SOLUTIONS TO CHAPTER 1

Problem 1.1

(a) Since the growth rate of a variable equals the time derivative of its log, as shown by equation (1.10) in the text, we can write

$$(1)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln\left[X(t)Y(t)\right]}{dt}.$$

Since the log of the product of two variables equals the sum of their logs, we have

$$(2)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\left[\ln X(t) + \ln Y(t)\right]}{dt} = \frac{d\ln X(t)}{dt} + \frac{d\ln Y(t)}{dt},$$

or simply

$$(3)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) Again, since the growth rate of a variable equals the time derivative of its log, we can write

$$(4)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln\left[X(t)/Y(t)\right]}{dt}.$$

Since the log of the ratio of two variables equals the difference in their logs, we have

$$(5)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\left[\ln X(t) - \ln Y(t)\right]}{dt} = \frac{d\ln X(t)}{dt} - \frac{d\ln Y(t)}{dt},$$

or simply

$$(6)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}.$$

(c) We have

$$(7)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)^{\alpha}]}{dt}.$$

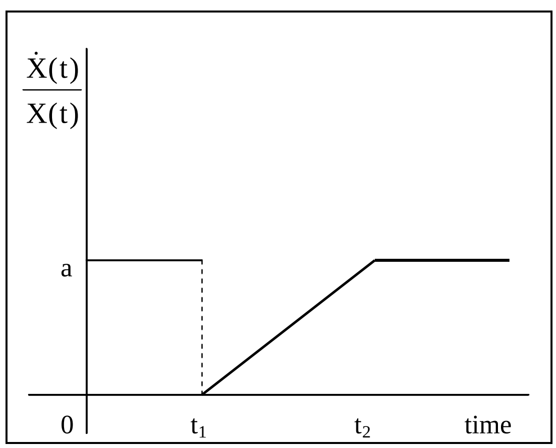
Using the fact that $\ln[X(t)^{\alpha}] = \alpha\ln X(t)$, we have

$$(8)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\left[\alpha\ln X(t)\right]}{dt} = \alpha\frac{d\ln X(t)}{dt} = \alpha\frac{\dot{X}(t)}{X(t)},$$

where we have used the fact that α is a constant.

Problem 1.2

(a) Using the information provided in the question, the path of the growth rate of X, $\dot{X}(t)/X(t)$, is depicted in the figure at right.



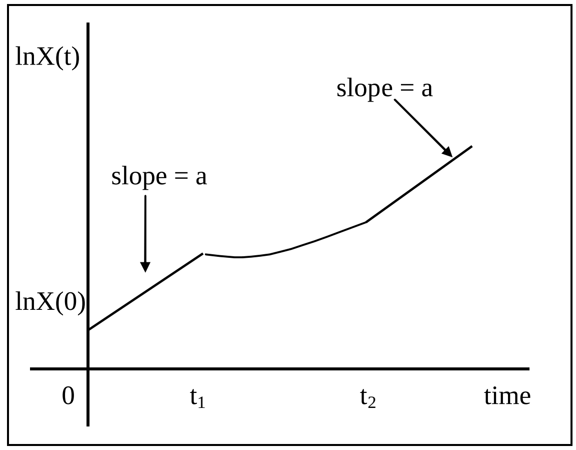
From time 0 to time t_1, the growth rate of X is constant and equal to $a > 0$. At time t_1, the growth rate of X drops to 0. From time t_1 to time t_2, the growth rate of X rises gradually from 0 to a. Note that we have made the assumption that $\dot{X}(t)/X(t)$ rises at a constant rate from t_1 to t_2. Finally, after time t_2, the growth rate of X is constant and equal to a again.

(b) Note that the slope of lnX(t) plotted against time is equal to the growth rate of X(t). That is, we know

$$\frac{d\ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

(See equation (1.10) in the text.)

From time 0 to time t_1 the slope of lnX(t) equals a > 0. The lnX(t) locus has an inflection point at t_1, when the growth rate of X(t) changes discontinuously from a to 0. Between t_1 and t_2, the slope of lnX(t) rises gradually from 0 to a. After time t_2 the slope of lnX(t) is constant and equal to a > 0 again.

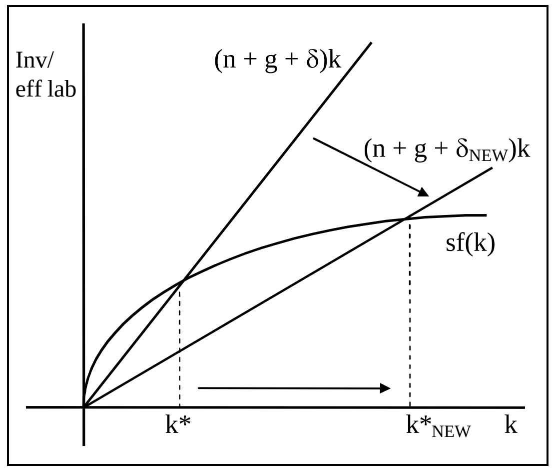


Problem 1.3

(a) The slope of the break-even investment line is given by (n + g + δ) and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, sf(k) is unaffected.

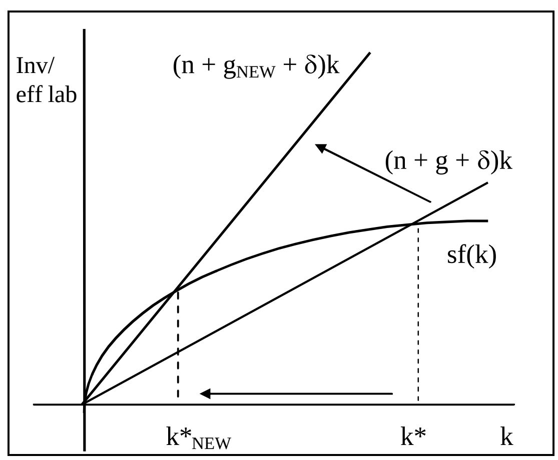
From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k* to k^*_{NEW} .

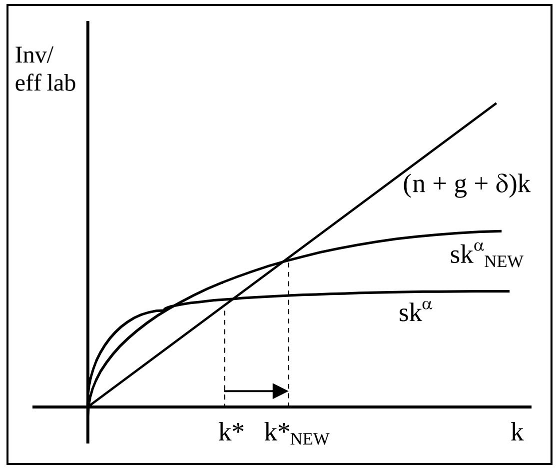


(b) Since the slope of the break-even investment line is given by (n + g + δ), a rise in the rate of technological progress, g, makes the break-even investment line steeper.

The actual investment curve, sf(k), is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor falls from k* to k^*_{NEW} .





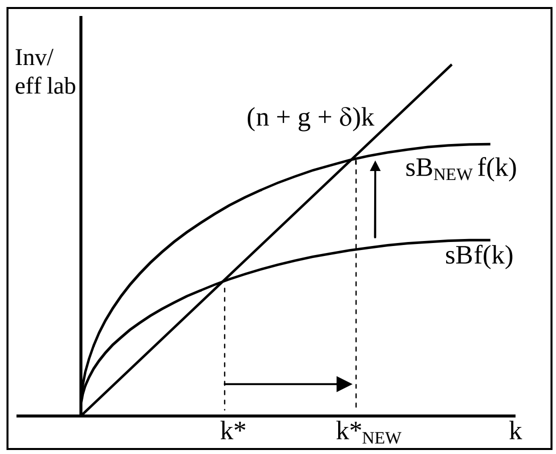
(c) The break-even investment line, $(n + g + \delta)k$, is unaffected by the rise in capital's share, α.

The effect of a change in α on the actual investment curve, sk^{α}, can be determined by examining the derivative $\partial(sk^{\alpha})/\partial\alpha$. It is possible to show that

(1) $\dfrac{\partial sk^{\alpha}}{\partial\alpha} = sk^{\alpha}\ln k$.

For $0 < \alpha < 1$, and for positive values of k, the sign of $\partial(sk^{\alpha})/\partial\alpha$ is determined by the sign of lnk. For $\ln k > 0$, or $k > 1$, $\partial sk^{\alpha}/\partial\alpha > 0$ and so the new actual investment curve lies above the old one. For $\ln k < 0$ or $k < 1$, $\partial sk^{\alpha}/\partial\alpha < 0$ and so the new actual investment curve lies below the old one. At $k = 1$, so that $\ln k = 0$, the new actual investment curve intersects the old one.

In addition, the effect of a rise in α on k* is ambiguous and depends on the relative magnitudes of s and $(n + g + \delta)$. It is possible to show that a rise in capital's share, α, will cause k* to rise if $s > (n + g + \delta)$. This is the case depicted in the figure above.



(d) Suppose we modify the intensive form of the production function to include a non-negative constant, B, so that the actual investment curve is given by $sBf(k)$, $B > 0$.

Then workers exerting more effort, so that output per unit of effective labor is higher than before, can be modeled as an increase in B. This increase in B shifts the actual investment curve up.

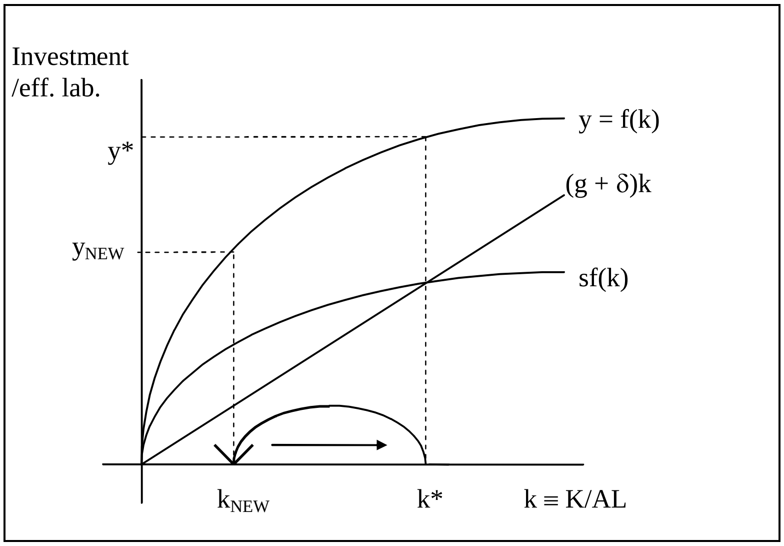
The break-even investment line, $(n + g + \delta)k$, is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k^* to k^*_{NEW} .

Problem 1.4

(a) At some time, call it t_0 , there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW} . We can see this by simply looking at the definition, $k \equiv K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y^* to y_{NEW} .

(b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g + \delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y*.



(c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k*. At k*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

Problem 1.5

(a) The equation describing the evolution of the capital stock per unit of effective labor is given by

(1) $\dot{k} = sf(k) - (n + g + \delta)k$.

Substituting in for the intensive form of the Cobb-Douglas, $f(k) = k^{\alpha}$, yields

(2) $\dot{k} = sk^{\alpha} - (n + g + \delta)k$.

On the balanced growth path, $\dot{k}$ is zero; investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k*, we have $sk^{*\alpha} = (n + g + \delta)k^*$. Rearranging to solve for k* yields

(3) $k^* = \left[s/(n + g + \delta)\right]^{1/(1-\alpha)}$.

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (3) into the intensive form of the production function, $y = k^{\alpha}$:

(4) $y^* = \left[s/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$. Substituting equation (4) into this expression yields

(5) $c^* = (1 - s)\left[s/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

(b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (3), which expresses the balanced-growth-path level of k, and rearrange it to solve for s:

(6) $s = (n + g + \delta)k^{*1-\alpha}$.

Now substitute equation (6) into equation (5):

(7) $c^* = \left[1 - (n + g + \delta)k^{*1-\alpha}\right]\left[(n + g + \delta)k^{*1-\alpha}/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

After some straightforward algebraic manipulation, this simplifies to

(8) $c^* = k^{*\alpha} - (n + g + \delta)k^*$.

Equation (8) states that consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor. On the balanced growth path, actual investment per unit of effective labor is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (8) to maximize c^* with respect to k^*. The first-order condition is given by

(9) $\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0$,

or simply

(10) $\alpha k^{*\alpha-1} = (n + g + \delta)$.

Note that equation (10) is just a specific form of the general condition that implicitly defines the golden-rule level of capital per unit of effective labor, given by $f'(k^*) = (n + g + \delta)$. Equation (10) has a graphical interpretation: it defines the level of k at which the slope of the intensive form of the production function is equal to the slope of the break-even investment line. Solving equation (10) for the golden-rule level of k yields

(11) $k^*_{GR} = [\alpha/(n + g + \delta)]^{1/(1-\alpha)}$.

(c) To get the saving rate that yields the golden-rule level of k, substitute equation (11) into (6):

(12) $s_{GR} = (n + g + \delta)[\alpha/(n + g + \delta)]^{(1-\alpha)/(1-\alpha)}$,

which simplifies to

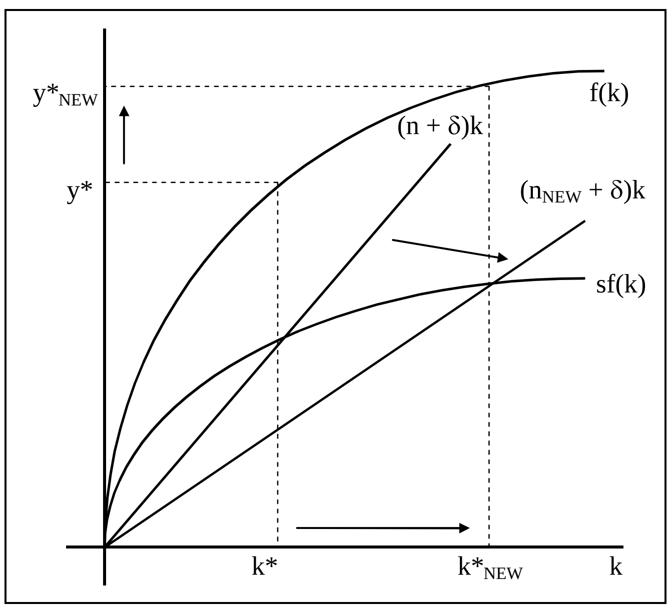
(13) $s_{GR} = \alpha$.

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

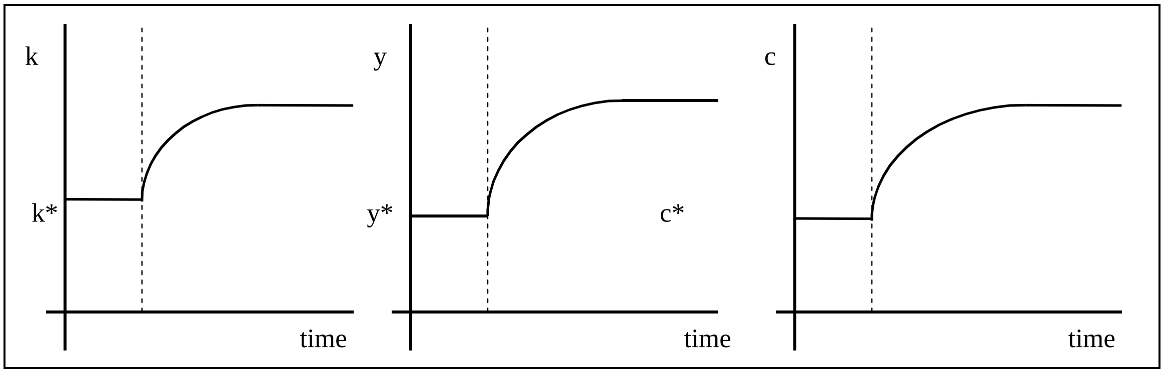
Problem 1.6

(a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus we can define $y \equiv Y/L$ and $k \equiv K/L$.

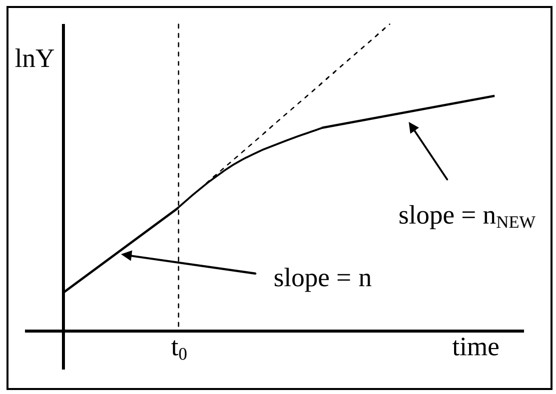
The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k, capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n – the economy was on a balanced growth path – the decrease in n causes $\dot{k}$ to become positive. At k*, actual investment per worker, sf(k*), now exceeds break-even investment per worker, $(n_{NEW} + \delta)k^*$. Thus k moves to a new higher balanced growth path level. See the figure at right.



As k rises, y – output per worker – also rises. Since a constant fraction of output is saved, c – consumption per worker – rises as y rises. This is summarized in the figures below.



(b) By definition, output can be written as $Y \equiv Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ – output per worker is constant – so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again – output per worker is constant again – and so $\dot{Y}/Y = \dot{L}/L = n_{NEW} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Examine the production function Y = F(K,AL). On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and that of AL (which is growing at n_{NEW}). Thus, during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally K, AL, and thus Y are all growing at the new lower n_{NEW}.

Problem 1.7

The derivative of $y^* = f(k^*)$ with respect to n is given by

(1) $\partial y^*/\partial n = f'(k^*)[\partial k^*/\partial n]$.

To find $\partial k^*/\partial n$, use the equation for the evolution of the capital stock per unit of effective labor, $\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, $k = k^*$ and thus $sf(k^*) = (n + g + \delta)k^*$. Taking the derivative of both sides of this expression with respect to n yields

(2) $sf'(k^*)\dfrac{\partial k^*}{\partial n} = (n + g + \delta)\dfrac{\partial k^*}{\partial n} + k^*$,

and rearranging yields

(3) $\dfrac{\partial k^*}{\partial n} = \dfrac{k^*}{sf'(k^*) - (n + g + \delta)}$.

Substituting equation (3) into equation (1) gives us

(4) $\dfrac{\partial y^*}{\partial n} = f'(k^*)\left[\dfrac{k^*}{sf'(k^*) - (n + g + \delta)}\right]$.

Rearranging the condition that implicitly defines k*, sf(k*) = (n + g + δ)k*, and solving for s yields
(5) $s = (n + g + \delta)k^*/f(k^*)$.
Substitute equation (5) into equation (4):
(6) $$\frac{\partial y^*}{\partial n} = \frac{f'(k^*)k^*}{[(n+g+\delta)f'(k^*)k^*/f(k^*)] - (n+g+\delta)}.$$
To turn this into the elasticity that we want, multiply both sides of equation (6) by n/y*:
(7) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = \frac{n}{(n+g+\delta)}\frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)] - 1}.$$
Using the definition that $\alpha_K(k^*) \equiv f'(k^*)k^*/f(k^*)$ gives us
(8) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{n}{(n+g+\delta)}\left[\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}\right].$$

Now, with $\alpha_K(k^*) = 1/3$, g = 2% and δ = 3%, we need to calculate the effect on y* of a fall in n from 2% to 1%. Using the midpoint of n = 0.015 to calculate the elasticity gives us
(9) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{0.015}{(0.015 + 0.02 + 0.03)}\left(\frac{1/3}{1-1/3}\right) \cong -0.12.$$
So this 50% drop in the population growth rate, from 2% to 1%, will lead to approximately a 6% increase in the level of output per unit of effective labor, since (-0.50)(-0.12) = 0.06. This calculation illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

Problem 1.8
(a) A permanent increase in the fraction of output that is devoted to investment from 0.15 to 0.18 represents a 20 percent increase in the saving rate. From equation (1.27) in the text, the elasticity of output with respect to the saving rate is
(1) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)},$$
where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

Substituting the assumption that $\alpha_K(k^*) = 1/3$ into equation (1) gives us
(2) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)} = \frac{1/3}{1-1/3} = \frac{1}{2}.$$
Thus the elasticity of output with respect to the saving rate is 1/2. So this 20 percent increase in the saving rate – from s = 0.15 to $s_{NEW} = 0.18$ – causes output to rise relative to what it would have been by about 10 percent. [Note that the analysis has been carried out in terms of output per unit of effective labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10 percent, output itself is also 10 percent higher than what it would have been.]

(b) Consumption rises less than output. Output ends up 10 percent higher than what it would have been. But the fact that the saving rate is higher means that we are now consuming a smaller fraction of output. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by
(3) $c^* = (1 - s)y^*$.
Taking the derivative with respect to s yields

(4) $\dfrac{\partial c^*}{\partial s} = -y^* + (1-s)\dfrac{\partial y^*}{\partial s}$.

To turn this into an elasticity, multiply both sides of equation (4) by s/c*:

(5) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-y^* s}{(1-s)y^*} + (1-s)\dfrac{\partial y^*}{\partial s}\dfrac{s}{(1-s)y^*}$,

where we have substituted c* = (1 - s)y* on the right-hand side. Simplifying gives us

(6) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-s}{(1-s)} + \dfrac{\partial y^*}{\partial s}\dfrac{s}{(1-s)y^*}$.

From part (a), the second term on the right-hand side of (6), the elasticity of output with respect to the saving rate, equals 1/2. We can use the midpoint between s = 0.15 and s_{NEW} = 0.18 to calculate the elasticity:

(7) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-0.165}{(1-0.165)} + 0.5 \cong 0.30$.

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate will cause consumption to be approximately 6% above what it would have been.

(c) The immediate effect of the rise in investment as a fraction of output is that consumption falls. Although y* does not jump immediately – it only begins to move toward its new, higher balanced-growth-path level – we are now saving a greater fraction, and thus consuming a smaller fraction, of this same y*. At the moment of the rise in s by 3 percentage points – since c = (1 - s)y* and y* is unchanged – c falls. In fact, the percentage change in c will be the percentage change in (1 - s). Now, (1 - s) falls from 0.85 to 0.82, which is approximately a 3.5 percent drop. Thus at the moment of the rise in s, consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine the length of time it takes for consumption to return to what it would have been without the increase in the saving rate. After the initial rise in s, s remains constant throughout. Since c = (1 - s)y, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y, after a linear approximation, is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta)$ equal to 6 percent per year and $\alpha_K = 1/3$, this yields a value for λ of about 4 percent. This means that k and y move about 4 percent of the remaining distance toward their balanced-growth-path values of k* and y* each year. Since c is proportional to y, c = (1 - s)y, it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.31) in the text, we could write

(8) $c(t) - c^* \cong e^{-(1-\alpha_K)(n+g+\delta)t}[c(0) - c^*]$,

or equivalently

(9) $e^{-\lambda t} = \dfrac{c(t) - c^*}{c(0) - c^*}$.

The term on the right-hand side of equation (9) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5 percent and will eventually be 6 percent higher than it would have been. Thus it must change by 9.5 percent on the way to the balanced growth path. It will therefore be equal to what it would have been about 36.8 percent (3.5%/9.5% ≅ 36.8%) of the way to the new balanced growth path. Equivalently, this is when the remaining distance to the new balanced growth path is 63.2 percent of the original distance. In order to determine the length of time this will take, we need to find a t* that solves

(10) $e^{-\lambda t^*} = 0.632$.

Taking the natural logarithm of both sides of equation (10) yields

(11) $-\lambda t^* = \ln(0.632)$.

Rearranging to solve for t gives us

(12) $t^* = 0.459/0.04$,

and thus

(13) $t^* \cong 11.5$ years.

It will take a fairly long time – over a decade – for consumption to return to what it would have been in the absence of the increase in investment as a fraction of output.

Problem 1.9

(a) Define the marginal product of labor to be $w \equiv \partial F(K,AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields

(1) $w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$,

as required.

(b) Define the marginal product of capital as $r \equiv [\partial F(K,AL)/\partial K] - \delta$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields

(2) $r \equiv [\partial Y/\partial K] - \delta = ALf'(k)[1/AL] - \delta = f'(k) - \delta$.

Substitute equations (1) and (2) into $wL + rK$:

(3) $wL + rK = A[f(k) - kf'(k)]L + [f'(k) - \delta]K = ALf(k) - f'(k)[K/AL]AL + f'(k)K - \delta K$.

Simplifying gives us

(4) $wL + rK = ALf(k) - f'(k)K + f'(k)K - \delta K = Alf(k) - \delta K \equiv ALF(K/AL, 1) - \delta K$.

Finally, since F is constant returns to scale, equation (4) can be rewritten as

(5) $wL + rK = F(ALK/AL, AL) - \delta K = F(K, AL) - \delta K$.

(c) As shown above, $r = f'(k) - \delta$. Since δ is a constant and since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r. In other words, on a balanced growth path, $\dot{r}/r = 0$. Thus the Solow model does exhibit the property that the return to capital is constant over time.

Since capital is paid its marginal product, the share of output going to capital is rK/Y. On a balanced growth path,

(6) $$\frac{\dot{\left(rK/Y\right)}}{\left(rK/Y\right)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n+g) - (n+g) = 0.$$

Thus, on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to one, this implies that the share of output going to labor is also constant on the balanced growth path.

We need to determine the growth rate of the marginal product of labor, w, on a balanced growth path. As shown above, $w = A[f(k) - kf'(k)]$. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

(7) $$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{\dot{[f(k) - kf'(k)]}}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

On a balanced growth path $\dot{k} = 0$ and so $\dot{w}/w = g$. That is, on a balanced growth path, the marginal product of labor rises at the rate of growth of the effectiveness of labor.

(d) As shown in part (c), the growth rate of the marginal product of labor is

$$(8)\quad \frac{\dot{w}}{w} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

If k < k*, then as k moves toward k*, $\dot{w}/w > g$. This is true because the denominator of the second term on the right-hand side of equation (8) is positive because f(k) is a concave function. The numerator of that same term is positive because k and $\dot{k}$ are positive and f '' (k) is negative. Thus, as k rises toward k*, the marginal product of labor grows faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path. As we move from k to k*, however, the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r, is

$$(9)\quad \frac{\dot{r}}{r} = \frac{\dot{\left[f'(k)\right]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}.$$

As k rises toward k*, this growth rate is negative since f ' (k) > 0, f '' (k) < 0 and $\dot{k}$ > 0. Thus, as the economy moves from k to k*, the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.

Problem 1.10

(a) By definition a balanced growth path occurs when all the variables of the model are growing at constant rates. Despite the differences between this model and the usual Solow model, it turns out that we can again show that the economy will converge to a balanced growth path by examining the behavior of k ≡ K/AL.

Taking the time derivative of both sides of the definition of k ≡ K/AL gives us

$$(1)\quad \dot{k} = \dot{\left(\frac{K}{AL}\right)} = \frac{\dot{K}(AL) - K\left[\dot{L}A - \dot{A}L\right]}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL}\left[\frac{\dot{L}A + \dot{A}L}{AL}\right] = \frac{\dot{K}}{AL} - k\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right).$$

Substituting the capital-accumulation equation, $\dot{K} = \left[\partial F(K,AL)/\partial K\right]K - \delta K$, and the constant growth rates of the labor force and technology, $\dot{L}/L = n$ and $\dot{A}/A = g$, into equation (1) yields

$$(2)\quad \dot{k} = \frac{\left[\partial F(K,AL)/\partial K\right]K - \delta K}{AL} - (n+g)k = \frac{\partial F(K,AL)}{\partial K}k - \delta k - (n+g)k.$$

Substituting ∂F(K,AL)/∂K = f '(k) into equation (2) gives us $\dot{k} = f'(k)k - \delta k - (n+g)k$ or simply

$$(3)\quad \dot{k} = \left[f'(k) - (n+g+\delta)\right]k.$$

Capital per unit of effective labor will be constant when $\dot{k} = 0$, i.e. when [f ' (k) - (n + g + δ)] k = 0. This condition holds if k = 0 (a case we will ignore) or f ' (k) - (n + g + δ) = 0. Thus the balanced-growth-path level of the capital stock per unit of effective labor is implicitly defined by f '(k*) = (n + g + δ). Since capital per unit of effective labor, k ≡ K/AL, is constant on the balanced growth path, K must grow at the same rate as AL, which grows at rate n + g. Since the production function has constant returns to capital and effective labor, which both grow at rate n + g on the balanced growth path, output must also grow at rate n + g on the balanced growth path. Thus we have found a balanced growth path where all the variables of the model grow at constant rates.

The next step is to show that the economy actually converges to this balanced growth path. At $k = k^*$, $f'(k) = (n + g + \delta)$. If $k > k^*$, $f'(k) < (n + g + \delta)$. This follows from the assumption that $f''(k) < 0$ which means that $f'(k)$ falls as k rises. Thus if $k > k^*$, we have $\dot{k} < 0$ so that k will fall toward its balanced-growth-path value. If $k < k^*$, $f'(k) > (n + g + \delta)$. Again, this follows from the assumption that $f''(k) < 0$ which means that $f'(k)$ rises as k falls. Thus if $k < k^*$, we have $\dot{k} > 0$ so that k will rise toward its balanced-growth-path value. Thus, regardless of the initial value of k (as long as it is not zero), the economy will converge to a balanced growth path at k*, where all the variables in the model are growing at constant rates.

(b) The golden-rule level of k – the level of k that maximizes consumption per unit of effective labor – is defined implicitly by $f'(k^{GR}) = (n + g + \delta)$. This occurs when the slope of the production function equals the slope of the break-even investment line. Note that this is exactly the level of k that the economy converges to in this model where all capital income is saved and all labor income is consumed.

In this model, we are saving capital's contribution to output, which is the marginal product of capital times the amount of capital. If that contribution exceeds break-even investment, $(n + g + \delta)k$, then k rises. If it is less than break-even investment, k falls. Thus k settles down to a point where saving, the marginal product of capital times k, equals break-even investment, $(n + g + \delta)k$. That is, the economy settles down to a point where $f'(k)k = (n + g + \delta)k$ or equivalently $f'(k) = (n + g + \delta)$.

Problem 1.11

We know that $\dot{y}$ is determined by k but since $k = g(y)$, where $g(\bullet) = f^{-1}(\bullet)$, we can write $\dot{y} = \dot{y}(y)$. When $k = k^*$ and thus $y = y^*$, $\dot{y} = 0$. A first-order Taylor-series approximation of $\dot{y}(y)$ around $y = y^*$ therefore yields

(1) $\dot{y} \cong \left[\left. \dfrac{\partial \dot{y}}{\partial y} \right|_{y=y^*} \right] (y - y^*)$.

Let λ denote $-\partial \dot{y}(y)/\partial y \big|_{y=y^*}$. With this definition, equation (1) becomes

(2) $\dot{y}(t) \cong -\lambda[y(t) - y^*]$.

Equation (2) implies that in the vicinity of the balanced growth path, y moves toward y* at a speed approximately proportional to its distance from y*. That is, the growth rate of $y(t) - y^*$ is approximately constant and equal to $-\lambda$. This implies

(3) $y(t) \cong y^* + e^{-\lambda t}[y(0) - y^*]$,

where y(0) is the initial value of y. We now need to determine λ.

Taking the time derivative of both sides of the production function,

(4) $y = f(k)$,

yields

(5) $\dot{y} = f'(k)\dot{k}$.

The equation of motion for capital is given by

(6) $\dot{k} = sf(k) - (n + g + \delta)k$.

Substituting equation (6) into equation (5) yields

(7) $\dot{y} = f'(k)[sf(k) - (n + g + \delta)k]$.

Equation (7) expresses $\dot{y}$ in terms of k. But $k = g(y)$ where $g(\bullet) = f^{-1}(\bullet)$. Thus we can write

(8) $\left.\dfrac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = \left[\left.\dfrac{\partial \dot{y}}{\partial k}\right|_{y=y^*}\right]\left[\left.\dfrac{\partial k}{\partial y}\right|_{y=y^*}\right]$.

Taking the derivative of $\dot{y}$ with respect to k gives us

(9) $\dfrac{\partial \dot{y}}{\partial k} = f''(k)[sf(k) - (n+g+\delta)k] + f'(k)[sf'(k) - (n+g+\delta)]$.

On the balanced growth path, $sf(k^*) = (n + g + \delta)k^*$ and thus

(10) $\left.\dfrac{\partial \dot{y}}{\partial k}\right|_{y=y^*} = f'(k^*)[sf'(k^*) - (n+g+\delta)]$.

Now, since $k = g(y)$ where $g(\bullet) = f^{-1}(\bullet)$,

(11) $\left.\dfrac{\partial k}{\partial y}\right|_{y=y^*} = \dfrac{1}{\left.\dfrac{\partial y}{\partial k}\right|_{y=y^*}} = \dfrac{1}{f'(k^*)}$.

Substituting equations (10) and (11) into equation (8) yields

(12) $\left.\dfrac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = f'(k^*)[sf'(k^*) - (n+g+\delta)]\dfrac{1}{f'(k^*)}$,

or simply

(13) $\left.\dfrac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = sf'(k^*) - (n+g+\delta)$.

And thus

(14) $\lambda \equiv -\left.\dfrac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = (n+g+\delta) - sf'(k^*)$.

Since $s = (n + g + \delta)k^*/f(k^*)$ on the balanced growth path, we can rewrite (14) as

(15) $\lambda \equiv -\left.\dfrac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = (n+g+\delta) - \dfrac{(n+g+\delta)k^* f'(k^*)}{f(k^*)}$.

Now use the definition that $\alpha_K \equiv kf'(k)/f(k)$ to rewrite (15) as

(16) $\lambda \equiv -\left.\dfrac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = [1 - \alpha_K(k^*)](n+g+\delta)$.

Thus y converges to its balanced-growth-path value at rate $[1 - \alpha_K(k^*)](n+g+\delta)$, the same rate at which k converges to its balanced-growth-path value.

Problem 1.12

(a) The production function with capital-augmenting technological progress is given by

(1) $Y(t) = [A(t)K(t)]^{\alpha} L(t)^{1-\alpha}$.

Dividing both sides of equation (1) by $A(t)^{\alpha/(1-\alpha)}L(t)$ yields

(2) $\dfrac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\dfrac{A(t)K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}\left[\dfrac{L(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{1-\alpha}$,

and simplifying:

(3) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}K(t)}{L(t)}\right]^{\alpha} A(t)^{-\alpha} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}A(t)^{-1}K(t)}{L(t)}\right]^{\alpha},$$

and thus finally

(4) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi \equiv \alpha/(1 - \alpha)$, $k(t) \equiv K(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(5) $y(t) = k(t)^{\alpha}$.

In order to analyze the dynamics of k(t), take the time derivative of both sides of $k(t) \equiv K(t)/A(t)^{\phi}L(t)$:

(6) $$\dot{k}(t) = \frac{\dot{K}(t)\left[A(t)^{\phi}L(t)\right] - K(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^2},$$

(7) $$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)^{\phi}L(t)} - \frac{K(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $k(t) \equiv K(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(8) $\dot{k}(t) = \dot{K}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t)$.

The evolution of the total capital stock is given by the usual

(9) $\dot{K}(t) = sY(t) - \delta K(t)$.

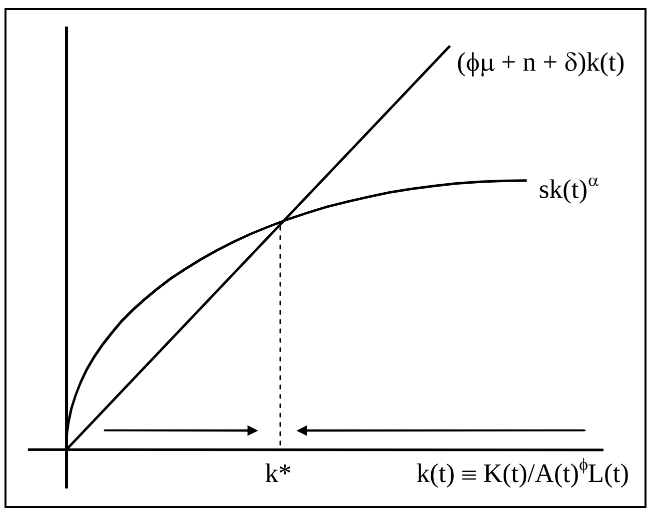
Substituting equation (9) into (8) gives us

(10) $\dot{k}(t) = sY(t)/A(t)^{\phi}L(t) - \delta K(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t) = sy(t) - (\phi\mu + n + \delta)k(t)$.

Finally, using equation (5), $y(t) = k(t)^{\alpha}$, we have

(11) $\dot{k}(t) = sk(t)^{\alpha} - (\phi\mu + n + \delta)k(t)$.

Equation (11) is very similar to the basic equation governing the dynamics of the Solow model with labor-augmenting technological progress. Here, however, we are measuring in units of $A(t)^{\phi}L(t)$ rather than in units of effective labor, A(t)L(t). Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{k}(t)$. See the figure at right.



When actual investment per unit of $A(t)^{\phi}L(t)$, $sk(t)^{\alpha}$, exceeds break-even investment per unit of $A(t)^{\phi}L(t)$, given by $(\phi\mu + n + \delta)k(t)$, k will rise toward k*. When actual investment per unit of $A(t)^{\phi}L(t)$ falls short of break-even investment per unit of $A(t)^{\phi}L(t)$, k will fall toward k*. Ignoring the case in which the initial level of k is zero, the economy will converge to a situation in which k is constant at k*. Since $y = k^{\alpha}$, y will also be constant when the economy converges to k*.

The total capital stock, K, can be written as $A^{\phi}Lk$. Thus when k is constant, K will be growing at the constant rate of $\phi\mu + n$. Similarly, total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$ as well. Since L and A grow at constant rates by assumption, we have found a balanced growth path where all the variables of the model grow at constant rates.

(b) The production function is now given by

(12) $Y(t) = J(t)^{\alpha} L(t)^{1-\alpha}$.

Define $\bar{J}(t) \equiv J(t)/A(t)$. The production function can then be written as

(13) $Y(t) = \left[A(t)\bar{J}(t)\right]^{\alpha} L(t)^{1-\alpha}$.

Proceed as in part (a). Divide both sides of equation (13) by $A(t)^{\alpha/(1-\alpha)}L(t)$ and simplify to obtain

(14) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)} L(t)} = \left[\frac{\bar{J}(t)}{A(t)^{\alpha/(1-\alpha)} L(t)}\right]^{\alpha}.$$

Now, defining $\phi \equiv \alpha/(1 - \alpha)$, $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(15) $y(t) = \bar{j}(t)^{\alpha}$.

In order to analyze the dynamics of $\bar{j}(t)$, take the time derivative of both sides of $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$:

(16) $$\dot{\bar{j}} = \frac{\dot{\bar{J}}(t)\left[A(t)^{\phi} L(t)\right] - \bar{J}(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi} L(t)\right]^2},$$

(17) $$\dot{\bar{j}}(t) = \frac{\dot{\bar{J}}(t)}{A(t)^{\phi} L(t)} - \frac{\bar{J}(t)}{A(t)^{\phi} L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(18) $\dot{\bar{j}}(t) = \dot{\bar{J}}(t)/A(t)^{\phi} L(t) - (\phi\mu + n)\bar{j}(t)$.

The next step is to get an expression for $\dot{\bar{J}}(t)$. Take the time derivative of both sides of $\bar{J}(t) \equiv J(t)/A(t)$:

(19) $$\dot{\bar{J}}(t) = \frac{\dot{J}(t)A(t) - J(t)\dot{A}(t)}{A(t)^2} = \frac{\dot{J}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)}\frac{J(t)}{A(t)}.$$

Now use $\bar{J}(t) \equiv J(t)/A(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$ to obtain

(20) $$\dot{J}(t) = \frac{sA(t)Y(t)}{A(t)} - \frac{\delta J(t)}{A(t)} - \mu\bar{J}(t),$$

or simply

(21) $\dot{\bar{J}}(t) = sY(t) - (\mu + \delta)\bar{J}(t)$.

Substitute equation (21) into equation (18):

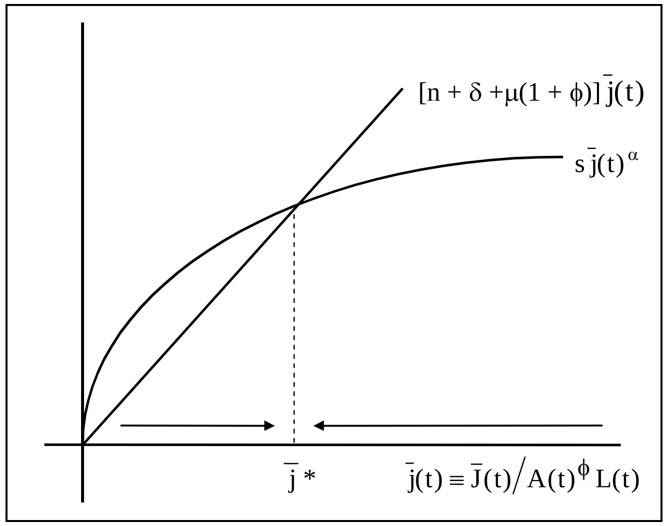
(22) $\dot{\bar{j}}(t) = sY(t)/A(t)^{\phi} L(t) - (\mu + \delta)\bar{J}(t)/A(t)^{\phi} L(t) - (\phi\mu + n)\bar{j}(t) = sy(t) - \left[n + \delta + \mu(1 + \phi)\right]\bar{j}(t)$.

Finally, using equation (15), $y(t) = \bar{j}(t)^{\alpha}$, we have

(23) $\dot{\bar{j}}(t) = s\bar{j}(t)^{\alpha} - \left[n + \delta + \mu(1 + \phi)\right]\bar{j}(t)$.

Using the same graphical technique as in the basic Solow model, we can graph both components of $\dot{\bar{j}}(t)$.

See the figure at right. Ignoring the possibility that the initial value of $\bar{j}$ is zero, the economy will converge to a situation where $\bar{j}$ is constant at $\bar{j}*$. Since $y = \bar{j}^{\alpha}$, y will also be constant when the economy converges to $\bar{j}*$.



The level of total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$.

By definition, $\bar{J} \equiv A^{\phi}L\bar{j}$. Once the economy converges to the situation where $\bar{j}$ is constant, $\bar{J}$ grows at the constant rate of $\phi\mu + n$. Since $J \equiv \bar{J}A$, the effective capital stock, J, grows at rate $\phi\mu + n + \mu$ or $n + \mu(1 + \phi)$. Thus the economy does converge to a balanced growth path where all the variables of the model are growing at constant rates.

(c) On the balanced growth path, $\dot{\bar{j}}(t) = 0$ and thus from equation (23):

(24) $s\bar{j}^{\alpha} = [n + \delta + \mu(1+\phi)]\bar{j} \qquad \Rightarrow \qquad \bar{j}^{1-\alpha} = s/[n + \delta + \mu(1+\phi)]$,

and thus

(25) $\bar{j}* = [s/(n + \delta + \mu(1+\phi))]^{1/(1-\alpha)}$.

Substitute equation (25) into equation (15) to get an expression for output per unit of $A(t)^{\phi}L(t)$ on the balanced growth path:

(26) $y* = [s/(n + \delta + \mu(1+\phi))]^{\alpha/(1-\alpha)}$.

Take the derivative of y* with respect to s:

(27) $$\frac{\partial y*}{\partial s} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right].$$

In order to turn this into an elasticity, multiply both sides by s/y* using the expression for y* from equation (26) on the right-hand side:

(28) $$\frac{\partial y*}{\partial s}\frac{s}{y*} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right]s\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{-\alpha/(1-\alpha)}.$$

Simplifying yields

(29) $$\frac{\partial y*}{\partial s}\frac{s}{y*} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{n+\delta+\mu(1+\phi)}{s}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right],$$

and thus finally

(30) $$\frac{\partial y*}{\partial s}\frac{s}{y*} = \frac{\alpha}{1-\alpha}.$$

(d) A first-order Taylor approximation of $\dot{y}$ around the balanced-growth-path value of y = y* will be of the form

(31) $\dot{y} \cong \partial\dot{y}/\partial y\big|_{y=y^*}\left[y - y^*\right]$.

Taking the time derivative of both sides of equation (15) yields

(32) $\dot{y} = \alpha\bar{j}^{\alpha-1}\dot{\bar{j}}$.

Substitute equation (23) into equation (32):

(33) $\dot{y} = \alpha\bar{j}^{\alpha-1}\left[s\bar{j}^{\alpha} - \left(n + \delta + \mu(1+\phi)\right)\bar{j}\right]$,

or

(34) $\dot{y} = s\alpha\bar{j}^{2\alpha-1} - \alpha\bar{j}^{\alpha}\left[n + \delta + \mu(1+\phi)\right]$.

Equation (34) expresses $\dot{y}$ in terms of $\bar{j}$. We can express $\bar{j}$ in terms of y: since y = $\bar{j}^{\alpha}$, we can write $\bar{j}$ = $y^{1/\alpha}$. Thus $\partial\dot{y}/\partial y$ evaluated at y = y* is given by

(35) $\left.\dfrac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = \left[\left.\dfrac{\partial\dot{y}}{\partial\bar{j}}\right|_{y=y^*}\right]\left[\left.\dfrac{\partial\bar{j}}{\partial y}\right|_{y=y^*}\right] = \left[s\alpha(2\alpha-1)\bar{j}^{2(\alpha-1)} - \alpha^2\bar{j}^{\alpha-1}\left(n + \delta + \mu(1+\phi)\right)\right]\left[\dfrac{1}{\alpha}y^{(1-\alpha)/\alpha}\right]$.

Now, $y^{(1-\alpha)/\alpha}$ is simply $\bar{j}^{\,1-\alpha}$ since y = $\bar{j}^{\alpha}$ and thus

(36) $\left.\dfrac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = s(2\alpha-1)\bar{j}^{2(\alpha-1)+(1-\alpha)} - \alpha\bar{j}^{\alpha-1+(1-\alpha)}\left[n + \delta + \mu(1+\phi)\right] = s(2\alpha-1)\bar{j}^{\alpha-1} - \alpha\left[n + \delta + \mu(1+\phi)\right]$.

Finally, substitute out for s by rearranging equation (25) to obtain $s = \bar{j}^{1-\alpha}\left[n + \delta + \mu(1+\phi)\right]$ and thus

(37) $\left.\dfrac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = \bar{j}^{1-\alpha}\left[n + \delta + \mu(1+\phi)\right](2\alpha-1)\bar{j}^{\alpha-1} - \alpha\left[n + \delta + \mu(1+\phi)\right]$,

or simply

(38) $\left.\dfrac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = -(1-\alpha)\left[n + \delta + \mu(1+\phi)\right]$.

Substituting equation (38) into equation (31) gives the first-order Taylor expansion:

(39) $\dot{y} \cong -(1-\alpha)\left[n + \delta + \mu(1+\phi)\right]\left[y - y^*\right]$.

Solving this differential equation (as in the text) yields

(40) $y(t) - y^* = e^{-(1-\alpha)\left[n+\delta+\mu(1+\phi)\right]}\left[y(0) - y^*\right]$.

This means that the economy moves fraction $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$ of the remaining distance toward y* each year.

(e) The elasticity of output with respect to s is the same in this model as in the basic Solow model. The speed of convergence is faster in this model. In the basic Solow model, the rate of convergence is given by $(1 - \alpha)[n + \delta + \mu]$, which is less than the rate of convergence in this model, $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$, since $\phi \equiv \alpha/(1 - \alpha)$ is positive.

<u>Problem 1.13</u>

(a) The growth-accounting technique of Section 1.7 yields the following expression for the growth rate of output per person:

(1) $\dfrac{\dot{Y}(t)}{Y(t)} - \dfrac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\dfrac{\dot{K}(t)}{K(t)} - \dfrac{\dot{L}(t)}{L(t)}\right] + R(t)$,

where α_K (t) is the elasticity of output with respect to capital at time t and R(t) is the Solow residual.

Now imagine applying this growth-accounting equation to a Solow economy that is on its balanced growth path. On the balanced growth path, the growth rates of output per worker and capital per worker are both equal to g, the growth rate of A. Thus equation (1) implies that growth accounting would attribute a fraction α_K of growth in output per worker to growth in capital per worker. It would attribute the rest – fraction $(1 - \alpha_K)$ – to technological progress, as this is what would be left in the Solow residual. So with our usual estimate of $\alpha_K = 1/3$, growth accounting would attribute about 67 percent of the growth in output per worker to technological progress and about 33 percent of the growth in output per worker to growth in capital per worker.

(b) In an accounting sense, the result in part (a) would be true, but in a deeper sense it would not: the <u>reason</u> that the capital-labor ratio grows at rate g on the balanced growth path is because the effectiveness of labor is growing at rate g. That is, the growth in the effectiveness of labor – the growth in A – raises output per worker through two channels. First, by directly raising output but also by (for a given saving rate) increasing the resources devoted to capital accumulation and thereby raising the capital-labor ratio. Growth accounting attributes the rise in output per worker through the second channel to growth in the capital-labor ratio, and not to its underlying source. Thus, although growth accounting is often instructive, it is not appropriate to interpret it as shedding light on the underlying determinants of growth.

<u>Problem 1.14</u>

(a) Ordinary least squares (OLS) yields a biased estimate of the slope coefficient of a regression if the explanatory variable is correlated with the error term. We are given that

(1) $\ln\left[\left(Y/N\right)_{1979}\right] - \ln\left[\left(Y/N\right)_{1870}\right]^* = a + b\ln\left[\left(Y/N\right)_{1870}\right]^* + \varepsilon$, and

(2) $\ln\left[\left(Y/N\right)_{1870}\right] = \ln\left[\left(Y/N\right)_{1870}\right]^* + u$,

where ε and u are assumed to be uncorrelated with each other and with the true unobservable 1870 income per person variable, $\ln[(Y/N)_{1870}]^*$.

Substituting equation (2) into (1) and rearranging yields

(3) $\ln\left[\left(Y/N\right)_{1979}\right] - \ln\left[\left(Y/N\right)_{1870}\right] = a + b\ln\left[\left(Y/N\right)_{1870}\right] + \left[\varepsilon - (1+b)u\right]$.

Running an OLS regression on model (3) will yield a biased estimate of b if $\ln[(Y/N)_{1870}]$ is correlated with the error term, $[\varepsilon - (1 + b)u]$. In general, of course, this will be the case since u is the measurement error that helps to determine the value of $\ln[(Y/N)_{1870}]$ that we get to observe. However, in the special case in which the true value of $b = -1$, the error term in model (3) is simply ε. Thus OLS will be unbiased since the explanatory variable will no longer be correlated with the error term.

(b) Measurement error in the dependent variable will not cause a problem for OLS estimation and is, in fact, one of the justifications for the disturbance term in a regression model. Intuitively, if the measurement error is in 1870 income per capita, the explanatory variable, there will be a bias toward finding convergence. If 1870 income per capita is overstated, growth is understated. This looks like convergence: a "high" initial income country growing slowly. Similarly, if 1870 income per capita is understated, growth is overstated. This also looks like convergence: a "low" initial income country growing quickly.

Suppose instead that it is only 1979 income per capita that is subject to random, mean-zero measurement error. When 1979 income is overstated, so is growth for a given level of 1870 income. When 1979 income is understated, so is growth for a given 1870 income. Either case is equally likely: overstating 1979 income for any given 1870 income is just as likely as understating it (or more precisely, measurement error is on average equal to zero). Thus there is no reason for this to systematically cause us to see more or less convergence than there really is in the data.

Problem 1.15

On a balanced growth path, K and Y must be growing at a constant rate. The equation of motion for capital, $\dot{K}(t) = sY(t) - \delta K(t)$, implies the growth rate of K is

(1) $\dfrac{\dot{K}(t)}{K(t)} = s\dfrac{Y(t)}{K(t)} - \delta.$

As in the model in the text, Y/K must be constant in order for the growth rate of K to be constant. That is, the growth rates of Y and K must be equal.

Taking logs of both sides of the production function, $Y(t) = K(t)^{\alpha} R(t)^{\beta} T(t)^{\gamma} [A(t)L(t)]^{1-\alpha-\beta-\gamma}$, yields

(2) $\ln Y(t) = \alpha \ln K(t) + \beta \ln R(t) + \gamma \ln T(t) + (1 - \alpha - \beta - \gamma)[\ln A(t) + \ln L(t)]$.

Differentiating both sides of (2) with respect to time gives us

(3) $g_Y(t) = \alpha g_K(t) + \beta g_R(t) + \gamma g_T(t) + (1 - \alpha - \beta - \gamma)[g_A(t) + g_L(t)]$.

Substituting in the facts that the growth rates of R, T, and L are all equal to n and the growth rate of A is equal to g gives us

(4) $g_Y(t) = \alpha g_K(t) + \beta n + \gamma n + (1 - \alpha - \beta - \gamma)(n + g)$.

Simplifying gives us

(5) $\begin{aligned} g_Y(t) &= \alpha g_K(t) + (\beta + \gamma)n + (1 - \alpha)n - (\beta + \gamma)n + (1 - \alpha - \beta - \gamma)g \\ &= \alpha g_K(t) + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g \end{aligned}$

Using the fact that g_Y and g_K must be equal on a balanced growth path leaves us with

(6) $g_Y = \alpha g_Y + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g$,

(7) $(1 - \alpha)g_Y = (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g$,

and thus the growth rate of output on the balanced growth path is given by

(8) $\tilde{g}_Y^{bgp} = \dfrac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$

The growth rate of output per worker on the balanced growth path is

(9) $\tilde{g}_{Y/L}^{bgp} = \tilde{g}_Y^{bgp} - \tilde{g}_L^{bgp}.$

Using equation (8) and the fact that L grows at rate n, we can write

(10) $\tilde{g}_{Y/L}^{bgp} = \dfrac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha} - n = \dfrac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g - (1 - \alpha)n}{1 - \alpha}.$

And thus finally

(11) $\tilde{g}_{Y/L}^{bgp} = \dfrac{(1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$

Equation (11) is identical to equation (1.50) in the text.

SOLUTIONS TO CHAPTER 2

Problem 2.1

(a) The firm's problem is to choose the quantities of capital, K, and effective labor, AL, in order to minimize costs, wAL + rK, subject to the production function, Y = ALf(k). Set up the Lagrangian:

(1) $L = \text{wAL} + \text{rK} + \lambda\left[\text{Y} - \text{ALf}(\text{K/AL})\right].$

The first-order condition for K is given by

(2) $\dfrac{\partial L}{\partial \text{K}} = \text{r} - \lambda\left[\text{ALf}'(\text{K/AL})(1/\text{AL})\right] = 0$,

which implies that

(3) $\text{r} = \lambda \text{f}'(\text{k})$.

The first-order condition for AL is given by

(4) $\dfrac{\partial L}{\partial(\text{AL})} = \text{w} - \lambda\left[\text{f}(\text{K/AL}) + \text{ALf}'(\text{K/AL})(-\text{K})/(\text{AL})^2\right] = 0$,

which implies that

(5) $\text{w} = \lambda\left[\text{f}(\text{k}) - \text{kf}'(\text{k})\right].$

Dividing equation (3) by equation (5) gives us

(6) $\dfrac{\text{r}}{\text{w}} = \dfrac{\text{f}'(\text{k})}{\text{f}(\text{k}) - \text{kf}'(\text{k})}$.

Equation (6) implicitly defines the cost-minimizing choice of k. Clearly this choice does not depend upon the level of output, Y. Note that equation (6) is the standard cost-minimizing condition: the ratio of the marginal cost of the two inputs, capital and effective labor, must equal the ratio of the marginal products of the two inputs.

(b) Since, as shown in part (a), each firm chooses the same value of k and since we are told that each firm has the same value of A, we can write the total amount produced by the N cost-minimizing firms as

(7) $\displaystyle\sum_{i=1}^{N} \text{Y}_i = \sum_{i=1}^{N} \text{AL}_i \text{f}(\text{k}) = \text{Af}(\text{k})\sum_{i=1}^{N} \text{L}_i = \text{A}\overline{\text{L}}\text{f}(\text{k})$,

where $\overline{\text{L}}$ is the total amount of labor employed.

The single firm also has the same value of A and would choose the same value of k; the choice of k does not depend on Y. Thus if it used all of the labor employed by the N cost-minimizing firms, $\overline{\text{L}}$, the single firm would produce Y = A$\overline{\text{L}}$f(k). This is exactly the same amount of output produced in total by the N cost-minimizing firms.

Problem 2.2

(a) The individual's problem is to maximize lifetime utility given by

(1) $\text{U} = \dfrac{\text{C}_1{}^{1-\theta}}{1-\theta} + \dfrac{1}{1+\rho}\dfrac{\text{C}_2{}^{1-\theta}}{1-\theta}$,

subject to the lifetime budget constraint given by

(2) $P_1C_1 + P_2C_2 = W$,

where W represents lifetime income.

Rearrange the budget constraint to solve for C_2 :

(3) $C_2 = W/P_2 - C_1P_1/P_2$.

Substitute equation (3) into equation (1):

(4) $$U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{\left[W/P_2 - C_1 P_1/P_2\right]^{1-\theta}}{1-\theta}.$$

Now we can solve the unconstrained problem of maximizing utility, as given by equation (4), with respect to first period consumption, C_1. The first-order condition is given by

(5) $$\partial U/\partial C_1 = C_1^{-\theta} + (1/1+\rho)C_2^{-\theta}(-P_1/P_2) = 0.$$

Solving for C_1 gives us

(6) $$C_1^{-\theta} = (1/1+\rho)\,(P_1/P_2)\,C_2^{-\theta},$$

or simply

(7) $$C_1 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2.$$

In order to solve for C_2, substitute equation (7) into equation (3):

(8) $$C_2 = W/P_2 - (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2(P_1/P_2).$$

Simplifying yields

(9) $$C_2[1 + (1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}] = W/P_2,$$

or simply

(10) $$C_2 = \frac{W/P_2}{\left[1 + (1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}.$$

Finally, to get the optimal choice of C_1, substitute equation (10) into equation (7):

(11) $$C_1 = \frac{(1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}(W/P_2)}{\left[1 + (1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}.$$

(b) From equation (7), the optimal ratio of first-period to second-period consumption is

(12) $$C_1/C_2 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}.$$

Taking the natural logarithm of both sides of equation (12) yields

(13) $$\ln(C_1/C_2) = (1/\theta)\ln(1+\rho) + (1/\theta)\ln(P_2/P_1).$$

The elasticity of substitution between C_1 and C_2, defined in such a way that it is positive, is given by

(14) $$\frac{\partial(C_1/C_2)}{\partial(P_2/P_1)}\frac{(P_2/P_1)}{(C_1/C_2)} = \frac{\partial[\ln(C_1/C_2)]}{\partial[\ln(P_2/P_1)]} = \frac{1}{\theta},$$

where we have used equation (13) to find the derivative. Thus higher values of θ imply that the individual is less willing to substitute consumption between periods.

<u>Problem 2.3</u>

(a) We can use analysis similar to the intuitive derivation of the Euler equation in Section 2.2 of the text. Think of the household's consumption at two moments of time. Specifically, consider a short (formally infinitesimal) period of time Δt from $(t_0 - \varepsilon)$ to $(t_0 + \varepsilon)$.

Imagine the household reducing consumption per unit of effective labor, c, at $(t_0 - \varepsilon)$ – an instant before the confiscation of wealth – by a small (again, infinitesimal) amount Δc. It then invests this additional saving and consumes the proceeds at $(t_0 + \varepsilon)$. If the household is optimizing, the marginal impact of this change on lifetime utility must be zero.

This experiment would have a utility cost of $u'(c_{before})\Delta c$. Ordinarily, since the instantaneous rate of return is r(t), c at time $(t_0 + \varepsilon)$ could be increased by $e^{[r(t)-n-g]\Delta t}\Delta c$. But here, half of that increase will be confiscated. Thus the utility benefit would be $[1/2]u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c$. Thus for the path of consumption to be utility-maximizing, it must satisfy

$$(1)\quad u'(c_{before})\Delta c = \frac{1}{2}u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c.$$

Rather informally, we can cancel the Δc's and allow $\Delta t \to 0$, leaving us with

$$(2)\quad u'(c_{before}) = \frac{1}{2}u'(c_{after}).$$

Thus there will be a discontinuous jump in consumption at the time of the confiscation of wealth. Specifically, consumption will jump down. Intuitively, the household's consumption will be high before t_0 because it will have an incentive not to save so as to avoid the wealth confiscation.

(b) In this case, from the viewpoint of an individual household, its actions will not affect the amount of wealth that is confiscated. For an individual household, essentially a predetermined amount of wealth will be confiscated at time t_0 and thus the household's optimization and its choice of consumption path would take this into account. The household would still prefer to smooth consumption over time and there will not be a discontinuous jump in consumption at time t_0 .

Problem 2.4

We need to solve the household's problem assuming log utility and in per capita terms rather than in units of effective labor. The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

$$(1)\quad U = \int_{t=0}^{\infty} e^{-\rho t}\ln C(t)\frac{L(t)}{H}dt,$$

subject to

$$(2)\quad \int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}A(t)w(t)\frac{L(t)}{H}dt.$$

Now let $W \equiv \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}A(t)w(t)\frac{L(t)}{H}dt$.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$(3)\quad \mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t}\ln C(t)\frac{L(t)}{H}dt + \lambda\left[W - \int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt\right].$$

The first-order condition is given by

$$(4)\quad \frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t}C(t)^{-1}\frac{L(t)}{H} - \lambda e^{-R(t)}\frac{L(t)}{H} = 0.$$

Canceling the L(t)/H yields

$$(5)\quad e^{-\rho t}C(t)^{-1} = \lambda e^{-R(t)},$$

which implies

$$(6)\quad C(t) = e^{\rho t}\lambda^{-1}e^{R(t)}.$$

Substituting equation (6) into the budget constraint given by equation (2) leaves us with

(7) $\int\limits_{t=0}^{\infty} e^{-R(t)}\left[e^{-\rho t}\lambda^{-1}e^{R(t)}\right]\frac{L(t)}{H}dt = W.$

Since $L(t) = e^{nt} L(0)$, this implies

(8) $\lambda^{-1}\frac{L(0)}{H}\int\limits_{t=0}^{\infty} e^{-(\rho-n)t}dt = W.$

As long as $\rho - n > 0$ (which it must be), the integral is equal to $1/(\rho - n)$ and thus λ^{-1} is given by

(9) $\lambda^{-1} = \frac{W}{L(0)/H}(\rho - n).$

Substituting equation (9) into equation (6) yields

(10) $C(t) = e^{R(t)-\rho t}\left[\frac{W}{L(0)/H}(\rho - n)\right].$

Initial consumption is therefore

(11) $C(0) = \frac{W}{L(0)/H}(\rho - n).$

Note that C(0) is consumption per person, W is wealth per household and L(0)/H is the number of people per household. Thus W/[L(0)/H] is wealth per person. This equation says that initial consumption per person is a constant fraction of initial wealth per person, and $(\rho - n)$ can be interpreted as the marginal propensity to consume out of wealth. With logarithmic utility, this propensity to consume is independent of the path of the real interest rate. Also note that the bigger is the household's discount rate ρ – the more the household discounts the future – the bigger is the fraction of wealth that it initially consumes.

Problem 2.5

The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int\limits_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt,$

subject to

(2) $\int\limits_{t=0}^{\infty} e^{-rt}C(t)\frac{L(t)}{H}dt = W,$

where W denotes the household's initial wealth plus the present value of its lifetime labor income, i.e. the right-hand side of equation (2.6) in the text. Note that the real interest rate, r, is assumed to be constant.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

(3) $\mathcal{L} = \int\limits_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt + \lambda\left[W - \int\limits_{t=0}^{\infty} e^{-rt}C(t)\frac{L(t)}{H}dt\right].$

The first-order condition is given by

(4) $\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t}C(t)^{-\theta}\frac{L(t)}{H} - \lambda e^{-rt}\frac{L(t)}{H} = 0.$

Canceling the L(t)/H yields

(5) $e^{-\rho t}C(t)^{-\theta} = \lambda e^{-rt}.$

Differentiate both sides of equation (5) with respect to time:

(6) $e^{-\rho t}\left[-\theta C(t)^{-\theta-1}\dot{C}(t)\right]-\rho e^{-\rho t}C(t)^{-\theta}+r\lambda e^{-rt}=0.$

This can be rearranged to obtain

(7) $-\theta\dfrac{\dot{C}(t)}{C(t)}e^{-\rho t}C(t)^{-\theta}-\rho e^{-\rho t}C(t)^{-\theta}+r\lambda e^{-rt}=0.$

Now substitute the first-order condition, equation (5), into equation (7):

(8) $-\theta\dfrac{\dot{C}(t)}{C(t)}\lambda e^{-rt}-\rho\lambda e^{-rt}+r\lambda e^{-rt}=0.$

Canceling the λe^{-rt} and solving for the growth rate of consumption, $\dot{C}(t)/C(t)$, yields

(9) $\dfrac{\dot{C}(t)}{C(t)}=\dfrac{r-\rho}{\theta}.$

Thus with a constant real interest rate, the growth rate of consumption is a constant. If $r > \rho$ – that is, if the rate that the market pays to defer consumption exceeds the household's discount rate – consumption will be rising over time. The value of θ determines the magnitude of consumption growth if r exceeds ρ. A lower value of θ – and thus a higher value of the elasticity of substitution, $1/\theta$ – means that consumption growth will be higher for any given difference between r and ρ.

We now need to solve for the path of C(t). First, note that equation (9) can be rewritten as

(10) $\dfrac{\partial \ln C(t)}{\partial t}=\dfrac{r-\rho}{\theta}.$

Integrate equation (10) forward from time $\tau = 0$ to time $\tau = t$:

(11) $\ln C(t)-\ln C(0)=\left[\left(r-\rho\right)/\theta\right]\tau\Big|_{\tau=0}^{\tau=t},$

which simplifies to

(12) $\ln\left[C(t)/C(0)\right]=\left[\left(r-\rho\right)/\theta\right]t.$

Taking the exponential function of both sides of equation (12) yields

(13) $C(t)/C(0)=e^{\left[\left(r-\rho\right)/\theta\right]t},$

and thus

(14) $C(t)=C(0)\,e^{\left[\left(r-\rho\right)/\theta\right]t}.$

We can now solve for initial consumption, C(0), by using the fact that it must be chosen to satisfy the household's budget constraint. Substitute equation (14) into equation (2):

(15) $\displaystyle\int_{t=0}^{\infty} e^{-rt}C(0)e^{\left[\left(r-\rho\right)/\theta\right]t}\,\frac{L(t)}{H}dt=W.$

Using the fact that $L(t)=L(0)e^{nt}$ yields

(16) $\displaystyle\frac{C(0)L(0)}{H}\int_{t=0}^{\infty} e^{-\left[\rho-r+\theta(r-n)\right]t/\theta}dt=W.$

As long as $[\rho - r + \theta(r - n)]/\theta > 0$, we can solve the integral:

(17) $\displaystyle\int_{t=0}^{\infty} e^{-\left[\rho-r+\theta(r-n)\right]t/\theta}dt=\frac{\theta}{\rho-r+\theta(r-n)}.$

Substitute equation (17) into equation (16) and solve for C(0):

(18) $C(0) = \dfrac{W}{L(0)/H}\left[\dfrac{(\rho - r)}{\theta} + (r - n)\right].$

Finally, to get an expression for consumption at each instant in time, substitute equation (18) into equation (14):

(19) $C(t) = e^{[(r-\rho)/\theta]t}\,\dfrac{W}{L(0)/H}\left[\dfrac{(\rho - r)}{\theta} + (r - n)\right].$

Problem 2.6

(a) The equation describing the dynamics of the capital stock per unit of effective labor is

(1) $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t).$

For a given k, the level of c that implies $\dot{k} = 0$ is given by c = f(k) - (n + g)k. Thus a fall in g makes the level of c consistent with $\dot{k} = 0$ higher for a given k. That is, the $\dot{k} = 0$ curve shifts up. Intuitively, a lower g makes break-even investment lower at any given k and thus allows for more resources to be devoted to consumption and still maintain a given k. Since (n + g)k falls proportionately more at higher levels of k, the $\dot{k} = 0$ curve shifts up more at higher levels of k. See the figure.

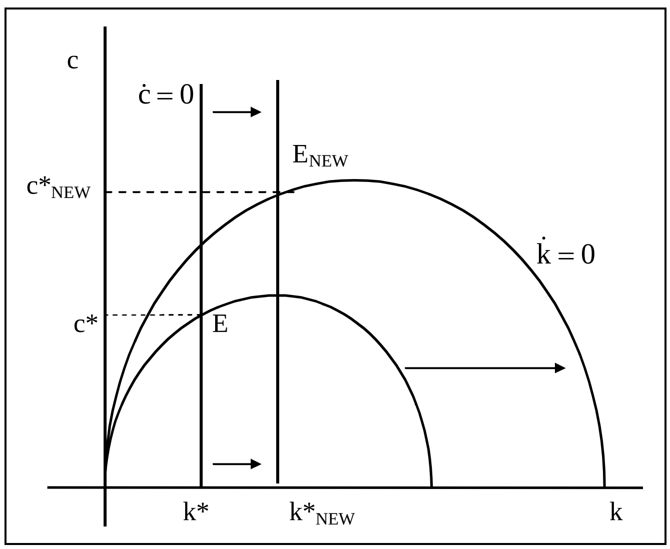
(b) The equation describing the dynamics of consumption per unit of effective labor is given by

(2) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}.$

Thus the condition required for $\dot{c} = 0$ is given by f ' (k) = ρ + θg. After the fall in g, f ' (k) must be lower in order for $\dot{c} = 0$. Since f " (k) is negative this means that the k needed for $\dot{c} = 0$ therefore rises. Thus the $\dot{c} = 0$ curve shifts to the right.

(c) At the time of the change in g, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.

In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time of the shock. In order for the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.



However, we cannot tell whether the new saddle path passes above or below the original point E. Thus we cannot tell whether c jumps up or down and in fact, if the new saddle path passes right through point E, c might even remain the same at the instant that g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

(d) On a balanced growth path, the fraction of output that is saved and invested is given by [f(k*) - c*]/f(k*). Since k is constant, or $\dot{k} = 0$ on a balanced growth path then, from equation (1), we can write f(k*) - c* = (n + g)k*. Using this, we can rewrite the fraction of output that is saved on a balanced growth path as

(3) $s = [(n + g)k^*]/f(k^*)$.
Differentiating both sides of equation (3) with respect to g yields

(4) $$\frac{\partial s}{\partial g} = \frac{f(k^*)\left[(n+g)(\partial k^*/\partial g) + k^*\right] - (n+g)k^* f'(k^*)(\partial k^*/\partial g)}{\left[f(k^*)\right]^2},$$

which simplifies to

(5) $$\frac{\partial s}{\partial g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)](\partial k^*/\partial g) + f(k^*)k^*}{\left[f(k^*)\right]^2}.$$

Since k^* is defined by $f'(k^*) = \rho + \theta g$, differentiating both sides of this expression gives us $f''(k^*)(\partial k^*/\partial g) = \theta$. Solving for $\partial k^*/\partial g$ gives us

(6) $\partial k^*/\partial g = \theta/f''(k^*) < 0$.

Substituting equation (6) into equation (5) yields

(7) $$\frac{\partial s}{\partial g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)]\theta + f(k^*)k^* f''(k^*)}{\left[f(k^*)\right]^2 f''(k^*)}.$$

The first term in the numerator is positive whereas the second is negative. Thus the sign of $\partial s/\partial g$ is ambiguous; we cannot tell whether the fall in g raises or lowers the saving rate on the new balanced growth path.

(e) When the production function is Cobb-Douglas, $f(k) = k^\alpha$, $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha - 1)k^{\alpha-2}$. Substituting these facts into equation (7) yields

(8) $$\frac{\partial s}{\partial g} = \frac{(n+g)[k^{*\alpha} - k^* \alpha k^{*\alpha-1}]\theta + k^{*\alpha}\, k^* \alpha(\alpha-1)k^{*\alpha-2}}{k^{*\alpha}\, k^{*\alpha}\, \alpha(\alpha-1)k^{*\alpha-2}},$$

which simplifies to

(9) $$\frac{\partial s}{\partial g} = \frac{(n+g)k^{*\alpha}(1-\alpha)\theta - (1-\alpha)k^{*\alpha}\, \alpha k^{*\alpha-1}}{[-(1-\alpha)k^{*\alpha}(\alpha k^{*\alpha-1})(\alpha k^{*\alpha-1})/\alpha]},$$

which implies

(10) $$\frac{\partial s}{\partial g} = -\alpha\frac{[(n+g)\theta - (\rho+\theta g)]}{(\rho+\theta g)^2}.$$

Thus, finally, we have

(11) $$\frac{\partial s}{\partial g} = -\alpha\frac{(n\theta - \rho)}{(\rho+\theta g)^2} = \alpha\frac{(\rho - n\theta)}{(\rho+\theta g)^2}.$$

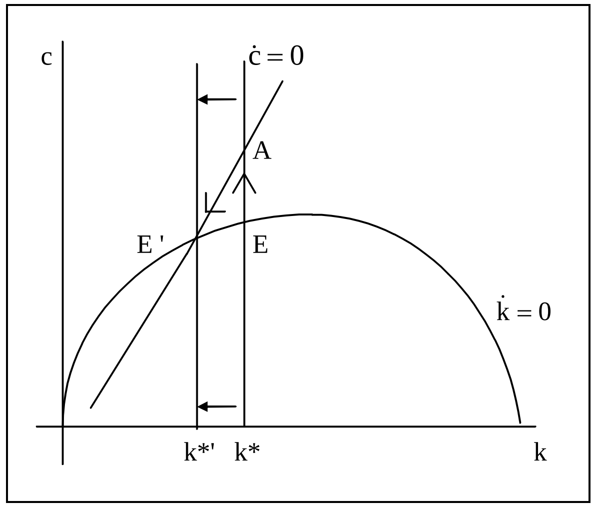
<u>Problem 2.7</u>

The two equations of motion are

(1) $$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta},$$

and

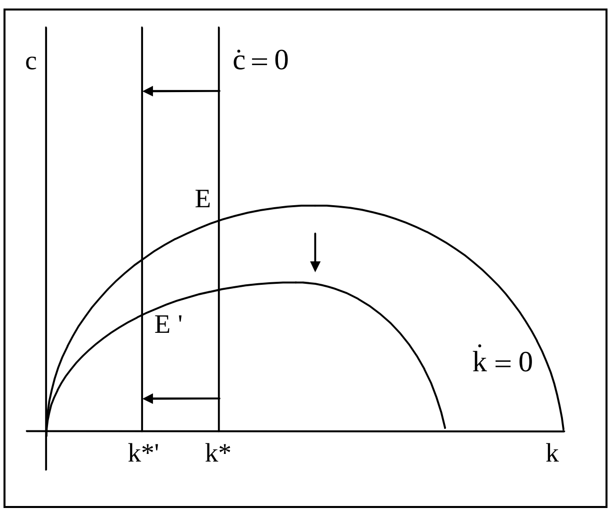
(2) $$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

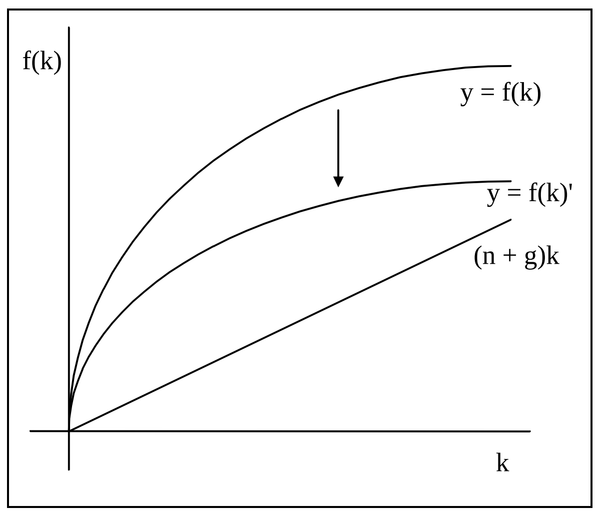


(a) A rise in θ or a fall in the elasticity of substitution, 1/θ, means that households become less willing to substitute consumption between periods. It also means that the marginal utility of consumption falls off more rapidly as consumption rises. If the economy is growing, this tends to make households value present consumption more than future consumption.

The capital-accumulation equation is unaffected. The condition required for $\dot{c} = 0$ is given by f ' (k)= ρ + θg. Since f " (k) < 0, the f ' (k) that makes $\dot{c} = 0$ is now higher. Thus the value of k that satisfies $\dot{c} = 0$ is lower. The $\dot{c} = 0$ locus shifts to the left. The economy moves up to point A on the new saddle path; people consume more now. Movement is then down along the new saddle path until the economy reaches point E '. At that point, c* and k* are lower than their original values.

(b) We can assume that a downward shift of the production function means that for any given k, both f(k) and f ' (k) are lower than before.



