Capital Accumulation and Growth: The Solow Model

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Steady Growth at the Frontier for 150 Years

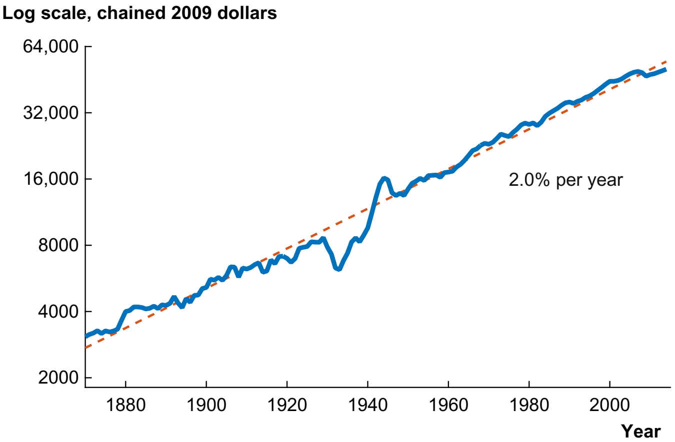
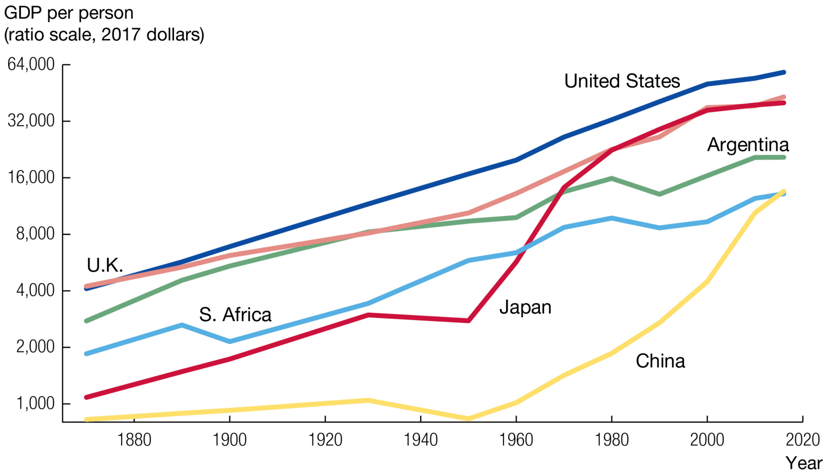


Fig. 1 GDP per person in the United States. Source: *Data for 1929–2014 are from the U.S. Bureau of Economic Analysis, NIPA table 7.1. Data before 1929 are spliced from Maddison, A. 2008. Statistics on world population, GDP and per capita GDP, 1-2006 AD. Downloaded on December 4, 2008 from http://www.ggdc.net/maddison/.*

Source: Jones (2016)

Uneven Growth Across the World



Source: The Maddison-Project, www.ggdc.net/maddison/. Observations are presented every decade after 1950 and less frequently before that as a way of smoothing the series.

Source: Jones (2021)

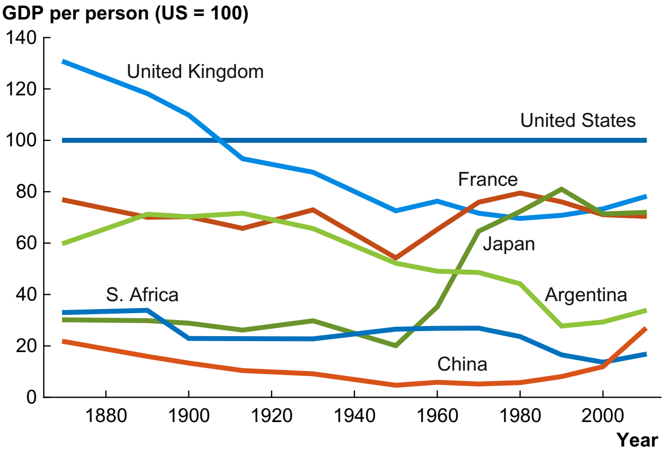
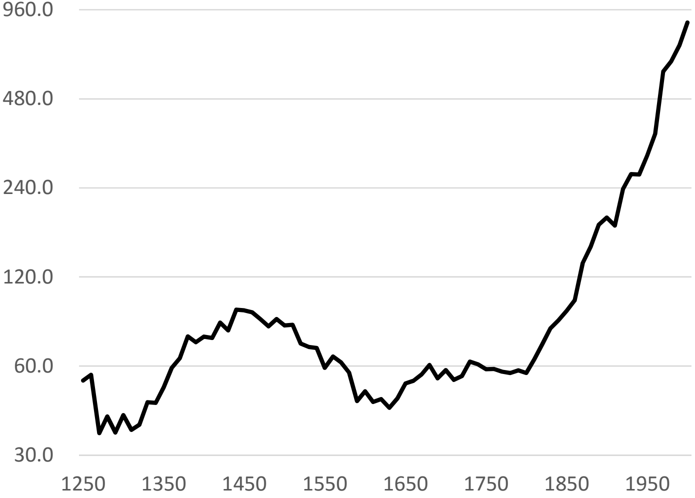


Fig. 22 The spread of economic growth since 1870. Source: *Bolt, J., van Zanden, J.L. 2014. The Maddison Project: collaborative research on historical national accounts. Econ. Hist. Rev. 67 (3), 627–651.*

Source: Jones (2016)

Growth Is a Recent Phenomenon!



Source: Clark (2010)

Importance of a Log Scale

- Figures like these often plotted on linear scale
  to make them more dramatic (hockey stick) Linear Scale
- This is misleading.
- Fluctuations before 1800 were large!

(Also Maddison data back thousands of years are “guestimates”)

Big Picture Questions about Growth

- What sustains growth at the frontier?

  (Will it continue in the future?)

- Why are some countries so far behind the frontier?

  (What might help them close the gap?)

- Why did growth begin?

- Why was there no growth before Industrial Revolution?

We will focus on first two questions. (210A in the spring covers later two.)

Mathusian Stagnation / Industrial Revolution

- Steinsson, J. (2021): "Malthus and Pre-Industrial Stagnation," draft textbook chapter.

  https://eml.berkeley.edu/~jsteinsson/teaching/malthus.pdf

- Steinsson, J. (2021): "How Did Growth Begin? The Industrial Revolution and Its Antecedents," draft textbook chapter.

  https://eml.berkeley.edu/~jsteinsson/teaching/originsofgrowth.pdf

Three Textbooks

- Romer, D. (2019): *Advanced Macroeconomics*, McGraw Hill, New York, NY.
- Acemoglu, D. (2009): *Introduction to Modern Economic Growth*, Princeton University Press, Princeton, NJ.
- Barro, R.J. and X. Sala-i-Martin (2004): *Economic Growth*, MIT Press, Cambridge, MA.

The Solow Model

Is Capital Accumulation Key to Growth?

