Section 01 : Mathematics of Dynamic Programming

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Outline

1. Dynamic Programming
    1. Sequence Problem to Functional Problem
2. Policy Function
3. When does the (FE) problem have a solution?
    - Contraction Mapping Theorem
    - Blackwell's Sufficient Condition
4. How to Solve the Problem?
    - Guess and Verify Method
    - Functional Euler Equation
5. Optimal Stopping Problem

Optimal Growth Problem, Sequence Problem

- **Problem 1 (SP)**

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$\text{s.t.} \quad c_t + k_{t+1} \leq f(k_t), \; c_t, k_{t+1} \geq 0, \quad \forall \; t$$
$$\text{given } k_0$$

which is equivalent to

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$
$$\text{s.t.} \quad 0 \leq k_{t+1} \leq f(k_t) \;\; \forall \; t$$
$$\text{given } k_0$$

From the Sequence Problem to Functional Equation

1. Sequence Problem (SP)
   - The problem in the prievous slide is referred to as the **sequence problem**.
   - We would like to choose an infinite sequence $\{k_{t+1}\}_{t=0}^{\infty}$ from some space of infinite sequences, $\{\tilde{k}_{t+1}\}_{t=0}^{\infty}$
2. **Dynamic Programming**: SP $\rightarrow$ FE
   - Here, we would like to transform a sequence problem to a functional equation.
   - Instead of finding a sequence, we would like to find a function, as this often gives better insights, similar to the logic of comparing today and tomorrow.
   - This formulation is sometimes easier to characterize analytically and numerically.

From the Sequence Problem to Functional Equation

$$V^*(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}), \quad \text{s.t.} \qquad k_{t+1} \in \Gamma(k_t) \;\; \forall \; t$$

$$V^*(k_0) = \max_{k_{t+1} \in \Gamma(k_t) \forall t} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

$$= \max_{k_1 \in \Gamma(k_0)} \left[U(f(k_0) - k_1) + \max_{k_{t+1} \in \Gamma(k_t) \forall t} \sum_{t=1}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \right]$$

$$= \max_{k_1 \in \Gamma(k_0)} \left[U(f(k_0) - k_1) + \max_{k_{(t+1)+1} \in \Gamma(k_{t+1}) \forall t} \sum_{t=0}^{\infty} \beta^{t+1} U(f(k_{t+1}) - k_{(t+1)+1}) \right]$$

$$= \max_{k_1 \in \Gamma(k_0)} \left[U(f(k_0) - k_1) + \beta \max_{k_{(t+1)+1} \in \Gamma(k_{t+1}) \forall t} \sum_{t=0}^{\infty} \beta^t U(f(k_{t+1}) - k_{(t+1)+1}) \right]$$

$$= \max_{k_1 \in \Gamma(k_0)} \left[U(f(k_0) - k_1) + \beta V^*(k_1) \right]$$

- **Problem 2 (FE)**

$$V(k_0) = \max_{k_1} \; \{U(f(k_0) - k_1) + \beta V(k_1)\}$$
$$\text{s.t.} \quad k_1 \in \underbrace{\Gamma(k_0)}_{=\text{Feasible Set}} = [0, f(k_0)], \quad \text{given } k_0$$

- We are now solving for a **function** V not a **sequence** $\{k_{t+1}\}_{t=0}^{\infty}$
- Here, instead of finding a sequence $\{k_{t+1}\}_{t=0}^{\infty}$, we would like to find a function $V(k)$, for all k, satisfying the Bellman equation defined as previous.
- Because the function V is defined **recursively**, this is often referred to as the **recursive formulation**.

Remarks on Bellman Equation

- **Bellman Equation**

$$V(k) = \max_{k'} \left\{ U(f(k) - k') + \beta V(k) \right\}$$
$$\text{s.t. } \; k' \in \Gamma(k) = [0, f(k)]$$

- Note that $V(k)$ is a function. $V : K \to \mathbb{R}$.
- V does NOT depend on t. This implies that the above equation is **stationary.**
  - Note that finite-horizon problem is **non-stationary**, *i.e.* V depends on t, as it depends on the remaining time periods.
- How should we put $V(K)$ into the computer?