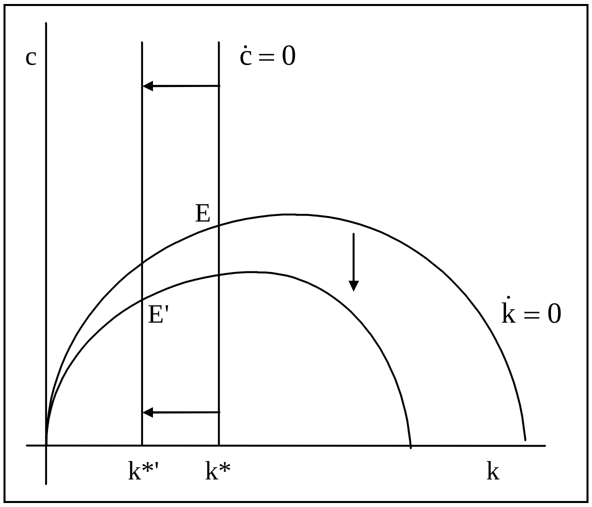


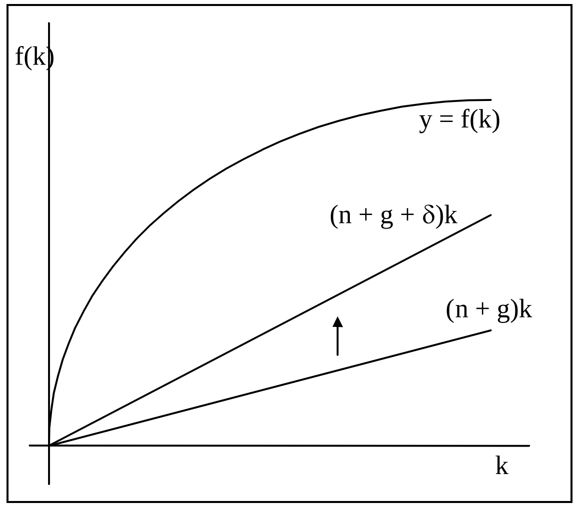
The condition required for $\dot{k} = 0$ is given by $c = f(k) - (n+g)k$. We can see from the figure on the right that the $\dot{k} = 0$ locus will shift down more at higher levels of k. Also, since for a given k, f ' (k) is lower now, the golden-rule k will be lower than before. Thus the $\dot{k} = 0$ locus shifts as depicted in the figure.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. For a given k, f ' (k) is now lower. Thus we need a lower k to keep f ' (k) the same and satisfy the $\dot{c} = 0$ equation. Thus the $\dot{c} = 0$ locus shifts left. The economy will eventually reach point E ' with lower c* and lower k*. Whether c initially jumps up or down depends upon whether the new saddle path passes above or below point E.

(c) With a positive rate of depreciation, δ > 0, the new capital-accumulation equation is

(3) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.





The level of saving and investment required just to keep any given k constant is now higher – and thus the amount of consumption possible is now lower – than in the case with no depreciation. The level of extra investment required is also higher at higher levels of k. Thus the $\dot{k} = 0$ locus shifts down more at higher levels of k.

In addition, the real return on capital is now f ' (k(t)) - δ and so the household's maximization will yield

(4) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$.

The condition required for $\dot{c} = 0$ is $f'(k) = \delta + \rho + \theta g$. Compared to the case with no depreciation, f ' (k) must be higher and k lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left. The economy will eventually wind up at point E ' with lower levels of c* and k*. Again, whether c jumps up or down initially depends upon whether the new saddle path passes above or below the original equilibrium point of E.

Problem 2.8

With a positive depreciation rate, δ > 0, the Euler equation and the capital-accumulation equation are given by

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.

We begin by taking first-order Taylor approximations to (1) and (2) around k = k* and c = c*. That is, we can write

(3) $\dot{c} \cong \dfrac{\partial \dot{c}}{\partial k}[k - k^*] + \dfrac{\partial \dot{c}}{\partial c}[c - c^*]$, and (4) $\dot{k} \cong \dfrac{\partial \dot{k}}{\partial k}[k - k^*] + \dfrac{\partial \dot{k}}{\partial c}[c - c^*]$,

where $\partial \dot{c}/\partial k$, $\partial \dot{c}/\partial c$, $\partial \dot{k}/\partial k$ and $\partial \dot{k}/\partial c$ are all evaluated at k = k* and c = c*.

Define $\tilde{c} \equiv c - c^*$ and $\tilde{k} \equiv k - k^*$. Since c* and k* are constants, $\dot{c}$ and $\dot{k}$ are equivalent to $\dot{\tilde{c}}$ and $\dot{\tilde{k}}$ respectively. We can therefore rewrite (3) and (4) as

(5) $\dot{\tilde{c}} \cong \dfrac{\partial \dot{c}}{\partial k}\tilde{k} + \dfrac{\partial \dot{c}}{\partial c}\tilde{c}$, and (6) $\dot{\tilde{k}} \cong \dfrac{\partial \dot{k}}{\partial k}\tilde{k} + \dfrac{\partial \dot{k}}{\partial c}\tilde{c}$.

Using equations (1) and (2) to compute these derivatives yields

(7) $\left.\dfrac{\partial\dot{c}}{\partial k}\right|_{bgp} = \dfrac{f''(k^*)c^*}{\theta}$, (8) $\left.\dfrac{\partial\dot{c}}{\partial c}\right|_{bgp} = \dfrac{f'(k^*)-\delta-\rho-\theta g}{\theta} = 0$,

(9) $\left.\dfrac{\partial\dot{k}}{\partial k}\right|_{bgp} = f'(k^*) - (n+g+\delta)$, (10) $\left.\dfrac{\partial\dot{k}}{\partial c}\right|_{bgp} = -1.$

Substituting equations (7) and (8) into (5) and equations (9) and (10) into (6) yields

(11) $\dot{\tilde{c}} \cong \dfrac{f''(k^*)c^*}{\theta}\tilde{k}$, and

(12) $$\begin{aligned}\dot{\tilde{k}} &\cong \left[f'(k^*) - (n+g+\delta)\right]\tilde{k} - \tilde{c}\\ &\cong \left[(\delta+\rho+\theta g) - (n+g+\delta)\right]\tilde{k} - \tilde{c}\\ &\cong \beta\tilde{k} - \tilde{c}.\end{aligned}$$

The second line of equation (12) uses the fact that (1) implies that $f'(k^*) = \delta + \rho + \theta g$. The third line uses the definition of $\beta \equiv \rho - n - (1-\theta)g$.

Dividing equation (11) by $\tilde{c}$ and dividing equation (12) by $\tilde{k}$ yields

(13) $\dfrac{\dot{\tilde{c}}}{\tilde{c}} \cong \dfrac{f''(k^*)c^*}{\theta}\dfrac{\tilde{k}}{\tilde{c}}$, and (14) $\dfrac{\dot{\tilde{k}}}{\tilde{k}} \cong \beta - \dfrac{\tilde{c}}{\tilde{k}}$.

Note that these are exactly the same as equations (2.32) and (2.33) in the text; adding a positive depreciation rate does not alter the expressions for the growth rates of $\tilde{c}$ and $\tilde{k}$. Thus equation (2.37), the expression for μ, the constant growth rate of both $\tilde{c}$ and $\tilde{k}$ as the economy moves toward the balanced growth path, is still valid. Thus

(15) $\mu_1 = \dfrac{\beta - \sqrt{\beta^2 - 4f''(k^*)c^*/\theta}}{2}$,

where we have chosen the negative growth rate so that c and k are moving toward c* and k*, not away from them.

Now consider the Cobb-Douglas production function, $f(k) = k^{\alpha}$. Thus

(16) $f'(k^*) = \alpha k^{*\alpha-1} \equiv r^* + \delta$, and (17) $f''(k^*) = \alpha(\alpha-1)k^{*\alpha-2}$.

Squaring both sides of equation (16) gives us

(18) $(r^*+\delta)^2 = \alpha^2 k^{*2\alpha-2}$,

and so equation (17) can be rewritten as

(19) $f''(k^*) = \dfrac{(r^*+\delta)^2(\alpha-1)}{\alpha k^{*\alpha}} = \dfrac{\alpha-1}{\alpha}\dfrac{(r^*+\delta)^2}{f(k^*)}$.

In addition, defining s* to be the saving rate on the balanced growth path, we can write the balanced-growth-path level of consumption as

(20) $c^* = (1 - s^*)f(k^*)$.

Substituting equations (19) and (20) into (15) yields

(21) $\mu_1 = \dfrac{\beta - \sqrt{\beta^2 - 4\left(\dfrac{\alpha-1}{\alpha}\right)\dfrac{(r^*+\delta)^2}{f(k^*)\theta}(1-s^*)f(k^*)}}{2}$.

Canceling the f(k*) and multiplying through by the minus sign yields

$$(22)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(r^*+\delta)^2(1-s^*)}}{2}.$$

On the balanced growth path, the condition required for $\dot{c} = 0$ is given by $r^* = \rho + \theta g$ and thus

(23) $r^* + \delta = \rho + \theta g + \delta$.

In addition, actual saving, $s^*f(k^*)$, equals break-even investment, $(n + g + \delta)k^*$, and thus

$$(24)\quad s^* = \frac{(n+g+\delta)k^*}{f(k^*)} = \frac{(n+g+\delta)}{k^{*\alpha-1}} = \frac{\alpha(n+g+\delta)}{(r^*+\delta)},$$

where we have used equation (16), $r^* + \delta = \alpha k^{*\alpha-1}$.

From equation (24), we can write

$$(25)\quad (1-s^*) = \frac{(r^*+\delta) - \alpha(n+g+\delta)}{(r^*+\delta)}.$$

Substituting equations (23) and (25) into equation (22) yields

$$(26)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(\rho+\theta g+\delta)\left[\rho+\theta g+\delta-\alpha(n+g+\delta)\right]}}{2}.$$

Equation (26) is analogous to equation (2.39) in the text. It expresses the rate of adjustment in terms of the underlying parameters of the model. Keeping the values in the text – $\alpha = 1/3$, $\rho = 4\%$, $n = 2\%$, $g = 1\%$ and $\theta = 1$ – and using $\delta = 3\%$ yields a value for μ_1 of approximately - 8.8%. This is faster convergence than the -5.4% obtained with no depreciation.

Problem 2.9

(a) We are given that

(1) $y(t) = k(t)^{\alpha}$.

From the textbook we know that when $\dot{c} = 0$, then $f'(k) = \rho + \theta g$. Substituting (1) and simplifying results in

$$(2)\quad k^* = \left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}}.$$

(b) Likewise, from the textbook, we know that when $\dot{k} = 0$ then $c^* = f(k) - (n+g)k$. Substituting (1) and simplifying results in

$$(3)\quad c^* = \left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{\alpha}{1-\alpha}} - (n+g)\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}}.$$

(c) Let $z(t) = k(t)/y(t)$ and $x(t) = c(t)/k(t)$. Substituting (1) into the first equation and simplifying, we get

(4) $k = z^{1/(1-\alpha)} \Leftrightarrow k^{1-\alpha} = z$.

Substituting (4) into the second equation above and simplifying results in

(5) $ck^{-\alpha} = xz$.

Now, using equation (4), take the time derivative of $z = k/y$, which results in

(6) $\dot{z} = (1-\alpha)k^{-\alpha}\dot{k}$.

Equation (2.25) in the textbook tells us that $\dot{k} = k^{\alpha} - c - (n+g)k$; therefore (6) becomes

(7) $\dot{z} = (1-\alpha)k^{-\alpha}(k^{\alpha} - c - (n+g)k)$.

Simplifying and substituting equations (4) and (5), equation (7) becomes

(8) $\dot{z} = (1-\alpha)(1 - xz - (n+g)z)$.

Now look at $x = c/k$. Taking logs and then the time derivative results in

(9) $\dot{x}/x = \dot{c}/c - \dot{k}/k$.

Using equation (2.24) and (2.25) from the textbook, equation (9) becomes

(10) $\dfrac{\dot{x}}{x} = \dfrac{\alpha k^{\alpha-1} - \rho - \theta g}{\theta} + \dfrac{-k^{\alpha} + c + (n+g)k}{k}$.

Again, substituting equations (4) and (5) into (10), and using the assumption that $\alpha = \theta$ gives us

(11) $\dot{x}/x = x + n - \rho/\alpha$.

(d)(i) From the conjecture that x is constant, equation (8) becomes

(12) $\dot{z} = (1-\alpha)(1 - (n + g + x^*)z)$.

To find the path of z, consider equation (12) and observe that it is a linear non-homogeneous ordinary differential equation. Therefore, our solution will consist of the complementary solution (z_c) and the particular solution (z_p). For simplification, let $\lambda = (1-\alpha)(n + g + x^*)$.

To solve for the complementary solution, we consider the homogeneous case in which $\dot{z} + \lambda z = 0$. Isolating for $\dot{z}/z$ yields a complementary solution of

(13) $z_c = A_1 e^{-\lambda t}$, where A_1 is a constant of integration.

To solve for the particular solution, we consider the non-homogeneous case, where $\dot{z} + \lambda z = (1-\alpha)$. Using an integrating factor, we find the solution to be

(14) $z_p = (1-\alpha)/\lambda + A_2 e^{-\lambda t}$, where A_2 is a constant of integration. Therefore,

(15) $z = z_p + z_c \Rightarrow z = (1-\alpha)/\lambda + (A_1 + A_2)e^{-\lambda t}$.

Using the initial condition z(0), we can substitute for $A_1 + A_2$ and (15) becomes

(16) $z = (1-\alpha)/\lambda + e^{-\lambda t}(z(0) - (1-\alpha)/\lambda)$.

Lastly, we can simplify the $(1-\alpha)/\lambda$ term by substituting equations (2) and (3) for x* and using equation (4), resulting in

(17) $z = z^* + e^{-\lambda t}(z(0) - z^*)$.

(ii) We want to find the path of y, so consider equation (1) and substitute in equation (4). We know the path value of z, so substitute in equation (17). This results in

(18) $y = (z^* + e^{-\lambda t}(z(0) - z^*))^{\alpha/(1-\alpha)}$.

Again, use equation (4) to put everything in terms of k and (18) becomes

(19) $y = ((k^*)^{1-\alpha} + e^{-\lambda t}(k(0)^{1-\alpha} - (k^*)^{1-\alpha}))^{\alpha/(1-\alpha)}$.

Now we would like to see whether the speed of convergence to the balanced growth path is constant. Using equation (19) and subtracting the balanced growth path y*, where

$y^* = (k^*)^{\alpha} = \left(\dfrac{\alpha}{\rho + \alpha g}\right)^{\alpha/(1-\alpha)}$ from equation (2), and taking logs results in

(20) $\ln(y - y^*) = \ln\left(z^{\alpha/(1-\alpha)} - \left(\frac{\alpha}{\rho + \theta g}\right)^{\alpha/(1-\alpha)} \right).$

Taking the time derivative, we get:

(21) $\dfrac{\partial \ln(y - y^*)}{\partial t} = \dfrac{\frac{\alpha}{1-\alpha} z^{\frac{\alpha}{1-\alpha}-1} \dot{z}}{z^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha}{\rho + \theta g}\right)^{\frac{\alpha}{1-\alpha}}}.$

Clearly this is not constant, so the speed of convergence is not constant.

(e) We want to see whether equations (2.24) and (2.25) from the textbook— $\dot{c}/c = (\alpha k^{1-\alpha} - \rho - \theta g)/\theta$ and $\dot{k} = k^{\alpha} - c - (n + g)$ —are satisfied. Because these equations were used to find $\dot{x}/x$, then $\dot{x}/x$ holds if and only if equations (2.24) and (2.25) also hold. However, we know that equation (2.25) holds because it was used previously to find $\dot{z}$. Thus, we can say that $\dot{x}/x = 0$ if and only if $\dot{c}/c = 0$. Assuming that $\dot{x}/x = 0$ and using equation (11) results in

(22) $x^* = \rho/\alpha - n$.

Using parts (a) and (b), the balanced growth path of x* is $(\rho + \alpha g)/g - (n + g)$ so (22) becomes

(23) $x^* = \rho/\alpha - n$.

Equation (23) is the same result as (11), so equations (2.24) and (2.25) are satisfied.

Problem 2.10

(a) The real after-tax rate of return on capital is now given by $(1 - \tau)f'(k(t))$. Thus the household's maximization would now yield the following expression describing the dynamics of consumption per unit of effective labor:

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{[(1-\tau)f'(k(t)) - \rho - \theta g]}{\theta}.$

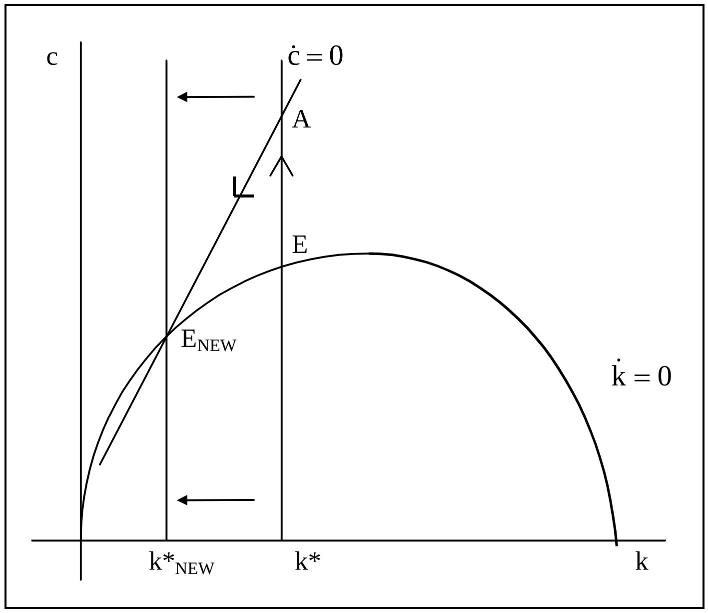
The condition required for $\dot{c} = 0$ is given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return must equal $\rho + \theta g$. Compared to the case without a tax on capital, $f'(k)$, the pre-tax rate of return on capital, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left.

The equation describing the dynamics of the capital stock per unit of effective labor is still given by

(2) $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$.

For a given k, the level of c that implies $\dot{k} = 0$ is given by $c(t) = f(k) - (n + g)k$. Since the tax is rebated to households in the form of lump-sum transfers, this $\dot{k} = 0$ locus is unaffected.

(b) At time 0, when the tax is put in place, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.



In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time that the tax is introduced. This jump in c is not inconsistent with the consumption-smoothing behavior implied by the household's optimization problem since the tax was unexpected and could not be prepared for.

In order for the economy to reach the new balanced growth path, it should be clear what must occur. At time 0, c jumps up so that the economy is on the new saddle path. In the figure, the economy jumps from point E to a point such as A. Since the return to saving and accumulating capital is now lower than before, people switch away from saving and into consumption.

After time 0, the economy will gradually move down the new saddle path until it eventually reaches the new balanced growth path at E_{NEW} .

(c) On the new balanced growth path at E_{NEW} , the distortionary tax on investment income has caused the economy to have a lower level of capital per unit of effective labor as well as a lower level of consumption per unit of effective labor.

(d) (i) From the analysis above, we know that the higher is the tax rate on investment income, τ, the lower will be the balanced-growth-path level of k*, all else equal. In terms of the above story, the higher is τ the more that the $\dot{c} = 0$ locus shifts to the left and hence the more that k* falls. Thus $\partial k^*/\partial\tau < 0$.

On a balanced growth path, the fraction of output that is saved and invested, the saving rate, is given by [f(k*) - c*]/f(k*). Since k is constant, or $\dot{k} = 0$, on a balanced growth path then from $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$ we can write f(k*) - c* = (n + g)k*. Using this we can rewrite the fraction of output that is saved on a balanced growth path as

(3) s = [(n + g)k*]/f(k*).

Use equation (3) to take the derivative of the saving rate with respect to the tax rate, τ:

$$(4)\quad \frac{\partial s}{\partial \tau} = \frac{(n+g)(\partial k^*/\partial\tau)\, f(k^*) - (n+g)k^* f'(k^*)(\partial k^*/\partial\tau)}{f(k^*)^2}.$$

Simplifying yields

$$(5)\quad \frac{\partial s}{\partial \tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau} - \frac{(n+g)}{f(k^*)}\frac{k^* f'(k^*)}{f(k^*)}\frac{\partial k^*}{\partial\tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau}\left[1 - \frac{k^* f'(k^*)}{f(k^*)}\right].$$

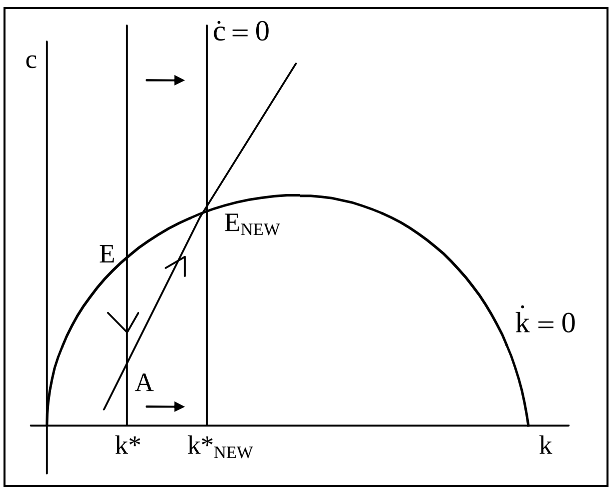
Recall that k*f ' (k*)/f(k*) ≡ α_K (k*) is capital's (pre-tax) share in income, which must be less than one. Since $\partial k^*/\partial\tau < 0$ we can write

(6) $\dfrac{\partial s}{\partial \tau} = \dfrac{(n+g)}{f(k^*)}\dfrac{\partial k^*}{\partial \tau}\left[1-\alpha_K(k^*)\right] < 0$.

Thus the saving rate on the balanced growth path is decreasing in τ.

(d) (ii) Citizens in low-τ, high-k^*, high-saving countries do not have the incentive to invest in low-saving countries. From part (a), the condition required for $\dot{c} = 0$ is $(1 - \tau)f'(k) = \rho + \theta g$. That is, the after-tax rate of return must equal $\rho + \theta g$. Assuming preferences and technology are the same across countries so that ρ, θ and g are the same across countries, the after-tax rate of return will be the same across countries. Since the after-tax rate of return is thus the same in low-saving countries as it is in high-saving countries, there is no incentive to shift saving from a high-saving to a low-saving country.

(e) Should the government subsidize investment instead and fund this with a lump-sum tax? This would lead to the opposite result from above and the economy would have higher c and k on the new balanced growth path.



The answer is no. The original market outcome is already the one that would be chosen by a central planner attempting to maximize the lifetime utility of a representative household subject to the capital-accumulation equation. It therefore gives the household the highest possible lifetime utility.

Starting at point E, the implementation of the subsidy would lead to a short-term drop in consumption at point A, but would eventually result in permanently higher consumption at point E_{NEW} .

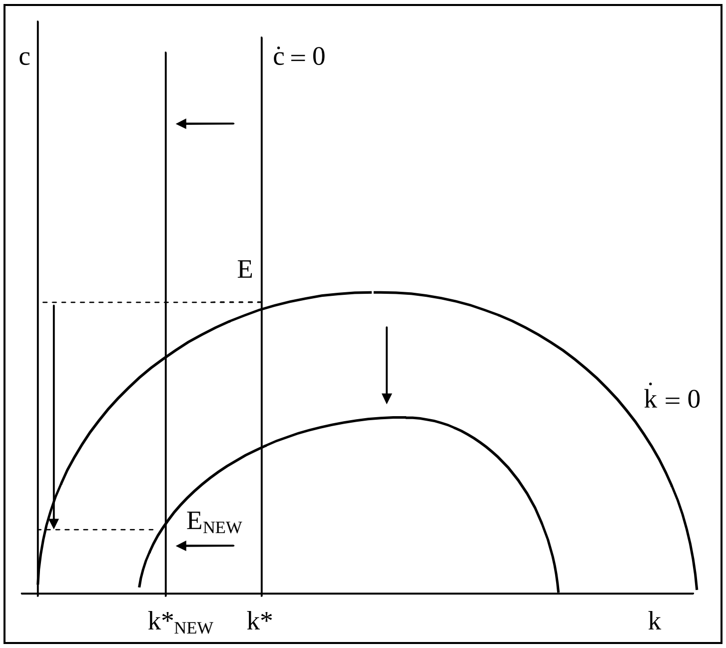
It would turn out that the utility lost from the short-term sacrifice would outweigh the utility gained in the long-term (all in present value terms, appropriately discounted).

This is the same type of argument used to explain the reason that households do not choose to consume at the golden-rule level. See Section 2.4 for a more complete description of the welfare implications of this model.

(f) Suppose the government does not rebate the tax revenue to households but instead uses it to make government purchases. Let G(t) represent government purchases per unit of effective labor. The equation describing the dynamics of the capital stock per unit of effective labor is now given by

(2 ') $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$.

The fact that the government is making purchases that do not add to the capital stock – it is assumed to be government consumption, not government investment – shifts down the $\dot{k} = 0$ locus.



After the imposition of the tax, the $\dot{c} = 0$ locus shifts to the left, just as it did in the case in which the government rebated the tax to households. In the end, k* falls to k^*_{NEW} just as in the case where the government rebated the tax.

Consumption per unit of effective labor on the new balanced growth path at E_{NEW} is lower than in the case where the tax is rebated by the amount of the government purchases, which is $\tau f'(k)k$.

Finally, whether the level of c jumps up or down initially depends upon whether the new saddle path passes above or below the original balanced-growth-path point of E.

Problem 2.11

(a) - (c) Before the tax is put in place, i.e. until time t_1 , the equations governing the dynamics of the economy are

(1) $$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta},$$

and

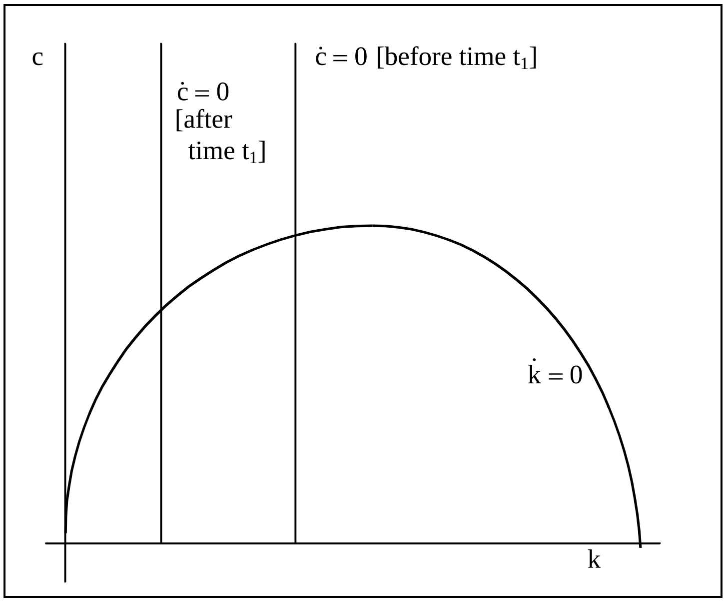
(2) $$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation is not affected when the tax is put in place at time t_1 since we are assuming that the government is rebating the tax, not spending it.

Since the real after-tax rate of return on capital is now $(1 - \tau)f'(k(t))$, the household's maximization yields the following growth rate of consumption:

(3) $$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)f'(k(t)) - \rho - \theta g}{\theta}.$$

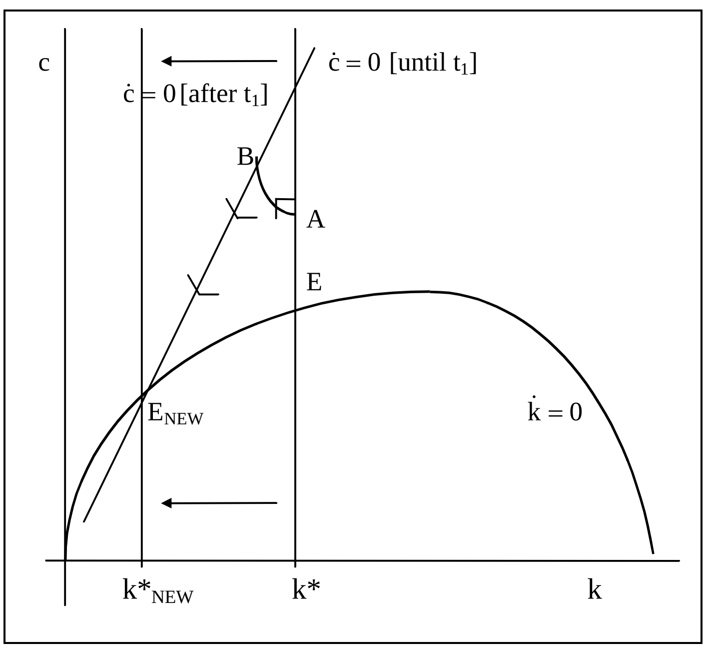
The condition required for $\dot{c} = 0$ is now given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return on capital must equal $\rho + \theta g$. Thus the pre-tax rate of return, $f'(k)$, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus at time t_1 , the $\dot{c} = 0$ locus shifts to the left.



The important point is that the dynamics of the economy are still governed by the original equations of motion until the tax is actually put in place. Between the time of the announcement and the time the tax is actually put in place, it is the original $\dot{c} = 0$ locus that is relevant.

When the tax is put in place at time t_1 , c cannot jump discontinuously because households know ahead of time that the tax will be implemented then. A discontinuous jump in c would be inconsistent with the consumption smoothing implied by the household's intertemporal optimization. The household would not want c to be low, and thus marginal utility to be high, a moment before t_1 knowing that c will jump up and be high, and thus marginal utility will be low, a moment after t_1. The household would like to smooth consumption between the two instants in time.

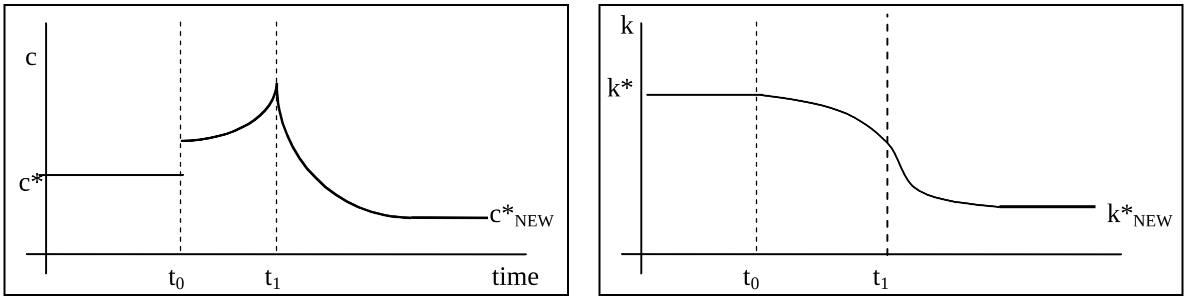
(d) We know that c cannot jump at time t_1 . We also know that if the economy is to reach the new balanced growth path at point E_{NEW} , it must be right on the new saddle path at the time that the tax is put in place. Thus when the tax is announced at time t_0 , c must jump up to a point such as A. Point A lies between the original balanced growth path at E and the new saddle path.



At A, c is too high to maintain the capital stock at k* and so k begins falling. Between t_0 and t_1 , the dynamics of the system are still governed by the original $\dot{c} = 0$ locus. The economy is thus to the left of the $\dot{c} = 0$ locus and so consumption begins rising.

The economy moves off to the northwest until at t_1 , it is right at point B on the new saddle path. The tax is then put in place and the system is governed by the new $\dot{c} = 0$ locus. Thus c begins falling. The economy moves down the new saddle path, eventually reaching point E_{NEW} .

(e) The story in part (d) implies the following time paths for consumption per unit of effective labor and capital per unit of effective labor.



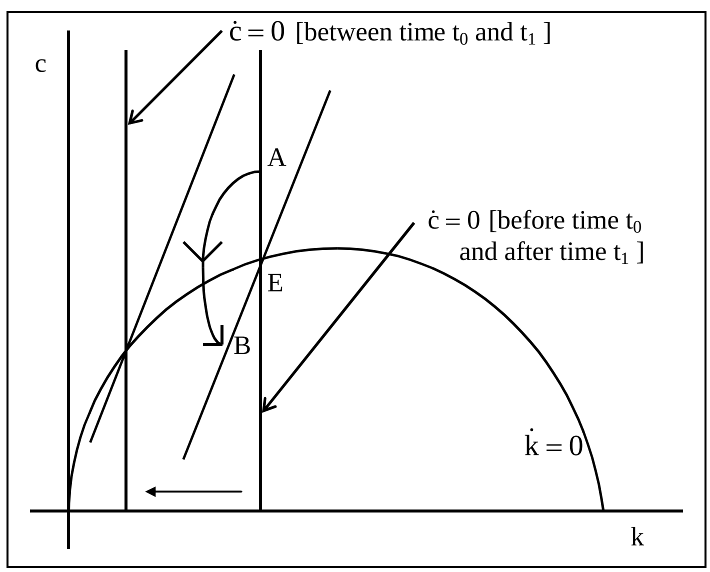
Problem 2.12

(a) The first point is that consumption cannot jump at time t_1. Households know ahead of time that the tax will end then and so a discontinuous jump in c would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus, for the economy to return to a balanced growth path, we must be somewhere on the original saddle path right at time t_1.

Before the tax is put in place – until time t_0 – and after the tax is removed – after time t_1 – the equations governing the dynamics of the economy are

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation, and thus the $\dot{k} = 0$ locus, is not affected by the tax. The $\dot{c} = 0$ locus is affected, however. Between time t_0 and time t_1, the condition required for $\dot{c} = 0$ is that the after-tax rate of return on capital equal $\rho + \theta g$ so that $(1 - \tau)f'(k) = \rho + \theta g$. Thus between t_0 and t_1, $f'(k)$ must be higher and so k must be lower in order for $\dot{c} = 0$. That is, between time t_0 and time t_1, the $\dot{c} = 0$ locus lies to the left of its original position.

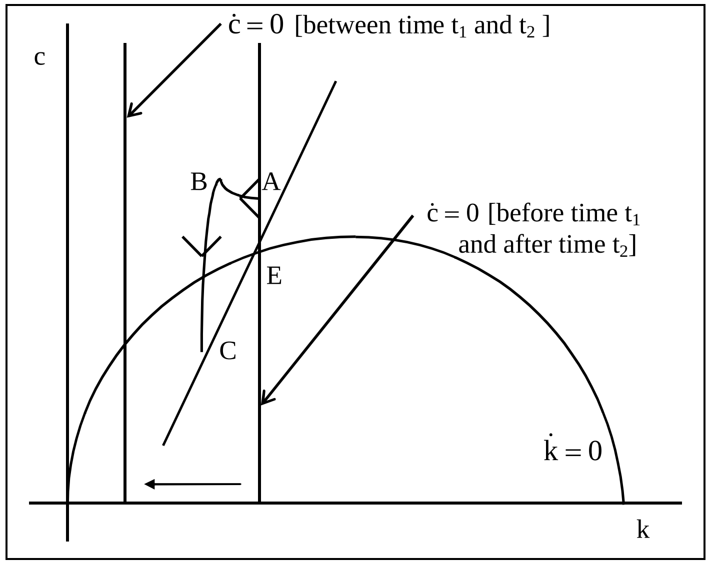


At time t_0, the tax is put in place. At point E, the economy is still on the $\dot{k} = 0$ locus but is now to the right of the new $\dot{c} = 0$ locus. Thus if c did not jump up to a point like A, c would begin falling. The economy would then be below the $\dot{k} = 0$ locus and so k would start rising. The economy would drift away from point E in the direction of the southeast and could not be on the original saddle path right at time t_1.

Thus at time t_0, c must jump up so that the economy is at a point like A. Thus, k and c begin falling. Eventually the economy crosses the $\dot{k} = 0$ locus and so k begins rising. This can be interpreted as households anticipating the removal of the tax on capital and thus being willing to accumulate capital again. Point A must be such that given the dynamics of the system, the economy is right at a point like B, on the original saddle path, at time t_1 when the tax is removed. After t_1, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

(b) The first point is that consumption cannot jump at either time t_1 or time t_2. Households know ahead of time that the tax will be implemented at t_1 and removed at t_2. Thus a discontinuous jump in c at either date would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization.

In order for the economy to return to a balanced growth path, the economy must be somewhere on the original saddle path right at time t_2.



Before the tax is put in place – until time t_1 – and after the tax is removed – after time t_2 – equations (1) and (2) govern the dynamics of the system. An important point is that even during the time between the announcement and the implementation of the tax – that is, between time t_0 and time t_1 – the original $\dot{c} = 0$ locus governs the dynamics of the system.

At time t_0, the tax is announced. Consumption must jump up so that the economy is at a point like A.

At A, the economy is still on the $\dot{c} = 0$ locus but is above the $\dot{k} = 0$ locus and so k starts falling. The economy is then to the left of the $\dot{c} = 0$ locus and so c starts rising. The economy drifts off to the northwest.

At time t_1, the tax is implemented, the $\dot{c} = 0$ locus shifts to the left and the economy is at a point like B. The economy is still above the $\dot{k} = 0$ locus but is now to the right of the relevant $\dot{c} = 0$ locus; k continues to fall and c stops rising and begins to fall.

Eventually the economy crosses the $\dot{k} = 0$ locus and k begins rising. Households begin accumulating capital again before the actual removal of the tax on capital income. Point A must be chosen so that given the dynamics of the system, the economy is right at a point like C, on the original saddle path, at time t_2 when the tax is removed.

After t_2, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

<u>Problem 2.13</u>

With government purchases in the model, the capital-accumulation equation is given by

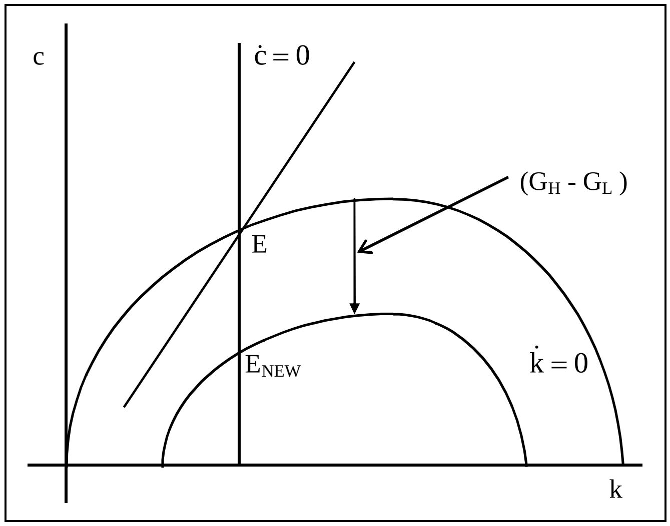
(1) $\dot{k}(t) = f\big(k(t)\big) - c(t) - G(t) - (n + g)k(t)$,

where G(t) represents government purchases in units of effective labor at time t.

Intuitively, since government purchases are assumed to be a perfect substitute for private consumption, changes in G will simply be offset one-for-one with changes in c. Suppose that G(t) is initially constant at some level G_L. The household's maximization yields

$$(2)\quad \frac{\dot{c}(t)}{c(t)+G_L} = \frac{f'(k(t)) - \rho - \theta g}{\theta}.$$

Thus the condition for constant consumption is still given by $f'(k) = \rho + \theta g$. Changes in the level of G_L will affect the level of c, but will not shift the $\dot{c} = 0$ locus.



Suppose the economy starts on a balanced growth path at point E. At some time t_0, G unexpectedly increases to G_H and households know this is temporary; households know that at some future time t_1, government purchases will return to G_L. At time t_0, the $\dot{k} = 0$ locus shifts down; at each level of k, the government is using more resources leaving less available for consumption. In particular, the $\dot{k} = 0$ locus shifts down by the amount of the increase in purchases, which is $(G_H - G_L)$.

The difference between this case, in which c and G are perfect substitutes, and the case in which G does not affect private utility, is that c can jump at time t_1 when G returns to its original value. In fact, at t_1, when G jumps down by the amount $(G_H - G_L)$, c must jump up by that exact same amount. If it did not, there would be a discontinuous jump in marginal utility that could not be optimal for households. Thus at t_1, c must jump up by $(G_H - G_L)$ and this must put the economy somewhere on the original saddle path. If it did not, the economy would not return to a balanced growth path. What must happen is that at time t_0, c falls by the amount $(G_H - G_L)$ and the economy jumps to point E_{NEW}. It then stays there until time t_1. At t_1, c jumps back up by the amount $(G_H - G_L)$ and so the economy jumps back to point E and stays there.

Why can't c jump down by less than $(G_H - G_L)$ at t_0? If it did, the economy would be above the new $\dot{k} = 0$ locus, k would start falling putting the economy to the left of the $\dot{c} = 0$ locus. Thus c would start rising and so the economy would drift off to the northwest. There would be no way for c to jump up by $(G_H - G_L)$ at t_1 and still put the economy on the original saddle path.

Why can't c jump down by more that $(G_H - G_L)$ at t_1? If it did, the economy would be below the new $\dot{k} = 0$ locus, k would start rising putting the economy to the right of the $\dot{c} = 0$ locus. Thus c would start falling and so the economy would drift off to the southeast. Again, there would be no way for c to jump up by $(G_H - G_L)$ at t_1 and still put the economy on the original saddle path.

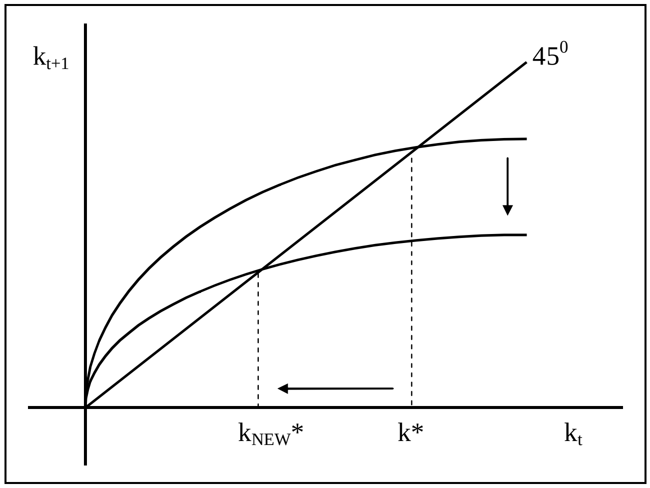
In summary, the capital stock and the real interest rate are unaffected by the temporary increase in G. At the instant that G rises, consumption falls by an equal amount. It remains constant at that level while G remains high. At the instant that G falls to its initial value, consumption jumps back up to its original value and stays there.

Problem 2.14

Equation (2.60) in the text describes the relationship between k_{t+1} and k_t in the special case of logarithmic utility and Cobb-Douglas production:

(2.60) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}.$$

(a) A rise in n shifts the k_{t+1} function down. From equation (2.60), a higher n means a smaller k_{t+1} for a given k_t . Since the fraction of their labor income that the young save does not depend on n, a given amount of capital per unit of effective labor and thus output per unit of effective labor in time t yields the same amount of saving in period t. Thus it yields the same amount of capital in period t + 1. However, the number of individuals increases more from period t to period t + 1 than it used to. So that capital is spread out among more individuals than it would have been in the absence of the increase in population growth and thus capital per unit of effective labor in period t + 1 is lower for a given k_t .



(b) With the parameter B added to the Cobb-Douglas production function, $f(k) = Bk^{\alpha}$, equation (2.60) becomes

(1) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}(1-\alpha)Bk_t^{\alpha}.$$

This fall in B causes the k_{t+1} function to shift down. See the figure from part (a). A lower B means that a given amount of capital per unit of effective labor in period t now produces less output per unit of effective labor in period t. Since the fraction of their labor income that the young save does not depend on B, this leads to less total saving and a lower capital stock per unit of effective labor in period t + 1 for a given k_t .

(c) We need to determine the effect on k_{t+1} for a given k_t , of a change in α. From equation (2.60):

(2) $$\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}\left[-k_t^{\alpha} + (1-\alpha)\frac{\partial k_t^{\alpha}}{\partial \alpha}\right].$$

We need to determine $\partial k_t^{\alpha}/\partial\alpha$. Define $f(\alpha) \equiv k_t^{\alpha}$ and note that $\ln f(\alpha) = \alpha \ln k_t$. Thus

(3) $\partial \ln f(\alpha)/\partial\alpha = \ln k_t$.

Now note that we can write

(4) $$\frac{\partial f(\alpha)}{\partial \alpha} = \frac{\partial f(\alpha)}{\partial \ln f(\alpha)}\frac{\partial \ln f(\alpha)}{\partial \alpha} = \frac{1}{\left[\partial \ln f(\alpha)/\partial f(\alpha)\right]}\frac{\partial \ln f(\alpha)}{\partial \alpha},$$

and thus finally

(5) $\partial f(\alpha)/\partial\alpha = f(\alpha)\ln k_t$.

Therefore, we have $\partial k_t^{\alpha}/\partial\alpha = k_t^{\alpha} \ln k_t$. Substituting this fact into equation (2) yields

(6) $$\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)}\frac{1}{2+\rho}\left[-k_t^{\alpha} + (1-\alpha)k_t^{\alpha}\ln k_t\right],$$

or simply

(7) $\dfrac{\partial k_{t+1}}{\partial \alpha} = \dfrac{1}{(1+n)(1+g)}\dfrac{1}{2+\rho}\left\{k_t{}^{\alpha}\left[(1-\alpha)\ln k_t - 1\right]\right\}.$

Thus, for $(1 - \alpha)\ln k_t - 1 > 0$, or $\ln k_t > 1/(1 - \alpha)$, an increase in α means a higher k_{t+1} for a given k_t and thus the k_{t+1} function shifts up over this range of k_t 's. However, for $\ln k_t < 1/(1 - \alpha)$, an increase in α means a lower k_{t+1} for a given k_t . Thus the k_{t+1} function shifts down over this range of k_t 's. Finally, right at $\ln k_t = 1/(1 - \alpha)$, the old and new k_{t+1} functions intersect.

Problem 2.15

(a) We need to find an expression for k_{t+1} as a function of k_t. Next period's capital stock is equal to this period's capital stock, plus any investment done this period, less any depreciation that occurs. Thus

(1) $K_{t+1} = K_t + sY_t - \delta K_t$.

To convert this into units of effective labor, divide both sides of equation (1) by $A_{t+1}L_{t+1}$:

(2) $\dfrac{K_{t+1}}{A_{t+1}L_{t+1}} = \dfrac{K_t(1-\delta)+sY_t}{A_{t+1}L_{t+1}} = \dfrac{K_t(1-\delta)+sY_t}{(1+n)(1+g)A_tL_t} = \dfrac{k_t(1-\delta)+sf(k_t)}{(1+n)(1+g)},$

which simplifies to

(3) $k_{t+1} = \left[\dfrac{1-\delta}{(1+n)(1+g)}\right]k_t + \left[\dfrac{s}{(1+n)(1+g)}\right]f(k_t).$

(b) We need to sketch k_{t+1} as a function of k_t .

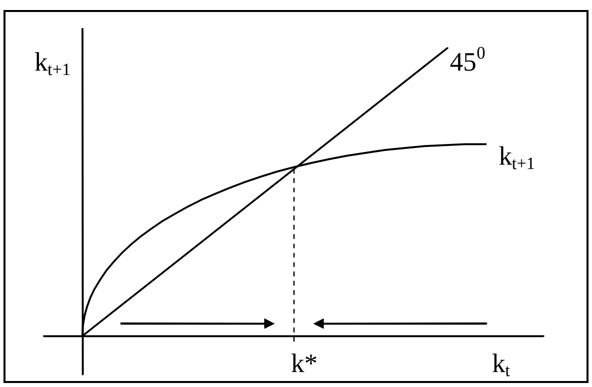
Note that

(4) $\dfrac{\partial k_{t+1}}{\partial k_t} = \dfrac{1-\delta}{(1+n)(1+g)} + \left[\dfrac{s}{(1+n)(1+g)}\right]f'(k_t) \;\; > 0,$ and (5) $\dfrac{\partial^2 k_{t+1}}{\partial k_t{}^2} = \dfrac{sf''(k_t)}{(1+n)(1+g)} \;\; < 0.$

The Inada conditions are given by

(6) $\lim\limits_{k\to 0}\dfrac{\partial k_{t+1}}{\partial k_t} = \infty$, and (7) $\lim\limits_{k\to\infty}\dfrac{\partial k_{t+1}}{\partial k_t} = \dfrac{1-\delta}{(1+n)(1+g)} < 1.$

Thus the function eventually has a slope of less than one and will therefore cross the 45 degree line at some point. Also, the function is well-behaved and will cross the 45 degree line only once.



As long as k starts out at some value other than 0, the economy will converge to k*. For example, if k starts out below k*, we see that k_{t+1} will be greater than k_t and the economy will move toward k*. Similarly, if k starts out above k*, we see that k_{t+1} will be below k_t and again the economy will move toward k*. At k*, y* = f(k*) is also constant and we have a balanced growth path.

(c) On a balanced growth path, $k_{t+1} = k_t \equiv k^*$ and thus from equation (3) we have

(8) $k^* = \left[\dfrac{1-\delta}{(1+n)(1+g)}\right]k^* + \left[\dfrac{s}{(1+n)(1+g)}\right]f(k^*),$

which simplifies to

(9) $k^*\left[\dfrac{1+n+g+ng-1+\delta}{(1+n)(1+g)}\right]=\left[\dfrac{s}{(1+n)(1+g)}\right]f(k^*).$

Thus on a balanced growth path:

(10) $k^*(n + g + ng + \delta) = sf(k^*)$.

Rearranging equation (10) to get an expression for s on the balanced growth path yields

(11) $s = (n + g + ng + \delta)k^*/f(k^*)$.

Consumption per unit of effective labor on the balanced growth path is given by

(12) $c^* = (1 - s)f(k^*)$.

Substitute equation (11) into equation (12):

(13) $c^*=\left[1-\dfrac{(n+g+ng+\delta)k^*}{f(k^*)}\right]f(k^*)=\left[\dfrac{f(k^*)-k^*(n+g+ng+\delta)}{f(k^*)}\right]f(k^*).$

Canceling the $f(k^*)$ yields

(14) $c^* = f(k^*) - (n + g + ng + \delta)k^*$.

To get an expression for the $f'(k^*)$ that maximizes consumption per unit of effective labor on the balanced growth path, we need to maximize c* with respect to k*. The first-order condition is given by

(15) $\partial c^*/\partial k^* = f'(k^*) - (n+g+ng+\delta) = 0.$

Thus the golden-rule capital stock is defined implicitly by

(16) $f'(k_{GR}) = (n + g + ng + \delta)$.

(d) (i) Substitute a Cobb-Douglas production function, $f(k_t) = k_t^{\alpha}$, into equation (3):

(17) $k_{t+1}=\left[\dfrac{1-\delta}{(1+n)(1+g)}\right]k_t+\left[\dfrac{s}{(1+n)(1+g)}\right]k_t^{\alpha}.$

(d) (ii) On a balanced growth path, $k_{t+1} = k_t \equiv k^*$. Thus from equation (17):

(18) $k^*=\left[\dfrac{1-\delta}{(1+n)(1+g)}\right]k^*+\left[\dfrac{s}{(1+n)(1+g)}\right]k^{*\alpha}.$

Simplifying yields

(19) $\left[\dfrac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)}\right]k^*=\left[\dfrac{s}{(1+n)(1+g)}\right]k^{*\alpha},$

or

(20) $k^{*1-\alpha} = s/(n + g + ng + \delta)$.

Thus, finally we have

(21) $k^* = [s/(n + g + ng + \delta)]^{1/(1-\alpha)}$.

(d) (iii) Using equation (17):

(22) $\left.\dfrac{dk_{t+1}}{dk_t}\right|_{k_t=k^*}=\dfrac{1-\delta}{(1+n)(1+g)}+\dfrac{\alpha s}{(1+n)(1+g)}k^{*\alpha-1}.$

Substituting the balanced-growth-path value of k* – equation (21) – into equation (22) yields

(23) $\left.\dfrac{dk_{t+1}}{dk_t}\right|_{k_t=k^*}=\dfrac{1-\delta}{(1+n)(1+g)}+\dfrac{\alpha s}{(1+n)(1+g)}\left[\dfrac{(n+g+ng+\delta)}{s}\right].$

It will be useful to write $(n + g + ng + \delta)$ as $(1 + n)(1 + g) - (1 - \delta)$:

(24) $$\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \frac{(1-\delta)+\alpha\left[(1+n)(1+g)-(1-\delta)\right]}{(1+n)(1+g)}.$$

Simplifying further yields

(25) $$\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \alpha + \frac{(1-\delta)(1-\alpha)}{(1+n)(1+g)}.$$

Replacing equation (17) by its first-order Taylor approximation around k = k* therefore gives us

(26) $k_{t+1} \cong k^* + \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]\left[k_t - k^*\right]$.

Since we can write this simply as

(27) $k_{t+1} - k^* \cong \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]\left[k_t - k^*\right]$,

equation (26) implies

(28) $k_t - k^* \cong \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]^t\left[k_0 - k^*\right]$.

Thus the economy moves fraction $1-\left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]$ of the way to the balanced growth path each period. Some simple algebra simplifies the expression for this rate of convergence to $(1-\alpha)(n+g+ng+\delta)/(1+n)(1+g)$. With α = 1/3, n = 1%, g = 2% and δ = 3%, this yields a rate of convergence of about 3.9%. This is slower than the rate of convergence found in the continuous-time Solow model.

Problem 2.16

(a) The individual's optimization problem is not affected by the depreciation which means that $r_t = f'(k_t) - \delta$. The household's problem is still to maximize utility as given by

(1) $$U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta},$$

subject to the budget constraint

(2) $$C_{1,t} + \frac{1}{1+r_{t+1}}C_{2,t+1} = A_t w_t.$$

As in the text, with no depreciation, the fraction of income saved, $s(r_{t+1}) \equiv (1-C_{1,t})A_t w_t$, is given by

(3) $$s(r_{t+1}) = \frac{1}{1+(1+\rho)^{1/\theta}(1+r_{t+1})^{(\theta-1)/\theta}}.$$

Thus the way in which the fraction of income saved depends on the real interest rate, r_{t+1}, is unchanged. The only difference is that the real interest rate itself is now $f'(k_{t+1}) - \delta$, rather than just $f'(k_{t+1})$.
The capital stock in period t + 1 equals the amount saved by young individuals in period t. Thus

(4) $K_{t+1} = S_t L_t$,

where S_t is the amount of saving done by a young person in period t. Note that $S_t \equiv s(r_{t+1})A_t w_t$; the amount of saving done is equal to the fraction of income saved times the amount of income. Thus equation (4) can be rewritten as

(5) $K_{t+1} = L_t s(r_{t+1})A_t w_t$.

To get this into units of time t + 1 effective labor, divide both sides of equation (5) by $A_{t+1}L_{t+1}$:

(6) $$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{A_t L_t}{A_{t+1}L_{t+1}}\left[s(r_{t+1})w_t\right].$$

Since $A_{t+1} = (1+g)A_t$, we have $A_t/A_{t+1} = 1/(1+g)$. Similarly, $L_t/L_{t+1} = 1/(1+n)$. In addition, $K_{t+1}/A_{t+1}L_{t+1} \equiv k_{t+1}$. Thus

(7) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s(r_{t+1})w_t\right].$$

Finally, substitute for $r_{t+1} = f'(k_{t+1}) - \delta$ and $w_t = f(k_t) - k_t f'(k_t)$:

(8) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\left(f'(k_{t+1}) - \delta\right)\right]\left[f(k_t) - k_t f'(k_t)\right].$$

This should be compared with equation (2.59) in the text, the analogous expression with no depreciation, which is

(2.59) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\left(f'(k_{t+1})\right)\right]\left[f(k_t) - k_t f'(k_t)\right].$$

Thus adding depreciation does alter the relationship between k_{t+1} and k_t. Whether k_{t+1} will be higher or lower for a given k_t depends on the way in which saving varies with r_{t+1}.

(b) With logarithmic utility, the fraction of income saved does not depend upon the rate of return on saving and in fact

(9) $s(r_{t+1}) = 1/(2 + \rho)$.

In addition, with Cobb-Douglas production, $y_t = k_t^{\alpha}$, the real wage is $w_t = k_t^{\alpha} - k_t \alpha k_t^{\alpha-1} = (1 - \alpha)k_t^{\alpha}$. Thus equation (8) becomes

(10) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}\right].$$

We need to compare this with equation (3) in the solution to Problem 2.15, the analogous expression in the discrete-time Solow model, with the additional assumption of 100% depreciation (i.e. $\delta = 1$).

The saving rate in this economy is total saving divided by total output. Note that this is not the same as $s(r_{t+1})$, which is simply the fraction of their labor income that the young save. Denote the economy's total saving rate as $\hat{s}$. Then $\hat{s}$ will equal the saving of the young plus the dissaving of the old, all divided by total output and in addition, all variables are measured in units of effective labor.

The saving of the young is $\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}$. Since there is 100% depreciation, the old do not get to dissave by the amount of the capital stock; there is no dissaving by the old. Thus

(11) $$\hat{s} = \frac{\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}}{k_t^{\alpha}} = \frac{1}{2+\rho}(1-\alpha).$$

Thus equation (10) can be rewritten as

(12) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\hat{s}k_t^{\alpha} = \left[\frac{\hat{s}}{(1+n)(1+g)}\right]f(k_t).$$

Note that this is exactly the <u>same</u> as the expression for k_{t+1} as a function of k_t in the discrete-time Solow model with $\delta = 1$. That is, it is equivalent to equation (3) in the solution to Problem 2.15 with δ set to one. Thus that version of the Solow model does have some microeconomic foundations, although the assumption of 100% depreciation is quite unrealistic.

<u>Problem 2.17</u>

(a) (i) The utility function is given by

(1) $\ln C_{1,t} + \left[1/(1+\rho)\right]\ln C_{2,t+1}$.

With the social security tax of T per person, the individual faces the following constraints (with g, the growth rate of technology, equal to 0, A is simply a constant throughout):

(2) $C_{1,t} + S_t = Aw_t - T$, and

(3) $C_{2,t+1} = (1 + r_{t+1})S_t + (1+n)T$,

where S_t represents the individual's saving in the first period. As far as the individual is concerned, the rate of return on social security is (1 + n); in general this will not be equal to the return on private saving which is ($1 + r_{t+1}$). From equation (3), $(1+r_{t+1})S_t = C_{2,t+1} - (1+n)T$. Solving for S_t yields

(4) $S_t = \dfrac{C_{2,t+1}}{1+r_{t+1}} - \dfrac{(1+n)}{(1+r_{t+1})}T.$

Now substitute equation (4) into equation (2):

(5) $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + \dfrac{(1+n)}{(1+r_{t+1})}T.$

Rearranging, we get the intertemporal budget constraint:

(6) $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - \dfrac{(r_{t+1}-n)}{(1+r_{t+1})}T.$

We know that with logarithmic utility, the individual will consume fraction $(1 + \rho)/(2 + \rho)$ of her lifetime wealth in the first period. Thus

(7) $C_{1,t} = \left(\dfrac{1+\rho}{2+\rho}\right)\left[Aw_t - \left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)T\right].$

To solve for saving per person, substitute equation (7) into equation (2):

(8) $S_t = Aw_t - \left(\dfrac{1+\rho}{2+\rho}\right)\left[Aw_t - \left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)T\right] - T$

Simplifying gives us

(9) $S_t = \left[1 - \left(\dfrac{1+\rho}{2+\rho}\right)\right]Aw_t - \left[1 - \left(\dfrac{1+\rho}{2+\rho}\right)\left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)\right]T,$

or

(10) $S_t = \left[1/(2+\rho)\right]Aw_t - \left[\dfrac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right]T.$

Note that if $r_{t+1} = n$, saving is reduced one-for-one by the social security tax. If $r_{t+1} > n$, saving falls less than one-for-one. Finally, if $r_{t+1} < n$, saving falls more than one-for-one.

Denote $Z_t \equiv \left[(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)\right]\big/(2+\rho)(1+r_{t+1})$ and thus equation (10) becomes

(11) $S_t = \left[1/(2+\rho)\right]Aw_t - Z_t T.$

It is still true that the capital stock in period t+1 will be equal to the total saving of the young in period t, hence

(12) $K_{t+1} = S_t L_t$.

Converting this into units of effective labor by dividing both sides of (12) by AL_{t+1} and using equation (11) yields

(13) $k_{t+1} = \left[1/(1+n)\right]\left[\left(1/(2+\rho)\right)w_t - Z_t T/A\right].$

With a Cobb-Douglas production function, the real wage is given by

(14) $w_t = (1 - \alpha)k_t^{\alpha}$.

Substituting (14) into (13) gives the new relationship between capital in period t + 1 and capital in period t, all in units of effective labor:

(15) $k_{t+1} = \left[1/(1+n)\right]\left[\left(1/(2+\rho)\right)(1-\alpha)k_t^{\alpha} - Z_t T/A\right].$

(a) (ii) To see what effect the introduction of the social security system has on the balanced-growth-path value of k, we must determine the sign of Z_t. If it is positive, the introduction of the tax, T, shifts down the k_{t+1} curve and reduces the balanced-growth-path value of k. We have

$$(16)\quad Z_t = \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} = \frac{(1+1+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})},$$

and simplifying further allows us to determine the sign of Z_t :

$$(17)\quad Z_t = \frac{(1+r_{t+1}) + (1+\rho)\left[(1+r_{t+1}) - (r_{t+1}-n)\right]}{(2+\rho)(1+r_{t+1})} = \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0.$$

Thus, the k_{t+1} curve shifts down, relative to the case without the social security, and k* is reduced.

(a) (iii) If the economy was initially dynamically efficient, a marginal increase in T would result in a gain to the old generation that would receive the extra benefits. However, it would reduce k* further below k_{GR} and thus leave future generations worse off, with lower consumption possibilities. If the economy was initially dynamically inefficient, so that $k^* > k_{GR}$, the old generation would again gain due to the extra benefits. In this case, the reduction in k* would actually allow for higher consumption for future generations and would be welfare-improving. The introduction of the tax in this case would reduce or possibly eliminate the dynamic inefficiency caused by the over-accumulation of capital.

(b) (i) Equation (3) becomes

$$(18)\quad C_{2,t+1} = (1+r_{t+1})S_t + (1+r_{t+1})T.$$

As far as the individual is concerned, the rate of return on social security is the same as that on private saving. We can now derive the intertemporal budget constraint. From equation (4),

$$(19)\quad S_t = C_{2,t+1}\big/\left(1+r_{t+1}\right) - T.$$

Substituting equation (19) into equation (2) yields

$$(20)\quad C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + T,$$

or simply

$$(21)\quad C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t.$$

This is just the usual intertemporal budget constraint in the Diamond model. Solving the individual's maximization problem yields the usual Euler equation:

$$(22)\quad C_{2,t+1} = \left[1/(1+\rho)\right](1+r_{t+1})C_{1,t}$$

Substituting this into the budget constraint, equation (21), yields

$$(23)\quad C_{1,t} = \left[(1+\rho)/(2+\rho)\right]Aw_t.$$

To get saving per person, substitute equation (23) into equation (2):

$$(24)\quad S_t = Aw_t - \left[(1+\rho)/(2+\rho)\right]Aw_t - T,$$

or simply

$$(25)\quad S_t = \left[1/(2+\rho)\right]Aw_t - T.$$

The social security tax causes a one-for-one reduction in private saving.

The capital stock in period t +1 will be equal to the sum of total private saving of the young plus the total amount invested by the government. Hence

(26) $K_{t+1} = S_t L_t + TL_t$.

Dividing both sides of (26) by AL_{t+1} to convert this into units of effective labor, and using equation (25) yields

(27) $k_{t+1} = \left(\frac{1}{1+n}\right)\left[\left(\frac{1}{2+\rho}\right)w_t - \frac{T}{A}\right] + \left(\frac{1}{1+n}\right)\frac{T}{A}$,

which simplifies to

(28) $k_{t+1} = \left[1/(1+n)\right]\left[1/(2+\rho)\right] w_t$.

Using equation (14) to substitute for the wage yields

(29) $k_{t+1} = \left[1/(1+n)\right]\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}$.

Thus the fully-funded social security system has no effect on the relationship between the capital stock in successive periods.

(b) (ii) Since there is no effect on the relationship between k_{t+1} and k_t , the balanced-growth-path value of k is the same as it was before the introduction of the fully-funded social security system. (Note that we have been assuming that the amount of the tax is not greater than the amount of saving each individual would have done in the absence of the tax). The basic idea is that total investment and saving is still the same each period; the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus individuals offset one-for-one any saving that the government does for them.

Problem 2.18

(a) In the decentralized equilibrium, there will be no intergenerational trade. Even if the young would like to trade goods this period for goods next period, the only people around to trade with are the old. Unfortunately, the old will be dead – and thus in no position to complete the trade – next period.

The individual's utility function is given by

(1) $\ln C_{1,t} + \ln C_{2,t+1}$.

The constraints are

(2) $C_{1,t} + F_t = A$, and (3) $C_{2,t+1} = xF_t$,

where F_t is the amount stored by the individual.

Substituting equation (3) into (2) yields the intertemporal budget constraint:

(4) $C_{1,t} + C_{2,t+1}/x = A$.

The individual's problem is to maximize lifetime utility, as given by equation (1), subject to the intertemporal budget constraint, as given by equation (4). Set up the Lagrangian:

(5) $L = \ln C_{1,t} + \ln C_{2,t+1} + \lambda\left[A - C_{1,t} - C_{2,t+1}/x\right]$.

The first-order conditions are given by

$\partial L/\partial C_{1,t} = 1/C_{1,t} - \lambda = 0 \Rightarrow 1/C_{1,t} = \lambda$, and (6)

$\partial L/\partial C_{2,t+1} = 1/C_{2,t+1} - \lambda/x = 0 \Rightarrow 1/C_{2,t+1} = \lambda/x$. (7)

Substitute equation (6) into equation (7) and rearrange to obtain

(8) $C_{2,t+1} = xC_{1,t}$.

Substitute equation (8) into the intertemporal budget constraint, equation (4), to obtain

(9) $C_{1,t} + xC_{1,t}/x = A$,

or simply

(10) $C_{1,t} = A/2$.

To obtain an expression for second-period consumption, substitute equation (10) into equation (8):

(11) $C_{2,t+1} = xA/2$.

When young, each individual consumes half of her endowment and stores the other half; that is, $f_t = 1/2$. This allows her to consume xA/2 when old. Note that with log utility, the fraction of her endowment that the individual stores does not depend upon the return to storage.

(b) What is consumption per unit of effective labor at time t? First, calculate total consumption at time t:

(12) $C_t = C_{1,t}L_t + C_{2,t}L_{t-1}$,

where there are L_t young and L_{t-1} old individuals alive at time t. Each young person consumes the fraction of her endowment that she does not store, (1 - f)A, and each old person gets to consume the gross return on the fraction of her endowment that she stored, fxA. Thus

(13) $C_t = (1-f)AL_t + fxAL_{t-1}$.

To convert this into units of time t effective labor, divide both sides by AL_t to get

(14) $C_t/AL_t = (1-f) + f\left[x/(1+n)\right]$.

Thus consumption per unit of time t effective labor is a weighted average of one and something less than one, since x < (1 + n). It will therefore be maximized when the weight on one is one; that is, when f = 0. (We could also carry out this analysis on consumption per person alive at time t which would not change the result here).

The decentralized equilibrium, with f = 1/2, is not Pareto efficient. Since intergenerational trade is not possible, individuals are "forced" into storage because that is the only way they can save and consume in old age. They must do this even if the return on storage, x, is low. However, at any point in time, a social planner could take one unit from each young person and give (1 + n) units to each old person since there are fewer old persons than young persons. With (1 + n) > x, this gives a better return than storage. Therefore, the social planner could raise welfare by taking the half of each generation's endowment that it was going to store and instead give it to the old. The planner could then do this each period. This allows individuals to consume A/2 units when young – the same as in the decentralized equilibrium – but it also allows them to consume (1 + n)A/2 units when old. This is greater than the xA/2 units of consumption when old that they would have had in the decentralized equilibrium with storage.

Problem 2.19

(a) The individual has a utility function given by

(1) $\ln C_{1,t} + \ln C_{2,t+1}$,

and constraints, expressed in units of money, given by

(2) $P_tC_{1,t} = P_tA - P_tF_t - M_t^d$, and

(3) $P_{t+1}C_{2,t+1} = P_{t+1}xF_t + M_t^d$,

where M_t^d is nominal money demand and F_t is the amount stored.

One way of thinking about the problem is the following. The individual has two decisions to make. The first is to decide how much of her endowment to consume and how much to "save". Then she must decide the way in which to save, through storage, by holding money or a combination of both. With log utility, we can separate the two decisions since the rate of return on "saving" will not affect the fraction of the first-period endowment that is saved. From the solution to Problem 2.18, we know she will consume half of the endowment in the first period, regardless of the rate of return on money or storage. Thus

(4) $C_{1,t} = A/2$.

What does she do with the other half? That will depend upon the gross rate of return on storage, x, relative to the gross rate of return on money, which is P_t/P_{t+1} . The gross rate of return on money is P_t/P_{t+1}

since the individual can sell one unit of consumption in period t and get P_t units of money. In period t + 1, one unit of consumption costs P_{t+1} units of money and thus one unit of money will buy $1/P_{t+1}$ units of consumption. Thus the individual's P_t units of money will buy P_t/P_{t+1} units of consumption in period t+1.

CASE 1 : $x > P_t/P_{t+1}$
She will consume half of her endowment, store the rest and not hold any money since the rate of return on money is less than the rate of return on storage. Thus

$$C_{1,t} = A/2 \qquad F_t = A/2 \qquad M_t^d/P_t = 0 \qquad C_{2,t+1} = xA/2.$$

CASE 2 : $x < P_t/P_{t+1}$
Now storage is dominated by holding money. She will consume half of her endowment and then sell the rest for money:

$$C_{1,t} = A/2 \qquad F_t = 0 \qquad M_t^d/P_t = A/2 \qquad C_{2,t+1} = [P_t/P_{t+1}][A/2].$$

CASE 3 : $x = P_t/P_{t+1}$
Money and storage pay the same rate of return. She will consume half of her endowment and is then indifferent as to how much of the other half to store and how much of it to sell for money. Let $\alpha \in [0,1]$ be the fraction of saving that is in the form of money. Thus

$$C_{1,t} = A/2 \qquad F_t = (1-\alpha)A/2 \qquad M_t^d/P_t = \alpha A/2 \qquad C_{2,t+1} = xA/2 = [P_t/P_{t+1}][A/2].$$

(b) Equilibrium requires that aggregate real money demand equal aggregate real money supply. We can derive expressions for both real money demand and supply in period t:
aggregate real money demand = $L_t[A/2]$, and
aggregate real money supply = $[L_0/(1+n)]M/P_t = [L_t/(1+n)^{t+1}]M/P_t$.
The expression for aggregate real money supply uses the fact that in period 0, each old person, and there are $[L_0/(1 + n)]$ of them, receives M units of money. The last step then uses the fact that since population grows at rate n, $L_t = (1 + n)^t L_0$ and thus $L_0 = L_t/(1 + n)^t$.
We can then use the equilibrium condition to solve for P_t :

$$L_t[A/2] = [L_t/(1+n)^{t+1}]M/P_t \qquad \Rightarrow \qquad P_t = 2M/[A(1+n)^{t+1}]. \quad (5)$$

We can similarly derive expressions for real money demand and supply in period t + 1:
aggregate real money demand = $L_{t+1}[A/2] = (1+n)L_t[A/2]$, and
aggregate real money supply = $[L_t/(1+n)^{t+1}]M/P_{t+1}$.
We can then use the equilibrium condition to solve for P_{t+1} :

$$(1+n)L_0[A/2] = [L_t/(1+n)^{t+1}]M/P_{t+1} \qquad \Rightarrow \qquad P_{t+1} = 2M/[A(1+n)^{t+2}]. \quad (6)$$

Dividing equation (6) by equation (5) yields

$$P_{t+1}/P_t = 1/(1+n) \qquad \Rightarrow \qquad P_{t+1} = P_t/(1+n).$$

This analysis holds for all time periods $t \geq 0$ and so $P_{t+1} = P_t/(1 + n)$ is an equilibrium. This shows that if money is introduced into a dynamically inefficient economy, storage will not be used. The monetary equilibrium will thus result in attainment of the "golden-rule" level of storage. See the solution to part (b) of Problem 2.18 for an explanation of the reason that zero storage maximizes consumption per unit of effective labor.

(c) This is the situation where $P_t/P_{t+1} = x$; the return on money is equal to the return on storage. In this case, individuals are indifferent as to how much of their saving to store and how much to hold in the form

of money. Let $\alpha_t \in [0,1]$ be the fraction of saving held in the form of money in period t. We can again derive expressions for aggregate real money demand and supply in period t:
aggregate real money demand = $L_t\alpha_t[A/2]$, and
aggregate real money supply = $[L_0/(1+n)]M/P_t = [L_t/(1+n)^{t+1}]M/P_t$.
We can then use the equilibrium condition to solve for P_t:

$$L_t\alpha_t[A/2] = [L_t/(1+n)^{t+1}]M/P_t \qquad \Rightarrow \qquad P_t = 2M/[\alpha_t A(1+n)^{t+1}]. \quad (7)$$

We can similarly derive expressions for real money demand and supply in period t + 1:
aggregate real money demand = $L_{t+1}\alpha_{t+1}[A/2] = (1+n)L_t\alpha_{t+1}[A/2]$, and
aggregate real money supply = $[L_t/(1+n)^{t+1}]M/P_{t+1}$.
We can then use the equilibrium condition to solve for P_{t+1}:

$$(1+n)\,L_0\alpha_{t+1}[A/2] = [L_t/(1+n)^{t+1}]M/P_{t+1} \qquad \Rightarrow \qquad P_{t+1} = 2M/[\alpha_{t+1}A(1+n)^{t+2}]. \quad (8)$$

Dividing equation (8) by (7) yields

$$P_{t+1}/P_t = [\alpha_t/\alpha_{t+1}]\,[1/(1+n)].$$

For $P_{t+1}/P_t = 1/x$, we need

$$[\alpha_t/\alpha_{t+1}]\,[1/(1+n)] = 1/x \qquad \Rightarrow \qquad [\alpha_{t+1}/\alpha_t] = [x/(1+n)] < 1.$$

Thus for all $t \geq 0$, $P_{t+1} = P_t/x$ will be an equilibrium for any path of α's that satisfies $\alpha_{t+1}/\alpha_t = x/(1+n)$.

(d) $P_t = \infty$ – money is worthless – is also an equilibrium. This occurs if the young generation at time 0 does not believe that money will be valued in the next period and thus that the generation one individuals will not accept money for goods. In that case, in period 0, the young simply consume half of their endowment and store the rest, and the old have some useless pieces of paper to go along with their endowment. This is an equilibrium with real money demand equal to zero and real money supply equal to zero as well. If no one believes the next generation will accept money for goods, this equilibrium continues for all future time periods.

This will be the only equilibrium if the economy ends at some date T. The young at date T will not want to sell any of their endowment. They will maximize the utility of their one-period life by consuming all of their endowment in period T. Thus, if the old at date T held any money, they would be stuck with it and it would be useless to them. Thus when they are young, in period T - 1, they will not sell any of their endowment for money, knowing that the money will be of no use to them when old. Thus, if the old at date T - 1 held any money, they would be stuck with it and it would be useless to them. Thus the old at T - 1 will not want any money when they are young and so on. Working backward, no one would ever want to sell goods for money and money would not be valued.

Problem 2.20

(a) (i) The individual has a utility function given by
(1) $U = \ln C_{1,t} + \ln C_{2,t+1}$,
and a lifetime budget constraint given by
(2) $Q_t C_{1,t} + Q_{t+1} C_{2,t+1} = Q_t (A - S_t) + Q_{t+1} xS_t$.
From Problem 2.18, we know that with log utility, the individual wants to consume A/2 in the first period. The way in which the individual accomplishes this depends on the gross rate of return on storage, x, relative to the gross rate of return on trading.

The individual can sell one unit of the good in period t for Q_t . In period t + 1, it costs Q_{t+1} to obtain one unit of the good or equivalently, it costs one to obtain $1/Q_{t+1}$ units of the good. Thus for Q_t , it is possible to obtain Q_t/Q_{t+1} units of the good. Thus selling a unit of the good in period t allows the individual to buy Q_t/Q_{t+1} units of the good in period t + 1. Thus the gross rate of return on trading is Q_t/Q_{t+1} .

Now, $Q_{t+1} = Q_t/x$ for all t > 0 is equivalent to $x = Q_t/Q_{t+1}$ for all t > 0. In other words, the rate of return on storage is equal to the rate of return on trading and hence the individual is indifferent as to the amount to store and the amount to trade. Let $\alpha_t \in [0, 1]$ represent the fraction of "saving", A/2, that the individual sells in period t. That is, the individual sells $\alpha_t(A/2)$ in period t. This allows the individual to buy the amount $\alpha_t(Q_t/Q_{t+1})(A/2)$ when she is old in period t + 1. The individual stores a fraction $(1 - \alpha_t)$ of her "saving". Thus

(3) $S_t = (1 - \alpha_t)(A/2)$.

Consumption in period t + 1 will be equal to the amount the individual buys plus the amount she has through storage. Thus

(4) $C_{2,t+1} = \alpha_t(Q_t/Q_{t+1})(A/2) + (1 - \alpha_t)x(A/2)$.

Since we are considering a case in which $Q_t/Q_{t+1} = x$, equation (4) can be rewritten as

(5) $C_{2,t+1} = \alpha_t x(A/2) + (1 - \alpha_t)x(A/2) = x(A/2)$.

Consider some period t + 1 and let L represent the total number of individuals born each period, which is constant. Aggregate supply in period t + 1 is equal to the total number of young individuals, L, multiplied by the amount that each young individual wishes to sell, $\alpha_{t+1}(A/2)$. Thus

(6) $\text{Aggregate Supply}_{t+1} = L\alpha_{t+1}(A/2)$.

Aggregate demand in period t + 1 is equal to the total number of old individuals, L, multiplied by the amount each old individual wishes to buy, $(Q_t/Q_{t+1})\alpha_t(A/2)$. Thus

(7) $\text{Aggregate Demand}_{t+1} = L(Q_t/Q_{t+1})\alpha_t(A/2)$.