- Seems plausible!
- Conventional wisdom in 1950s: Yes!
- See discussion in Easterly (2002)
- Solow (1956) tackled this question

The Production Function

$$Y(t) = F[K(t), A(t)L(t)]$$

- $Y(t)$: Output at time t
- $K(t)$: Capital stock at time t
- $L(t)$: Labor supply at time t
- $A(t)$: "effectiveness of labor" at time t (aka "productivity")

Production Function

$$Y(t) = F[K(t), A(t)L(t)]$$

- The model is dynamics
- Time is continuous
- Time only enters production function through inputs
- Productivity is "labor augmenting" (Harrod neutral)
- This last point is important for getting "balanced growth"

Balanced Growth: Kaldor Facts

Kaldor (1963): As per capita income has risen

- The capital-output ratio has been roughly constant
- Real interest rates have no trend
- The labor and capital share of production have been roughly constant

Roughly Constant Capital-Output Ratio

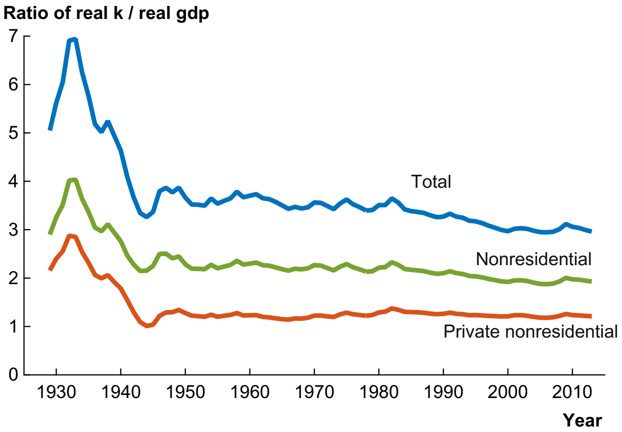
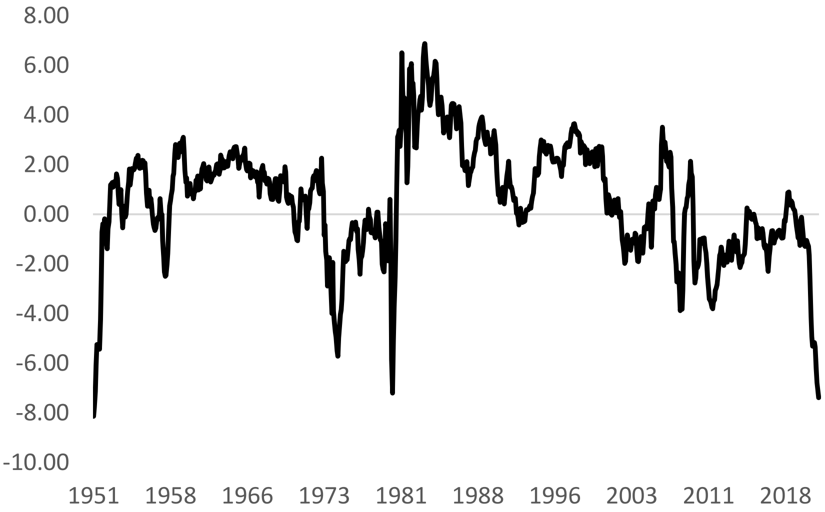


Fig. 3 The ratio of physical capital to GDP. Source: *Burea of Economic Analysis Fixed Assets tables 1.1 and 1.2. The numerator in each case is a different measure of the real stock of physical capital, while the denominator is real GDP.*

Source: Jones (2016)

Ex Post Real Interest Rate



Source: FRED. 3 month T-bill rate minus 12-month CPI inflation.

Roughly Constant Labor and Capital Shares

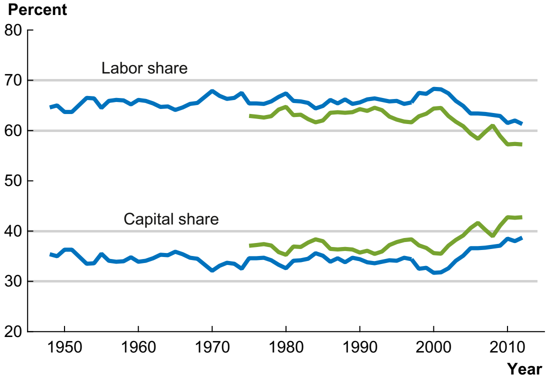


Fig. 6 Capital and labor shares of factor payments, United States. Source: *The series starting in 1975 are from Karabarbounis, L., Neiman, B. 2014. The global decline of the labor share. Q. J. Econ. 129 (1), 61–103. http://ideas.repec.org/a/oup/qjecon/v129y2014i1p61-103.html and measure the factor shares for the corporate sector, which the authors argue is helpful in eliminating issues related to self-employment. The series starting in 1948 is from the Bureau of Labor Statistics* Multifactor Productivity Trends, *August 21, 2014, for the private business sector. The factor shares add to 100%.*

Source: Jones (2016)

Forms of Technical Progress

- Hicks Neutral:

$$A(t)F[K(t), L(t)]$$

(Ratio of marginal products remains constant for a given K/L ratio)

- Harrod Neutral / Labor-Augmenting:

$$F[K(t), A(t)L(t)]$$

(Ratio of input shares ($F_K K/F_L L$) remain constant for a given K/Y ratio)

- Solow Neutral / Capital-Augmenting:

$$F[A(t)K(t), L(t)]$$

(Ratio of input shares ($F_K K/F_L L$) remain constant for a given L/Y ratio)

- In general: Some combination of all three

UZAWA'S (1961) THEOREM

Roughly speaking:

- Balanced growth in the long run is only possible if all technical progress is labor augmenting

  (See Acemoglu (2009, sec. 2.7) and Barro-Sala-I-Martin (2004, sec. 1.2.12) for details)

Why balanced growth:

- Empirically: We see a stable K/Y ratio and relatively stable factor shares
- Theoretically: Very convenient because model will have a steady state when technical progress is constant

Uzawa's (1961) Theorem

Acemoglu (2009, p. 59):

> *This result is very surprising and troubling, since there are no compelling reasons for why technological progress should take this form. [i.e., be labor augmenting]*

EXAMPLE: COBB-DOUGLAS PRODUCTION FUNCTION

The Cobb-Douglas production function satisfies all three properties

- Hicks Neutral:

$$A(t)K(t)^{\alpha}L(t)^{1-\alpha}$$

- Harrod Neutral:

$$K(t)^{\alpha}[\tilde{A}(t)L(t)]^{1-\alpha} \text{ where } \tilde{A}(t) = A(t)^{1/(1-\alpha)}$$

- Solow Neutral:

$$[\check{A}(t)K(t)]^{\alpha}L(t)^{1-\alpha} \text{ where } \check{A}(t) = A(t)^{1/\alpha}$$

Definition: A function f is homogeneous of degree m in x and y if

$$f(\lambda x, \lambda y, z) = \lambda^m f(x, y, z)$$

- $m < 1$: decreasing returns to scale
- $m = 1$: constant returns to scale
- $m > 1$: increasing returns to scale

EULER'S THEOREM

Euler's Theorem: If f is homogeneous of degree m in x and y:

$$mf(x,y,z) = \frac{\partial}{\partial x} f(x,y,z)x + \frac{\partial}{\partial y} f(x,y,z)y$$

(See Acemoglu (2009, p. 29) for a more careful statement of this theorem.)

