t) = -\mu(t)\left[ak(t)^{a-1} - \delta\right] \tag{50}$$

Similarly to before, take logs and derivatives of equation 49 to get:

$$-\rho - \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)} \tag{51}$$

Then using equation 50, we have that:

$$\frac{\dot{c}(t)}{c(t)} = ak(t)^{a-1} - \delta - \rho \tag{52}$$

Look at this! The growth rate in consumption is the same as the net interest rate less the discount rate. We could proceed from here to find steady-state optimal levels of consumption and capital. We could also linearize the resulting nonlinear system an examine its stability and quantitative behavior.

Exercise 6 *Consider the following consumption saving problem as in the Ramsey-Cass-Koopman model from class:*[7]

$$\max_{C(t)} \int_0^\infty e^{-\rho t} \left(\frac{C(t)^{1-\theta} L(t)}{1-\theta} \right) dt \tag{53}$$

$$s.t.\ \dot{B(t)} = rB(t) - C(t)L(t) + W(t)L(t) \tag{54}$$

$$\lim_{t\to\infty} e^{-rt} B(t) \geq 0 \tag{55}$$

a. *Write out the Hamiltonian above. Then write the FONCs for H_x and H_y*

b. *Use these FONCs to derive an expression for consumption growth in terms of the model parameters (also known as the Euler Equation)*

For part a: the hamiltonian is

$$H \equiv e^{-\rho t} \left(\frac{C(t)^{1-\theta} L(t)}{1-\theta} \right) + \lambda(t) \left(rB(t) - C(t)L(t) + W(t)L(t) \right)$$

And the key FONCs are:

$$H_y = 0 \quad \Rightarrow \quad e^{-\rho t} C(t)^{-\theta} = \lambda(t) \tag{56}$$

$$H_x = -\dot{\lambda}(t) \quad \Rightarrow \quad r\lambda(t) = -\dot{\lambda}(t) \tag{57}$$

For part b: start by taking the derivative of consumption w.r.t. time, and manipulate:

$$\begin{aligned}
-\rho e^{-\rho t} C(t)^{-\theta} - \theta e^{-\rho t} C(t)^{-\theta-1} \dot{C}(t) &= \dot{\lambda}(t) \\
e^{-\rho t} C(t)^{-\theta} \left(\rho + \theta C(t)^{-1} \dot{C}(t) \right) &= -\dot{\lambda}(t) \\
e^{-\rho t} C(t)^{-\theta} \left(\rho + \theta \frac{\dot{C}(t)}{C(t)} \right) &= -\dot{\lambda}(t)
\end{aligned}$$

[7] This is as presented in POG's notes, but with $r(t) = r$ for simplicity.

Now noting that the outside term is equal to $\lambda(t)$, divide across

$$\begin{aligned} \rho + \theta \frac{\dot{C}(t)}{C(t)} &= -\frac{\dot{\lambda}(t)}{\lambda(t)} \\ \frac{\dot{C}(t)}{C(t)} &= \frac{r-\rho}{\theta} \end{aligned}$$

This is the continuous time version of the euler equation! Note that the parameter θ, the inverse of the elasticity of intertemporal substitution, governs how the growth rate of consumption responds to changes in r.