For the market to clear, aggregate supply must equal aggregate demand or

(8) $L\alpha_{t+1}(A/2) = L(Q_t/Q_{t+1})\alpha_t(A/2)$,

or simply

(9) $\alpha_{t+1} = (Q_t/Q_{t+1})\alpha_t$.

Since the proposed price path has $Q_{t+1} = Q_t/x$, the equilibrium condition given by equation (9) can also be written as

(10) $\alpha_{t+1} = x\alpha_t$.

Now consider the situation in period 0. The old individuals simply consume their endowment. Thus we must have α_0 equal to zero in order for the market to clear in period 0. Thus equation (10) implies that we must have $\alpha_t = 0$ for all $t \geq 0$.

The resulting equilibrium is the same as that in part (a) of Problem 2.18. The individual consumes half of her endowment in the first period of life, stores the rest and consumes xA/2 in the second period of life. Note that with x < 1 + n here (since n = 0 and x < 1), this equilibrium is dynamically inefficient. Thus eliminating incomplete markets by allowing individuals to trade before the start of time does not eliminate dynamic inefficiency.

(a) (ii) Suppose the auctioneer announces $Q_{t+1} < Q_t/x$ or equivalently $x < (Q_t/Q_{t+1})$ for some date t. This means that trading dominates storage for the young at date t. This means that the young at date t will want to sell all of their saving – $\alpha_t = 1$ so that they want to sell A/2 – and not store anything. Thus aggregate supply in period t is equal to L(A/2). For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to buy when old on Q_t/Q_{t-1} which was equal to x. Thus as described in part (a) (i), old individuals were not planning to buy anything. Thus aggregate demand in period t is zero. Thus aggregate demand will be less than aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

Suppose instead that the auctioneer announces $Q_{t+1} > Q_t /x$ or equivalently $x > (Q_t /Q_{t+1})$ for some date t. This means that storage dominates trading for the young at date t. This means that the young at date t will want to store their entire endowment and will want to buy A/2. For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to trade when old on Q_t /Q_{t-1} which was equal to x. Thus each old individual was not planning to buy or sell anything. Thus aggregate demand exceeds aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

(b) Consider the social planner's problem. The planner can divide the resources available for consumption between the young and the old in any matter. The planner can take, for example, one unit of each young person's endowment and transfer it to the old. Since there are the same number of old and young people in this model, this increases the consumption of each old person by one. With x < 1, this method of transferring from the young to the old provides a better return than storage. If the economy did not end at some date T, the planner could prevent this change from making anyone worse off by requiring the next generation of young to make the same transfer in the following period. However, if the economy ends at some date T, the planner cannot do this. Taking anything from the young at date T would make them worse off since the planner cannot give them anything in return the next period; there is no next period. Thus the planner cannot make some generations better off without making another generation worse off. Thus the decentralized equilibrium is Pareto-efficient.

(c) It is infinite duration that is the source of the dynamic inefficiency. Allowing individuals to trade before the start of time requires a price path that results in an equilibrium which is equivalent to the situation where such a market does not exist. This equilibrium is not Pareto-efficient; a social planner could raise welfare by doing the procedure described in part (b). However, removing infinite duration also removes the social planner's ability to Pareto improve the decentralized equilibrium, as explained in part (b).

Problem 2.21

(a) The individual has utility function given by

(1) $$\frac{C_{1,t}{}^{1-\theta}}{1-\theta}+\frac{C_{2,t+1}{}^{1-\theta}}{1-\theta} \qquad \theta<1.$$

The constraints expressed in units of money are

(2) $P_t C_{1,t} = P_t A - M_t^d$, and (3) $P_{t+1}C_{2,t+1} = M_t^d$.

Combining equations (2) and (3) yields the lifetime budget constraint:

(4) $P_t C_{1,t} + P_{t+1}C_{2,t+1} = P_t A$.

Note that $\theta < 1$ means that the elasticity of substitution, $1/\theta$, is greater than one. Thus when the rate of return on saving increases, the substitution effect dominates; the individual will consume less now and save more. This is essentially what we will be showing here. As the rate of return on holding money, which is P_t /P_{t+1}, rises, the individual wishes to hold more money.

The individual's problem is to maximize (1) subject to (4). Set up the Lagrangian:

(5) $$L=\frac{C_{1,t}{}^{1-\theta}}{1-\theta}+\frac{C_{2,t+1}{}^{1-\theta}}{1-\theta}+\lambda\left[P_t A - P_t C_{1,t} - P_{t+1}C_{2,t+1}\right]$$

The first-order conditions are

$$\frac{\partial L}{\partial C_{1,t}} = C_{1,t}{}^{-\theta} - \lambda P_t = 0 \Rightarrow \lambda = \frac{C_{1,t}{}^{-\theta}}{P_t}, \text{ and} \qquad (6)$$

$$\frac{\partial L}{\partial C_{2,t+1}} = C_{2,t+1}{}^{-\theta} - \lambda P_{t+1} = 0 \Rightarrow C_{2,t+1}{}^{-\theta} = \lambda P_{t+1}. \quad (7)$$

Substitute (6) into (7) to obtain

(8) $C_{2,t+1}{}^{-\theta} = C_{1,t}{}^{-\theta} P_{t+1} / P_t$,

Taking both sides of equation (8) to the exponent –1/θ gives us

(9) $C_{2,t+1} = \left(P_t / P_{t+1}\right)^{1/\theta} C_{1,t}$.

This is the Euler equation, which can now be substituted into the budget constraint given by equation (4):

(10) $P_t C_{1,t} + P_{t+1}\left(P_t / P_{t+1}\right)^{1/\theta} C_{1,t} = P_t A$.

Dividing by P_t yields

(11) $C_{1,t} + \left(P_{t+1} / P_t\right)\left(P_t / P_{t+1}\right)^{1/\theta} C_{1,t} = A$,

and simplifying yields

(12) $C_{1,t} + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta} C_{1,t} = A$

or

(13) $C_{1,t}\left[1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}\right] = A$.

Thus consumption when young is given by

(14) $C_{1,t} = \dfrac{A}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}$.

To get the amount of her endowment that the individual sells for money (in real terms), we can use equation (2), expressed in real terms

(15) $M_t^d / P_t = A - C_{1,t}$.

Substitute equation (14) into (15) to obtain

(16) $\dfrac{M_t^d}{P_t} = A - \dfrac{A}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}$,

which can be rewritten as

(17) $\dfrac{M_t^d}{P_t} = A\left[1 - \dfrac{1}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}\right]$.

Simplifying by getting a common denominator yields

(18) $\dfrac{M_t^d}{P_t} = A \dfrac{\left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}$.

Dividing the top and bottom of the right-hand side of equation (18) by $\left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}$ yields

(19) $\dfrac{M_t^d}{P_t} = \dfrac{A}{\left(P_t / P_{t+1}\right)^{(\theta-1)/\theta} + 1}$.

Thus the fraction of her endowment that the individual sells for money is

(20) $h_t = \dfrac{1}{\left(P_t / P_{t+1}\right)^{(\theta-1)/\theta} + 1}$.

Taking the derivative of h_t with respect to (P_t/P_{t+1}) allows us to shows that the fraction of her endowment that the agent sells for money is an increasing function of the rate of return on holding money:

(21) $$\frac{\partial h_t}{\partial\left(P_t/P_{t+1}\right)} = \frac{-[(\theta-1)/\theta]\left(P_t/P_{t+1}\right)^{[(\theta-1)/\theta]-1}}{\left[\left(P_t/P_{t+1}\right)^{(\theta-1)/\theta}+1\right]^2} > 0 \text{ for } \theta < 1.$$

We can also show that as the rate of return on money goes to zero, the amount of her endowment that the individual sells for money goes to zero. Use equation (18) to rewrite h_t as

(22) $$h_t = \frac{\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}{\left[1+\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}\right]},$$

and thus

(23) $$\lim_{\left(P_t/P_{t+1}\right)\to 0} h_t = 0/(1+0) = 0.$$

(b) The constraints expressed in real terms are

(24) $C_{1,t} = A - M_t^d/P_t$, and (25) $C_{2,t+1} = M_t^d/P_{t+1}$.

Since there is no population growth, we can normalize the population to one without loss of generality. From (25), a generation born at time t plans to buy M^d_t/P_{t+1} units of the good when it is old. Thus, the generation born at time 0 plans to buy M^d_0/P_1 units when it is old (in period 1). Use equation (19) to find M^d_0, substituting t = 0:

(26) $$M_0^d = \frac{P_0 A}{(P_0/P_1)^{(\theta-1)/\theta}+1},$$

which, expressed in real terms, is

(27) $$\frac{M_0^d}{P_1} = \frac{[P_0/P_1]A}{(P_0/P_1)^{(\theta-1)/\theta}+1}.$$

From equation (24), a generation born at time t plans to sell M^d_t/P_t units of the good for money. Thus the generation born at time 1 plans to sell M^d_1/P_1 units of the good. Substituting t = 1 into equation (19) gives us

(28) $$\frac{M_1^d}{P_1} = \frac{A}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1}.$$

In order for the amount of the consumption good that generation 0 wishes to buy with its money, given by equation (27), to be equal to the amount of the consumption good that generation 1 wishes to sell for money, given by equation (28), we need

(29) $$\frac{[P_0/P_1]A}{(P_0/P_1)^{(\theta-1)/\theta}+1} = \frac{A}{(P_1/P_2)^{(\theta-1)/\theta}+1},$$

or

(30) $$\frac{(P_0/P_1)^{(\theta-1)/\theta}+1}{(P_1/P_2)^{(\theta-1)/\theta}+1} = \frac{P_0}{P_1}.$$

Now with $P_0/P_1 < 1$, this means that we need

(31) $$(P_0/P_1)^{(\theta-1)/\theta}+1 < (P_1/P_2)^{(\theta-1)/\theta}+1,$$

which is equivalent to

(32) $$(P_0/P_1)^{(\theta-1)/\theta} < (P_1/P_2)^{(\theta-1)/\theta}.$$

Since (θ - 1)/θ is negative, this implies

(33) $\frac{P_1}{P_2} < \frac{P_0}{P_1} < 1.$

(c) Iterating this reasoning forward, the rate of return on money will have to be falling over time. That is,

$$1 > \frac{P_0}{P_1} > \frac{P_1}{P_2} > \frac{P_2}{P_3} > \ldots \quad \text{and so} \quad \frac{P_t}{P_{t+1}} \to 0.$$

As shown in part (a), this means that the fraction of the endowment that is sold for money will also go to zero. The economy approaches the situation in which individuals consume their entire endowment in the first period. This is an equilibrium path in the sense that every time period, markets will clear. Each period, the real money demand by the young will be equal to real money supplied by the old and they will both go to zero as t gets large.

(d) If $P_0/P_1 > 1$, we obtain the opposite result for the path of prices. That is, P_t/P_{t+1} will rise over time:

$$1 < \frac{P_0}{P_1} < \frac{P_1}{P_2} < \frac{P_2}{P_3} < \ldots \quad \text{and so} \quad \frac{P_t}{P_{t+1}} \to \infty.$$

From equation (20) we can see that this means that the fraction of the endowment sold for money will go to one. In other words, the economy approaches the situation in which no one consumes anything in the first period and individuals sell their entire endowment for money. Thus, total real money demand will go to A, the endowment of the young (we have normalized the population to one). But with this path of prices, the price level goes to zero, which means that real money supplied by the old goes to infinity. Thus this cannot represent an equilibrium path for the economy because there will be a time period when real money supply will exceed real money demand and the market will not clear.

SOLUTIONS TO CHAPTER 3

Problem 3.1

The production functions for output and new knowledge are given by

(1) $Y(t) = A(t)(1 - a_L)L(t)$,

and

(2) $\dot{A}(t) = Ba_L{}^{\gamma} L(t)^{\gamma} A(t)^{\theta}, \qquad \theta < 1$.

(a) On a balanced growth path,

(3) $\dot{A}(t)/A(t) = g_A{}^* = \gamma n/(1 - \theta)$.

Dividing both sides of equation (2) by A(t) yields

(4) $\dot{A}(t)/A(t) = Ba_L{}^{\gamma} L(t)^{\gamma} A(t)^{\theta-1}$.

Equating (3) and (4) yields

(5) $Ba_L{}^{\gamma} L(t)^{\gamma} A(t)^{\theta-1} = \gamma n/(1-\theta) \quad \Rightarrow \quad A(t)^{\theta-1} = \gamma n \big/ (1-\theta) Ba_L{}^{\gamma} L(t)^{\gamma}$.

Simplifying and solving for A(t) yields

(6) $A(t) = \left[(1-\theta)Ba_L{}^{\gamma} L(t)^{\gamma} \big/ \gamma n\right]^{1/(1-\theta)}$.

(b) Substitute equation (6) into equation (1):

$Y(t) = \left[(1-\theta)Ba_L{}^{\gamma} L(t)^{\gamma} \big/ \gamma n\right]^{1/(1-\theta)} (1-a_L)L(t) = \left[(1-\theta)B/\gamma n\right]^{1/(1-\theta)} a_L{}^{\gamma/(1-\theta)} (1-a_L)L(t)^{[\gamma/(1-\theta)]+1}$.

We can maximize the log of output with respect to a_L or maximize

(7) $\ln Y(t) = \left[1/(1-\theta)\right] \ln\left[(1-\theta)B/\gamma n\right] + \left[\gamma/(1-\theta)\right] \ln a_L + \ln(1-a_L) + \left[\left(\gamma/(1-\theta)\right)+1\right] \ln L(t)$.

The first-order condition is given by

(8) $\dfrac{\partial \ln Y(t)}{\partial a_L} = \dfrac{\gamma}{(1-\theta)} \dfrac{1}{a_L} - \dfrac{1}{1-a_L} = 0$.

Solving for an expression for $a_L{}^*$ yields

(9) $a_L{}^* = \dfrac{\gamma}{(1-\theta)+\gamma}$.

The higher is θ, the importance of knowledge in the production of new knowledge, and the higher is γ, the importance of labor in the production of new knowledge, the more of the labor force that should be employed in the knowledge sector.

Problem 3.2

Substituting the production function, $Y_i(t) = K_i(t)^{\theta}$, into the capital-accumulation equation, $\dot{K}_i(t) = s_i Y_i(t)$, yields

(1) $\dot{K}_i(t) = s_i K_i(t)^{\theta}, \quad \theta > 1$.

Dividing both sides of equation (1) by $K_i(t)$ gives an expression for the growth rate of the capital stock, $g_{K,i}$:

(2) $g_{K,i}(t) \equiv \dot{K}_i(t)/K_i(t) = s_i K_i(t)^{\theta-1}$.

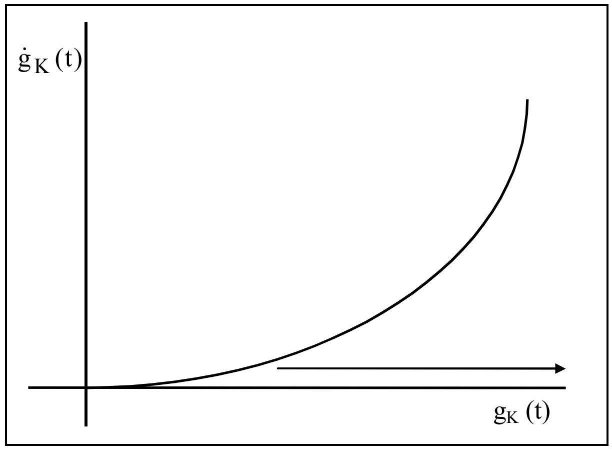
Taking the time derivative of the log of equation (2) yields an expression for the growth rate of the growth rate of capital:

(3) $\dot{g}_{K,i}(t)/g_{K,i}(t) = (\theta-1)g_{K,i}(t)$,

and thus

(4) $\dot{g}_{K,i}(t) = (\theta-1)g_{K,i}(t)^2$.

Equation (4) is plotted at right. With $\theta > 1$, $g_{K,i}$ will be always increasing. The initial value of $g_{K,i}$ is determined by the initial capital stock and the saving rate; see equation (2).



Since both economies have the same K(0) but one has a higher saving rate, then from equation (2), the economy with the higher s will have the higher initial $g_{K,i}(0)$.

From equation (3), the growth rate of $g_{K,i}$ is increasing in $g_{K,i}$. Thus the growth rate of the capital stock in the high-saving economy will always exceed the growth rate of the capital stock in the low-saving economy. That is, we have $g_{K,1}(t) > g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two growth rates will be increasing over time.

More formally, using the production function, we can write the ratio of output in the high-saving country, country 1, to output in the low-saving country, country 2, as

(5) $Y_1(t)/Y_2(t) = \left[K_1(t)/K_2(t)\right]^{\theta}$.

Taking the time derivative of the log of equation (5) yields an expression for the growth rate of the ratio of output in the high-saving economy to output in the low-saving economy:

$$(6)\quad \frac{\left[Y_1(t)/\dot{\ } Y_2(t)\right]}{\left[Y_1(t)/Y_2(t)\right]} = \theta\left[\frac{\dot{K}_1(t)}{K_1(t)} - \frac{\dot{K}_2(t)}{K_2(t)}\right] = \theta\left[g_{K,1}(t) - g_{K,2}(t)\right] > 0.$$

As explained above, $g_{K,1}(t)$ will exceed $g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two will be increasing over time. Thus the growth rate of the output ratio will be positive and increasing over time. That is, the ratio of output in the high-saving economy to output in the low-saving economy will be continually rising, and rising at an increasing rate.

Problem 3.3

The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines are given by

(1) $\dot{g}_K = 0 \quad \Rightarrow \quad g_K = g_A + n$,

and

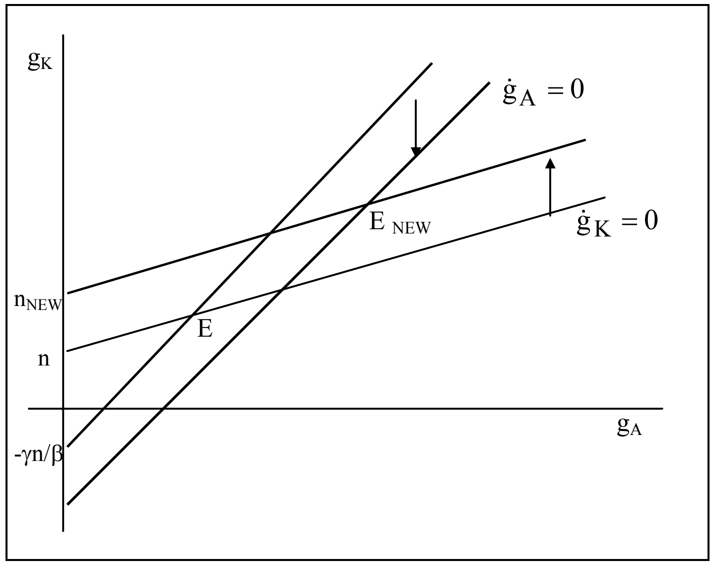
(2) $\dot{g}_A = 0 \quad \Rightarrow \quad g_K = \dfrac{(1-\theta)g_A - \gamma n}{\beta}$.

The expressions for the growth rates of capital and knowledge are

(3) $g_K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha} \qquad c_K \equiv s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$

(4) $g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1} \qquad c_A \equiv Ba_K{}^{\beta} a_L{}^{\gamma}$.

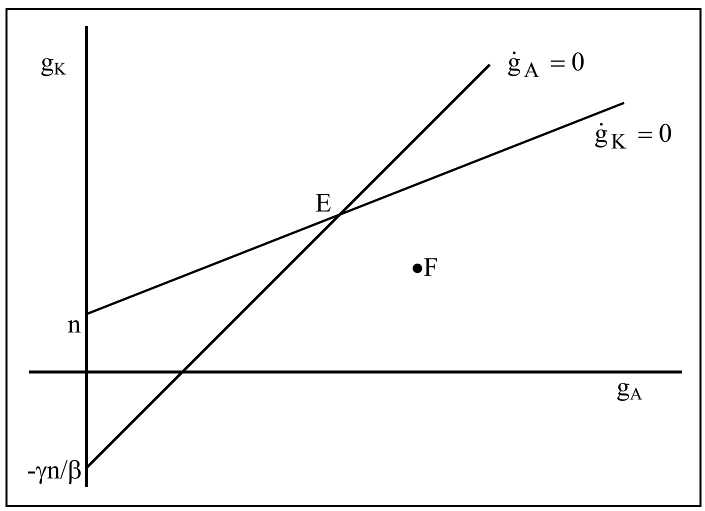
(a) From equation (1), for a given g_A, the value of g_K that satisfies $\dot{g}_K = 0$ is now higher as a result of the rise in population growth from n to n_{NEW}. Thus the $\dot{g}_K = 0$ locus shifts up. From equation (2), for a given g_A, the value of g_K that satisfies $\dot{g}_A = 0$ is now lower. Thus the $\dot{g}_A = 0$ locus shifts down.



Since n does not appear in equation (3), there is no jump in the value of g_K at the moment of the increase in population growth.

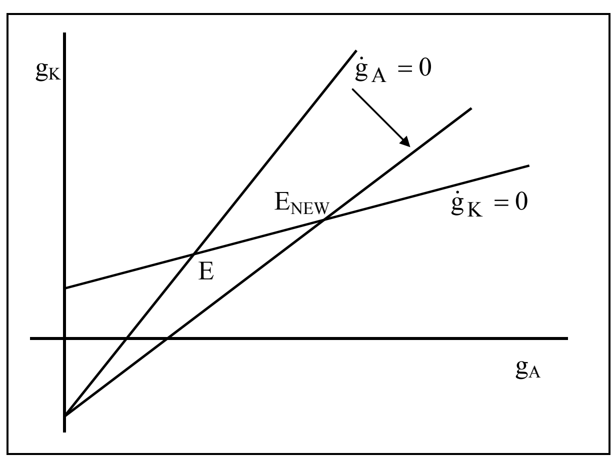
Similarly, since n does not appear in equation (4), there is no jump in the value of g_A at the moment of the rise in population growth.

(b) Note that a_K does not appear in equation (1), the $\dot{g}_K = 0$ line, or in equation (2), the $\dot{g}_A = 0$ line. Thus neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ line shifts as a result of the increase in the fraction of the capital stock used in the knowledge sector from a_K to a_K^{NEW}.



From equation (3), the rise in a_K causes the growth rate of capital, g_K, to jump down. From equation (4), the growth rate of knowledge, g_A, jumps up at the instant of the rise in a_K. Thus the economy moves to a point such as F in the figure.

(c) Since θ does not appear in equation (1), there is no shift of the $\dot{g}_K = 0$ locus as a result of the rise in θ, the coefficient on knowledge in the knowledge production function. From equation (2), the $\dot{g}_A = 0$ locus has slope $(1 - \theta)/\beta$ and therefore becomes flatter after the rise in θ. See the figure.



Since θ does not appear in equation (3), the growth rate of capital, g_K, does not jump at the time of the rise in θ. θ does appear in equation (4) and thus we need to determine the effect that the rise in θ has on the growth rate of knowledge. It turns out that g_A may jump up, jump down or stay the same at the instant of the change in θ. Taking the log of both sides of equation (4) gives us

(5) $\ln g_A(t) = \ln c_A + \beta \ln K(t) + \gamma \ln L(t) + (\theta - 1)\ln A(t)$.

Taking the derivative of both sides of equation (5) with respect to θ yields

(6) $\partial \ln g_A(t)/\partial\theta = \ln A(t)$.

So if A(t) is less than one, so that $\ln A(t) < 0$, the growth rate of knowledge jumps down at the instant of the rise in θ. However, if A(t) is greater than one, so that $\ln A(t) > 0$, the growth rate of knowledge jumps up at the instant of the rise in θ. Finally, if A(t) is equal to one at the time of the change in θ, there is no initial jump in g_A . This means the dynamics of the adjustment to E_{NEW} may differ depending on the value of g_A at the time of the change in θ, but the end result is the same.

Problem 3.4

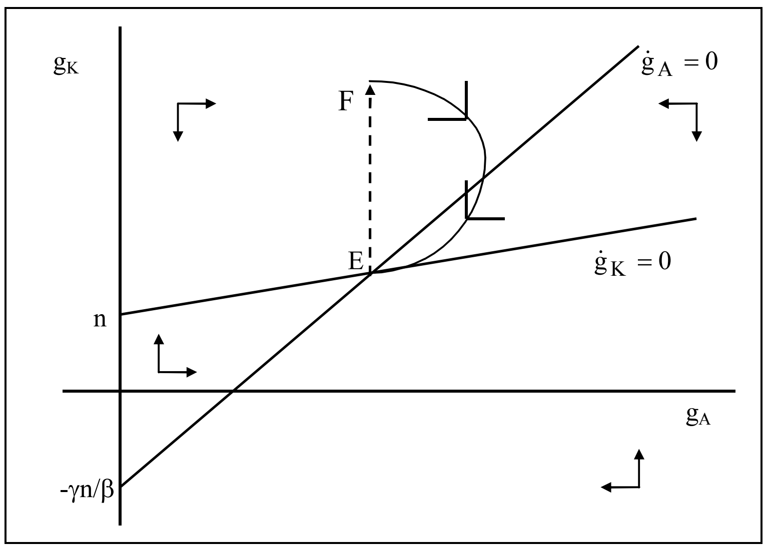
The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci are

(1) $\dot{g}_K = 0 \quad\Rightarrow\quad g_K = g_A + n$, and (2) $\dot{g}_A = 0 \quad\Rightarrow\quad g_K = \dfrac{(1-\theta)g_A - \gamma n}{\beta}$.

The equations defining the growth rates of capital and knowledge at any point in time are

(3) $g_K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha}$ $\qquad c_K \equiv s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$

(4) $g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$ $\qquad c_A \equiv B a_K{}^{\beta} a_L{}^{\gamma}$.



(a) Since the saving rate, s, does not appear in equations (1) or (2), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ locus shifts when s increases. From equation (4), the growth rate of knowledge, g_A, does not change at the moment that s increases. However, from equation (3), a rise in s causes an upward jump in the growth rate of capital, g_K. In the figure, the economy jumps from its balanced growth path at E to a point such as F at the moment that s increases.

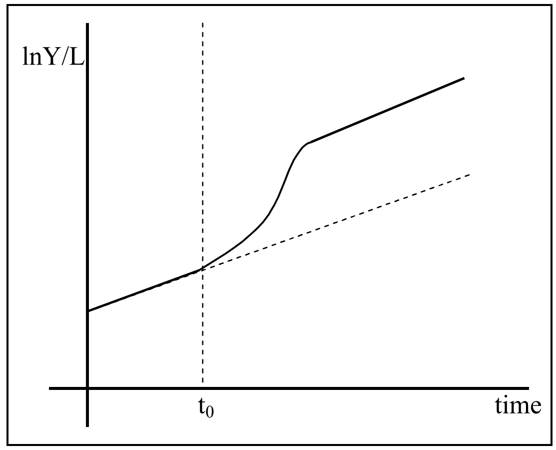
(b) At point F, the economy is above the $\dot{g}_A = 0$ locus and thus g_A is rising. Due to the increase in s, the growth rate of capital is higher than it would have been – the amount of capital going into the production of knowledge is higher than it would have been – and so the growth rate of knowledge begins to rise above what it would have been. Also at point F, the economy is above the $\dot{g}_K = 0$ locus and so g_K is falling. The economy drifts to the southeast and eventually crosses the $\dot{g}_A = 0$ locus at which point g_A begins to fall as well. Since there are decreasing returns to capital and knowledge in the production of new knowledge – θ + β < 1 – the increase in s does not have a permanent effect on the growth rates of K and A. The economy eventually returns to point E.

The production function is given by

(5) $Y(t) = \left[(1-a_K)K(t)\right]^{\alpha}\left[A(t)(1-a_L)L(t)\right]^{1-\alpha}$.

Taking the time derivative of the log of equation (5) will yield the growth rate of total output:

(6) $\dfrac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1-\alpha)\left[g_A(t) + n\right].$



On the initial balanced growth path, from equation (1), $g_K^* = g_A^* + n$. From equation (6), this means that total output is also growing at rate $g_K^* = g_A^* + n$ on the initial balanced growth path. Thus output per person, Y(t)/L(t), is initially growing at rate g_A^*. During the transition period, both g_K and g_A are higher than on the balanced growth path and so output per worker must be growing at a rate greater than its balanced-growth-path value of g_A^*. Whether the growth rate of output per worker is rising or falling will depend, among other things, on the value of α since there is a period of time when g_K is falling and g_A is rising. The figure shows the growth rate of output per worker initially rising and then falling, but the important point is that during the entire transition, the growth rate itself is higher than its balanced-growth-path value of g_A^*. In the end, once the economy returns to point E, output per worker is again growing at rate g_A^*, which has not changed.

(c) Note that the effects of an increase in s in this model are qualitatively similar to the effects in the Solow model. Since there are net decreasing returns to the produced factors of production here – $\theta + \beta < 1$ – the increase in s has only a level effect on output per worker. The path of output per worker lies above the path it would have taken but there is no permanent effect on the growth rate of output per worker, which on the balanced growth path is equal to the growth rate of knowledge. This is the same effect that a rise in s has in the Solow model in which there are diminishing returns to the produced factor, capital. Quantitatively, the effect is larger than in the Solow model (for a given set of parameters). This is due to the fact that, here, A rises above the path it would have taken whereas that is not true in the Solow model.

Problem 3.5

(a) From equations (3.14) and (3.16) in the text, the growth rates of capital and knowledge are given by

(1) $g_K(t) \equiv \dot{K}(t)/K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha}$, where $c_K \equiv s[1 - a_K]^{\alpha}[1 - a_L]^{1-\alpha}$, and

(2) $g_A(t) \equiv \dot{A}(t)/A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$, where $c_A \equiv B a_K^{\beta} a_L^{\gamma}$.

With the assumptions of $\beta + \theta = 1$ and $n = 0$, these equations simplify to

(3) $g_K(t) = [c_K L^{1-\alpha}][A(t)/K(t)]^{1-\alpha}$, and (4) $g_A(t) = [c_A L^{\gamma}][K(t)/A(t)]^{\beta}$.

Thus given the parameters of the model and the population (which is constant), the ratio A/K determines both growth rates. The two growth rates, g_K and g_A, will be equal when

(5) $[c_K L^{1-\alpha}][A(t)/K(t)]^{1-\alpha} = [c_A L^{\gamma}][K(t)/A(t)]^{\beta}$,

or when

(6) $[A(t)/K(t)]^{1-\alpha+\beta} = [c_A/c_K]L^{\gamma-(1-\alpha)}$.

Thus the value of A/K that yields equal growth rates of capital and knowledge is given by

(7) $A(t)/K(t) = \left[(c_A/c_K)L^{\gamma-(1-\alpha)}\right]^{1/(1-\alpha+\beta)}$.

(b) In order to find the growth rate of A and K when $g_K = g_A \equiv g^*$, substitute equation (7) into (3):

(8) $g^* = \left[c_K L^{1-\alpha}\right]\left[(c_A/c_K)\, L^{\gamma-(1-\alpha)}\right]^{(1-\alpha)/(1-\alpha+\beta)}$.

Simplifying the exponents yields

(9) $g^* = \left[c_K{}^{(1-\alpha+\beta)-(1-\alpha)} c_A{}^{1-\alpha} L^{(1-\alpha)+\gamma(1-\alpha)-(1-\alpha)^2}\right]^{1/(1-\alpha+\beta)}$,

or simply

(10) $g^* = \left[c_K{}^{\beta} c_A{}^{1-\alpha} L^{(1-\alpha)(\gamma+\alpha)}\right]^{1/(1-\alpha+\beta)}$.

(c) In order to see the way in which an increase in s affects the long-run growth rate of the economy, substitute the definitions of c_K and c_A into equation (10):

(11) $g^* = \left[s^{\beta}(1-a_K)^{\alpha\beta}(1-a_L)^{(1-\alpha)\beta} B^{1-\alpha} a_K{}^{\beta(1-\alpha)} a_L{}^{\gamma(1-\alpha)} L^{(1-\alpha)\cdot(\gamma+\alpha)}\right]^{1/(1-\alpha+\beta)}$.

Taking the natural logarithm of both sides of equation (11) gives us

(12) $\ln g^* = \left[1/(1-\alpha+\beta)\right]\{\beta \ln s + (1-\alpha)(\gamma+\alpha)\ln L + (1-\alpha)\ln B + \beta[\alpha \ln(1-a_K) + (1-\alpha)\ln a_K] +$
$(1-\alpha)[\beta \ln(1-a_L) + \gamma \ln a_L]\}$.

Using equation (12), the elasticity of the long-run growth rate of the economy with respect to the saving rate is

(13) $\partial \ln g^*/\partial \ln s = \beta/(1 - \alpha + \beta) > 0$.

Thus an increase in the saving rate increases the long-run growth rate of the economy This is essentially because it increases the resources devoted to physical capital accumulation and in this model, we have constant returns to the produced factors of production.

(d) We can maximize lng* with respect to a_K to determine the fraction of the capital stock that should be employed in the R&D sector in order to maximize the long-run growth rate of the economy.

The first-order condition is

(14) $\dfrac{\partial \ln g^*}{\partial a_K} = \dfrac{\beta}{(1-\alpha+\beta)}\left[\dfrac{-\alpha}{(1-a_K)} + \dfrac{(1-\alpha)}{a_K}\right] = 0.$

Solving for the optimal a_K* yields

(15) $\alpha/(1 - a_K) = (1 - \alpha)/a_K$.

Simplifying gives us

(16) $\alpha a_K = 1 - a_K + \alpha a_K - \alpha$,

and thus

(17) $a_K^* = (1 - \alpha)$.

Thus the optimal fraction of the capital stock to employ in the R&D sector is equal to effective labor's share in the production of output. Note that β, capital's share in the production function for new knowledge, does not affect the optimal allocation of capital to the R&D sector. The reason for this is that an increase in β has two effects. It makes capital more important in the R&D sector, thereby tending to raise the a_K that maximizes g*. A rise in β also makes the production of new capital more valuable, and new capital is produced when there is more output to be saved and invested. This tends to lower the a_K that maximizes g* since it implies that more resources should be devoted to the production of output rather than knowledge. In the case we are considering, these two effects exactly cancel each other out.

Problem 3.6

(a) From equation (3.15) in the text, the growth rate of the growth rate of capital is given by

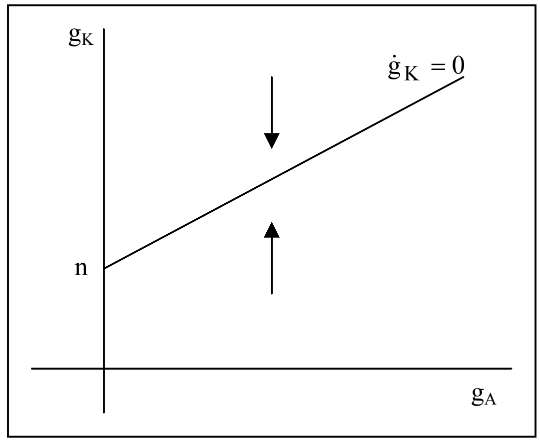
(1) $\dfrac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) + n - g_K(t)]$.

Thus the equation of the $\dot{g}_K = 0$ locus is

(2) $g_K = g_A + n$.

The slope of the $\dot{g}_K = 0$ locus in (g_A, g_K) space is therefore equal to 1 and the vertical intercept is equal to n.

In addition, g_K is rising when $g_A + n - g_K > 0$ or when $g_K < g_A + n$ (below the $\dot{g}_K = 0$ line) and g_K is falling when $g_A + n - g_K < 0$ or when $g_K > g_A + n$ (above the $\dot{g}_K = 0$ line).



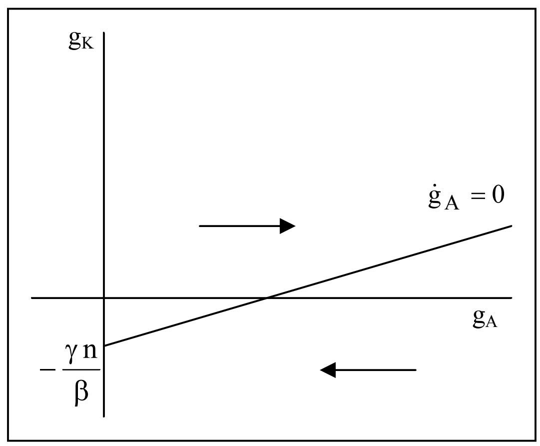
This information is summarized in the figure at right.

From equation (3.17) in the text, the growth rate of the growth rate of knowledge is given by

(3) $\dfrac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1)g_A(t)$.

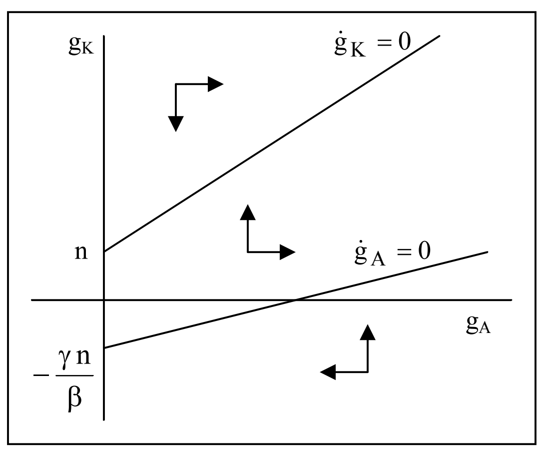
Thus the equation of the $\dot{g}_A = 0$ locus is

(4) $g_K = \dfrac{(1-\theta)}{\beta} g_A - \dfrac{\gamma n}{\beta}$.



The slope of the $\dot{g}_A = 0$ locus is therefore $(1 - \theta)/\beta$ and the vertical intercept is $-\gamma n/\beta$. In addition, g_A is rising when $\beta g_K + \gamma n + (\theta - 1)g_A$ is positive or when $g_K > [(1 - \theta)g_A - \gamma n]/\beta$ (above the $\dot{g}_A = 0$ line). Similarly, g_A is falling when $\beta g_K + \gamma n + (\theta - 1)g_A$ is negative or when $g_K < [(1 - \theta)g_A - \gamma n]/\beta$ (below the $\dot{g}_A = 0$ line).

Putting the two loci together gives us the phase diagram depicted in the figure at right. With $\theta + \beta$ greater than 1, or $\beta > (1 - \theta)$, the slope of the $\dot{g}_A = 0$ locus is less than the slope of the $\dot{g}_K = 0$ locus, which is 1. Thus the two lines diverge as shown in the figure.



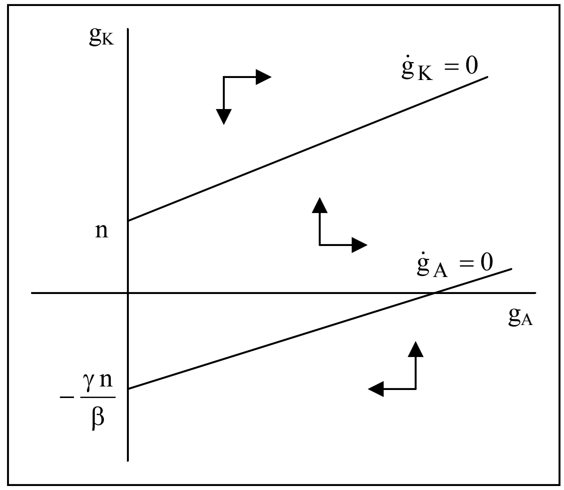
(b) The initial values of g_A and g_K are determined by the parameters of the model and by the initial values of A, K, and L. As the phase diagram from part (a) shows, regardless of where the economy starts, it eventually enters the region between the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines. Once this occurs, we can see that the growth rates of both A and K increase continually. Since output is given by

(5) $Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)]^{1-\alpha}$,
the growth rate of output is

(6) $\dfrac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1-\alpha)\left[g_A(t) + n\right]$.

Thus when g_A and g_K increase continually, so does output.

(c) In this case, $(1 - \theta)/\beta$ equals 1, and thus the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci have the same slope. Since $n > 0$, the $\dot{g}_K = 0$ line lies above the $\dot{g}_A = 0$ line. The dynamics of the economy are similar to the case where $\beta + \theta > 1$; see the phase diagram at right.



Regardless of where the economy starts, it eventually enters the region between the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines. Once this occurs, the growth rates of capital, knowledge, and output increase continually.

Problem 3.7
The relevant equations are

(1) $Y(t) = K(t)^{\alpha}A(t)^{1-\alpha}$, (2) $\dot{K}(t) = sY(t)$, and (3) $\dot{A}(t) = BY(t)$.

(a) Substituting equation (1) into equation (2) yields $\dot{K}(t) = sK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by K(t) allows us to obtain the following expression for the growth rate of capital, $g_K(t)$:

(4) $g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1}A(t)^{1-\alpha}$.

Substituting equation (1) into (3) gives us $\dot{A}(t) = BK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by A(t) allows us to obtain the following expression for the growth rate of knowledge, $g_A(t)$:

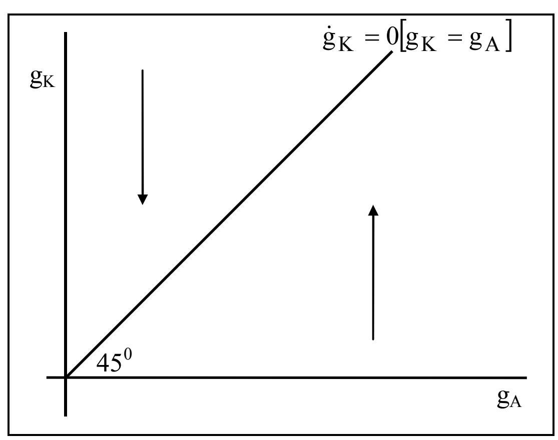
(5) $g_A(t) \equiv \dot{A}(t)/A(t) = BK(t)^{\alpha}A(t)^{-\alpha}$.

(b) Taking the time derivative of (4) yields the following growth rate of the growth rate of capital:

(6) $\dfrac{\dot{g}_K(t)}{g_K(t)} = (\alpha - 1)\dfrac{\dot{K}(t)}{K(t)} + (1-\alpha)\dfrac{\dot{A}(t)}{A(t)}$,

or

(7) $\dot{g}_K(t)/g_K(t) = (1-\alpha)\left[g_A(t) - g_K(t)\right]$.



From equation (7), g_K will be constant when $g_A = g_K$. Thus the $\dot{g}_K = 0$ locus is a 45^0 line in (g_A, g_K) space. Also, g_K will be rising when $g_A > g_K$. Thus g_K is rising below the $\dot{g}_K = 0$ line. Lastly, g_K will fall when $g_A < g_K$. Thus g_K is falling above the $\dot{g}_K = 0$ line.

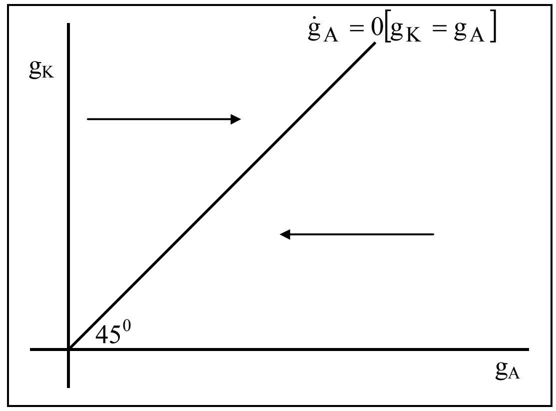
Taking the time derivative of the log of equation (5) yields the following growth rate of the growth rate of knowledge:

(8) $\dfrac{\dot{g}_A(t)}{g_A(t)} = \alpha\dfrac{\dot{K}(t)}{K(t)} - \alpha\dfrac{\dot{A}(t)}{A(t)}$,

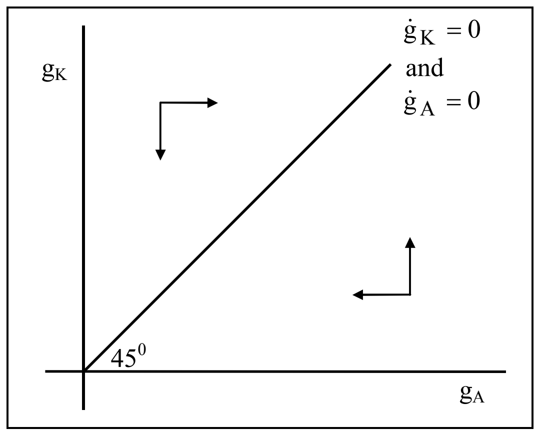
or

(9) $\dot{g}_A(t)/g_A(t) = \alpha\left[g_K(t) - g_A(t)\right]$.

From equation (9), g_A will be constant when $g_K = g_A$. Thus the $\dot{g}_A = 0$ locus is also a 45^0 line in (g_A, g_K) space. Also, g_A will be rising when $g_K > g_A$. Thus above the $\dot{g}_A = 0$ line, g_A will be rising. Finally, g_A will be falling when $g_K < g_A$. Thus below the $\dot{g}_A = 0$ line, g_A will be falling.



(c) We can put the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci into one diagram.



Although we can see that the economy will eventually arrive at a situation where $g_K = g_A$ and they are constant, we still do not have enough information to determine the unique balanced growth path. Rewriting equations (4) and (5) gives us

(4) $g_K(t) = sK(t)^{\alpha-1}A(t)^{1-\alpha} = s\left[A(t)/K(t)\right]^{1-\alpha}$,

and

(5) $g_A(t) = BK(t)^{\alpha}A(t)^{-\alpha} = B\left[A(t)/K(t)\right]^{-\alpha}$.

At any point in time, the growth rates of capital and knowledge are linked because they both depend on the ratio of knowledge to capital at that point in time. It is therefore possible to write one growth rate as a function of the other.

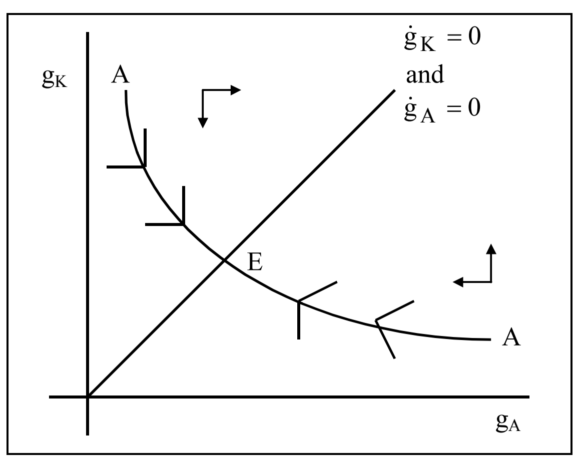
From equation (5), $\left[A(t)/K(t)\right]^{\alpha} = B/g_A(t)$ or simply

(10) $A(t)/K(t) = \left[B/g_A(t)\right]^{1/\alpha}$.

Substituting equation (10) into equation (4) gives us

(11) $g_K(t) = s\left[B/g_A(t)\right]^{(1-\alpha)/\alpha}$.

It must be the case that g_K and g_A lie on the locus satisfying equation (11), which is labeled AA in the figure. Regardless of the initial ratio of A/K the economy starts somewhere on this locus and then moves along it to point E. Thus the economy does converge to a unique balanced growth path at E.

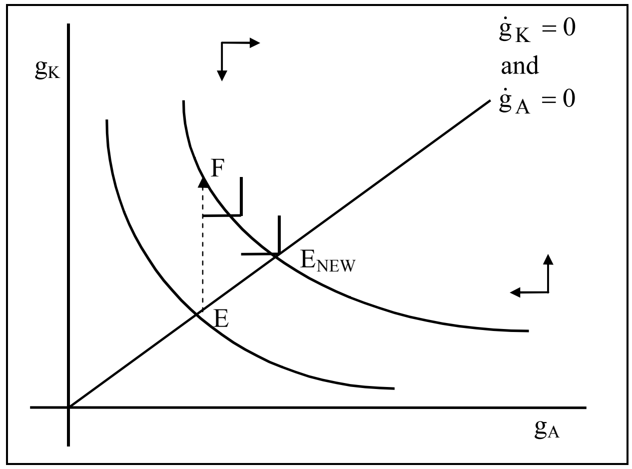


To calculate the growth rates of capital and knowledge on the balanced growth path, note that at point E we are on the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci

where $g_K = g_A$. Letting g* denote this common growth rate, then from equation (11),
$g^* = s\left[B/g^*\right]^{(1-\alpha)/\alpha}$.
Rearranging to solve for g* yields
(12) $g^* = s^{\alpha}B^{1-\alpha}$.

Taking the time derivative of the log of the production function, equation (1), yields the growth rate of real output, $\dot{Y}(t)/Y(t) = \alpha g_K(t) + (1-\alpha)g_A(t)$. On the balanced growth path, $g_K = g_A \equiv g^*$, and thus
(13) $\dot{Y}(t)/Y(t) = \alpha g^* + (1-\alpha)g^* = g^* \equiv s^{\alpha}B^{1-\alpha}$.
On the balanced growth path, capital, knowledge and output all grow at rate g*.

(d) Clearly, from equation (12), a rise in the saving rate, s, raises g* and thus raises the long-run growth rates of capital, knowledge and output.



From equations (7) and (9), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ lines shift when s changes since s does not appear in either equation. From equation (4), a rise in s causes g_K to jump up. Also, the locus given by equation (11) shifts out. So at the moment that s rises, the economy moves from its balanced growth path at point E to a point such as F. It then moves down along the AA locus given by equation (11) until it reaches a new balanced growth path at point E_{NEW}.

<u>Problem 3.8</u>

Using the constant-relative-risk aversion equation (2.2) and (2.20) from the text, we have

$$U(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \quad \text{and} \quad \frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}.$$

Solving for r(t) gives us

(1) $r(t) = \rho + \theta\dfrac{\dot{C}(t)}{C(t)}$.

All output is consumed and all individuals are the same and thus choose the same consumption path so equilibrium in the goods market requires that $C(t)\bar{L} = Y(t)$ and so consumption grows at the same rate as output. Output grows at $[(1-\phi)/\phi]BL_A$ and so equation (1) becomes

(2) $r(t) = \rho + \theta\left(\dfrac{1-\phi}{\phi}\right)BL_A$,

which is analogous to equation (3.40).

From the textbook, we know that the present value of profits is now given by

$$(3)\quad \pi(t) = \frac{\frac{1-\phi}{\phi}(\bar{L}-L_A)\frac{w(t)}{A(t)}}{\rho+\theta\left(\frac{1-\phi}{\phi}\right)BL_A-\left(\frac{1-2\phi}{\phi}\right)BL_A} = \frac{1-\phi}{\phi}\frac{w(t)}{A(t)}\left(\frac{\bar{L}-L_A}{\rho+BL_A\left(\frac{\theta(1-\phi)}{\phi}-\frac{1-2\phi}{\phi}\right)}\right),$$

and that the cost of an invention is w(t)/BA(t). At equilibrium the present value of profits must equal the costs of the invention, or

$$(4)\quad \frac{w(t)}{BA(t)} = \frac{1-\phi}{\phi}\frac{w(t)}{A(t)}\left(\frac{\bar{L}-L_A}{\rho+BL_A\left(\frac{\theta(1-\phi)}{\phi}-\frac{1-2\phi}{\phi}\right)}\right).$$

Solving equation (4) we can find the optimal level of labor in the R&D sector, L_A:

$$(5)\quad \bar{L}-L_A = \frac{\phi}{1-\phi}\frac{1}{B}\left(\rho+BL_A\left(\frac{\theta(1-\phi)}{\phi}-\frac{1-2\phi}{\phi}\right)\right),$$

which simplifies to

$$(6)\quad \bar{L}-\frac{\phi\rho}{(1-\phi)B} = L_A\left(\theta-\left(\frac{1-2\phi}{1-\phi}\right)+1\right).$$

Solving for L_A gives us

$$(7)\quad L_A = \left(\bar{L}-\frac{\phi\rho}{(1-\phi)B}\right)\left(\frac{1-\phi}{\phi+\theta-\phi\theta}\right),$$

or simply

$$(8)\quad L_A = \frac{1}{\phi+\theta-\phi\theta}\left((1-\phi)\bar{L}-\frac{\phi\rho}{B}\right).$$

The resulting optimal level of L_A is similar to the result in equation (3.43). However, the level of labor cannot be less than 0, so we rewrite equation (8) as

$$(9)\quad L_A = \max\left\{\frac{1}{\phi+\theta-\phi\theta}\left((1-\phi)\bar{L}-\frac{\phi\rho}{B}\right),0\right\}.$$

Problem 3.9

(a) The patent price is now $\delta w(t)/\phi$, where δ satisfies $\phi \le \delta \le 1$. Each patent-holder's profits at time t are now given by

$$(1)\quad \pi(t) = \frac{\bar{L}-L_A}{A(t)}\left[\frac{\delta w(t)}{\phi}-w(t)\right],$$

or simply

$$(2)\quad \pi(t) = \frac{\delta-\phi}{\phi}\frac{\bar{L}-L_A}{A(t)}w(t).$$

Comparing equation (2) with equation (3.39) in the text, we can see that the growth rate of profits from a given invention will be unchanged from the standard model and equal to $[(1-2\varphi)/\varphi]BL_A$. In addition, the real interest rate remains equal to $\rho + [(1-\varphi)/\varphi]BL_A$. The present value of the profits earned from the discovery of a new idea at time t is therefore

(3) $$\pi(t) = \frac{\dfrac{\delta-\phi}{\phi}\dfrac{\bar{L}-L_A}{A(t)}w(t)}{\rho + \dfrac{1-\phi}{\phi}BL_A - \dfrac{1-2\phi}{\phi}BL_A},$$

which simplifies to

(4) $$\pi(t) = \frac{\delta-\phi}{\phi}\frac{\bar{L}-L_A}{\rho+BL_A}\frac{w(t)}{A(t)}.$$

If the amount of R&D is strictly positive, equilibrium requires the present value of profits from an invention to equal the costs of an invention, w(t)/[BA(t)]. Thus, equilibrium requires

(5) $$\frac{\delta-\phi}{\phi}\frac{\bar{L}-L_A}{\rho+BL_A}\frac{w(t)}{A(t)} = \frac{w(t)}{BA(t)}.$$

We must now solve equation (5) for the equilibrium value of L_A. We can rewrite (5) as

(6) $$\bar{L}-L_A = \frac{1}{B}\frac{\phi}{\delta-\phi}(\rho+BL_A).$$

Isolating the terms in L_A yields

(7) $$L_A + \frac{\phi}{\delta-\phi}L_A = \bar{L} - \frac{\phi}{\delta-\phi}\frac{\rho}{B},$$

or simply

(8) $$\frac{\delta}{\delta-\phi}L_A = \bar{L} - \frac{\phi}{\delta-\phi}\frac{\rho}{B}.$$

Thus, L_A is given by

(9) $$L_A = \frac{\delta-\phi}{\delta}\bar{L} - \frac{\phi}{\delta}\frac{\rho}{B}.$$

To allow for the possibility of a corner solution, we need to modify (9) to

(10) $$L_A = \max\left\{\frac{\delta-\phi}{\delta}\bar{L} - \frac{\phi}{\delta}\frac{\rho}{B}, 0\right\}.$$

Since the growth rate of output is $[(1-\varphi)/\varphi]BL_A$, we have

(11) $$\frac{\dot{Y}(t)}{Y(t)} = \max\left\{\frac{\delta-\phi}{\delta}\frac{1-\phi}{\phi}B\bar{L} - \frac{1-\phi}{\delta}\rho, 0\right\}.$$

In order to see the effects of a change in δ on the equilibrium growth rate, take the derivative of $\dot{Y}(t)/Y(t)$ with respect to δ in the case where output growth is strictly positive:

(12) $$\frac{\partial \dot{Y}(t)/Y(t)}{\partial\delta} = \frac{1-\phi}{\phi}B\bar{L}\left[\frac{\delta-(\delta-\phi)}{\delta^2}\right] + \frac{1-\phi}{\delta^2}\rho,$$

or simply

(13) $$\frac{\partial \dot{Y}(t)/Y(t)}{\partial\delta} = \frac{\phi}{\delta^2}\frac{1-\phi}{\phi}B\bar{L} + \frac{1-\phi}{\delta^2}\rho > 0.$$

Thus, in the situation where output growth is positive, a decrease in the value of δ reduces the equilibrium growth rate of output.

(b) It is true that setting $\delta = \varphi$ would eliminate the monopoly distortion: if patent-holders were forced to charge marginal cost, the present value of the monopolist's profits would be zero. But the profits from a new invention represent the incentive to innovate in this model. From the production function for new ideas, it takes 1/[BA(t)] units of labor to produce a new idea and so it costs w(t)/[BA(t)] to produce a new idea. Marginal-cost pricing would imply that innovators would not be able to recoup this cost of producing an idea and there would be no incentive to innovate in this model. This would result in no R&D, which is not socially optimal.

Problem 3.10

(a) Equation (3.47) from the textbook is

$$(3.47)\quad U = \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} e^{\frac{1-\phi}{\phi} BL_A} e^{t} \right) dt,$$

which can be rewritten as

$$(1)\quad U = \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt + \int_{t=0}^{\infty} te^{-\rho t} \frac{1-\phi}{\phi} BL_A dt.$$

Let's look at the first integral. We can write

$$(2)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \lim_{z\to\infty} \int_{t=0}^{z} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt,$$

which simplifies to

$$(3)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \lim_{z\to\infty} -\frac{1}{\rho} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) \Bigg|_{t=0}^{t=z}.$$

Thus, the first integral is given by

$$(4)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \frac{1}{\rho} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right),$$

or simply

$$(5)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \frac{1}{\rho} \left(\ln \frac{\bar{L} - L_A}{\bar{L}} + \frac{1-\phi}{\phi} \ln A(0) \right).$$

Now let's look at the second integral. We can write

$$(6)\quad \int_{t=0}^{\infty} te^{-\rho t} \frac{1-\phi}{\phi} BL_A dt = \lim_{z\to\infty} \int_{t=0}^{z} te^{-\rho t} \frac{1-\phi}{\phi} BL_A dt = \frac{1-\phi}{\phi} BL_A \lim_{z\to\infty} \int_{t=0}^{z} te^{-\rho t} dt.$$

Integration by parts states that $\int u dv = uv - \int v du$. Here, let $u = t$ and $dv = e^{-\rho t}$. Then, $\frac{du}{dt} = 1$ and $v = -\frac{1}{\rho} e^{-\rho t}$ so we can rewrite equation (6) as

$$(7)\quad \int_{t=0}^{\infty} te^{-\rho t} \frac{1-\phi}{\phi} BL_A dt = \frac{1-\phi}{\phi} BL_A \lim_{z\to\infty} \left[-\frac{t}{\rho} e^{-\rho t} + \int_{t=0}^{z} \frac{1}{\rho} e^{-\rho t} dt \right].$$

Solving the integral gives us

(8) $\int_{t=0}^{\infty} te^{-\rho t}\frac{1-\phi}{\phi}BL_A dt = \frac{1-\phi}{\phi}BL_A \lim_{z\to\infty}\left[-\frac{t}{\rho}e^{-\rho t} - \frac{1}{\rho^2}e^{-\rho t}\Big|_{t=0}^{t=z}\right]$,

or simply

(9) $\int_{t=0}^{\infty} te^{-\rho t}\frac{1-\phi}{\phi}BL_A dt = \frac{1-\phi}{\phi}BL_A \lim_{z\to\infty}\left[-\frac{z}{\rho}e^{-\rho z} - \frac{1}{\rho^2}e^{-\rho z} + \frac{1}{\rho^2}\right]$.

We cannot solve the first term on the right-hand-side of equation (9) because of its indeterminant form. Using L'Hôpital's Rule, we get:

(10) $\lim_{z\to\infty} -\frac{z}{\rho e^{\rho z}} = \lim_{z\to\infty} -\frac{1}{\rho^2 e^{\rho z}} = 0$.

Therefore, the limit of the expression on the right-hand-side of (9) simplifies to $1/\rho^2$ and so we have

(11) $\int_{t=0}^{\infty} te^{-\rho t}\frac{1-\phi}{\phi}BL_A dt = \frac{1-\phi}{\phi}\frac{1}{\rho^2}BL_A$.

Therefore, using equations (5) and (11), equation (1) becomes

(12) $U = \frac{1}{\rho}\left(\ln\frac{\bar{L}-L_A}{\bar{L}} + \frac{1-\phi}{\phi}\ln A(0)\right) + \frac{1-\phi}{\phi}\frac{1}{\rho^2}BL_A$,

which is equivalent to equation (3.48) in the text.

(b) To arrive at the socially optimal level of L_A given by equation (3.49), we must maximize the expression in equation (3.48) for lifetime utility with respect to L_A. We can rewrite equation (3.48) as

(1) $$\begin{aligned} U &= \frac{1}{\rho}\left(\ln\frac{\bar{L}-L_A}{\bar{L}} + \frac{1-\phi}{\phi}\ln A(0) + \frac{1-\phi}{\phi}\frac{BL_A}{\rho}\right) \\ &= \frac{1}{\rho}\ln(\bar{L}-L_A) - \frac{1}{\rho}\ln\bar{L} + \frac{1-\phi}{\phi}\frac{1}{\rho}\ln A(0) + \frac{1}{\rho^2}\frac{1-\phi}{\phi}BL_A \end{aligned}.$$

The first order-condition is given by

(2) $\frac{\partial U}{\partial L_A} = \frac{\partial}{\partial L_A}\left\{\frac{1}{\rho}\ln(\bar{L}-L_A) - \frac{1}{\rho}\ln\bar{L} + \frac{1-\phi}{\phi}\frac{1}{\rho}\ln A(0) + \frac{1}{\rho^2}\frac{1-\phi}{\phi}BL_A\right\} = 0$.

Taking the partial derivative yields

(3) $-\frac{1}{\rho}\frac{1}{\bar{L}-L_A} + \frac{B}{\rho^2}\frac{1-\phi}{\phi} = 0$.

Solving for L_A gives us

(4) $\frac{1}{\rho}\frac{1}{\bar{L}-L_A} = \frac{B}{\rho^2}\frac{1-\phi}{\phi}$,

or simply

(5) $\bar{L}-L_A = \frac{\phi}{1-\phi}\frac{\rho}{B}$,

and thus finally the optimal choice of L_A is given by

(6) $L_A^{OPT} = \bar{L} - \frac{\phi}{1-\phi}\frac{\rho}{B}$.

However, the optimal level of labor in R&D cannot be negative, so we rewrite equation (6) as

(7) $L_A^{OPT} = \max\left\{\bar{L} - \frac{\phi}{1-\phi}\frac{\rho}{B}, 0\right\}$.

Now we need to check that L_A^{OPT} is indeed the maximum. By taking the second derivative, we get

(8) $\frac{\partial^2 U}{\partial L_A^2} = -\frac{1}{\rho}\frac{1}{(\bar{L} - L_A)^2} < 0$.

The second derivative is negative, so L_A^{OPT} is a maximum. Therefore, equation (7) provides the socially optimal level of L_A and is equivalent to equation (3.49) in the textbook.

Problem 3.11

(a) (i) Substituting the assumption that A(t) = BK(t) into the expression for firm i's output, $Y_i(t) = K_i(t)^\alpha (A(t)L_i(t))^{1-\alpha}$, we get $Y_i(t) = K_i(t)^\alpha (BK(t)L_i(t))^{1-\alpha}$.

To find the private marginal products of capital and labor, we take the first derivative of output with respect to the firm's choice of capital and labor assuming that the firm takes the aggregate capital stock, K, as given. The private marginal product of capital is therefore

(1) $\partial Y_i(t)/\partial K_i(t) = \alpha K_i(t)^{\alpha-1}(BK(t)L_i(t))^{1-\alpha}$

or simply

(2) $\partial Y_i(t)/\partial K_i(t) = \alpha B^{1-\alpha}K(t)^{1-\alpha}(K_i(t)/L_i(t))^{-(1-\alpha)}$.

The private marginal product of labor is given by

(3) $\partial Y_i(t)/\partial L_i(t) = (1-\alpha)L_i(t)^{-\alpha}K_i(t)^\alpha (BK(t))^{1-\alpha}$,

or simply

(4) $\partial Y_i(t)/\partial L_i(t) = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}(K_i(t)/L_i(t))^\alpha$.

(ii) Because factor markets are competitive, at equilibrium the private marginal product of capital and labor cannot differ across firms. We can see from equations (2) and (4) that this implies the capital-labor ratio will be the same for all firms. Therefore,

(5) $K_i(t)/L_i(t) = K(t)/L(t)$,

for all firms.

(iii) With no depreciation, the real interest rate must equal the private marginal product of capital. From equation (2), this implies

(6) $r(t) = \partial Y_i(t)/\partial K_i(t) = \alpha B^{1-\alpha}K(t)^{1-\alpha}(K_i(t)/L_i(t))^{-(1-\alpha)}$.

Using the fact that the capital-labor ratio is the same across firms, we can substitute equation (5) into equation (6) to obtain

(6) $r(t) = \alpha B^{1-\alpha}K(t)^{1-\alpha}(K(t)/L)^{-(1-\alpha)}$,

which simplifies to

(7) $r(t) = \alpha B^{1-\alpha}L^{1-\alpha} = \alpha b$,

where $b \equiv B^{1-\alpha}L^{1-\alpha}$.

With no population growth, L is constant, and thus so is the real interest rate.

The real wage must equal the marginal product of labor. From equation (4) this implies

(8) $w(t) = \partial Y_i(t)/\partial L_i(t) = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}(K_i(t)/L_i(t))^\alpha$.

Again, using the fact that the capital-labor ratio is the same across firms, we can substitute equation (5) into equation (8) to obtain

(9) $w(t) = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}(K(t)/L)^{\alpha}$,

which simplifies to

(10) $w(t) = (1-\alpha)B^{1-\alpha}K(t)L^{-\alpha} = (1-\alpha)B^{1-\alpha}L^{1-\alpha}(K(t)/L) = (1-\alpha)b(K(t)/L)$,

or simply

(11) $w(t) = (1-\alpha)b(K(t)/L)$.

(b) Using the hint, since utility of the representative household takes the constant-relative-risk-aversion-form, consumption growth in equilibrium will be

(12) $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{r(t)-\rho}{\theta}$.

Substituting equation (7) for the real interest rate into equation (12) yields

(13) $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{\alpha b-\rho}{\theta}$,

where $b \equiv B^{1-\alpha}L^{1-\alpha}$. Note that with no population growth so that L is constant, consumption growth is constant as well.

Using the zero-profit condition, we can write output as

(14) $Y(t) = r(t)K(t) + w(t)L$.

Substituting equations (7) and (11) into equation (14) gives us

(15) $Y(t) = \alpha bK(t) + (1-\alpha)b(K(t)/L)L$,

which simplifies to

(16) $Y(t) = bK(t)$.

Since b is a constant then output and capital grow at the same rate. Capital accumulation is then given by

(17) $\dot{K}(t) = sbK(t)$,

where s is the saving rate. Thus, the growth rate of the capital stock is given by

(18) $\dfrac{\dot{K}(t)}{K(t)} = sb$,

and so the growth rate of output also equals sb. Since C = (1 – s)Y, we can write the saving rate as

(19) $s = 1 - \dfrac{C}{Y}$.

Thus, we can write the growth rate of output as

(20) $\dfrac{\dot{Y}(t)}{Y(t)} = b\left(1 - \dfrac{C(t)}{Y(t)}\right)$.

If output growth were less than consumption growth, C/Y would rise over time. Output growth and capital growth would turn negative, which is not an allowable path. If output growth were greater than consumption growth, C/Y would fall to 0 over time. Output growth and capital growth would approach b. This implies that growth would eventually exceed the real interest rate, which is αb, and so this is also not an allowable path. Thus, the equilibrium growth rates of output and consumption must be equal.

(c) (i) We can take the derivative of the growth rate of output (which equals the growth rate of consumption) with respect to B to obtain

$$\frac{\partial\,[\dot{Y}(t)/Y(t)]}{\partial B} = \frac{\partial\left[\dfrac{\alpha B^{1-\alpha}L^{1-\alpha}-\rho}{\theta}\right]}{\partial B} = \frac{\alpha(1-\alpha)L^{1-\alpha}}{\theta B^{\alpha}} > 0.$$

Thus an increase in B increases long-run growth.

(ii) We can take the derivative of the growth rate of output with respect to ρ to obtain

$$\frac{\partial\,[\dot{Y}(t)/Y(t)]}{\partial \rho} = \frac{\partial\left[\dfrac{\alpha B^{1-\alpha}L^{1-\alpha}-\rho}{\theta}\right]}{\partial \rho} = -\frac{1}{\theta} < 0.$$

Thus an increase in ρ decreases long-run growth.

(iii) We can take the derivative of the growth rate of output with respect to L to obtain

$$\frac{\partial\,[\dot{Y}(t)/Y(t)]}{\partial L} = \frac{\partial\left[\dfrac{\alpha B^{1-\alpha}L^{1-\alpha}-\rho}{\theta}\right]}{\partial L} = \frac{\alpha(1-\alpha)B^{1-\alpha}}{\theta L^{\alpha}} > 0.$$

Thus an increase in L increases long-run growth.

(d) The equilibrium growth rate is less than the socially optimal growth rate. A social planner would internalize the knowledge spillovers and would set the growth rate of consumption dependent on the social return to capital, not the private return. We know that the private marginal product of capital is αb and the social marginal product is b (returns to capital are constant at the social level). Therefore, unless $\alpha = 1$, the growth rate set by the social planner would be greater than the decentralized equilibrium growth rate.

<u>Problem 3.12</u>

The production functions, after the normalization of T = 1, are given by

(1) $C(t) = K_C(t)^{\alpha}$, and (2) $\dot{K}(t) = BK_K(t)$.

(a) The return to employing an additional unit of capital in the capital-producing sector is given by $\partial\dot{K}(t)/\partial K_K(t) = B$. This has value $P_K(t)B$ in units of consumption goods. The return from employing an additional unit of capital in the consumption-producing sector is $\partial C(t)/\partial[K_C(t)] = \alpha[K_C(t)]^{\alpha-1}$. Equating these returns gives us

(3) $P_K(t)B = \alpha\left[K_C(t)\right]^{\alpha-1}$.

Taking the time derivative of the log of equation (3) yields the growth rate of the price of capital goods relative to consumption goods,

(4) $\dfrac{\dot{P}_K(t)}{P_K(t)} + \dfrac{\dot{B}}{B} = \dfrac{\dot{\alpha}}{\alpha} + (\alpha-1)\left[\dfrac{\dot{K}_C(t)}{K_C(t)}\right]$.

Using the fact that B and α are constants leaves us with

(5) $\dfrac{\dot{P}_K(t)}{P_K(t)} = (\alpha-1)\dfrac{\dot{K}_C(t)}{K_C(t)}$

Now since $K_C(t)$ is growing at rate $g_K(t)$ and denoting the growth rate of $P_K(t)$ as $g_P(t)$, we have
(6) $g_P(t) = (\alpha - 1)g_K(t)$.

(b) (i) The growth rate of consumption is given by
(7) $g_C(t) \equiv \dot{C}(t)/C(t) = \left[r(t) - \rho\right]/\sigma = \left[B + g_p(t) - \rho\right]/\sigma = \left[B + (\alpha - 1)g_K(t) - \rho\right]/\sigma,$
where we have used equation (6) to substitute for $g_P(t)$.

(b) (ii) Taking the time derivative of the log of the consumption production function, equation(1), yields
(8) $g_C(t) \equiv \dot{C}(t)/C(t) = \alpha\left[\dot{K}_C(t)/K_C(t)\right] = \alpha g_K(t).$
Equating the two expressions for the growth rate of consumption, equations (7) and (8), yields
(9) $\alpha g_K(t) = \left[B + (\alpha - 1)g_K(t) - \rho\right]/\sigma \quad \Rightarrow \quad \alpha\sigma g_K(t) + (1 - \alpha)g_K(t) = B - \rho.$
Thus in order for C to be growing at rate $g_C(t)$, $K_C(t)$ must be growing at the following rate:
(10) $g_K(t) = (B - \rho)/\left[\alpha\sigma + (1 - \alpha)\right].$

(b) (iii) We have already solved for $g_K(t)$ in terms of the underlying parameters. To solve for $g_C(t)$, substitute equation (10) into equation (8):
(11) $g_C(t) = \alpha(B - \rho)/\left[\alpha\sigma + (1 - \alpha)\right].$

(c) The real interest rate is now $(1 - \tau)(B + g_P)$. Thus equation (7) becomes

(12) $$g_C(t) = \frac{(1-\tau)\left[B + g_p(t)\right] - \rho}{\sigma} = \frac{(1-\tau)\left[B + (\alpha - 1)g_K(t)\right] - \rho}{\sigma},$$

where we have used equation (6) – which is unaffected by the imposition of the tax – to substitute for $g_P(t)$. Equating the two expressions for the growth rate of consumption, equations (12) and (8), yields

(13) $$\alpha g_K(t) = \frac{(1-\tau)\left[B + (\alpha - 1)g_K(t)\right] - \rho}{\sigma},$$

which implies
(14) $\alpha\sigma g_K(t) + (1 - \tau)(1 - \alpha)g_K(t) = (1 - \tau)B - \rho,$
and thus

(15) $$g_K(t) = \frac{(1-\tau)B - \rho}{\left[\alpha\sigma + (1-\tau)(1-\alpha)\right]}.$$

Substituting equation (15) into equation (8) yields the following expression for the growth rate of consumption as a function of the underlying parameters of the model:

(16) $$g_C(t) = \alpha\left[\frac{(1-\tau)B - \rho}{\alpha\sigma + (1-\tau)(1-\alpha)}\right].$$

In order to see the effects of the tax, take the derivative of $g_C(t)$ with respect to τ:

(17) $$\frac{\partial g_C(t)}{\partial\tau} = -\alpha\left\{\frac{B\left[\alpha\sigma + (1-\tau)(1-\alpha)\right] - \left[(1-\tau)B - \rho\right](1-\alpha)}{\left[\alpha\sigma + (1-\tau)(1-\alpha)\right]^2}\right\} = -\alpha\left\{\frac{B\alpha\sigma + \rho(1-\alpha)}{\left[\alpha\sigma + (1-\tau)(1-\alpha)\right]^2}\right\} < 0$$

Thus an increase in the tax rate τ causes the growth rate of consumption to fall.

Problem 3.13
(a) Note that the model of the northern economy is simply the Solow model with a constant growth rate of technology equal to $g = Ba_{LN}L_N$. From our analysis of the Solow model in Chapter 1, we know that

the long-run growth rate of northern output per worker will be equal to that constant growth rate of technology.

(b) Taking the time derivative of both sides of the definition, $Z(t) \equiv A_S(t)/A_N(t)$, yields

(1) $\dot{Z}(t) = \dfrac{A_N(t)\dot{A}_S(t) - A_S(t)\dot{A}_N(t)}{A_N(t)^2}$.

Substituting the expressions for $\dot{A}_S(t)$ and $\dot{A}_N(t)$ into equation (1) gives us

(2) $\dot{Z}(t) = \dfrac{A_N(t)\left[\mu a_{LS}L_S\left(A_N(t) - A_S(t)\right)\right] - A_S(t)\left[Ba_{LN}L_NA_N(t)\right]}{A_N(t)^2}$.

Simplifying yields

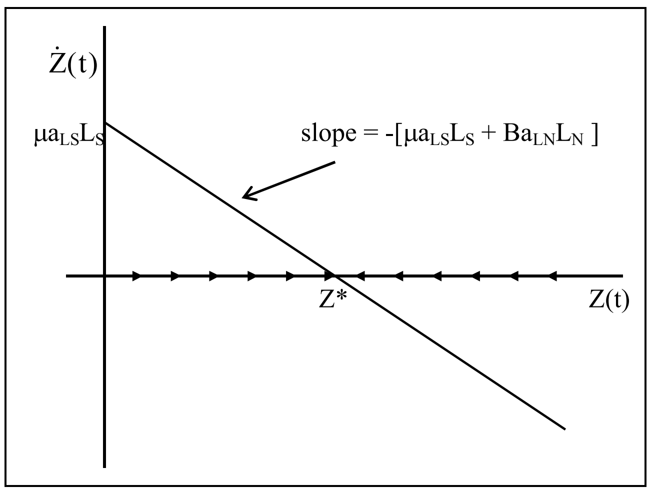
(3) $\dot{Z}(t) = \left[\mu a_{LS}L_S\left(1 - A_S(t)/A_N(t)\right)\right] - \left[A_S(t)/A_N(t)\right]\left[Ba_{LN}L_N\right]$.

Substituting the definition of $Z(t) \equiv A_S(t)/A_N(t)$ into equation (3) gives us

(4) $\dot{Z}(t) = \mu a_{LS}L_S - \mu a_{LS}L_SZ(t) - Ba_{LN}L_NZ(t)$.

Collecting terms yields

(5) $\dot{Z}(t) = \mu a_{LS}L_S - \left[\mu a_{LS}L_S + Ba_{LN}L_N\right]Z(t)$.



The phase diagram implied by equation (5) is depicted at right. Note that equation (5) and the accompanying phase diagram do not apply for the case of $Z \geq 1$, since $\dot{A}_S(t) = 0$ for $A_S(t) \geq A_N(t)$.

The relationship between $\dot{Z}(t)$ and $Z(t)$ is linear with slope equal to $-[\mu a_{LS}L_S + Ba_{LN}L_N] < 0$. From the phase diagram, if $Z < Z^*$, then $\dot{Z}(t) > 0$. Thus if Z begins to the left of Z*, it rises toward Z* over time. Similarly if $Z > Z^*$, then $\dot{Z}(t) < 0$. Thus if Z begins to the right of Z*, it falls toward Z* over time. Thus Z, the ratio of technology in the south to technology in the north, does converge to a stable value. To solve for Z*, set $\dot{Z}(t) = 0$:

(6) $0 = \mu a_{LS}L_S - [\mu a_{LS}L_S + Ba_{LN}L_N]Z^*$.