Bellman Equation

- **Bellman Equation**

$$V(k) = \max_{k'} \ \left\{ U(f(k) - k') + \beta V(k) \right\}$$
$$\text{s.t.} \quad k' \in \Gamma(k) = [0, f(k)]$$

  - We are now solving for a **function** V(k) not a **sequence** $\{k_{t+1}\}_{t=0}^{\infty}$
  - Here, instead of finding a sequence $\{k_{t+1}\}_{t=0}^{\infty}$, we would like to find a function $V(k)$, for all k, satisfying the Bellman equation defined as previous.
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Policy Function

- This problem generates the solution, which is often called **policy function**, $G : K \to K$, determining the value of k' given k.

$$G(k) = \arg\max_{k' \in \Gamma(k)} \left[U(f(k) - k') + \beta V(k') \right]$$

$$V(k) = U(f(k) - G(k)) + \beta V(G(k)), \quad \forall \ k \in K$$

Policy Function

Are the solutions of the two problems equivalent?

$$V^*(k_0) = \max_{\{0 \leq k_{t+1} \leq f(k_t)\}_{t=0^\infty}} \sum_{t=0}^{\infty} \beta^t \underbrace{F(k_t, k_{t+1})}_{=U(f(k_t) - k_{t+1}))}, \quad \forall \;\; t = 0, 1, \ldots, \text{ given } k_0$$

$$V(k) = \max_{k' \in \Gamma(K)} \{F(k, k') + \beta V(k')\}$$

Want: Under what conditions, the two are equivalent?

Equivalence of Value Function and Policy Function

Want: Under what conditions, the two are equivalent?

To answer this, we'll need **some definitions**:

- A set of **feasible sequences** or plans with initial value of k_0:

$$\Pi(k_0) = \{\{k_{t+1}\}_{t=0}^{\infty} : k_{t+1} \in \Gamma(k_t),\ t = 0, 1, \cdots ,\}$$

- $\mathbf{k} = \{k_0, k_1, \cdots\} \in \Pi(k_0)$.
- Let $V^*(k_0)$ be the maximum in Problem 1. (SP).
- Let $V(k_0)$ be the solution of the Problem 2. (FE).

Equivalence of Value Function and Policy Function

Equivalence Between Value Functions

- **(Assumption 1)**
  - $\Gamma(k)$ is nonempty for all $k \in K$.
  - For all $k_0 \in K$, and $\mathbf{k} \in \Pi(k_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(k_t, k_{t+1})$ exists.
- Let K, Γ, F, and β satisfy (Assumption 1). Then

$$V^*(k_0) = \sup_{\mathbf{k}\in\Pi(k_0)} \sum_{t=1}^{\infty} \beta^t F(k_t, k_{t+1})$$

  is the **unique** solution, to Problem 2 (SP), though there can be many $\mathbf{k}$ that achieve this.

Equivalence of Value Function and Policy Function

Equivalence Between Value Functions

- Let K, Γ, F, and β satisfy (Assumption 1). If V (which is **a** solution of (FE)) satisfies

$$\lim_{n \to \infty} \beta^T V(k_T) = 0, \quad \text{for all } \{k_0, k_1, \cdots\} \in \Pi(k_0), \ \ \forall\, k_0 \in K,$$

then $V = V^*$.

Equivalence of Value Function and Policy Function

Equivalence Between Policy Functions (Principle of Optimality)

- Let K, Γ, F, and β satisfy (Assumption 1). Let $\mathbf{k}^* \in \Pi(k_0)$ be a feasible plan that attains the maximum in Problem 1 (SP), starting with k_0. Then

$$V^*(k_t^*) = F(k_t^*, k_{t+1}^*) + \beta V^*(k_{t+1}^*), \quad t = 0, 1, 2 \ldots$$

