Economics 202A: Section 4

OLG Models

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Outline for Today:

1. Review unanticipated/anticipated shocks in the Ramsey model's phase diagram (continuous time)

2. Review of Diamond OLG and Derivation of Dynamic Inefficiency (discrete time)

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

1 Returning to Ramsey

For reference, the equations of the Ramsey model are

$$\frac{\dot{c}}{c} = \frac{(f'(k) - \delta - \rho - \theta g)}{\theta} \qquad (1)$$

$$\dot{k} = f(k) - c - (\delta + n + g)k \qquad (2)$$

→ Euler.

10M K_t

along with the initial and transversality conditions.

Exercise from last section: Suppose initially the Ramsey economy is in the steady state. Using phase diagrams, show how the economy responds to (i.e. how c, $\dot{c}$, k, and $\dot{k}$ react, do any loci shift, etc.)

1. A one-time unanticipated reduction in capital stock.
2. A one-time anticipated reduction in capital stock.
3. A one-time, unanticipated reduction in the depreciation rate.
4. A one-time, anticipated reduction in the depreciation rate.

permanent

For the next two questions suppose that you start from a point below the BGP

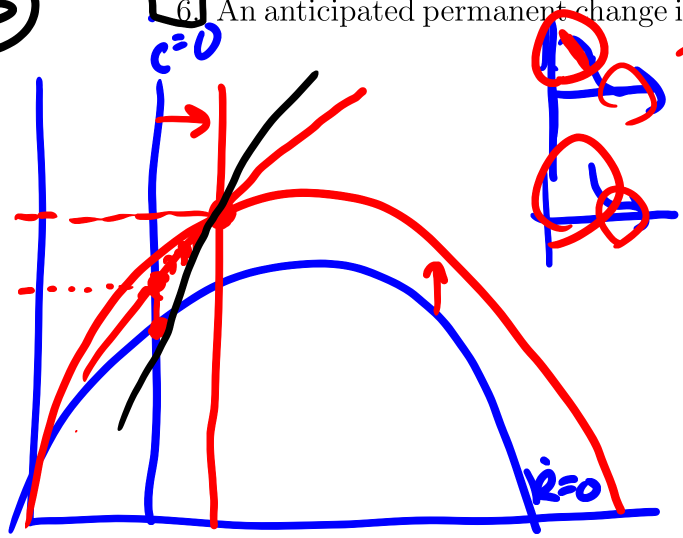
5. An unanticipated permanent change in consumption-smoothing preferences (θ goes down).