Constant Returns to Scale

- We assume that the production function is constant returns to scale:

$$F(cK, cAL) = cF(k, AL)$$

- Why?

Constant Returns to Scale

- We assume that the production function is constant returns to scale:

$$F(cK, cAL) = cF(k, AL)$$

- Why?
  - Economy large enough that each establishment has reached efficient size (micro returns to scale and gains from specialization exhausted)
  - Fixed factors (e.g., land) unimportant
  - Positive and negative externalities between establishments unimportant
  - $A(t)$ non-rival (can be used many times)
  - Replication argument: Can build a second identical establishment with double the inputs

Intensive Form

- Since

$$F(cK, cAL) = cF(k, AL)$$

- we can write production function in intensive form:

$$\frac{Y}{AL} = \frac{1}{AL}F(K, AL) = F\left(\frac{K}{AL}, 1\right)$$

- Define:
  - $k = K/AL$: Capital per effective worker
  - $y = Y/AL$: Output per effective worker
- Also define: $f(k) = F(k, 1)$
- Then we have:

$$y = f(k)$$

(Why do this? ... Will become clear in a few slides.)

Returns to Capital

What do we want to assume about returns to capital?

Returns to Capital

What do we want to assume about returns to capital?

Returns to capital are ...

- Positive: $f'(k) > 0$
- Diminishing: $f''(k) < 0$

Also ...

- f(0) = 0
- Inada conditions:

$$\lim_{k \to 0} f'(k) = \infty \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0$$

Neoclassical Production Function

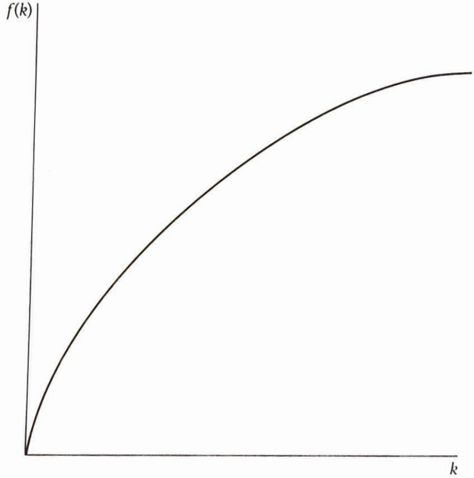


FIGURE 1.1 An example of a production function

Source: Romer (2019)

Example: Cobb-Douglas

- If the production function is Cobb-Douglas, we have

$$y = \frac{Y}{AL} = \frac{1}{AL} K^{\alpha} (AL)^{1-\alpha} = \left(\frac{K}{AL} \right)^{\alpha} = k^{\alpha}.$$

  So, we have:

$$y = k^{\alpha}$$

- This function satisfies all the conditions we have specified on previous slides

Cambridge Capital Controversy

- Early post-WWII debate between (mostly) British and (mostly) US economists
- Does it make sense to talk about aggregate capital?
- Do lower interest rates lead to higher capital/labor ratios?
- Outcome:
  - Various pathologies possible
  - Similar to Giffen goods in consumption theory
  - Not clear any of this is practically important

Cambridge Capital Controversy

Cambridge, U.K.:

- Harcourt, G.C. (1969): "Some Cambridge Controversies in the Theory of Capital," *Journal of Economic Literature*, 7(2), 369-405.
- Cohen, A.J. and G.C. Harcourt (2003): "Whatever Happened to the Cambridge Capital Theory Controversies?" *Journal of Economic Perspectives*, 17(1), 199-214.

Cambridge, U.S.:

- Samuelson, P.A. (1966): "A Summing Up," *Quarterly Journal of Economics*, 80(4), 568-583.
- Stiglitz, J.E. (1974): "The Cambridge-Cambridge Controversy in the Theory of Capital: A View from New Haven," *Journal of Political Economy*, 82(4), 893-903.

What Happens to the Output?

- Output is divided between consumption and investment:

$$Y(t) = C(t) + I(t)$$

- How much is invested?

- Simplifying assumption: Constant savings rate

$$I(t) = sY(t)$$

(We will introduce optimizing households in Ramsey model)

Evolution of Capital

$$\begin{aligned}\dot{K}(t) &= I(t) - \delta K(t) \\ &= sY(t) - \delta K(t)\end{aligned}$$

$(\dot{K}(t) = dK(t)/dt)$

- Each instant:
  - New investment adds to capital stock
  - Existing capital depreciates by some fraction (per unit time)
- Change in capital stock is the difference between these two

Labor and Productivity Exogenous

- Labor and productivity grow at constant rates:

$$\dot{L}(t) = nL(t)$$

$$\dot{A}(t) = gA(t)$$

- Notice that

$$\frac{d \log X(t)}{dt} = \frac{d \log X(t)}{dX(t)} \frac{dX(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

where log denotes the natural log

$$\frac{d \log L(t)}{dt} = \frac{\dot{L}(t)}{L(t)} = n$$

$$\log L(t) = \log L(0) + nt$$

$$L(t) = L(0)e^{nt}$$

and similarly for $A(t)$.

Full Solow Model

$$Y(t) = F[K(t), A(t)L(t)]$$

$$Y(t) = C(t) + I(t)$$

$$I(t) = sY(t)$$

$$\dot{K}(t) = I(t) - \delta K(t)$$

$$\dot{L}(t) = nL(t)$$

$$\dot{A}(t) = gA(t)$$

- Initial Conditions: $K(0)$, $A(0)$, $L(0)$ given
- Goal: Solve for evolution of $K(t)$, $Y(t)$, $C(t)$, $I(t)$, $L(t)$, $A(t)$

About Models

- Solow model is a gross simplification
- Not necessarily a defect
- Real world is fully realistic, but too complicated to understand
- Simple models can provide insight about specific issues
- But may cause "theory-induced blindness"
- Kahneman: "Once you have accepted a theory, it is extraordinarily difficult to notice its flaws."
- Fully realistic model not insightful but would allow for calculation of counterfactuals and the analysis of policy experiments

About Models

Two uses of models:

- Provide insight about mechanisms
  - Such models must be (relatively) simple
  - Unlikely to be good guides to real-world counterfactuals
- Provide a basis for policy evaluation
  - Such models need not be insightful
  - But they must be “realistic”

Important to keep this distinction clear

Finding a Steady State

- When solving a dynamic system of equations, often useful to find a steady state
- A stable steady state is a point the system stays at if unperturbed and returns to if perturbed by a small amount
- Since $L(t)$ and $A(t)$ are growing, no steady state in the original variables
- Key to finding a steady state to work with transformed variables:

$$y(t) = \frac{Y(t)}{A(t)L(t)} \qquad\qquad k(t) = \frac{K(t)}{A(t)L(t)}$$

Dynamics of $k(t)$

- Using the chain rule we have that

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}[\dot{A}(t)L(t) + A(t)\dot{L}(t)]$$
$$= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{L}(t)}{L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{A}(t)}{A(t)}$$