Moreover, if any $\mathbf{k}^* \in \Pi(k_0)$ satisfies the above, then it attains the optimal value in

$$V(k^*) = F(k^*, k'^*) + \beta V(k'^*)$$

Equivalence of Value Function and Policy Function

Equivalence Between Policy Functions (Principle of Optimality)

$$\begin{aligned}
V(k_0^*) &\geq F(k_0^*, k_1^*) + \beta V(k_1^*) \ \ \text{(FE)} \\
&\geq \sum_{t=1}^{T} \beta^t F(k_t^*, k_{t+1}^*) + \beta^{T+1} V(k_{T+1}^*) \\
\lim_{T\to\infty} V(k_0^*) &\geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(k_t^*, k_{t+1}^*) + \underbrace{\lim_{T\to\infty} \beta^{T+1} V(k_{T+1}^*)}_{=0 \ \ \text{By Assumption}} \\
&= \sum_{t=0}^{\infty} \beta^t F(k_t^*, k_{t+1}^*) \\
&= V^*(k_0^*) \ \ \text{(SP)}
\end{aligned}$$

- Must be the same solution since $V^*(k_0^*)$ is unique

When Does a Solution Exist?

- To that end, let's formulate the Bellman Equation as a Mapping:

$$W(k) = T\left[V(k)\right]$$
$$T[V(k)] = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\}$$

  which maps function V to a new function W.

- Then, a **Fixed Point** of T, $T[V] = V$, will be a solution to this above problem.

When Does a Solution Exist?

- To see the existence of the solution, we need more assumption.
- **(Assumption 2)**
  - K is a compact subset of $\mathbb{R}^K$.
  - Γ is nonempty, compact-valued and continuous.
  - Moreover, let $A = \{(k, y) \in K \times K : y \in \Gamma(k)\}$ and $F : A \to \mathbb{R}$ be bounded and continuous.
- Remarks on Assumption 2
  - This assumption allows us to focus on the space of **bounded functions**, which implies that $V^*(K)$ is bounded as well.
  - Therefore, we can focus our attention on value functions in the space of $\mathbf{B}(K)$ of continuous bounded functions defined on K, with the natural norm on this space, the maximum norm (sup norm), $||f|| = \max_{k \in K} |f(K)|$.

When Does a Solution Exist?

Let's look at the **Result** first:

- Let K, Γ, F, and β satisfy (Assumption 1) and (Assumption 2), and let $\mathbf{B}(X)$ be the space of bounded continuous functions $f : K \to \mathbb{R}$ with the sup (maximum) norm. Then the operator T maps $\mathbf{B}(K)$ into itself, *i.e.*

  $$T : \mathbf{B}(X) \to \mathbf{B}(X),$$

  and has a unique fixed point, $V \in \mathbf{B}(K)$, satisfying

  $$V(k) = \max_{k' \in \Gamma(k)} \left[F(k, k') + \beta V(k') \right]$$
  $$T[V(k)] = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\}$$

When Does a Solution Exist?

- To get to this **Result**, we need to know two (actually three) things:
  1. Berge's Maximum Theorem
     - Is T well-defined?
     - *i.e.* Does the **maximization problem** of the RHS have a solution?
  2. Contraction Mapping Theorem
     - Contraction Mapping
     - Blackwell's Sufficient Condition

Berge's Maximum Theorem

Let (X, d_x), (Y, d_Y) be metric spaces. Consider the maximization problem:

$$\max_{y \in Y} f(x, y) \;\; \text{s.t. } y \in \Gamma(x)$$

where $\Gamma : X \rightrightarrows Y$ and $f : X \times Y \to \mathbb{R}$. Suppose that f is continuous and Γ is compact-valued and continuous at x. Then

1. $M(x) = \max\limits_{y \in Y} \{f(x, y) : y \in \Gamma(x)\}$ exists and is conintuous at x, and
2. $G(x) = \arg\max\limits_{y \in Y} \{f(x, y) : y \in \Gamma(x)\}$ is non-empty-valued, compact-valued, and upper hemicontinuous.
    - Note that upper-hemicontinuity is similar concept to examine continuity properties with *correspondence.*
    - Note that if $G(\cdot)$ is a function, then it is a continuous function since upper hemicontinuous functions are continuous.