t=0 → learn about the shock

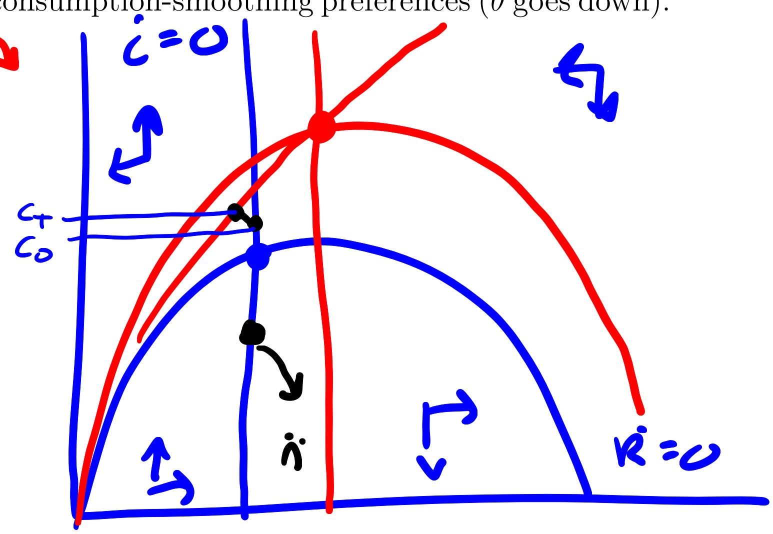
t=T → shock hits

6. An anticipated permanent change in consumption-smoothing preferences (θ goes down).

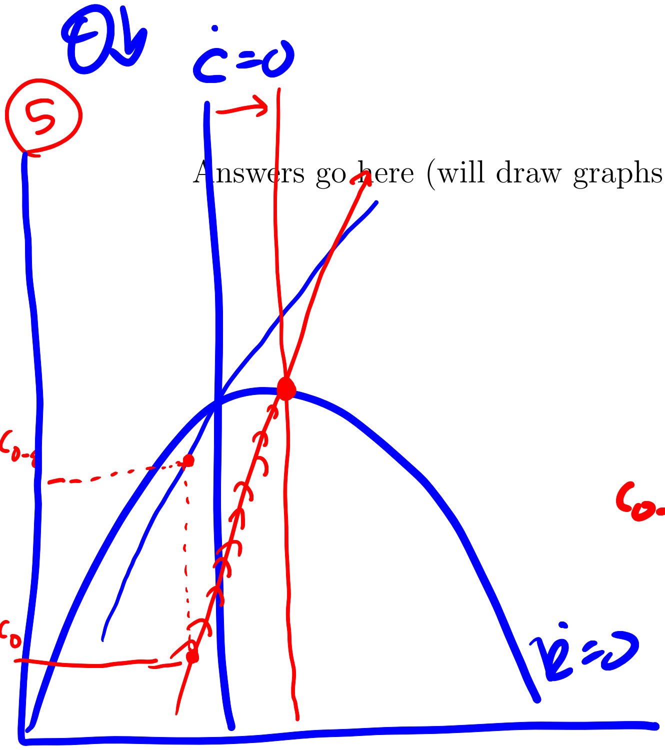
③

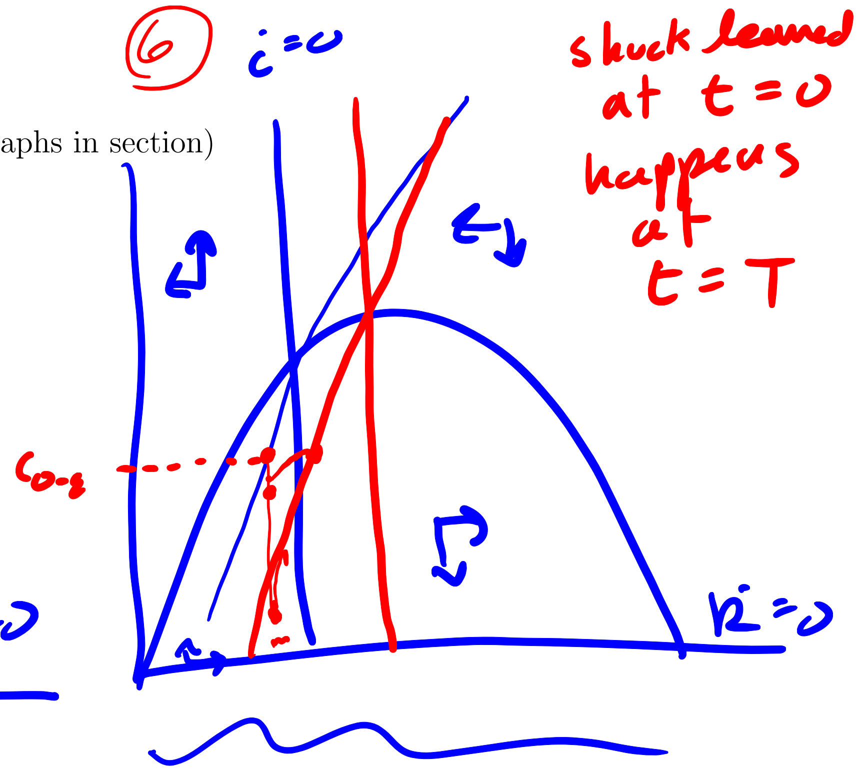


④



Answers go here (will draw graphs in section)