- Using $\dot{L}/L = n$, $\dot{A}/A = g$, and $\dot{K} = sY - \delta K$ we have that

$$\dot{k}(t) = \frac{sY(t) - \delta K(t)}{A(t)L(t)} - nk(t) - gk(t)$$

- Using $y = f(k)$ we have that

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)$$

DYNAMICS OF $k(t)$

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)$$

- Rate of change of $k(t)$ difference between:
  - Actual investment: $sf(k(t))$
  - Break-even investment: $(n + g + \delta)k(t)$
- Notice that break-even investment determined by:
  - Population growth: n
  - Productivity growth: g
  - Depreciation: δ
- Intuition: capital per effective worker must keep up with amount of effective labor (which is growing due to n and g)

ACTUAL AND BREAK-EVEN INVESTMENT

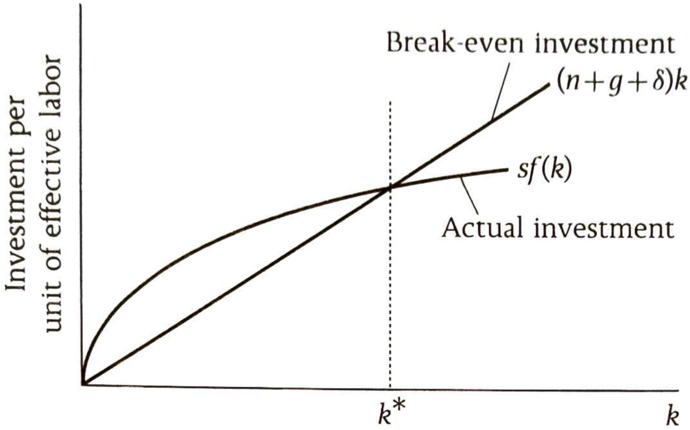


FIGURE 1.2 Actual and break-even investment

Source: Romer (2019)

Phase Diagram for $k(t)$

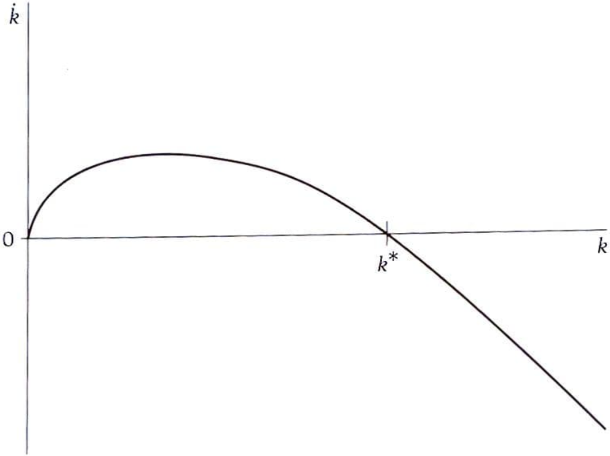


FIGURE 1.3 The phase diagram for k in the Solow model

Source: Romer (2019)

Economy Converges to a Steady State k^*

- Inada conditions and $f''(k) < 0$ imply that actual investment and break-even investment lines cross once
  (with actual investment crossing from above)
- This point is denoted k^*
- k^* is a steady state for $k(t)$
- Economy converges to k^* globally
  (i.e., from any (positive) starting point)

Balanced Growth Path

- At steady state $k(t)$ is constant
- This implies that $K = ALk$ grows at a rate $n + g$
- Since both K and AL grow at $n + g$, Y also grows at rate $n + g$
- Furthermore, K/L and Y/L grow at rate g
- Economy converges to a balanced growth path

These conclusion flow from fact that the growth rate of the product of two variables is the sum of their growth rates. See, Problem 1.1 in Romer (2019).

First Lesson from Solow Model

- Capital accumulation cannot serve as a source of long-run growth in living standards
    - If $g = 0$, growth in Y/L is zero
- Why?

First Lesson from Solow Model

- Capital accumulation cannot serve as a source of long-run growth in living standards
  - If $g = 0$, growth in Y/L is zero
- Why? Because of diminishing returns to capital.
  - Diminishing returns mean actual investment eventually cannot keep up with break-even investment
  - This gives rise to a steady state with property listed above
- Long-run growth must come from $A(t)$

Effects of Changes in Fundamental Parameters

- One can use the Solow model to think about changes in:
  - The savings rate s
  - The population growth rate n
  - The growth rate of technology g
  - The depreciation rate δ
- Such exercises are “other things equal” type exercises
- Let’s consider a permanent increase in the savings rate

Increase in the Savings Rate

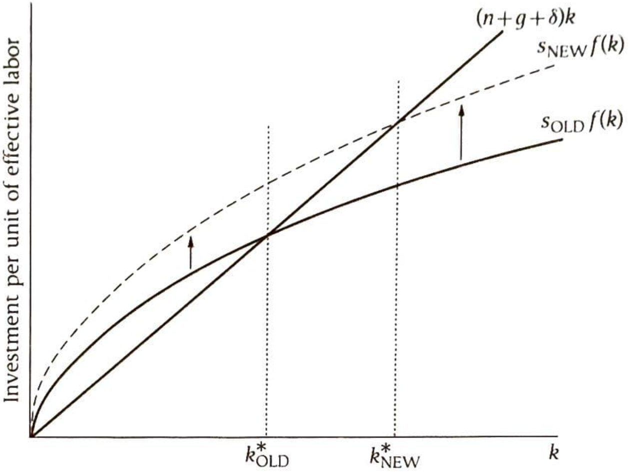
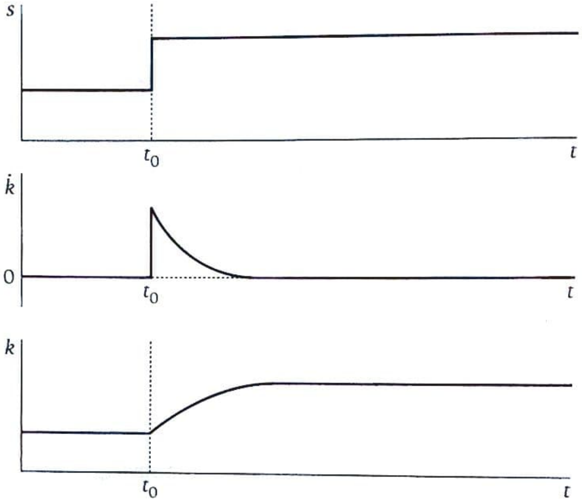


FIGURE 1.4 The effects of an increase in the saving rate on investment

Source: Romer (2019)

Increase in the Savings Rate



Source: Romer (2019)