Solving for Z* yields

(7) $Z^* = \dfrac{\mu a_{LS}L_S}{\mu a_{LS}L_S + Ba_{LN}L_N}$.

The next step is to determine the long-run growth rate of southern output per worker. We have just shown that $Z(t) \equiv A_S(t)/A_N(t)$ converges to a constant. Thus in the long-run, $A_S(t)$ must be growing at the same rate as $A_N(t)$. In the long-run, then, the south is a Solow economy with a growth rate of technology equal to $Ba_{LN}L_N$. Thus the long-run growth rate of southern output per worker is equal to that growth rate.

Note that in the long-run, the growth rate of southern output per worker is the same as that in the north. This means that a_{LS}, the fraction of the south's labor force that is engaged in learning the technology of the north, does not affect the south's long-run growth rate. That growth rate is entirely determined by the number of people the north has working to produce new technology.

(c) Dividing the northern production function, $Y_N(t) = K_N(t)^{\alpha}[A_N(t)(1 - a_{LN})L_N]^{1-\alpha}$, by the quantity of effective labor, $A_N(t)L_N$, yields

(8) $$\frac{Y_N(t)}{A_N(t)L_N} = \left[\frac{K_N(t)}{A_N(t)L_N}\right]^{\alpha}\left[\frac{A_N(t)(1-a_{LN})L_N}{A_N(t)L_N}\right]^{1-\alpha}.$$

Defining output and capital per unit of effective labor as $y_N(t) \equiv Y_N(t)/A_N(t)L_N$ and $k_N(t) \equiv K_N(t)/A_N(t)L_N$ respectively, we can rewrite equation (8) as

(9) $y_N(t) = k_N(t)^{\alpha}(1 - a_{LN})^{1-\alpha}$.

Now we can use the technique employed to solve the Solow model to show that on the balanced growth path, $k_S^* = k_N^*$. Taking the time derivative of both sides of the definition of $k_N(t) \equiv K_N(t)/A_N(t)L_N$ yields

(10) $$\dot{k}_N(t) = \frac{\dot{K}_N(t)}{A_N(t)L_N} - \frac{K_N(t)}{A_N(t)L_N}\frac{\dot{A}_N(t)}{A_N(t)}.$$

Substituting the capital-accumulation equation, $\dot{K}_N(t) = s_N Y_N(t)$, into equation (10) gives us

(11) $$\dot{k}_N(t) = \frac{s_N Y_N(t)}{A_N(t)L_N} - \frac{\dot{A}_N(t)}{A_N(t)}\frac{K_N(t)}{A_N(t)L_N} = s_N y_N(t) - Ba_{LN}L_N k_N(t),$$

where we have used the definitions of $y_N(t)$ and $k_N(t)$ and have substituted for the growth rate of northern technology. Finally, using equation (9) to substitute for $y_N(t)$ yields

(12) $$\dot{k}_N(t) = s_N k_N(t)^{\alpha}(1-a_{LN})^{1-\alpha} - Ba_{LN}L_N k_N(t).$$

An analogous derivation for the south would yield

(13) $$\dot{k}_S(t) = s_S k_S(t)^{\alpha}(1-a_{LS})^{1-\alpha} - Ba_{LN}L_N k_S(t),$$

where we have used the fact that in the long-run, the growth rate of technology in the south is $Ba_{LN}L_N$.

Using the facts that $s_N = s_S$ and $a_{LN} = a_{LS}$, we can see that the equations for the dynamics of k are the same for the two economies. Thus we know that the balanced-growth-path values of k and y will be the same for the two economies. That is, we know that $k_S^* = k_N^*$ and $y_S^* = y_N^*$. This implies

(14) $y_S^*/y_N^* = 1$.

Using the definitions of y_S and y_N, this implies that

(15) $$\frac{Y_S/A_SL_S}{Y_N/A_NL_N} = 1,$$

or equivalently,

(16) $$\frac{Y_S/L_S}{Y_N/L_N} = \frac{A_S}{A_N}.$$

Equation (16) states that, on the balanced growth path, the ratio of output per worker in the south to output per worker in the north is equal to the ratio of technology in the south to technology in the north. From part (b), we know that A_S/A_N converges to Z* in the long-run. Using equation (7) for Z* to substitute into equation (16) leaves us with

(17) $$\frac{Y_S/L_S}{Y_N/L_N} = \frac{\mu a_{LS}L_S}{\mu a_{LS}L_S + Ba_{LN}L_N}.$$

Note that with $Ba_{LN}L_N > 0$, this ratio must be less than one; output per person in the south will be lower than output per person in the north. Also note that on the balanced growth path, the ratio of output per person in the south to that in the north does depend on a_{LS}, the fraction of southern workers engaged in learning the north's technology. In fact, the higher is a_{LS}, the closer will be the path of output per person in the south to that in the north.

Problem 3.14

(a) We need to find a value of τ such that $[Y_N(t)/L_N]/[Y_S(t)/L_S]$, the ratio of output per worker in the north to that in the south, is equal to 10. From the northern production function,

(1) $Y_N(t)/L_N = A_N(t)(1 - a_L)$.

Taking the time derivative of the natural log of equation (1) yields an expression for the growth rate of northern output per worker:

(2) $$\frac{\dot{\left[Y_N(t)/L_N\right]}}{Y_N(t)/L_N} = \frac{\dot{A}_N(t)}{A_N(t)} = 0.03,$$

where we have used the information given in the problem that the growth rate of northern output per worker, and thus of northern knowledge, is 3% per year. Since $\dot{A}_N(t)/A_N(t) = 0.03$ then

(3) $A_N(t) = e^{0.03\tau}A_N(t - \tau)$.

From the southern production function,

(4) $Y_S(t)/L_S = A_S(t)$.

Dividing equation (3) by equation (4) yields an expression for the ratio of output per worker in the north to that in the south:

(5) $$\frac{Y_N(t)/L_N}{Y_S(t)/L_S} = \frac{A_N(t)(1-a_L)}{A_S(t)} \approx \frac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau},$$

where we have used the fact that $a_L \approx 0$, that $A_S(t) = A_N(t - \tau)$, and equation (3).

For output per person in the north to exceed that in the south by a factor of 10, we need a τ such that $e^{0.03\tau} = 10$, or $0.03\tau = \ln(10)$, which implies that τ must be approximately 76.8 years. Thus, attributing realistic cross-country differences in income per person to slow transmission of knowledge to poor countries requires the transmission to be very slow. Poor countries would need to be using technology that the rich countries developed in the late 1920s in order to explain a 10-fold difference in income per person.

(b) (i) Recall that in the Solow model, the balanced-growth-path value of $k \equiv K/AL$ is defined implicitly by the condition that actual investment, $sf(k^*)$, equal break-even investment, $(n + g + \delta)k^*$. Thus for the north, k_N^* is implicitly defined by

(6) $sf(k_N^*) = (n + g + \delta)k_N^*$,

where $g = \dot{A}_N(t)/A_N(t)$.

We are told that s, n, δ and the function $f(\bullet)$ are the same for the north and the south. The only possible source of difference is the growth rate of southern knowledge. However, it is straightforward to show that $\dot{A}_S(t)/A_S(t) = g$ also.

We are told that the knowledge used in the south at time t is the knowledge that was used in the north at time $t - \tau$. That is,

(7) $A_S(t) = A_N(t - \tau)$.

Taking the time derivative of equation (7) yields

(8) $\dot{A}_S(t) = \dot{A}_N(t - \tau)$.

Dividing equation (8) by equation (7) yields

(9) $\dfrac{\dot{A}_S(t)}{A_S(t)} = \dfrac{\dot{A}_N(t-\tau)}{A_N(t-\tau)}.$

The growth rate of northern knowledge is constant and equal to g at all points in time and thus

(10) $\dot{A}_S(t)/A_S(t) = g.$

Therefore, for the south, k_S* is implicitly defined by

(11) $sf(k_S^*) = (n + g + \delta)k_S^*$.

Since k_N* and k_S* are implicitly defined by the same equation, they must be equal.

(b) (ii) Introducing capital will not change the answer to part (a). Since $k_N^* = k_S^*$, output per unit of effective labor on the balanced growth path will also be equal in the north and the south. That is, $y_N^* = y_S^*$ where $y_i^* = [Y_i / A_iL_i]^*$. We can write the balanced-growth-path value of output per worker in the north as

(12) $Y_N(t)/L_N(t) \equiv A_N(t)y_N^*$.

Similarly, the balanced-growth-path value of output per worker in the south is

(13) $Y_S(t)/L_S(t) \equiv A_S(t)y_S^*$.

Dividing equation (12) by equation (13) yields

(14) $\dfrac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \dfrac{A_N(t)y_N^*}{A_S(t)y_S^*} = \dfrac{A_N(t)}{A_S(t)} = \dfrac{A_N(t)}{A_N(t-\tau)}.$

The second-to-last step uses the fact that $y_N^* = y_S^*$. The last step uses $A_S(t) = A_N(t - \tau)$. Using equation (3), we again have

(15) $\dfrac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \dfrac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau}.$

The same calculation as in part (a) would yield a value of $\tau = 76.8$ years in order for $[Y_N(t)/L_N]/[Y_S(t)/L_S] = 10$.

SOLUTIONS TO CHAPTER 4

Problem 4.1

(a) From equation (4.11) in the text, output per person on the balanced growth path with the assumption that $G(E) = e^{\phi E}$ is given by

(1) $$\left(\frac{Y}{N}\right)^{bgp} = y^* A(t) e^{\phi E} \frac{e^{-nE} - e^{-nT}}{1 - e^{-nT}},$$

where $y^* = f(k^*)$ which is output per unit of effective labor services on the balanced growth path. We can maximize the natural log of $(Y/N)^{bgp}$ with respect to E, noting that y^* and $A(t)$ are not functions of E. The log of output per person on the balanced growth path is

(2) $$\ln\left(\frac{Y}{N}\right)^{bgp} = \ln y^* + \ln A(t) + \phi E + \ln\left[e^{-nE} - e^{-nT}\right] - \ln\left[1 - e^{-nT}\right],$$

and so the first-order condition is given by

(3) $$\frac{\partial \ln(Y/N)^{bgp}}{\partial E} = \phi + \frac{1}{e^{-nE} - e^{-nT}} e^{-nE}(-n) = 0,$$

or

(4) $$\phi\left(e^{-nE} - e^{-nT}\right) = ne^{-nE}.$$

Collecting the terms in e^{-nE} gives us

(5) $(\phi - n)e^{-nE} = \phi e^{-nT}$,

or simply

(6) $$e^{-nE} = \frac{\phi}{\phi - n} e^{-nT}.$$

Taking the natural log of both sides of equation (6) yields

(7) $- nE = [\ln\phi - \ln(\phi - n)] - nT$.

Multiplying both sides of (7) by $- 1/n$ gives us the following golden-rule level of education:

(8) $$E^* = T - \frac{1}{n}\ln\left[\frac{\phi}{\phi - n}\right].$$

(b) (i) Taking the derivative of E* with respect to T gives us

(9) $$\frac{\partial E^*}{\partial T} = 1.$$

So a rise in T – an increase in lifespan – raises the golden-rule level of education one for one.

(b) (ii) Showing that a fall in n increases the golden-rule level of education is somewhat complicated. From equation (6), we can write

(10) $$e^{-n(T-E^*)} = \frac{\phi - n}{\phi},$$

or

(11) $$1 - e^{-n(T-E^*)} = \frac{n}{\phi}.$$

Multiplying both sides of equation (11) by ϕ/n gives us

(12) $$\frac{\phi}{n}[1 - e^{-n(T-E^*)}] = 1.$$

Now note that the left-hand side of equation (12) is equivalent to

(13) $V \equiv \phi \int_{s=0}^{T-E^*} e^{-ns} ds.$

Thus, totally differentiating equation (12) gives us

(14) $\dfrac{\partial V}{\partial n} dn + \dfrac{\partial V}{\partial E^*} dE^* = 0,$

and so

(15) $\dfrac{dE^*}{dn} = -\dfrac{\partial V/\partial n}{\partial V/\partial E^*}.$

Now note that

(16) $\dfrac{\partial V}{\partial n} = \phi \int_{s=0}^{T-E^*} -se^{-ns} ds < 0,$

and

(17) $\dfrac{\partial V}{\partial E^*} = -\phi e^{-n(T-E^*)} < 0.$

Thus dE*/dn < 0 and so a fall in n raises the golden-rule level of education.

Problem 4.2

(a) In general, the present discounted value, at time zero, of the worker's lifetime earnings is

(1) $Y = \int_{t=E}^{T} e^{-\bar{r}t} w(t)L(t)dt.$

We can normalize L(t) to one and we are assuming that $w(t) = be^{gt}e^{\phi E}$. Thus (1) becomes

(2) $Y = \int_{t=E}^{T} e^{-\bar{r}t} be^{gt} e^{\phi E} dt = be^{\phi E} \int_{t=E}^{T} e^{-(\bar{r}-g)t} dt.$

Solving the integral in (2) gives us

(3) $Y = be^{\phi E} \left[\dfrac{-1}{(\bar{r}-g)} e^{-(\bar{r}-g)t} \Big|_{t=E}^{T} \right] = \dfrac{be^{\phi E}}{\bar{r}-g} \left[-e^{-(\bar{r}-g)T} + e^{-(\bar{r}-g)E} \right],$

which can be rewritten as

(4) $Y = \dfrac{b}{\bar{r}-g} \left[-e^{\phi E-(\bar{r}-g)T} + e^{[\phi-(\bar{r}-g)]E} \right].$

(b) The first-order condition for the choice of E is given by

(5) $\dfrac{\partial Y}{\partial E} = \dfrac{b}{\bar{r}-g} \left[-\phi e^{\phi E-(\bar{r}-g)T} + [\phi - (\bar{r}-g)] e^{[\phi-(\bar{r}-g)]E} \right] = 0.$

This can be rewritten as

(6) $[\phi - (\bar{r}-g)] e^{[\phi-(\bar{r}-g)]E} = \phi e^{\phi E-(\bar{r}-g)T}.$

Dividing both sides by $e^{\phi E}$ and rearranging yields

(7) $e^{-(\bar{r}-g)(E-T)} = \dfrac{\phi}{\phi - (\bar{r}-g)}.$

Taking the natural log of both sides of equation (7) gives us

(8) $-(\bar{r}-g)(E-T) = \ln\left[\dfrac{\phi}{\phi - (\bar{r}-g)} \right].$

Dividing both sides of (8) by $-(\bar{r} - g)$ and then adding T to both sides of the resulting expression gives us

$$(9)\quad E^* = T - \frac{1}{\bar{r} - g}\ln\left[\frac{\phi}{\phi - (\bar{r} - g)}\right].$$

(c) (i) From equation (9),

$$(10)\quad \frac{\partial E^*}{\partial T} = 1.$$

Thus an increase in lifespan increases the optimal amount of education. Intuitively, a longer lifespan provides a longer working period over which to receive the higher wages yielded by more education.

(c) (ii) & (iii) The interest rate, $\bar{r}$, and the growth rate, g, enter the optimal choice of education through their difference, $(\bar{r} - g)$. Intuitively, it should be clear that a rise in $\bar{r}$, and thus a rise in $(\bar{r} - g)$, will cause the individual to choose less education. Getting marginally more education foregoes current earnings for higher future earnings. A higher interest rate means that the higher future wages due to increased education will be worth less in present-value terms and hence the individual chooses less education.

Showing this formally is somewhat complicated, however. Taking the inverse of both sides of equation (7) gives us

$$(11)\quad e^{-(\bar{r}-g)(T-E^*)} = \frac{\phi - (\bar{r} - g)}{\phi},$$

or

$$(12)\quad 1 - e^{-(\bar{r}-g)(T-E^*)} = \frac{(\bar{r} - g)}{\phi}.$$

Multiplying both sides of equation (12) by $\phi/(\bar{r} - g)$ gives us

$$(13)\quad \frac{\phi}{(\bar{r} - g)}[1 - e^{-(\bar{r}-g)(T-E^*)}] = 1.$$

Now note that the left-hand side of equation (13) is equivalent to

$$(14)\quad V \equiv \phi \int_{s=0}^{T-E^*} e^{-(\bar{r}-g)s} ds.$$

Thus, totally differentiating equation (13) gives us

$$(15)\quad \frac{\partial V}{\partial(\bar{r} - g)} d(\bar{r} - g) + \frac{\partial V}{\partial E^*} dE^* = 0,$$

and so

$$(16)\quad \frac{dE^*}{d(\bar{r} - g)} = -\frac{\partial V/\partial(\bar{r} - g)}{\partial V/\partial E^*}.$$

Now note that

$$(17)\quad \frac{\partial V}{\partial(\bar{r} - g)} = \phi \int_{s=0}^{T-E^*} -se^{-(\bar{r}-g)s} ds < 0,$$

and

$$(18)\quad \frac{\partial V}{\partial E^*} = -\phi e^{-(\bar{r}-g)(T-E^*)} < 0.$$

Thus $dE^*/d(\bar{r} - g) < 0$. So a rise in $\bar{r}$ decreases the optimal choice of education; a rise in g increases the optimal choice of education.

Problem 4.3

(a) The country's output is described by $Y_i = A_i Q_i e^{\phi E_i} L_i$ and the quality of education is described by $Q_i = B_i (Y_i / L_i)^{\gamma}$. Solving for output per worker and taking logs, we have

(1) $\ln(Y_i/L_i) = \ln A_i + \ln B_i + \gamma \ln(Y_i/L_i) + \phi E_i$.

Thus, the difference in log output per worker between two countries would be

(2) $\ln(Y_2/L_2) - \ln(Y_1/L_1) = \ln A_2 - \ln A_1 + \ln B_2 - \ln B_1 + \gamma(\ln(Y_2/L_2) - \ln(Y_1/L_1)) + \phi(E_2 - E_1)$.

Intuitively, attributing amount $\phi(E_2 - E_1)$ of the difference in log output per worker to education would capture only the direct effect of a higher level of education on output per worker. It would miss the fact that a higher level of education leads to higher output per worker, which results in a higher quality of education that in turn increases output per worker even more.

(b) We can solve equation (2) for the difference in log output per worker, giving us

(3) $\ln(Y_2/L_2) - \ln(Y_1/L_1) = \dfrac{1}{1-\gamma}(\ln A_2 - \ln A_1) + \dfrac{1}{1-\gamma}(\ln B_2 - \ln B_1) + \dfrac{\phi}{1-\gamma}(E_2 - E_1)$.

For a more accurate measure, we should attribute $\dfrac{\phi}{1-\gamma}(E_2 - E_1)$ of $\ln(Y_2/L_2) - \ln(Y_1/L_1)$ to education. Note that the larger is γ—the greater is the effect of output per worker on the quality of education—the larger is φ/(1-γ) and the more we would understate the true effect of education by simply using $\phi(E_2 - E_1)$.

Problem 4.4

(a) We know the production function is given by $Y = K^{\alpha}(e^{\phi E} L)^{1-\alpha}$ and that the marginal product of capital is the first derivative of output with respect to capital. Therefore, we get

(1) $\partial Y/\partial K = \alpha K^{\alpha-1}(e^{\phi E} L)^{1-\alpha}$.

(b) There is perfect capital mobility so the marginal product of capital equals the world rate of return. That is, $\partial Y/\partial K = r^*$. Setting the marginal product of capital equal to r* in equation (1) and solving for K, we have

(2) $K = (r^*/\alpha)^{1/\alpha-1} L e^{\phi E}$.

(c) We would like to find an expression for $\partial \ln Y/\partial E$ so substitute the expression for K in equation (2) into our production function and take the natural log of both sides. Simplifying we get

(3) $\ln Y = \dfrac{\alpha}{1-\alpha} \ln(r^*/\alpha) + \ln L + \phi E$.

Taking the partial derivative with respect to E results in

(4) $\partial \ln Y/\partial E = \phi$.

(d) From our result in equation (4) we can see that capital mobility increases the impact of the change in E on output. With perfect capital mobility, more education increases the marginal product of capital as we can see in equation (1). That implies that for the marginal product to remain equal to the world rate of return, the capital stock must increase to offset this. This is reflected by the fact that K is increasing in E in equation (2). The increase in the level of the capital stock causes output to rise more. Note that without the response of the capital stock, $\partial \ln Y/\partial E = (1-\alpha)\phi$, which is less than φ.

Problem 4.5

(a) Substituting the expression for output given by equation (1),

(1) $Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}$,

into the definition of output per unit of effective labor, $y(t) \equiv [Y(t)/A(t)L(t)]$, gives us

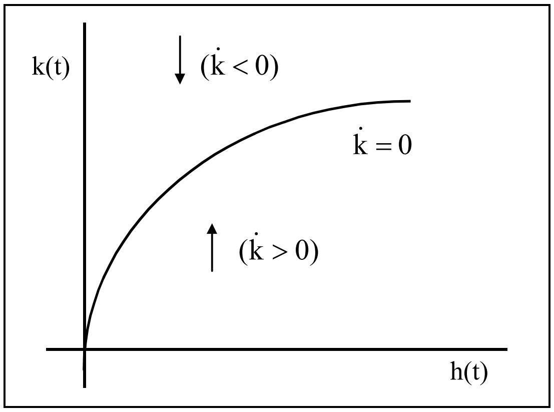
(2) $y(t) = \dfrac{K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}}{A(t)L(t)}$.

Now using the definitions of physical and human capital per unit of effective labor, $k(t) \equiv [K(t)/A(t)L(t)]$ and $h(t) \equiv [H(t)/A(t)L(t)]$, we can rewrite equation (2) as

(3) $y(t) = \dfrac{(A(t)L(t)k(t))^{\alpha} (A(t)L(t)(h(t))^{\beta} (A(t)L(t))^{1-\alpha-\beta}}{A(t)L(t)}$.

Noting that we have $A(t)L(t)$ in both the numerator and denominator, this simplifies to

(4) $y(t) = k(t)^{\alpha} h(t)^{\beta}$.



(b) Differentiating both sides of the definition of $k(t) \equiv K(t)/A(t)L(t)$ with respect to time yields

(5) $\dot{k}(t) = \dfrac{\dot{K}(t)A(t)L(t) - K(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}$.

Using the definition of $k(t) \equiv K(t)/A(t)L(t)$, equation (5) can be rewritten as

(6) $\dot{k}(t) = \dfrac{\dot{K}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]k(t)$.

Substituting the capital-accumulation equation, $\dot{K}(t) = s_k Y(t) - \delta K(t)$ $\dot{K}(t) = sY(t) - \delta_K K(t)$, as well as the constant growth rates of knowledge and labor into equation (6) gives us

(7) $\dot{k}(t) = \dfrac{s_k Y(t) - \delta K(t)}{A(t)L(t)} - (n+g)k(t)$,

which simplifies to

(8) $\dot{k}(t) = s_k y(t) - (n+g+\delta)k(t)$.

Substituting our expression for output per unit of effective labor given by equation (4) into (8) yields

(9) $\dot{k}(t) = s_k k(t)^{\alpha} h(t)^{\beta} - (n+g+\delta)k(t)$.

In order to derive the $\dot{k} = 0$ locus, set the right-hand-side of equation (9) to 0 to obtain

(10) $s_k k(t)^{\alpha} h(t)^{\beta} = (n+g+\delta)k(t)$.

Solving for k(t) gives us

(11) $k(t)^{\alpha-1} = \dfrac{n+g+\delta}{s_k}\dfrac{1}{h(t)^{\beta}}$,

and thus finally

(12) $k(t) = \left(\dfrac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} h(t)^{\frac{\beta}{1-\alpha}}$.

In order to describe the $\dot{k} = 0$ locus, we can take the following derivatives:

(16) $dk(t)/dh(t)\big|_{\dot{k}=0} = c_k \dfrac{\beta}{1-\alpha} h(t)^{\frac{\alpha+\beta-1}{1-\alpha}} > 0$, and

(17) $d^2k(t)/dh(t)^2\big|_{\dot{k}=0} = c_k \dfrac{\beta}{1-\alpha}\dfrac{\alpha+\beta-1}{1-\alpha} h(t)^{\frac{\alpha+\beta-1}{1-\alpha}-1} < 0$,

where we have defined $c_k \equiv \left(\dfrac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} > 0$. The second derivative is negative because $\alpha + \beta < 1$.

Thus, the $\dot{k} = 0$ locus is upward sloping and concave in (k, h) space. From equation (9) we can see that $\dot{k}$ is increasing in h(t) and so to the right of the $\dot{k} = 0$ locus, $\dot{k}$ is positive and so k is rising To the left of the $\dot{k} = 0$ locus, $\dot{k}$ is negative and so k is falling. See the figure above.

(c) Differentiating both sides of the definition of $h(t) \equiv H(t)/A(t)L(t)$ with respect to time yields

(18) $\dot{h}(t) = \dfrac{\dot{H}(t)A(t)L(t) - H(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}$.

Equation (18) can be simplified to

(19) $\dot{h}(t) = \dfrac{\dot{H}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right] h(t)$.

Substituting $\dot{H}(t) = s_h Y(t) - \delta H(t)$, the human-capital-accumulation equation, as well as the constant growth rates of knowledge and labor into equation (19) gives us

(20) $\dot{h}(t) = \dfrac{s_h Y(t) - \delta H(t)}{A(t)L(t)} - (n+g)h(t)$,

which simplifies to

(21) $\dot{h}(t) = s_h y(t) - (n+g+\delta)h(t)$.

Substituting our expression for output per unit of effective labor given by equation (4) into (21) yields

(22) $\dot{h}(t) = s_h k(t)^{\alpha} h(t)^{\beta} - (n+g+\delta)h(t)$.

In order to derive the $\dot{h} = 0$ locus, set the right-hand-side of equation (22) to 0 to obtain

(23) $s_h k(t)^{\alpha} h(t)^{\beta} = (n+g+\delta)h(t)$.

Solving for k(t) gives us

(24) $k(t)^{\alpha} = \dfrac{n+g+\delta}{s_h} h(t)^{1-\beta}$,

and thus finally

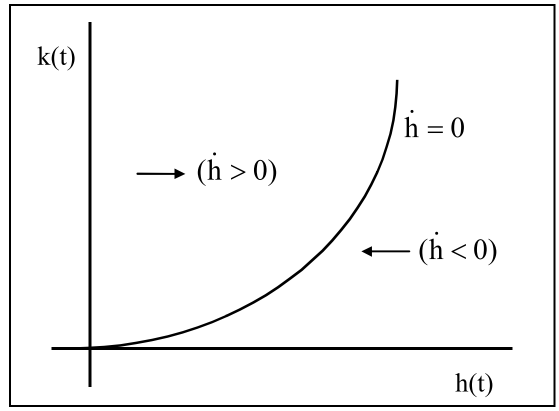
(25) $k(t) = \left(\dfrac{n+g+\delta}{s_h}\right)^{\frac{1}{\alpha}} h(t)^{\frac{1-\beta}{\alpha}}$.

In order to describe the $\dot{h} = 0$ locus, we can take the following derivatives:

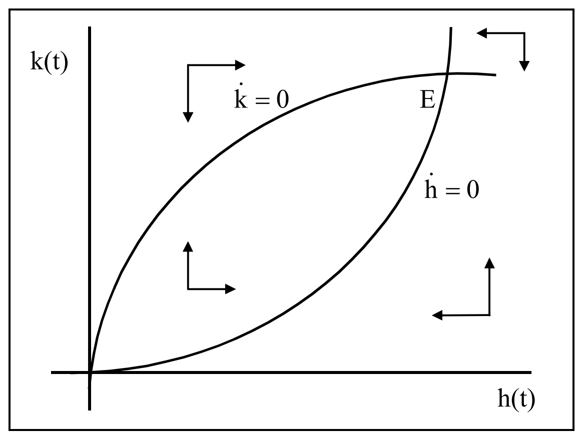
(26) $dk(t)/dh(t)\big|_{\dot{h}=0} = c_h \dfrac{1-\beta}{\alpha} h(t)^{\frac{1-\alpha-\beta}{\alpha}} > 0$, and

(27) $d^2k(t)/dh(t)^2\big|_{\dot{h}=0} = c_h \dfrac{1-\beta}{\alpha}\dfrac{1-\alpha-\beta}{\alpha} h(t)^{\frac{1-\alpha-\beta}{\alpha}-1} > 0$,

where we have defined $c_h \equiv \left(\dfrac{n+g+\delta}{s_h}\right)^{\frac{1}{\alpha}} > 0$. The second derivative is positive because $\alpha + \beta < 1$. Thus, the $\dot{h} = 0$ locus is upward sloping and convex in (k, h) space. From equation (9) we can see that $\dot{h}$ is increasing in k(t) and so above the $\dot{h} = 0$ locus, $\dot{h}$ is positive and so h is rising Below the $\dot{h} = 0$ locus, $\dot{h}$ is negative and so h is falling. See the figure at right.



(d) Putting the $\dot{k} = 0$ and $\dot{h} = 0$ loci together, we can see that the economy will converge to a stable balanced growth path at point E. This stable balanced growth path is unique (as long as we ignore the origin with k = h = 0). From the figure, physical capital per unit of effective labor, k(t) ≡ K(t)/A(t)L(t), is constant on a balanced growth path. Thus physical capital per person, K(t)/L(t) ≡ k(t)A(t), must grow at the same rate as knowledge, which is g. Similarly, human capital per unit of effective labor, h(t) ≡ H(t)/A(t)L(t), is constant on the balanced growth path. Thus human capital per person, H(t)/L(t) ≡ H(t)A(t), must also grow at the same rate as knowledge, which is g.



Dividing the production function, equation (1) by L(t) gives us an expression for output per person:

(28) $\dfrac{Y(t)}{L(t)} = \left(\dfrac{K(t)}{L(t)}\right)^{\alpha}\left(\dfrac{H(t)}{L(t)}\right)^{\beta} A(t)^{1-\alpha-\beta}$

Since K(t)/L(t), H(t)/L(t), and A(t) all grow at rate g on the balanced growth path and since the production function is constant returns to scale, output per person also grows at rate g on the balanced growth path.

Problem 4.6

(a) To solve for the balanced-growth-path values of k* and h*, we can use equations (12) and (25) to write

(1) $\left(\dfrac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} h^{*\frac{\beta}{1-\alpha}} = \left(\dfrac{n+g+\delta}{s_h}\right)^{\frac{1}{\alpha}} h^{*\frac{1-\beta}{\alpha}}$.

Solving for h* gives us

(2) $h^{*\frac{1-\beta}{\alpha}-\frac{\beta}{1-\alpha}} = \left(\dfrac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}\left(\dfrac{s_h}{n+g+\delta}\right)^{\frac{1}{\alpha}}$.

The exponent on h* simplifies to $(1-\alpha-\beta)/[\alpha(1-\alpha)]$. Taking both sides of (2) to the inverse of that exponent then gives us

$$(3)\quad h^* = s_K^{\frac{\alpha}{1-\alpha-\beta}} s_H^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}}.$$

In order to find k*, we can then substitute the expression for h* into equation (12) to obtain

$$(4)\quad k^* = s_k^{\frac{1}{1-\alpha}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \left[s_K^{\frac{\alpha}{1-\alpha-\beta}} s_H^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}} \right]^{\frac{\beta}{1-\alpha}}.$$

Simplifying gives us

$$(5)\quad k^* = s_k^{\frac{1}{1-\alpha}+\frac{\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha}+\frac{\beta}{(1-\alpha)(1-\alpha-\beta)}}.$$

The exponents then simplify, leaving us with

$$(6)\quad k^* = s_k^{\frac{1-\alpha-\beta+\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1-\alpha-\beta+\beta}{(1-\alpha)(1-\alpha-\beta)}},$$

and thus, finally

$$(7)\quad k^* = s_k^{\frac{1-\beta}{1-\alpha-\beta}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}},$$

where we have used the fact that $(1-\alpha-\beta+\alpha\beta) = (1-\alpha)(1-\beta)$ to simplify the exponent on s_k.

(b) Since technology is the same in both countries, we can compare output per unit of effective labor. From equation (4) in the solution to problem 4.5, the ratio of output on a balanced growth path in country A to that in country B is given by

$$(8)\quad \frac{y_A^*}{y_B^*} = \left(\frac{k_A^*}{k_B^*}\right)^{\alpha} \left(\frac{h_A^*}{h_B^*}\right)^{\beta}.$$

We know that $\alpha = 1/3$ and $\beta = 1/2$. Substituting these values into equations (7) and (3) yields

$$(9)\quad k^* = s_k^3 s_h^3 \left(\frac{1}{n+g+\delta}\right)^6,$$

and

$$(10)\quad h^* = s_K^2 s_H^4 \left(\frac{1}{n+g+\delta}\right)^6.$$

Substituting equations (9) and (10) into equation (8) gives us

$$(11)\quad \frac{y_A^*}{y_B^*} = \left(\frac{s_{k,A}^3 s_{h,A}^3}{s_{k,B}^3 s_{h,B}^3}\right)^{\frac{1}{3}} \left(\frac{s_{k,A}^2 s_{h,A}^4}{s_{k,B}^2 s_{h,B}^4}\right)^{\frac{1}{2}} = \left(\frac{s_{k,A} s_{h,A}}{s_{k,B} s_{h,B}}\right) \left(\frac{s_{k,A} s_{h,A}^2}{s_{k,B} s_{h,B}^2}\right).$$

We also know that $s_{k,A} = 2s_{k,B}$ and $s_{h,A} = 2s_{h,B}$. Everything else is identical between Countries A and B. Thus, equation (11) becomes

(12) $\dfrac{y_A^*}{y_B^*} = 2\cdot 2\cdot 2\cdot 2^2 = 32$.

Thus, output per worker in country A will be 32 times higher than in country B.

(c) We can compare the amount of skills per unit of effective labor since technology is the same across the two countries. Using equation (10), the ratio of skills in country A to those in country B is given by

(13) $\dfrac{h_A^*}{h_B^*} = \left(\dfrac{s_{k,A}^2 s_{h,A}^4}{s_{k,B}^2 s_{h,B}^4}\right)$.

Substituting $s_{k,A} = 2s_{k,B}$ and $s_{h,A} = 2s_{h,B}$ into equation (13) yields

(14) $h_A^* / h_B^* = 2^2 \cdot 2^4 = 64$.

The level of skills per worker on the balanced growth path will be 64 times higher in country A than in country B.

Problem 4.7

(a) Since G(E) does not enter the expression for the evolution of the capital stock, $\dot{K}$, the balanced-growth-path values of capital and output per unit of effective labor services, k and y, are not affected by changes in E. We can write output per worker as

(1) Y/L = AG(E)y.

Thus, the growth rate of output per worker is given by

(2) $\dfrac{\dot{Y/L}}{Y/L} = \dfrac{\dot{A}}{A} + \dfrac{G'(E)\dot{E}}{G(E)} + \dfrac{\dot{y}}{y}$.

On a balanced growth path, output per unit of effective labor services is constant so the last term is zero. Technology grows at rate g. Since $G(E) = e^{\phi E}$, where $\dot{E} = m$, we can therefore rewrite equation (2) as

(3) $\dfrac{\dot{Y/L}}{Y/L} = g + \dfrac{e^{\phi E}\phi\dot{E}}{e^{\phi E}} = g + \phi m$.

(b) We are given that $\phi = 0.1$, m = 1/15, and that growth of output per worker is 0.02. Therefore, from equation (3), the ratio of education's contribution to the overall growth rate of output per worker is given by [(0.1)(1/15)]/(0.02), which equals one-third.

(c) The lifespan of an individual is fixed at T years, so years of education have to be less than or equal to T. Therefore, $\dot{E}(t)$ cannot equal m forever since years of schooling will eventually be longer than one's lifespan. However, that is not to say that $\dot{E}(t)$ cannot equal m ever. For periods of time that $\dot{E}(t) = m$, output per worker growth would be larger, allowing for convergence for poorer economies.

Problem 4.8

(a) Differentiating both sides of the definition of k(t) ≡ K(t)/A(t)L(t) with respect to time yields

(1) $\dot{k}(t) = \dfrac{\dot{K}(t)A(t)L(t) - K(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}$.

Using the definition of k(t) ≡ K(t)/A(t)L(t), equation (1) can be rewritten as

(2) $\dot{k}(t) = \dfrac{\dot{K}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]k(t)$.

Substituting the capital-accumulation equation, $\dot{K}(t) = sY(t) - \delta_K K(t)$, as well as the constant growth rates of knowledge and labor into equation (2) gives us

(3) $\dot{k}(t) = \dfrac{sY(t) - \delta_K K(t)}{A(t)L(t)} - (n+g)\,k(t)$.

Substituting the production function, $Y(t) = [(1-a_K)K(t)]^{\alpha}[(1-a_H)H(t)]^{1-\alpha}$, into equation (3) yields

(4) $\dot{k}(t) = s\left[\dfrac{(1-a_K)K(t)}{A(t)L(t)}\right]^{\alpha}\left[\dfrac{(1-a_H)H(t)}{A(t)L(t)}\right]^{1-\alpha} - (n+g+\delta_K)k(t)$.

Finally, defining $c_K \equiv s(1 - a_K)^{\alpha}(1 - a_H)^{1-\alpha}$ and using $k(t) \equiv K(t)/A(t)L(t)$ as well as $h(t) \equiv H(t)/A(t)L(t)$, equation (4) can be rewritten as

(5) $\dot{k}(t) = c_K k(t)^{\alpha} h(t)^{1-\alpha} - (n+g+\delta_K)k(t)$.

Differentiating both sides of the definition of $h(t) \equiv H(t)/A(t)L(t)$ with respect to time yields

(6) $\dot{h}(t) = \dfrac{\dot{H}(t)A(t)L(t) - H(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}$.

Equation (6) can be simplified to

(7) $\dot{h}(t) = \dfrac{\dot{H}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]h(t)$.

Substituting $\dot{H}(t) = B[a_K K(t)]^{\gamma}[a_H H(t)]^{\phi}[A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t)$, the human-capital-accumulation equation, as well as the constant growth rates of knowledge and labor into equation (7) gives us

(8) $\dot{h}(t) = B\left[\dfrac{a_K K(t)}{A(t)L(t)}\right]^{\gamma}\left[\dfrac{a_H H(t)}{A(t)L(t)}\right]^{\phi}\left[\dfrac{A(t)L(t)}{A(t)L(t)}\right]^{1-\gamma-\phi} - (n+g+\delta_H)k(t)$.

Finally, defining $c_H \equiv Ba_K^{\gamma}a_H^{\phi}$ allows us to rewrite equation (8) as

(9) $\dot{h}(t) = c_H k(t)^{\gamma} h(t)^{\phi} - (n+g+\delta_H)h(t)$.

(b) To find the combinations of h and k such that $\dot{k} = 0$, set the right-hand side of equation (5) equal to zero and solve for k as a function of h:

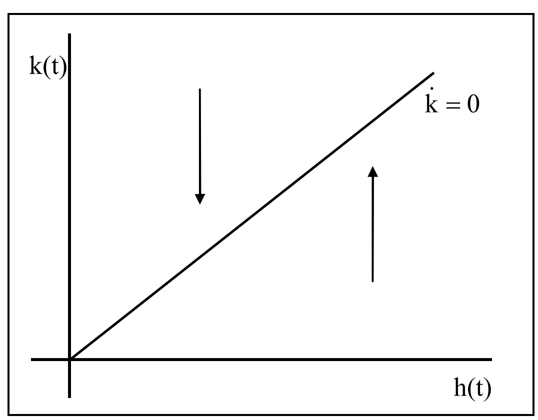
(10) $c_K\, k(t)^{\alpha}h(t)^{1-\alpha} = (n + g + \delta_K)k(t)$,

or

(11) $k(t)^{1-\alpha} = c_K\, h(t)^{1-\alpha}/(n + g + \delta_K)$,

and thus finally

(12) $k(t) = [c_K/(n + g + \delta_K)]^{1/(1-\alpha)}\, h(t)$.



The $\dot{k} = 0$ locus, as defined by equation (12), is a straight line with slope $[c_K/(n + g + \delta_K)]^{1/(1-\alpha)} > 0$ that passes through the origin. See the figure at right. From equation (5), we can see that $\dot{k}(t)$ is increasing in h(t). Thus to the right of the $\dot{k} = 0$ locus, $\dot{k} > 0$ and so k(t) is rising. To the left of the $\dot{k} = 0$ locus, $\dot{k} < 0$ and so k(t) is falling.

To find the combinations of h and k such that $\dot{h} = 0$, set the right-hand side of equation (9) equal to zero and solve for k as a function of h:

(13) $c_H\, k(t)^{\gamma}\, h(t)^{\phi} = (n + g + \delta_H)h(t)$,

or

(14) $k(t)^{\gamma} = [(n + g + \delta_H)/c_H]h(t)^{1-\phi}$,

and thus finally

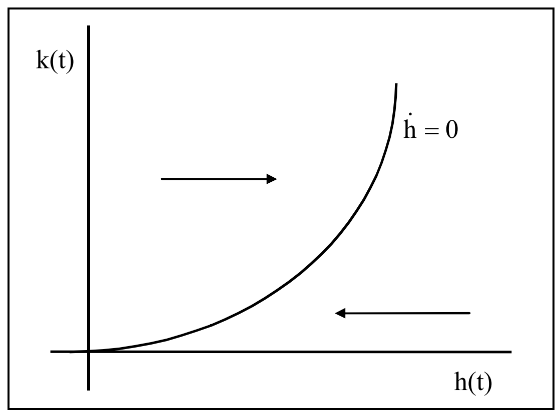
(15) $k(t) = [c_H/(n + g + \delta_H)]^{\gamma}\, h(t)^{(1-\phi)/\gamma}$.

The following derivatives will be useful:

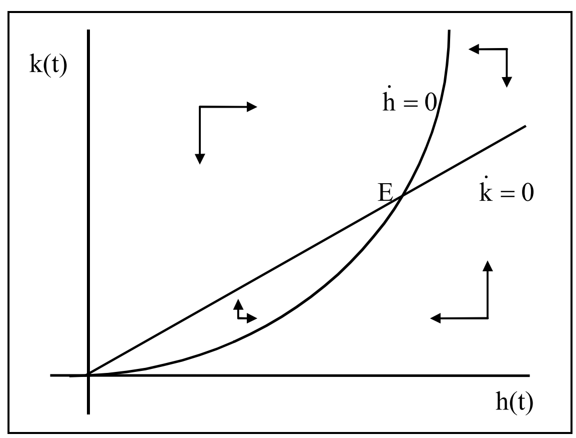
(16) $dk(t)/dh(t)\big|_{\dot{h}=0} = [(1-\phi)/\gamma][c_H/(n+g+\delta_H)]^{\gamma}\, h(t)^{(1-\phi-\gamma)/\gamma} > 0$, and

(17) $d^2k(t)/dh(t)^2\big|_{\dot{h}=0} = [(1-\phi-\gamma)/\gamma][(1-\phi)/\gamma][c_H/(n+g+\delta_H)]^{\gamma}\, h(t)^{(1-\phi-2\gamma)/\gamma} > 0$.

The $\dot{h} = 0$ locus, as defined by equation (15), is upward-sloping with a positive second derivative. See the figure at right. From equation (9), we can see that $\dot{h}(t)$ is increasing in k(t). Therefore, above the $\dot{h} = 0$ locus, $\dot{h} > 0$ and so h(t) is increasing. Below the $\dot{h} = 0$ locus, $\dot{h} < 0$ and so h(t) is falling.



(c) Putting the $\dot{k} = 0$ and $\dot{h} = 0$ loci together, we can see that the economy will converge to a stable balanced growth path at point E. This stable balanced growth path is unique (as long as we ignore the origin with k = h = 0). From the figure, physical capital per unit of effective labor, $k(t) \equiv K(t)/A(t)L(t)$, is constant on a balanced growth path. Thus physical capital per person, $K(t)/L(t) \equiv k(t)A(t)$, must grow at the same rate as knowledge, which is g. Similarly, human capital per unit of effective labor, $h(t) \equiv H(t)/A(t)L(t)$, is constant on the balanced growth path. Thus human capital per person, $H(t)/L(t) \equiv H(t)A(t)$, must also grow at the same rate as knowledge, which is g.



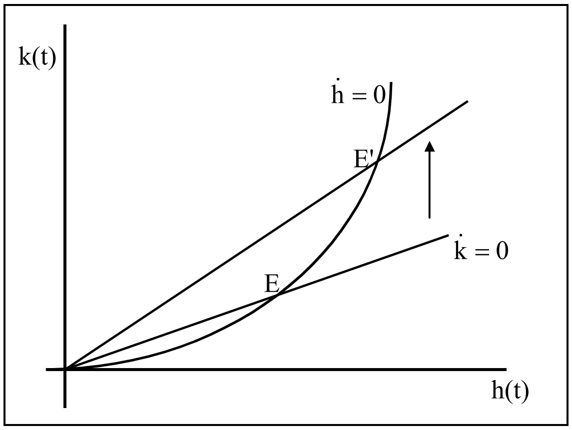
Dividing the production function by L(t) gives us an expression for output per person:

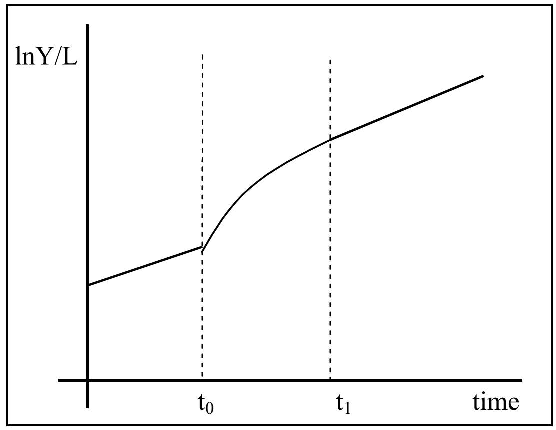
(18) $Y(t)/L(t) = [(1 - a_K)K(t)/L(t)]^{\alpha}\, [(1 - a_H)H(t)/L(t)]^{1-\alpha}$.

Since K(t)/L(t) and H(t)/L(t) both grow at rate g on the balanced growth path and since the production function is constant returns to scale, output per person also grows at rate g on the balanced growth path.

(d) From equation (12), the slope of the $\dot{k} = 0$ locus is $[c_K/(n + g + \delta_K)]^{1/(1-\alpha)}$, where we have defined

$c_K \equiv s(1 - a_K)^{\alpha} (1 - a_H)^{1-\alpha}$. Thus a rise in s will make the $\dot{k} = 0$ locus steeper. Since s does not appear in equation (15), the $\dot{h} = 0$ locus is unaffected. See the figure on the left. The economy will move from its old balanced growth path at E to a new balanced growth path at E '.





Output per person grows at rate g until the time that s rises (denoted time t_0 in the figure on the right). During the transition from E to E ', both h(t) and k(t) are rising. Thus human capital per person and physical capital per person grow at a rate greater than g during the transition. From equation (18), this means that output per person grows at a rate greater than g during the transition as well. Once the economy reaches the new balanced growth path (at time t_1 in the diagram), h(t) and k(t) are constant again. Thus human and physical capital per person grow at rate g again. Thus output per person grows at rate g again on the new balanced growth path. A permanent rise in the saving rate has only a level effect on output per person, not a permanent growth rate effect.

Problem 4.9

The relevant equations are

(1) $Y(t) = K(t)^{\alpha}\left[\left(1 - a_H\right) H(t)\right]^{\beta}$, (2) $\dot{H}(t) = Ba_H H(t)$, and (3) $\dot{K}(t) = sY(t)$,

where $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta > 1$.

(a) To get the growth rate of human capital – which turns out to be constant – divide equation (2) by H(t):

(4) $g_H \equiv \dot{H}(t)/H(t) = Ba_H$.

(b) Substitute the production function, equation (1), into the expression for the evolution of the physical capital stock, equation (3), to obtain

(5) $\dot{K}(t) = sK(t)^{\alpha}\left[\left(1 - a_H\right) H(t)\right]^{\beta}$.

To get the growth rate of physical capital, divide equation (5) by K(t):

(6) $g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1}\left[\left(1 - a_H\right) H(t)\right]^{\beta}$.

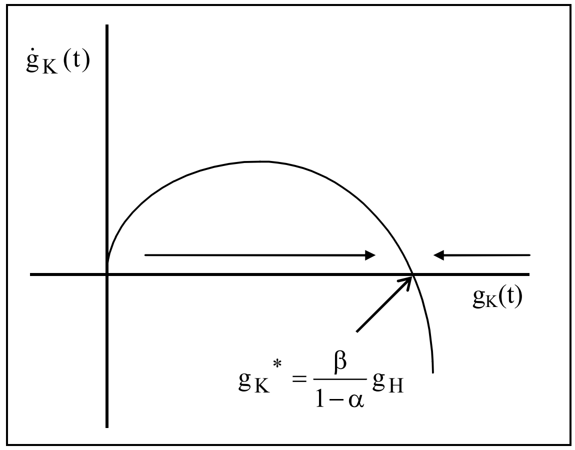
We need to examine the dynamics of the growth rate of physical capital. Taking the time derivative of the log of equation (6) yields the following growth rate of the growth rate of physical capital:

(7) $\dot{g}_K(t)/g_K(t) = (\alpha - 1)\,\dot{K}(t)/K(t) + \beta\,\dot{H}(t)/H(t) = (\alpha - 1)g_K(t) + \beta g_H$.

Now we can plot the change in the growth rate of capital, $\dot{g}_K(t)$, as a function of the growth rate of capital itself, $g_K(t)$. Multiplying both sides of equation (7) by $g_K(t)$ gives us

(8) $\dot{g}_K(t) = (\alpha - 1)\,g_K(t)^2 + \beta g_H g_K(t)$.

Note that we are assuming that $\alpha < 1$ which means that there are decreasing returns to physical capital alone. The phase diagram implied by equation (8) is depicted in the figure at right. Note that $g_K(t)$ is constant when $\dot{g}_K(t) = 0$ or when $(\alpha - 1)g_K(t) + \beta g_H = 0$. Solving this expression for $g_K(t)$ yields $g_K^* = [\beta/(1 - \alpha)]g_H$.



Note that $g_K^* > g_H$ since $\alpha + \beta > 1$ or $\beta > 1 - \alpha$. To the left of g_K^*, from the phase diagram, $\dot{g}_K(t) > 0$ and so $g_K(t)$ rises toward g_K^*.

Similarly, to the right of g_K^*, $\dot{g}_K(t) < 0$ and so $g_K(t)$ falls toward g_K^*. Thus the growth rate of capital converges to a constant value of g_K^* and a balanced growth path exists.

Taking the time derivative of the log of equation (1) yields the growth rate of output:

(9) $\dot{Y}(t)/Y(t) = \alpha\, \dot{K}(t)/K(t) + \beta\, \dot{H}(t)/H(t) = \alpha g_K(t) + \beta g_H$.

On the balanced growth path, $g_K(t) = g_K^* = [\beta/(1 - \alpha)]g_H$ and so

(10) $\dfrac{\dot{Y}(t)}{Y(t)} = \dfrac{\alpha\beta}{(1-\alpha)} g_H + \beta g_H = \dfrac{\alpha\beta + \beta - \alpha\beta}{(1-\alpha)} g_H = \dfrac{\beta}{(1-\alpha)} g_H \equiv g_K^*$.

On the balanced growth path, output grows at the same rate as physical capital, which in turn is greater than the constant growth rate of human capital, $g_H = Ba_H$.

Problem 4.10

(a) The true relationship between social infrastructure and log income per person is expressed by $y_i(t) = a + bSI_i + e_i$. However, of the two different components of social infrastructure, we only have data for SI_i^A. Therefore, we substitute for SI_i and the equation becomes

(1) $y_i = a + bSI_i^A + bSI_i^B + e_i$.

(b) We would like to run an OLS regression on (1) but we cannot observe SI_i^B, so we run an OLS regression on

(2) $y_i = \alpha + \beta SI_i^A + e_i$.

We know that $\beta = \dfrac{\text{cov}(SI_i^A, y_i)}{\text{var}(SI_i^A)}$, where y_i represents the true log income per person, shown in equation (1). Substituting in the true log income per person and simplifying, we get

(3) $\beta = \dfrac{\text{cov}(SI_i^A, a + bSI_i^A + bSI_i^B + e_i)}{\text{var}(SI_i^A)} = b + \dfrac{b\,\text{cov}(SI_i^A, SI_i^B)}{\text{var}(SI_i^A)} = b\left(1 + \dfrac{\text{cov}(SI_i^A, SI_i^B)}{\text{var}(SI_i^A)}\right)$.

Remember that SI_i^A is uncorrelated with e_i so $\text{cov}(SI_i^A, e_i) = 0$.

(i) If SI_i^A and SI_i^B are uncorrelated, then $\text{cov}(SI_i^A, SI_i^B) = 0$. Equation (3) shows that $\beta = b$, such that the OLS regression for equation (2) would not produce a bias. Therefore, we would see the true impact of social infrastructure on log income per person.

(ii) If SI_i^A and SI_i^B are positively correlated, then $\text{cov}(SI_i^A, SI_i^B) > 0$. Equation (3) shows that $\beta > b$, such that the OLS regression for equation (2) would produce an upward bias. Therefore, we would be overestimating the impact of the component of social infrastructure that we observe. Intuitively, since the unobserved component of social infrastructure varies positively with the observed component, we would be incorrectly attributing some of the impact of the unobserved component to the observed component.

Problem 4.11

(a) False. Using this OLS method could result in bias. Assume we would like to run an OLS regression on $y_i(t) = a + bSI_i + e_i$, where y_i is income per person, SI_i is social infrastructure, and e_i is an error term that captures everything other than social infrastructure that has an impact on income per person. For an unbiased estimate, social infrastructure cannot be correlated with the error term; otherwise, the estimate will have an upward or downward bias (see problem 4.10). As discussed in the text, it is in fact likely that social infrastructure is correlated with other factors affecting income per person such as cultural factors or geography.

(b) False. Instrumenting a variable requires that the variable is uncorrelated with the error. Since we can only guarantee that the instrumental variable is uncorrelated with social infrastructure, the condition could be violated. Then, the result would be biased, and we would not measure the true effect of social infrastructure on output per person.

(c) False. The coefficient of determination measures the correlation between variables, so a high R^2 would show a good "fit" for the data. However, a high R^2 does not help determine whether the model reflects reality (that is, that the model captures the true effect of social infrastructure on output per person) or whether we have accounted for common problems like omitted variable bias.

Problem 4.12

(a) (i) We have

(1) $$\frac{dy_i(t)}{dt} = -\lambda[y_i(t) - y^*].$$

Since y^* is a constant, the derivative of $y_i(t)$ with respect to time is the same as the derivative of $y_i(t) - y^*$ with respect to time and so equation (1) is equivalent to

(2) $$\frac{d[y_i(t) - y^*]}{dt} = -\lambda[y_i(t) - y^*],$$

which implies that $y_i(t) - y^*$ grows at rate $-\lambda$. Thus

(3) $y_i(t) - y^* = e^{-\lambda t}[y_i(0) - y^*].$

Rearranging equation (3) to solve for $y_i(t)$ gives us

(4) $y_i(t) = (1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0).$

(a) (ii) Adding a mean-zero, random disturbance to $y_i(t)$ gives us

(5) $y_i(t) = (1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0) + u_i(t).$

Consider the cross-country growth regression given by

(6) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \varepsilon_i$.

Using the hint in the question, the coefficient on $y_i(0)$ in this regression equals the covariance of $y_i(t) - y_i(0)$ and $y_i(0)$ divided by the variance of $y_i(0)$. Thus the estimate of β is given by

$$(7)\quad \beta = \frac{\text{cov}[y_i(t) - y_i(0), y_i(0)]}{\text{var}[y_i(0)]}.$$

(If the sample size is large enough, we can treat sample parameters as equivalent to their population counterparts.) Now, use the fact that for any two random variables, X and Y, cov[(X - Y), Y] = cov[X, Y] - var[Y] and so

$$(8)\quad \beta = \frac{\text{cov}[y_i(t), y_i(0)] - \text{var}[y_i(0)]}{\text{var}[y_i(0)]} = \frac{\text{cov}[y_i(t), y_i(0)]}{\text{var}[y_i(0)]} - 1.$$

Using equation (5),

$$(9)\quad \text{cov}[y_i(t), y_i(0)] = \text{cov}[(1 - e^{-\lambda t})y^* + e^{-\lambda t} y_i(0) + u_i(t), y_i(0)].$$

Since y^* is a constant, and $u_i(t)$ and $y_i(0)$ are assumed to be uncorrelated, we have

$$(10)\quad \text{cov}[y_i(t), y_i(0)] = e^{-\lambda t}\,\text{var}[y_i(0)].$$

Substituting equation (10) into equation (8) gives us

$$(11)\quad \beta = \frac{e^{-\lambda t}\,\text{var}[y_i(0)]}{\text{var}[y_i(0)]} - 1,$$

or

$$(12)\quad e^{-\lambda t} = 1 + \beta.$$

Taking the natural log of both sides of equation (12) and solving for λ gives us

$$(13)\quad \lambda = -\frac{\ln(1+\beta)}{t}.$$

Thus, given an estimate of β, equation (13) could be used to calculate an estimate of the rate of convergence, λ.

(a) (iii) From equation (5), the variance of $y_i(t)$ is given by

$$(14)\quad \text{var}[y_i(t)] = e^{-2\lambda t}\,\text{var}[y_i(0)] + \text{var}[u_i(t)].$$

From equation (13), if $\beta < 0$ then $\lambda > 0$. This does not, however, ensure that $\text{var}[y_i(t)] < \text{var}[y_i(0)]$, so that the variance of cross-country income is falling. The is due to the variance of the random shocks to output, represented by the $\text{var}[u_i(t)]$ term in equation (14). Thus the effect of $\beta < 0$ or $\lambda > 0$, which tends to reduce the dispersion of income, can be offset by the random shocks to output, which tend to raise income dispersion.

If $\beta > 0$ then $\lambda < 0$. From equation (14), we can see that this means $\text{var}[y_i(t)]$ will be greater than $\text{var}[y_i(0)]$. In this case, the effect of $\beta < 0$ or $\lambda > 0$ is to increase income dispersion, and thus this works in the same direction as the random shocks which also tend to increase income dispersion.

(b) (i) Since y_i^* is time-invariant, analysis equivalent to that in part (a) (i) would yield

$$(15)\quad y_i(t) = (1 - e^{-\lambda t})(a + bX_i) + e^{-\lambda t} y_i(0),$$

where we have used the fact that $y_i^* = a + bX_i$.

(b) (ii) We will determine the value of λ implied by an estimate of β in this model and compare it to the value implied by using the formula from part (a) (ii). In the cross-country growth regression given by

$$(16)\quad y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \varepsilon_i,$$

again we have

(17) $\beta = \dfrac{\text{cov}[y_i(t), y_i(0)] - \text{var}[y_i(0)]}{\text{var}[y_i(0)]} = \dfrac{\text{cov}[y_i(t), y_i(0)]}{\text{var}[y_i(0)]} - 1.$

Then, since

(18) $y_i(t) = (1 - e^{-\lambda t})y_i^* + e^{-\lambda t}y_i(0) + e_i,$

we have

(19) $\text{cov}[y_i(t), y_i(0)] = (1 - e^{-\lambda t})\,\text{cov}[y_i^*, y_i(0)] + e^{-\lambda t}\,\text{var}[y_i(0)].$

Since

(20) $y_i(0) = y_i^* + u_i = a + bX_i + u_i$,

we have

(21) $\text{var}[y_i(0)] = b^2\,\text{var}[X_i] + \text{var}[u_i]$,

and

(22) $\text{cov}[y_i^*, y_i(0)] = \text{cov}[a + bX_i, a + bX_i + u_i] = b^2\,\text{var}[X_i]$,

since X_i and u_i are assumed to be uncorrelated. Substituting equations (21) and (22) into equation (19) gives us

(23) $\text{cov}[y_i(t), y_i(0)] = (1 - e^{-\lambda t})b^2\,\text{var}[X_i] + b^2 e^{-\lambda t}\,\text{var}[X_i] + e^{-\lambda t}\,\text{var}[u_i],$

or simply

(24) $\text{cov}[y_i(t), y_i(0)] = b^2\,\text{var}[X_i] + e^{-\lambda t}\,\text{var}[u_i].$

Substituting equations (21) and (24) into (17) gives us

(25) $\beta = \dfrac{b^2\,\text{var}[X_i] + e^{-\lambda t}\,\text{var}[u_i]}{b^2\,\text{var}[X_i] + \text{var}[u_i]} - 1 = \dfrac{-(1 - e^{-\lambda t})\,\text{var}[u_i]}{b^2\,\text{var}[X_i] + \text{var}[u_i]}.$

We can now solve for the value of λ implied by equation (25) and compare it to the one we would calculate if we used equation (13). Equation (25) implies

(26) $e^{-\lambda t} = 1 + \dfrac{b^2\,\text{var}[X_i] + \text{var}[u_i]}{\text{var}[u_i]}\beta.$

Taking the natural log of both sides of (26) and solving for λ gives us

(27) $\lambda = \dfrac{-\ln\left[1 + \dfrac{b^2\,\text{var}[X_i] + \text{var}[u_i]}{\text{var}[u_i]}\beta\right]}{t}.$

Since $(b^2\,\text{var}[X_i] + \text{var}[u_i])/\text{var}[u_i] > 1$, using the formula given by equation (13) would lead us to calculate an estimate for λ that is too small in absolute value. That is, if $\lambda > 0$, using the method of part (a) (ii) would yield an underestimate of the rate of convergence.

(b) (iii) Subtracting $y_i(0)$ from both sides of equation (18) gives us

(28) $y_i(t) - y_i(0) = (1 - e^{-\lambda t})y_i^* - (1 - e^{-\lambda t})y_i(0) + e_i.$

Substituting equation (20) into (28) yields

(29) $y_i(t) - y_i(0) = (1 - e^{-\lambda t})y_i^* - (1 - e^{-\lambda t})[y_i^* + u_i] + e_i,$

which simplifies to

(30) $y_i(t) - y_i(0) = (e^{-\lambda t} - 1)u_i + e_i.$

Defining $Q \equiv (e^{-\lambda t} - 1)$, we can see that the regression given by

(31) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \gamma X_i + \varepsilon_i$

is equivalent to projecting $Qu_i + e_i$ on a constant, $y_i(0)$, and X_i , where e_i is simply a mean-zero, random error that is uncorrelated with the right-hand side variables. Rearranging $y_i(0) = a + bX_i + u_i$ to solve for u_i gives us

(32) $u_i = -a + y_i(0) - bX_i$,
and so
(33) $Qu_i = -Qa + Qy_i(0) - QbX_i$.
Thus, in the regression given by (31), an estimate of β provides an estimate of Q and an estimate of γ provides an estimate of -Qb. Thus, we can construct an estimate of b by taking the negative of the estimate of γ, divided by the estimate of β, or

(34) $-\dfrac{\gamma}{\beta} = -\dfrac{-Qb}{Q} = b.$

SOLUTIONS TO CHAPTER 5

Problem 5.3

(a) The equations describing the evolution of technology are given by

(1) $\ln A_t = \overline{A} + gt + \tilde{A}_t$,

and

(2) $\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}$,

where $-1 < \rho_A < 1$. From equation (1) and letting $\ln A_0$ denote the value of $\ln A$ in period 0, we have $\ln A_0 = \overline{A} + g(0) + \tilde{A}_0$. Rearranging to solve for $\tilde{A}_0$ gives us

(3) $\tilde{A}_0 = \ln A_0 - \overline{A}$.

Using equations (1) and (2) we can write the following for period 1:

(4) $\ln A_1 = \overline{A} + g + \tilde{A}_1$,

and

(5) $\tilde{A}_1 = \rho_A \tilde{A}_0 + \varepsilon_{A,1}$.

Substituting equation (3) into equation (5) yields

(6) $\tilde{A}_1 = \rho_A \left(\ln A_0 - \overline{A}\right) + \varepsilon_{A,1}$.

Finally, substituting equation (6) into equation (4) gives us

(7) $\ln A_1 = \overline{A} + g + \rho_A \left(\ln A_0 - \overline{A}\right) + \varepsilon_{A,1}$.

Using equations (1) and (2), we can write the following for period 2:

(8) $\ln A_2 = \overline{A} + 2g + \tilde{A}_2$,

and

(9) $\tilde{A}_2 = \rho_A \tilde{A}_1 + \varepsilon_{A,2}$. Substituting equation (6) into equation (9) yields

(10) $\tilde{A}_2 = \rho_A \left[\rho_A \left(\ln A_0 - \overline{A}\right) + \varepsilon_{A,1}\right] + \varepsilon_{A,2} = \rho_A{}^2 \left(\ln A_0 - \overline{A}\right) + \rho_A \varepsilon_{A,1} + \varepsilon_{A,2}$.

Finally, substituting equation (10) into equation (8) gives us

(11) $\ln A_2 = \overline{A} + 2g + \rho_A{}^2 \left(\ln A_0 - \overline{A}\right) + \rho_A \varepsilon_{A,1} + \varepsilon_{A,2}$.

Again using equations (1) and (2), we can write the following for period 3:

(12) $\ln A_3 = \overline{A} + 3g + \tilde{A}_3$,

and

(13) $\tilde{A}_3 = \rho_A \tilde{A}_2 + \varepsilon_{A,3}$.

Substituting equation (10) into equation (13) yields

(14) $\tilde{A}_3 = \rho_A \left[\rho_A{}^2 \left(\ln A_0 - \overline{A}\right) + \rho_A \varepsilon_{A,1} + \varepsilon_{A,2}\right] + \varepsilon_{A,3} = \rho_A{}^3 \left(\ln A_0 - \overline{A}\right) + \rho_A{}^2 \varepsilon_{A,1} + \rho_A \varepsilon_{A,2} + \varepsilon_{A,3}$.

Finally, substituting equation (14) into equation (12) gives us

(15) $\ln A_3 = \overline{A} + 3g + \rho_A{}^3 \left(\ln A_0 - \overline{A}\right) + \rho_A{}^2 \varepsilon_{A,1} + \rho_A \varepsilon_{A,2} + \varepsilon_{A,3}$.

(b) Using equation (7) to find the expected value of $\ln A_1$ yields

(16) $E[\ln A_1] = \overline{A} + g + \rho_A \left(\ln A_0 - \overline{A}\right)$,

since $E[\varepsilon_{A,1}] = 0$.

Using equation (11) to find the expected value of $\ln A_2$ yields

(17) $E[\ln A_2] = \overline{A} + 2g + \rho_A{}^2 \left(\ln A_0 - \overline{A}\right)$,

since $E[\rho_A \varepsilon_{A,1}] = \rho_A E[\varepsilon_{A,1}] = 0$, $E[\varepsilon_{A,2}] = 0$.

Using equation (15) to find the expected value of $\ln A_3$ yields

(18) $E[\ln A_3] = \overline{A} + 3g + \rho_A{}^3\left(\ln A_0 - \overline{A}\right)$,

since $E[\rho_A{}^2\varepsilon_{A,1}] = \rho_A{}^2 E[\varepsilon_{A,1}] = 0$, $E[\rho_A\varepsilon_{A,2}] = \rho_A E[\varepsilon_{A,2}] = 0$, $E[\varepsilon_{A,3}] = 0$.

Problem 5.4

(a) We need to solve the household's one period problem assuming no initial wealth and normalizing the size of the household to one. Thus the problem is given by

(1) $\max\limits_{c,l} \ \ln c + b(1 - l)^{1-\gamma}/(1 - \gamma)$,

subject to the budget constraint,

(2) $c = wl$.

Set up the Lagrangian:

(3) $L = \ln c + b(1 - l)^{1-\gamma}/(1 - \gamma) + \lambda[wl - c]$.

The first-order conditions are

(4) $\partial L/\partial c = (1/c) - \lambda = 0$,

and

(5) $\partial L/\partial l = -b(1 - l)^{-\gamma} + \lambda w = 0$.

Substituting the budget constraint into equation (4) yields

(6) $\lambda = 1/c = 1/(wl)$.

Substituting equation (6) into equation (5) yields

(7) $-b(1 - l)^{-\gamma} + w/(wl) = 0$,

and simplifying gives us

(8) $1/l = b/(1 - l)^{\gamma}$.

Although labor supply, l, is only implicitly defined by equation (8), we can see that it will not depend upon the real wage.

(b) We want a formula for relative leisure in the two periods, $(1 - l_1)/(1 - l_2)$. Assume that the household lives for two periods, has no initial wealth, has size $N_t/H = 1$ for both periods and finally that there is no uncertainty. Thus the problem can be formalized as

(9) $$\max \ \ln c_1 + b\frac{(1-l_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\ln c_2 + e^{-\rho}b\frac{(1-l_2)^{1-\gamma}}{1-\gamma},$$

subject to the intertemporal budget constraint given by

(10) $$c_1 + \frac{c_2}{1+r} = w_1 l_1 + \frac{w_2 l_2}{1+r}.$$

Set up the Lagrangian:

(11) $$L = \ln c_1 + b\frac{(1-l_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\ln c_2 + e^{-\rho}b\frac{(1-l_2)^{1-\gamma}}{1-\gamma} + \lambda\left[w_1 l_1 + \frac{w_2 l_2}{1+r} - c_1 - \frac{c_2}{1+r}\right].$$

The first-order conditions are given by the following four equations:

(12) $\partial L/\partial c_1 = (1/c_1) - \lambda = 0$, and (13) $\partial L/\partial c_2 = (e^{-\rho}/c_2) - [\lambda/(1 + r)] = 0$,

(14) $\partial L/\partial l_1 = -b(1 - l_1)^{-\gamma} + \lambda l_1 = 0$, and (15) $\partial L/\partial l_2 = -e^{-\rho}b(1 - l_2)^{-\gamma} + [\lambda l_2/(1 + r)] = 0$.

Rearranging equation (14) yields one expression for λ : $\lambda = b(1 - l_1)^{-\gamma}/l_1$.

Rearranging (15) yields another expression for λ : $\lambda = [e^{-\rho}b(1 - l_2)^{-\gamma}(1 + r)]/l_2$. Equating these two expressions for λ yields

$$(16)\quad \frac{e^{-\rho}b(1+r)}{(1-l_2)^{\gamma}w_2} = \frac{b}{(1-l_1)^{\gamma}w_1} \Rightarrow \frac{(1-l_1)^{\gamma}}{(1-l_2)^{\gamma}} = \frac{1}{e^{-\rho}(1+r)}\frac{w_2}{w_1},$$

and thus

$$(17)\quad \frac{(1-l_1)}{(1-l_2)} = \left[\frac{1}{e^{-\rho}(1+r)}\frac{w_2}{w_1}\right]^{1/\gamma}.$$

If w_2/w_1 rises, then $(1 - l_1)/(1 - l_2)$ rises. That is, suppose the real wage in the second period rises relative to the real wage in the first period. Then the individual increases first-period leisure relative to second-period leisure, or reduces first-period labor supply relative to second-period labor supply. We can calculate the elasticity, denoting – for ease of notation – $(1 - l_1)/(1 - l_2) \equiv l^*$ and $w_2/w_1 \equiv w^*$:

$$(18)\quad \frac{\partial l^*}{\partial w^*}\frac{w^*}{l^*} = \frac{1}{\gamma}\frac{\left[1/e^{-\rho}(1+r)\right]^{1/\gamma} w^{*(1/\gamma)-1}\, w^*}{l^*}.$$

Substitute in the denominator for $l^* \equiv (1 - l_1)/(1 - l_2)$ from equation (17) to yield

$$(19)\quad \frac{\partial l^*}{\partial w^*}\frac{w^*}{l^*} = \frac{1}{\gamma}\frac{\left[1/e^{-\rho}(1+r)\right]^{1/\gamma} w^{*1/\gamma}}{\left[\left(1/e^{-\rho}(1+r)\right)w^*\right]^{1/\gamma}} = \frac{1}{\gamma}.$$

Thus the smaller is γ – or the bigger is $1/\gamma$ – the more the individual will adjust relative labor supply in response to a change in relative real wages.

From equation (17), we can also see that if r rises then $(1 - l_1)/(1 - l_2)$ falls. That is, suppose that there is a rise in the real interest rate. Then the individual reduces first-period leisure relative to second-period leisure, or increases first-period labor supply relative to second-period labor supply. It is straightforward to show that

$$(20)\quad \frac{\partial\left[(1-l_1)/(1-l_2)\right]}{\partial(1+r)}\frac{(1+r)}{\left[(1-l_1)/(1-l_2)\right]} = -\frac{1}{\gamma}.$$

Thus the smaller is γ – or the bigger is $1/\gamma$ – the more the individual will respond to a change in the real interest rate. Note that with log utility, where $\gamma = 1$, this elasticity is equal to one.

Intuitively, a low value of γ means that utility is not very sharply curved in l. This means that l responds a lot to changes in wages and the interest rate.

<u>**Problem 5.5**</u>

(a) The problem is to maximize utility as given by

(1) $\ln c_1 + b\ln(1 - l_1) + e^{-\rho}[\ln c_2 + b\ln(1 - l_2)]$,

subject to the following lifetime budget constraint:

$$(2)\quad c_1 + \frac{1}{1+r}c_2 = w_1 l_1 + \frac{1}{1+r}w_2 l_2$$

Set up the Lagrangian:

$$(3)\quad L = \ln c_1 + b\ln(1-l_1) + e^{-\rho}\left[\ln c_2 + b\ln(1-l_2)\right] + \lambda\left[w_1 l_1 + \frac{1}{1+r}w_2 l_2 - c_1 - \frac{1}{1+r}c_2\right].$$

The four first-order conditions are given by

(4) $\dfrac{\partial L}{\partial c_1} = \dfrac{1}{c_1} - \lambda = 0$;

(5) $\dfrac{\partial L}{\partial c_2} = \dfrac{e^{-\rho}}{c_2} - \dfrac{\lambda}{1+r} = 0$;

(6) $\dfrac{\partial L}{\partial l_1} = \dfrac{-b}{(1-l_1)} + \lambda w_1 = 0$; and

(7) $\dfrac{\partial L}{\partial l_2} = \dfrac{-e^{-\rho}b}{(1-l_2)} + \dfrac{\lambda w_2}{1+r} = 0$.

Equation (4) implies that

(8) $c_1 = \dfrac{1}{\lambda}$.

Equation (5) implies that

(9) $c_2 = \dfrac{e^{-\rho}(1+r)}{\lambda}$.

Equation (6) implies that

(10) $l_1 = 1 - \dfrac{b}{\lambda w_1}$.

Equation (7) implies that

(11) $l_2 = 1 - \dfrac{(1+r)e^{-\rho}b}{\lambda w_2}$.

Now substitute equations (8) - (11) into the lifetime budget constraint, equation (2), to obtain

(12) $\dfrac{1}{\lambda} + \dfrac{e^{-\rho}(1+r)}{\lambda(1+r)} = w_1\left[1 - \dfrac{b}{\lambda w_1}\right] + \dfrac{w_2}{1+r}\left[1 - \dfrac{(1+r)e^{-\rho}b}{\lambda w_2}\right]$.

Multiplying both sides of equation (12) by λ gives us

(13) $1 + e^{-\rho} = \lambda w_1\left[\dfrac{\lambda w_1 - b}{\lambda w_1}\right] + \dfrac{\lambda w_2}{1+r}\left[\dfrac{\lambda w_2 - (1+r)e^{-\rho}b}{\lambda w_2}\right]$.

Simplifying further allows us to obtain

(14) $1 + e^{-\rho} = \lambda w_1 - b + \dfrac{\lambda w_2}{1+r} - e^{-\rho}b$.

Now use the fact that $1 + e^{-\rho} + b + e^{-\rho}b = (1 + e^{-\rho})(1 + b)$ to solve for λ:

(15) $\lambda = \dfrac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]}$.

To obtain an expression for first-period labor supply, substitute equation (15) into equation (10) to obtain

(16) $l_1 = 1 - \dfrac{b}{\dfrac{(1+e^{-\rho})(1+b)}{\left[w_1 + w_2/(1+r)\right]}w_1} = 1 - \dfrac{b\left[w_1 + w_2/(1+r)\right]}{(1+e^{-\rho})(1+b)w_1}$.

Dividing the top and bottom of the second term on the right-hand side of (16) by w_1 yields

(17) $l_1 = 1 - \dfrac{b\left[1 + (w_2/w_1)(1/(1+r))\right]}{(1+e^{-\rho})(1+b)}$.

Note that l_1 is a function of the relative wage, w_2/w_1. Thus any change in w_1 and w_2 that leaves w_2/w_1 unchanged will leave l_1 unchanged.

To obtain an expression for second-period labor supply, substitute equation (15) into equation (11) to obtain

$$(18)\quad l_2 = 1 - \frac{(1+r)e^{-\rho}b}{\dfrac{(1+e^{-\rho})(1+b)}{[w_1 + w_2/(1+r)]}w_2} = 1 - \frac{(1+r)e^{-\rho}b[w_1 + w_2/(1+r)]}{(1+e^{-\rho})(1+b)w_2}.$$

Dividing the top and bottom of the second term on the right-hand side of (18) by w_2 yields

$$(19)\quad l_2 = 1 - \frac{(1+r)e^{-\rho}b[(w_1/w_2) + 1/(1+r)]}{(1+e^{-\rho})(1+b)}.$$

Again, note that l_2 depends only on the relative wage, w_1/w_2. Thus any change in w_1 and w_2 that leaves w_1/w_2 unchanged will leave l_2 unchanged.

