Problem 1

Expand $\dot{\ln k}$ around $\ln k^*$ using a first-order Taylor series approximation.

$$\dot{\ln k} \approx \dot{\ln k}|_{\ln k^*} + \left.\frac{\partial \dot{\ln k}}{\partial \ln k}\right|_{\ln k^*} (\ln k - \ln k^*) \tag{1}$$

$$\dot{\ln k}|_{\ln k^*} = \left.\frac{\dot{k}}{k}\right|_{k^*} = 0 \tag{2}$$

$$\dot{\ln k} = \frac{\partial \ln k}{\partial t} = \frac{\partial \ln k}{\partial k}\frac{\partial k}{\partial t} = \frac{1}{k}\cdot \dot{k} = \frac{sf(k)}{k} - (n+g+\delta) \tag{3}$$

$$k = e^{\ln k} \implies \dot{\ln k} = \frac{sf(e^{\ln k})}{e^{\ln k}} - (n+g+\delta) \tag{4}$$

Let $y = \dot{\ln k}$ and $x = \ln k$, then

$$y = se^{-x}f(e^x) - (n+g+\delta) \tag{5}$$

$$\frac{\partial \dot{\ln k}}{\partial \ln k} = \frac{\partial y}{\partial x} = s(-e^{-x}f(e^x) + e^{-x}e^x f'(e^x)) = s\left(f'(k) - \frac{1}{k}f(e^x)\right) \tag{6}$$

So (1) becomes

$$\dot{\ln k} \approx \ln\left(\frac{k}{k^*}\right) s\left(f'(k^*) - \frac{1}{k^*}f(k^*)\right) \tag{7}$$

And at k^*, $\dot{k} = 0$ so we know $\frac{f(k^*)}{k^*} = \frac{(n+g+\delta)}{s}$, so

$$= \ln\left(\frac{k}{k^*}\right)(sf'(k^*) - (n+g+\delta)) \tag{8}$$

Problem 2

Does consumption change discontinuously if we know in advance that have our wealth will be confiscated at t_0?

Yes. Because we have the ability to influence the amount of our wealth that is confiscated, a utility-maximizing household would increase consumption before t_0. Since decreasing their consumption right before t_0 to meet the consumption at t_0 would not provide any benefit, it is rational to have discontinuous jump at t_0, while consumption would rise back to the the utility optimizing path after t_0 as if the household has begun time with the amount of wealth they have left over at t_0.

From the Euler equation and the definition of a derivative, we have

$$\frac{\partial \dot{\ln C}}{\partial t} = \frac{1}{\theta}(f'(k) - (n+g+\delta)) \tag{9}$$

$$\implies \ln\left(\frac{C(t+\delta)}{C(t-d)}\right) = \frac{2\delta}{\theta}(f'(k) - (n+g+\delta)) \tag{10}$$

for some small $\delta > 0$

Does consumption change discontinuously if we know in advance that a lump sum equaling half the average household wealth will be confiscated at t_0?

No. Because the household knows from the beginning that a fixed amount will be taken at t_0 (an amount they do not have control over, since they have essentially no significant contribution to the average), they can plan out their lifetime consumption to be smooth.

Problem 3

(a)
- Assume that $G(t)$ is continuous.
- Assume that $G_0(t) = 0$ is not the utility maximizing path.
- Then there exists $G_1(t)$ with $G_1(t_1) \neq 0$ for some $t_1 \in \mathbb{R}$.
- Then $G_1(t_1)^2 > 0$ and $\int_0^\infty e^{-\rho t}\left[-\frac{a}{2}G_1(t)^2\right] dt < 0$.
- But $\int_0^\infty e^{-\rho t}\left[-\frac{a}{2}0\right] dt = 0$.
- So $G_0(t) = 0$ would result in a larger objective function.
- Thus, $G(t) = G_0(t) = 0$ would always result in the maximized objective function.

(b) $\mathcal{H}(t) = -\dfrac{a}{2}G(t)^2 + \mu(t)G(t)$

(c)

First condition:

$$\frac{\partial H}{\partial G} = 0 = -aG(t) + \mu(t) \implies \mu(t) = aG(t) \tag{11}$$

Second condition:

$$\frac{\partial H}{\partial T} = \rho\mu - \dot{\mu} = \frac{\partial H}{\partial t}\frac{\partial t}{\partial T} = \frac{\partial H}{\partial t}\left(\dot{T}\right)^{-1} = \frac{\partial H}{\partial t}\frac{1}{G} \tag{12}$$

$$\frac{\partial H}{\partial t} = -aG\dot{G} + \mu G + \dot{\mu}G \tag{13}$$

Substituting in the result from the first condition...

$$= -aG\dot{G} + aG\dot{G} + aG\dot{G} = aG\dot{G} \tag{14}$$

$$\text{So } \rho\mu - \dot{\mu} = [aG\dot{G}]\frac{1}{G} \tag{15}$$

$$\rho aG - a\dot{G} = [aG\dot{G}]\frac{1}{G} \tag{16}$$

$$\implies \dot{G} = \frac{\rho}{2}G \tag{17}$$

$$\implies G(t) = ce^{\frac{\rho}{2}t} \text{ for some } c \in \mathbb{R}. \tag{18}$$