Berge's Maximum Theorem

What does this mean?

- When solving a constrained optimization problem, if
  - The objective function is continuous, and
  - The correspondence defining the constraint set is continuous, compact, and non-empty,

  then,

  - The problem has a solution.
  - The optimized function is continuous.
  - The function defining the optimal choice set is continuous.

  $\Rightarrow$ So, the RHS of the (FE) is well-defined.

Contraction Mapping

We say that (S, d) is a metric space if S is a space and d is a metric defined over this space with the usual properties (loosely corresponding to "distance" between elements of S).

- **A Contraction Mapping**: Let (S, d) be a metric space and $T : S \to S$ be an operator mapping S into itself. If for some $\beta \in (0, 1)$,

$$d(Tv_1, Tv_2) \leq \beta d(v_1, v_2) \quad \forall \quad v_1, v_2 \in S,$$

then T is a contraction mapping (with modulus β).

Contraction Mapping Theorem

Recall that a metric space (S, d) is complete if every Cauchy sequence in S converges to an element in S, and a Cauchy sequence, $\{k_i\}$ is the one that satisfies for any $\epsilon > 0$, there exists a number M such that

$$\forall \quad n, l \geq M, \quad d(k_n, k_l) < \epsilon,$$

- **(Contraction Mapping Theorem)** Let (S, d) be a complete metric space and suppose that $T : S \to S$ is a contraction. Then T has a unique fixed point, $\hat{v}$; that is there exists a unique $\hat{v} \in S$ such that

$$T\hat{v} = \hat{v}.$$

Blackwell's Sufficient Conditions for a Contraction

- **(Blackwell's Sufficient Conditions for a Contraction)** Let $X \subseteq \mathbb{R}^K$, and $\mathbf{B}(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ defined on X equipped with the max norm $||\cdot||$. Suppose that $T : \mathbf{B}(X) \to \mathbf{B}(X)$ be an operator satisfying the following two conditions:

  1. Monotonicity: For any $f, g \in \mathbf{B}(X)$, $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$; and
  2. Discounting: There exists $\beta \in (0,1)$ such that

$$[T(f+c)](x) \leq (Tf)(x) + \beta c(x)$$

$$\text{for all} \;\; f \in \mathbf{B}(X), \;\; c(x) = c \geq 0, \;\; \text{and} \;\; x \in X.$$

  Then T is a contraction mapping with modulus β on $\mathbf{B}(X)$.

Remarks on Blackwell's Sufficient Conditions

- **Berge's Maximum Theorem** (Birges) tells us that the RHS maximization problem has a solution, T is defined
- **Blackwell's Sufficient Conditions** (Blackwells) tells us that, as long as $T : \mathbf{B}(X) \to \mathbf{B}(X)$ satisfies the two conditions (actually three conditions), then T is a contraction mapping.
- Then, from the **Contraction Mapping Theorem** (CMT), this T has a *unique fixed point*, which is a solution of the Bellman equation.

Application - **Step1:** Berge's Maximum Theorem

$$V(k) = \max_{k' \in \Gamma(k)} \left[F(k, k') + \beta V(k') \right]$$

$$T[V(k)] = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\}$$

- Note that $k \in \Gamma(k) = [0, f(k)]$ is a compact set and continuous at k, and $F(k, k') = U(f(k) - k')$ is a bounded, continuous function, with regular utility functions and production functions.
  - What utility function $U()$ or production function $f()$ could violate this condition?
- Therefore, the (RHS) has a solution, and continuous at k. So the above mapping, T is well-defined.