(b) (i) The fact that the household has initial wealth of $Z > 0$ will not affect equation (5.23) in the text – the Euler equation – that relates consumption in one period to expectations of consumption the following period. The fact that the household has initial wealth does not change the marginal utility lost from reducing current consumption by a small amount today nor does it change the expected marginal utility gained by using the resulting greater wealth to increase consumption next period above what it otherwise would have been. That is, it does not affect the experiment by which we informally derived the Euler equation. The budget constraint, just as in the Ramsey model, only becomes important when determining the level of consumption each period.

(b) (ii) The result in part (a) will not continue to hold if the household has initial wealth. The new lifetime budget constraint is given by

$$(20)\quad c_1 + \frac{1}{1+r}c_2 = Z + w_1 l_1 + \frac{1}{1+r}w_2 l_2.$$

The addition of a constant to lifetime wealth does not affect the four first-order conditions. Take those first-order conditions, equations (8) through (11), and substitute them into this new budget constraint:

$$(21)\quad \frac{1}{\lambda} + \frac{e^{-\rho}(1+r)}{\lambda(1+r)} = Z + w_1\left[1 - \frac{b}{\lambda w_1}\right] + \frac{w_2}{1+r}\left[1 - \frac{(1+r)e^{-\rho}b}{\lambda w_2}\right].$$

Following the same algebra steps as in part (a) now yields

$$(22)\quad \lambda = \frac{(1+e^{-\rho})(1+b)}{[Z + w_1 + w_2/(1+r)]}.$$

To obtain an expression for first-period labor supply, substitute equation (22) into equation (10) to obtain

$$(23)\quad l_1 = 1 - \frac{b}{\dfrac{(1+e^{-\rho})(1+b)}{[Z + w_1 + w_2/(1+r)]}w_1} = 1 - \frac{b[Z + w_1 + w_2/(1+r)]}{(1+e^{-\rho})(1+b)w_1}.$$

Dividing the top and bottom of the second term on the right-hand side of (23) by w_1 yields

$$(24)\quad l_1 = 1 - \frac{b[(Z/w_1) + 1 + (w_2/w_1)(1/(1+r))]}{(1+e^{-\rho})(1+b)}.$$

Taking the derivative of l_1 with respect to w_1 – imposing the condition that w_2/w_1 remains constant – yields

$$(25)\quad \frac{\partial l_1}{\partial w_1} = \frac{bZ/w_1^2}{(1+e^{-\rho})(1+b)} > 0.$$

Thus a change in w_1, even if it is accompanied by a change in w_2 such that relative wages remain constant, does affect first-period labor supply. In fact, a rise in the first-period wage increases first-period labor supply.

To obtain an expression for second-period labor supply, substitute equation (22) into equation (11) to obtain

$$(26)\quad l_2 = 1 - \frac{(1+r)e^{-\rho}b}{\dfrac{(1+e^{-\rho})(1+b)}{[Z+w_1+w_2/(1+r)]}w_2} = 1 - \frac{(1+r)e^{-\rho}b[Z+w_1+w_2/(1+r)]}{(1+e^{-\rho})(1+b)w_2}.$$

Dividing the top and bottom of the second term on the right-hand side of (26) by w_2 yields

$$(27)\quad l_2 = 1 - \frac{(1+r)e^{-\rho}b[(Z/w_2)+(w_1/w_2)+1/(1+r)]}{(1+e^{-\rho})(1+b)}.$$

Taking the derivative of l_2 with respect to w_2 – imposing the condition that w_1/w_2 remains constant – yields

$$(28)\quad \frac{\partial l_2}{\partial w_2} = \frac{(1+r)e^{-\rho}bZ/w_2^2}{(1+e^{-\rho})(1+b)} > 0.$$

Thus a change in w_2, even if it is accompanied by a change in w_1 such that relative wages remain constant, does affect second-period labor supply. In fact, a rise in the second-period wage increases second-period labor supply.

Problem 5.6

(a) The real interest rate is potentially random, so let $r = E[r] + \varepsilon$ where ε is a mean-zero random error. The individual maximizes expected utility as given by

$$(1)\quad U = \ln C_1 + E[\ln C_2].$$

Substituting in for C_2 yields

$$(2)\quad U = \ln C_1 + E[\ln((1+E[r]+\varepsilon)(Y_1 - C_1))].$$

Set the derivative of equation (2) with respect to C_1 equal to zero to obtain the first-order condition:

$$(3)\quad \partial U/\partial C_1 = 1/C_1 + E[(-1)(1+E[r]+\varepsilon)/(1+E[r]+\varepsilon)(Y_1 - C_1)] = 0,$$

or simplifying

$$(4)\quad 1/C_1 - E[1/(Y_1 - C_1)] = 0.$$

Since $1/(Y_1 - C_1)$ is not random, $E[1/(Y_1 - C_1)] = 1/(Y_1 - C_1)$ and thus we can write

$$(5)\quad C_1 = Y_1/2.$$

In this case, the choice of C_1 is not affected by whether r is certain or not. Even if r is random, the individual simply consumes half of first-period income and saves the rest.

(b) Now the individual does not receive any first-period income but receives income Y_2 in period 2. So the individual's problem is to maximize expected utility as given by equation (1), subject to

$$(6)\quad C_1 = B_1,$$

and

(7) $C_2 = Y_2 - (1 + E[r] + \varepsilon)B_1 = Y_2 - (1 + E[r] + \varepsilon)C_1$,

where B_1 represents the amount of borrowing the individual does in the first period. Substituting (7) into the expected utility function (1) yields

(8) $U = \ln C_1 + E[\ln(Y_2 - (1 + E[r] + \varepsilon)C_1)]$.

Set the derivative of equation (8) with respect to C_1 equal to zero to find the first-order condition:

(9) $\partial U/\partial C_1 = 1/C_1 - E\left[(1 + E[r] + \varepsilon)/C_2\right] = 0$.

Use the formula for the expected value of the product of 2 random variables – E[XY] = E[X]E[Y] + cov(X,Y) – to obtain

(10) $1/C_1 = (1 + E[r])E[1/C_2] + \text{cov}\left(1 + E[r] + \varepsilon, 1/C_2\right)$.

The covariance term is positive. Intuitively, a higher ε means the individual has to pay more interest on her borrowing which forces her to have lower C_2 and thus higher $1/C_2$.

If r is not random – so that r = E[r] with certainty – we can rewrite equation (10) as

(11) $1/C_1 = (1 + E[r])(1/C_2) = (1 + E[r])/\left[Y_2 - (1 + E[r])C_1\right]$,

which implies that

(12) $Y_2 - (1 + E[r])C_1 = (1 + E[r])C_1$.

Solving for C_1 yields

(13) $C_1 = Y_2/2(1 + E[r])$.

In the case where r is random, equation (10) can be written as

(14) $1/C_1 = E[1 + E[r] + \varepsilon]\,E[1/C_2] + \text{cov}\left(1 + E[r] + \varepsilon, 1/C_2\right)$.

Since $1/C_2$ is a convex function of C_2 , then by Jensen's inequality we have $E[1/C_2] > 1/E[C_2]$. In addition, because the covariance term is positive, we can write

(15) $1/C_1 = (1 + E[r])E[1/C_2] + \text{cov}\left(1 + E[r] + \varepsilon, 1/C_2\right) > (1 + E[r])\left[1/E[C_2]\right]$.

Substituting into this inequality the fact that $E[C_2] = Y_2 - (1 + E[r])C_1$ yields

(16) $1/C_1 > (1 + E[r])/\left[Y_2 - (1 + E[r])C_1\right]$,

which implies that

(17) $Y_2 - (1 + E[r])C_1 > (1 + E[r])C_1$.

The inequality in (17) can be rewritten as

(18) $2(1 + E[r])C_1 < Y_2$

or simply

(19) $C_1 < Y_2/2(1 + E[r])$.

Note from equation (13) that the right-hand side of (19) is the optimal choice of C_1 under certainty. Thus we have shown that if r becomes random with no change in the expected value of r, the optimal choice of C_1 becomes smaller. Essentially, if there is some uncertainty about how much interest the individual will have to pay in the second period, she is more cautious in her decision as to how much to borrow and consume in the first period.

Problem 5.7

(a) Imagine the household increasing its labor supply per member in period t by a small amount Δl. Suppose it then uses the resulting greater wealth to allow less labor supply per member in the next period and allowing for consumption per member to be the same in both periods as it otherwise would have been. If the household is behaving optimally, a marginal change of this type must leave expected lifetime utility unchanged.

Household utility and the instantaneous utility function of the representative member of the household are given by

(1) $U = \sum_{t=0}^{t=\infty} e^{-\rho t} u(c_t, 1 - \ell_t) N_t / H$,

and

(2) $u_t = \ln c_t + b\ln(1 - \ell_t)$.

From equations (1) and (2), the marginal disutility of working in period t is given by

(3) $-\partial U/\partial \ell_t = e^{-\rho t}(N_t/H)[b/(1 - \ell_t)]$.

Thus increasing labor supply per member by $\Delta\ell$ has the following utility cost for the household:

(4) Utility Cost = $e^{-\rho t}(N_t/H)[b/(1 - \ell_t)]\Delta\ell$.

This change in labor supply raises income per member in period t by $w_t\Delta\ell$. Note that the household has e^n times as many members in period t + 1 as in period t. Thus the increase in wealth per member in period t + 1 is $e^{-n}[(1 + r_{t+1})w_t\Delta\ell]$.

We need to determine how much this change in labor supply in period t allows labor supply per member in period t + 1 to fall, if the path of consumption is to be unaffected. In period t + 1, giving up one unit of labor per member costs w_{t+1} in lost income per member. Thus giving up $1/w_{t+1}$ units of labor per member implies lost income of one per member. Or, giving up $[e^{-n}(1 + r_{t+1})w_t\Delta\ell]/w_{t+1}$ units of labor results in lost income per member of $e^{-n}(1 + r_{t+1})w_t\Delta\ell$. That is exactly equal to the extra wealth per member the household has from working more last period. Thus we have determined that labor supply per member can fall below what it otherwise would have been by the amount $[e^{-n}(1 + r_{t+1})w_t\Delta\ell]/w_{t+1}$ while still allowing consumption to be the same as it otherwise would have been. This allowable drop in labor supply per member results in the following expected utility benefit as of period t:

(5) Expected Utility Benefit = $E_t\left[e^{-\rho(t+1)}\frac{N_{t+1}}{H}\frac{b}{(1-\ell_{t+1})}\frac{e^{-n}(1+r_{t+1})w_t\Delta\ell}{w_{t+1}}\right]$.

Equating the cost and expected benefit yields

(6) $e^{-\rho t}\frac{N_t}{H}\frac{b}{(1-\ell_t)}\Delta\ell = E_t\left[e^{-\rho(t+1)}\frac{N_{t+1}}{H}\frac{b}{(1-\ell_{t+1})}\frac{e^{-n}(1+r_{t+1})w_t\Delta\ell}{w_{t+1}}\right]$.

Since $e^{-\rho(t+1)}(N_{t+1}/H)e^{-n}$ is not uncertain and since $N_{t+1} = N_t e^n$, this simplifies to

(7) $\frac{b}{(1-\ell_t)} = e^{-\rho}E_t\left[\frac{b(1+r_{t+1})w_t}{(1-\ell_{t+1})w_{t+1}}\right]$.

(b) Consider the household in period t. Suppose it reduces its current consumption per member by a small amount Δc and then uses the resulting greater wealth to increase consumption per member in the next period above what it otherwise would have been. The following equation, (5.23) in the text, gives the condition this experiment implies, assuming the household is behaving optimally:

(5.23) $\frac{1}{c_t} = e^{-\rho}E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right]$.

Now imagine the household increasing its labor supply per member in period t by a small amount $\Delta\ell$ and using the resulting income to increase its consumption in that period. The following equation, (5.26) in the text, gives the condition that this experiment implies, assuming that the household is behaving optimally:

(5.26) $\dfrac{c_t}{1-l_t} = \dfrac{w_t}{b}$.

Solving for $1/c_t$ gives us

(8) $\dfrac{1}{c_t} = \dfrac{b}{(1-l_t)w_t}$.

Note that equations (5.26) and (8) hold in every period. Thus for period t + 1, we can write

(9) $\dfrac{1}{c_{t+1}} = \dfrac{b}{(1-l_{t+1})w_{t+1}}$.

Substituting equations (8) and (9) into equation (5.23) yields

(10) $\dfrac{b}{(1-l_t)w_t} = e^{-\rho}E_t\left[\dfrac{b(1+r_{t+1})}{(1-l_{t+1})w_{t+1}}\right]$.

Multiplying both sides of equation (10) by w_t , and using the fact that $E_t[w_t] = w_t$ gives us

(11) $\dfrac{b}{(1-l_t)} = e^{-\rho}E_t\left[\dfrac{b(1+r_{t+1})w_t}{(1-l_{t+1})w_{t+1}}\right]$.

This is the same condition obtained from the experiment in part (a).

Problem 5.8

(a) To obtain the first-order condition or Euler equation, we can use the informal perturbation method. The experiment is to suppose the individual reduces period-t consumption by ΔC. She then uses the resulting greater wealth in period t + 1 to increase consumption above what it otherwise would have been. The utility cost in period t of doing so is given by the following expression:

(1) Utility Cost = $[1/(1+\rho)]^t u'(C_t)\Delta C = [1/(1+\rho)]^t[1-2\theta C_t]\Delta C$,

where we have used the instantaneous utility function, $u(C_t) = C_t - \theta C_t^2$, to calculate $u'(C_t)$.

The following expression gives the expected utility gain in period t + 1 from the above experiment:

(2) Exp. Utility Gain = $E_t\left[(1/(1+\rho))^{t+1}u'(C_{t+1})(1+A)\Delta C\right] = [1/(1+\rho)]^{t+1}E_t[1-2\theta C_{t+1}](1+A)\Delta C$,

where A is the real interest rate. Equation (2) simplifies to

(3) Exp. Utility Gain = $[1/(1+\rho)]^{t+1}[1-2\theta E_t[C_{t+1}]](1+A)\Delta C$.

If the individual is optimizing, the utility cost from this perturbation must equal the expected utility gain:

(4) $[1/(1+\rho)]^t[1-2\theta C_t]\Delta C = [1/(1+\rho)]^{t+1}[1-2\theta E_t[C_{t+1}]](1+A)\Delta C$,

or simply

(5) $1-2\theta C_t = [1/(1+\rho)](1+A)[1-2\theta E_t[C_{t+1}]]$.

Using the fact that $\rho = A$ and simplifying yields

(6) $C_t = E_t[C_{t+1}]$.

Consumption follows a random walk. The expected value of consumption next period is simply equal to today's actual realization of consumption.

(b) We will guess that consumption takes the form

(7) $C_t = \alpha + \beta K_t + \gamma e_t$.

Substitute equation (7) and the production function, $Y_t = AK_t + e_t$, into the capital-accumulation equation, $K_{t+1} = K_t + Y_t - C_t$, to obtain

(8) $K_{t+1} = K_t + AK_t + e_t - \alpha - \beta K_t - \gamma e_t$,

or simply

(9) $K_{t+1} = -\alpha + \left(1 + A - \beta\right)K_t + (1-\gamma)e_t$.

(c) Substitute equation (7) and equation (7) lagged forward one period into the first-order condition, equation (1):

(10) $\alpha + \beta K_t + \gamma e_t = E_t\left[\alpha + \beta K_{t+1} + \gamma e_{t+1}\right]$.

Substituting equation (9) into equation (10) yields

(11) $\alpha + \beta K_t + \gamma e_t = \alpha + \beta E_t\left[-\alpha + (1 + A - \beta)K_t + (1-\gamma)e_t\right] + \gamma E_t\left[e_{t+1}\right]$.

Noting that $E_t\left[e_{t+1}\right] = E_t\left[\phi e_t + \varepsilon_{t+1}\right] = \phi e_t$, we can collect terms to obtain

(12) $\alpha + \beta K_t + \gamma e_t = \alpha(1-\beta) + \beta(1 + A - \beta)K_t + \left[\beta + \gamma(\phi - \beta)\right]e_t$.

In order for equation (12) to hold, we need the coefficients on K_t and e_t , as well as the constant term, to be the same on both sides. Equating the coefficients on K_t gives us

(13) $\beta = \beta(1 + A - \beta)$,

or simply

(14) $\beta = A$.

Equating the coefficients on e_t gives us

(15) $\gamma = \beta + \gamma(\phi - \beta)$.

Using equation (14) and simplifying yields

(16) $\gamma(1 - \phi + A) = A$,

or simply

(17) $\gamma = \dfrac{A}{1 - \phi + A}$.

Finally, equating the constant terms yields

(18) $\alpha = \alpha(1-\beta)$.

Unless $\beta = A = 1$, this requires

(19) $\alpha = 0$.

Note that we are also ignoring the case in which $\beta = 0$ and $\gamma = 0$ with no restriction on α.

(d) Substituting equations (14), (17), and (19) into the guess for consumption, equation (7), and the capital-accumulation equation, equation (9), yields

(20) $C_t = AK_t + \left(\dfrac{A}{1 - \phi + A}\right)e_t$,

and

(21) $K_{t+1} = K_t + \left(\dfrac{1 - \phi}{1 - \phi + A}\right)e_t$.

To keep the analysis simple, and without loss of generality, we can assume that ε, and thus e, both equal 0 until some period t. In period t, there is a one-time, positive realization of $\varepsilon_t = 1 - \phi + A$. From period t + 1 forward, $\varepsilon = 0$ again. In what follows, the change in a variable refers to the difference between its actual value and the value it would have had in the absence of the one-time shock (i.e. if ε and e had remained at 0 forever).

In period t, K_t is unaffected. From equation (21), we can see that K_t is determined by last period's capital stock and last period's realization of e. From the production function, $Y_t = AK_t + e_t$, we have

(22) $\Delta Y_t = A\Delta K_t + \Delta e_t = 0 + (1 - \phi + A)$.

Thus output in the period of the shock is higher by $(1-\phi+A)$. From equation (20), the change in consumption is given by

(23) $\Delta C_t = A\Delta K_t + \left(\frac{A}{1-\phi+A}\right)\Delta e_t = 0 + \left(\frac{A}{1-\phi+A}\right)(1-\phi+A) = A$.

Thus consumption in the period of the shock is higher by A.

In period t + 1, even though ε_{t+1} is assumed to be 0 again, e_{t+1} is different than it would have been in the absence of the one-time shock due to the autoregressive form of the e terms. More precisely

(24) $\Delta e_{t+1} = \phi\Delta e_t = \phi(1-\phi+A)$.

From equation (21), the change in the capital stock is given by

(25) $\Delta K_{t+1} = \Delta K_t + \left(\frac{1-\phi}{1-\phi+A}\right)\Delta e_t = 0 + \left(\frac{1-\phi}{1-\phi+A}\right)(1-\phi+A) = (1-\phi)$.

Intuitively, last period, output rose by $(1-\phi+A)$ but consumption rose only by A. The rest of the increase in output, $(1-\phi)$, was devoted to investment and hence the rise in this period's capital stock by an equal amount (we are assuming no depreciation). From the production function, $Y_{t+1} = AK_{t+1} + e_{t+1}$, the change in output is

(26) $\Delta Y_{t+1} = A\Delta K_{t+1} + \Delta e_{t+1} = A(1-\phi) + \phi(1-\phi) + A = A - \phi A + \phi + \phi A - \phi^2 = A + \phi(1-\phi)$.

From equation (20), the change in consumption is given by

(27) $\Delta C_{t+1} = A\Delta K_{t+1} + \left(\frac{A}{1-\phi+A}\right)\Delta e_{t+1} = A(1-\phi) + \left(\frac{A}{1-\phi+A}\right)\phi(1-\phi+A) = A - \phi A + \phi A = A$.

Thus there are no further dynamics for consumption. It remains higher than it would have been in the absence of the shock by the amount A.

Similarly, we can calculate these changes for period t+2:

(28) $\Delta e_{t+2} = \phi\Delta e_{t+1} = \phi^2(1-\phi+A)$;

(29) $\Delta K_{t+2} = \Delta K_{t+1} + \left(\frac{1-\phi}{1-\phi+A}\right)\Delta e_{t+1} = (1-\phi) + \left[\frac{(1-\phi)\phi(1-\phi+A)}{1-\phi+A}\right] = (1-\phi) + \phi(1-\phi) = 1-\phi^2$;

(30) $\Delta Y_{t+2} = A\Delta K_{t+2} + \Delta e_{t+2} = A(1-\phi) + A\phi(1-\phi) + \phi^2(1-\phi+A) = A + \phi^2(1-\phi)$; and

(31) $\Delta C_{t+2} = A\Delta K_{t+2} + \left(\frac{A}{1-\phi+A}\right)\Delta e_{t+2} = A(1-\phi)^2 + \left(\frac{A}{1-\phi+A}\right)\phi^2(1-\phi+A) = A - \phi^2 A + \phi^2 A = A$.

The pattern can now be inferred. Suppose there is a one-time shock of $\varepsilon_t = 1-\phi+A$. In the period of the shock, consumption rises by A and permanently stays at that new level with no further dynamics. In addition, n periods after the shock, the change in output is

(32) $\Delta Y_{t+n} = A + \phi^n(1-\phi)$,

and the change in the capital stock is

(33) $\Delta K_{t+n} = 1-\phi^n$.

The nature of the dynamics of Y and K depends on the value of ϕ. In the special case in which ϕ is equal to 0, so that there is no persistence in the technology shock, there are no further dynamics after period t + 1. The period after the shock, and in all those thereafter, capital is higher by one and output is higher by A.

For the case of $0 < \phi < 1$, the capital stock rises by $(1-\phi)$ the period after the shock. It then increases more each period until it asymptotically approaches its new long-run level that is one higher than it would have been in the absence of the shock. Output rises by $(1-\phi+A)$ the period of the shock. It then decreases each period until it asymptotically approaches its new long-run level that is A higher than it would have been in the absence of the shock.

For the case of $-1 < \phi < 0$, capital and output oscillate – alternating above and below their new long-run levels in successive periods – and gradually settle down to be one and A higher, respectively.

Problem 5.9

(a) To obtain the first-order condition or Euler equation, we can use the informal perturbation method. The experiment is to suppose the individual reduces period-t consumption by ΔC. She then uses the resulting greater wealth in period t + 1 to increase consumption above what it otherwise would have been. The utility cost in period t of doing so is given by the following expression:

(1) Utility Cost = $[1/(1+\rho)]^t u'(C_t)\Delta C = [1/(1+\rho)]^t [1-2\theta(C_t+\nu_t)]\Delta C$,

where we have used the instantaneous utility function, $u(C_t) = C_t - \theta(C_t + \nu_t)^2$, to calculate $u'(C_t)$.

The expected utility gain in period t + 1 from the above experiment is given by the following:

(2) Expected Utility Gain = $E_t\left[(1/(1+\rho))^{t+1} u'(C_{t+1})(1+A)\Delta C\right]$

$$= [1/(1+\rho)]^{t+1} E_t[1-2\theta(C_{t+1}+\nu_{t+1})](1+A)\Delta C,$$

where A is the real interest rate. Since ν is white noise, $E_t[\nu_{t+1}] = 0$, and thus

(3) Expected Utility Gain = $[1/(1+\rho)]^{t+1}[1-2\theta E_t[C_{t+1}]](1+A)\Delta C$.

If the individual is optimizing, the utility cost from this perturbation must equal the expected utility gain:

(4) $[1/(1+\rho)]^t[1-2\theta(C_t+\nu_t)]\Delta C = [1/(1+\rho)]^{t+1}[1-2\theta E_t[C_{t+1}]](1+A)\Delta C$,

or simply

(5) $1-2\theta(C_t+\nu_t) = [1/(1+\rho)](1+A)[1-2\theta E_t[C_{t+1}]]$.

Using the fact that $\rho = A$ and simplifying yields

(6) $C_t + \nu_t = E_t[C_{t+1}]$.

(b) We will guess that consumption takes the form

(7) $C_t = \alpha + \beta K_t + \gamma \nu_t$.

Substitute equation (7) and the production function, $Y_t = AK_t$, into the capital-accumulation equation, $K_{t+1} = K_t + Y_t - C_t$, to obtain

(8) $K_{t+1} = K_t + AK_t - \alpha - \beta K_t - \gamma \nu_t$,

or simply

(9) $K_{t+1} = -\alpha + (1+A-\beta)K_t - \gamma\nu_t$.

(c) Substitute equation (7) and equation (7) lagged forward one period into the first-order condition, equation (6):

(10) $\alpha + \beta K_t + \gamma\nu_t + \nu_t = E_t[\alpha + \beta K_{t+1} + \gamma\nu_{t+1}]$.

Noting that $E_t[V_{t+1}] = 0$, we have

(11) $\alpha + \beta K_t + (\gamma+1)\nu_t = \alpha + \beta E_t[K_{t+1}]$.

Substitute equation (9) into equation (11). Since K_{t+1} is a function of K_t and ν_t which are both known at time t, we have

(12) $\alpha + \beta K_t + (\gamma + 1)\nu_t = \alpha + \beta\left[-\alpha + \left(1 + A - \beta\right)K_t - \gamma\nu_t\right]$.

Simplifying yields

(13) $\alpha + \beta K_t + (\gamma + 1)\nu_t = \alpha\left(1 - \beta\right) + \beta\left(1 + A - \beta\right)K_t - \beta\gamma\nu_t$.

In order for equation (13) to hold, we need the coefficients on K_t , ν_t ,and the constant term to be the same on both sides. That is, we need

(14) $\beta = \beta\left(1 + A - \beta\right)$,

which implies that

(15) $\beta = A$.

In addition, we require that

(16) $\gamma + 1 = -\beta\gamma$,

or simply

(17) $\gamma = -1/(1 + \beta)$.

Substituting equation (15) into equation (17) gives us

(18) $\gamma = -1/(1 + A)$.

Finally, we require that

(19) $\alpha\left(1 - \beta\right) = \alpha$.

If β does not equal 0, this implies that

(20) $\alpha = 0$.

There is another set of parameter values that satisfies equation (13): $\beta = 0$ and $\gamma = -1$ with no restriction on α. This second solution is economically unappealing, however, since $\beta = 0$ implies that consumption does not depend on the capital stock. This is not realistic since consumption depends on output which in turn is determined by the capital stock. Thus we can, on economic grounds, ignore this second solution.

(d) Substituting equations (15), (18) and (20) into the guess for consumption, equation (7), and the capital-accumulation equation, equation (9), yields

(21) $C_t = AK_t - \left[1/(1 + A)\right]\nu_t$,

and

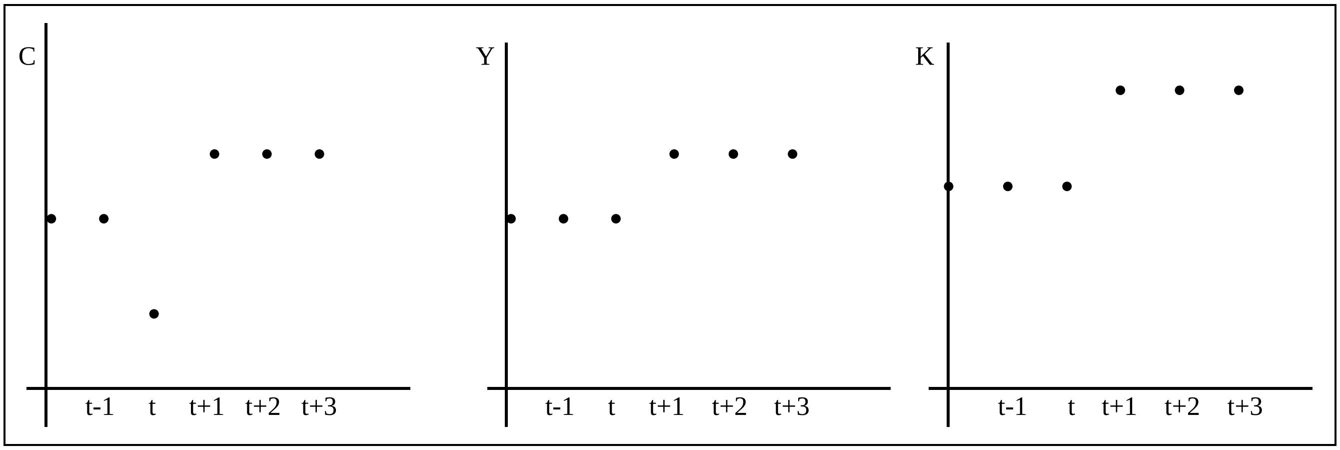
(22) $K_{t+1} = K_t + \left[1/(1 + A)\right]\nu_t$.

Without loss of generality, we can assume that $\nu = 0$ until some period t when there is a one-time positive realization of ν_t . To keep the analysis simple, assume that $\nu_t = (1 + A)$. From period t + 1 forward, $\nu = 0$ again.

In period t, K_t is unaffected. It is determined by last period's capital stock and last period's saving. From the production function, $Y_t = AK_t$, Y_t is unaffected since K_t is unaffected. From equation (21), we can see that consumption in period t, C_t , is lower by $\left[1/(1 + A)\right]\nu_t = \left[1/(1 + A)\right](1 + A) = 1$.

In period t + 1, we can see from equation (22) that K_{t+1} is higher by $\left[1/(1 + A)\right]\nu_t = \left[1/(1 + A)\right](1 + A) = 1$. Intuitively, last period's drop in consumption by one, with unchanged output, meant an increase in saving of one. This in turn means an increase in this period's capital stock by one. Through the production function, since K_{t+1} is higher by one, output is higher by A. Finally, with ν_{t+1} assumed to be 0, then since K_{t+1} is higher by one, C_{t+1} must be higher than it was in period t - 1 (before the shock) by A. This last fact can be seen from equation (21).

From period t + 2 forward, assuming $\nu = 0$ forever, there will be no further dynamics. K stays at its new higher level: one higher than in period t - 1. Y stays at its new higher level: A higher than in period t - 1. C stays at its new higher level: A higher than in period t - 1. All of this is depicted in the figure below.



Problem 5.10

(a) From the Solow, Ramsey, and Diamond models it is clear that on the balanced growth path without shocks, the growth rates of Y, K and C are all equal to n + g. In addition, the growth rate of w is g, the growth rate of L is n, and the growth rates of ℓ and r are zero. Given the logarithmic structure here, "growth rate" means the change in the logarithm of the variable; the fact that the growth rate of K is n + g means that $\ln(K_{t+1}) - \ln(K_t) = n + g$.

Dividing both sides of the production function, $Y_t = K_t^{\alpha}[A_tL_t]^{1-\alpha}$, by A_tL_t yields

(1) $Y_t/A_tL_t = K_t^{\alpha}[A_tL_t]^{-\alpha} = [K_t/A_tL_t]^{\alpha}$.

Since y* and k* are the balanced-growth-path values of Y/AL and K/AL respectively, we have

(2) $y^* = k^{*\alpha}$.

Similarly, dividing both sides of the capital-accumulation equation, $K_{t+1} = K_t + Y_t - C_t - G_t - \delta K_t$, by A_tL_t gives us

(3) $\dfrac{K_{t+1}}{A_tL_t} = \dfrac{K_t}{A_tL_t} + \dfrac{Y_t}{A_tL_t} - \dfrac{C_t}{A_tL_t} - \dfrac{G_t}{A_tL_t} - \dfrac{\delta K_t}{A_tL_t}$.

Using the notation given in the question and the fact that $K_{t+1} = e^n e^g K_t$ on the balanced growth path and thus that $K_{t+1}/A_tL_t = e^n e^g K_t/A_tL_t$ yields

(4) $e^n e^g k^* = k^* + y^* - c^* - G^* - \delta k^*$.

Dividing both sides of the equation giving the real wage, $w_t = (1 - \alpha)[K_t/A_tL_t]^{\alpha}A_t$, by A_t yields

(5) $w_t/A_t = (1 - \alpha)[K_t/A_tL_t]^{\alpha}$.

Denoting the value of w/A on the balanced growth path as w* gives us

(6) $w^* = (1 - \alpha)k^{*\alpha}$.

From equation (5.4) in the text giving the real interest rate, we have on a balanced growth path

(7) $r^* = \alpha k^{*-(1-\alpha)} - \delta$.

We need to transform textbook equation (5.26), which relates the trade-off between current consumption and current labor supply, into an expression concerning the balanced growth path without shocks. Note that in equation (5.26), $c_t/(1 - \ell_t) = w_t/b$, c is consumption per person, C/N. We are interested in c* which is consumption per unit of effective labor, C/AL. Since C/N = (C/AL)(L/N)A, on the balanced growth path it is true that $c = c^*\ell^*A$. Using this fact and dividing both sides of equation (5.26) by A, we obtain

(8) $\dfrac{c^* l^* A/A}{(1-l^*)} = \dfrac{w/A}{b}$.

Since w* = W/A, we have

(9) $\dfrac{c^* l^*}{(1-l^*)} = \dfrac{w^*}{b}$.

Finally, we need to transform textbook equation (5.23), which relates the tradeoff between current and future consumption. First, since there is no uncertainty without any shocks, we can write equation (5.23), $1/c_t = e^{-\rho}E_t\,[(1 + r_{t+1})/c_{t+1}]$, as

(10) $1/c_t = e^{-\rho}\,(1 + r_{t+1})/c_{t+1}$.

Then multiply both sides of equation (10) by c_{t+1}:

(11) $c_{t+1}/c_t = e^{-\rho}(1 + r_{t+1})$.

On the balanced growth path, consumption per person grows at rate g and thus $c_{t+1} = c_t\,e^g$ or $c_{t+1}/c_t = e^g$. Thus we have

(12) $1 + r^* = e^{\rho+g}$.

We now have six equations in the following six variables: y*, k*, c*, w*, l*, and r*.

(b) We need to assume the following parameter values: $\alpha = 1/3$, g = 0.005, n = 0.0025, $\delta = 0.025$, r* = 0.015, and l* = 1/3. Note that these are quarterly values for n, g and r*.

From equation (7), we can obtain an expression for capital per unit of effective labor on the balanced growth path, k*:

(13) $k^* = [\alpha/(r^* + \delta)]^{1/(1-\alpha)}$.

Substituting the given parameter values yields

(14) $k^* = [(1/3)/(0.015 + 0.025)]^{1/(1-1/3)} = 24.0563$.

Substituting this value for k* into equation (2) gives us a value for quarterly output per unit of effective labor on the balanced growth path:

(15) $y^* = k^{*\alpha} = (24.0563)^{1/3} = 2.8868$.

We are told that the ratio of government purchases to output on the balanced growth path is (G/Y)* = 0.2. This means that:

(16) [G/AL]/[Y/AL] = 0.2

and substituting for the value of y* calculated above gives us

(17) $G/AL \equiv G^* = (0.2)(2.8868) = 0.5774$.

From equation (4), we can solve for consumption per unit of effective labor on the balanced growth path, c*:

(18) $c^* = k^* + y^* - G^* - \delta k^* - e^n e^g k^* = (1 - \delta - e^n e^g)k^* + y^* - G^*$.

Substituting for the values we know yields

(19) $c^* = (1 - 0.025 - e^{0.0025}e^{0.005})(24.0563) + 2.8868 - 0.5774 = 1.5269$.

It is then straightforward to use these values for c* and y* to solve for the share of output devoted to consumption on the balanced growth path:

(20) $C/Y = [C/AL]/[Y/AL] \equiv c^*/y^* = 1.5269/2.8868 = 0.5289$.

Thus consumption's share in output is approximately 53 percent. Since output is devoted to consumption, investment, or government purchases, we know that

(21) I/Y = 1 - C/Y - G/Y = 1 - 0.5289 - 0.2 = 0.2711,

and thus investment's share in output is roughly 27%. Compared to actual figures for the U.S. this is giving slightly too much weight to investment and slightly too little weight to consumption. Finally, the implied ratio of capital to annual output on the balanced growth path is

(22) $K/4Y = [K/AL]/[4Y/AL] \equiv k^*/4y^* = 24.0563/[(4)(2.8868)] = 2.083$.

Problem 5.11
Before invoking the simplifying assumptions, the model here is the RBC model with no government and 100-percent depreciation, given by
(1) $Y_t = K_t^{\alpha}[A_t L_t]^{1-\alpha}$, (2) $K_{t+1} = Y_t - C_t$, (3) $\ln A_t = \overline{A} + gt + \tilde{A}_t$,
(4) $\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}$, (5) $\ln N_t = \overline{N} + nt$, and (6) $u_t = \ln c_t + b\ln(1 - l_t)$.
In this question, we are simplifying by assuming $n = g = \overline{A} = \overline{N} = 0$. This results in the following adjustments to the model. Population is given by
(7) $\ln N_t = 0$,
which implies that
(8) $N_t = 1$.
Since we have normalized the population to one, labor supply per person, l_t, will be the same as total labor supply, L_t. Thus we can rewrite the production function as
(9) $Y_t = K_t^{\alpha}[A_t l_t]^{1-\alpha}$.
With respect to technology, since g and $\overline{A}$ are equal to 0, we have $\ln A_t = \tilde{A}_t$. Using equation (4) to rewrite this yields
(10) $\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_{A,t}$.

(a) Define the value function at time t as

$$(11)\quad V_t = \max E_t\left[\sum_{s=t}^{\infty} e^{-\rho(s-t)}\left[\ln C_s + b\ln(1 - l_s)\right]\right].$$

Since we are solving the social planner's problem, the maximization is subject to the constraints given by the production function, equation (9), the capital-accumulation equation (2), and the technology equation (10). Thus the value function at time t is the expected present value of lifetime utility, from time t forward, evaluated at all the optimal choices of consumption and labor supply. The technique we will use here allows us to reduce what looks like a complicated multiperiod problem down to a two-period problem. That is because the value function must satisfy Bellman's Equation given by

$$(12)\quad V_t(K_t, A_t) = \max_{C,l}\left\{[\ln C_t + b\ln(1 - l_t)] + e^{-\rho}E_t\left[V_{t+1}(K_{t+1}, A_{t+1})\right]\right\}.$$

Equation (12) says that the value function at time t is equal to utility at time t, evaluated at the optimal C_t and l_t, plus the discounted expected value as of time t of next period's value function. That is, the expected value of maximized lifetime utility is maximized lifetime utility "today" plus "today's" expectation of maximized lifetime utility from "tomorrow" on, appropriately discounted.

(b) We will guess that the value function is of the form
(13) $V_t(K_t, A_t) = \beta_0 + \beta_K \ln K_t + \beta_A \ln A_t$.
Substituting this guess into equation (12), the Bellman equation, yields

$$(14)\quad V_t(K_t, A_t) = \max_{C,l}\left\{[\ln C_t + b\ln(1 - l_t)] + e^{-\rho}E_t\left[\beta_0 + \beta_K \ln K_{t+1} + \beta_A \ln A_{t+1}\right]\right\}.$$

Taking logs and then expectations of both sides of equation (2), the capital-accumulation equation, yields
(15) $E_t[\ln K_{t+1}] = E_t[\ln(Y_t - C_t)] = \ln(Y_t - C_t)$,
since Y_t and C_t are both known as of time t. Taking the expected value of both sides of equation (10) yields
(16) $E_t[\ln A_{t+1}] = \rho_A \ln A_t$,
since the ε shocks have mean zero. Substituting equations (15) and (16) into equation (12) yields

$$(17)\quad V_t(K_t, A_t) = \max_{C,l}\left\{[\ln C_t + b\ln(1 - l_t)] + e^{-\rho}\left[\beta_0 + \beta_K \ln(Y_t - C_t) + \beta_A \rho_A \ln A_t\right]\right\}.$$

The first-order condition for C_t is

(18) $0 = 1/C_t + e^{-\rho}\beta_K(-1)/(Y_t - C_t)$,

which implies that

(19) $1/C_t = e^{-\rho}\beta_K/(Y_t - C_t)$.

Equation (19) can be rewritten as

(20) $Y_t - C_t = e^{-\rho}\beta_K C_t$,

or simply

(21) $C_t(1 + e^{-\rho}\beta_K) = Y_t$.

Thus consumption is given by

(22) $C_t = [1/(1 + e^{-\rho}\beta_K)]Y_t$.

The ratio of consumption to output is given by

(23) $C_t/Y_t = 1/(1 + e^{-\rho}\beta_K)$.

Equation (23) shows that the ratio of consumption to output does not depend on K_t or A_t.

(c) The first-order condition for l_t (noting that $L_t = l_t$) is

(24) $0 = -b/(1 - l_t) + [e^{-\rho}\beta_K/(Y_t - C_t)](1 - \alpha)K_t^{\alpha}A_t^{1-\alpha}l_t^{-\alpha}$.

Simplifying yields

(25) $b/(1 - l_t) = [e^{-\rho}\beta_K/(Y_t - C_t)](1 - \alpha)(Y_t/l_t)$.

Substituting equation (20) into equation (25) yields

(26) $b/(1 - l_t) = [e^{-\rho}\beta_K/e^{-\rho}\beta_K C_t](1 - \alpha)(Y_t/l_t) = (Y_t/C_t)[(1 - \alpha)/l_t]$.

Substituting equation (23) into equation (26) yields

(27) $b/(1 - l_t) = (1 - \alpha)(1 + e^{-\rho}\beta_K)/l_t$.

Multiplying both sides of equation (27) by $(1 - l_t)l_t$ gives us

(28) $b\,l_t = (1 - l_t)(1 - \alpha)(1 + e^{-\rho}\beta_K)$.

Further simplification allows us to obtain

(29) $l_t[(1 - \alpha)(1 + e^{-\rho}\beta_K) + b] = (1 - \alpha)(1 + e^{-\rho}\beta_K)$,

and thus

(30) $$l_t = \frac{(1-\alpha)}{(1-\alpha) + [b/(1 + e^{-\rho}\beta_K)]}\,.$$

Thus l_t, labor supply per person, does not depend on K_t or A_t either. In addition, with some simple algebra, it is possible to solve for optimal leisure, an expression which will be useful later on:

(31) $(1 - l_t) = b/[(1-\alpha)(1 + e^{-\rho}\beta_K) + b]$.

(d) Now take these optimal choices of consumption and leisure, as well as the production function, and substitute them all into the value function. It will turn out that the original guess that the value function is loglinear in capital and technology is valid.

Formally, substitute equations (20), (22) and (31) into equation (17) to obtain

(32) $$\begin{aligned} &V_t(K_t, A_t) = \ln[Y_t/(1 + e^{-\rho}\beta_K)] + b\ln\{b/[(1-\alpha)(1 + e^{-\rho}\beta_K) + b]\} + \\ &e^{-\rho}\{\beta_0 + \beta_K \ln[e^{-\rho}\beta_K Y_t/(1 + e^{-\rho}\beta_K)] + \beta_A\rho_A \ln A_t\} \end{aligned}$$

Substituting the production function, equation (9), into equation (32) and expanding some of the logarithms yields

$$\begin{aligned}&V_t(K_t, A_t) = \alpha \ln K_t + (1-\alpha)\ln A_t + (1-\alpha)\ln l_t - \ln(1+e^{-\rho}\beta_K) + b\ln\{b/[(1-\alpha)(1+e^{-\rho}\beta_K)+b]\}\\ (33)\quad &+ e^{-\rho}\beta_0 + e^{-\rho}\beta_K\{\ln(e^{-\rho}\beta_K) - \ln(1+e^{-\rho}\beta_K) + \alpha\ln K_t + (1-\alpha)\ln A_t + (1-\alpha)\ln l_t\}\\ &+ e^{-\rho}\beta_A\rho_A \ln A_t.\end{aligned}$$

There is no need to substitute in for l_t since we already know that it does not depend on K_t or A_t and it is really the coefficients on $\ln K_t$ and $\ln A_t$ that we are interested in. It is possible to rewrite equation (33) as

(34) $V_t(K_t, A_t) = \beta_0' + \beta_K' \ln K_t + \beta_A' \ln A_t$,

where $\beta_K' \equiv \alpha(1 + e^{-\rho}\beta_K)$, $\beta_A' \equiv (1-\alpha)(1 + e^{-\rho}\beta_K) + e^{-\rho}\beta_A\rho_A$, and $\beta_0' \equiv$ the rest of the terms that do not depend upon K_t or A_t.

(e) In order for our original guess to be correct, we need the coefficient on $\ln K_t$ in equation (34) to be equal to β_K. That is, we need $\beta_K = \alpha(1 + e^{-\rho}\beta_K)$. Solving for β_K yields

(35) $\beta_K = \alpha/(1 - \alpha e^{-\rho})$.

We also need the coefficient on $\ln A_t$ in equation (34) to be equal to β_A. That is, we need

(36) $\beta_A = (1-\alpha)(1 + e^{-\rho}\beta_K) + e^{-\rho}\beta_A\rho_A$.

Substituting the expression for β_K, equation (35), into equation (36) yields

(37) $\beta_A = (1-\alpha)\{1 + [\alpha e^{-\rho}/(1-\alpha e^{-\rho})]\} + e^{-\rho}\beta_A\rho_A$.

Collecting the terms in β_A and simplifying yields

(38) $\beta_A(1 - e^{-\rho}\rho_A) = (1-\alpha)/(1-\alpha e^{-\rho})$,

and thus finally, β_A is given by

(39) $\beta_A = (1-\alpha)/[(1-\alpha e^{-\rho})(1-\rho_A e^{-\rho})]$.

(f) Substitute the value of β_K that was derived above into the earlier solutions for Y_t/C_t and l_t. That is, substitute equation (35) into equation (23):

$$(40)\quad \frac{C_t}{Y_t} = \frac{1}{1+\left[\alpha e^{-\rho}/\left(1-\alpha e^{-\rho}\right)\right]} = \frac{1}{\left[\left(1-\alpha e^{-\rho}+\alpha e^{-\rho}\right)/\left(1-\alpha e^{-\rho}\right)\right]},$$

or simply

(41) $C_t/Y_t = 1 - \alpha e^{-\rho}$.

This is the same ratio of consumption to output that was obtained by deriving the competitive solution to this model, with the additional assumption of n = 0 incorporated.

Similarly for labor supply, substitute equation (35) into equation (30):

$$(42)\quad l_t = \frac{(1-\alpha)}{(1-\alpha) + b/[1+(\alpha e^{-\rho}/(1-\alpha e^{-\rho}))]}.$$

We have already worked on an expression like the one in the denominator just above and thus

(43) $l_t = (1-\alpha)/[(1-\alpha) + b(1-\alpha e^{-\rho})]$.

This expression for labor supply is the same as the one that was obtained when deriving the competitive solution to the model, with the additional assumption of n = 0 incorporated.

Problem 5.12

The derivation of a constant saving rate, $s_t = \hat{s}$, and a constant labor supply per person, $l_t = \hat{l}$, does not depend on the behavior of technology. As the text points out, it is the combination of logarithmic utility, Cobb-Douglas production, and 100 percent depreciation that causes movements in both technology and capital to have offsetting income and substitution effects on saving. These assumptions allow us to derive an expression for the saving rate that does not depend on technology. See equation (5.31) in the text. Once a constant saving rate is established, technology plays no role in the derivation that labor supply is also constant. That relies on the form of the utility function and on Cobb-Douglas production. The latter is necessary so that labor's share in income is a constant.

Problem 5.13

(a) Imagine the household increasing its labor supply per member in period t by a small amount Δl and using the resulting income to increase its consumption in that period. Household utility and the instantaneous utility function are given by

(1) $U = \sum_{t=0}^{t=\infty} e^{-\rho t} u(c_t, 1 - l_t) N_t / H$,

and

(2) $u_t = \ln c_t + b(1 - l_t)^{1-\gamma}/(1 - \gamma)$.

From equations (1) and (2), the marginal disutility of working in period t is given by

(3) $-\partial U/\partial l_t = e^{-\rho t}(N_t/H)b(1 - l_t)^{-\gamma}$.

Thus increasing labor supply per member by Δl has the following utility cost for the household:

(4) Utility Cost $= e^{-\rho t}(N_t/H)b(1 - l_t)^{-\gamma}\Delta l$.

Since the change raises consumption by $w_t\Delta l$, it has the following utility benefit for the household:

(5) Utility Benefit $= e^{-\rho t}(N_t/H)(1/c_t)w_t\Delta l$.

If the household is behaving optimally, a marginal change of this type must leave expected lifetime utility unchanged. Thus the utility cost must equal the utility benefit; equating these two expressions gives us

(6) $e^{-\rho t}\dfrac{N_t}{H}\dfrac{b}{(1 - l_t)^{\gamma}}\Delta l = e^{-\rho t}\dfrac{N_t}{H}\dfrac{1}{c_t}w_t\Delta l$,

or simply

(7) $\dfrac{c_t}{(1 - l_t)^{\gamma}} = \dfrac{w_t}{b}$.

Equation (7) relates current leisure and consumption given the wage.

(b) With this change to the model, the saving rate is still constant. The derivation of a constant saving rate begins from the condition relating current consumption to expectations of future consumption, $1/c_t = e^{-\rho}E_t[(1 + r_{t+1})/c_{t+1}]$. That relation is not affected by this change to the instantaneous utility function. The rest of the derivation depends on Cobb-Douglas production and 100 percent depreciation, but not on the way that utility is affected by leisure. Thus equation (5.33) in the text, $\hat{s} = \alpha e^{n-\rho}$, continues to hold.

(c) Leisure per person is still constant as well. Note that c_t in equation (7) is consumption per person. It can be written as $c_t \equiv C_t/N_t = (1 - \hat{s})Y_t/N_t$, where $\hat{s}$ is the constant saving rate. Taking the natural logarithm of both sides of equation (7) and substituting for c_t yields

(8) $\ln[(1 - \hat{s})Y_t/N_t] - \gamma\ln(1 - l_t) = \ln w_t - \ln b$.

Since the production function is Cobb-Douglas, labor's share of output is $(1 - \alpha)$ and thus

$w_t l_t N_t = (1 - \alpha)Y_t$. Note that we have used $L_t \equiv l_t N_t$; the total amount of labor, L_t, is equal to labor supply per person, l_t, multiplied by the number of people, N_t. Rearranging, we have $w_t = (1 - \alpha)Y_t / l_t N_t$. Substituting this fact into equation (8) yields

(9) $\ln(1 - \hat{s}) + \ln Y_t - \ln N_t - \gamma\ln(1 - l_t) = \ln(1 - \alpha) + \ln Y_t - \ln l_t - \ln N_t - \ln b$.

Canceling terms and rearranging gives us

(10) $\ln l_t - \gamma\ln(1 - l_t) = \ln(1 - \alpha) - \ln(1 - \hat{s}) - \ln b$.

Taking the exponential function of both sides of equation (10) yields

(11) $\dfrac{l_t}{(1 - l_t)^{\gamma}} = \dfrac{(1-\alpha)}{b(1-\hat{s})}$.

Equation (11) implicitly defines leisure per person as a function of the constants γ, α, b, and $\hat{s}$. Thus leisure per person is also a constant.

Problem 5.14

(a) Taking logs of the production function, $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, gives us

(1) $\ln Y_t = \alpha\ln K_t + (1 - \alpha)(\ln A_t + \ln L_t)$.

In the model of Section 5.5 it was shown that labor supply and the saving rate were constant so that $L_t = \hat{l} N_t$ and $K_t = \hat{s}Y_{t-1}$. Thus we can write

(2) $\ln Y_t = \alpha\ln\hat{s} + \alpha\ln Y_{t-1} + (1 - \alpha)(\ln A_t + \ln\hat{l} + \ln N_t)$.

Finally we can use the equations for the evolution of technology and population, $\ln A_t = \bar{A} + gt + \tilde{A}_t$ and $\ln N_t = \bar{N} + nt$, to obtain

(3) $\ln Y_t = \alpha\ln\hat{s} + \alpha\ln Y_{t-1} + (1-\alpha)(\bar{A} + gt) + (1-\alpha)\tilde{A}_t + (1-\alpha)[\ln\hat{l} + \bar{N} + nt]$.

We need to solve for the path that log output would settle down to if there were no technology shocks. Start by subtracting (n + g)t from both sides of equation (3):

(4) $\ln Y_t - (n+g)t = \alpha\ln\hat{s} + \alpha\ln Y_{t-1} - \alpha(n+g)t + (1-\alpha)[\bar{A} + \ln\hat{l} + \bar{N} + \tilde{A}_t]$.

Add and subtract $\alpha(n + g)$ to the right-hand side of equation (4) to yield

(5) $\ln Y_t - (n+g)t = \alpha\ln\hat{s} + (1-\alpha)[\bar{A} + \ln\hat{l} + \bar{N}] - \alpha(n+g) + \alpha[\ln Y_{t-1} - (n+g)(t-1)] + (1-\alpha)\tilde{A}_t$.

Now define $Q \equiv \alpha\ln\hat{s} + (1-\alpha)[\bar{A} + \ln\hat{l} + \bar{N}] - \alpha(n+g)$ and use this to rewrite equation (5) as

(6) $\ln Y_t - (n+g)t = Q + \alpha[\ln Y_{t-1} - (n+g)(t-1)] + (1-\alpha)\tilde{A}_t$.

On a balanced growth path with no shocks to technology, the $\tilde{A}$s are uniformly 0. In addition, we know that output will simply grow at rate n + g. With this logarithmic structure that means $\ln Y_t - \ln Y_{t-1} = n + g$ or $\ln Y_{t-1} = \ln Y_t - (n + g)$. Substituting these facts into equation (6) yields

(7) $\ln Y_t - (n + g)t = Q + \alpha[\ln Y_t - (n + g) - (n + g)(t - 1)] = Q + \alpha[\ln Y_t - (n + g)t]$.

Further simplification yields

(8) $[\ln Y_t - (n + g)t](1 - \alpha) = Q$,

or simply

(9) $\ln Y_t^* = Q/(1 - \alpha) + (n + g)t$.

Equation (9) gives an expression for $\ln Y_t^*$, the path that log output would settle down to if there were never any technology shocks.

(b) By definition, $\tilde{Y}_t \equiv \ln Y_t - \ln Y_t{}^*$, where $\ln Y_t^*$ is the path found in part (a). Thus $\tilde{Y}_t$ gives us the difference between what log output actually is in any period and what it would have been in the complete absence of any technology shocks. Substituting for $\ln Y_t^*$ from equation (9) yields

(10) $\tilde{Y}_t = \ln Y_t - Q/(1 - \alpha) - (n + g)t$.

Note that equation (10) holds every period and so we can write

(11) $\tilde{Y}_{t-1} = \ln Y_{t-1} - Q/(1 - \alpha) - (n + g)(t - 1)$.

Multiplying both sides of equation (11) by α and solving for $\alpha \ln Y_{t-1}$ yields

(12) $\alpha \ln Y_{t-1} = \alpha\tilde{Y}_{t-1} + [\alpha/(1 - \alpha)]Q + \alpha(n + g)(t - 1)$.

Substituting equation (12) into equation (3) and then substituting the resulting expression into equation (10) yields

$$(13)\quad \begin{aligned}\tilde{Y}_t &= \alpha \ln \hat{s} + \alpha\tilde{Y}_{t-1} + [\alpha/(1-\alpha)]Q + \alpha(n+g)(t-1) \\ &+ (1-\alpha)\left[\overline{A} + gt + \tilde{A}_t + \ln \hat{l} + \overline{N} + nt\right] - Q/(1-\alpha) - (n+g)t.\end{aligned}$$

Simplification yields

(14) $\tilde{Y}_t = \alpha(n + g)t + (1 - \alpha)(n + g)t - (n + g)t + \alpha\tilde{Y}_{t-1} + (1 - \alpha)\tilde{A}_t$,

and thus finally

(15) $\tilde{Y}_t = \alpha\tilde{Y}_{t-1} + (1 - \alpha)\tilde{A}_t$.

Equation (15) is identical to equation (5.40) in the text.

Problem 5.15

(a) (i) The equation of motion for capital is given by

(1) $K_{t+1} = K_t + Y_t - C_t - G_t - \delta K_t$,

or substituting the production function into equation (1), we have

(2) $K_{t+1} = K_t + K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - G_t - \delta K_t$.

Using equation (1), $\partial \ln K_{t+1} / \partial \ln K_t$ (holding A_t, L_t, C_t, and G_t fixed) is

$$(3)\quad \frac{\partial \ln K_{t+1}}{\partial \ln K_t} = \frac{\partial K_{t+1}}{\partial K_t}\frac{K_t}{K_{t+1}} = \left[1 + \frac{\partial Y_t}{\partial K_t} - \delta\right]\frac{K_t}{K_{t+1}}.$$

By definition, since factors are paid their marginal products, the real interest rate is $r_t = \partial Y_t / \partial K_t - \delta$ and thus

$$(4)\quad \frac{\partial \ln K_{t+1}}{\partial \ln K_t} = (1 + r_t)\frac{K_t}{K_{t+1}}.$$

(a) (ii) On the balanced growth path without shocks, capital grows at rate n + g, so that $K_{t+1} = e^{n+g}K_t$. In addition, using r* to denote the balanced-growth-path value of the real interest rate, equation (4) can be rewritten as

$$(5)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln K_t}\right|_{bgp} = (1 + r^*)\frac{K_t}{e^{n+g}K_t} = \frac{1 + r^*}{e^{n+g}}.$$

(b) Using equation (2), $\partial \ln K_{t+1} / \partial \ln A_t$ (holding K_t, C_t, G_t and L_t fixed) is

$$(6)\quad \frac{\partial \ln K_{t+1}}{\partial \ln A_t} = \frac{\partial K_{t+1}}{\partial A_t}\frac{A_t}{K_{t+1}} = (1-\alpha)K_t{}^{\alpha}A_t{}^{-\alpha}L_t{}^{1-\alpha}\left(\frac{A_t}{K_{t+1}}\right) = \frac{(1-\alpha)K_t{}^{\alpha}(A_t L_t)^{1-\alpha}}{K_{t+1}}.$$

Using $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, equation (6) becomes

$$(7)\quad \frac{\partial \ln K_{t+1}}{\partial \ln A_t} = \frac{(1-\alpha)Y_t}{K_{t+1}}.$$

On the balanced growth path without shocks, $K_{t+1} = e^{n+g} K_t$. Since the production function is Cobb-Douglas, the amount of income going to capital – which is the marginal product of capital multiplied by the amount of capital, $(r^* + \delta)K_t$ – is equal to αY_t. Thus $Y_t = (r^* + \delta)K_t/\alpha$. Substituting these two facts into equation (7) yields

$$(8)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln A_t}\right|_{bgp} = \frac{(1-\alpha)(r^*+\delta)K_t}{\alpha e^{n+g}K_t} = \frac{(1-\alpha)(r^*+\delta)}{\alpha e^{n+g}}.$$

Using equation (2), $\partial\ln K_{t+1}/\partial \ln L_t$ (holding K_t, C_t, G_t and A_t fixed) is

$$(9)\quad \frac{\partial \ln K_{t+1}}{\partial \ln L_t} = \frac{\partial K_{t+1}}{\partial L_t}\frac{L_t}{K_{t+1}} = (1-\alpha)K_t^{\alpha}A_t^{1-\alpha}L_t^{-\alpha}\left(\frac{L_t}{K_{t+1}}\right) = \frac{(1-\alpha)K_t^{\alpha}(A_tL_t)^{1-\alpha}}{K_{t+1}}.$$

Comparing equation (9) to equation (6), we can see that $\partial\ln K_{t+1}/\partial\ln L_t = \partial\ln K_{t+1}/\partial \ln A_t$. Thus by performing the same manipulations as above, we can write

$$(10)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln L_t}\right|_{bgp} = \frac{(1-\alpha)(r^*+\delta)}{\alpha e^{n+g}}.$$

Using equation (2), $\partial\ln K_{t+1}/\partial\ln G_t$ (holding K_t, C_t, L_t and A_t fixed) is

$$(11)\quad \frac{\partial \ln K_{t+1}}{\partial \ln G_t} = \frac{\partial K_{t+1}}{\partial G_t}\frac{G_t}{K_{t+1}} = -\frac{G_t}{K_{t+1}}.$$

Multiplying and dividing the right-hand side of equation (11) by Y_t gives us

$$(12)\quad \frac{\partial \ln K_{t+1}}{\partial \ln G_t} = -\frac{Y_t(G_t/Y_t)}{K_{t+1}}.$$

As explained above, on the balanced growth path without shocks, $K_{t+1} = e^{n+g} K_t$ and $Y_t = (r^* + \delta)K_t/\alpha$. Substituting these two facts into equation (12) and using (G^*/Y) to denote the ratio of G to Y on the balanced growth path yields

$$(13)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln G_t}\right|_{bgp} = -\frac{(r^*+\delta)K_t(G^*/Y)}{\alpha e^{n+g}K_t} = -\frac{(r^*+\delta)(G^*/Y)}{\alpha e^{n+g}}.$$

Using equation (2), $\partial\ln K_{t+1}/\partial\ln C_t$ (holding K_t, G_t, L_t and A_t fixed) is

$$(14)\quad \frac{\partial \ln K_{t+1}}{\partial \ln C_t} = \frac{\partial K_{t+1}}{\partial C_t}\frac{C_t}{K_{t+1}} = -\frac{C_t}{K_{t+1}}.$$

Using hint (2), we can write this derivative evaluated at the balanced growth path as

$$(15)\quad \left.\frac{\partial \ln K_{t+1}}{\partial \ln C_t}\right|_{bgp} = -\frac{\left[Y_t - G^* - \delta K_t - (e^{n+g}-1)K_t\right]}{K_{t+1}}.$$

Defining $\lambda_1 \equiv (1 + r^*)/e^{n+g}$, $\lambda_2 \equiv (1 - \alpha)(r^* + \delta)/(\alpha e^{n+g})$ and $\lambda_3 \equiv -(r^* + \delta)(G^*/Y)/(\alpha e^{n+g})$, we need to show that $\partial\ln K_{t+1}/\partial\ln C_t$ evaluated at the balanced growth path is equal to $1 - \lambda_1 - \lambda_2 - \lambda_3$. By definition,

$$(16)\quad 1-\lambda_1-\lambda_2-\lambda_3 = 1 - \frac{1+r^*}{e^{n+g}} - \frac{(1-\alpha)(r^*+\delta)}{\alpha e^{n+g}} + \frac{(r^*+\delta)(G^*/Y)}{\alpha e^{n+g}},$$

or simply

$$(17)\quad 1-\lambda_1-\lambda_2-\lambda_3 = 1 - \frac{1+r^*}{e^{n+g}} + \frac{(r^*+\delta)\left[(G^*/Y)-(1-\alpha)\right]}{\alpha e^{n+g}}.$$

Note that we can write $(r^* + \delta)$ as

(18) $(r^* + \delta) = \partial Y_t / \partial K_t = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} = \alpha Y_t / K_t$.
Substituting equation (18) into equation (17) gives us

(19) $$1-\lambda_1-\lambda_2-\lambda_3 = 1-\frac{1+r^*}{e^{n+g}}+\frac{\alpha Y_t\left[(G^*/Y_t)-(1-\alpha)\right]}{\alpha K_t e^{n+g}} = 1-\frac{1+r^*}{e^{n+g}}+\frac{G^*-(1-\alpha)Y_t}{e^{n+g}K_t}$$

Obtaining a common denominator and using the fact that on the balanced growth path without shocks, $K_{t+1} = e^{n+g} K_t$, we have

(20) $$1-\lambda_1-\lambda_2-\lambda_3 = \frac{e^{n+g}K_t - (1+r^*)K_t + G^* - Y_t + \alpha Y_t}{K_{t+1}}.$$

From equation (18), $\alpha Y_t = (r^* + \delta)K_t$. Substituting this into equation (20) yields

(21) $$1-\lambda_1-\lambda_2-\lambda_3 = \frac{e^{n+g}K_t - K_t - r^*K_t + G^* - Y_t + r^*K_t + \delta K_t}{K_{t+1}}.$$

Collecting terms yields

(22) $$1-\lambda_1-\lambda_2-\lambda_3 = -\frac{\left[Y_t - G^* - \delta K_t - (e^{n+g}-1)K_t\right]}{K_{t+1}}.$$

Comparing equations (15) and (22), we have shown that

(23) $$\left.\frac{\partial \ln K_{t+1}}{\partial \ln C_t}\right|_{bgp} = 1-\lambda_1-\lambda_2-\lambda_3.$$

The log linearization is of the form

(24) $$\begin{aligned}\tilde{K}_{t+1} \cong &\left[\left.\frac{\partial \ln K_{t+1}}{\partial \ln K_t}\right|_{bgp}\right]\tilde{K}_t + \left[\left.\frac{\partial \ln K_{t+1}}{\partial \ln A_t}\right|_{bgp}\right]\tilde{A}_t + \left[\left.\frac{\partial \ln K_{t+1}}{\partial \ln L_t}\right|_{bgp}\right]\tilde{L}_t \\ &+ \left[\left.\frac{\partial \ln K_{t+1}}{\partial \ln G_t}\right|_{bgp}\right]\tilde{G}_t + \left[\left.\frac{\partial \ln K_{t+1}}{\partial \ln C_t}\right|_{bgp}\right]\tilde{C}_t\end{aligned}.$$

Using equations (4), (8), (10), (13), and (23) as well as the definitions of λ_1, λ_2 and λ_3 to substitute for the derivatives, we have

(25) $$\tilde{K}_{t+1} \cong \lambda_1 \tilde{K}_t + \lambda_2 (\tilde{A}_t + \tilde{L}_t) + \lambda_3 \tilde{G}_t + (1-\lambda_1-\lambda_2-\lambda_3)\tilde{C}_t.$$

(c) Substituting equation (5.43), $\tilde{C}_t \cong a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t$, and equation (5.44), $\tilde{L}_t \cong a_{LK}\tilde{K}_t + a_{LA}\tilde{A}_t + a_{LG}\tilde{G}_t$, into equation (25) yields

(26) $$\begin{aligned}\tilde{K}_{t+1} \cong &\lambda_1 \tilde{K}_t + \lambda_2 \tilde{A}_t + \lambda_2(a_{LK}\tilde{K}_t + a_{LA}\tilde{A}_t + a_{LG}\tilde{G}_t) + \lambda_3 \tilde{G}_t \\ &+ (1-\lambda_1-\lambda_2-\lambda_3)(a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t).\end{aligned}$$

Collecting terms gives us

(27) $$\begin{aligned}\tilde{K}_{t+1} \cong &\left[\lambda_1 + \lambda_2 a_{LK} + (1-\lambda_1-\lambda_2-\lambda_3)a_{CK}\right]\tilde{K}_t + \left[\lambda_2 + \lambda_2 a_{LA} + (1-\lambda_1-\lambda_2-\lambda_3)a_{CA}\right]\tilde{A}_t \\ &+ \left[\lambda_2 a_{LG} + \lambda_3 + (1-\lambda_1-\lambda_2-\lambda_3)a_{CG}\right]\tilde{G}_t.\end{aligned}$$

Defining $b_{KK} \equiv \lambda_1 + \lambda_2 a_{LK} + (1 - \lambda_1 - \lambda_2 - \lambda_3)a_{CK}$, $b_{KA} \equiv \lambda_2 (1 + a_{LA}) + (1 - \lambda_1 - \lambda_2 - \lambda_3)a_{CA}$ and $b_{KG} \equiv \lambda_2 a_{LG} + \lambda_3 + (1 - \lambda_1 - \lambda_2 - \lambda_3)a_{CG}$, equation (27) can be rewritten as

(28) $$\tilde{K}_{t+1} \cong b_{KK}\tilde{K}_t + b_{KA}\tilde{A}_t + b_{KG}\tilde{G}_t.$$

Equation (28) is identical to equation (5.53) in the text.

SOLUTIONS TO CHAPTER 6

Problem 6.1

(a) The IS curve in Figure 6.1 is given by

(1) $\ln Y_t = \ln Y_{t+1} - \dfrac{1}{\theta} r_t$.

The slope of the IS curve is given by dr/dY for a given Y_{t+1}. From equation (1)

(2) $\dfrac{1}{Y_t} dY_t = -\dfrac{1}{\theta} dr_t$,

and thus the slope is given by

(3) $\left.\dfrac{dr_t}{dY_t}\right|_{IS} = -\dfrac{\theta}{Y_t}$.

Thus, an increase in θ increases the slope in absolute value and thus makes the IS curve steeper. Intuitively, an increase in the coefficient of relative risk aversion, θ, means that households are less willing to substitute consumption across time. Thus, any given change in r results in a smaller change in consumption and thus output along the IS curve.

The LM curve is given by the combinations of Y and r that satisfy the following equation for a given level of real money holdings:

(4) $\dfrac{M_t}{P_t} = Y_t^{\theta/\nu}\left(\dfrac{1+r_t}{r_t}\right)^{1/\nu}$.

Taking the natural log of both sides gives us

(5) $\ln\left(\dfrac{M_t}{P_t}\right) = \dfrac{\theta}{\nu}\ln Y_t + \dfrac{1}{\nu}\ln(1+r_t) - \dfrac{1}{\nu}\ln r_t$.

The slope of the LM curve is given by dr/dY for a given M/P. From equation (5)

(6) $0 = \dfrac{\theta}{\nu}\dfrac{1}{Y_t} dY_t + \dfrac{1}{\nu}\dfrac{1}{1+r_t} dr_t - \dfrac{1}{\nu}\dfrac{1}{r_t} dr_t$.

Solving for dr/dY gives us

(7) $\left.\dfrac{dr_t}{dY_t}\right|_{LM} = \dfrac{\theta r_t (1+r_t)}{Y_t}$.

Thus, an increase in θ also makes the LM curve steeper.

(b) The IS curve does not depend on ν at all, so it is unchanged. From equation (7) we can see that a change in ν does not affect the slope of the LM curve. From equation (4) we can see that a decrease in ν increases the demand for real money balances at a given level of output and the real interest rate. This implies that the LM curve must shift up. Intuitively, with a fixed real money supply, any given level of output would now require a higher real interest rate in order for the demand for money to equal the fixed supply. Thus, the new LM curve associated with the lower value of ν must lie above the old LM curve.

(c) Again, the IS curve does not depend on Γ(•) at all, so it does not change. As in the text, consider a balanced budget change in M_t/P_t and C_t. Specifically, suppose the household raises M_t/P_t by dm and lowers C_t by $[i_t/(1 + i_t)]$dm. At the margin, this change must not affect utility. The utility cost of this change is still $U'(C_t)[i_t/(1 + i_t)]$dm. With our modification to the utility function, the utility benefit is now given by B $\Gamma'(M_t/P_t)$dm. Thus, the first-order condition for optimal money holdings is now

(8) $B\Gamma'\left(\frac{M_t}{P_t}\right) = \frac{i_t}{1+i_t}U'(C_t)$.

Given the form of $\Gamma(\bullet)$ and $U(\bullet)$, and the fact that $C_t = Y_t$ this implies

(9) $B\left(\frac{M_t}{P_t}\right)^{-\nu} = \frac{i_t}{1+i_t}Y_t^{-\theta}$.

Dividing both sides of equation (9) by B and then taking both sides of the resulting expression to the exponent -1/ ν, and then using the fact that with the price level fixed, the real and nominal interest rates are equivalent, we have

(10) $\frac{M_t}{P_t} = B^{1/\nu}Y_t^{\theta/\nu}\left(\frac{1+r_t}{r_t}\right)^{1/\nu}$.

Intuitively, a decrease in B reduces the utility from holding any given quantity of real money balances. Thus, as we can see from equation (10), it reduces optimal money holdings for any given level of output and the real interest rate. This implies that the LM curve must shift down. With the lower value of B, any given level of output must now be associated with a lower real interest rate in order for real money demand to remain equal to the fixed real money supply.

<u>Problem 6.2</u>

(a) We need to find the average cost per unit time of conversions plus foregone interest, which we will denote AC. The cost of conversions in nominal terms is P times C and the number of conversions per unit time is $1/\tau$. Average foregone interest per unit time is average money holdings, $\alpha YP\tau/2$ multiplied by the nominal interest rate, i. Thus,

(1) $AC = \frac{PC}{\tau} + \frac{\alpha YP\tau}{2}i$.

The first-order condition for τ is

(2) $\frac{\partial AC}{\partial \tau} = -\frac{PC}{\tau^2} + \frac{\alpha YP}{2}i = 0$.

This simplifies to

(3) $\frac{C}{\tau^2} = \frac{\alpha Yi}{2}$.

Solving for the optimal choice, τ^* yields

(4) $\tau^* = \left(\frac{2C}{\alpha Yi}\right)^{1/2}$.

Note that $\partial^2 AC/\partial \tau^2 = 2PC/\tau^3 > 0$ and so τ^* is a minimum.

(b) Average real money holdings are given by

(5) $\frac{M}{P} = \frac{\alpha Y\tau}{2}$.

Substituting the optimal choice of τ^* given by equation (4) into equation (5) yields

(6) $\frac{M}{P} = \frac{\alpha Y}{2}\left(\frac{2C}{\alpha Yi}\right)^{1/2}$.

This simplifies to

(7) $\frac{M}{P} = \left(\frac{\alpha CY}{2i}\right)^{1/2}$.

Taking the natural log of both sides of equation (7) gives us

(8) $\ln(M/P) = 1/2[\ln\alpha + \ln Y + \ln C - \ln 2 - \ln i]$.

Differentiating both sides of (8) with respect to i yields

(9) $\frac{1}{M/P}\frac{\partial[M/P]}{\partial i} = -\frac{1}{2}\frac{1}{i}$.

Thus, the elasticity of real money holdings with respect to i is given by

(10) $\frac{\partial[M/P]}{\partial i}\frac{i}{M/P} = -\frac{1}{2}$.

Differentiating both sides of (8) with respect to Y yields

(11) $\frac{1}{M/P}\frac{\partial[M/P]}{\partial Y} = \frac{1}{2}\frac{1}{Y}$.

Thus, the elasticity with respect to Y is given by

(12) $\frac{\partial[M/P]}{\partial Y}\frac{Y}{M/P} = \frac{1}{2}$.

Thus, average real money holdings are decreasing in i and increasing in Y.

Problem 6.3

(a) Substituting the consumption function, $C_t = a + bY_{t-1}$, and the assumption about investment, $I_t = K_t^* - cY_{t-2}$, into the equation for output, $Y_t = C_t + I_t + G_t$, yields

(1) $Y_t = a + bY_{t-1} + K_t^* - cY_{t-2} + G_t$.

Substituting for the desired capital stock, $K_t^* = cY_{t-1}$, and for the constant level of government purchases, $G_t = \overline{G}$, yields

(2) $Y_t = a + bY_{t-1} + cY_{t-1} - cY_{t-2} + \overline{G}$.

Collecting terms in Y_{t-1} gives us output in period t as a function of Y_{t-1}, Y_{t-2} and the parameters of the model:

(3) $Y_t = a + (b + c)Y_{t-1} - cY_{t-2} + \overline{G}$.

(b) With the assumptions of b = 0.9 and c = 0.5, output in period t is given by

(4) $Y_t = a + 1.4Y_{t-1} - 0.5Y_{t-2} + \overline{G}$.

Throughout the following, the change in a variable represents the change from the path that variable would have taken if G had simply remained constant at $\overline{G}$.

In period t,

(5) $Y_t = a + 1.4Y_{t-1} - 0.5Y_{t-2} + \overline{G} + 1$,

and thus the change in output from the path it would have taken is given by

(6) $\Delta Y_t = +1$.

In period t + 1, using the fact that equation (4) will hold in all future periods,

(7) $\Delta Y_{t+1} = 1.4\Delta Y_t - 0.5\Delta Y_{t-1} = 1.4(+1) - 0.5(0) = +1.4$.

In period t + 2,

(8) $\Delta Y_{t+2} = 1.4\Delta Y_{t+1} - 0.5\Delta Y_t = 1.4(+1.4) - 0.5(+1) = +1.46$.

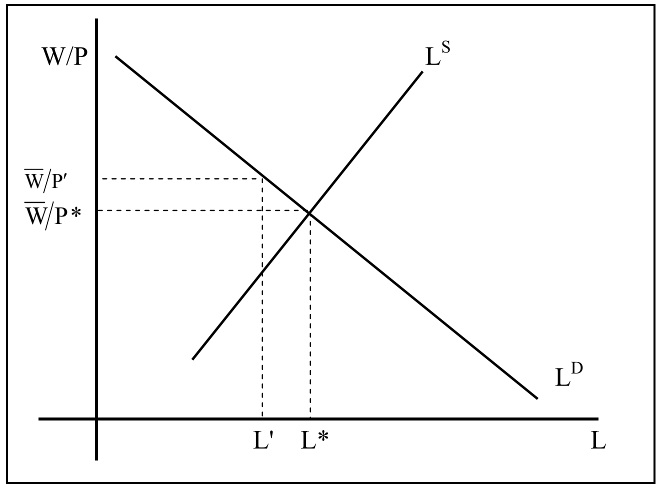
In period t + 3,

(9) $\Delta Y_{t+3} = 1.4\Delta Y_{t+2} - 0.5\Delta Y_{t+1} = 1.4(+1.46) - 0.5(+1.4) = +1.344$.

With similar calculations, one can show that $\Delta Y_{t+4} = +1.15$, $\Delta Y_{t+5} = 0.938$ and so on. Thus output follows a "hump-shaped" response to the one-time increase in government purchases of one. The maximum effect is felt two periods after the increase in G and the effect then goes to 0 over time.

Problem 6.4

(a) The *short-side rule* implies that the level that generates the maximum output is the intersection of the labor supply and labor demand curves, where there is no unemployment. Thus, a price level of P*, seen at the right, maximizes output.



(b) A price level (here given by P') that is above the level that generates maximum output will cause more labor to be supplied than is demanded, thus causing unemployment and output that is lower than the maximum. See figure at the right.

Problem 6.5

Suppose the increase in g occurs in time period t and define $\Delta g \equiv g^H - g^L > 0$. In what follows, the change in a variable refers to the difference between its actual value and the value it would have had in the absence of the rise in g.

(a) In this case, unemployment does not need to change in order for price inflation to remain constant. Wage inflation simply rises by Δg since it is given by

(1) $\pi_t^w = \pi_t + g_t$.