Increase in the Savings Rate

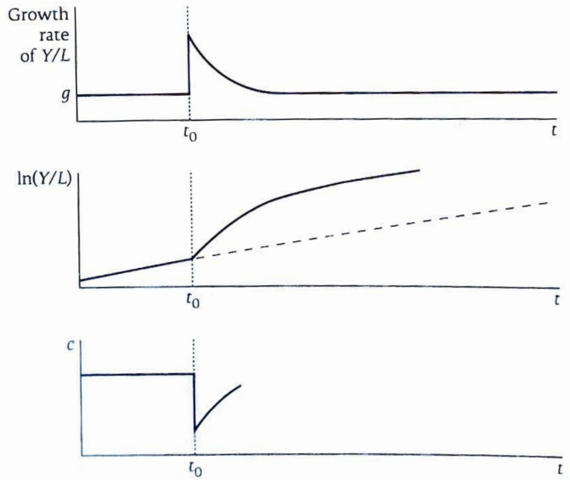


FIGURE 1.5 The effects of an increase in the saving rate

Source: Romer (2019)

Increase in Savings Rate

- Increase in savings rate has a "level effect" on per capita output
- It does not have a "growth effect"

Transition Dynamics

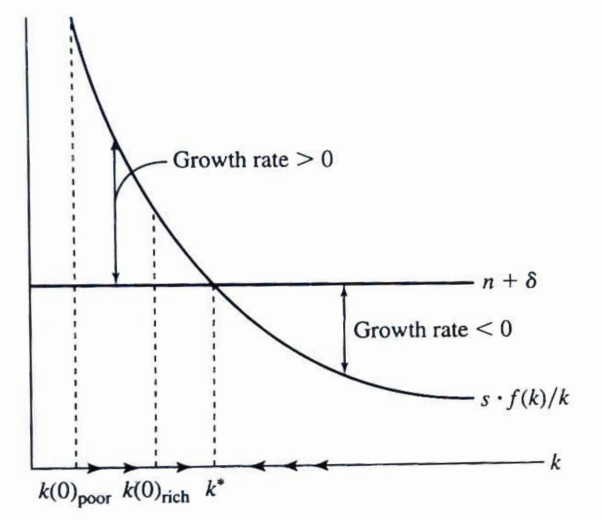
Transition Dynamics in the Solow Model

- Our focus has been on long run effects
- Solow model also has interesting implications about "short run"

Transition Dynamics in the Solow Model

- Start with
$$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t)$$
- Divide by $k(t)$:
$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (n+g+\delta)$$
- Left-hand-side is growth rate of capital
- $(n+g+\delta)$ is constant as a function of $k(t)$
- While
$$\lim_{k\to 0} \frac{sf(k(t))}{k(t)} = \infty \qquad \lim_{k\to\infty} \frac{sf(k(t))}{k(t)} = 0$$
$$\frac{d}{dk}\frac{sf(k)}{k} = -\frac{f(k)/k - f'(k)}{k} < 0$$
(numerator is average product of capital minus marginal product of capital)

Transition Dynamics



Source: Barro and Sala-I-Martin (2004). Figure is for $g = 0$. Adding $g > 0$ would just shift up horizontal line.

Transitional Growth Rates

- Differentiate $y(t) = f(k(t))$ with respect to t

$$\dot{y}(t) = f'(k(t))\dot{k}(t)$$

- Divide through by $y(t)$:

$$\frac{\dot{y}(t)}{y(t)} = \frac{f'(k(t))k(t)}{f(k(t))}\frac{\dot{k}(t)}{k(t)}$$

- Let g_x denote the growth rate of x_t and $\alpha_K(k(t)) = f'(k(t))k(t)/f(k(t))$

$$g_y = \alpha_K(k(t))g_k$$

($\alpha_K(k(t))$ is the elasticity of output with respect to capital.)

- Growth rate of output is proportional to growth rate of capital

Second Lesson from Solow Model

- Countries that are below **their** steady state level of capital/output should grow faster than countries that are above **their** steady state.
- If countries share same fundamentals, Solow model predicts *absolute convergence*
- More generally, Solow model predicts *conditional convergence*

BAUMOL (1986)

- Analyzed data for 16 industrialized countries for which long historical data were available
- Estimated:

$$\log \tilde{y}_{i,1979} - \log \tilde{y}_{i,1870} = a + b \log \tilde{y}_{i,1870} + \epsilon_i$$

  where $\tilde{y}_{i,t}$ denotes output per person in country i at time t

- Negative b indicates convergence (initial poor grow faster)

BAUMOL (1986)

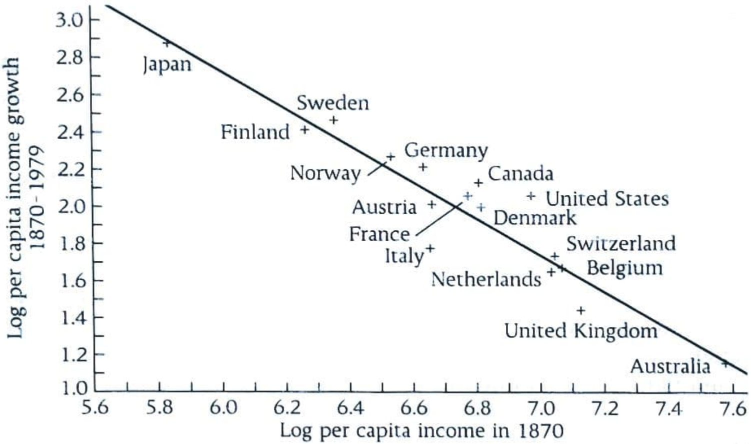


FIGURE 1.7 Initial income and subsequent growth in Baumol's sample (from DeLong, 1988; used with permission)

Source: Romer (2019)

De Long (1988)

- De Long (1988) presented two important critiques of Baumol (1986)
- Sample selection:
  - Baumol chose countries that were ex post rich
  - Any difference in initial conditions will yield convergence
  - Data more likely to be available for ex post successful countries
  - De Long selects countries based on initial GDP per capita
- Measurement error:
  - Initial income shows up both on LHS and RHS
  - Measurement error in initial income creates bias toward convergence

De Long (1988)

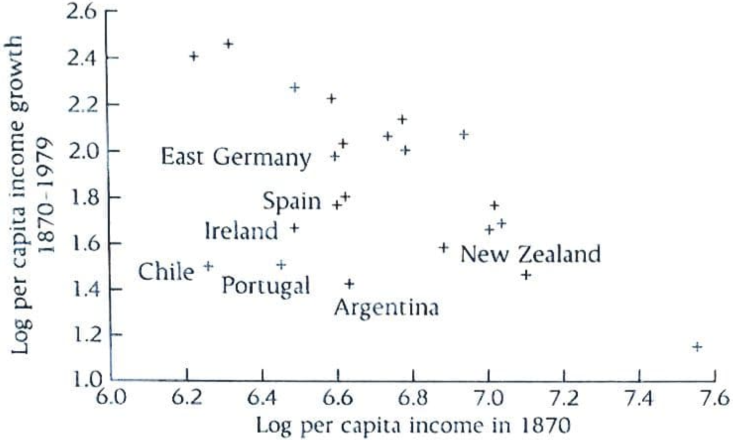


FIGURE 1.8 Initial income and subsequent growth in the expanded sample (from DeLong, 1988; used with permission)