(d)

Transversality condition:

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)T(t) = 0 = \lim_{t\to\infty} e^{-\rho t}aG(t)T(t) \tag{19}$$

$$T(t) = \int_0^t G = \frac{2c}{\rho}(e^{\frac{\rho}{2}t} - 1) \tag{20}$$

$$\lim_{t\to\infty} e^{-\rho t}aG(t)T(t) = \lim_{t\to\infty} e^{-\rho t}ace^{\frac{\rho}{2}t}\frac{2c}{\rho}(e^{\frac{\rho}{2}t} - 1) \tag{21}$$

$$= \lim_{t\to\infty}\frac{2ac^2}{\rho}(1 - e^{\frac{\rho}{2}t}) \tag{22}$$

Since $a, \rho > 0$, this can only go to 0 if $c = 0 \implies G(t) = 0$

(e) The class of solutions $G(t) = ce^{\frac{\rho}{2}t}$ presents a rate of garbage production where the individual gets the same utility from producing a unit of garbage in every time period (since the optimizing function results in a constant value). Utility-smoothing is often an optimizing behavior. But the Transversality Condition tells us either the rate of garbage accumulation (G) or the entire stock of garbage (T) goes to 0, and because we are limiited to the family of increasing exponentions, both must be 0.

Problem 4

Flow Budget Constraint

BC:

4 $A(0)=0,\quad Y=\bar{Y}>0,\quad r=\bar{r}>0,\quad \dot{A}(t)=\bar{r}\cdot A(t)+\bar{Y}-C(t)$

$$\int_0^\infty e^{-rt}C(t)\,dt \le \int_0^\infty e^{-\rho t}\bar{Y}\,dt + \int_0^\infty A(0)(1+r)^t$$

$$\int_{t=0}^{\infty} e^{-\rho t}\left[u(C(t)) + v(A(t))\right]dt;\quad u',v'>0;\quad u'',v''<0;\quad \rho>0$$

a $\rho=\bar{r}$: indifferent between consuming now and saving to consume later.
but the maximizing decision will depend on u' and v'. Given the Inada conditions though, I would expect both consumption and savings to be >0 at $t=0$ since their marginal utility is so large at $t=0$.

Thus, $C(0)>0$ and $\dot{A}(0)>0$ so $C(0)<\bar{Y}$.

b $\mathcal{H}(t) = u(C(t)) + v(A(t)) + \mu(t)\cdot\dot{A}(t)$

$= u(C) + v(A) + \mu\left[\bar{r}A + \bar{Y} - C\right]$

$\boxed{\bar{Y}+\bar{r}A(t) = \dot{A}(t) + C(t)}$ State Variable : $A(t)$

$C(t) = \bar{Y} + \bar{r}A(t) - \dot{A}(t)\qquad C \perp\!\!\!\perp A$

state and control var. are ind. so $\frac{\partial A}{\partial c}=0$

c $\frac{\partial H}{\partial c}(t)=0\ \forall t\qquad \frac{\partial H}{\partial c} = u'(c) + \mu\frac{\partial \dot{A}}{\partial c} = u'(c) - \mu = 0 \Rightarrow \boxed{u'(c)=\mu} \Rightarrow \dot{\mu} = \frac{\partial u'(c(t))}{\partial t} = \frac{\partial u'}{\partial c}\frac{\partial c}{\partial t} = u''(c)\dot{c}$

$\frac{\partial \mu}{\partial t} = \frac{\partial}{\partial t}\frac{\partial u}{\partial c} =$

$\frac{\partial H}{\partial A} = \rho\mu - \dot{\mu}\qquad \frac{\partial H}{\partial A} = v'(A) + \mu\frac{\partial \dot{A}}{\partial A} = v'(A) + \mu\bar{r} = \rho\mu - \dot{\mu} = \rho\cdot u'(c) - u''(c)\cdot\dot{c}$

$v'(A) + u'(c)\bar{r} = \rho u'(c) - u''(c)\cdot\dot{c}\qquad -u''(c)\cdot\dot{c} = u'(c)(\bar{r}-\rho) + v'(A)$

$\lim_{t\to\infty} e^{-\rho t}\mu(t)A(t) = 0 = \lim_{t\to\infty} e^{-\rho t}u'(c(t))A(t)\qquad \frac{\dot{c}}{c} = \frac{1}{c\cdot u''(c)}\left[u'(c)(\rho-\bar{r}) - v'(A)\right]$

Acknowledgements: Although I worked through all problems on my own, I then worked worked several others to compare reasoning and find math errors: thanks to Lucy Hackett, Simon Greenhill, Alice Huang, Kendra Marcoux, Shreya Chandra, Shreya Sarkar, Trevor Woolley, and Liyuan Zhang for thoughts and long discussions.