2 The Intertemporal budget constraint and Ricardian Equivalence

Here we review discrete time optimization and digress on "Ricardian Equivalence" in the Ramsey model So far we have formulated our optimization problems in terms of the *flow* budget constraint that maps current period income and expenditures into the change in assets carried into the next period. For our savings problem in discrete time, the budget constraint was

$$a_{t+1} - a_t = ra_t + \bar{y} - c_t \tag{3}$$

where a denotes savings (assets), r is the *net* real interest rate, y denotes income, and c denotes consumption. In continuous time, this can be written as

$$\dot{a} = ra(t) + \bar{y} - c(t) \tag{4}$$

When we solved this problem in the discrete time case, we iteratively substituted for a_{t+1} in this equation and applied the transversality condition to get

$$0 = \sum_{t=o}^{\infty}(1+r)^{-t}(\bar{y} - c_t) + (1+r)a_0 \tag{5}$$

which we can rearrange to get

$$(1+r)a_0 = \sum_{t=o}^{\infty}(1+r)^{-t}(c_t - \bar{y}) \tag{6}$$

This is the consumer's *intertemporal budget constraint* equating current assets to the present discounted value of the excess of consumption over income. Note that any path for c_t that satisfies this constraint is a feasible path, since every sequential budget constraint and the No-Ponzi scheme are satisfied. Thus we have replaced an infinite series of budget constraints with a single budget constraint involving an infinite discounted sum of consumption. We could now go on to solve the reformulated problem

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{7}$$

$$s.t.\ (1+r)a_0 = \sum_{t=o}^{\infty} (1+r)^{-t}(c_t - \bar{y}) \tag{8}$$

using the usual Lagrangian methods, and you can check that we would get the same answer as before.[1]

The assumption behind this formulation is that the *only* restriction on the pattern of consumer borrowing and spending is that, *in the long run*, they do not consume more than their income plus initial assets. There is no restriction on how much debt the consumer can run up in the short term. If the consumer did face such short term borrowing constraints we would have to include those as side constraints and they could potentially affect the solution.

The idea of Ricardian equivalence is that any transfer that affects both sides of equation (8) equally will leave the consumer facing the same constraint as before, and therefore will have no effect on the solution to the maximization problem. The canonical example is when the government gives you d_0 in bonds at time 0, which pay interest rate r, but will hit you with a sequence of taxes τ_t such that the bond is eventually paid off. How does this change the budget constraint (8)? We add d_0 to the LHS and $\sum_{t=0}^{\infty}(1+r)^{-t}\tau_t$ to the RHS to get

$$d_0 + (1+r)a_0 = \sum_{t=o}^{\infty} (1+r)^{-t}(c_t - \bar{y}) + \sum_{t=0}^{\infty} (1+r)^{-t}\tau_t \tag{10}$$

But since $d_0 = \sum_{t=0}^{\infty}(1+r)^{-t}\tau_t$ we could just subtract d_0 from both sides and we would just have our original constraint. Since the optimization problem has not changed, the optimal consumption path has not changed.

What if the consumer faced a non-negativity constraint on assets, i.e. no borrowing, and

[1]In continuous time the intertemporal budget constraint takes the form

$$a(0) = \int_0^{\infty} e^{-rt}(c(t) - \bar{y})dt \tag{9}$$

Make sure you know how to solve a linear differential equation so you can derive this on an exam.

suppose further that $y_0 < y_t = \bar{y} \; \forall t > 0$ and that $a_0 = 0$? We know that if the non-negativity constraints on assets do not bind, the Euler equation tells us that $c_t = \bar{c}$. If we went ahead and solved the problem assuming this constraint did not bind, we would find

$$\bar{c} = \frac{ry_0 + \bar{y}}{1 + r} \tag{11}$$

But this would require borrowing in period 0 because $\bar{c} > y_0$, so this is not feasible. So the optimal policy is to consume y_0 in the first period and $\bar{y}$ thereafter.