Source: Romer (2019)

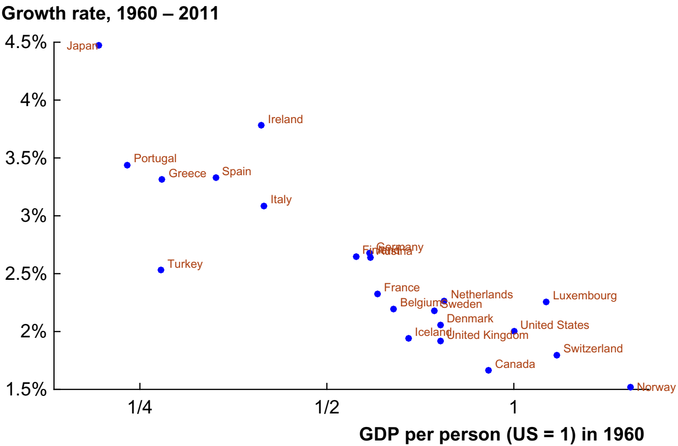
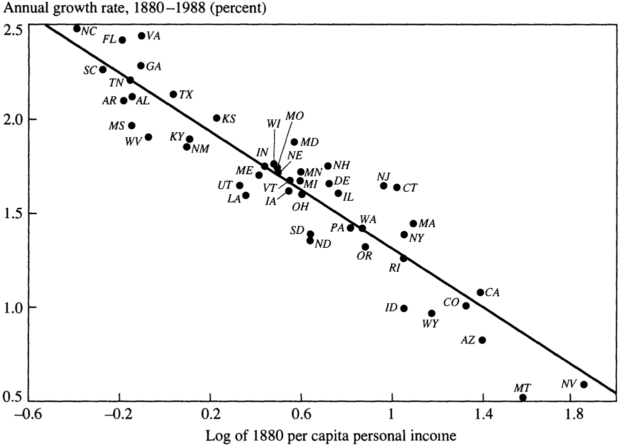


Fig. 25 Convergence in the OECD. Source: *The Penn World Tables 8.0. Countries in the OECD as of 1970 are shown.*

Source: Jones (2016)

U.S. States Post-1880

Figure 1. Convergence of Personal Income across U.S. States: 1880 Income and Income Growth from 1880 to 1988



Sources: Bureau of Economic Analysis (1984), Easterlin (1960a, 1960b), and *Survey of Current Business*, various issues. The postal abbreviation for each state is used to plot the figure. Oklahoma, Alaska, and Hawaii are excluded from the analysis.

Source: Barro and Sala-I-Martin (1991)

ALL COUNTRIES POST-1960

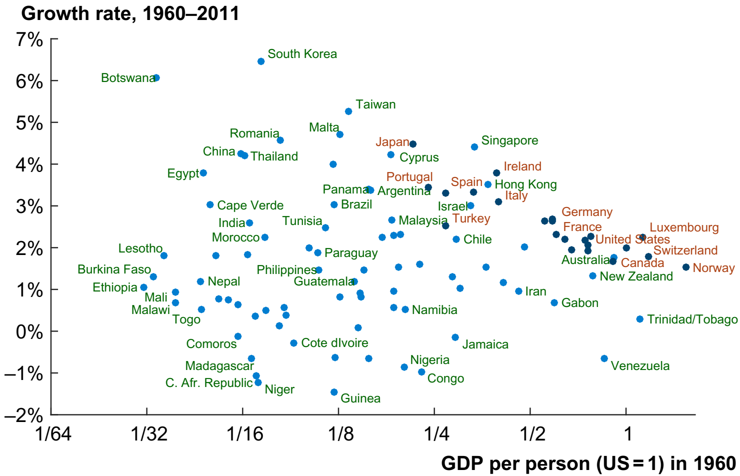


Fig. 26 The lack of convergence worldwide. Source: *The Penn World Tables 8.0.*

Source: Jones (2016)

Conditional Convergence

- Solow model implies:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)$$

- If $f(k(t)) = k(t)^{\alpha}$, steady state:

$$k^* = \left(\frac{s}{n + g + \delta} \right)^{1/(1-\alpha)}$$

Conditional Convergence

- But $k = K/AL$ is not observable (A is not observable)
- Let's rewrite the steady state in terms of K/L

$$\left(\frac{K}{L}\right)^* = \left(\frac{sA}{n+g+\delta}\right)^{1/(1-\alpha)}$$

- Model implies convergence conditional on: A, s, n, g, δ

Mankiw, Romer, and Weil (1992)

TABLE III
TESTS FOR UNCONDITIONAL CONVERGENCE

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	−0.266	0.587	3.69
	(0.380)	(0.433)	(0.68)
ln(Y60)	0.0943	−0.00423	−0.341
	(0.0496)	(0.05484)	(0.079)
$\overline{R}^2$	0.03	−0.01	0.46
s.e.e.	0.44	0.41	0.18
Implied λ	−0.00360	0.00017	0.0167
	(0.00219)	(0.00218)	(0.0023)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960.

Source: Mankiw, Romer, Weil (1992)

Mankiw, Romer, and Weil (1992)

TABLE IV

Tests for Conditional Convergence

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	1.93	2.23	2.19
	(0.83)	(0.86)	(1.17)
ln(Y60)	−0.141	−0.228	−0.351
	(0.052)	(0.057)	(0.066)
ln(I/GDP)	0.647	0.644	0.392
	(0.087)	(0.104)	(0.176)
$\ln(n + g + \delta)$	−0.299	−0.464	−0.753
	(0.304)	(0.307)	(0.341)
$\overline{R}^2$	0.38	0.35	0.62
s.e.e.	0.35	0.33	0.15
Implied λ	0.00606	0.0104	0.0173
	(0.00182)	(0.0019)	(0.0019)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

Source: Mankiw, Romer, Weil (1992)

Mankiw, Romer, and Weil (1992)

TABLE V
TESTS FOR CONDITIONAL CONVERGENCE

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	3.04	3.69	2.81
	(0.83)	(0.91)	(1.19)
ln(Y60)	−0.289	−0.366	−0.398
	(0.062)	(0.067)	(0.070)
ln(I/GDP)	0.524	0.538	0.335
	(0.087)	(0.102)	(0.174)
$\ln(n + g + \delta)$	−0.505	−0.551	−0.844
	(0.288)	(0.288)	(0.334)
ln(SCHOOL)	0.233	0.271	0.223
	(0.060)	(0.081)	(0.144)
$\overline{R}^2$	0.46	0.43	0.65
s.e.e.	0.33	0.30	0.15
Implied λ	0.0137	0.0182	0.0203
	(0.0019)	(0.0020)	(0.0020)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Source: Mankiw, Romer, Weil (1992)

Determinants of Growth

$$\left(\frac{K}{L}\right)^* = \left(\frac{sA}{n+g+\delta}\right)^{1/(1-\alpha)}$$