Since price inflation is given by

(2) $\pi_t = \pi_{t-1} - \phi(u_t - \bar{u})$,

π then remains constant and u remains equal to $\bar{u}$.

(b) Since price inflation is given by

(3) $\pi_t = \pi_t^w - g_t$,

then

(4) $\Delta\pi_t = \Delta\pi_t^w - \Delta g$,

and so wage inflation must rise by Δg in order for price inflation to remain constant. Since wage inflation is given by

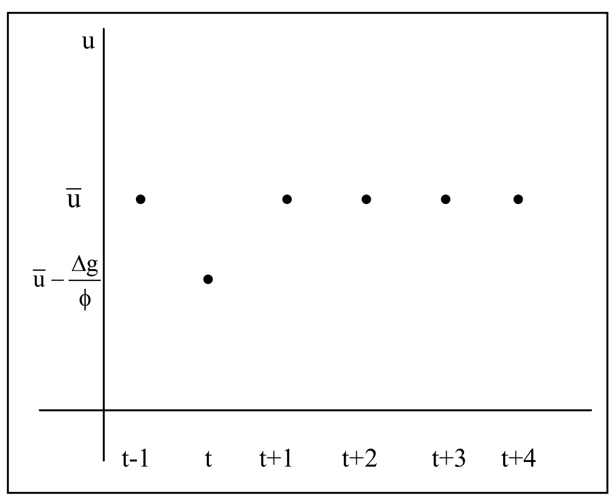
(5) $\pi_t^w = \pi_{t-1}^w - \phi(u_t - \bar{u})$,

this means that unemployment must fall in period t. More formally, we need $\Delta\pi_t^w = \Delta g$ and with $\Delta\pi_{t-1}^w = 0$ and $\Delta\bar{u} = 0$, equation (5) implies

(6) $\Delta g_t = -\phi(\Delta u_t)$,

or simply

(7) $\Delta u_t = -\frac{1}{\phi}\Delta g$.



In period t + 1 we again require π_{t+1}^w to be Δg

higher than it would have been since g is Δg higher than it would have been. Note that π_t^w is Δg higher, and so from

(8) $\pi_{t+1}^{w} = \pi_t^{w} - \phi(u_{t+1} - \bar{u})$,

we can see that π_{t+1}^{w} will be Δg higher if $u_{t+1} = \bar{u}$. Thus the period after the shock, unemployment must return to $\bar{u}$ if price inflation is to remain constant. The required path of the unemployment rate is shown in the figure at right. The rise in g causes a one-period drop in u.

(c) As in part (b), π_t^{w} must rise by Δg in order for price inflation to be constant. Since wage inflation is given by

(9) $\pi_t^{w} = \pi_{t-1} - \phi(u_t - \bar{u})$,

this means that unemployment must fall in period t. As in part (b), the required change in the unemployment rate is

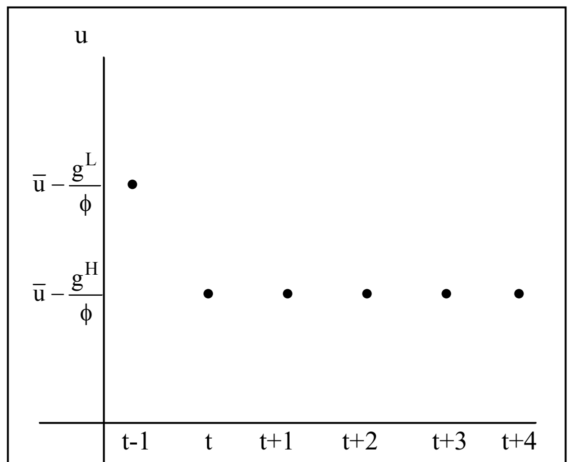
(10) $\Delta u_t = -\dfrac{1}{\phi}\Delta g$.

In period t + 1 we require π_{t+1}^{w} to be Δg high since g is Δg higher. Equation (9) holds for all periods and so in period t + 1, we can write

(11) $\pi_{t+1}^{w} = \pi_t - \phi(u_{t+1} - \bar{u})$.

Since $\Delta\pi_t = 0$, in order for $\pi_{t+1}^{w} = \Delta g$, we again require unemployment to be lower than it would have been and

(12) $\Delta u_{t+1} = -\dfrac{1}{\phi}\Delta g$.



The same is true in each following period. Thus in this case, the rise in g requires a permanent drop in the unemployment rate in order for price inflation to remain constant. Note that in the absence of the shock, the unemployment rate is $u_t = \bar{u} - (1/\phi)g^L$. That is because we assume that unemployment is initially at the level that causes price inflation to be constant. Here, if g is not changing, that requires wage inflation to be constant. Substituting equation (3) lagged one period into equation (9) gives us

(13) $\pi_t^{w} = \pi_{t-1}^{w} - g_{t-1} - \phi(u_t - \bar{u})$.

And thus, in order for wage inflation to be constant when g equals g_L, as it does in period (t – 1), the unemployment rate must equal $\bar{u} - (1/\phi)g^L$. In period t, when u falls by (1/ϕ)Δg, the level of the unemployment rate becomes $\bar{u} - (1/\phi)g^L - (1/\phi)\Delta g = \bar{u} - (1/\phi)(g^L + g^H - g^L)$ or simply $\bar{u} - (1/\phi)(g^H)$. It then remains at that level thereafter.

(d) As in parts (b) and (c), π_t^{w} must rise by Δg in order for price inflation to be constant. In order to see what this implies for unemployment in period t, we first need to calculate the change in $\hat{g}_t$ due to the change in g. Since

(14) $\hat{g}_t = \rho\hat{g}_{t-1} + (1-\rho)g_t$,

we can write

(15) $\Delta\hat{g}_t = (1-\rho)\Delta g$.

Wage inflation in period t is given by

(16) $\pi_t^w = \pi_{t-1} + \hat{g}_t - \phi(u_t - \bar{u})$.

Equations (15) and (16), along with the fact that we require $\pi_t^w = \Delta g$ imply

(17) $\Delta g = (1-\rho)\Delta g - \phi\Delta u_t$.

Thus the change in unemployment required for price inflation to be constant in period t is

(18) $\Delta u_t = -\dfrac{1}{\phi}[1-(1-\rho)]\Delta g$.

In period t + 1, constant price inflation again requires that π_{t+1}^w be Δg higher. Since equation (14) holds in all periods, the change in $\hat{g}$ for period t + 1 is

(19) $\Delta\hat{g}_{t+1} = \rho\Delta\hat{g}_t + (1-\rho)\Delta g$.

Substituting equation (15) into equation (19) yields

(20) $\Delta\hat{g}_{t+1} = \rho(1-\rho)\Delta g + (1-\rho)\Delta g$,

or simply

(21) $\Delta\hat{g}_{t+1} = [(1-\rho) + \rho(1-\rho)]\Delta g$.

Wage inflation in period t + 1 is given by

(22) $\pi_{t+1}^w = \pi_t + \hat{g}_{t+1} - \phi(u_{t+1} - \bar{u})$.

Equations (21) and (22), along with the fact that we require $\pi_{t+1}^w = \Delta g$ imply

(23) $\Delta g = [(1-\rho) + \rho(1-\rho)]\Delta g - \phi\Delta u_{t+1}$.

Thus the change in the unemployment rate required for constant price inflation in period t + 1 is given by

(24) $\Delta u_{t+1} = -\dfrac{1}{\phi}[1-(1-\rho)-\rho(1-\rho)]\Delta g$.

Similar analysis for period t + 1 would yield the following required change in the unemployment rate:

(25) $\Delta u_{t+2} = -\dfrac{1}{\phi}[1-(1-\rho)-\rho(1-\rho)-\rho^2(1-\rho)]\Delta g$.

And in general, in period t + s, the change in the unemployment rate required for constant price inflation is given by

(26) $\Delta u_{t+s} = -\dfrac{1}{\phi}[1-(1-\rho)-\rho(1-\rho)-\rho^2(1-\rho)-\ldots-\rho^s(1-\rho)]\Delta g$.

Allowing s to go to infinity, we can write

(27) $1-(1-\rho)-\rho(1-\rho)-\rho^2(1-\rho)-\ldots = 1-[(1-\rho)(1+\rho+\rho^2+\ldots)]$.

Since $0 < \rho < 1$, the infinite series $1 + \rho + \rho^2 + \ldots$ converges and we can write

(28) $1-(1-\rho)-\rho(1-\rho)-\rho^2(1-\rho)-\ldots = 1-[(1-\rho)\dfrac{1}{(1-\rho)}] = 1-1 = 0$.

Thus

(29) $\lim\limits_{s\to\infty} \Delta u_{t+s} = 0$.

In this case, if price inflation is to remain constant when g rises, the unemployment rate must fall at time t and then rise back until it asymptotically approaches its original value.

Problem 6.6
(a) Substitute the IS equation,
(1) $Y_t = -r_t/\theta$
into the money-market equilibrium condition,
(2) $m - p = L(r + \pi^e, Y)$,
to obtain
(3) $m - p = L(r + \pi^e, \frac{-r_t}{\theta})$.
With $P = \overline{P}$ and $\pi^e = 0$, equation (3) simplifies to
(4) $m - \overline{p} = L(r, \frac{-r_t}{\theta})$,
where $\overline{p} = \ln(\overline{P})$. Differentiating both sides of equation (4) with respect to m gives us
(5) $1 = L_r \frac{dr}{dm} + L_Y (\frac{-1}{\theta}) \frac{dr}{dm}$.
Solving for dr/dm yields
(6) $\frac{dr}{dm} = \frac{1}{L_r + L_Y(\frac{-1}{\theta})}$.

Since $L_r < 0$, $L_Y > 0$, and $\frac{-1}{\theta} < 0$, $dr/dm < 0$. Thus an increase in the nominal money supply does lower the real interest rate.

(b) Substituting the assumption that $\pi^e = 0$ into equation (3) gives us
(7) $m - p = L(r, \frac{-r_t}{\theta})$.
Differentiating both sides of equation (7) with respect to m yields
(8) $1 - \frac{dp}{dm} = L_r \frac{dr}{dm} + L_Y (\frac{-1}{\theta}) \frac{dr}{dm}$.
Solving for dr/dm leaves us with
(9) $\frac{dr}{dm} = \frac{1 - dp/dm}{L_r + L_Y(\frac{-1}{\theta})}$.
As shown in part (a), the denominator of dr/dm is negative. With $0 < dp/dm < 1$, the numerator is strictly between 0 and 1. Thus $dr/dm < 0$ and so an increase in the nominal money supply again reduces the real interest rate. Comparing equations (6) and (9), we can see that, since $0 < dp/dm < 1$, dr/dm is smaller in absolute value here. Thus a given change in m causes a smaller change in r when p is not completely fixed. Or conversely, a larger change in m is required for any given change in r.

(c) Differentiate both sides of equation (3) with respect to m to yield
(10) $1 - \frac{dp}{dm} = L_{r+\pi^e} \frac{dr}{dm} + L_{r+\pi^e} \frac{d\pi^e}{dm} + L_Y (\frac{-1}{\theta}) \frac{dr}{dm}$.
Solving for dr/dm gives us
(11) $\frac{dr}{dm} = \frac{1 - dp/dm}{L_{r+\pi^e} + L_Y(\frac{-1}{\theta})} - \frac{L_{r+\pi^e}\, d\pi^e/dm}{L_{r+\pi^e} + L_Y(\frac{-1}{\theta})}$.

As shown in part (b), the first term on the right-hand side of equation (11) is negative. The denominator of the second term on the right-hand side is negative, as shown previously. Since $L_{r+\pi^e} < 0$ and $d\pi^e/dm > 0$, the numerator is also negative and thus that second term is positive. Thus $dr/dm < 0$ again, meaning that an increase in the nominal money supply, m, lowers the real interest rate. Comparing equations (9) and (11) we can see that dr/dm is larger in absolute value here. Thus, if expected inflation increases when the nominal money supply rises – as it does here – a given change in m has a larger effect on the real interest rate. Thus a given change in r requires a smaller change in m than in part (b).

(d) We can substitute the assumptions that $dp/dm = 1$ and $d\pi^e/dm = 0$ into equation (10) to obtain

(12) $1-1 = L_{r+\pi^e}\dfrac{dr}{dm} + L_Y\left(\dfrac{-1}{\theta}\right)\dfrac{dr}{dm}$.

And thus

(13) $\dfrac{dr}{dm} = 0$.

With complete and instantaneous price adjustment, a change in the nominal money supply does not affect the real interest rate.

Problem 6.7

(a) Recall that the AD curve is derived from the IS and MP curves. The IS curve is given by

(1) $y(t) = -[i(t) - \pi(t)]/\theta$,

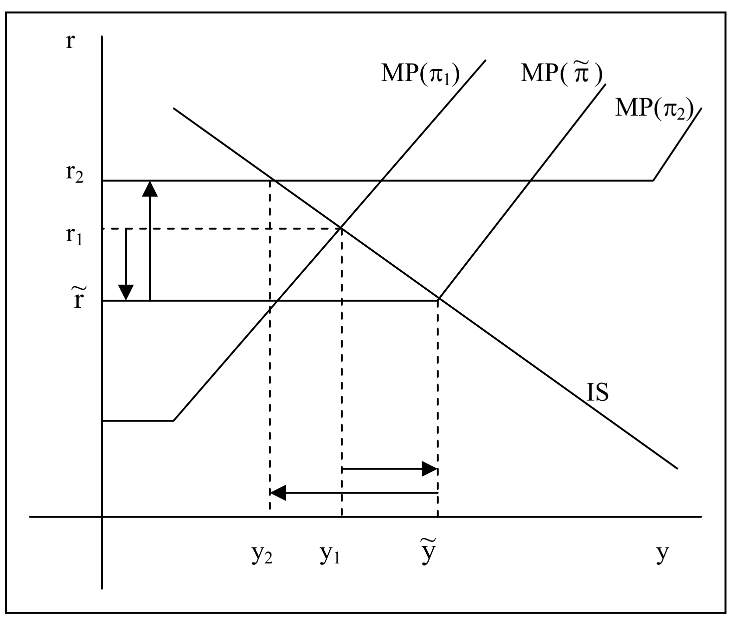
with $y'(\bullet) < 0$ so that the IS curve is downward-sloping in (y, r) space. The monetary-policy rule is given by

(2) $r(t) = r(y(t) - \bar{y}(t), \pi(t))$,

which, with the assumption that $\bar{y}(t) = 0$, simplifies to

(3) $r(t) = r(y(t), \pi(t))$,

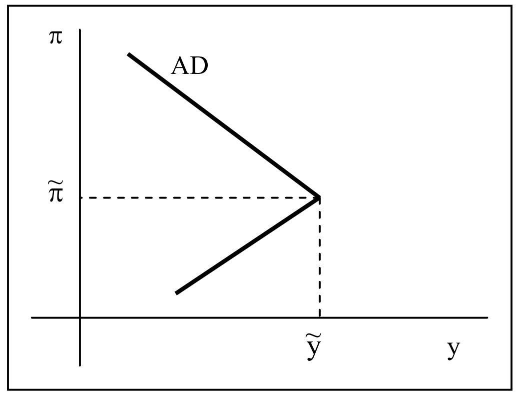
with the constraint that the nominal interest rate, $i(t) = r(t) + \pi(t)$, cannot be negative. Given this constraint, we can treat the MP curve as horizontal over the range of output for which the Federal Reserve's desired value of r would result in a negative value of i; that is, over the range of y for which $r(y(t),\pi(t)) + \pi < 0$ or $r(y(t),\pi(t)) < -\pi$.



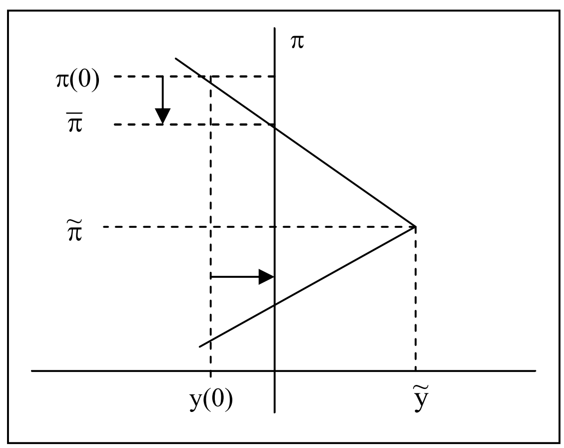
See the figure at right. It is drawn for an inflation rate π_1 at which the non-negative nominal interest rate constraint is not binding. The flat portion of the MP(π_1) curve is horizontal at a real interest rate equal to $-\pi_1$. At the intersection of the IS curve and the MP(π_1) curve, the initial real interest rate is r_1 and the initial level of output is y_1.

Now consider a fall in the inflation rate to $\tilde{\pi} < \pi_1$. The IS curve is unaffected. The upward-sloping portion of the MP curve shifts down since $r > 0$; the Fed would like to set a lower real interest rate at lower levels of inflation. But the horizontal portion of the MP curve shifts up; the real interest rate at which the constraint of a non-negative nominal interest rate becomes binding is now higher (at a 5 percent rate of inflation, the Fed cannot achieve a real interest rate below negative 5 percent; at a 2 percent rate of inflation, the Fed cannot achieve an r below negative 2 percent).

The new MP curve is labeled MP($\tilde{\pi}$) in the figure. As drawn, at the level of output where planned and actual expenditures are equal given the Federal Reserve's choice of r, the constraint is just binding; that is, $r(\tilde{y},\tilde{\pi})+\tilde{\pi}=0$ or $r(\tilde{y},\tilde{\pi})=-\tilde{\pi}$. In terms of the figure, this means that the IS curve intersects the kink in the MP curve. The fall in inflation raises output from y_1 to $\tilde{y}$. Thus over the range of inflation rates exceeding $\tilde{\pi}$, a drop in inflation increases output along the AD curve as it does in our standard model.



Now suppose that inflation falls farther to $\pi_2 < \tilde{\pi}$. Again, the upward-sloping portion of the MP curve shifts down and the horizontal portion shifts up. The new MP curve is labeled MP(π_2) in the figure above. At the intersection of IS and this MP(π_2) curve, we can see that the real interest rate rises to r_2 and output falls to y_2. Intuitively, the nominal interest rate, $i = r + \pi$, is already at zero at an inflation rate of $\tilde{\pi}$. Thus at lower inflation rates such as π_2, the real interest rate must be higher since i will still equal zero. That rise in the real interest rate reduces the level of output at which planned and actual expenditures are equal. Thus as inflation falls below $\tilde{\pi}$, output falls below $\tilde{y}$. The AD curve is backward-bending as shown in the second figure.



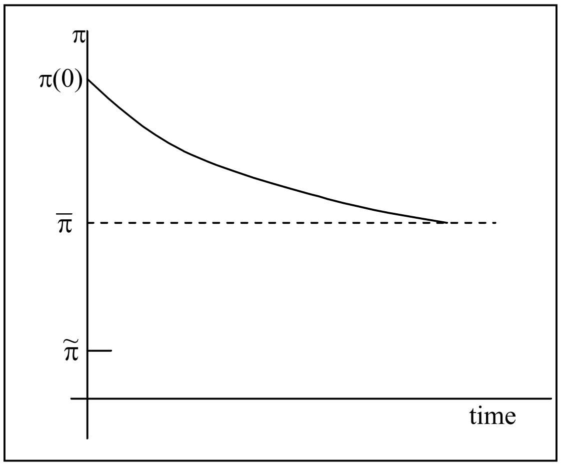
(b) (i) The initial situation is depicted in the figure at right. The initial level of output is given by y(0) and the initial inflation rate is given by π(0). The initial level of output is less than the value that would generate stable inflation.

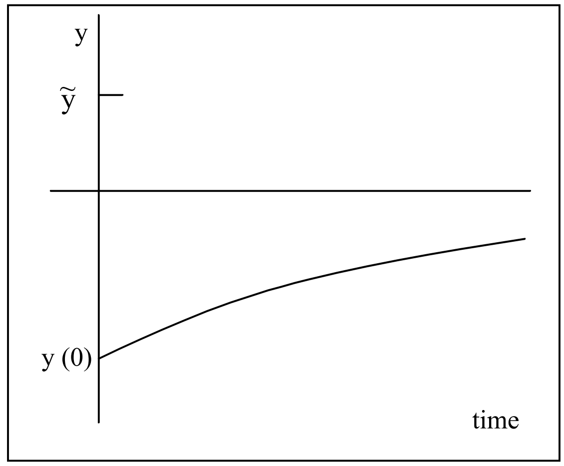
Since $y(0) < \tilde{y}$ and the dynamics of inflation are given by

(4) $\dot{\pi}(t) = \lambda[y(t) - \bar{y}]$,

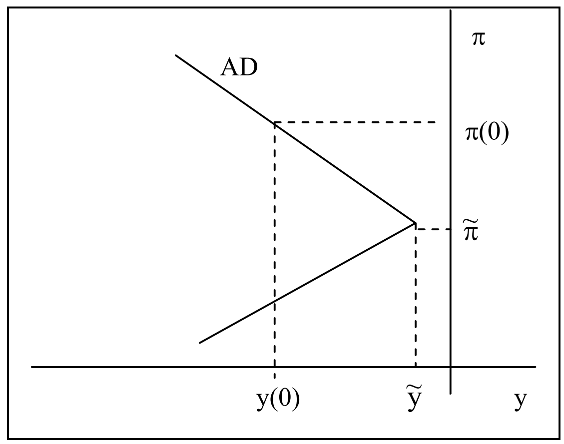
where $\lambda > 0$, inflation falls over time. As inflation falls, the economy moves down along the AD curve and so output rises from y(0) toward 0. Since $\tilde{y} > 0$, the constraint that the nominal interest rate cannot be negative never becomes binding. As output approaches 0, inflation approaches a value of $\bar{\pi}$.

See the figures below.

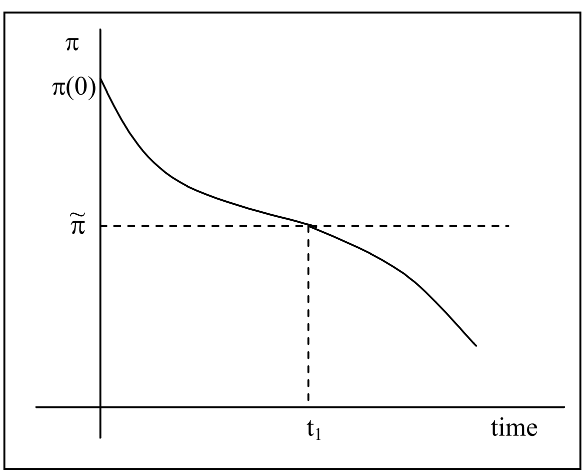


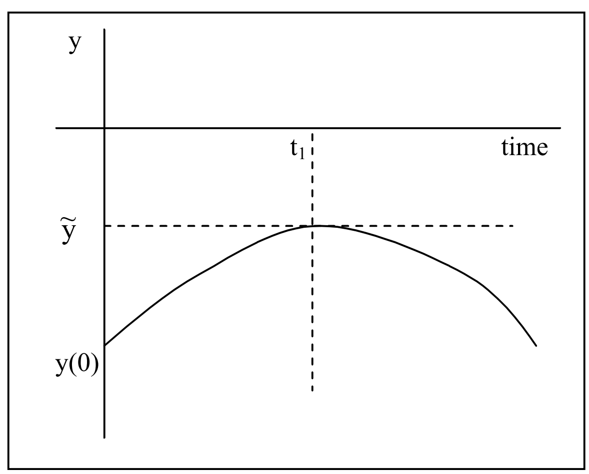


(b) (ii) Since the initial value of inflation, π(0), is greater than $\tilde{\pi}$ and $\tilde{y} < 0$, it must be true that the initial value of y is less than 0. See the figure at right.

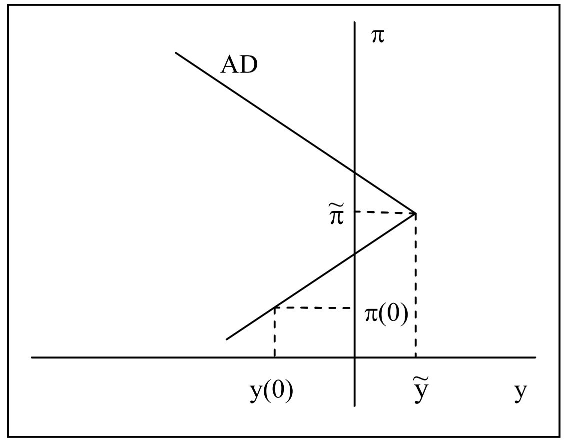


The initial level of output is less than the value that would generate stable inflation. Since y(0) < 0, inflation falls over time. As inflation falls, the economy moves down along the AD curve with output rising and inflation falling. At some time t_1, the economy reaches $\tilde{\pi}$ and $\tilde{y}$. At that point, however, output is still less than 0. Thus inflation continues to fall. In addition, the nominal interest rate is at zero and cannot go any lower. Thus as inflation falls below $\tilde{\pi}$, the real interest rate rises, making output fall. The economy moves down the backward-bending portion of the AD curve with inflation and output continuing to fall. See the figures below.

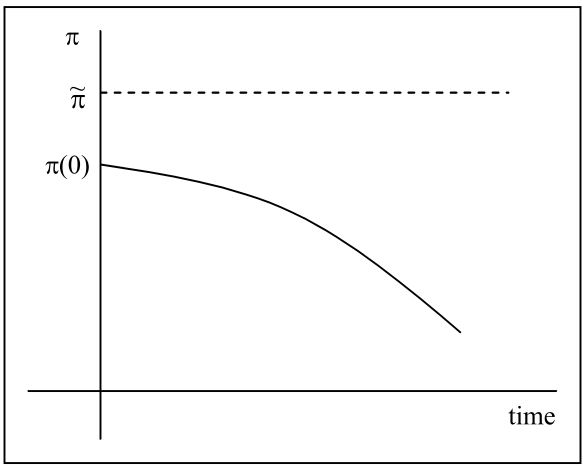


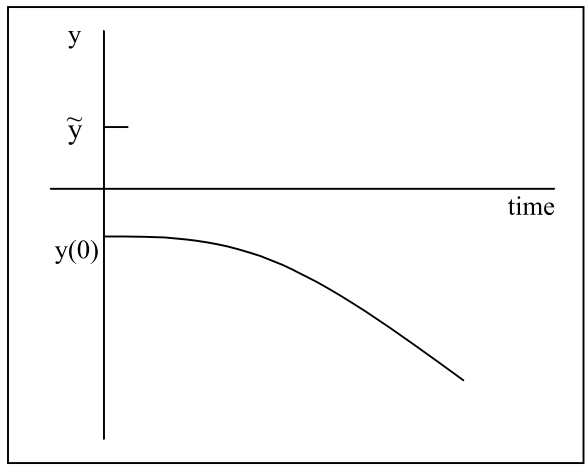


(b) (iii) The initial value of inflation, $\pi(0)$, is less than $\tilde{\pi}$ and the initial value of output, $y(0)$, is less than 0 which, in turn, is less than $\tilde{y}$. We are on the backward-bending portion of the AD curve and the nominal interest rate is zero. See the figure at right.



Since the initial level of output is less than the value that would generate stable inflation, inflation falls over time. As inflation falls, the economy moves down along the backward-bending portion of the AD curve and therefore output falls as well. Intuitively, since the nominal interest rate is already at zero and cannot go any lower, as inflation falls,





the real interest rate rises. Thus output must fall. See the figures above.

Problem 6.8

The MP equation is now given by

(1) $r_t = by_t + u_t^{MP}$,

where

(2) $u_t^{MP} = \rho_{MP} u_{t-1}^{MP} + e_t^{MP}$.

The IS curve no longer contains a shock term and is given by

(3) $y_t = E_t[y_{t+1}] - \dfrac{1}{\theta} r_t$.

Substituting equation (1) into (3) gives us

(4) $y_t = E_t[y_{t+1}] - \dfrac{1}{\theta}[by_t + u_t^{MP}]$,

which simplifies to

(5) $y_t = \dfrac{\theta}{\theta + b} E_t[y_{t+1}] - \dfrac{1}{\theta + b} u_t^{MP}$.

Note that equation (5) holds in all future periods. Denoting $\varphi \equiv \theta/(\theta + b)$, this means we can write

(6) $y_{t+j} = \phi E_{t+j}[y_{t+j+1}] - \frac{\phi}{\theta} u^{MP}_{t+j}$ for j = 1,2,3,…

Taking expectations of both sides of (6) as of time t, and using equation (2) and the law of iterated projections gives us

(7) $E_t[y_{t+j}] = \phi E_t[y_{t+j+1}] - \frac{\phi}{\theta}\rho^{j}_{MP} u^{MP}_{t}$.

We can now iterate equation (5) forward. First, substitute for $E_t[Y_{t+1}]$ to obtain

(8) $y_t = -\frac{\phi}{\theta} u^{MP}_{t} + \phi(\phi E_t[y_{t+2}] - \frac{\phi}{\theta}\rho_{MP} u^{MP}_{t})$.

Now substitute for $E_t[Y_{t+2}]$ to yield

(9) $y_t = -\frac{\phi}{\theta} u^{MP}_{t} - \frac{\phi^2}{\theta}\rho_{MP} u^{MP}_{t} + \phi^2 (\phi E_t[y_{t+3}] - \frac{\phi}{\theta}\rho^{2}_{MP} u^{MP}_{t})$,

and so on. We can therefore rewrite (9) as

(10) $y_t = (1 + \phi\rho_{MP} + \phi^2\rho^2_{MP} + \ldots)\left(-\frac{\phi}{\theta} u^{MP}_{t}\right) + \lim_{n\to\infty} \phi^n E_t[y_{t+n}]$.

Since φ < 1, the limit would fail to converge to 0 only if $E_t[Y_{t+n}]$ diverged. Since this cannot happen—agents cannot expect output to diverge—then the limit converges to 0. Thus, equation (10) becomes

(11) $y_t = \frac{-\phi/\theta}{1 - \phi\rho_{MP}} u^{MP}_{t}$.

Using the definition of φ ≡ θ/(θ + b) we can write (11) as

(12) $y_t = \frac{-\frac{1}{\theta + b}}{\frac{\theta + b - \theta\rho_{MP}}{\theta + b}} u^{MP}_{t}$,

or simply

(13) $y_t = \frac{-1}{\theta + b - \theta\rho_{MP}} u^{MP}_{t}$.

Problem 6.9

(a) Substituting the MP equation given by

(1) $r_t = by_t$,

into the IS equation given by

(2) $y_t = E_t[y_{t+1}] - \frac{1}{\theta} r_t + u^{IS}_{t}$,

gives us

(3) $y_t = E_t[y_{t+1}] - \frac{b}{\theta} y_t + u^{IS}_{t}$.

Equation (3) simplifies to

(4) $y_t = \frac{\theta}{\theta + b} E_t[y_{t+1}] + \frac{\theta}{\theta + b} u^{IS}_{t}$.

If we conjecture that the solution is of the form

(5) $y_t = Au^{IS}_{t}$,

which implies that

(6) $E_t[y_{t+1}] = AE_t[u^{IS}_{t+1}]$,

then equation (4) becomes

(7) $Au_t^{IS} = \frac{\theta}{\theta + b} AE_t[u_{t+1}^{IS}] + \frac{\theta}{\theta + b} u_t^{IS}$.

Because $u_t^{IS} = \rho_{IS} u_{t-1}^{IS} + e_t^{IS}$ holds for each period and because e is white noise, then we can write

(8) $E_t[u_{t+1}^{IS}] = \rho_{IS} u_t^{IS}$.

Substituting equation (8) into (7) yields

(9) $Au_t^{IS} = \frac{\theta}{\theta + b} A\rho_{IS} u_t^{IS} + \frac{\theta}{\theta + b} u_t^{IS}$.

Dividing both sides of equation (9) by u_t^{IS} and collecting the terms in A gives us

(10) $\left[1 - \frac{\theta}{\theta + b}\rho_{IS}\right] A = \frac{\theta}{\theta + b}$.

Equation (10) simplifies to

(11) $\left[\frac{\theta + b - \theta\rho_{IS}}{\theta + b}\right] A = \frac{\theta}{\theta + b}$,

or

(12) $A = \frac{\theta}{\theta + b - \theta\rho_{IS}}$.

Thus, our solution is

(13) $y_t = \frac{\theta}{\theta + b - \theta\rho_{IS}} u_t^{IS}$,

which is the same as the solution given by equation (6.35) in the text.

(b) Substituting our conjectures for inflation and output in period t into the AS equation gives us our first equation:

(14) $Cu_t^{IS} + D\pi_{t-1} = \pi_{t-1} + \lambda[Au_t^{IS} + B\pi_{t-1}]$.

Substituting for inflation and output into the new form of the MP equation gives us our second equation:

(15) $r_t = b(Au_t^{IS} + B\pi_{t-1}) + c(Cu_t^{IS} + D\pi_{t-1})$.

Substituting for output in the IS equation yields our third equation:

(16) $Au_t^{IS} + B\pi_{t-1} = E_t[Y_{t+1}] - (1/\theta)r_t + u_t^{IS}$.

Finally, our conjecture that the solution is of the form

(17) $y_t = Au_t^{IS} + B\pi_{t-1}$,

implies that

(18) $E_t[y_{t+1}] = AE_t[u_{t+1}^{IS}] + B\pi_t$.

Substituting equation (8) for $E_t[u_{t+1}^{IS}]$ and our conjecture for π_t yields our fourth equation:

(19) $E_t[y_{t+1}] = A\rho_{IS} u_t^{IS} + B(Cu_t^{IS} + D\pi_{t-1})$.

Problem 6.10

(a) Substituting the expression for aggregate demand, y = m - p, into the equation that defines the optimal price for firms, $p^* = p + \phi y$, yields $p^* = p + \phi(m - p)$ or simply

(1) $p^* = (1 - \phi)p + \phi m$.

Substituting the aggregate price level, $p = fp^*$, and $m = m'$ into equation (1) yields

(2) $p^* = (1 - \phi)fp^* + \phi m'$.

Solving for p* gives us

(3) $p^* = \dfrac{\phi}{1-(1-\phi)f} m'$.

Now substitute equation (3) into the expression for the aggregate price level, p = fp*, to obtain

(4) $p = \dfrac{\phi f}{1-(1-\phi)f} m'$.

Substituting equation (4) and m = m' into the expression for aggregate demand, y = m - p, yields

(5) $y = m' - \dfrac{\phi f}{1-(1-\phi)f} m' = \left[\dfrac{1-f+\phi f-\phi f}{1-(1-\phi)f}\right] m'$,

or simply

(6) $y = \dfrac{(1-f)}{1-(1-\phi)f} m'$.

(b) Substituting equation (3), the expression for a firm's optimal price, into the expression describing the firm's incentive to adjust its price, Kp^{*2}, yields

(7) $Kp^{*2} = K\left[\dfrac{\phi m'}{1-(1-\phi)f}\right]^2$.

We need to plot this incentive to change price as a function of f, the fraction of firms that change their price. The following derivatives will be useful:

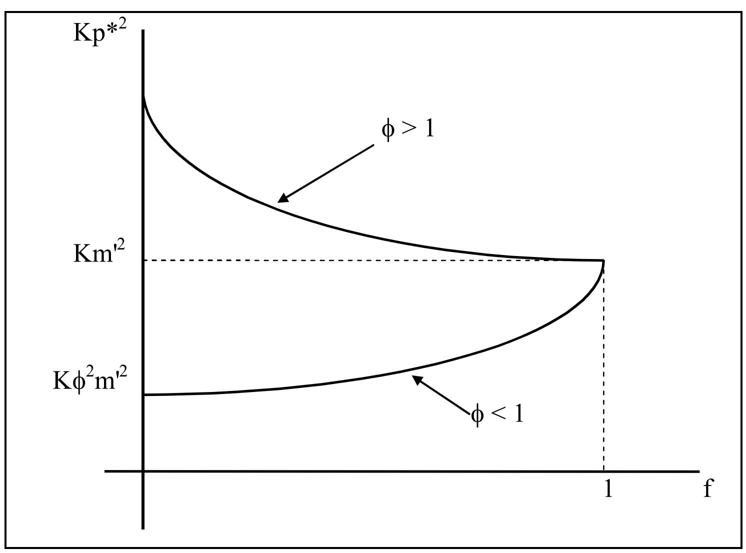
(8) $\dfrac{\partial\left[Kp^{*2}\right]}{\partial f} = \dfrac{2K(1-\phi)(\phi m')^2}{\left[1-(1-\phi)f\right]^3}$,

and

(9) $\dfrac{\partial^2\left[Kp^{*2}\right]}{\partial f^2} = \dfrac{6K(1-\phi)^2(\phi m')^2}{\left[1-(1-\phi)f\right]^4}$.

When $\phi < 1$, $\partial[Kp^{*2}]/\partial f > 0$ and $\partial^2[Kp^{*2}]/\partial f^2 > 0$. From equation (7), at f = 1, $Kp^{*2} = K[\phi m']^2/\phi^2 = Km'^2$. At f = 0, $Kp^{*2} = K\phi^2 m'^2 < Km'^2$ when $\phi < 1$.

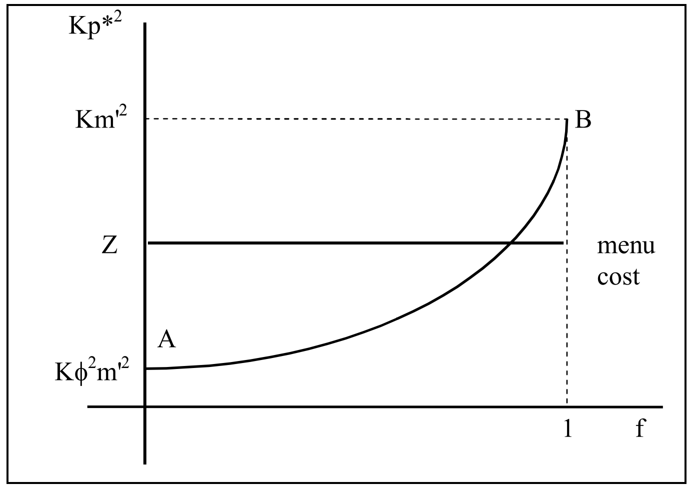
Thus when $\phi < 1$, the incentive for a firm to adjust its price is an increasing function of how many other firms change their price. See the figure at right.



When $\phi > 1$, $\partial[Kp^{*2}]/\partial f < 0$ and $\partial^2[Kp^{*2}]/\partial f^2 > 0$. From equation (7), at f = 1, $Kp^{*2} = K[\phi m']^2/\phi^2 = Km'^2$. At f = 0, $Kp^{*2} = K\phi^2 m'^2 > Km'^2$ when $\phi > 1$.

Thus when $\phi > 1$, the incentive for a firm to adjust its price is a decreasing function of how many other firms change their price. See the figure.

(c) In the case of $\phi < 1$, there can be a situation where both adjustment by all firms and adjustment by no firms are equilibria.

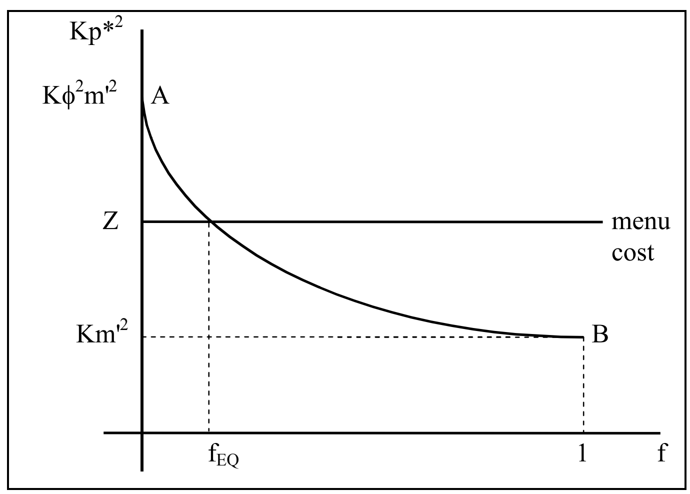


See the figure at right where the menu cost, Z, is assumed to be such that $K\phi^2m'^2 < Z < Km'^2$.

Point A is an equilibrium with f = 0. Consider the situation of a representative firm at point A. If no one else is changing their price, the profits a firm loses by not changing its price, which are given by $Kp^{*2} = K\phi^2m'^2$, are less than the menu cost of Z. Thus it is optimal for the representative firm not to change its price. This is true for all firms and thus no one changing price is an equilibrium.

Point B is also an equilibrium with f = 1. Consider the situation of a representative firm at point B. If everyone else is changing their price, the profits a firm loses by not changing its price, $Kp^{*2} = Km'^2$, exceed the menu cost of Z. Thus it is optimal for the representative firm to change its price. This is true for all firms and thus everyone changing price is also an equilibrium.

In the case of $\phi > 1$, there can be a situation where neither adjustment by all firms nor adjustment by no firms are equilibria. See the figure at right where the menu cost, Z, is assumed to be such that $Km'^2 < Z < K\phi^2m'^2$.



Consider the situation of f = 0 at point A. If no one else is changing their price, the profits that a representative firm would lose by not changing price, $Kp^{*2} = K\phi^2m'^2$, exceed the menu cost Z. Thus it is optimal for the firm to change its price. This is true for all firms and thus it cannot be an equilibrium for no one to change their price.

Now consider the situation of f = 1 at point B. If everyone else is changing their price, the profits that a representative firm would lose by not changing its price, $Kp^{*2} = Km'^2$, are less than the menu cost of Z. Thus it is optimal for the representative firm not to change its price. This is true for all firms and thus it cannot be an equilibrium for all firms to change their price.

From this discussion, we can see that the equilibrium in this case is for fraction f_{EQ} of firms to change their price. If fraction f_{EQ} of firms are changing their price, the profit that a representative firm would lose by not changing its price is exactly equal to the menu cost, Z. Thus the representative firm is indifferent and there is no tendency for the economy to move away from this point where fraction f_{EQ} of firms are changing their price.

Problem 6.11

(a) $\pi(y_1, r^*(y_1))$ is the profit a firm receives at aggregate output level y_1, if it charges the profit-maximizing real price, $r^*(y_1)$. $\pi(y_1, r^*(y_0))$ is the profit a firm receives at aggregate output level y_1 if it continues to charge a real price of $r^*(y_0)$, which was the optimal price to charge when aggregate output was y_0. Thus $G = \pi(y_1, r^*(y_1)) - \pi(y_1, r^*(y_0))$ is the additional profit a firm would receive if, when aggregate output changes from y_0 to y_1, the firm changes its price to its new profit-maximizing level. This represents, therefore, the firm's incentive to change its price in the face of a change in aggregate real output.

(b) The second-order Taylor approximation will be of the form

(1) $$G \cong G\Big|_{y_1=y_o} + \left[\frac{\partial G}{\partial y_1}\bigg|_{y_1=y_0}\right][y_1 - y_0] - \frac{1}{2}\left[\frac{\partial^2 G}{\partial y_1^{\,2}}\bigg|_{y_1=y_0}\right][y_1 - y_0]^2.$$

Clearly, G evaluated at $y_1 = y_0$ is equal to zero. In addition

(2) $\partial G/\partial y_1 = \pi_1(y_1, r^*(y_1)) + \pi_2(y_1, r^*(y_1))[r^*\,'(y_1)] - \pi_1(y_1, r^*(y_0))$.

Evaluating this derivative at $y_1 = y_0$ gives us

(3) $\partial G/\partial y_1 |_{y1=y0} = \pi_1(y_0, r^*(y_0)) + \pi_2(y_0, r^*(y_0))[r^*\,'(y_0)] - \pi_1(y_0, r^*(y_0)) = 0$.

Since $r^*(y_0)$ is defined implicitly by $\pi_2(y_0, r^*(y_0)) = 0$, the right-hand side of equation (3) is equal to zero.

Using equation (2) to find the second derivative of G with respect to y_1 gives us

(4) $\partial^2 G/\partial y_1^2 = \pi_{11}(y_1, r^*(y_1)) + \pi_{12}(y_1, r^*(y_1))[r^*\,'(y_1)] +$
$[\pi_{21}(y_1, r^*(y_1)) + \pi_{22}(y_1, r^*(y_1))r^*\,'(y_1)][r^*\,'(y_1)] + \pi_2(y_1, r^*(y_1))[r^*\,''(y_1)] -$
$\pi_{11}(y_1, r^*(y_0))$.

Using the fact that $\pi_2(y_1, r^*(y_1)) = 0$ and $\pi_{12}(y_1, r^*(y_1)) = \pi_{21}(y_1, r^*(y_1))$, equation (4) becomes

(5) $\partial^2 G/\partial y_1^2 = \pi_{11}(y_1, r^*(y_1)) + 2\pi_{12}(y_1, r^*(y_1))[r^*\,'(y_1)] + \pi_{22}(y_1, r^*(y_1))[r^*\,'(y_1)]^2 - \pi_{11}(y_1, r^*(y_0))$.

Evaluating this derivative at $y_1 = y_0$ leaves us with

(6) $\partial^2 G/\partial y_1^2 |_{y1=y0} = 2\pi_{12}(y_0, r^*(y_0))[r^*\,'(y_0)] + \pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)]^2$.

Now differentiate both sides of the equation that implicitly defines $r^*(y_0)$, $\pi_2(y_0, r^*(y_0)) = 0$, with respect to y_0 to obtain

(7) $\pi_{21}(y_0, r^*(y_0)) + \pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)] = 0$,

and thus

(8) $\pi_{21}(y_0, r^*(y_0)) = -\pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)]$.

Substituting equation (8) into equation (6) yields

(9) $\partial^2 G/\partial y_1^2 |_{y1=y0} = -2\pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)]^2 + \pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)]^2$
$= -\pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)]^2$.

Thus, since G and $\partial G/\partial y_1$ evaluated at $y_1 = y_0$ are both equal to zero, substituting equation (9) into equation (1) gives us the second-order Taylor approximation:

(10) $G \cong -\pi_{22}(y_0, r^*(y_0))[r^*\,'(y_0)]^2[y_1 - y_0]^2/2$.

(c) The $[r^*\,'(y_0)]^2$ component reflects the degree of real rigidity. It tells us how much the firm's profit-maximizing real price responds to changes in aggregate real output. The $\pi_{22}(y_0, r^*(y_0))$ component reflects insensitivity of the profit function. It tells us the curvature of the profit function and thus the cost in lost profits from the firm allowing its real price to differ from its profit-maximizing value.

Problem 6.12

(a) Substituting the expression for the nominal wage, $w = \theta p$, into the aggregate price equation, $p = w + (1 - \alpha)\ell - s$, yields $p = \theta p + (1 - \alpha)\ell - s$. Solving for p yields

(1) $p = [(1 - \alpha)\ell - s]/(1 - \theta)$.

Substituting the aggregate output equation, $y = s + \alpha\ell$, and equation (1) for the price level into the aggregate demand equation, $y = m - p$, yields

(2) $s + \alpha\ell = m - [(1 - \alpha)\ell - s]/(1 - \theta)$.

Collecting the terms in ℓ leaves us with

(3) $\alpha\ell + [(1 - \alpha)\ell/(1 - \theta)] = m + [s/(1 - \theta)] - s$.

Obtaining a common denominator and simplifying gives us

(4) $[\alpha(1 - \theta) + (1 - \alpha)]\ell/(1 - \theta) = m + [1 - (1 - \theta)]s/(1 - \theta)$,

or

(5) $(1 - \alpha\theta)\ell/(1 - \theta) = m + [\theta s/(1 - \theta)]$,

and thus finally, employment is given by

(6) $$\ell = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)}.$$

Substituting equation (6) into equation (1) yields

(7) $$p = \frac{(1-\alpha)\,[(1-\theta)m + \theta s]}{(1-\theta)(1-\alpha\theta)} - \frac{s}{(1-\theta)}.$$

Simplifying gives us

(8) $$p = \frac{(1-\alpha)(1-\theta)m + (1-\alpha)\theta s - (1-\alpha\theta)s}{(1-\theta)(1-\alpha\theta)} = \frac{(1-\alpha)(1-\theta)m - (1-\theta)s}{(1-\theta)(1-\alpha\theta)}.$$

Thus, the aggregate price level is given by

(9) $$p = \frac{(1-\alpha)m - s}{(1-\alpha\theta)}.$$

Substituting equation (6) into the aggregate output equation, $y = s + \alpha\ell$, and simplifying yields

(10) $$y = s + \frac{\alpha(1-\theta)m + \alpha\theta s}{(1-\alpha\theta)} = \frac{s - \alpha s + \alpha(1-\theta)m + \alpha\theta s}{(1-\alpha\theta)}.$$

And therefore, output is given by

(11) $$y = \frac{s + \alpha(1-\theta)m}{(1-\alpha\theta)}.$$

Finally, to get an expression for the nominal wage, substitute equation (9) into $w = \theta p$:

(12) $$w = \frac{\theta\left[(1-\alpha)m - s\right]}{(1-\alpha\theta)}.$$

The next step is to see the way in which the degree of indexation affects the responsiveness of employment to monetary shocks. First, use equation (6) to find how employment varies with m:

(13) $$\frac{\partial \ell}{\partial m} = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)}.$$

Taking the derivative of both sides of equation (13) with respect to θ gives us

$$(14)\quad \frac{\partial\left[\partial\ell/\partial \mathrm{m}\right]}{\partial\,\theta}=\frac{(-1)[1-\alpha\theta]-(1-\theta)(-\alpha)}{(1-\alpha\theta)^2}=\frac{(\alpha-1)}{(1-\alpha\theta)^2}<0\,.$$

Thus an increase in the degree of indexation, θ, reduces the amount that employment will change due to a given monetary shock.

The next step is to examine the way in which the degree of indexation affects the responsiveness of employment to supply shocks. First, use equation (6) to find how employment varies with s:

$$(15)\quad \frac{\partial\ell}{\partial \mathrm{s}}=\frac{\theta}{(1-\alpha\theta)}\,.$$

Taking the derivative of both sides of equation (15) with respect to θ gives us

$$(16)\quad \frac{\partial\left[\partial\ell/\partial \mathrm{s}\right]}{\partial\,\theta}=\frac{(1)[1-\alpha\theta]-(\theta)(-\alpha)}{(1-\alpha\theta)^2}=\frac{1}{(1-\alpha\theta)^2}>0\,.$$

Thus an increase in the degree of wage indexation, θ, increases the amount that employment will change due to a given supply shock.

(b) From equation (6), the variance of employment is given by

$$(17)\quad V_\ell=\left[\frac{(1-\theta)}{(1-\alpha\theta)}\right]^2 V_m+\left[\frac{\theta}{(1-\alpha\theta)}\right]^2 V_s\,,$$

where we have used the fact that m and s are independent random variables with variances V_m and V_s. We need to find the value of θ that minimizes this variance of employment. The first-order condition for this minimization is

$$(18)\quad \frac{\partial V_\ell}{\partial\,\theta}=2\left[\frac{(1-\theta)}{(1-\alpha\theta)}\right]\left[\frac{(\alpha-1)}{(1-\alpha\theta)^2}\right]V_m+2\left[\frac{\theta}{(1-\alpha\theta)}\right]\left[\frac{1}{(1-\alpha\theta)^2}\right]V_s=0\,.$$

Equation (18) simplifies to

(19) $0 = (1 - \theta)(\alpha - 1)V_m + \theta V_s$.

Collecting the terms in θ gives us

(20) $\theta[(1 - \alpha)V_m + V_s\,] = (1 - \alpha)V_m$.

Thus the optimal degree of wage indexation is

$$(21)\quad \theta=\frac{(1-\alpha)V_m}{(1-\alpha)V_m+V_s}$$

Given the result in part (a) – that indexation reduces the impact on employment of monetary shocks but increases the impact from supply shocks – equation (21) is intuitive. First, if $V_s = 0$ – so that there are no supply shocks – the optimal degree of indexation is one. In addition, the larger is the variance of the supply shocks relative to the variance of the monetary shocks, the lower is the optimal degree of indexation.

(c) (i) As stated in the problem:

(22) $y_i = y - \phi(w_i - w)$,

where $\phi \equiv \alpha\eta/[\alpha + (1 - \alpha)\eta]$. Since $w = \theta p$ and $w_i = \theta_i p$, equation (22) becomes

(23) $y_i = y - \phi(\theta_i\, p - \theta p) = y - (\theta_i - \theta)\phi p$.

From the production function, $y_i = s + \alpha l_i$ and $y = s + \alpha l$ and thus we can write

(24) $y_i - y = \alpha(l_i - l)$.

Solving equation (24) for employment at firm i yields

(25) $l_i = l + (1/\alpha)(y_i - y)$.
Substituting equation (23) for $y_i - y$ into equation (25) gives us
(26) $l_i = l - (1/\alpha)(\theta_i - \theta)\phi p$.
Substituting equation (6) for aggregate employment and equation (9) for the price level into equation (26) gives us

$$(27)\ \ \ell_i = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)} - \frac{(\theta_i - \theta)\phi\,[(1-\alpha)m - s]}{\alpha(1-\alpha\theta)} = \frac{1}{\alpha(1-\alpha\theta)}\left[\alpha(1-\theta)m + \alpha\theta s - (\theta_i - \theta)\phi[(1-\alpha)m - s]\right],$$

which implies

$$(28)\ \ \ell_i = \frac{1}{\alpha(1-\alpha\theta)}\left\{m\left[\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)\right] + s\left[\alpha\theta + (\theta_i - \theta)\phi\right]\right\}.$$

(c) (ii) From equation (28), the variance of employment at firm i is given by

$$(29)\ \ \mathrm{Var}(\ell_i) = \left[\frac{\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)}{\alpha(1-\alpha\theta)}\right]^2 V_m + \left[\frac{\alpha\theta + (\theta_i - \theta)\phi}{\alpha(1-\alpha\theta)}\right]^2 V_s .$$

The first-order condition for the value of the degree of wage indexation at firm i, θ_i, that minimizes the variance of employment at firm i is

$$(30)\ \ \frac{\partial\,\mathrm{Var}(\ell_i)}{\partial\,\theta_i} = 2\left[\frac{\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)}{\alpha(1-\alpha\theta)}\right]\left[-\phi(1-\alpha)\right]V_m + 2\left[\frac{\alpha\theta + (\theta_i - \theta)\phi}{\alpha(1-\alpha\theta)}\right]\phi V_s = 0 .$$

Equation (30) simplifies to
(31) $\{\alpha(1 - \theta) - \theta_i[\phi(1 - \alpha)] + \theta\phi(1 - \alpha)\}\phi(1 - \alpha)V_m = (\alpha\theta + \theta_i\phi - \theta\phi)\phi V_s$,
which implies
(32) $\theta_i\phi^2 V_s + \theta_i[\phi(1 - \alpha)]^2 V_m = [\alpha(1 - \theta) + \theta\phi(1 - \alpha)]\phi(1 - \alpha)V_m - (\alpha\theta - \theta\phi)\phi V_s$.
Thus θ_i is given by

$$(33)\ \ \theta_i = \frac{[\alpha(1-\theta) + \theta\phi(1-\alpha)]\phi(1-\alpha)V_m - [\theta(\phi - \alpha)]\phi V_s}{\phi^2 V_s + [\phi(1-\alpha)]^2 V_m}.$$

(c) (iii) We need to find a value of θ such that the first-order condition given by equation (31) holds when $\theta_i = \theta$. That is, we need to find a value of θ such that if economy-wide indexation is given by θ, the representative firm, in order to minimize its employment fluctuations, wishes to choose θ as well.
Setting $\theta_i = \theta$ in equation (31) gives us
(34) $\alpha(1 - \theta)\phi(1 - \alpha)V_m = \alpha\theta\phi V_s$,
which implies
(35) $\theta\,[V_s + (1 - \alpha)V_m] = (1 - \alpha)V_m$.
Thus the Nash-equilibrium value of θ is

$$(36)\ \ \theta^{EQ} = \frac{(1-\alpha)V_m}{(1-\alpha)V_m + V_s}.$$

This is exactly the same value of θ we found in part (b); see equation (21). The value of θ that minimizes the variance of aggregate fluctuations in employment is also a Nash equilibrium. Given that other firms are choosing θ^{EQ} as their degree of wage indexation, it is optimal for any individual firm to choose θ^{EQ} as well.

Problem 6.13
(a) We can use the intuitive reasoning employed to explain equation (10.27) in Chapter 10. Consider an asset that "pays" -c when the individual climbs a palm tree and pays $\overline{u}$ when an individual trades and eats another's coconut. Assume that this asset is being priced by risk-neutral investors with required rate of

return equal to r, the individual's discount rate. Since the expected present value of this asset is the same as the individual's expected value of lifetime utility, the asset must have price V_P while the individual is looking for palm trees and price V_C while the individual is looking for other people with coconuts.

For the asset to be held, it must provide an expected rate of return of r. That is, its dividends per unit time plus any expected capital gains or losses per unit time, must equal rV_P. When the individual is looking for palm trees, there are no dividends per unit time. There is a probability b per unit time of a capital "gain" of $(V_C - V_P) - c$; if the individual finds a palm tree and climbs it, the difference in the price of the asset is $V_C - V_P$ and the asset "pays" $-c$ at that time. Thus we have
(1) $rV_P = b(V_C - V_P - c)$.

(b) The asset must have price V_C while the individual is looking for others with coconuts and must provide an expected rate of return of r. Thus its dividends per unit time plus any expected capital gains or losses per unit time, must equal rV_C. When the individual is looking for others with coconuts, there are no dividends per unit time. There is a probability aL per unit time of a capital gain of $(V_P - V_C) + \bar{u}$; if the individual finds someone else with a coconut, trades and eats that coconut, the change in the price of the asset is $(V_P - V_C)$ and the asset pays $\bar{u}$ at that time. Thus we have
(2) $rV_C = aL(V_P - V_C + \bar{u})$.

(c) Solving for V_P in equation (2) gives us
(3) $V_P = (rV_C/aL) + V_C - \bar{u}$.
Substituting equation (3) into equation (1) yields
(4) $r[(rV_C/aL + V_C - \bar{u}] = b[V_C - (rV_C/aL) - V_C + \bar{u} - c]$.
Collecting terms in V_C gives us
(5) $V_C[(r^2/aL) + r + (br/aL)] = r\bar{u} + b\bar{u} - bc$.
Equation (5) can be rewritten as
(6) $V_C[r(r + aL + b)]/aL = \bar{u}(r + b) - bc$.
Thus finally, the value of being in state C is given by

(7) $$V_C = \frac{aL[\bar{u}(r+b) - bc]}{r(r + aL + b)}.$$

Substituting equation (7) into equation (3) yields the following value of being in state P:

(8) $$V_P = \frac{\bar{u}(r+b) - bc}{r + aL + b} + \frac{aL[\bar{u}(r+b) - bc]}{r(r + aL + b)} - \bar{u}.$$

Subtracting equation (8) from equation (7) gives us

(9) $$V_C - V_P = -\left[\frac{\bar{u}(r+b) - bc}{r + aL + b}\right] + \bar{u} = \frac{-\bar{u}r - \bar{u}b + bc + \bar{u}r + \bar{u}aL + \bar{u}b}{r + aL + b},$$

or simply

(10) $$V_C - V_P = \frac{bc + \bar{u}aL}{r + aL + b}.$$

(d) For a steady state in which L—the total number of people carrying coconuts—is constant, the flows out of state C must always equal the flows into state C. That is, the number of people finding a trading partner and eating their coconut per unit time must equal the number of people finding and climbing a tree per unit time.

The number of people leaving state C per unit time is given by the probability of finding a trading partner, aL, multiplied by the number of people with coconuts and looking for a trading partner, L. The number of people entering state C per unit time is given by the probability of finding a tree, b, multiplied by the number of people looking for a tree, (N - L). For a steady state, these two must be equal. That is, a steady state requires

(11) $(aL)L = b(N - L)$.

Rearranging equation (11), we have the following quadratic equation in L:

(12) $aL^2 + bL - bN = 0$.

Using the quadratic formula gives us

(13) $$L = \frac{-b \pm \sqrt{b^2 + 4abN}}{2a} = \frac{-b \pm \sqrt{9b^2}}{2a} = \frac{b}{a},$$

where we have used the given condition that $aN = 2b$. Also note that we can ignore the solution with $L = -2b/a < 0$.

(e) For such a steady-state equilibrium, the gain to an individual from climbing a tree, $V_C - V_P$ – going from having the value of being in state P to the value of being in state C – must be greater than or equal to the cost to the individual of climbing the tree, c. That is, for a steady-state equilibrium where everyone who finds a palm tree actually climbs it, we require

(14) $V_C - V_P \geq c$.

Substituting the steady-state value of $L = b/a$ from equation (13) into the expression for $V_C - V_P$ given in equation (10), we have

(15) $$V_C - V_P = \frac{bc + \bar{u}a(b/a)}{r + a(b/a) + b} = \frac{bc + b\bar{u}}{r + 2b}.$$

Substituting equation (15) into inequality (14) yields

(16) $$\frac{bc + b\bar{u}}{r + 2b} \geq c,$$

which implies that

(17) $bc + b\bar{u} \geq c(r + 2b)$,

or

(18) $c(r + 2b - b) \leq b\bar{u}$.

Thus the cost of climbing a tree must be such that

(19) $c \leq b\bar{u}/(r + b)$.

Note that the maximum possible cost for which it is optimal to always climb a tree when one is found (as long as everyone else is doing so) is increasing in the utility gained from eating a coconut and decreasing in the individual's discount rate.

(f) The situation in which no one who finds a tree climbs it is a steady-state equilibrium for any $c > 0$. If no one else is climbing a tree when they find one, it is optimal for an individual not to climb a tree when she finds one. If the individual were to climb a tree and pick a coconut, she would lose c units of utility with no hope of ever trading with someone else. If she does not climb the tree, she loses no utility. Thus it is optimal not to climb the tree. The decision process is the same for every individual who comes across a tree. Thus no one climbing a tree, or $L = 0$, is a steady-state equilibrium for any $c > 0$. This implies that for $0 < c \leq b\bar{u}/(r + b)$, there is more than one steady-state equilibrium. We have shown two: $L = 0$ and $L = b/a$.

In the situation of multiple equilibria, the one with $L = b/a$ involves higher welfare than the one with $L = 0$. We have shown that in part (e), with $c \leq b\bar{u}/(r + b)$, individuals end up gaining utility each time they climb a tree. That is why they do it. They know that the utility they will eventually receive by trading their coconut outweighs the cost of climbing the tree to obtain their coconut. Thus the equilibrium in which people go through a cycle of searching, climbing, searching, trading and eating etc., generates positive utility for the individual. In the equilibrium with L = 0, people never achieve any positive utility since they never trade and obtain the $\bar{u}$ units of utility from eating another person's coconut.

Problem 6.14

(a) The individual's problem is to choose labor supply, L_i, to maximize expected utility, conditional on the realization of P_i . Since $L_i = Y_i$, the problem is

(1) $\max_{Y_i} E\left[\left(C_i - (1/\gamma)Y_i^{\gamma}\right)\middle| P_i\right]$.

Substituting $C_i = P_iQ_i /P$ and $Q_i = L_i$ gives us

(2) $\max_{Y_i} E\left[\left(\frac{P_iY_i}{P} - \frac{1}{\gamma}Y_i^{\gamma}\right)\middle| P_i\right]$.

Since only P is uncertain, this can be rewritten as

(3) $\max_{Y_i} E\left[(P_i/P)\middle|P_i\right]Y_i - (1/\gamma)Y_i^{\gamma}$.

The first-order condition is given by

(4) $E[(P_i /P)|P_i] - Y_i^{\gamma-1} = 0$,

or

(5) $Y_i^{\gamma-1} = E[(P_i /P)|P_i]$.

Thus optimal labor supply is given by

(6) $Y_i = \left\{E\left[(P_i/P)\middle|P_i\right]\right\}^{1/(\gamma-1)}$.

Taking the log of both sides of equation (6) and defining $y_i \equiv \ln Y_i$ yields

(7) $y_i = [1/(\gamma - 1)]\ln E[(P_i /P)|P_i]$.

(b) The amount of labor the individual supplies if she follows the certainty-equivalence rule is given by (in logs)

(8) $l_i = [1/(\gamma - 1)]E[\ln(P_i /P)|P_i]$.

Since $\ln(P_i/P)$ is a concave function of (P_i/P), by Jensen's inequality $\ln E[(P_i/P)|P_i] > E[\ln(P_i/P)|P_i]$. Thus the amount of labor the individual supplies if she follows the certainty-equivalence rule is <u>less</u> than the optimal amount derived in part (a).

(c) We are given that

(9) $\ln(P_i /P) = E[\ln(P_i /P)|P_i] + u_i$, $\quad u_i \sim N(0,V_u)$.

Taking the exponential function of both sides of equation (9) yields

(10) $P_i/P = e^{E[\ln(P_i/P)\,|\,P_i]}e^{u_i}$.

Now take the expected value, conditional on P_i , of both sides of equation (10):

(11) $E\left[(P_i/P)\middle|P_i\right] = e^{E[\ln(P_i/P)\,|\,P_i]}E[e^{u_i}|P_i]$.

Taking the natural log of both sides of equation (11) yields

(12) $\ln E\left[(P_i/P)\middle|P_i\right] = E\left[\ln(P_i/P)\middle|P_i\right] + \ln E\left[e^{u_i}\middle|P_i\right]$.

Note that $\ln E[e^{u_i}|P_i]$ is just a constant that is independent of P_i. Substituting equation (12) into equation (7), the expression for the optimal amount of (log) labor supply, gives us

(13) $\ell_i = [1/(\gamma-1)][E[\ln(P_i/P)|P_i] + \ln E[e^{u_i}|P_i]]$,

or simply

(14) $\ell_i = [1/(\gamma-1)]E[\ln(P_i/P)|P_i] + [1/(\gamma-1)][\ln E[e^{u_i}|P_i]]$.

The first term on the right-hand side of equation (14), $[1/(\gamma - 1)]E[\ln(P_i/P)|P_i]$, is the certainty-equivalence choice of (log) labor supply and the second term is a constant. Thus the l_i that maximizes expected utility differs from the certainty-equivalence rule only by a constant.

Problem 6.15

(a) Model (i) is given by

(1) $y_t = a' z_{t-1} + be_t + v_t$.

This model says that only the unexpected component of money, e_t, affects output. Model (ii) is given by

(2) $y_t = \alpha' z_{t-1} + \beta m_t + v_t$.

This model says that all money matters for output.

Substituting the assumption about monetary policy, $m_t = c' z_{t-1} + e_t$, into equation (2) yields

(3) $y_t = \alpha' z_{t-1} + \beta[c' z_{t-1} + e_t] + v_t$,

and collecting terms in z_{t-1} gives us

(4) $y_t = (\alpha' + \beta c') z_{t-1} + \beta e_t + v_t$.

The models given by equations (1) and (4) cannot be distinguished from one another. Given some a' and b, $\alpha' = a' - \beta c'$ and $\beta = b$ have the same predictions. Intuitively, it is not possible to separate the direct effect of the z's on output from any possible indirect effect they may have through monetary policy. So it could be the case that only unexpected money matters and the effect of the z's on output that we observe is simply their direct effect. However, it could also be the case that the expected component of money affects output and thus the effect of the z's that we observe consists of both the direct and indirect effects.

(b) Substituting the new assumption about monetary policy, $m_t = c' z_{t-1} + \gamma' w_{t-1} + e_t$, into model (ii) yields

(5) $y_t = \alpha' z_{t-1} + \beta[c' z_{t-1} + \gamma' w_{t-1} + e_t] + v_t$,

or collecting the z_{t-1} terms gives us

(6) $y_t = (\alpha' + \beta c') z_{t-1} + \beta\gamma' w_{t-1} + \beta e_t + v_t$.

In this case, it is possible to distinguish between the two theories. Model (i), only unexpected money matters, predicts that the coefficients on the w's should be zero. Model (ii), all money matters, does not predict this. Intuitively, since the w's do not directly affect output, if they are correlated with output it must be due to their indirect effect through their impact on the money supply.

Problem 6.16

(a) Substitute the aggregate price level, $p = qp^r + (1 - q)p^f$, into the expression for the price set by flexible-price firms, $p^f = (1 - \phi)p + \phi m$, to yield

(1) $p^f = (1 - \phi)[qp^r + (1 - q)p^f] + \phi m$.

Solving for p^f yields

(2) $p^f [1 - (1 - \phi)(1 - q)] = (1 - \phi)qp^r + \phi m$.

Since $1 - (1 - \phi)(1 - q) = q + \phi - \phi q = \phi + (1 - \phi)q$, equation (2) can be rewritten as

(3) $p^f [\phi + (1 - \phi)q] = (1 - \phi)qp^r + \phi m$,

and thus finally

(4) $$p^f = \frac{(1-\phi)q}{\phi + (1-\phi)q} p^r + \frac{\phi}{\phi + (1-\phi)q} m = p^r + \frac{\phi}{\phi + (1-\phi)q}(m - p^r) .$$

(b) Since rigid-price firms set $p^r = (1 - \phi)Ep + \phi Em$, we need to solve for Ep, the expectation of the aggregate price level. Taking the expected value of both sides of $p = qp^r + (1 - q)p^f$ gives us

(5) $Ep = qp^r + (1 - q)Ep^f$.

Thus we have

(6) $p^r = (1 - \phi)[qp^r + (1 - q)Ep^f] + \phi Em$.
The rigid-price firms know how the flexible-price firms will set their price. That is, they know that flexible-price firms will use equation (4) to set their prices. Thus the rational expectation of the price set by the flexible-price firms is

$$(7)\ \ Ep^f = p^r + \frac{\phi}{\phi + (1-\phi)q}(Em - p^r).$$

Substituting equation (7) into equation (6) yields

$$(8)\ \ p^r = (1-\phi)\left\{qp^r + (1-q)\left[p^r + \frac{\phi}{\phi + (1-\phi)q}(Em - p^r)\right]\right\} + \phi Em,$$

which implies

$$(9)\ \ p^r = (1-\phi)p^r + \phi Em + \frac{(1-\phi)(1-q)\phi}{\phi + (1-\phi)q}(Em - p^r).$$

Defining $C \equiv [(1 - \phi)(1 - q)\phi]/[\phi + (1 - \phi)q]$, we can rewrite equation (9) as
(10) $p^r = [1 - (1 - \phi) + C] = (\phi + C)Em$,
or
(11) $p^r (\phi + C) = (\phi + C)Em$,
and thus finally
(12) $p^r = Em$.
Rigid-price firms simply set their prices equal to the expected value of the nominal money stock.

(c) The aggregate price level is given by
(13) $p = qp^r + (1 - q)p^f$.
Substituting equation (4) for p^f into equation (13) yields

$$(14)\ \ p = qp^r + (1-q)\left[p^r + \frac{\phi}{\phi + (1-\phi)q}(m - p^r)\right] = p^r + \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - p^r).$$

Finally, from equation (12), we know that $p^r = Em$. Thus the aggregate price level is

$$(15)\ \ p = Em + \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - Em).$$

We know that $y = m - p$. Adding and subtracting Em on the right-hand side of this expression yields
(16) $y = Em + (m - Em) - p$.
Substituting equation (15) into equation (16) yields

$$(17)\ \ y = (m - Em) - \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - Em) = \frac{\phi + (1-\phi)q - (1-q)\phi}{\phi + (1-\phi)q}(m - Em),$$

which simplifies to

$$(18)\ \ y = \frac{q}{\phi + (1-\phi)q}(m - Em).$$

(c) (i) From equations (15) and (18), we can see that <u>anticipated</u> changes in m affect only prices. Specifically, consider the effects of an upward shift in the entire distribution of m, with the realization of $m - Em$ held fixed. From equation (18) we can see that this will have no effects on real output. In this case, rigid-price firms get to set their price knowing that m has changed and thus incorporate it into their price-setting decision.

(c) (ii) <u>Unanticipated</u> changes in m affect real output. That is, a higher value of m given its distribution—that is, given Em—does raise y as we can see from equation (18). In this case, the rigid-price firms do not get to observe the higher realization of m and cannot incorporate it into their price-setting decision and hence the economy does not achieve the flexible-price equilibrium.

In addition, flexible-price firms are reluctant to allow their real prices to change. One can show that

$$(19)\ \frac{\partial y}{\partial \phi} = \frac{-(1-q)q}{\left[\phi + (1-\phi)q\right]^2}\left[m - Em\right].$$

The derivative in equation (19) is negative for m > Em.

Thus a lower value of ϕ —a higher degree of "real rigidity"—leads to a higher level of output for any given positive realization of m - Em. This means that the impact on real output of an unanticipated increase in aggregate demand is larger the larger is the degree of real rigidity or the more reluctant are flexible-price firms to allow their real prices to vary.

SOLUTIONS TO CHAPTER 7

Problem 7.1

When t is even, the price level is given by

(1) $p_t = fp_t^1 + (1 - f)p_t^2$,

where p_t^1 denotes the price set for t by the fraction f of firms who set their prices in the odd period t – 1, and p_t^2 denotes the price set for t by the fraction (1 – f) of firms who set their prices in the even period t – 2. Now, p_t^1 equals the expectation as of period t – 1 of p_{it}^* and thus

(2) $p_t^1 = E_{t-1}\, p_{it}^* = E_{t-1}\, [\phi m_t + (1 - \phi)p_t\,]$.

Substituting equation (1) into equation (2) and using the fact that p_t^2 is already known when p_1^t is set and is thus not uncertain, yields

(3) $p_t^1 = \phi E_{t-1}\, m_t + (1 - \phi)\, [fp_t^1 + (1 - f)p_t^2\,]$.

Some simple algebra allows us to solve for p_t^1 :

(4) $$p_t^1 = \frac{\phi}{1-(1-\phi)f} E_{t-1} m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f} p_t^2.$$

Now, p_t^2 equals the expectation as of period t - 2 of p_{it}^* and thus

(5) $p_t^2 = E_{t-2}\, p_{it}^* = E_{t-2}\, [\phi m_t + (1 - \phi)p_t\,]$.

Substituting equation (1) into equation (5) yields

(6) $p_t^2 = \phi E_{t-2}\, m_t + (1 - \phi)\, [fE_{t-2}\, p_t^1 + (1 - f)p_t^2\,]$.

We need to find $E_{t-2}\, p_t^1$. Since the left- and right-hand sides of equation (4) are equal and since expectations are rational, the expectation as of period t - 2 of these two expressions must be equal. That is, we have

(7) $$E_{t-2} p_t^1 = \frac{\phi}{1-(1-\phi)f} E_{t-2} m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f} p_t^2,$$

where we have used the law of iterated projections so that $E_{t-2}\, E_{t-1}\, m_t = E_{t-2}\, m_t$. Substituting equation (7) into equation (6) yields

(8) $$p_t^2 = \phi E_{t-2} m_t + (1-\phi)\left[\frac{\phi f}{1-(1-\phi)f} E_{t-2} m_t + \frac{(1-\phi)(1-f)f}{1-(1-\phi)f} p_t^2 + (1-f)p_t^2\right].$$

Collecting terms gives us

(9) $$p_t^2 = \left[\frac{\phi - \phi(1-\phi)f + (1-\phi)\phi f}{1-(1-\phi)f}\right] E_{t-2} m_t + (1-\phi)\left[\frac{(1-\phi)(1-f)f + (1-f) - (1-\phi)(1-f)f}{1-(1-\phi)f}\right] p_t^2,$$

which simplifies to

(10) $$p_t^2 = \frac{\phi}{1-(1-\phi)f} E_{t-2} m_t + \frac{(1-\phi)(1-f)}{1-(1-\phi)f} p_t^2.$$

Collecting the terms in p_t^2 gives us

(11) $$\left[\frac{1-(1-\phi)f - (1-\phi)(1-f)}{1-(1-\phi)f}\right] p_t^2 = \frac{\phi}{1-(1-\phi)f} E_{t-2} m_t,$$

or simply

(12) $$\frac{\phi}{1-(1-\phi)f} p_t^2 = \frac{\phi}{1-(1-\phi)f} E_{t-2} m_t.$$

And thus the price set in period t - 2 is given by

(13) $p_t^2 = E_{t-2}\, m_t$.

Now to solve for the price set for t by those setting price in t - 1, substitute equation (13) into equation (4):

(14) $p_t^1 = \dfrac{\phi}{1-(1-\phi)f} E_{t-1}m_t + \dfrac{(1-\phi)(1-f)}{1-(1-\phi)f} E_{t-2}m_t$.

Adding and subtracting $E_{t-2}\, m_t$ to the right-hand side of equation (14) yields

(15) $p_t^1 = E_{t-2}m_t + \dfrac{\phi}{1-(1-\phi)f} E_{t-1}m_t + \left[\dfrac{(1-\phi)(1-f)-1+(1-\phi)f}{1-(1-\phi)f}\right] E_{t-2}m_t$.

Since $(1 - \phi)(1 - f) - 1 + (1 - \phi)f = -(1 - \phi)f + (1 - \phi) - 1 + (1 - \phi)f = -\phi$, equation (15) can be rewritten as

(16) $p_t^1 = E_{t-2}m_t + \dfrac{\phi}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)$.

To get an expression for the aggregate price level, substitute the formulas for p_t^1 and p_t^2, equations (16) and (13), into equation (1):

(17) $p_t = f\left[E_{t-2}m_t + \dfrac{\phi}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)\right] + (1-f)E_{t-2}m_t$.

Simplifying yields

(18) $p_t = E_{t-2}m_t + \dfrac{\phi f}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)$.

To solve for output in period t, substitute equation (18) into the expression for aggregate demand, $y_t = m_t - p_t$:

(19) $y_t = m_t - E_{t-2}m_t - \dfrac{\phi f}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)$.