Now suppose the government implements its scheme and that $d_0 \geq \bar{y} - y_0$. The consumer may now implement her unconstrained optimal plan of consuming $\bar{c}$ from equation (11) because she has enough assets in period 0 to finance her extra consumption in that period. Thus the timing of government debt and taxes can potentially affect whether or not borrowing constraints bind, and thus affect consumption behavior through this channel. Government debts provide "liquidity" in this model. The role of government debt in providing market liquidity is an active area of research in macroeconomics.

3 Basics of the OLG model

Here we review the Diamond Overlapping Generations (OLG) model. Each period, L_t young are born, each endowed with one unit of labor income. L_t grows at $L_t = (1+n)L_{t-1}$. People live for two periods, but only supply labor when young. Thus young consumers in every period solve

$A_t = (1+g)A_{t-1}$

$$\max_{c_t^y, c_{t+1}^o} = u(\tilde{c}_t^y) + \frac{1}{1+\rho}u(\tilde{c}_{t+1}^o) \tag{12}$$

$$s.t. \quad c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} = w_t A_t \tag{13}$$

whereas the old simply consume their interest income and their capital stock in every period (after the capital has been used to produce output).[2] Output is produced using the production function $f(k_t)$ and factors earn their marginal product.

How can we analyze this model? Does a BGP exist, like in the Solow model?

1 $K_t = (W_{t-1} - C^y_{t-1})L_{t-1}$ denote $\boxed{s(r_t) \equiv \frac{W_{t-1} - C^y_{t-1}}{W_{t-1}}}$

claims

$\Rightarrow \frac{K_t}{A_t L_t} = \frac{s(r_t) W_{t-1} L_{t-1}}{A_t L_t} \cdot \frac{A_{t-1}}{A_{t-1}}$

Resource Constraint $\left\{ \boxed{k_t = \frac{s(r_t)\,\omega_{t-1}}{(1+g)(1+n)}} \right.$ where $\omega_t \equiv \frac{W_t}{A_t}$

Firm optimality $f'(k_t) = r_t$ and $f(k) - f'(k)k_t = \omega_t$

HH optimality $\mathcal{L}: u(c^y_t) + \frac{1}{1+\rho}u(c^o_{t+1}) - \lambda\left(c^y_t + \frac{c^o_{t+1}}{1+r_{t+1}} - w_t A_t\right)$

FOC c^y_t: $u'(c^y_t) = \lambda$

FOC c^o_{t+1}: $\frac{u'(c^o_{t+1})}{1+\rho} = \frac{\lambda}{1+r_{t+1}}$

$\Rightarrow \boxed{u'(c^y_t) = \frac{1+r_{t+1}}{1+\rho}u'(c^o_{t+1})}$ EE

FOC: $\lambda \Rightarrow c^y_t + \frac{c^o_{t+1}}{1+r_{t+1}} = w_t A_t$

CRRA $c_t^{y\,-\theta} = (\cdot)\, c_{t+1}^{o\,-\theta}$

Assume: $u(c) = \frac{c^{1-\theta}}{1-\theta}$ $\longrightarrow$ $\frac{c^o_{t+1}}{c^y_t} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}$

[2]Notice also the assumption on interest rates. Here we assume interest is acrued on savings at the start of the period, which is obvious if you write the budget constraint as: $c^o_{t+1} = (w_t - c^y_t)(1+r_{t+1})$.

plug (EE) into the BC:

$$c_t^y + \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \frac{c_t^y}{1+r_{t+1}} = w_t A_t$$

$$\Rightarrow c_t^y \left(1 + \frac{(1+r_{t+1})^{\frac{1}{\theta}-1}}{(1+\rho)^{\frac{1}{\theta}}}\right) = W_t$$