- Mankiw-Romer-Weil 92 condition on s, n, schooling
- But what about A?
- What is A anyway?
- Potentially many things that contribute to efficiency of labor
- Barro (1991) explores correlates of growth

Regressions for per capita growth rate

Independent variable	(1)	(2)
Log(GDP)	−.0254 (.0031)	−.0225 (.0032)
Male secondary and higher schooling	.0118 (.0025)	.0098 (.0025)
Log(life expectancy)	.0423 (.0137)	.0418 (.0139)
Log(GDP) ∗ male schooling	−.0062 (.0017)	−.0052 (.0017)
Log(fertility rate)	−.0161 (.0053)	−.0135 (.0053)
Government consumption ratio	−.136 (.026)	−.115 (.027)
Rule of law index	.0293 (.0054)	.0262 (.0055)
Terms of trade change	.137 (.030)	.127 (.030)
Democracy index	.090[a] (.027)	.094 (.027)
Democracy index squared	−.088 (.024)	−.091 (.024)
Inflation rate	−.043 (.008)	−.039 (.008)
Sub-Saharan Africa dummy		−.0042[b] (.0043)
Latin America dummy		−.0054 (.0032)
East Asia dummy		.0050 (.0041)
R^2	.58, .52, .42	.60, .52, .47
Number of observations	80, 87, 84	80, 87, 84

Source: Barro (1998)

Cross-Country Growth Regressions

- Cross-country growth regressions hard to interpret
- Reverse causation: Growth may cause other things like democracy / rule of law / government spending
  - This is called “modernization theory”
- Both growth and variable of interest may be caused by a third factor
- But including these does result in conditional convergence

Speed of Convergence

- What does Solow model imply about speed of convergence?
- If speed of convergence is fast:
  - Most countries will be close to steady state (already mostly converged)
  - We can focus on steady state analysis
- Also interesting as a possible test of the model

Speed of Convergence

- Start with:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)$$

- So $\dot{k}(t)$ is a function of $k(t)$
- Let's write this as $\dot{k}(k)$ (dropping dependence on t for notational simplicity)
- A first-order Taylor series approximation of $\dot{k}(k)$ around k^* is:

$$\dot{k} \simeq \left[\left. \frac{\partial \dot{k}(k)}{\partial k} \right|_{k=k^*} \right] (k - k^*)$$

- Let's denote $\lambda = -\partial \dot{k}(k)/\partial k|_{k=k^*}$ which means we have

$$\dot{k}(t) \simeq -\lambda(k(t) - k^*)$$

Speed of Convergence

- Linear first-order differential equation:

$$\dot{k}(t) \simeq -\lambda(k(t) - k^*)$$

- Solution:

$$k(t) - k^* \simeq e^{-\lambda t}[k(0) - k^*]$$

- So, λ is rate of convergence
- Half-life:

$$0.5 = e^{-\lambda t}$$

$$t = -\log(0.5)/\lambda \simeq 0.69/\lambda$$

Speed of Convergence

- Using:

$$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t)$$

we get that

$$\begin{aligned}\lambda = \left[\left. \frac{\partial \dot{k}(k)}{\partial k} \right|_{k=k^*} \right] &= -[sf'(k^*) - (n+g+\delta)] \\ &= (n+g+\delta) - \frac{(n+g+\delta)k^* f'(k^*)}{f(k^*)} \\ &= [1-\alpha_K(k^*)](n+g+\delta)\end{aligned}$$

- Speed of convergence of output is the same as capital

Speed of Convergence

- Solow model implies that speed of convergence is

$$\lambda = [1 - \alpha_K(k^*)](n + g + \delta)$$

  Rough calibration:

  - Technological growth: $g = 0.02$
  - Population growth: $n = 0.01$
  - Depreciation: $\delta = 0.04$
  - Capital share: $\alpha_K(k^*) = 1/3$

$$\lambda = \frac{2}{3}(0.01 + 0.02 + 0.05) = 0.053$$

- This implies a half-life of 13 years
- Very fast convergence!!

Speed of Convergence in the Data

- To measure speed of convergence in the data, must run convergence regressions in terms of annual growth rates
- Barro and Sala-I-Martin (1991,1992) consider:

$$\frac{1}{T}\log\left(\frac{y_{i,t}}{t_{i,t-T}}\right) = a + (1 - e^{-\beta T})\frac{1}{T}\log y_{i,t-T} + \text{other variables}$$

- In this case, β is the annual rate of convergence

Table 1. Regressions for Personal Income across U.S. States, 1880–1988

Period	*Basic equation*		*Equation with regional dummies*		*Equation with regional dummies and sectoral variables*[a]	
	$\hat{\beta}$	$R^2\,[\hat{\sigma}]$	$\hat{\beta}$	$R^2\,[\hat{\sigma}]$	$\hat{\beta}$	$R^2\,[\hat{\sigma}]$
1880–1900	0.0101	0.36	0.0224	0.62	0.0268	0.65
	(0.0022)	[0.0068]	(0.0040)	[0.0054]	(0.0048)	[0.0053]
1900–20	0.0218	0.62	0.0209	0.67	0.0269	0.71
	(0.0032)	[0.0065]	(0.0063)	[0.0062]	(0.0075)	[0.0060]
1920–30	−0.0149	0.14	−0.0122	0.43	0.0218	0.64
	(0.0051)	[0.0132]	(0.0074)	[0.0111]	(0.0112)	[0.0089]
1930–40	0.0141	0.35	0.0127	0.36	0.0119	0.46
	(0.0030)	[0.0073]	(0.0051)	[0.0075]	(0.0072)	[0.0071]
1940–50	0.0431	0.72	0.0373	0.86	0.0236	0.89
	(0.0048)	[0.0078]	(0.0053)	[0.0057]	(0.0060)	[0.0053]
1950–60	0.0190	0.42	0.0202	0.49	0.0305	0.66
	(0.0035)	[0.0050]	(0.0052)	[0.0048]	(0.0054)	[0.0041]
1960–70	0.0246	0.51	0.0135	0.68	0.0173	0.72
	(0.0039)	[0.0045]	(0.0043)	[0.0037]	(0.0053)	[0.0036]
1970–80	0.0198	0.21	0.0119	0.36	0.0042	0.46
	(0.0062)	[0.0060]	(0.0069)	[0.0056]	(0.0070)	[0.0052]
1980–88	−0.0060	0.00	−0.0005	0.51	0.0146	0.76
	(0.0130)	[0.0142]	(0.0114)	[0.0103]	(0.0099)	[0.0075]
Nine periods combined[b]						
β restricted	0.0175	. . .	0.0189	. . .	0.0224	. . .
	(0.0013)		(0.0019)		(0.0022)	
Likelihood-ratio statistic[c]	65.6	. . .	32.1	. . .	12.4	. . .
P-value	0.000		0.000		0.134	

Source: Barro and Sala-I-Martin (1991)

Speed of Convergence

TABLE 3

Comparison of Regressions across Countries and U.S. States

Sample	$\hat{\beta}$	Additional Variables	R^2	$\hat{\sigma}$
1. 98 countries, 1960–85	−.0037 (.0018)	no	.04	.0183
2. 98 countries, 1960–85	.0184 (.0045)	yes	.52	.0133
3. 20 OECD countries, 1960–85	.0095 (.0028)	no	.45	.0051
4. 20 OECD countries, 1960–85	.0203 (.0068)	yes	.69	.0046
5. 48 U.S. states, 1963–86	.0218 (.0053)	no	.38	.0040
6. 48 U.S. states, 1963–86	.0236 (.0013)	yes	.61	.0033

Source: Barro and Sala-I-Martin (1992)

Speed of Convergence in the Data

- Barro's "iron law of convergence": 2% per year
- This implies a half-life of 35 years
- Takes 115 years for 90% of convergence to occur
- Convergence is very slow in practice!!