Application - **Step2**: T is a Contraction Mapping

$$T[V(k)] = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\}$$

i. Bounded $T : \mathbf{B}(K) \to \mathbf{B}(K)$:
As long as $F(k, k')$ bounded, V bounded, giving TV being bounded.

ii. Monotonicity: Consider $V, W \in \mathbf{B}(K)$, with $W(k) \leq V(k)$ for all $k \in K$, and let $G_w(k)$ be the optimal policy corresponding to W for all $k \in K$. Then

$$\begin{aligned} T[W(k)] &= F(k, G_w(k)) + \beta W(G_w(k)) \\ &\leq F(k, G_w(k)) + \beta V(G_w(k)) \\ &\leq \max_{k' \in \Gamma(k)} \{F(k, k') + \beta V(k')\} = T[V(k)]. \end{aligned}$$

Application - **Step2**: T is a Contraction Mapping

$$T[V(k)] = \max_{k \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\}$$

iii. Discounting: For $c(x) = c \geq 0$, and $k \in K$,

$$\begin{aligned}
[T(V + c)](k) &= \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta(V(k') + c) \right\} \\
&= \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta V(k') \right\} + \beta c \\
&= [TV](k) + \beta c.
\end{aligned}$$

$\Rightarrow$ So T satisfies Blackwell's sufficient conditions, meaning that T is a contraction mapping. Therefore, a contraction mapping theorem says that a unique $V \in \mathbf{B}(K)$ satisfying the problem exists.

Other Useful Theorems

- **(Assumption 3)** For each k', $F(\cdot, k')$ is strictly increasing in each of its first K arguments, and Γ is monotone in a sense that $k_1 \leq k_2$ implies $\Gamma(k_1) \subseteq \Gamma(k_2)$.
- **(Properties of V)** Let K, Γ, F, and β satisfy (Assumption 2) and (Assumption 3), and let V be the unique solution to the above mapping. Then V is strictly increasing.
- **(Assumption 4)** F is strictly concave, and Γ is convex.
- **(Properties of V and k)** Let K, Γ, F, and β satisfy (Assumption 2), (Assumption 3), and (Assumption 4), and let V be the unique solution to the above mapping. Then V is strictly concave and G is continuous, single-valued function.

Remarks

- So far, we have examined properties of the Bellman Equations in a non-stochastic setting.
- The problems we see in the papers, however, are usually *stochastic*.
- Luckily, with a slight modification of all the assumptions here, most of the results still hold in a *stochastic* setting as well.
    - $\Rightarrow$ Please see the Chapter 16 of Acemoglu (2005) for the assumptions and theorems in a stochastic setting.

How to Solve the Bellman Equation?

Now that we know the solution exists, let's solve the problem.

1. Guess and Verify $\Rightarrow$ Today
2. Numerical Solution $\Rightarrow$ (Section 2 and 3)
   - Value Function Iteration
   - Policy Function Iteration
   - Howard Policy Improvement

Let's Take A Breather

7-10 Minute break to ask questions or just relax

Topics we've discussed so far:

- Sequential Problem vs. Functional Equation
- Value Function vs. Policy Function
- Berge's Maximum Theorem
- Contraction Mapping Theorem
- Blackwell's Sufficient Conditions

Food for Thought: Why are we using these techniques rather than the tools from growth (Hamiltonian, etc.)?

Guess and Verify

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ \log(k^\alpha - k') + \beta V(k') \right\}$$

Guess that $V(k) = A + B \log k$.

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ \log(k^\alpha - k') + \beta \left(A + B \log k' \right) \right\}$$

Then the first order condition is

$$0 = -\frac{1}{k^\alpha - k'} + \beta \frac{B}{k'} \quad \Rightarrow \quad k' = \frac{\beta B}{1 + \beta B} k^\alpha.$$

Guess and Verify

$$
\begin{aligned}
V(k) &= \log(k^\alpha - \frac{\beta B}{1+\beta B}k^\alpha) + \beta A + \beta B \log\left(\frac{\beta B}{1+\beta B}k^\alpha\right) \\
&= \log \frac{1}{1+\beta B}k^\alpha + \beta A + \beta B \log(\frac{\beta B}{1+\beta B}k^\alpha) \\
A + B \log k &= \log \frac{1}{1+\beta B}k^\alpha + \beta A + \beta B \log(\frac{\beta B}{1+\beta B}k^\alpha) \\
(1-\beta)A + B \log k &= \log \frac{(\beta B)^{(\beta B)}}{(1+\beta B)^{(1+\beta B)}} + \alpha(1+\beta B)\log k
\end{aligned}
$$