[Extra space if needed]

$$\Rightarrow \frac{c_t^{y_t}}{w_t} = \frac{1}{(\cdot)}$$

$$\Rightarrow 1 - s(r_{t+1}) = \left(1 + \frac{(1+r_{t+1})^{\frac{1}{\theta}-1}}{(1+\rho)^{\frac{1}{\theta}}}\right)^{-1}$$

note: $s(r_t) = \frac{w_{t-1} - c_{t-1}^y}{w_{t-1}}$

$s(r_t) = 1 - \frac{c_{t-1}^y}{w_{t-1}}$

$1 - s(r_t) = \frac{c_{t-1}^y}{w_{t-1}}$

assume log utility $\theta \to 1 \Rightarrow s(r_{t+1}) = \frac{1}{2+\rho}$

Assume: $f = k^\alpha$

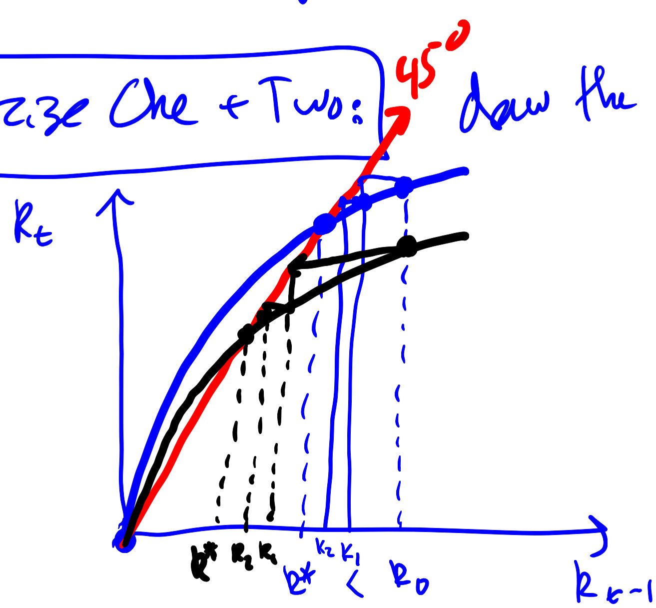
$$\Rightarrow k_t = \frac{w_{t-1}}{(2+\rho)(1+n)(1+g)} = \frac{f(k_{t-1}) - f'(k_{t-1})k_{t-1}}{(2+\rho)(1+n)(1+g)}$$

$$\Rightarrow k_t = \frac{k_{t-1}^\alpha - \alpha k_{t-1}^{\alpha-1} \cdot k_{t-1}}{(\cdot)}$$

$$\Rightarrow \boxed{k_t = \frac{(1-\alpha)k_{t-1}^\alpha}{(2+\rho)(1+n)(1+g)}}$$

② → draw the phase diagram!

Exercise One + Two: draw the diagram:



Consider an permanent unanticipated increase in ρ.

start at same $k_0 > k^*$!

Exercise 1 *Draw the phase diagram for the OLG model with log utility and Cobb-Douglass production, find the steady state and characterize the dynamics of the system.*

We can use this diagram to analyze the impact of changes in parameters, similar to the Ramsey phase diagram.

Exercise 2 (Midterm 07) *Consider an economy described by the Diamond overlapping generations model in the special case of logarithmic utility and Cobb-Douglas production. Assume that initially k is above its balanced-growth-path level. Now suppose there is an unexpected permanent increase in agents' discount factor ρ. Sketch the resulting paths of k, and what that path would have been if ρ had not changed. Explain your answers.*

Exercise 3 (Midterm 06) *In the Diamond OLG model with log utility and Cobb-Douglas production, analyze how changes in n,g,α and ~~β~~ ρ affect the likelihood of an economy becoming dynamically inefficient. Give the intuition for the impact of each change.*

Hint: proceed in two steps. First, calculate the Golden Rule capital stock in this model (try working from $K_{t+1} = F(K_t, A_tL_t) - C_t + K_t$ where C is aggregate consumption across both generations). You should get a condition for the marginal product of capital. Then, compare this condition to the marginal product of capital on the BGP of this model.