Collecting the terms in $E_{t-2}\, m_t$ as well as adding and subtracting $E_{t-1}\, m_t$ to the right-hand side of equation (19) gives us

(20) $y_t = m_t - E_{t-1}m_t + \left[\dfrac{\phi f - 1 + (1-\phi)f}{1-(1-\phi)f}\right]E_{t-2}m_t + \left[\dfrac{1-(1-\phi)f - \phi f}{1-(1-\phi)f}\right]E_{t-1}m_t$.

Since $\phi f - 1 + (1 - \phi)f = -(1 - f)$ and $1 - (1 - \phi)f - \phi f = (1 - f)$, equation (20) can be rewritten as

(21) $y_t = \dfrac{(1-f)}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right) + \left(m_t - E_{t-1}m_t\right)$.

Equations (18) and (21) give equilibrium price and output for an even period.

The analysis for the case of t odd is identical to the preceding analysis, except that the roles of f and $(1 - f)$ are reversed. Thus derivations analogous to those used to obtain (18) and (21) yield

(22) $p_t = E_{t-2}m_t + \dfrac{\phi(1-f)}{1-(1-\phi)(1-f)}\left(E_{t-1}m_t - E_{t-2}m_t\right)$,

and

(23) $y_t = \dfrac{f}{1-(1-\phi)(1-f)}\left(E_{t-1}m_t - E_{t-2}m_t\right) + \left(m_t - E_{t-1}m_t\right)$.

Equations (22) and (23) give equilibrium price and output for an odd period.

Problem 7.2

We will deal first with firms that set price in odd periods. Suppose that period t is an even period. Then p_{it} is the price set for an even period by a firm that sets prices in odd periods and p_{it+1} is the price set for an odd period by a firm that sets prices in odd periods. From equation (16) in the solution to Problem 7.1, p_{it} is given by

(1) $p_{it} = E_{t-2}m_t + \dfrac{\phi}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)$.

With the assumption that m is a random walk, we have $E_{t-2}\, m_t = m_{t-2}$ and $E_{t-1}\, m_t = m_{t-1}$. Thus

(2) $p_{it} = m_{t-2} + \dfrac{\phi}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right)$.

As usual, the optimal price for period t is given by

(3) $p_{it}^* = \phi m_t + (1 - \phi)p_t$.

From equation (18) in the solution to Problem 7.1, the aggregate price level in an even period is

(4) $p_t = E_{t-2}m_t + \dfrac{\phi f}{1-(1-\phi)f}\left(E_{t-1}m_t - E_{t-2}m_t\right)$.

Again, since m follows a random walk, this is equivalent to

(5) $p_t = m_{t-2} + \dfrac{\phi f}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right)$.

Substituting equation (5) into equation (3) yields the optimal price in period t:

(6) $p_{it}^* = \phi m_t + (1-\phi)m_{t-2} + \dfrac{(1-\phi)\phi f}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right)$.

Thus, the amount of profit a firm expects to lose in period t is proportional to $E_t[(p_{it} - p_{it}^*)^2]$ or

$$E_t\left[\left(m_{t-2} + \frac{\phi}{1-(1-\phi)f}(m_{t-1} - m_{t-2}) - \phi m_t - (1-\phi)m_{t-2} + \frac{(1-\phi)\phi f}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right)\right)^2\right]$$

Collecting terms yields

(7) $E_t[(p_{it} - p_{it}^*)^2] = E_t\left[\left(-\phi m_t + \dfrac{\phi[1-(1-\phi)f]}{1-(1-\phi)f}\left(m_{t-1} - m_{t-2}\right) + \phi m_{t-2}\right)^2\right]$.

Simplifying gives us

(8) $E_t\,[(p_{it} - p_{it}^*)^2\,] = E_t\,[(-\phi m_t + \phi m_{t-1} - \phi m_{t-2} + \phi m_{t-2}\,)^2]$,

or

(9) $E_t\,[(p_{it} - p_{it}^*)^2\,] = \phi^2\, E_t\,[(m_{t-1} - m_t\,)^2]$.

Now, the price set for an odd period by a firm that sets prices in odd periods is given by

(9) $p_{it+1} = E_{t-1}\, m_{t+1}$.

Since m follows a random walk, $E_{t-1}\, m_{t+1} = m_{t-1}$ and thus

(10) $p_{it+1} = m_{t-1}$.

The optimal price for period t + 1 is given by

(11) $p_{it+1}^* = \phi m_{t+1} + (1 - \phi)p_{t+1}$.

From equation (22) in the solution to Problem 7.1, the aggregate price level in an odd period is

(12) $p_{t+1} = E_{t-1}m_{t+1} + \dfrac{\phi(1-f)}{1-(1-\phi)(1-f)}\left(E_t m_{t+1} - E_{t-1}m_{t+1}\right)$.

Since $E_{t-1}\, m_{t+1} = m_{t-1}$ and $E_t\, m_{t+1} = m_t$, we have

(13) $p_{t+1} = m_{t-1} + \dfrac{\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_t - m_{t-1}\right)$.

Substituting equation (13) into equation (11) yields the optimal price in period t + 1:

(14) $p_{it+1}^* = \phi m_{t+1} + (1-\phi)m_{t-1} + \dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_t - m_{t-1}\right)$.

Thus the amount of profit a firm expects to lose in period t +1 is proportional to

(15) $E_t[(p_{it+1} - p_{it+1}^*)^2] = E_t\left[\left(m_{t-1} - \phi m_{t+1} - (1-\phi)m_{t-1} - \dfrac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\left(m_t - m_{t-1}\right)\right)^2\right]$.

Collecting terms gives us

$$(16)\quad E_t[(p_{it+1} - p_{it+1}{*})^2] = E_t\left[\left(\phi(m_{t-1} - m_{t+1}) + \frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}(m_{t-1} - m_t)\right)^2\right].$$

Expanding the right-hand side of equation (16) yields

$$(17)\quad \begin{aligned} E_t[(p_{it+1} - p_{it+1}{*})^2] &= \phi^2 E_t[(m_{t-1} - m_{t+1})^2] + \frac{2\phi(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} E_t[(m_{t-1} - m_{t+1})(m_{t-1} - m_t)] + \\ &\left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\right]^2 E_t[(m_{t-1} - m_t)^2]. \end{aligned}$$

Since m follows a random walk, we have $E_t[(m_{t-1} - m_{t+1})(m_{t-1} - m_t)] = E_t[m_{t-1} - m_{t+1}]E_t[m_{t-1} - m_t] = (m_{t-1} - m_t)(m_{t-1} - m_{t-1}) = 0$. Thus, the second term on the right-hand side of equation (17) is equal to 0. Using this fact, we can add equations (8) and (17). Thus, the total amount of profit a firm setting prices in odd periods expects to lose is proportional to $E_t[(p_{it} - p_{it}{*})^2] + E_t[(p_{it+1} - p_{it+1}{*})^2]$ or

(18) Exp. Loss_{odd} =

$$\phi^2 E_t[(m_{t-1} - m_t)^2] + \phi^2 E_t[(m_{t-1} - m_{t+1})^2] + \left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\right]^2 E_t[(m_{t-1} - m_t)^2].$$

Now we will deal with firms that set price in even periods. When period t is an odd period, analysis parallel to that used to derive (8) shows that the amount of profit a firm expects to lose in period t + 1 is proportional to

(19) $E_t(p_{it} - p_{it}{*})^2 = \phi^2 E_t(m_{t-1} - m_t)^2$.

Analysis parallel to that used to derive (17) shows that the amount of profit a firm expects to lose in period t + 1 is proportional to

$$(20)\quad \begin{aligned} E_t[(p_{it+1} - p_{it+1}{*})^2] &= \phi^2 E_t[(m_{t-1} - m_{t+1})^2] + \frac{2\phi(1-\phi)\phi f}{1-(1-\phi)f} E_t[(m_{t-1} - m_{t+1})(m_{t-1} - m_t)] + \\ &\left[\frac{(1-\phi)\phi f}{1-(1-\phi)f}\right]^2 E_t[(m_{t-1} - m_t)^2]. \end{aligned}$$

Note that this is identical to (17) except that the roles of f and (1 – f) are reversed. Proceeding as above, since m follows a random walk, we have $E_t[(m_{t-1} - m_{t+1})(m_{t-1} - m_t)] = E_t[m_{t-1} - m_{t+1}]E_t[m_{t-1} - m_t] = (m_{t-1} - m_t)(m_{t-1} - m_{t-1}) = 0$. Thus, the second term on the right-hand side of equation (20) is equal to 0. Using this fact, we can add equations (19) and (20). Thus, the total amount of profit a firm setting prices in even periods expects to lose is proportional to $E_t[(p_{it} - p_{it}{*})^2] + E_t[(p_{it+1} - p_{it+1}{*})^2]$ or

$$(21)\quad \text{Exp. Loss}_{even} = \phi^2 E_t[(m_{t-1} - m_t)^2] + \phi^2 E_t[(m_{t-1} - m_{t+1})^2] + \left[\frac{(1-\phi)\phi f}{1-(1-\phi)f}\right]^2 E_t[(m_{t-1} - m_t)^2].$$

We need to compare the right-hand sides of equations (18) and (21). Recall that f is the fraction of firms that set prices in odd periods. Note that with f < ½ –more firms setting prices in even periods than in odd periods—we have (1 - f) > f. Using this and the fact that $\phi < 1$, we can say that

$$(22)\quad \left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)}\right]^2 > \left[\frac{(1-\phi)\phi f}{1-(1-\phi)f}\right]^2,$$

since $(1 - \phi)\phi(1 - f) > (1 - \phi)\phi f$ and $1 - (1 - \phi)(1 - f) < 1 - (1 - \phi)f$.

Thus, the right-hand side of equation (18) is greater than the right-hand side of equation (21). This means that the profit a firm expects to lose if it sets prices in odd periods exceeds the profit a firm expects to lose if it sets prices in even periods. Thus it is not optimal to set prices in odd periods and firms would like to switch to setting prices in even periods. This means that with $\phi < 1$, if we start with $f < 1/2$, we would expect to see f go to zero; no firms will set price in odd periods.

By reasoning analogous to that above, we could show that if $f > 1/2$, the inequality in (22) is reversed. Firms setting prices in even periods expect to lose more than firms setting prices in odd periods. Thus it is not optimal to set price in even periods and firms would like to switch to setting prices in odd periods. This means that with $\phi < 1$, if we start with $f > 1/2$, we would expect to see f go to one; all firms will set price in odd periods.

Thus if $\phi < 1$, staggered price setting with $f = 1/2$ is not a stable equilibrium. If the economy starts with anything other than $f = 1/2$, staggered price setting will not prevail. The economy will move to a situation in which all firms set price in the same period.

Problem 7.3

(a) The representative individual will set her price equal to the average of the optimal price for t and the expected optimal price for t + 1. Thus

(1) $x_t = (p_{it}^* + E_t\, p_{it+1}^*)/2$.

Since $p_{it}^* = \phi m_t + (1 - \phi)p_t$ and this holds for all periods, we have

(2) $x_t = [(\phi m_t + (1 - \phi)p_t) + (\phi E_t\, m_{t+1} + (1 - \phi)E_t\, p_{t+1})]/2$.

(b) With synchronization, $p_t = x_t$ and $p_{t+1} = x_t$ and thus

(3) $x_t = [(\phi m_t + (1 - \phi)x_t) + (\phi E_t\, m_{t+1} + (1 - \phi)x_t]/2$.

Simplifying yields

(4) $x_t = [2(1 - \phi)x_t + \phi(m_t + E_t\, m_{t+1})]/2$.

Collecting the terms in x_t yields

(5) $[1 - (1 - \phi)]x_t = \phi(m_t + E_t\, m_{t+1})/2$,

and thus

(6) $x_t = (m_t + E_t\, m_{t+1})/2$.

Firms set their price equal to the average of this period's value of m and the expected value of next period's value of m.

(c) Substitute $p_t = x_t = (m_t + E_t\, m_{t+1})/2$ into the aggregate demand equation to obtain

(7) $y_t = m_t - (m_t + E_t\, m_{t+1})/2$.

Simplifying gives us

(8) $y_t = (m_t - E_t\, m_{t+1})/2$.

Assuming that m follows a random walk so that $E_t\, m_{t+1} = m_t$, we have

(9) $y_t = (m_t - m_t)/2$,

or simply

(10) $y_t = 0$.

Now, substituting $p_{t+1} = x_t = (m_t + E_t\, m_{t+1})/2$ into the aggregate demand equation for period t + 1 gives us

(11) $y_{t+1} = m_{t+1} - [(m_t + E_t\, m_{t+1})/2]$.

Assuming that m follows a random walk so that $E_t\, m_{t+1} = m_t$, we have

(12) $y_{t+1} = m_{t+1} - (m_t + m_t)/2$,

or simply

(13) $y_{t+1} = m_{t+1} - m_t$.

The central result of the Taylor model does not continue to hold. Nominal disturbances that occur in periods when firms are not setting prices feed through one-for-one into output; see equation (13). However, once firms set prices again, output returns to its normal value of 0; see equation (10).

Intuitively, we have removed the mechanism by which the Taylor model generates long-lasting effects of nominal disturbances. In the Taylor model, once firms get to adjust price, they do not adjust fully to a nominal disturbance because they know that not all firms are adjusting at that time. Thus fully adjusting will change their relative price, which they are reluctant to do. But here, firms know that all firms are also adjusting their price at the same time. Thus firms' real prices will not change if they fully adjust and thus they do so.

Problem 7.4

The price set by firms in period t is

(1) $x_t = (p_{it}^* + E_t\, p_{i,t+1}^*)/2 = [(\phi m_t + (1 - \phi)p_t) + (\phi E_t\, m_{t+1} + (1 - \phi)E_t\, p_{t+1})]/2$,

where we have used the fact that $p^* = \phi m + (1 - \phi)p$. Since $p_t = (x_t + x_{t-1})/2$ and $E_t\, m_{t+1} = 0$, equation (1) can be rewritten as

(2) $$x_t = \phi m_t + \frac{(1-\phi)(x_t + x_{t-1})}{2} + \frac{\left[(1-\phi)(x_t + E_t x_{t+1})/2\right]}{2},$$

which simplifies to

(3) $$x_t = \frac{\phi m_t}{2} + \frac{(1-\phi)(x_{t-1} + 2x_t + E_t x_{t+1})}{4}.$$

Solving for x_t yields

(4) $x_t = A(x_{t-1} + E_t\, x_{t+1}) + [(1 - 2A)/2]m_t$,

where $A \equiv \frac{1}{2}\frac{1-\phi}{1+\phi}$.

We need to eliminate $E_t\, x_{t+1}$ from the expression in (4). As in the text, it is reasonable to guess that x_t is a linear function of x_{t-1} and m_t, or

(5) $x_t = \mu + \lambda x_{t-1} + \nu m_t$.

We need to determine whether there are actually values of μ, λ, and ν that solve the model. As explained in the text, each price equals m in the flexible-price equilibrium. In light of this, consider a situation in which x_{t-1} and m_t both equal zero. If period-t price-setters also set their prices to $m_t = 0$, the economy is at its flexible-price equilibrium. In addition, since m is white noise, the period-t price-setters have no reason to expect m_{t+1} to be on average either more or less than zero, and hence no reason to expect x_{t+1} to depart on average from zero. Thus in this situation, p_{it}^* and $E_t\, p_{it+1}^*$ are both equal to zero and so price-setters would choose $x_t = 0$. In summary, it is reasonable to guess that when $x_{t-1} = m_t = 0$, $x_t = 0$. In terms of equation (5), this condition is

(6) $0 = \mu + \lambda(0) + \nu(0)$,

or simply $\mu = 0$. Thus equation (5) becomes

(7) $x_t = \lambda x_{t-1} + \nu m_t$.

Our goal is to find values of λ and ν that solve the model. Since equation (7) holds each period, it implies that $x_{t+1} = \lambda x_t + \nu m_{t+1}$. Thus the expectation as of time t of x_{t+1} is λx_t since $E_t\, m_{t+1} = 0$. Using equation (7) to substitute for x_t yields

(8) $E_t\, x_{t+1} = \lambda^2 x_{t-1} + \lambda\nu m_t$.

Substituting equation (8) into equation (4) gives us

(9) $x_t = A(x_{t-1} + \lambda^2 x_{t-1} + \lambda\nu m_t) + [(1 - 2A)/2]m_t$,

which implies

(10) $x_t = (A + A\lambda^2)x_{t-1} + \{A\lambda\nu + [(1 - 2A)/2]\}m_t$.

The coefficients on x_{t-1} and m_t must be the same in equations (7) and (10). This requires

(11) $A + A\lambda^2 = \lambda$,

and

(12) $A\lambda\nu + (1 - 2A)/2 = \nu$.

Equation (11) is the same as equation (7.38) in the text for the version of the model in which m follows a random walk. The solution to this quadratic is thus given by

(13) $\lambda = \dfrac{1-\sqrt{\phi}}{1+\sqrt{\phi}}$.

Solving for ν in equation (12) yields

(14) $\nu(1 - A\lambda) = (1 - 2A)/2$.

From equation (11), dividing through by λ and rearranging, gives us $1 - A\lambda = A\lambda$. Substituting this expression into equation (14) gives us

(15) $\nu = \dfrac{1-2A}{2}\dfrac{\lambda}{A}$.

Substituting equation (15) into equation (7) yields

(16) $x_t = \lambda x_{t-1} + \dfrac{1-2A}{2}\dfrac{\lambda}{A} m_t$.

Thus equation (16) with λ given by equation (13) solves the model.

We can now describe the behavior of output. Using the definitions of A and λ, some simple algebra allows us to rewrite equation (16) as

(17) $x_t = \lambda x_{t-1} + \dfrac{2\phi}{(1+\sqrt{\phi})^2} m_t = \lambda x_{t-1} + C m_t$,

where we have defined $C \equiv \dfrac{2\phi}{(1+\sqrt{\phi})^2}$.

Since $y_t = m_t - p_t$ and $p_t = (x_t + x_{t-1})/2$ we have

(18) $y_t = m_t - [(x_t + x_{t-1})]/2$.

Substituting equation (17), and equation (17) lagged one period, into equation (18) yields

(19) $y_t = m_t - [(\lambda x_{t-1} + Cm_t + \lambda x_{t-2} + Cm_{t-1})/2]$,

or simply

(20) $y_t = m_t - \lambda p_{t-1} - [(C/2)m_t] - [(C/2)m_{t-1}]$,

where we have used the fact that $p_{t-1} = (x_{t-1} + x_{t-2})/2$. Now since $y_{t-1} = m_{t-1} - p_{t-1}$, this implies

(21) $y_t = m_t + \lambda y_{t-1} - \lambda m_{t-1} - (C/2)m_t - (C/2)m_{t-1}$.

Collecting terms yields

(22) $y_t = \left(1 - \dfrac{C}{2}\right) m_t - \left(\lambda + \dfrac{C}{2}\right) m_{t-1} + \lambda y_{t-1}$.

Finally, since

(23) $1 - \dfrac{C}{2} = 1 - \dfrac{\phi}{(1+\sqrt{\phi})^2} = \dfrac{1 + 2\sqrt{\phi} + \phi - \phi}{(1+\sqrt{\phi})^2} = \dfrac{1 + 2\sqrt{\phi}}{(1+\sqrt{\phi})^2}$,

and

(24) $\lambda + \dfrac{C}{2} = \dfrac{1-\sqrt{\phi}}{1+\sqrt{\phi}} + \dfrac{\phi}{(1+\sqrt{\phi})^2} = \dfrac{(1-\sqrt{\phi})(1+\sqrt{\phi}) + \phi}{(1+\sqrt{\phi})^2} = \dfrac{1 - \phi + \phi}{(1+\sqrt{\phi})^2} = \dfrac{1}{(1+\sqrt{\phi})^2}$.

Using (23) and (24), equation (22) can be rewritten as

(25) $y_t = \lambda y_{t-1} + \dfrac{1+2\sqrt{\phi}}{(1+\sqrt{\phi})^2}\varepsilon_t - \dfrac{1}{(1+\sqrt{\phi})^2}\varepsilon_{t-1}$,

where we have substituted for $m_t = \varepsilon_t$ and $m_{t-1} = \varepsilon_{t-1}$. Thus if the money stock is white noise, output is an ARMA(1,1) process rather than an AR(1) process.

Problem 7.5

We could proceed as in the text and obtain equation (7.50), which holds for a general process for m, and which is given by

(7.50) $x_t = \lambda x_{t-1} + \dfrac{\lambda}{A}\dfrac{1-2A}{2}[m_t + (1+\lambda)(E_t m_{t+1} + \lambda E_t m_{t+2} + \lambda^2 E_t m_{t+3} + \ldots)]$.

When the money stock is white noise, $E_t m_{t+s} = 0$ for all s > 0. Thus equation (7.50) simplifies to

(1) $x_t = \lambda x_{t-1} + \dfrac{\lambda}{A}\dfrac{1-2A}{2}m_t$.

Note that equation (1) is identical to equation (16) in the solution to Problem 7.4. As in that solution, we could now proceed to determine the behavior of output.

Problem 7.6

(a) This conjecture is reasonable. If λ is between 0 and 1, the price level will be positive. The price level in this formula is proportional to the money supply. As t gets larger, $1 - \lambda^t$ will increase if λ is between 0 and 1 and thus the price level will rise. As $t \to \infty$, $p_t \to m_1$; i.e. the price level approaches the money supply in the long run. The fact that price adjustment is Calvo, with a constant fraction of prices being adjusted each period, makes it reasonable to conjecture that the form of adjustment of the price level involves the price level moving a constant fraction of the remaining distance to its long-run value each period.

(b) Our conjecture is that

(1) $p_t = (1-\lambda^t)m_1$.

Substituting equation (1) and the fact that $m_t = m_1$ into $y_t = m_t - p_t$ gives us

(2) $y_t = m_1 - (1-\lambda^t)m_1$,

which simplifies to

(3) $y_t = \lambda^t m_1$.

Substituting equation (1) applied to periods t and t – 1 into the definition of inflation, $\pi_t = p_t - p_{t-1}$, yields

(4) $\pi_t = (1-\lambda^t)m_1 - (1-\lambda^{t-1})m_1$,

which simplifies to

(5) $\pi_t = \lambda^{t-1}(1-\lambda)m_1$.

From equation (7.60) in the text, the new Keynesian Phillips curve is given by

(6) $\pi_t = \kappa y_t + \beta E_t \pi_{t+1}$,

where $\kappa = \alpha[1-(1-\alpha)\beta]\phi/(1-\alpha)$. Substituting equation (3) for y_t and the expectation of π_{t+1} , using equation (5), gives us

(7) $\pi_t = \kappa\lambda^t m_1 + \beta\lambda^t(1-\lambda)m_1$.

Then we can set the right-hand sides of equations (5) and (7) equal to each other to obtain

(8) $\lambda^{t-1}(1-\lambda)m_1 = \kappa\lambda^t m_1 + \beta\lambda^t(1-\lambda)m_1$.

Dividing both sides by $\lambda^{t-1}m_1$ gives us

(9) $1-\lambda = \kappa\lambda + \beta\lambda(1-\lambda)$.

After rearranging the terms, we have the following quadratic in λ:

(10) $\beta\lambda^2 - (1+\kappa+\beta)\lambda + 1 = 0$.

The solutions to this quadratic equation are

(11) $$\lambda_1 = \frac{(1+\kappa+\beta) + \sqrt{(1+\kappa+\beta)^2 - 4\beta}}{2\beta}$$

and

(12) $$\lambda_2 = \frac{(1+\kappa+\beta) - \sqrt{(1+\kappa+\beta)^2 - 4\beta}}{2\beta}.$$

Only one of the two solutions produces reasonable results for the possible range of parameter values. For example, consider $\alpha = 0.5$, $\beta = 1$, and $\varphi = 1$. This implies that $\kappa = 0.5$, $\lambda_1 = 2$, and $\lambda_2 = 0.5$. The value of 2 for λ_1 is not possible because it leads to a negative price, which we can see from equation (1). Thus, we will choose λ_2 as our solution.

(c) Taking the first derivative of λ_2 with respect to α gives us

(13) $$\frac{\partial\lambda_2}{\partial\alpha} = \frac{1}{2\beta}\left[\frac{\partial\kappa}{\partial\alpha} - \frac{1}{2}\frac{2(1+\kappa+\beta)(\partial\kappa/\partial\alpha)}{\sqrt{(1+\kappa+\beta)^2 - 4\beta}}\right],$$

which simplifies to

(14) $$\frac{\partial\lambda_2}{\partial\alpha} = \frac{1}{2\beta}\frac{\partial\kappa}{\partial\alpha}\left[1 - \frac{1+\kappa+\beta}{\sqrt{(1+\kappa+\beta)^2 - 4\beta}}\right].$$

Note that $1+\kappa+\beta > \sqrt{(1+\kappa+\beta)^2 - 4\beta}$ and so the term in square brackets is negative. In addition, $1/2\beta > 0$. Thus, we must determine the sign of $\partial\kappa/\partial\alpha$. We have

(15) $$\begin{aligned}\kappa &= \frac{\alpha\phi[1-(1-\alpha)\beta]}{(1-\alpha)} \\ &= \frac{\alpha}{1-\alpha}\phi - \alpha\beta\phi \quad .\end{aligned}$$

Taking the derivative with respect to α gives us

(16) $$\frac{\partial\kappa}{\partial\alpha} = \frac{\phi(1-\alpha) - \alpha\phi(-1)}{(1-\alpha)^2} - \beta\phi.$$

Equation (16) simplifies to

(17) $$\begin{aligned}\frac{\partial\kappa}{\partial\alpha} &= \frac{\phi}{(1-\alpha)^2} - \beta\phi \\ &= \phi\left[\frac{1}{(1-\alpha)^2} - \beta\right].\end{aligned}$$

Since α is between 0 and 1, $1/(1-\alpha)^2 > 1$. Then, since β is less than 1, the term in square brackets is positive. Since $\varphi > 0$, the derivative $\partial\kappa/\partial\alpha$ is positive. Thus, finally, we have

(18) $$\frac{\partial\lambda_2}{\partial\alpha} = \frac{1}{2\beta}\frac{\partial\kappa}{\partial\alpha}\left[1 - \frac{1+\kappa+\beta}{\sqrt{(1+\kappa+\beta)^2 - 4\beta}}\right] < 0.$$

An increase in the fraction of firms that set new prices each period is associated with a decrease in λ, which implies faster adjustment of the price level to the change in the money supply, according to equation (1).

Next, we first take the first derivative of λ_2 with respect to β to yield

$$(19)\quad \frac{\partial \lambda_2}{\partial \beta} = \frac{\left[\frac{\partial \kappa}{\partial \beta} + 1 - \frac{1}{2}\frac{2(1+\kappa+\beta)[(\partial\kappa/\partial\beta)+1]}{\sqrt{(1+\kappa+\beta)^2-4\beta}}\right](2\beta) - \left[(1+\kappa+\beta) - \sqrt{(1+\kappa+\beta)^2-4\beta}\right](2)}{4\beta^2},$$

which simplifies to

$$(20)\quad \frac{\partial \lambda_2}{\partial \beta} = \frac{\beta\left[\frac{\partial \kappa}{\partial \beta}+1\right]\left[1-\frac{(1+\kappa+\beta)}{\sqrt{(1+\kappa+\beta)^2-4\beta}}\right] - \left[(1+\kappa+\beta)-\sqrt{(1+\kappa+\beta)^2-4\beta}\right]}{2\beta^2}.$$

From equation (15), we have

$$(21)\quad \frac{\partial \kappa}{\partial \beta} + 1 = 1 - \alpha\phi > 0.$$

As explained above, since $1+\kappa+\beta > \sqrt{(1+\kappa+\beta)^2-4\beta}$, we know that

$$(22)\quad \left[1-\frac{(1+\kappa+\beta)}{\sqrt{(1+\kappa+\beta)^2-4\beta}}\right] < 0, \text{ and}$$

$$(23)\quad \left[(1+\kappa+\beta)-\sqrt{(1+\kappa+\beta)^2-4\beta}\right] > 0.$$

Finally, since β is positive, we know that

$$(24)\quad \frac{\partial \lambda_2}{\partial \beta} = \frac{\beta\left[\frac{\partial \kappa}{\partial \beta}+1\right]\left[1-\frac{(1+\kappa+\beta)}{\sqrt{(1+\kappa+\beta)^2-4\beta}}\right] - \left[(1+\kappa+\beta)-\sqrt{(1+\kappa+\beta)^2-4\beta}\right]}{2\beta^2} < 0.$$

An increase in households' discount factor is associated with a decrease in λ, which implies faster adjustment of the price level to the change in the money supply.

Finally, taking the derivative of λ_2 with respect to φ gives us

$$(22)\quad \frac{\partial \lambda_2}{\partial \phi} = \frac{1}{2\beta}\left[\frac{\partial \kappa}{\partial \phi} - \frac{1}{2}\frac{2(1+\kappa+\beta)(\partial\kappa/\partial\phi)}{\sqrt{(1+\kappa+\beta)^2-4\beta}}\right],$$

which simplifies to

$$(23)\quad \frac{\partial \lambda_2}{\partial \phi} = \frac{1}{2\beta}\frac{\partial \kappa}{\partial \phi}\left[1-\frac{1+\kappa+\beta}{\sqrt{(1+\kappa+\beta)^2-4\beta}}\right].$$

As explained above, the term in square brackets is negative and 1/2β is positive. We therefore need to determine the sign of $\partial\kappa/\partial\phi$. Taking the derivative gives us

$$(24)\quad \begin{aligned}\frac{\partial \kappa}{\partial \phi} &= \frac{\alpha}{1-\alpha} - \alpha\beta \\ &= \alpha\left[\frac{1}{1-\alpha} - \beta\right]\end{aligned} .$$

Since α is between 0 and 1, $1/(1-\alpha) > 1$. Then, since β is less than 1, the term in square brackets is positive. Since α is also positive, the derivative $\partial\kappa/\partial\alpha$ is positive. Thus, finally, we have

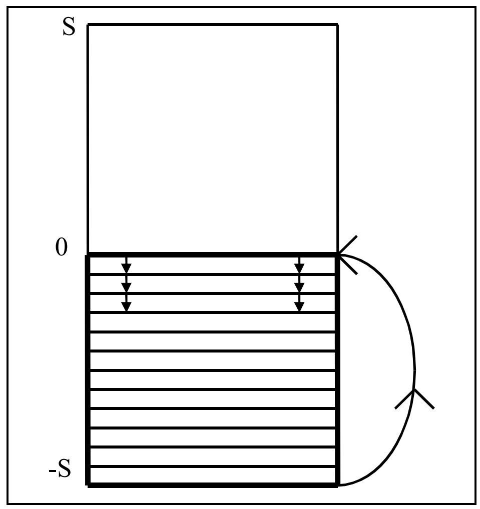
$$(18)\quad \frac{\partial \lambda_2}{\partial \phi} = \frac{1}{2\beta}\frac{\partial \kappa}{\partial \phi}\left[1 - \frac{1+\kappa+\beta}{\sqrt{(1+\kappa+\beta)^2 - 4\beta}}\right] < 0 .$$

A decrease in φ is associated with an increase in λ. Since a decrease in φ means a higher degree of real rigidity, this implies that greater real rigidity leads to slower price adjustment in response to the increase in the money supply.

Problem 7.7

(a) Suppose first that the elevator is not at the top or bottom of the shaft. Now assume that the money supply rises by a small (formally, infinitesimal) amount dm. Since $p_i - p_i^*$ does not equal S or -S for anyone, no prices change. All the $(p_i - p_i^*)$'s fall by dm. The elevator moves down the shaft by dm and stays of height S. Similarly, if m falls, no prices change. The elevator moves up the shaft by dm and stays of height S.

Now suppose the elevator is at the bottom of the shaft. Assume that the money supply rises by dm. Firms that initially have $p_i - p_i^*$ "just above" -S reach the barrier. They therefore raise their price so that $p_i - p_i^* = 0$. Everyone else moves down the shaft by dm. Since the height of the elevator was S, the top of the elevator was initially at zero.



In the figure at right, the horizontal lines represent "slices" of the elevator of infinitesimal height dm. Essentially, the firms at the bottom of the elevator jump up to the top and everyone else moves down by dm. So the elevator does not move or change shape. Thus, with an infinitesimal change in m, the distribution of $p_i - p_i^*$ is unchanged, just as in the Caplin-Spulber model.

The situation in which the elevator is at the bottom of the shaft and m falls is similar to the case in which the elevator is not at the top or bottom of the shaft. It simply moves up by dm.

Finally, the case in which the elevator is at the top of the shaft is the reverse of the case in which it starts at the bottom. If m falls, the elevator does not move. If m rises, the elevator moves down by dm.

(b) For an increase in m, average price is unchanged except if the elevator is at the bottom of the shaft. In this case, the average price rises exactly as much as m. Thus, on average, increases in the money supply increase output.

Problem 7.8

(a) The price set by individuals at time t is

$$(1)\quad x(t)=\frac{1}{T}\int_{\tau=0}^{T} E_t[m(t+\tau)]dt=\frac{1}{T}\int_{\tau=0}^{T}(t+\tau)gd\tau,$$

where we have substituted for the fact that m(t) = gt and thus that E_t [m(t + τ)] = g(t + τ). Solving the integral in (1) gives us

$$(2)\quad \int_{\tau=0}^{T}(t+\tau)gd\tau=\left[gt\tau\Big|_{\tau=0}^{\tau=T}\right]+\left[\frac{1}{2}\tau^2\Big|_{\tau=0}^{\tau=T}\right]=gtT+\frac{1}{2}gT^2.$$

Substituting equation (2) back into equation (1) gives us the price set by individuals at time t, which is

(3) x(t) = gt + (gT/2).

The aggregate price level is the average of the prices set over the last interval of length T. Thus

$$(4)\quad p(t)=\frac{1}{T}\int_{\tau=0}^{T}x(t-\tau)d\tau.$$

Substituting equation (3) into equation (4) yields

$$(5)\quad p(t)=\frac{1}{T}\int_{\tau=0}^{T}\left[g(t-\tau)+\frac{1}{2}gT\right]d\tau.$$

Solving the integral in (5) gives us

$$(6)\quad \int_{\tau=0}^{T}\left[g(t-\tau)+\frac{1}{2}gT\right]d\tau=\left[gt\tau-\frac{1}{2}g\tau^2+\frac{1}{2}gT\tau\right]\Bigg|_{\tau=0}^{\tau=T}=gtT-\frac{1}{2}gT^2+\frac{1}{2}gT^2=gtT.$$

Substituting equation (6) back into equation (5) gives us the price level at time t, which is

(7) p(t) = gt.

Substituting m(t) = gt and p(t) = gt into y(t) = m(t) - p(t) gives us

(8) y(t) = 0.

(b) (i) Suppose that x(t) = gT/2 for all t > 0. Then for t > T, since p(t) is just the average of the x's set over the last interval of length T, p(t) = gT/2. Now we know that for t > T, m(t) = gT/2. Thus for t > T, we do have p(t) = m(t). From y(t) = m(t) - p(t), this means that for all t > T, y(t) = 0, which would have been its value in the absence of the change in policy.

Now consider the situation for some time t between time 0 and time T. From time 0 to time t, we are assuming that individuals set price equal to gT/2. From equation (3), we know that before time 0, individuals set price equal to gt + (gT/2). The aggregate price at time t, which is the average of the prices set by individuals over the past interval of length T, is therefore given by

$$(9)\quad p(t)=\frac{1}{T}\left[\int_{\tau=t-T}^{0}\left(g\tau+\frac{gT}{2}\right)d\tau+\int_{\tau=0}^{t}\left(\frac{gT}{2}\right)d\tau\right]=\frac{1}{T}\left[\int_{\tau=t-T}^{0}\left(g\tau+\frac{gT}{2}\right)d\tau+\frac{gTt}{2}\right].$$

Solving the remaining integral in equation (9) gives us

$$(10)\quad \int_{\tau=t-T}^{0}\left(g\tau+\frac{gT}{2}\right)d\tau=\left[\frac{g\tau^2}{2}+\frac{gT\tau}{2}\right]\Bigg|_{\tau=t-T}^{\tau=0}=-\frac{g(t-T)^2}{2}-\frac{gT(t-T)}{2}.$$

Substituting equation (10) back into equation (9) and expanding yields

$$(11)\quad p(t)=\frac{1}{T}\left[-\frac{g(t-T)^2}{2}-\frac{gT(t-T)}{2}+\frac{gTt}{2}\right]=\frac{1}{T}\left[\frac{-gt^2+2gtT-gT^2-gTt+gT^2+gTt}{2}\right],$$

which implies

(12) $p(t) = gt - \dfrac{gt^2}{2T} = gt\left[1 - \dfrac{t}{2T}\right]$.

Thus p(t) = m(t) for t between 0 and T as well. Thus if x(t) = gT/2 for all t > 0, then p(t) = m(t) for all t > 0, and thus output is the same as it would be without the change in policy.

(b) (ii) At time t, individuals set their prices equal to the average of the expected money supply over the next interval of length T. We know that m(t) = gT/2 for t ≥ T, but it is strictly less than gT/2 for t < T. Thus individuals setting prices at some time t before T are going to set their prices less than gT/2. Why? They will be averaging some m's equal to gT/2 with some m's less than gT/2 and so we must have x(t) < gT/2 for 0 < t < T. For T ≤ t < 2T, the money supply is expected to be constant at gT/2 and thus individuals set prices equal to this constant money supply. Thus x(t) = gT/2 for T ≤ t < 2T.

We have shown in part (b) (i), that <u>if</u> everyone sets prices to gT/2, p(t) = m(t) and thus y(t) = 0, which is its value in the absence of the change in policy. But as we have just explained, individuals actually set prices less than gT/2 for 0 < t < T. Thus the aggregate price level will be less than m(t) over the interval 0 < t < 2T. Since y(t) = m(t) - p(t), this means that output will be greater than zero during this interval. Thus this steady reduction in money growth actually causes output to be higher than it would have been in the absence of the policy change.

<u>Problem 7.9</u>

(a) From equation (7.76), the new Keynesian Phillips curve with indexation is

(1) $\pi_t = \dfrac{1}{1+\beta}\pi_{t-1} + \dfrac{\beta}{1+\beta}E_t\pi_{t+1} + \chi y_t$,

where $\chi \equiv \dfrac{1}{1+\beta}\dfrac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi$. With perfect foresight and β = 1, equation (1) simplifies to

(2) $\pi_t = \dfrac{1}{2}\pi_{t-1} + \dfrac{1}{2}\pi_{t+1} + \chi y_t$.

Using the definition of inflation, $\pi_t = p_t - p_{t-1}$, and the usual aggregate demand equation, $y_t = m_t - p_t$, we can rewrite equation (2) as

(3) $p_t - p_{t-1} = \dfrac{1}{2}(p_{t-1} - p_{t-2}) + \dfrac{1}{2}(p_{t+1} - p_t) + \chi(m_t - p_t)$.

After rearranging the terms, we have an expression for p_{t+1} in terms of its lagged values and m_t:

(4) $p_{t+1} = (3+2\chi)p_t - 3p_{t-1} + p_{t-2} - 2\chi m_t$.

(b) Consider an anticipated, permanent, one-time increase in m: $m_t = 0$ for t < 0 and $m_t = 1$ for all t ≥ 0. Using the lag operator, we can write equation (4) as

(5) $p_{t+1} = (3+2\chi)Lp_{t+1} - 3L^2p_{t+1} + L^3p_{t+1} - 2\chi m_t$.

Collecting the p_{t+1} terms gives us

(6) $[I - (3+2\chi)L + 3L^2 - L^3]p_{t+1} = -2\chi m_t$,

or

(7) $p_{t+1} = [I - (3+2\chi)L + 3L^2 - L^3]^{-1}(-2\chi m_t)$.

Then we can factor $[I - (3+2\chi)L + 3L^2 - L^3]$ as $(I - a_1L)(I - a_2L)(I - a_3L)$ and therefore write

(8) $p_{t+1} = [(I - a_1L)(I - a_2L)(I - a_3L)]^{-1}(-2\chi m_t)$,

or simply

(9) $p_{t+1} = (I - a_1 L)^{-1}(I - a_2 L)^{-1}(I - a_3 L)^{-1}(-2\chi m_t)$.

Then we can express each of $(I - a_1 L)^{-1}, (I - a_2 L)^{-1}, (I - a_3 L)^{-1}$ as an infinite sum of the form $1 + a_i L + a_i^2 L^2 + \ldots$ for i = 1, 2, and 3. Thus, we can rewrite equation (9) as

(10) $p_{t+1} = (1 + a_1 L + a_1^2 L^2 + \ldots)(1 + a_2 L + a_2^2 L^2 + \ldots)(1 + a_3 L + a_3^2 L^2 + \ldots)(-2\chi m_t)$.

Then after distributing the terms for multiplication and some simplification, we find the resulting path of p_t is given by

(11) $p_{t+1} = [1 + (a_1 + a_2 + a_3)L + (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1^2 + a_2^2 + a_3^2)L^2 + \ldots](-2\chi m_t)$.

Problem 7.10

We will first derive the hint given in the problem. With partial indexation, the average (log) price in period t of firms that do not review their prices is $p_{t-1} + \gamma\pi_{t-1}$. Since fraction (1 – α) of firms do not review their price and fraction α set their price equal to x_t, the average price in period t is given by

(1) $p_t = \alpha x_t + (1 - \alpha)(p_{t-1} + \gamma\pi_{t-1})$.

Subtracting p_{t-1} from both sides of equation (1) gives us

(2) $p_t - p_{t-1} = \alpha(x_t - p_{t-1}) + (1 - \alpha)\gamma\pi_{t-1}$.

Adding and subtracting αp_t to the right-hand-side of equation (2) yields

(3) $p_t - p_{t-1} = \alpha[(x_t - p_t) + (p_t - p_{t-1})] + (1 - \alpha)\gamma\pi_{t-1}$.

Since $\pi_t \equiv p_t - p_{t-1}$, we can rewrite (3) as

(4) $\pi_t = \alpha(x_t - p_t) + \alpha\pi_t + (1 - \alpha)\gamma\pi_{t-1}$.

Collecting the terms in π_t gives us

(5) $\alpha(x_t - p_t) = (1 - \alpha)\pi_t - (1 - \alpha)\gamma\pi_{t-1}$.

Thus, finally, we arrive at the hint:

(6) $x_t - p_t = \dfrac{1 - \alpha}{\alpha}\left[\pi_t - \gamma\pi_{t-1}\right]$.

Equation (6) represents one key equation in our solution. The second key equation will be an expression analogous to equation (7.75). Intuitively, from period t to period t + 1, the firm will adjust "γ for 1" to inflation in period t. The firm would prefer not to adjust to π_t at all, though, because given p_t, π_t has no effect on the profit-maximizing price. Also, from t to t + 1, the firm will not adjust at all to expected inflation in period t + 1. The firm would like to adjust one-for-one, though, because given p_t , π_{t+1} moves the proft-maximizing price one-for-one. Thus, we can conjecture that the analogue to equation (7.75) will be:

(7) $x_t - p_t = [1 - \beta(1 - \alpha)]\phi y_t + \beta(1 - \alpha)\left\{E_t[x_{t+1} - p_{t+1}] + E_t[\pi_{t+1}] - \gamma\pi_t\right\}$.

In order to derive (7), we need to generalize equation (7.13) to discounting and to the fact that the price changes between reviews. We will denote the price the firm sets as x_t and allow $z_{t+j}(x_t)$ to denote the price of a firm setting x_t at time t and not getting a chance to review before period t + j. Thus, the analogue to equation (7.13) is

(8) $\min\limits_{x_t} \quad \sum\limits_{j=0}^{\infty}\beta^j q_j \frac{1}{2} E_t\left[\left(z_{t+j}(x_t) - p^*_{t+j}\right)^2\right]$.

Since fraction (1 – α) of firms get to review their price each period, $q_j = (1 - \alpha)^j$. With partial indexation, we have

$$(9)\quad z_{t+j}(x_j) = \begin{cases} x_t & \text{for} \quad j = 0 \\ x_t + \sum\limits_{\tau=0}^{j-1} \gamma \pi_{t+\tau} & \text{for} \quad j \geq 1. \end{cases}$$

The profit-maximizing price in period t + j is given by

$$(10)\quad p^*_{t+j} = \begin{cases} p_t + \phi y_t & \text{for} \quad j = 0 \\ p_t + \sum\limits_{\tau=1}^{j} \pi_{t+\tau} + \phi y_{t+j} & \text{for} \quad j \geq 1. \end{cases}$$

Thus, we can rewrite the minimization problem as

$$(11)\quad \min_{x_t} \quad \frac{1}{2}[x_t - (p_t + \phi y_t)]^2 + \sum_{j=1}^{\infty} \beta^j (1-\alpha)^j \frac{1}{2} E_t\left[\left(x_t + \sum_{\tau=0}^{j-1} \gamma \pi_{t+\tau} - p_t - \sum_{\tau=1}^{j} \pi_{t+\tau} - \phi y_{t+j}\right)^2\right].$$

The first-order condition for x_t is given by

$$(12)\quad x_t - (p_t + \phi y_t) + \sum_{j=1}^{\infty} \beta^j (1-\alpha)^j E_t\left[x_t + \sum_{\tau=0}^{j-1} \gamma \pi_{t+\tau} - p_t - \sum_{\tau=1}^{j} \pi_{t+\tau} - \phi y_{t+j}\right] = 0.$$

We can collect the $(x_t - p_t)$ terms and rewrite equation (12) as

$$(13)\quad \sum_{j=0}^{\infty} \beta^j (1-\alpha)^j (x_t - p_t) = \phi y_t + \sum_{j=1}^{\infty} \beta^j (1-\alpha)^j E_t\left[-\sum_{\tau=0}^{j-1} \gamma \pi_{t+\tau} + \sum_{\tau=1}^{j} \pi_{t+\tau} + \phi y_{t+j}\right].$$

Since $\beta(1 - \alpha) < 1$, the sum on the left-hand-side of equation (13) converges to $\{1/[1-\beta(1-\alpha)]\}(x_t - p_t)$ and so we can write

$$(14)\quad x_t - p_t = [1-\beta(1-\alpha)]\left\{\phi y_t + \sum_{j=1}^{\infty} \beta^j (1-\alpha)^j E_t\left[-\sum_{\tau=0}^{j-1} \gamma \pi_{t+\tau} + \sum_{\tau=1}^{j} \pi_{t+\tau} + \phi y_{t+j}\right]\right\}.$$

Equation (14) holds for all periods. Expressing it for period t + 1, multiplying both sides by $\beta(1 - \alpha)$, and taking the expectation of the resulting expression as of period t gives us

$$(15)\quad \begin{aligned} &\beta(1-\alpha)E_t[x_{t+1} - p_{t+1}] = \\ &[1-\beta(1-\alpha)]\left\{\beta(1-\alpha)\phi E_t[y_{t+1}] + \sum_{j=1}^{\infty} \beta(1-\alpha)\beta^j (1-\alpha)^j E_t\left[-\sum_{\tau=0}^{j-1} \gamma \pi_{t+1+\tau} + \sum_{\tau=1}^{j} \pi_{t+1+\tau} + \phi y_{t+1+j}\right]\right\}. \end{aligned}$$

Changing the subscripts on the summations allows us to write

$$(16)\quad \begin{aligned} &\beta(1-\alpha)E_t[x_{t+1} - p_{t+1}] = \\ &[1-\beta(1-\alpha)]\left\{\beta(1-\alpha)\phi E_t[y_{t+1}] + \sum_{j=2}^{\infty} \beta^j (1-\alpha)^j E_t\left[-\sum_{\tau=1}^{j-1} \gamma \pi_{t+\tau} + \sum_{\tau=2}^{j} \pi_{t+\tau} + \phi y_{t+j}\right]\right\}. \end{aligned}$$

We now want to expand equation (14) so that we express $x_t - p_t$ as several terms plus the expression given by equation (16). That is, we can write equation (14) as

$$(17)\quad \begin{aligned} x_t - p_t = [1-\beta(1-\alpha)]\Bigg\{&\phi y_t + \beta(1-\alpha)E_t\left[-\sum_{\tau=0}^{0} \gamma \pi_{t+\tau}\right] + \sum_{j=2}^{\infty} \beta^j (1-\alpha)^j E_t\left[-\sum_{\tau=0}^{0} \gamma \pi_{t+\tau}\right] + \\ &\beta(1-\alpha)\sum_{\tau=1}^{1} \pi_{t+\tau} + \sum_{j=2}^{\infty} \beta^j (1-\alpha)^j E_t\left[\sum_{\tau=1}^{1} \pi_{t+\tau}\right]\Bigg\} + \beta(1-\alpha)E_t[x_{t+1} - p_{t+1}]. \end{aligned}$$

Since $\beta(1 - \alpha) < 1$, we can write equation (17) as

(18) $x_t - p_t = [1-\beta(1-\alpha)]\left\{\phi y_t + \dfrac{\beta(1-\alpha)}{1-\beta(1-\alpha)}(-\gamma\pi_t) + \dfrac{\beta(1-\alpha)}{1-\beta(1-\alpha)} E_t[\pi_{t+1}]\right\} + \beta(1-\alpha)E_t[x_{t+1} - p_{t+1}]$.

Finally, equation (18) simplifies to equation (7), as desired:

(7) $x_t - p_t = [1-\beta(1-\alpha)]\phi y_t + \beta(1-\alpha)\{E_t[x_{t+1} - p_{t+1}] + E_t[\pi_{t+1}] - \gamma\pi_t\}$.

We can now use equations (6) and (7) to solve the problem. Setting the right-hand sides of (6) and (7) equal to each other gives us

(19) $\dfrac{1-\alpha}{\alpha}[\pi_t - \gamma\pi_{t-1}] = [1-\beta(1-\alpha)]\phi y_t + \beta(1-\alpha)\{E_t[x_{t+1} - p_{t+1}] + E_t[\pi_{t+1}] - \gamma\pi_t\}$.

Expressing equation (6) for period t + 1 and taking the expectation as of period t gives us

(20) $E_t[x_{t+1} - p_{t+1}] = \dfrac{1-\alpha}{\alpha}[E_t[\pi_{t+1}] - \gamma\pi_t]$.

Substituting equation (20) into equation (19) yields

(21) $\dfrac{1-\alpha}{\alpha}[\pi_t - \gamma\pi_{t-1}] = [1-\beta(1-\alpha)]\phi y_t + \beta(1-\alpha)\left\{\dfrac{1-\alpha}{\alpha}[E_t[\pi_{t+1}] - \gamma\pi_t] + E_t[\pi_{t+1}] - \gamma\pi_t\right\}$.

Equation (21) simplifies to

(22) $\dfrac{1-\alpha}{\alpha}[\pi_t - \gamma\pi_{t-1}] = [1-\beta(1-\alpha)]\phi y_t + \dfrac{\beta(1-\alpha)}{\alpha}[E_t[\pi_{t+1}] - \gamma\pi_t]$.

Multiplying both sides of equation (22) by $\alpha/(1-\alpha)$ and adding $\gamma\pi_{t+1}$ to both sides of the resulting expression yields

(23) $\pi_t = \gamma\pi_{t-1} + \dfrac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi y_t + \beta[E_t[\pi_{t+1}] - \gamma\pi_t]$.

Collecting terms in π_t gives us

(24) $(1+\beta\gamma)\pi_t = \gamma\pi_{t-1} + \beta E_t[\pi_{t+1}] + \dfrac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi y_t$.

Finally, the new Keynesian Phillips curve with partial indexation is given by

(25) $\pi_t = \dfrac{1}{1+\beta\gamma}\gamma\pi_{t-1} + \dfrac{\beta}{1+\beta\gamma}E_t[\pi_{t+1}] + \dfrac{1}{1+\beta\gamma}\dfrac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi y_t$.

In the special case in which $\gamma = 0$, equation (25) simplifies to

(26) $\pi_t = \beta E_t[\pi_{t+1}] + \dfrac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi y_t$,

which is equivalent to equation (7.60), the new Keynesian Phillips curve.

In the special case in which $\gamma = 1$, equation (25) simplifies to

(27) $\pi_t = \dfrac{1}{1+\beta}\pi_{t-1} + \dfrac{\beta}{1+\beta}E_t[\pi_{t+1}] + \dfrac{1}{1+\beta}\dfrac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi y_t$,

which is equivalent to equation (7.76), the new Keynesian Phillips curve with full indexation.

Problem 7.11

(a) Since $e^{-\alpha i}$ is the probability that a price path set at time t is still being followed at time t + i, then we can let $\lambda(i) = 1 - e^{-\alpha i}$ be the probability that a firm has had an opportunity to change its price path between t and t + i. Then similar to equation (7.80) we can write

(1) $\lambda(i)[(1-\phi)a(i) + \phi] = a(i)$.

Solving for a(i) gives us

(2) $a(i) = \dfrac{\phi\lambda(i)}{1-(1-\phi)\lambda(i)}$.

Substituting our expression for λ(i) into equation (2) yields

(3) $a(i) = \dfrac{\phi(1-e^{-\alpha i})}{1-(1-\phi)(1-e^{-\alpha i})}$.

(b) (i) Reasoning analogous to that used to derive equation (7.83) gives us

(4) $y(t) = [(1-a(t)](-gt)$.

Substituting equation (3) for a(t) into equation (4) gives us

(5) $y(t) = \left[1 - \dfrac{\phi(1-e^{-\alpha t})}{1-(1-\phi)(1-e^{-\alpha t})}\right](-gt)$.

Equation (5) simplifies to

(6) $y(t) = -\dfrac{e^{-\alpha t}}{1-(1-\phi)(1-e^{-\alpha t})}\,gt$.

(b) (ii) We need to find an expression for inflation, $\dot{p}(t)$, for $t \geq 0$. Our usual aggregate demand equation

(7) y(t) = m(t) – p(t) ,

can be rewritten in terms of the price level as

(8) p(t) = m(t) – y(t).

Because of the assumption that all firms expect m(t) = 0 for t > 0, this simplifies to

(9) p(t) = -y(t).

Taking the time derivative of both sides of equation (9) implies

(10) $\dot{p}(t) = -\dot{y}(t)$.

We need to take the time derivative of output, which is given by equation (6). Using the quotient rule, we have

(11) $\dot{p}(t) = -\left\{-\dfrac{(-\alpha e^{-\alpha t}gt + ge^{-\alpha t})[1-(1-\phi)(1-e^{-\alpha t})] - e^{-\alpha t}gt[-(1-\phi)\alpha e^{-\alpha t}]}{[1-(1-\phi)(1-e^{-\alpha t})]^2}\right\}$.

After rearranging the terms and simplifying, we arrive at the following expression for inflation:

(12) $\dot{p}(t) = \dfrac{ge^{-\alpha t}[(1-\alpha\phi\, t) - (1-\phi)(1-e^{-\alpha t})]}{[1-(1-\phi)(1-e^{-\alpha t})]^2}$.

Since $ge^{-\alpha t}$ and $[1-(1-\phi)(1-e^{-\alpha t})]^2$ are both positive, inflation will be negative if $[(1-\alpha\phi\, t) - (1-\phi)(1-e^{-\alpha t})]$ is negative. Since $0 < \alpha \leq 1$, then $(1-e^{-\alpha t}) > 0$ for large enough t. In addition, since $(1-\phi) \geq 0$, then $(1-\phi)(1-e^{-\alpha t}) > 0$ for large t, or equivalently, $-(1-\phi)(1-e^{-\alpha t}) < 0$ for large t. Since $(1-\alpha\phi\, t) < 0$ for $t > 1/\alpha\phi$, the answer to this question is yes and inflation during the transition to the new steady state will be negative for large enough t.

(b) (iii) With the assumption that φ = 1, equation (6) becomes

(13) $y(t) = -e^{-\alpha t}gt$.

To find the time at which output reaches its lowest level, we need to set dy(t)/dt = 0 and solve for t. Thus, we have

(14) $\frac{dy(t)}{dt} = \alpha e^{-\alpha t} gt - e^{-\alpha t} g = 0$,
which implies
(15) $ge^{-\alpha t}(\alpha t - 1) = 0$.
Because $ge^{-\alpha t}$ is nonzero, our solution is
(16) $t^* = \frac{1}{\alpha}$.
To check whether this is a minimum, we take the second derivative
(17) $\left.\frac{d^2 y(t)}{dt^2}\right|_{t=\frac{1}{\alpha}} = -\alpha^2 e^{-\alpha t} gt + \alpha e^{-\alpha t} g + \alpha e^{-\alpha t} g\Big|_{t=\frac{1}{\alpha}}$,
which simplifies to
(18) $\left.\frac{d^2 y(t)}{dt^2}\right|_{t=\frac{1}{\alpha}} = \alpha e^{-\alpha t} g(2 - \alpha t)\Big|_{t=\frac{1}{\alpha}}$,
and, finally
(19) $\left.\frac{d^2 y(t)}{dt^2}\right|_{t=\frac{1}{\alpha}} = \alpha e^{-1} g > 0$.
Thus, output does reach a minimum at $t^* = 1/\alpha$.

To determine the time at which inflation reaches its lowest level, we need to take the first derivative of expression (10) with respect to time, set it to zero, and solve for t. That is, we need to solve
(20) $\frac{d\dot{p}(t)}{dt} = -\frac{d\dot{y}(t)}{dt} = 0$.
From equation (18), we already know that $d\dot{y}(t)/dt = \alpha e^{-\alpha t} g(2 - \alpha t)$ and so we have to solve
(21) $-\alpha e^{-\alpha t} g(2 - \alpha t) = 0$.
Because $\alpha e^{-\alpha t} g$ is nonzero, our solution is
(22) $t^* = \frac{2}{\alpha}$.
To check whether this is a minimum, we know from equation (21) that
(22) $\frac{d\dot{p}(t)}{dt} = -\alpha e^{-\alpha t} g(2 - \alpha t)$,
which we can rewrite as
(23) $\frac{d\dot{p}(t)}{dt} = -2\alpha e^{-\alpha t} g + \alpha^2 e^{-\alpha t} gt$.
Taking the second derivative of $\dot{p}(t)$ with respect to time gives us

(24) $\left.\frac{d^2 \dot{p}(t)}{dt^2}\right|_{t=\frac{2}{\alpha}} = 2\alpha^2 e^{-\alpha t} g + \alpha^2 g e^{-\alpha t} + \alpha^2 gt(-\alpha) e^{-\alpha t}\Big|_{t=\frac{2}{\alpha}}$,
which simplifies to

(25) $$\left.\frac{d^2\dot{p}(t)}{dt^2}\right|_{t=\frac{2}{\alpha}} = \alpha^2 g e^{-\alpha t}(3-\alpha t)\Big|_{t=\frac{2}{\alpha}}.$$

Thus, we have

(26) $$\left.\frac{d^2\dot{p}(t)}{dt^2}\right|_{t=\frac{2}{\alpha}} = \alpha^2 g e^{-2} > 0.$$

Thus, inflation does reach a minimum at $t^* = 2/\alpha$.

Problem 7.12

(a) For the case of white-noise disturbances, to find expressions analogous to (7.92)-(7.94), we start with the three core equations given by

(1) $$y_t = E_t[y_{t+1}] - \frac{1}{\theta} r_t + u_t^{IS}, \qquad \theta > 0,$$

and

(2) $$\pi_t = \beta E_t[\pi_{t+1}] + \kappa y_t + u_t^{\pi}, \qquad 0 < \beta < 1, \quad \kappa > 0,$$

and

(3) $$r_t = \phi_\pi \pi_t + \phi_y y_t + u_t^{MP}, \qquad \phi_\pi > 0, \quad \phi_y \geq 0.$$

First, we substitute equation (3) for r_t into equation (1) to yield

(4) $$y_t = E_t[y_{t+1}] - \frac{1}{\theta}(\phi_\pi \pi_t + \phi_y y_t + u_t^{MP}) + u_t^{IS}.$$

Substituting equation (2) for π_t into equation (4) gives us

(5) $$y_t = E_t[y_{t+1}] - \frac{1}{\theta}\left[\phi_\pi \beta E_t[\pi_{t+1}] + \phi_\pi \kappa y_t + \phi_\pi u_t^{\pi} + \phi_y y_t + u_t^{MP}\right] + u_t^{IS}.$$

We need to solve for y_t. Collecting terms gives us

(6) $$\left[1 + \frac{\phi_\pi \kappa + \phi_y}{\theta}\right] y_t = E_t[y_{t+1}] - \frac{1}{\theta}\left[\phi_\pi \beta E_t[\pi_{t+1}] + \phi_\pi u_t^{\pi} + u_t^{MP}\right] + u_t^{IS}.$$

Multiplying both sides of equation (6) by $\theta/(\theta + \phi_\pi \kappa + \phi_y)$ gives us

(7) $$y_t = \frac{\theta}{\theta + \phi_\pi \kappa + \phi_y} E_t[y_{t+1}] - \frac{1}{\theta + \phi_\pi \kappa + \phi_y}\left[\phi_\pi \beta E_t[\pi_{t+1}] + \phi_\pi u_t^{\pi} + u_t^{MP}\right] + \frac{\theta}{\theta + \phi_\pi \kappa + \phi_y} u_t^{IS},$$

which we can rewrite as

(8) $$y_t = \frac{1}{\theta + \phi_\pi \kappa + \phi_y}\left[\theta E_t[y_{t+1}] - \phi_\pi \beta E_t[\pi_{t+1}] - \phi_\pi u_t^{\pi} - u_t^{MP} + \theta u_t^{IS}\right].$$

Next, we incorporate the fundamental solution of $E_t[y_{t+1}] = 0$ and $E_t[\pi_{t+1}] = 0$, giving us

(9) $$y_t = \frac{1}{\theta + \phi_\pi \kappa + \phi_y}\left[\theta u_t^{IS} - \phi_\pi u_t^{\pi} - u_t^{MP}\right].$$

Now substitute equation (8) for y_t into equation (2) to yield the following expression for inflation:

(10) $$\pi_t = \beta E_t[\pi_{t+1}] + \frac{\kappa}{\theta + \phi_\pi \kappa + \phi_y}\left[\theta E_t[y_{t+1}] - \phi_\pi \beta E_t[\pi_{t+1}] - \phi_\pi u_t^{\pi} - u_t^{MP} + \theta u_t^{IS}\right] + u_t^{\pi}.$$

Collecting the terms in $E_t[\pi_{t+1}]$ and u_t^{π} gives us

$$(11)\quad \pi_t = \left[1 - \frac{\phi_\pi \kappa}{\theta + \phi_\pi \kappa + \phi_y}\right] \beta E_t[\pi_{t+1}] + \frac{\kappa}{\theta + \phi_\pi \kappa + \phi_y}\left[\theta E_t[y_{t+1}] - u_t^{MP} + \theta u_t^{IS}\right] + \left[1 - \frac{\phi_\pi \kappa}{\theta + \phi_\pi \kappa + \phi_y}\right] u_t^{\pi}.$$

Since

$$(12)\quad \left[1 - \frac{\phi_\pi \kappa}{\theta + \phi_\pi \kappa + \phi_y}\right] = \frac{\theta + \phi_y}{\theta + \phi_\pi \kappa + \phi_y},$$

we can rewrite equation (11) as

$$(13)\quad \pi_t = \frac{1}{\theta + \phi_\pi \kappa + \phi_y}\left[\kappa \theta E_t[y_{t+1}] - \kappa u_t^{MP} + \kappa \theta u_t^{IS} + (\theta + \phi_y)(\beta E_t[\pi_{t+1}] + u_t^{\pi})\right].$$

Substituting the fundamental solution of $E_t[y_{t+1}] = 0$ and $E_t[\pi_{t+1}] = 0$ into equation (13) yields

$$(14)\quad \pi_t = \frac{1}{\theta + \phi_\pi \kappa + \phi_y}\left[\kappa \theta u_t^{IS} + (\theta + \phi_y) u_t^{\pi} - \kappa u_t^{MP}\right].$$

Next, we can substitute equations (9) and (14) for output and inflation into equation (3) to obtain an expression for the real interest rate, given by

$$(15)\quad r_t = \frac{\phi_\pi}{\theta + \kappa \phi_\pi + \phi_y}\left[\kappa \theta u_t^{IS} + (\theta + \phi_y) u_t^{\pi} - \kappa u_t^{MP}\right] + \frac{\phi_y}{\theta + \kappa \phi_\pi + \phi_y}\left[\theta u_t^{IS} - \phi_\pi u_t^{\pi} - u_t^{MP}\right] + u_t^{MP}.$$

Collecting terms gives us

$$(16)\quad r_t = \frac{1}{\theta + \kappa \phi_\pi + \phi_y}\left[(\kappa \phi_\pi \theta + \phi_y \theta) u_t^{IS} + (\phi_\pi \theta + \phi_\pi \phi_y - \phi_\pi \phi_y) u_t^{\pi} + (-\kappa \phi_\pi - \phi_y + \theta + \phi_y + \kappa \phi_\pi) u_t^{MP}\right],$$

which simplifies to

$$(17)\quad r_t = \frac{1}{\theta + \kappa \phi_\pi + \phi_y}\left[\theta(\kappa \phi_\pi + \phi_y) u_t^{IS} + \phi_\pi \theta u_t^{\pi} + \theta u_t^{MP}\right].$$

Equations (9), (14), and (17) are analogous to (7.92)-(7.94).

Because we are assuming that all constants are positive, we obtain the following signs on the impacts of the unfavorable inflation shock:

$$(18)\quad \frac{\partial y_t}{\partial u_t^{\pi}} = -\frac{\phi_\pi}{\theta + \kappa \phi_\pi + \phi_y} < 0;$$

$$(19)\quad \frac{\partial \pi_t}{\partial u_t^{\pi}} = \frac{\theta + \phi_y}{\theta + \kappa \phi_\pi + \phi_y} > 0;$$

and

$$(20)\quad \frac{\partial r_t}{\partial u_t^{\pi}} = \frac{\phi_\pi \theta}{\theta + \kappa \phi_\pi + \phi_y} > 0.$$

Thus, an unfavorable inflation shock reduces output and increases both inflation and the real interest rate.

(b) The shocks follow independent AR-1 processes:

$$(21)\quad u_t^{IS} = \rho_{IS} u_{t-1}^{IS} + e_t^{IS}, \qquad -1 < \rho_{IS} < 1;$$

$$(22)\quad u_t^{\pi} = \rho_{\pi} u_{t-1}^{\pi} + e_t^{\pi}, \qquad -1 < \rho_{\pi} < 1;$$

and

$$(23)\quad u_t^{MP} = \rho_{MP} u_{t-1}^{MP} + e_t^{MP}, \qquad -1 < \rho_{MP} < 1,$$

where e^{IS}, e^{π}, and e^{MP} are white-noise disturbances that are uncorrelated with one another. To solve the model using the method of undetermined coefficients, we guess that the endogenous variables are linear functions of the disturbances. We can therefore write output and inflation as

(24) $y_t = a_{IS}u_t^{IS} + a_{\pi}u_t^{\pi} + a_{MP}u_t^{MP}$,

and

(25) $\pi_t = b_{IS}u_t^{IS} + b_{\pi}u_t^{\pi} + b_{MP}u_t^{MP}$.

Since (24) and (25) hold for all t, we can express output and inflation in period t + 1 as

(26) $y_{t+1} = a_{IS}u_{t+1}^{IS} + a_{\pi}u_{t+1}^{\pi} + a_{MP}u_{t+1}^{MP}$,

and

(27) $\pi_{t+1} = b_{IS}u_{t+1}^{IS} + b_{\pi}u_{t+1}^{\pi} + b_{MP}u_{t+1}^{MP}$.

Now substitute equations (21) through (23), expressed in period t + 1, into equations (26) and (27) and take the expectation of the resulting expressions as of period t to yield

(28) $E_t[y_{t+1}] = a_{IS}\rho_{IS}u_t^{IS} + a_{\pi}\rho_{\pi}u_t^{\pi} + a_{MP}\rho_{MP}u_t^{MP}$,

and

(29) $E_t[\pi_{t+1}] = b_{IS}\rho_{IS}u_t^{IS} + b_{\pi}\rho_{\pi}u_t^{\pi} + b_{MP}\rho_{MP}u_t^{MP}$.

We can now substitute the expressions for $E_t[y_{t+1}]$ and $E_t[\pi_{t+1}]$, along with equations (24) and (25) into equations (8) and (13). We obtain the following expression from equation (8):

$$\begin{aligned} a_{IS}u_t^{IS} + a_{\pi}u_t^{\pi} + a_{MP}u_t^{MP} &= \frac{\theta}{\theta+\phi_{\pi}\kappa+\phi_y}(a_{IS}\rho_{IS}u_t^{IS} + a_{\pi}\rho_{\pi}u_t^{\pi} + a_{MP}\rho_{MP}u_t^{MP}) \\ &- \frac{\phi_{\pi}\beta}{\theta+\phi_{\pi}\kappa+\phi_y}(b_{IS}\rho_{IS}u_t^{IS} + b_{\pi}\rho_{\pi}u_t^{\pi} + b_{MP}\rho_{MP}u_t^{MP}) + \frac{1}{\theta+\phi_{\pi}\kappa+\phi_y}\left[\theta u_t^{IS} - \phi_{\pi}u_t^{\pi} - u_t^{MP}\right]. \end{aligned} \tag{30}$$

We also obtain the following expression from equation (13):

$$\begin{aligned} b_{IS}u_t^{IS} + b_{\pi}u_t^{\pi} + b_{MP}u_t^{MP} &= \frac{\kappa\theta}{\theta+\phi_{\pi}\kappa+\phi_y}(a_{IS}\rho_{IS}u_t^{IS} + a_{\pi}\rho_{\pi}u_t^{\pi} + a_{MP}\rho_{MP}u_t^{MP}) + \\ &\frac{(\theta+\phi_y)\beta}{\theta+\phi_{\pi}\kappa+\phi_y}(b_{IS}\rho_{IS}u_t^{IS} + b_{\pi}\rho_{\pi}u_t^{\pi} + b_{MP}\rho_{MP}u_t^{MP}) + \frac{1}{\theta+\phi_{\pi}\kappa+\phi_y}\left[\kappa\theta u_t^{IS} + (\theta+\phi_y)u_t^{\pi} - \kappa u_t^{MP}\right]. \end{aligned} \tag{31}$$

The two sides of (30) must be equal for all values of u_t^{IS}, u_t^{π}, and u_t^{MP}. Thus the coefficients on u_t^{IS}, u_t^{π}, and u_t^{MP} on the two sides must be equal. Similarly for equation (31). Once we have solved for the a's and b's, equations (24) and (25) tell us the behavior of output and inflation. We can then substitute those solutions for π_t and y_t into equation (3) to solve for the real interest rate.

SOLUTIONS TO CHAPTER 8

Problem 8.1

(a) The present value of lifetime consumption must be less than or equal to the present value of lifetime income (the individual has no initial wealth). Thus we have

(1) $\int_{t=0}^{T} C(t)dt \le \int_{t=0}^{T} Y(t)dt$.

Since the individual's income is $Y(t) = Y_0 + gt$ for $0 \le t < R$ and $Y(t) = 0$ for $R \le t \le T$, the present value of lifetime income is

(2) $\int_{t=0}^{T} Y(t)dt = \int_{t=0}^{R} (Y_0 + gt)dt$.

Solving this integral gives us

(3) $\int_{t=0}^{R} (Y_0 + gt)dt = [Y_0 t + \frac{1}{2}gt^2]\Big|_{t=0}^{R}$,

or simply

(4) $\int_{t=0}^{R} (Y_0 + gt)dt = RY_0 + \frac{1}{2}gR^2$.

Substituting equation (4) into equation (1) gives us the lifetime budget constraint:

(5) $\int_{t=0}^{T} C(t)dt \le RY_0 + \frac{1}{2}gR^2$.

(b) Since $u''(\bullet) < 0$ and the interest rate and discount rate are equal to zero here, the analysis in Section 8.1 implies that utility-maximization requires consumption to be constant. The budget constraint then implies that consumption at each point in time is equal to lifetime resources divided by the length of life. As shown in part (a), lifetime resources are given by $RY_0 + \frac{1}{2}gR^2$ and thus the constant level of consumption is given by

(6) $\overline{C} = \frac{R}{T}[Y_0 + \frac{1}{2}gR]$.

Note that equation (6) can be derived more formally using the calculus of variations, the informal Lagrangian approach introduced in Chapter 2, or an Euler-equation argument.

(c) The individual's wealth at any time t is the sum of saving from time 0 to time t, or

(7) $W(t) = \int_{\tau=0}^{t} S(\tau)\,d\tau$,

where S represents the individual's saving and W represents her wealth. Saving at time t is the difference between income and consumption, or

(8) $S(t) = Y(t) - C(t)$.

Thus saving is given by

(9) $S(t) = \begin{cases} Y_0 + gt - \overline{C} & \text{for} \quad 0 \le t < R \\ -\overline{C} & \text{for} \quad R \le t \le T. \end{cases}$

Intuitively, the individual dissaves when current income is less than mean lifetime income (as it is in retirement when current income is zero) and saves when current income is greater than mean lifetime income.

For $0 \le t < R$, wealth is given by

$$(10)\quad W(t) = \int_{\tau=0}^{t} S(\tau)\,d\tau = \int_{\tau=0}^{t} [Y_0 + g\tau - \overline{C}]\,d\tau\,.$$

Solving the integral in (18) gives us

$$(11)\quad W(t) = [Y_0\tau + \frac{1}{2}g\tau^2 - \overline{C}\tau]\Big|_{\tau=0}^{t}\,,$$

or

$$(12)\quad W(t) = Y_0 t + \frac{1}{2}gt^2 - \overline{C}t\,.$$

Equation (12) can be rewritten as

$$(13)\quad W(t) = t\,[Y_0 + \frac{1}{2}gt - \overline{C}]\,.$$

For $R \le t \le T$, wealth is equal to

$$(14)\quad W(t) = \int_{\tau=R}^{t} S(\tau)\,d\tau + W(R)\,,$$

where W(R) is wealth at the time the individual retires. We can substitute t = R into equation (13) to determine wealth at retirement. This gives us

$$(15)\quad W(R) = R\,[Y_0 + \frac{1}{2}gR - \overline{C}]\,.$$

Since $\overline{C} = (R/T)[Y_0 + (1/2)gR]$, equation (15) can be rewritten as

$$(16)\quad W(R) = R\,[\frac{T}{R}\overline{C} - \overline{C}]\,,$$

which simplifies to

$$(17)\quad W(R) = (T - R)\overline{C}\,.$$

Equation (17) is intuitive: since the individual receives no income in retirement, consumption when retired must be financed by the wealth built up during the individual's working life. Since (T – R) is the amount of time spent in retirement, and since the individual consumes $\overline{C}$ per unit time, wealth at retirement must equal $(T - R)\overline{C}$.

Substituting equation (17) and the fact that $S(t) = -\overline{C}$ for $R \le t \le T$ into equation (14) yields

$$(18)\quad W(t) = (T - R)\overline{C} - \int_{\tau=R}^{t} \overline{C}\,d\tau\,.$$

Solving the integral in equation (18) leaves us with

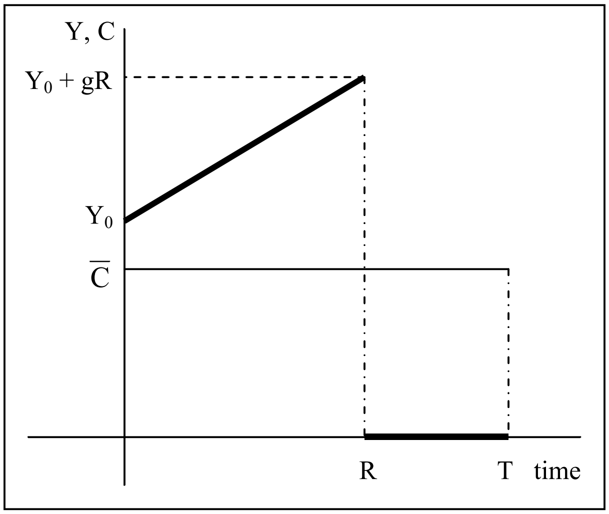
$$(19)\quad W(t) = (T - R)\overline{C} - \overline{C}(t - R)\,,$$

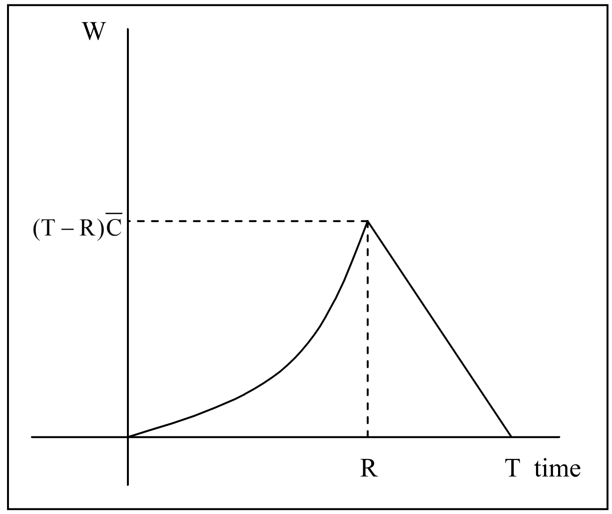
or simply

$$(20)\quad W(t) = (T - t)\overline{C}\,.$$

Thus the individual begins positive saving once current income exceeds mean lifetime income. Wealth is maximized at retirement, after which wealth is run down to finance consumption until – from equation (20) – wealth equals zero at time T, the end of life.

The pattern implied by our analysis is shown in the figures below.





The left-hand figure depicts income and consumption as functions of time, assuming that income at time 0 exceeds the constant level of consumption. The bold line shows income which equals $Y_0 + gt$ until retirement and equals 0 thereafter. Consumption is constant at the level $\overline{C}$.