Guess and Verify

Equation Coefficients generates

$$
\begin{aligned}
B &= \alpha(1+\beta B) \\
\Rightarrow B &= \frac{\alpha}{1-\alpha\beta}. \\
A(1-\beta) &= \log \frac{(\beta B)^{(\beta B)}}{(1+\beta B)^{(1+\beta B)}} \\
\Rightarrow A &= \frac{\log(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\log(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}.
\end{aligned}
$$

Policy Function

$$
\begin{aligned}
k' &= \frac{\beta B}{1+\beta B} k^{\alpha} \\
\Rightarrow k' &= \alpha\beta k^{\alpha}
\end{aligned}
$$

The Functional Euler Equation

$$V(k) = \max_{k' \in \Gamma(k)} \left\{F(k, k') + \beta V(k')\right\}$$

$$V(k) = F(k, G(k)) + \beta V(G(k)), \quad k' = G(k).$$

Note that from the FOC of the first equation, we have

$$0 = F_2(k, k') + \beta V'(k').$$

This implies that with $k' = G(k)$,

$$0 = F_2(k, G(k)) + \beta V'(G(k)).$$

Here, the problem is that $V'(\cdot)$ is NOT known.

The Functional Euler Equation

$$V'(k) = F_1(k, G(k)) + G'(k)\underbrace{\left(F_2(k, G(k)) + \beta V'(G(k))\right)}_{=0}$$

This is an application of the EVT (Envelop Theorem).

$$V'(k) = F_1(k, G(k)) \quad \Rightarrow \quad V'(k') = F_1(k', G(k')) = F_1(G(k), G(G(k))).$$

$$F_2(k, k') + \beta F_1(k', k") = 0, \quad \forall \;\; k.$$

With $F(k, k') = U(f(k) - k') = \log(k^\alpha - k')$, we have

$$\frac{1}{k^\alpha - G(k)} = \alpha\beta G(k)^{\alpha-1}\frac{1}{G(k)^\alpha - G(G(k))}.$$

The Functional Euler Equation

Now guess that $G(k) = \sigma k^\alpha$, we have

$$\frac{1}{(1-\sigma)k^\alpha} = \frac{\alpha\beta(\sigma k^\alpha)^{\alpha-1}}{(\sigma k^\alpha)^\alpha - \sigma(\sigma k^\alpha)^\alpha}$$

generating that

$$\sigma = \alpha\beta,$$

which is the same as previous.

Flexibility of the DP: Optimal Stopping Problem

- Consider a simpler version of the Eat-the-Pie Problem in Pset 2.
- Assume that $R = 1$, and that the pie must be eaten in one period. This is an example of **optimal stopping problem** as in problem 3 of the Pset 2. The common element in all these problems is the timing of a single event: when to eat the pie, or when to take a job.
- Assume also that there is a taste shock, ϵ, which affects the satisfaction of eating the pie. So the utility over consumption is now given by

$$\epsilon U(C).$$

- Assume also that this taste shock takes on only two values$\epsilon \in \{\epsilon_h, \epsilon_l\}$, and follows a first-order Markov process, with $\pi_{ij} = Prob(\epsilon' = \epsilon_j | \epsilon = \epsilon_i)$.
- Assume also that now the pie depreciates at rate $\rho < 1$..

Flexibility of the DP: Optimal Stopping Problem

Let $V^E(A, \epsilon)$ and $V^N(A, \epsilon)$ be the value of eating the size A pie now (E) and waiting (N), respectively, given the current taste shock, $\epsilon \in \{\epsilon_h, \epsilon_l\}$. Then,

$$
\begin{aligned}
V^E(A, \epsilon) &= \epsilon U(A) \\
V^N(A, \epsilon) &= \beta \mathbb{E}_{\epsilon' | \epsilon} V(\rho A, \epsilon'), \\
V(A, \epsilon) &= \max \ \{V^E(A, \epsilon), V^N(A, \epsilon)\}.
\end{aligned}
$$