Reconciling Model and Data

- Convergence in basic Solow model way too fast:

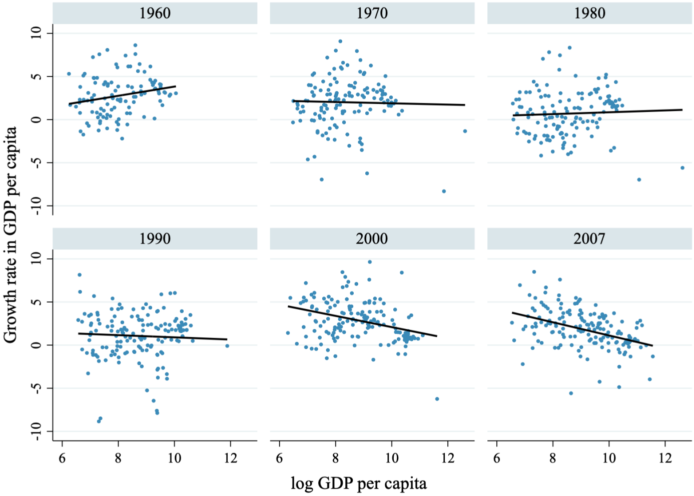
$$\lambda = [1 - \alpha_K(k^*)](n + g + \delta)$$

- One way to reconcile model and data is to raise the value of $\alpha_K(k^*)$
- if $\alpha_K(k^*) \simeq 0.75$ then convergence will be close to 2% per year
- $\alpha_K(k^*)$ is the capital share (if markets are competitive)
- High $\alpha_K(k^*)$ may make sense if one includes human capital

Kremer, Willis, You (2021)

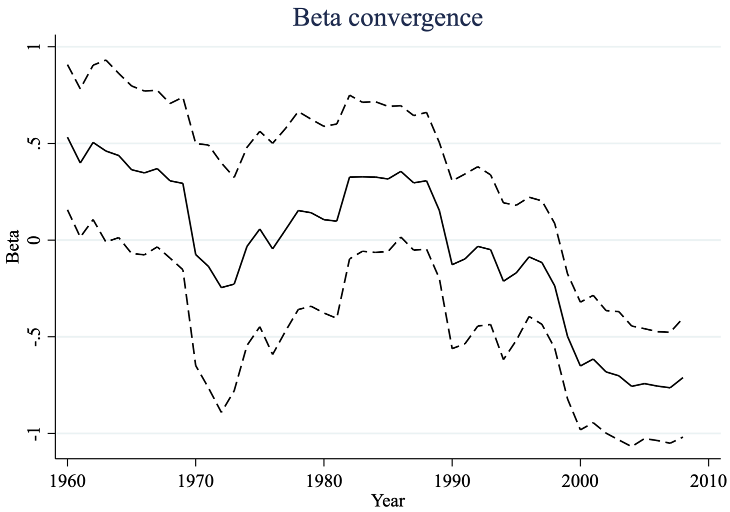
- Revisit convergence after 25 years
- Absolute convergence since 2000
- Why? Proximate answer: Fundamentals have converged (i.e., A, s, n, etc.)
- Leaves deeper question of why fundamentals have converged

Converging to Convergence



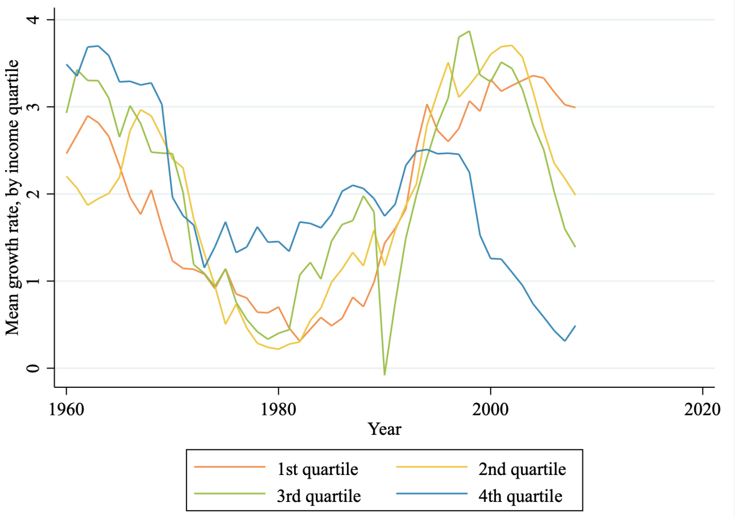
Source: Kremer, Willis, You (2021)

Converging to Convergence



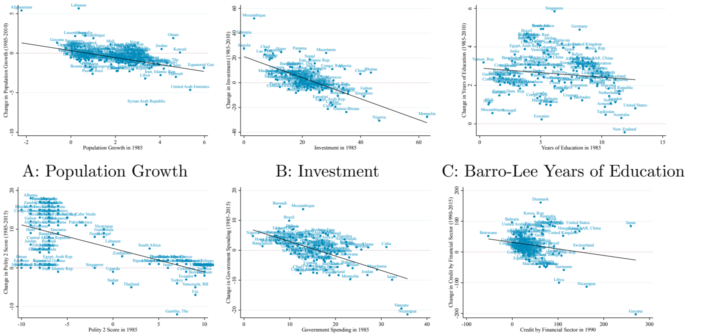
Source: Kremer, Willis, You (2021)

Converging to Convergence



Source: Kremer, Willis, You (2021)

Figure 4: Convergence in growth correlates: level in 1985 versus change 1985-2015



A: Population Growth | B: Investment | C: Barro-Lee Years of Education

D: Polity 2 | E: Government Spending | F: Credit

Notes: This figure plots β-convergence for growth six representative correlates (potential determinants of steady-state income) from 1985 (or the earliest available year) to 2015 against the baseline correlate level in 1985. We include six of the correlates which are comparable over time, for illustration: Population growth rate (%), Investment rate (% of GDP), Barro-Lee average years of education among 20-60-year-olds, Polity 2 score, government spending (% of GDP), credit by the financial sector. The sample for each figure is the complete set of countries for which the relevant data is available in 1985 and 2015.

Source: Kremer, Willis, You (2021)

Appendix

Beware the Linear Scale!

Source: Clark (2010) Back