(1) $$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{F(K_t, A_tL_t) - C_t + K_t}{A_{t+1}L_{t+1}}$$

$$k_{t+1} = \frac{f(k_t) - c_t + k_t}{(1+n)(1+g)}$$

$$(1+n)(1+g)k^* = f(k^*) - c^* + k^*$$

$$\Rightarrow \quad c^* = f(k^*) + \big(1-(1+n)(1+g)\big)k^*$$

so k^{gr}: $$0 = f'(k_{gr}) + \big(1-(1+n)(1+g)\big) \Leftrightarrow \boxed{f'(k_{gr}) = (1+n)(1+g) - 1}$$

now: k^* in diamond: $$k_t = \frac{(1-\alpha)}{(2+\rho)(1+n)(1+g)} k_{t-1}^{\alpha} \quad \text{on a BGP: } (k^*)^{1-\alpha} = \frac{1-\alpha}{(2+\rho)(1+n)(1+g)}$$

$$\Rightarrow \quad k^* = \left(\frac{1-\alpha}{(2+\rho)(1+n)(1+g)}\right)^{\frac{1}{1-\alpha}}$$

or: $$f'(k^*) = \alpha(k^*)^{\alpha-1} = \alpha\left(\frac{1-\alpha}{(\cdot)}\right)^{\frac{\alpha-1}{1-\alpha}} = \boxed{\frac{\alpha}{1-\alpha}(2+\rho)(1+n)(1+g) = f'(k^*)}$$

$$k^* > k_{gr} \Leftrightarrow f'(k^*) < f'(k_{gr})$$

$$\Leftrightarrow \frac{\alpha}{1-\alpha}(2+\rho)(1+n)(1+g) < (1+n)(1+g) - 1$$

if $$\boxed{\frac{\alpha}{1-\alpha}(2+\rho) < 1 - \frac{1}{(1+n)(1+g)}}$$ then

diamond is dynamically inefficient

Exercise 4 *In the OLG model with no depreciation, which of the following is not correct:*

1. *If the growth rate of population and exogenous technical change are zero, the standard competitive equilibrium allocation will not be dynamically inefficient.*
2. *If the production function is Cobb-Douglas, a lower capital elasticity makes it more likely that the competitive equilibrium will overaccumulate capital.*
3. *If individuals do not discount over time, the standard competitive equilibrium allocation will not be dynamically inefficient.*

4 Money and Taxes in the OLG Model

The original OLG models of Allais and Samuelson were even simpler than the Diamond version of the model, and can also be used to illustrate the potential inefficiency of equilibria and the scope for monetary and fiscal policy to improve the allocation.

- Assume log utility, no discounting ($\rho = 0$), and no production: each young person is endowed with A units of the consumption good in period t, which they may either consume or store. Initial old are endowed with Z units of the good.
- storing S of the good yields xS of the good the next period.
- Population grows at rate n, and assume $x < 1 + n$.[3]

Exercise 5 *Solve the optimization problem solved by the young, and answer the following:*

1. *Describe the decentralized equilibrium of the economy. Is it efficient?*
2. *Describe a tax and transfer scheme carried out by an infinitely lived government that improves welfare.*

[3] This corresponds to the Diamond model with $g = 0$ and $F(K, L) = AL + xK$.

Exercise 6 *(Extra) For this problem, assume that the initial old are also endowed with a quantity M of a storable good called money that does not yield utility.*

1. *Suppose money can be exchanged for goods at price P_t per good. What is an individual's behavior as a function of P_t/P_{t+1}?*
2. *Show there is an equilibrium with $P_{t+1} = P_t/(1+n)$ for all t. How much is stored? Is this equilibrium efficient?*
3. *Show there is an equilibrium with $P_{t+1} = P_t/x$ for all t. How much is stored? Is this equilibrium efficient?*
4. *Show there is an equilibrium with $P_t = \infty$, that is, money is worthless.*
5. *Which of these equilibria will actually occur?*

OFFICE HOURS

$D(t) =$

$$\boxed{\theta_t \equiv \frac{B_t}{D_t}}$$

$$\boxed{D(t) = B(t) + K(t)}$$

$$D(t) = \theta(t)\,D(t) + K(t)$$

$$(1-\theta(t))\,D(t) = K(t)$$

$$\boxed{\dot{D}(t) = r^K(1-\theta(t)) + r^B\,\theta(t)\,D(t)}$$

$$+w(\theta)\,\ell(t) - c(t)$$