The right-hand figure depicts wealth as a function of time. The slope of the wealth curve equals saving which in turn is equal to the difference between income and consumption in the figure on the left. Wealth rises (at an increasing rate) during working life since income exceeds consumption (by an increasing amount); wealth reaches a maximum of $(T - R)\overline{C}$ at retirement. During retirement – between periods R and T – wealth declines at a constant rate until it reaches zero at the end of life. Given the shape of the wealth function, this pattern of wealth accumulation over the life cycle is known as "hump saving".

Problem 8.2

Since transitory income is on average equal to zero, we can interpret average income as average permanent income. Thus we are told that, on average, farmers have lower permanent income than nonfarmers do, or $\overline{Y}_F^P < \overline{Y}_{NF}^P$. We can interpret the fact that farmers' incomes fluctuate more from year to year as meaning that the variance of transitory income for farmers is larger than the variance of transitory income for nonfarmers, or $\text{var}(Y^T{}_F) > \text{var}(Y^T{}_{NF})$.

Consider the following regression model:

(1) $C_i = a + bY_i + e_i$,

where C_i is current consumption – which according to the permanent-income hypothesis is determined entirely by Y^P so that $C = Y^P$ – and Y_i is current income, which is assumed to be the sum of permanent income and transitory income so that $Y = Y^P + Y^T$. From equation (8.8) in the text, the Ordinary Least Squares (OLS) estimator of b takes the form

(2) $$\hat{b} = \frac{\text{var}(Y^P)}{\text{var}(Y^P) + \text{var}(Y^T)}.$$

As long as $\text{var}(Y^P)$ is the same across the two groups, the fact that $\text{var}(Y^T_F) > \text{var}(Y^T_{NF})$ means that the estimated slope coefficient should be smaller for farmers than it is for nonfarmers. This means that the estimated impact on consumption of a marginal increase in current income is smaller for farmers than for nonfarmers. According to the permanent-income hypothesis, this is because the increase is much more likely to be due to transitory income for farmers than for nonfarmers. Thus it can be expected to have a smaller impact on consumption for farmers than for nonfarmers.

From equation (8.9) in the text, the OLS estimator of the constant term takes the form

(3) $\hat{a} = (1 - \hat{b})\overline{Y}^P$.

The fact that farmers, on average, have lower permanent incomes than nonfarmers tends to make the estimated constant term smaller for farmers. However, as was just explained, $\hat{b}$ is smaller for farmers than it is for nonfarmers. This tends to make the estimated constant term bigger for farmers than for nonfarmers. Thus the effect on the estimated constant term is ambiguous.

We can, say, however, that at the average level of permanent income for farmers, the estimated consumption function for farmers is expected to lie below the one for nonfarmers. Thus if the two estimated consumption functions do cross, they cross at a level of income less than $\overline{Y}^P{}_F$. Why?

Consider a member of each group whose income equals the average income among farmers. Since there are many more nonfarmers with permanent incomes above this level than there are with permanent incomes below it, the nonfarmer's permanent income is much more likely to be greater than her current income than less. As a result, nonfarmers with this current income have on average higher permanent income; thus on average they consume more than their income. For the farmer, in contrast, her permanent income is about as likely to be more than current income as it is to be less; as a result, farmers with this current income on average have the same permanent income, and thus on average they consume their income. Thus the consumption function for farmers is expected to lie below the one for nonfarmers at the average level of income for farmers.

Problem 8.3

(a) We need to find an expression for $[(C_{t+2} + C_{t+3})/2] - [(C_t + C_{t+1})/2]$. We can write C_{t+1}, C_{t+2} and C_{t+3} in terms of C_t and the e's. Specifically, we can write

(1) $C_{t+1} = C_t + e_{t+1}$,

(2) $C_{t+2} = C_{t+1} + e_{t+2} = C_t + e_{t+1} + e_{t+2}$, and

(3) $C_{t+3} = C_{t+2} + e_{t+3} = C_t + e_{t+1} + e_{t+2} + e_{t+3}$,

where we have used equation (1) in deriving (2) and equation (2) in deriving (3). Using equations (1) through (3), the change in measured consumption from one two-period interval to the next is

(4) $$\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} = \frac{(C_t + e_{t+1} + e_{t+2}) + (C_t + e_{t+1} + e_{t+2} + e_{t+3})}{2} - \frac{C_t + (C_t + e_{t+1})}{2},$$

which simplifies to

(5) $$\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} = \frac{e_{t+3} + 2e_{t+2} + e_{t+1}}{2}.$$

(b) Through similar manipulations as in part (a), the previous value of the change in measured consumption would be

(6) $$\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2} = \frac{e_{t+1} + 2e_t + e_{t-1}}{2}.$$

Using equations (5) and (6), the covariance between successive changes in measured consumption is

(7) $$\text{cov}\left[\left(\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2}\right), \left(\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2}\right)\right] =$$

$$\operatorname{cov}\left[\left(\frac{e_{t+3}+2e_{t+2}+e_{t+1}}{2}\right),\left(\frac{e_{t+1}+2e_t+e_{t-1}}{2}\right)\right].$$

Since the e's are uncorrelated with each other and since e_{t+1} is the only value of e that appears in both expressions, the covariance reduces to

$$(8)\quad \operatorname{cov}\left[\left(\frac{C_{t+2}+C_{t+3}}{2}-\frac{C_t+C_{t+1}}{2}\right),\left(\frac{C_t+C_{t+1}}{2}-\frac{C_{t-2}+C_{t-1}}{2}\right)\right]=\frac{\sigma_e^{\,2}}{4},$$

where σ_e^2 denotes the variance of the e's. So the change in measured consumption is correlated with its previous value. Since the covariance is positive, this means that if measured consumption in the two-period interval (t, t + 1) is greater than measured consumption in the two-period interval (t - 2, t - 1), measured consumption in (t + 2, t + 3) will tend to be greater than measured consumption in (t, t + 1). When a variable follows a random walk, successive changes in the variable are uncorrelated. For example, with actual consumption in this model, we have $C_t - C_{t-1} = e_t$ and $C_{t+1} - C_t = e_{t+1}$. Since e_t and e_{t+1} are uncorrelated, successive changes in actual consumption are uncorrelated. Thus if C_t were bigger than C_{t-1} , it would not mean that C_{t+1} would tend to be higher than C_t . Since successive changes in measured consumption are correlated, measured consumption is not a random walk. The change in measured consumption today does provide us with some information as to what the change in measured consumption is likely to be tomorrow.

(c) From equation (5), the change in measured consumption from (t, t + 1) to (t + 2, t + 3) depends on e_{t+1} , the innovation to consumption in period t + 1. But this is known as of t + 1, which is part of the first two-period interval. Thus the change in consumption from one two-period interval to the next is not uncorrelated with everything known as of the first two-period interval. However, it is uncorrelated with everything known in the two-period interval immediately preceding (t, t + 1). From equation (5), e_{t+3} ,e_{t+2} and e_{t+1} are all unknown as of the two-period interval (t - 2, t - 1).

(d) We can write C_{t+3} as a function of C_{t+1} and the e's. Specifically, we can write

(9) $C_{t+3} = C_{t+2} + e_{t+3} = C_{t+1} + e_{t+2} + e_{t+3}$.

Thus the change in measured consumption from one two-period interval to the next is

(10) $C_{t+3} - C_{t+1} = C_{t+1} + e_{t+2} + e_{t+3} - C_{t+1} = e_{t+2} + e_{t+3}$.

The same calculations would yield the previous value of the change in measured consumption,

(11) $C_{t+1} - C_{t-1} = e_t + e_{t+1}$.

And so the covariance between successive changes in measured consumption is

(12) $\operatorname{cov}[(C_{t+3} - C_{t+1}), (C_{t+1} - C_{t-1})] = \operatorname{cov}[(e_{t+2} + e_{t+3}), (e_t + e_{t+1})]$.

Since the e's are uncorrelated with each other, the covariance is zero. Thus measured consumption is a random walk in this case. The amount that C_{t+1} differs from C_{t-1} does not provide any information about what the difference between C_{t+1} and C_{t+3} will be.

Problem 8.4

The uncertainty about future income – although it does not affect consumption – does affect expected lifetime utility. From equation (8.10) in the text, expected lifetime utility is given by

$$(1)\quad E_1[U]=E_1\left[\sum_{t=1}^{T}(C_t-\frac{a}{2}C_t^{\,2})\right],$$

where a > 0. This can be rewritten as

$$(2)\quad E_1[U]=\sum_{t=1}^{T}(E_1[C_t]-\frac{a}{2}E_1[C_t^{\,2}]).$$

Since the expected value of consumption in all periods after t is equal to C_1, or

(3) $E_1[C_t] = C_1$,
we can write
(4) $C_t = C_1 + e_t$,
where $E_1[e_t] = 0$ and $\mathrm{var}(e_t) = \sigma_{e_t}^2$. Equation (4) holds for all periods and so substituting (4) into (2) yields

(5) $$E_1[U] = \sum_{t=1}^{T}(E_1[C_1 + e_t] - \frac{a}{2}E_1[(C_1 + e_t)^2]).$$

Since $E_1[C_1] = C_1$ and $E_1[e_t] = 0$, this becomes

(6) $$E_1[U] = \sum_{t=1}^{T}(C_1 - \frac{a}{2}C_1{}^2 - \frac{a}{2}E_1[e_t{}^2]).$$

Since $E_1[e_t{}^2] = \mathrm{var}(e_t) = \sigma_{e_t}^2$, equation (6) can be written as

(7) $$E_1[U] = \sum_{t=1}^{T}(C_1 - \frac{a}{2}C_1{}^2 - \frac{a}{2}\sigma_{e_t}^2).$$

If $C_t = C_1$ with certainty so that $e_t = 0$ and $\mathrm{var}(e_t) = \sigma_{e_t}^2 = 0$, lifetime utility would be

(8) $$U = \sum_{t=1}^{T}(C_1 - \frac{a}{2}C_1{}^2).$$

Since C_1 is the same whether or not there is uncertainty, we can see by comparing equations (7) and (8) that if there is uncertainty – so that $\mathrm{var}(e_t) = \sigma_{e_t}^2 > 0$ – expected lifetime utility is lower.

Problem 8.5

(a) Consider the usual experiment of a decrease in consumption by a small (formally, infinitesimal) amount dC in period t. With the CRRA utility function given by
(1) $u(C_t) = C_t^{1-\theta}/(1 - \theta)$,
the marginal utility of consumption in period t is $C_t^{-\theta}$. Thus the change has a utility cost of
(2) utility cost $= C_t^{-\theta}\,dC$.
The marginal utility of consumption in period t + 1 is $C_{t+1}^{-\theta}$. With a real interest rate of r, the individual gets to consume an additional (1 + r)dC in period t + 1. This has a discounted expected utility benefit of

(3) expected utility benefit $= \dfrac{1}{1+\rho}E_t\left[C_{t+1}{}^{-\theta}(1+r)dC\right]$.

If the individual is optimizing, a marginal change of this type does not affect expected utility. This means that the utility cost must equal the expected utility benefit or

(4) $$C_t{}^{-\theta} = \frac{1+r}{1+\rho}E_t\left[C_{t+1}{}^{-\theta}\right],$$

where we have (rather informally) canceled the dC's. Equation (4) is the Euler equation.

(b) For any variable x, $e^{\ln x} = x$, and so we can write

(5) $$E_t\left[C_{t+1}{}^{-\theta}\right] = E_t\left[e^{-\theta \ln C_{t+1}}\right].$$

Using the hint in the question -- if $x \sim N(\mu, V)$ then $E[e^x] = e^{\mu}e^{V/2}$ -- then since the log of consumption is distributed normally, we have

$$\text{(6)}\quad \begin{aligned} E_t\left[C_{t+1}{}^{-\theta}\right] &= E_t\left[e^{-\theta E_t \ln C_{t+1}}\, e^{\theta^2\sigma^2/2}\right] \\ &= e^{-\theta E_t \ln C_{t+1}}\, e^{\theta^2\sigma^2/2}. \end{aligned}$$

In the first line, we have used the fact that conditional on time t information, the variance of log consumption is σ^2. In addition, we have written the mean of log consumption in period t + 1, conditional on time t information, as $E_t \ln C_{t+1}$. Finally, in the last line we have used the fact that $e^{-\theta E_t \ln C_{t+1}}\, e^{\theta^2\sigma^2/2}$ is simply a constant.

Substituting equation (6) back into equation (2) and taking the log of both sides yields

(7) $-\theta \ln C_t = \ln(1 + r) - \ln(1 + \rho) - \theta E_t \ln C_{t+1} + \theta^2 \sigma^2/2$.

Dividing both sides of equation (7) by ($-\theta$) leaves us with

(8) $\ln C_t = E_t \ln C_{t+1} + [\ln(1 + \rho) - \ln(1 + r)]/\theta - \theta\sigma^2/2$.

(c) Rearranging equation (8) to solve for $E_t \ln C_{t+1}$ gives us

(9) $E_t \ln C_{t+1} = \ln C_t + [\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$.

Equation (9) implies that consumption is expected to change by the constant amount $[\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$ from one period to the next. Changes in consumption other than this deterministic amount are unpredictable. By the definition of expectations we can write

(10) $E_t \ln C_{t+1} = \ln C_t + [\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2 + u_{t+1}$,

where the u's have mean zero and are serially uncorrelated. Thus log consumption follows a random walk with drift where $[\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$ is the drift parameter.

(d) From equation (9), expected consumption growth is

(11) $E_t [\ln C_{t+1} - \ln C_t] = [\ln(1 + r) - \ln(1 + \rho)]/\theta + \theta\sigma^2/2$.

Clearly, a rise in r raises expected consumption growth. We have

$$\text{(12)}\quad \frac{\partial E_t\left[\ln C_{t+1} - \ln C_t\right]}{\partial r} = \frac{1}{\theta}\frac{1}{(1+r)} > 0.$$

Note that the smaller is θ the – the bigger is the elasticity of substitution, $1/\theta$ – the more that consumption growth increases due to a given increase in the real interest rate.

An increase in σ^2 also increases consumption growth since

$$\text{(13)}\quad \frac{\partial E_t\left[\ln C_{t+1} - \ln C_t\right]}{\partial \sigma^2} = \frac{\theta}{2} > 0.$$

It is straightforward to verify that the CRRA utility function has a positive third derivative. From equation (1), $u'(C_t) = C_t^{-\theta}$ and $u''(C_t) = -\theta C_t^{-\theta-1}$. Thus

(14) $u'''(C_t) = -\theta(-\theta - 1)C_t^{-\theta-2} = (\theta^2 + \theta)C_t^{-\theta-2} > 0$.

So an individual with a CRRA utility function exhibits the precautionary saving behavior explained in Section 8.6. A rise in uncertainty (as measured by σ^2, the variance of log consumption) increases saving and thus expected consumption growth.

Problem 8.6

(a) Substituting the expression for consumption in period t, which is

$$(1)\quad C_t = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}\right],$$

into the expression for wealth in period t + 1, which is

(2) $A_{t+1} = (1 + r)[A_t + Y_t - C_t]$,

gives us

$$(3)\quad A_{t+1} = (1+r)\left[A_t + Y_t - \frac{r}{1+r}A_t - \frac{r}{1+r}\left(Y_t + \frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right)\right].$$

Obtaining a common denominator of (1 + r) and then canceling the (1 + r)'s gives us

$$(4)\quad A_{t+1} = A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right).$$

Since equation (1) holds in all periods, we can write consumption in period t + 1 as

$$(5)\quad C_{t+1} = \frac{r}{1+r}\left[A_{t+1} + \sum_{s=0}^{\infty} \frac{E_{t+1}[Y_{t+1+s}]}{(1+r)^s}\right].$$

Substituting equation (4) into equation (5) yields

$$(6)\quad C_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right) + \left(E_{t+1}Y_{t+1} + \frac{E_{t+1}Y_{t+2}}{1+r} + \ldots\right)\right].$$

Taking the expectation, conditional on time t information, of both sides of equation (6) gives us

$$(7)\quad E_tC_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right) + \left(E_tY_{t+1} + \frac{E_tY_{t+2}}{1+r} + \ldots\right)\right],$$

where we have used the law of iterated projections so that for any variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. If this did not hold, individuals would be expecting to revise their estimate either upward or downward and thus their original expectation could not have been rational. Collecting terms in equation (7) gives us

$$(8)\quad E_tC_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - \left(1 - \frac{r}{1+r}\right)E_tY_{t+1} + \left(\frac{1}{1+r} - \frac{r}{(1+r)^2}\right)E_tY_{t+2} + \ldots\right],$$

which simplifies to

$$(9)\quad E_tC_{t+1} = \frac{r}{1+r}\left[A_t + Y_t + \frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right].$$

Using summation notation, and noting that $E_t Y_t = Y_t$, we have

$$(10)\quad E_tC_{t+1} = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty} \frac{E_tY_{t+s}}{(1+r)^s}\right].$$

The right-hand sides of equations (1) and (10) are equal and thus

(11) $E_t C_{t+1} = C_t$.

Consumption follows a random walk; changes in consumption are unpredictable.

Since consumption follows a random walk, the best estimate of consumption in any future period is simply the value of consumption in this period. That is, for any $s \geq 0$, we can write

(12) $E_t C_{t+s} = C_t$.

Using equation (12), we can write the present value of the expected path of consumption as

(13) $\sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = \sum_{s=0}^{\infty} \frac{C_t}{(1+r)^s} = C_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s}$.

Since 1/(1 + r) < 1, the infinite sum on the right-hand side of (13) converges to 1/[1 - 1/(1+r)] = (1 + r)/r and thus

(14) $\sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = \frac{1+r}{r} C_t$.

Substituting equation (1) for C_t into the right-hand side of equation (14) yields

(15) $\sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = \frac{r}{1+r}\left(\frac{1+r}{r}\right)\left[A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}\right] = A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}$.

Equation (15) states that the present value of the expected path of consumption equals initial wealth plus the present value of the expected path of income.

(b) Taking the expected value, as of time t - 1, of both sides of equation (1) yields

(16) $E_{t-1}C_t = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty} \frac{E_{t-1}[Y_{t+s}]}{(1+r)^s}\right]$,

where we have used the fact that $A_t = (1 + r)[A_{t-1} + Y_{t-1} - C_{t-1}]$ is not uncertain as of t - 1. In addition, we have used the law of iterated projections so that $E_{t-1} E_t [Y_{t+s}] = E_{t-1} [Y_{t+s}]$. Subtracting equation (16) from equation (1) gives us the innovation in consumption:

(17) $C_t - E_{t-1}C_t = \frac{r}{1+r}\left[\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s} - \sum_{s=0}^{\infty} \frac{E_{t-1}[Y_{t+s}]}{(1+r)^s}\right] = \frac{r}{1+r}\left[\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}\left[Y_{t+s}\right]}{(1+r)^s}\right]$.

The innovation in consumption will be fraction r/(1 + r) of the present value of the change in expected lifetime income.

The next step is to determine the present value of the change in expected lifetime income. That is, we need to determine

(18) $\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}\left[Y_{t+s}\right]}{(1+r)^s} = \left[Y_t - E_{t-1}Y_t\right] + \left[\frac{E_tY_{t+1} - E_{t-1}Y_{t+1}}{1+r}\right] + \left[\frac{E_tY_{t+2} - E_{t-1}Y_{t+2}}{(1+r)^2}\right] + \dots$.

In what follows, "expected to be higher" means "expected, as of period t, to be higher than it was, as of period t - 1". We are told that $u_t = 1$ and thus

(19) $Y_t - E_{t-1} Y_t = 1$.

In period t + 1, since $\Delta Y_{t+1} = \phi\Delta Y_t + u_{t+1}$, the change in Y_{t+1} is expected to be $\phi\Delta Y_t = \phi$ higher. Thus the level of Y_{t+1} is expected to be higher by $1 + \phi$. Thus

(20) $\frac{E_tY_{t+1} - E_{t-1}Y_{t+1}}{1+r} = \frac{1+\phi}{1+r}$.

In period t + 2, since $\Delta Y_{t+2} = \phi\Delta Y_{t+1} + u_{t+2}$, the change in Y_{t+2} is expected to be higher by $\phi\Delta Y_{t+1} = \phi^2$. Thus the level of Y_{t+2} is expected to be higher by $1 + \phi + \phi^2$. Therefore, we have

(21) $\frac{E_tY_{t+2} - E_{t-1}Y_{t+2}}{(1+r)^2} = \frac{1+\phi+\phi^2}{(1+r)^2}$.

The pattern should be clear. We have

(22) $\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}\left[Y_{t+s}\right]}{(1+r)^s} = 1 + \frac{1+\phi}{1+r} + \frac{1+\phi+\phi^2}{(1+r)^2} + \frac{1+\phi+\phi^2+\phi^3}{(1+r)^3} + \dots$.

Note that this infinite series can be rewritten as

$$(23)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]-E_{t-1}\left[Y_{t+s}\right]}{(1+r)^s}=\left[1+\frac{1}{1+r}+\frac{1}{(1+r)^2}+\ldots\right]+\left[\frac{\phi}{1+r}+\frac{\phi}{(1+r)^2}+\frac{\phi}{(1+r)^3}\right]+\left[\frac{\phi^2}{(1+r)^2}+\frac{\phi^2}{(1+r)^3}+\ldots\right]+\ldots\,.$$

For ease of notation, define $\gamma \equiv 1/(1 + r)$. Then the first sum on the right-hand side of (23) converges to $1/(1 - \gamma)$. The second sum converges to $\phi\gamma/(1 - \gamma)$. The third sum converges to $\phi^2 \gamma^2 /(1 - \gamma)$. And so on. Thus equation (23) can be rewritten as

$$(24)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]-E_{t-1}\left[Y_{t+s}\right]}{(1+r)^s}=\frac{1}{1-\gamma}\left[1+\phi\gamma+\phi^2\gamma^2+\ldots\right]=\frac{1}{(1-\gamma)}\frac{1}{(1-\phi\gamma)}.$$

Using the definition of γ to rewrite equation (24) yields

$$(25)\quad \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]-E_{t-1}\left[Y_{t+s}\right]}{(1+r)^s}=\frac{1}{1-[1/(1+r)]}\frac{1}{1-[\phi/(1+r)]}=\frac{(1+r)}{r}\frac{(1+r)}{(1+r-\phi)}.$$

Substituting equation (25) into equation (24) gives us the following change in consumption:

$$(26)\quad C_t-E_{t-1}C_t=\frac{r}{(1+r)}\left[\frac{(1+r)}{r}\frac{(1+r)}{(1+r-\phi)}\right]=\frac{(1+r)}{(1+r-\phi)}.$$

(c) The variance of the innovation in consumption is

$$(27)\quad \mathrm{var}\left(C_t-E_{t-1}C_t\right)=\mathrm{var}\left[\frac{(1+r)}{(1+r-\phi)}u_t\right]=\left[\frac{(1+r)}{(1+r-\phi)}\right]^2\mathrm{var}(u_t)>\mathrm{var}(u_t)$$

Since $(1 + r)/(1 + r - \phi) > 1$, the variance of the innovation in consumption is greater than the variance of the innovation in income. Intuitively, an innovation to income means that, on average, the consumer will experience further changes in income in the same direction in future periods.

It is not clear whether consumers use saving and borrowing to smooth consumption relative to income. Income is not stationary, so it is not obvious what it means to smooth it.

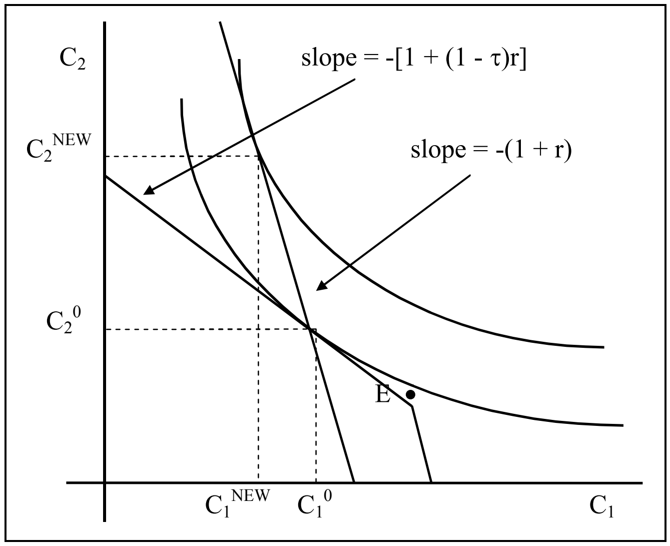
Problem 8.7

(a) The present value of the lump-sum taxes is $T_1 + [T_2 /(1 + r)]$. The present value of the tax on interest income is $[r/(1 + r)]\tau(Y_1 - C_1^0)$, where τ is the tax rate on interest income. The government must choose T_1 and T_2 so that these two quantities are equal, or

$$(1)\quad T_1+\frac{T_2}{1+r}=\frac{r}{1+r}\tau(Y_1-C_1{}^0).$$

(b) Suppose the new taxes satisfy condition (1). This means that at the point where the individual consumes C_1^0, she pays the same with the new lump-sum tax as she did with the old tax on interest income. That is, right at C_1^0, the individual's after-tax lifetime income is the same under both tax schemes. Thus at C_1^0, the individual has just enough to consume C_2^0 in the second period under both tax schemes. This means that the new budget line must go through (C_1^0, C_2^0) just as the old one did. Since (C_1^0, C_2^0) lies right on the new budget line, it is just affordable.

(c) First-period consumption must fall. Consider the figure at right. Point E represents the endowment, (Y_1, Y_2). The budget line under the tax on interest income has slope - [1 + (1 - τ)r] for $C_1 < Y_1$; for $C_1 > Y_1$ there is no positive saving and therefore no tax on interest income so that the slope equals - (1 + r).



As explained in part (b), the budget line with revenue-neutral, lump-sum taxes goes through the initial optimum consumption bundle, (C_1^0, C_2^0). It has slope equal to - (1 + r). With saving no longer taxed, then for any $C_1 < Y_1$, giving up one unit of period-one consumption yields more units of period-two consumption. Specifically, it yields (1 + r) rather than [1 + (1 - τ)r]. From the figure, we can see that the new tangency must involve lower consumption in the first period.

Intuitively, the government has set the tax rate so that there is no income effect from the change in policy, only a substitution effect. Thus, since the rate of return on saving increases, the individual chooses to save more and consume less in the first period.

Problem 8.8

(a) Consider the following experiment. In period t, the individual reduces consumption by a small (formally, infinitesimal) amount dC and uses the proceeds to purchase stock. Since one unit of stock costs P_t , dC will buy the individual dC/P_t units of stock. This change has a utility cost of dC since utility is linear in consumption.

In period t + 1, the individual will receive D_{t+1} [dC/P_t] in dividends which she can consume. She can then sell the stock, receiving P_{t+1} [dC/P_t] which she can also consume. The discounted expected utility benefit of doing this is E_t [[1/(1 + r)][D_{t+1} + P_{t+1}][dC/P_t]]. If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. Thus the utility cost must equal the expected utility benefit, or

$$(1)\quad dC = E_t\left[\left(\frac{1}{1+r}\right)(D_{t+1} + P_{t+1})\frac{dC}{P_t}\right].$$

Canceling the dC's , which is somewhat informal, and multiplying both sides of the resulting expression by P_t yields

$$(2)\quad P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right].$$

(b) Equation (2) holds in all periods and so we can write

$$(3)\quad P_{t+1} = E_{t+1}\left[\frac{D_{t+2} + P_{t+2}}{1+r}\right].$$

Substituting equation (3) into equation (2) gives us

$$(4)\quad P_t = E_t\left[\frac{D_{t+1}}{1+r}\right] + E_t E_{t+1}\left[\frac{D_{t+2} + P_{t+2}}{(1+r)^2}\right].$$

Now we can use the law of iterated projections. For a variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. Equation (4) then becomes

$$(5)\ \ P_t = E_t\left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2}\right] + E_t\left[\frac{P_{t+2}}{(1+r)^2}\right].$$

We could now substitute for P_{t+2} and then P_{t+3} and so on. We would have

$$(6)\ \ P_t = E_t\left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2} + \ldots + \frac{D_{t+s}}{(1+r)^s}\right] + E_t\left[\frac{P_{t+s}}{(1+r)^s}\right].$$

Imposing the no-bubbles condition that $\lim_{s\to\infty} E_t\,[P_{t+s}/(1+r)^s] = 0$, we can write P_t as

$$(7)\ \ P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right].$$

Equation (7) says that the price of the stock is the present value of the stream of expected future dividends.

Problem 8.9

(a) (i) With the bubble term, the price of the stock in period t is now

$$(1)\ \ P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + (1+r)^t b.$$

We need to see if such a price path satisfies the individual's first-order condition, which is given by

$$(2)\ \ P_t = E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right].$$

Specifically, then, we need to see if the right-hand sides of equations (1) and (2) are equivalent. Since equation (1) holds every period, we can write the price of the stock in period t + 1 as

$$(3)\ \ P_{t+1} = \sum_{s=1}^{\infty} E_{t+1}\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + (1+r)^{t+1} b.$$

Dividing both sides of equation (3) by (1 + r) and then taking the time-t expectation of both sides of the resulting expression gives us

$$(4)\ \ E_t\left[\frac{P_{t+1}}{1+r}\right] = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^{s+1}}\right] + (1+r)^t b,$$

where we have used the law of iterated projections so that $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$ for any variable x. Now add $E_t\,[D_{t+1}/(1+r)]$ to both sides of equation (4) to obtain

$$(5)\ \ E_t\left[\frac{D_{t+1} + P_{t+1}}{1+r}\right] = E_t\left[\frac{D_{t+1}}{1+r}\right] + \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^{s+1}}\right] + (1+r)^t b = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + (1+r)^t b.$$

Thus the right-hand sides of equations (1) and (2) are equivalent and so the proposed price path satisfies the individual's first-order condition. In this case, consumers are willing to pay more than the present value of the stream of expected future dividends. That is because they anticipate the price of the stock will keep rising so that they can enjoy capital gains that exactly offset the premium they are paying.

(a) (ii) If b were negative, then as $t \to \infty$, the bubble term, $(1 + r)^t b$, would go to minus infinity. Thus the price of the stock would eventually become negative and go to minus infinity. But that is not possible. The stock would never sell for a negative price. The strategy of just holding on to the stock and never selling it would avoid the capital loss from selling at a negative price. Or even more simply, an individual could just throw her stock certificate away rather than sell it for a negative price. Thus b cannot be negative.

(b) (i) With this bubble term, the price of the stock in period t is

$$(6)\quad P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + q_t\,,$$

where q_t equals $(1 + r)q_{t-1}/\alpha$ with probability α and equals zero with probability $(1 - \alpha)$. Again, we need to see if the right-hand side of equation (6) is equivalent to the right-hand side of equation (2), the first-order condition. Since equation (6) holds in every period, we can write the price of the stock in period t + 1 as

$$(7)\quad P_{t+1} = \sum_{s=1}^{\infty} E_{t+1}\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + q_{t+1}\,.$$

Taking the time t expectation of both sides of equation (7) and using the law of iterated projections, we have

$$(8)\quad E_t\left[P_{t+1}\right] = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + \frac{(1+r)q_t}{\alpha}\alpha + (0)(1-\alpha) = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^s}\right] + (1+r)q_t\,.$$

Dividing both sides of equation (8) by (1 + r) and then adding $E_t\,[D_{t+1}/(1 + r)]$ to both sides of the resulting expression gives us

$$(9)\quad E_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right] = E_t\left[\frac{D_{t+1}}{1+r}\right] + \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+1+s}}{(1+r)^{s+1}}\right] + q_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right] + q_t\,.$$

Thus the right-hand sides of equations (6) and (2) are equivalent and so the proposed price path satisfies the individual's first-order condition.

(b) (ii) The probability that the bubble has burst by time t + s is the probability that it bursts in t + 1 plus the probability that it bursts in t + 2, given that it did not burst in t + 1, plus the probability that it bursts in t + 3, given that it did not burst in t + 1 or t + 2 and so on. The probability that the bubble bursts in period t + 1 is $(1 - \alpha)$. The probability that the bubble bursts in t + 2, given that it did not burst in period t + 1 is given by $\alpha(1 - \alpha)$. The probability that the bubble bursts in t + 3, given that it did not burst in periods t + 1 or t +2 is $\alpha^2 (1 - \alpha)$. And so on, up to the probability that the bubble bursts in period s, given that it has not burst in any previous period, which is $\alpha^{s-1} (1 - \alpha)$. Thus the probability that the bubble has burst by time t + s is given by the sum of all these probabilities, or

(10) Prob(burst by t + s) = $(1 - \alpha)(1 + \alpha + \alpha^2 + ... + \alpha^{s-1})$.

As we allow s to go to infinity, then since $\alpha < 1$, $1 + \alpha + \alpha^2 + ... + \alpha^{s-1}$ converges to $1/(1 - \alpha)$. Thus the probability that the bubble has burst by time t + s, as s goes to infinity is $(1 - \alpha)/(1 - \alpha)$ or simply one.

(c) (i) The price of the stock in period t, in the absence of bubbles, is given by

$$(11)\quad P_t = \sum_{s=1}^{\infty} E_t\left[\frac{D_{t+s}}{(1+r)^s}\right].$$

If dividends follow a random walk, then $E_t D_{t+s} = D_t$ for any $s \geq 0$. Since changes in dividends are unpredictable, the best estimate of dividends in any future period is what dividends are today. Thus P_t can be written as

$$(12)\quad P_t = \sum_{s=1}^{\infty} \frac{D_t}{(1+r)^s} = D_t \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} = D_t\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots\right].$$

With $1/(1 + r) < 1$, we have

$$(13)\quad \frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots = \frac{1/(1+r)}{1-[1/(1+r)]} = \frac{1/(1+r)}{r/(1+r)} = \frac{1}{r}.$$

Substituting equation (13) into equation (12) gives us the following price of the stock in period t:

(14) $P_t = D_t/r$.

(c) (ii) With the bubble term, the price of the stock in period t is given by
(15) $P_t = (D_t /r) + b_t = (D_t /r) + (1 + r)b_{t-1} + ce_t$.
We need to see if the right-hand side of equation (15) is equivalent to the right-hand side of equation (2), the first-order condition. Since equation (15) holds every period, we can write the price of the stock in period t + 1 as
(16) $P_{t+1} = (D_{t+1} /r) + (1 + r)b_t + ce_{t+1} = [(D_t + e_{t+1})/r] + (1 + r)b_t +ce_{t+1}$,
where we have used the fact that $D_{t+1} = D_t + e_{t+1}$. Dividing both sides of equation (16) by (1 + r) and taking the time-t expectation of the resulting expression gives us

$$(17)\quad E_t\left[\frac{P_{t+1}}{1+r}\right]=\frac{D_t}{r(1+r)}+b_t.$$

Adding $E_t [D_{t+1} /(1 + r)]$ to both sides of equation (17) gives us

$$(18)\quad E_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right]=E_t\left[\frac{D_{t+1}}{1+r}\right]+\frac{D_t}{r(1+r)}+b_t=\frac{rD_t+D_t}{r(1+r)}+b_t=\frac{(1+r)D_t}{r(1+r)}+b_t,$$

and thus finally

$$(19)\quad E_t\left[\frac{D_{t+1}+P_{t+1}}{1+r}\right]=\frac{D_t}{r}+b_t.$$

Therefore, the right-hand sides of equations (15) and (2) are equivalent and thus the first-order condition is satisfied. With this formulation, the innovation to dividends, e, gets built into the bubble. Thus a positive realization of e does not just raise the expected path of dividends, it also raises the path of the bubble and the current price responds to both of these changes. It is in this sense that the price of the stock overreacts to changes in dividends.

Problem 8.10

(a) Suppose the individual reduces her consumption by a small (formally infinitesimal) amount dC in period t. The utility cost of doing this equals the marginal utility of consumption in period t, $1/C_t$, times dC. Thus we have
(1) utility cost = dC/C_t .
This reduction in consumption allows the individual to purchase dC/P_t trees in period t. In period t + 1, the individual receives the extra output from her additional holdings of trees. She gets to consume an extra $[dC/P_t]Y_{t+1}$. The individual then sells her additional holdings of trees for $[dC/P_t]P_{t+1}$ and consumes the proceeds. Thus her total extra consumption in period t + 1 is given by $[dC/P_t]Y_{t+1}$ + $[dC/P_t]P_{t+1}$. The marginal utility of consumption in period t + 1 is $1/C_{t+1}$. Thus the expected discounted utility benefit from this action is

$$(2)\quad \text{expected utility benefit} = E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\left(\frac{dC}{P_t}Y_{t+1}+\frac{dC}{P_t}P_{t+1}\right)\right].$$

If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. This means that the utility cost must equal the expected utility benefit, or

$$(3)\quad \frac{dC}{C_t}=E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\frac{dC}{P_t}\left(Y_{t+1}+P_{t+1}\right)\right].$$

Canceling the dC's (which is somewhat informal) gives us

$$(4)\quad \frac{1}{C_t}=E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\frac{1}{P_t}\left(Y_{t+1}+P_{t+1}\right)\right].$$

We can now solve equation (4) for P_t in terms of Y_t and expectations involving Y_{t+1} , P_{t+1} and C_{t+1} . Note that we can replace C_t with Y_t and that P_t is not uncertain at time t. Using these facts, equation (4) can be rewritten as

$$(5)\quad \frac{1}{Y_t} = \frac{1}{P_t} E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\left(Y_{t+1} + P_{t+1}\right)\right].$$

Solving equation (5) for the price of a tree in period t gives us

$$(6)\quad P_t = \frac{Y_t}{1+\rho} E_t\left[\frac{Y_{t+1} + P_{t+1}}{C_{t+1}}\right].$$

(b) Since $C_{t+s} = Y_{t+s}$ for all $s \geq 0$, equation (6) can be written as

$$(7)\quad P_t = \frac{Y_t}{1+\rho} E_t\left[1 + \frac{P_{t+1}}{Y_{t+1}}\right] = \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho} E_t\left[\frac{P_{t+1}}{Y_{t+1}}\right].$$

Equation (7) holds for all periods and so we can write the price of a tree in period t + 1 as

$$(8)\quad P_{t+1} = \frac{Y_{t+1}}{1+\rho} + \frac{Y_{t+1}}{1+\rho} E_{t+1}\left[\frac{P_{t+2}}{Y_{t+2}}\right].$$

Substituting equation (8) into equation (7) yields

$$(9)\quad P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho} E_t\left[\frac{1}{1+\rho} + \frac{1}{1+\rho} E_{t+1}\left(\frac{P_{t+2}}{Y_{t+2}}\right)\right].$$

Now use the law of iterated projections that states that for any variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$, to obtain

$$(10)\quad P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \frac{Y_t}{(1+\rho)^2} E_t\left[\frac{P_{t+2}}{Y_{t+2}}\right].$$

After repeated substitutions, we will have

$$(11)\quad P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \ldots + \frac{Y_t}{(1+\rho)^s} + \frac{Y_t}{(1+\rho)^s} E_t\left[\frac{P_{t+s}}{Y_{t+s}}\right].$$

Imposing the no-bubbles condition that $\lim_{s\to\infty} E_t\,[(P_{t+s}/Y_{t+s})/(1+\rho)^s] = 0$, the price of a tree in period t can be written as

$$(12)\quad P_t = Y_t\left[\frac{1}{1+\rho} + \frac{1}{(1+\rho)^2} + \ldots\right].$$

Since $1/(1 + \rho) < 1$, the sum converges and we can write

$$(13)\quad P_t = Y_t\left[\frac{1/(1+\rho)}{1-[1/(1+\rho)]}\right] = Y_t\left[\frac{1/(1+\rho)}{\rho/(1+\rho)}\right].$$

Thus, finally, the price of a tree in period t is

(14) $P_t = Y_t/\rho$.

(c) There are two effects of an increase in the expected value of dividends at some future date. First, at a given marginal utility of consumption, the higher expected dividends increase the attractiveness of owning trees. This tends to raise the current price of a tree. Since consumption equals dividends in this model, however, higher expected dividends in that future period mean higher consumption and lower marginal utility of consumption in that future period. This tends to reduce the attractiveness of owning trees – the tree is going to pay off more in a time when marginal utility is expected to be low – and thus tends to lower the current price of a tree. In the case of logarithmic utility, these two forces exactly offset each other, leaving the current price of a tree unchanged in the face of a rise in expected future dividends.

(d) The path of consumption is equivalent to the path of output. Thus if output follows a random walk, so does consumption. But if output does not follow a random walk, then consumption does not either.

Problem 8.11

(a) Suppose the individual reduces her holdings of the good-state asset by a small (formally, infinitesimal) amount dA_G. This change means that if the good state occurs – which it will, with probability 1/2 – the individual loses dA_G times the marginal utility of consumption in the good state, which is $U'(1)$. Thus

(1) expected utility loss = $U'(1)dA_G/2$.

Since p represents the relative price of the bad-state asset to the good-state asset, selling dA_G of the good-state asset allows the individual to purchase dA_G/p of the bad-state asset. This means that if the bad state occurs – which it will, with probability 1/2 – the individual gains dA_G/p times the expected marginal utility of consumption in the bad state. Fraction λ of the population consumes $1 - (\phi/\lambda)$ in the bad state and fraction $(1 - \lambda)$ consumes one. Thus the expected marginal utility of consumption in the bad state is $\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)$. Putting all of this together, we have

(2) expected utility benefit = $[dA_G/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]/2$.

If the individual is optimizing, this change in holdings of the two assets must leave expected utility unchanged. Thus the expected utility loss must equal the expected utility gain, or

(3) $U'(1)dA_G/2 = [dA_G/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]/2$.

(b) From equation (3), canceling the 1/2's and the dA_G's (which is somewhat informal), we have

(4) $U'(1) = [1/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]$.

Solving equation (4) for p gives us

(5) $$p = \frac{\lambda U'(1-(\phi/\lambda)) + (1-\lambda)U'(1)}{U'(1)}.$$

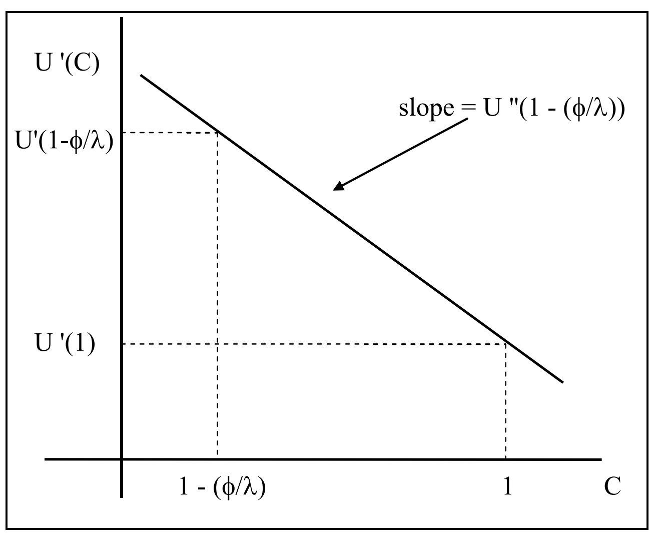
(c) The change in the equilibrium relative price of the bad-state asset to the good-state asset due to a change in λ is

(6) $$\frac{\partial p}{\partial \lambda} = U'(1-(\phi/\lambda)) + \lambda U''(1-(\phi/\lambda))(\phi/\lambda^2) - U'(1),$$

which simplifies to

(7) $$\frac{\partial p}{\partial \lambda} = U'(1-(\phi/\lambda)) - U'(1) + U''(1-(\phi/\lambda))(\phi/\lambda).$$

(d) If utility is quadratic, then $U'(C)$ is a linear function of C since $U''(C)$ is a constant. See the figure at right. We can calculate the slope of the $U'(C)$ line as



(8) $$\text{slope} = \frac{U'(1-(\phi/\lambda)) - U'(1)}{1-(\phi/\lambda)-1},$$

or

(9) $$\text{slope} = \frac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda}.$$

We also know that the slope of this line is $U''(C)$ at any value of C and in particular it equals $U''(1 - (\phi/\lambda))$. Equating these two expressions for the slope gives us

(10) $$\frac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda} = U''(1-(\phi/\lambda)),$$

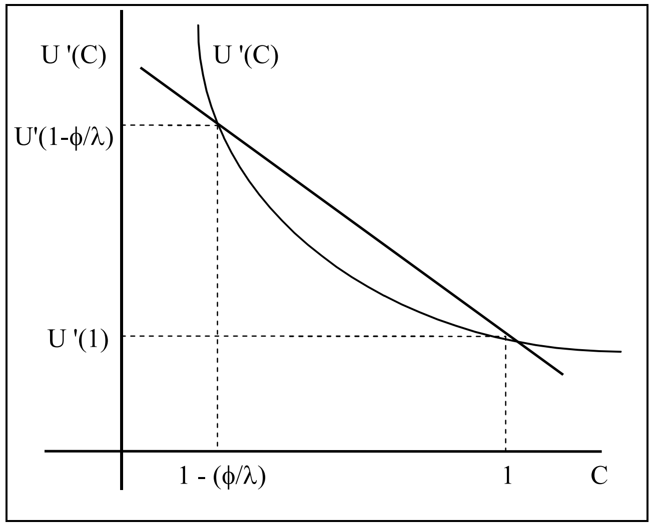
and hence

(11) $U'(1-(\phi/\lambda)) - U'(1) + (\phi/\lambda)U''(1-(\phi/\lambda)) = 0.$

The left-hand side of equation (11) is $\partial p/\partial\lambda$ and thus it equals zero as required. With quadratic utility, a marginal change in the concentration of aggregate shocks has no effect on the relative price of the bad-state asset to the good-state asset.

(e) If U '''(•) > 0 everywhere, then U '(C) is a convex function of C; U ''(C) rises, or becomes less negative, as C rises. Just as in part (d), we can draw in a straight line that goes through [1 - ϕ/λ, U '(1 - ϕ/λ)] as well as [1, U '(1)]. Again, the line will have slope equal to

(9) $\text{slope} = \dfrac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda}.$



However, from the figure, we can see that the slope of this line is less negative, or greater, than the slope of U '(C) at 1 - (ϕ/λ), which is U ''(1 - (ϕ/λ)). That is, we have

(12) $\dfrac{U'(1-(\phi/\lambda)) - U'(1)}{-\phi/\lambda} > U''(1-(\phi/\lambda)),$

or simply

(13) $U'(1-(\phi/\lambda)) - U'(1) + (\phi/\lambda)U''(1-(\phi/\lambda)) < 0.$

The left-hand side of equation (13) is $\partial p/\partial\lambda$ and thus $\partial p/\partial\lambda < 0$. If U '''(•) is everywhere positive, a fall in λ causes p to rise. That is, a marginal increase in the concentration of the aggregate shock – a fall in λ – causes the relative price of the bad-state asset to the good-state asset to rise.

Problem 8.12

(a) The change in purchases in period t, dX_t, must leave the present value of spending unchanged, so that

(1) $dX_t + dX_{t+1} + dX_{t+2} = 0.$

In addition, it must leave consumption in period t + 2 unchanged, or

(2) $(1 - \delta)^2 dX_t + (1 - \delta)dX_{t+1} + dX_{t+2} = 0.$

To see why equation (2) must hold, note that we can write the change in C_t as $dC_t = dX_t$. The change in C_{t+1} is $dC_{t+1} = (1 - \rho)dC_t + dX_{t+1}$ or substituting for dC_t, we have $dC_{t+1} = (1 - \rho)dX_t + dX_{t+1}$. The change in C_{t+2} is $dC_{t+2} = (1 - \delta)dC_{t+1} + dX_{t+2}$ or substituting for dC_{t+1}, we have $dC_{t+2} = (1 - \delta)^2 dX_t + (1 - \delta)dX_{t+1} + dX_{t+2}$. Thus if C_{t+2} is not to change, equation (2) must hold.

Thus we have two equations in two unknowns. Solving equation (1) for dX_{t+2} gives us

(3) $dX_{t+2} = -dX_t - dX_{t+1}$.

Substituting equation (3) into equation (2) yields

(4) $(1 - \delta)^2 dX_t + (1 - \delta)dX_{t+1} - dX_t - dX_{t+1} = 0.$

Expanding and collecting terms gives us

(5) $dX_t [1 - 2\delta + \delta^2 - 1] + [1 - \delta - 1]dX_{t+1} = 0,$

and thus

(6) $dX_{t+1} = (\delta - 2)dX_t$.

Substituting equation (6) into equation (3) yields

(7) $dX_{t+2} = -dX_t - (\delta - 2)dX_t$,

and thus

(8) $dX_{t+2} = (1 - \delta)dX_t$.

(b) Since $C_t = (1 - \delta)C_{t-1} + X_t$, then
(9) $dC_t = dX_t$.
Since $C_{t+1} = (1 - \delta)C_t + X_{t+1}$, then
(10) $dC_{t+1} = (1 - \delta)dC_t + dX_{t+1}$.
Substituting equations (9) and (6) into equation (10) gives us
(11) $dC_{t+1} = (1 - \delta)dX_t + (\delta - 2)dX_t = -dX_t$.

Since only C_t and C_{t+1} are changed – C_{t+2} is unchanged by construction – we only need to look at expected utility in periods t and t + 1. Since instantaneous utility is quadratic, the marginal utility of consumption in period t is $1 - aC_t$. Thus the change in utility in period t is $(1 - aC_t)(dX_t)$. The marginal utility of consumption in period t + 1 is given by $1 - aC_{t+1}$. Since $dC_{t+1} = -dX_t$, the change in expected utility in period t + 1 is the expected value of $(1 - aC_{t+1})(-dX_t)$.

(c) For this change in expected utility to be zero – as it must be, if the individual is optimizing – we require
(12) $(1 - aC_t)(dX_t)$ + expected value of $[(1 - aC_{t+1})(-dX_t)] = 0$.
Canceling the dX_t 's (which is somewhat informal), subtracting one from both sides and then dividing both sides by (- a) yields
(13) expected value of $C_{t+1} = C_t$.
Thus consumption follows a random walk since changes in consumption are unpredictable. The best estimate of consumption in period t + 1 is simply what consumption equals this period.

(d) Rearranging $C_t = (1 - \delta)C_{t-1} + X_t$ to solve for X_t gives us
(14) $X_t = C_t - (1 - \delta)C_{t-1}$.
Equation (14) holds for all periods and so we can write
(15) $X_{t-1} = C_{t-1} - (1 - \delta)C_{t-2}$.
Subtracting equation (15) from equation (14) gives us
(16) $X_t - X_{t-1} = C_t - (1 - \delta)C_{t-1} - C_{t-1} + (1 - \delta)C_{t-2}$,
which implies
(17) $X_t - X_{t-1} = (C_t - C_{t-1}) - (1 - \delta)(C_{t-1} - C_{t-2})$.
Since consumption is a random walk, we can write
(18) $C_t = C_{t-1} + u_t$,
where u_t is a variable whose expectation as of t - 1 is zero. Using equation (18), and the fact that (18) holds in all periods, equation (17) can be rewritten as
(19) $X_t - X_{t-1} = u_t - (1 - \delta)u_{t-1}$.

Equation (19) states that the change in purchases from t - 1 to t has a predictable component – a component that is known as of t - 1 – which is u_{t-1} , the innovation to consumption in period t - 1. Thus purchases of durable goods will not follow a random walk.

As explained in Section 8.2, any change in expected lifetime resources is spread out equally among consumption in each remaining period of the individual's life. Although we are simplifying by using a discount rate of zero, the basic ideas are general.

Now suppose that in period t - 1, the individual's estimate of lifetime resources changes in such a way that C_{t-1} is one unit higher than C_{t-2} , that is, $u_{t-1} = 1$. This also means that expected consumption in all future periods is one unit higher than it used to be. In order to get C_{t-1} up by one, purchases in period t - 1 must be one higher than they were expected to be. But now look at the change in purchases from t - 1 to t. From equation (19), the expectation (as of period t - 1) of the change in purchases from t - 1 to t is $-(1 - \delta)$, since u_{t-1} is assumed to equal one.

Intuitively, some of the new goods purchased in period t - 1 will still be around in period t. Thus to keep expected consumption in period t at the new higher path – one higher than it was before – it is not expected to be necessary to buy one unit of goods all over again. The individual only has to purchase enough to replace the fraction of the extra t - 1 purchases that depreciated, which is fraction δ. Thus purchases in period t are expected to be less than purchases in t - 1. Specifically, they are expected to be lower by the amount that does not depreciate, which is $(1 - \delta)$. Thus, as of period t - 1, part of the change in purchases between t - 1 and t is predictable and thus purchases do not follow a random walk.

Now consider what happens if $\delta = 0$, the case of no depreciation. Then from equation (19), the expectation (as of period t - 1) of the change in purchases from t - 1 to t is -1. Now all of the new goods purchased in period t - 1 will still be around in period t. Thus to keep expected consumption at its new higher path – one higher than it was before – it is not expected to be necessary to purchase anything new in period t. Thus purchases are expected to fall by the whole amount of the innovation in purchases the previous period.

Problem 8.13

(a) (i) Suppose $Z_{it} = C_{t-1}^{\phi}$, where $\phi \in [0,1]$ so the reference level of consumption is aggregate consumption in the previous period, which an individual takes as given. To find the Euler equation using the perturbation approach, we know that

(1) $U'(C_{it})dC = [1/(1+\rho)](1+r)U'(C_{i,t+1})dC$.

Using the utility function to find the marginal utilities in periods t and t+1 gives us

(2) $$\left(\frac{C_{it}}{Z_{it}}\right)^{-\theta}\frac{1}{Z_{it}} = \frac{1+r}{1+\rho}\left(\frac{C_{i,t+1}}{Z_{i,t+1}}\right)^{-\theta}\frac{1}{Z_{i,t+1}},$$

which simplifies to

(3) $$\left(\frac{C_{it}}{Z_{it}}\right)^{-\theta}\frac{Z_{i,t+1}}{Z_{it}} = \frac{1+r}{1+\rho}\left(\frac{C_{i,t+1}}{Z_{i,t+1}}\right)^{-\theta}.$$

Solving for the ratio of consumption in the two periods yields

(4) $$\frac{C_{i,t+1}}{C_{it}} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\theta}}\left(\frac{Z_{i,t+1}}{Z_{it}}\right)^{\frac{\theta-1}{\theta}}.$$

Substituting the assumption that $Z_{it} = C_{t-1}^{\phi}$ gives us

(5) $$\frac{C_{i,t+1}}{C_{it}} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\theta}}\left(\frac{C_t}{C_{t-1}}\right)^{\phi\frac{\theta-1}{\theta}}.$$

(a) (ii) We know that in equilibrium $C_t = C_{it}$ for all t. Substituting this into our Euler equation yields

(6) $$\frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\theta}}\left(\frac{C_t}{C_{t-1}}\right)^{\phi\frac{\theta-1}{\theta}}.$$

Taking logs gives us

(7) $\ln C_{t+1} - \ln C_t = \frac{1}{\theta}[\ln(1+r) - \ln(1+\rho)] + \frac{\theta-1}{\theta}\phi[\ln C_t - \ln C_{t-1}]$.

When θ = 1 so that we have log utility, the growth rate of consumption is thus given by

(8) $\ln C_{t+1} - \ln C_t = \ln(1+r) - \ln(1+\rho)$.

Note that for small values of x, we can use the approximation that $\ln(1+x) \cong x$ allowing us to write

(9) $\ln C_{t+1} - \ln C_t \cong r - \rho$.

With log utility, consumption growth depends only on the interest rate and the rate of time preference; habit formation does not affect the behavior of consumption. In the utility function, the individual's consumption choice and the reference level of consumption are separable. The reference level does not affect the individual's marginal utility of consumption because she assumes her choice of consumption does not affect the reference level and so the reference level does not impact the individual's choice of consumption. In contrast, when θ > 1, the marginal utility of consumption is affected by the reference level of consumption, which is aggregate consumption in the previous period. If the individual chooses to consume less today, the utility cost depends on aggregate consumption yesterday. The utility benefit of consuming more tomorrow then depends on aggregate consumption today. From equation (8) we can see that the implication is that consumption growth this period depends positively on consumption growth last period.

(b) (i) Suppose $Z_t = C_{i,t-1}$ so that φ = 1 and the reference level of consumption is determined by the individual's own level of past consumption so individuals internalize their habit. Individuals maximize

(12) $$U = \sum_{t=1}^{T}\left(\frac{1}{1+\rho}\right)^t \frac{1}{1-\theta}\left(\frac{C_{it}}{C_{i,t-1}}\right)^{1-\theta}.$$

The perturbation method would then imply the marginal utility lost today by giving up a small amount of consumption would equal the discounted marginal utility gain of increasing consumption tomorrow by (1 + r) times the small change in consumption or

(13) $U'(C_{it})dC = [1/(1+\rho)](1+r)U'(C_{i,t+1})dC$.

Since consumption today affects both utility today and utility tomorrow (through habit formation), the marginal utility loss in period t is given by

(14) $$U'(C_{it}) = \left(\frac{C_{it}}{C_{i,t-1}}\right)^{-\theta}\frac{1}{C_{i,t-1}} + \frac{1}{1+\rho}\left(\frac{C_{i,t+1}}{C_{it}}\right)^{-\theta}\left(-\frac{C_{i,t+1}}{C_{it}^2}\right),$$

which simplifies to

(15) $$U'(C_{it}) = \left(\frac{C_{it}}{C_{i,t-1}}\right)^{-\theta}\frac{1}{C_{i,t-1}} - \frac{1}{1+\rho}\left(\frac{C_{i,t+1}}{C_{it}}\right)^{1-\theta}\left(\frac{1}{C_{it}}\right).$$

Similarly, the marginal utility gain in period t + 1 is given by

(16) $$U'(C_{i,t+1}) = \left(\frac{C_{i,t+1}}{C_{it}}\right)^{-\theta}\frac{1}{C_{it}} - \frac{1}{1+\rho}\left(\frac{C_{i,t+2}}{C_{i,t+1}}\right)^{1-\theta}\left(\frac{1}{C_{i,t+1}}\right).$$

The Euler equation is then

(17) $$\left[\left(\frac{C_{it}}{C_{i,t-1}}\right)^{-\theta}\frac{C_{it}}{C_{i,t-1}} - \frac{1}{1+\rho}\left(\frac{C_{i,t+1}}{C_{it}}\right)^{1-\theta}\right]\frac{1}{C_{it}} = \frac{1+r}{1+\rho}\left[\left(\frac{C_{i,t+1}}{C_{it}}\right)^{-\theta}\frac{C_{i,t+1}}{C_{it}} - \frac{1}{1+\rho}\left(\frac{C_{i,t+2}}{C_{i,t+1}}\right)^{1-\theta}\right]\frac{1}{C_{i,t+1}},$$

which simplifies to

$$(18)\quad \left[\left(\frac{C_{it}}{C_{i,t-1}}\right)^{1-\theta} - \frac{1}{1+\rho}\left(\frac{C_{i,t+1}}{C_{it}}\right)^{1-\theta}\right]\frac{C_{i,t+1}}{C_{it}} = \frac{1+r}{1+\rho}\left[\left(\frac{C_{i,t+1}}{C_{it}}\right)^{1-\theta} - \frac{1}{1+\rho}\left(\frac{C_{i,t+2}}{C_{i,t+1}}\right)^{1-\theta}\right].$$

(b) (ii) Let $g_t \equiv (C_t / C_{t-1}) - 1$ denote consumption growth from t-1 to t. Assuming consumption growth is close to zero, we can approximate expressions of the form $(C_t / C_{t-1})^{\gamma}$ with $1 + \gamma g_t$, and since we are assuming that r = ρ = 0, expression (18) becomes

(19) $\{[1 + (1-\theta)g_t] - [1 + (1-\theta)g_{t+1}]\}(1 + g_{t+1}) \cong [1 + (1-\theta)g_{t+1}] - [1 + (1-\theta)g_{t+2}]$.

After rearranging the terms we obtain

(20) $(1-\theta)(g_t - g_{t+1})(1 + g_{t+1}) \cong (1-\theta)(g_{t+1} - g_{t+2})$.

Multiplying both sides by -1/(1 – θ), we arrive at

(21) $(g_{t+2} - g_{t+1}) \cong (g_{t+1} - g_t)(1 + g_{t+1})$.

Finally, assuming consumption growth is small, we can ignore the interaction term $(g_{t+1} - g_t)g_{t+1}$ leaving us with

(22) $(g_{t+2} - g_{t+1}) \cong (g_{t+1} - g_t)$

In this model, the change in the growth rate of consumption between successive periods is (approximately) equal and so the growth rate of consumption increases over time. To smooth marginal utilities of consumption across time, since there is a habit term that agents internalize, they must have an increasing consumption growth rate. This allows them to keep up with the habit and thus attain constant marginal utilities.

Problem 8.14

(a) Taking the derivative of the instantaneous utility function,

(1) $u(C) = -e^{-\gamma C}$

with respect to C gives us

(2) $u'(C) = -e^{-\gamma C}(-\gamma) = \gamma e^{-\gamma C}$.

The second derivative is given by

(3) $u''(C) = \gamma e^{-\gamma C}(-\gamma) = -\gamma^2 e^{-\gamma C}$.

The third derivative is

(4) $u'''(C) = -\gamma^2 e^{-\gamma C}(-\gamma) = \gamma^3 e^{-\gamma C} > 0$.

The third derivative is positive since γ > 0 and the exponential function is everywhere positive. Thus the constant-absolute-risk-aversion utility function implies the precautionary-saving behavior described in Section 8.6.

(b) Solving the individual's lifetime budget constraint for second-period consumption gives us

(5) $C_2 = Y_1 + Y_2 - C_1$.

Substituting equation (5) into the two-period utility function yields

(6) $U = -e^{-\gamma C_1} - e^{-\gamma(Y_1 + Y_2 - C_1)} = -e^{-\gamma C_1} - e^{-\gamma Y_1}e^{-\gamma Y_2}e^{\gamma C_1}$.

Since only Y_2 is uncertain, the expected value of lifetime utility is given by

(7) $E[U] = -e^{-\gamma C_1} - e^{-\gamma Y_1}e^{\gamma C_1}E[e^{-\gamma Y_2}]$.

We are told that $Y_2 \sim N(\overline{Y}_2, \sigma^2)$ and so $-\gamma Y_2 \sim N(-\gamma\overline{Y}_2, \gamma^2\sigma^2)$. Using the hint in Problem 8.5, part (b) that states that if a variable x is normally distributed with mean μ and variance V, $E[e^x]=e^{\mu}e^{V/2}$, we can write

(8) $E[e^{-\gamma Y_2}] = e^{-\gamma\overline{Y}_2}e^{\gamma^2\sigma^2/2}$.

Substituting equation (8) into equation (7) yields

(9) $E[U] = -e^{-\gamma C_1} - e^{-\gamma Y_1} e^{\gamma C_1} e^{-\gamma \overline{Y}_2} e^{\gamma^2 \sigma^2 / 2}$.

Equation (9) gives the individual's expected lifetime utility as a function of C_1 and the exogenous parameters Y_1, $\overline{Y}_2$, σ^2, and γ.

(c) The individual chooses C_1 in order to maximize expected lifetime utility as given by equation (9). The first-order condition is

(10) $\dfrac{\partial E[U]}{\partial C_1} = -e^{-\gamma C_1}(-\gamma) - e^{-\gamma Y_1} e^{\gamma C_1} e^{-\gamma \overline{Y}_2} e^{\gamma^2 \sigma^2 / 2}(\gamma) = 0$.

This simplifies to

(11) $e^{-\gamma C_1} = e^{-\gamma Y_1} e^{\gamma C_1} e^{-\gamma \overline{Y}_2} e^{\gamma^2 \sigma^2 / 2}$.

Taking the natural logarithm of both sides of equation (11) gives us

(12) $-\gamma C_1 = -\gamma Y_1 + \gamma C_1 - \gamma \overline{Y}_2 + \gamma^2 \sigma^2 / 2$.

Dividing both sides of equation (12) by γ and then subtracting C_1 from both sides yields

(13) $-2C_1 = -Y_1 - \overline{Y}_2 + \gamma \sigma^2 / 2$.

The optimal choice of C_1 is thus given by

(14) $C_1 = \dfrac{Y_1 + \overline{Y}_2}{2} - \dfrac{\gamma \sigma^2}{4}$.

If there is no uncertainty so that $\sigma^2 = 0$ and thus $Y_2 = \overline{Y}_2$ with certainty, first-period consumption is given by

(15) $C_1 = \dfrac{Y_1 + \overline{Y}_2}{2}$.

In this case, the individual simply consumes half of lifetime income in the first period (recall that the interest rate is zero and the individual has no initial wealth). To see how an increase in uncertainty affects first-period consumption, take the derivative of C_1 with respect to σ^2:

(16) $\dfrac{\partial C_1}{\partial \sigma^2} = -\dfrac{\gamma}{4} < 0.$

An increase in uncertainty lowers first-period consumption and thus increases saving. As expected, the individual undertakes precautionary saving in the face of uncertainty.

<u>Problem 8.15</u>

(a) (i) Under commitment, the individual chooses c_1 , c_2 , and c_3 to maximize

(1) $U_1 = \ln c_1 + \delta \ln c_2 + \delta \ln c_3$,

where $0 < \delta < 1$. With a real interest rate of zero and wealth of W, the individual's consumption choices must satisfy $c_1 + c_2 + c_3 = W$. We can solve an unconstrained optimization; $c_3 = W - c_1 - c_2$ can be substituted into equation (1) yielding

(2) $U_1 = \ln c_1 + \delta \ln c_2 + \delta \ln(W - c_1 - c_2)$.

Thus the individual chooses c_1 and c_2 to maximize (2) with c_3 determined by $c_3 = W - c_1 - c_2$. The first-order conditions are

(3) $\dfrac{\partial U_1}{\partial c_1} = \dfrac{1}{c_1} - \dfrac{\delta}{W - c_1 - c_2} = 0$, and

(4) $\dfrac{\partial U_1}{\partial c_2} = \dfrac{\delta}{c_2} - \dfrac{\delta}{W - c_1 - c_2} = 0$.

From equation (3) we have

(5) $\delta c_1 = W - c_1 - c_2$,
or
(6) $(1 + \delta)c_1 = W - c_2$,
and thus

(7) $c_1 = \dfrac{1}{1+\delta}(W - c_2)$.

From equation (4) we have
(8) $c_2 = W - c_1 - c_2$,
or simply
(9) $c_2 = (W - c_1)/2$.

Substituting equation (9) into equation (7) yields

(10) $c_1 = \dfrac{1}{1+\delta}\left[W - (W - c_1)/2\right] = \dfrac{1}{2(1+\delta)}(W + c_1)$.

Solving for c_1 gives us
(11) $c_1\,[2(1 + \delta) - 1] = W$,
or simply

(12) $c_1 = \dfrac{1}{1+2\delta}W$.

Equation (12) gives the individual's optimal choice of first-period consumption under commitment. To solve for second-period consumption, substitute equation (12) into (9):

(13) $c_2 = \dfrac{1}{2}\left(W - \dfrac{1}{1+2\delta}W\right) = \dfrac{1}{2(1+2\delta)}(W + 2\delta W - W)$,

which simplifies to

(14) $c_2 = \dfrac{\delta}{1+2\delta}W$.

It should be clear from the first-period objective function that c_2 and c_3 will be equal but to verify this, substitute equations (12) and (14) into the constraint, $c_3 = W - c_1 - c_2$, to obtain

(15) $c_3 = W - \dfrac{1}{1+2\delta}W - \dfrac{\delta}{1+2\delta}W = \dfrac{1+2\delta-1-\delta}{1+2\delta}W$,

which simplifies to

(16) $c_3 = \dfrac{\delta}{1+2\delta}W$.

(a) (ii) In period 2, the individual chooses c_2 taking her choice of c_1 – which was made last period – as given and with the constraint that $c_3 = W - c_1 - c_2$. Thus the individual chooses c_2 to maximize
(17) $U_2 = \ln c_2 + \delta \ln[W - c_1 - c_2]$.
The first-order condition is given by

(18) $\dfrac{\partial U_2}{\partial c_2} = \dfrac{1}{c_2} + \dfrac{\delta}{W - c_1 - c_2}(-1) = 0$.

Solving for c_2 as a function of W and c_1 yields
(19) $\delta c_2 = W - c_1 - c_2$,
or simply

(20) $c_2 = \dfrac{1}{1+\delta}(W - c_1)$.

This means that third-period consumption as a function of W and the choice of c_1 is given by

(21) $c_3 = W - c_1 - \dfrac{1}{1+\delta}(W - c_1) = \dfrac{(1+\delta)W - (1+\delta)c_1 - W + c_1}{1+\delta}$,

or simply

$$(22)\quad c_3 = \frac{\delta}{1+\delta}(W - c_1).$$

The individual chooses c_1 in period 1 just as she did under commitment since she (wrongly) believes she will choose c_2 in the same way as under commitment. Thus, again we have

$$(23)\quad c_1 = \frac{1}{1+2\delta}W.$$

Substituting equation (23) into equation (20) yields

$$(24)\quad c_2 = \frac{1}{1+\delta}\left[W - \frac{1}{1+2\delta}W\right] = \frac{1}{1+\delta}\left[\frac{1+2\delta-1}{1+2\delta}W\right],$$

or simply

$$(25)\quad c_2 = \frac{2\delta}{(1+\delta)(1+2\delta)}W.$$

Finally, we can obtain third-period consumption by substituting equation (23) into equation (22):

$$(26)\quad c_3 = \frac{\delta}{1+\delta}\left[W - \frac{1}{1+2\delta}W\right] = \frac{\delta}{1+\delta}\left[\frac{1+2\delta-1}{1+2\delta}W\right],$$

or simply

$$(27)\quad c_3 = \frac{2\delta^2}{(1+\delta)(1+2\delta)}W.$$

(a) (iii) Now, in period 1, the individual chooses c_1 realizing that her choices of c_2 and c_3 – which will be functions of her choice of c_1 – will be given by equations (20) and (22). Thus we can substitute (20) and (22) into the period-1 objective function:

$$(28)\quad U_1 = \ln c_1 + \delta \ln\left[\frac{1}{1+\delta}(W - c_1)\right] + \delta \ln\left[\frac{\delta}{1+\delta}(W - c_1)\right].$$

The first-order condition for the optimal choice of period-1 consumption is

$$(29)\quad \frac{\partial U_1}{\partial c_1} = \frac{1}{c_1} + \frac{\delta}{[1/(1+\delta)](W - c_1)}\left[\frac{(-1)}{1+\delta}\right] + \frac{\delta}{[\delta/(1+\delta)](W - c_1)}\left[\frac{(-\delta)}{1+\delta}\right] = 0,$$

which simplifies to

$$(30)\quad \frac{1}{c_1} = \frac{2\delta}{W - c_1}.$$

Solving for c_1 yields

$$(31)\quad 2\delta c_1 = W - c_1,$$

or simply

$$(32)\quad c_1 = \frac{1}{1+2\delta}W.$$

Note that the choice of period-1 consumption is the same here as it was under "naiveté". Since c_2 and c_3 will be chosen the same way as under "naiveté", they will be the same also and are once again given by equations (25) and (27).

(b) (i) The individual's preferences are time-inconsistent because the optimal choice of period-2 consumption that is made in the first period is no longer the optimal choice once period 2 actually arrives. This is illustrated by the fact that if the individual does not commit to period-2 consumption in the first period, then when period 2 arrives she chooses

(25) $c_2 = \dfrac{2\delta}{(1+\delta)(1+2\delta)} W,$

rather than the choice she had originally made in the first period, which was

(14) $c_2 = \dfrac{\delta}{1+2\delta} W.$

And, in fact, since $2/(1 + \delta) > 1$, she chooses a higher value of period-2 consumption once period 2 actually arrives.

We can see from the period-1 objective function that in the first period the individual is indifferent between period-2 and period-3 consumption; they are both discounted by δ. But when period 2 actually occurs, we can see from the period-2 objective function that the individual then prefers period-2 consumption over period-3 consumption.

(b) (ii) The key to the result that sophistication does not affect behavior is the assumption of log utility. The intuition behind this result is very similar to the intuition behind the version of the Tabellini-Alesina model with logarithmic utility that is presented in Section 12.6.

Think of a sophisticated individual contemplating a marginal decrease in c_1 , relative to what a naive individual would do. The naive individual believes she will allocate the increase in saving equally between c_2 and c_3 and that marginal utility will be the same in the two future periods. The sophisticated individual realizes that she will, in fact, devote most of the increase in saving to c_2 and that c_2 will be high. The individual does not particularly value c_2 thus marginal utility in period 2 will be low. This tends to make the increase in saving look relatively less attractive to the sophisticated individual than to the naive individual.

But the sophisticated individual also realizes that some of the increase in saving will be devoted to c_3 , which will be low. The individual values c_3 as much as c_2 and thus marginal utility will be high in period 3. This tends to make the increase in saving look relatively more attractive to the sophisticated individual than to the naive individual.

With log utility, these two effects exactly offset each other. With a general utility function, a sophisticated individual can consume either more or less in the first period than a naive individual.

SOLUTIONS TO CHAPTER 9

Problem 9.1

(a) Given K and the fixed quantity demanded, Y, the firm will hire enough labor to meet that demand. Given the production function

(1) $Y = K^{\alpha} L^{1-\alpha}$,

the firm will hire

(2) $L = Y^{1/(1-\alpha)} K^{-\alpha/(1-\alpha)}$.

(b) Substituting equation (2) for the firm's choice of L into the profit function, $\pi = PY - WL - r_K K$, we have

(3) $\pi = PY - W[Y^{1/(1-\alpha)} K^{-\alpha/(1-\alpha)}] - r_K K$.

(c) The first-order condition for the firm's choice of K is

(4) $\dfrac{\partial \pi}{\partial K} = \dfrac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{[-\alpha/(1-\alpha)]-1} - r_K = 0$,

or simply

(5) $\dfrac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{-1/(1-\alpha)} = r_K$.

In order for the value of K in equation (5) to be a maximum, we require $\partial^2\pi/\partial K^2$ to be negative. This derivative is

(6) $\dfrac{\partial^2 \pi}{\partial K^2} = \left(\dfrac{-1}{1-\alpha}\right)\dfrac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{(\alpha-2)/(1-\alpha)} < 0$.

So the second-order condition is satisfied since $\alpha < 1$.

(d) Solving equation (5) for K gives us

(7) $K^{1/(1-\alpha)} = \left(\dfrac{\alpha}{1-\alpha}\right)\dfrac{W Y^{1/(1-\alpha)}}{r_K}$.

Taking both sides of equation (7) to the exponent $(1 - \alpha)$ gives us the firm's choice of K:

(8) $K = Y\left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\dfrac{W}{r_K}\right)^{(1-\alpha)}$.

Thus changes in the price of the firm's product do not directly affect the profit-maximizing choice of K, although changes in P likely change Y. The elasticity of K with respect to the wage, W, is $(1 - \alpha)$, which is positive. Its elasticity with respect to the rental price of capital, r_K, is $-(1 - \alpha)$, which is negative. Finally, the elasticity of K with respect to the quantity demanded is one.

Problem 9.2

(a) At each point in time from t to t + T, the firm is allowed to deduct P_K/T from its taxable income. This allows it to save $\tau(P_K/T)$ in taxes at each point in time from t to t + T, where τ is the marginal tax rate. If i is the constant interest rate, the present value of the reduction in the firm's corporate tax liabilities, denoted X, is given by the following expression:

(1) $X = \displaystyle\int_{s=t}^{t+T} e^{-i(s-t)} \tau (P_K/T) ds$,

which implies

(2) $X = \tau(P_K/T)\int_{s=t}^{t+T} e^{-i(s-t)}ds = \tau(P_K/T)\left[-\frac{1}{i}e^{-i(s-t)}\Big|_{s=t}^{s=t+T}\right] = \tau(P_K/T)\left[\frac{1-e^{-iT}}{i}\right].$

Since the after-tax price of the capital good, denoted P_K^{AT}, is its pretax price, P_K, minus the present value of the tax saving that results, we have

(3) $P_K^{AT} = P_K - \tau(P_K/T)\left[\frac{1-e^{-iT}}{i}\right] = P_K\left[1-(\tau/T)\left(\frac{1-e^{-iT}}{i}\right)\right].$

(b) An increase in inflation, π, without a change in the real interest rate, r, increases the nominal interest rate, i. From equation (1), the present value of the reduction in the firm's corporate tax liabilities as a result of purchasing the capital good is

(4) $X = \tau(P_K/T)\int_{s=t}^{t+T} e^{-i(s-t)}ds.$

The change in the present value of the reduction in the firm's corporate tax liabilities due to a change in the nominal interest rate is therefore

(5) $\frac{\partial X}{\partial i} = \tau(P_K/T)\int_{s=t}^{t+T} -(s-t)e^{-i(s-t)}ds = -\tau(P_K/T)\int_{s=t}^{t+T}(s-t)e^{-i(s-t)}ds < 0.$

The increase in i reduces the present value of the tax savings from purchasing the capital good. Therefore, it increases the after-tax price of the capital good.

Problem 9.3

From equation (9.4) in the text, the real user cost of capital is

(1) $r_K(t) = [r(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t),$

where r(t) is the relevant real interest rate, δ is the rate of depreciation and $p_K(t)$ is the real price of capital. Here, capital refers to owner-occupied housing. The after-tax real interest rate for owner-occupied housing is $r(t) - \tau i(t)$ where τ is the marginal tax rate. This is due to the fact that nominal interest payments are tax deductible.

Intuitively, if an individual foregoes selling her home, she does lose $r(t)p_K(t)$ – the interest she could obtain by selling it and saving the proceeds – but she does get the bonus of deducting her nominal interest payments from her income. Thus she receives τ times $i(t)p_K(t)$ in tax savings by holding on to her home, which reduces the user cost of capital. Thus for owner-occupied housing, equation (1) becomes

(2) $r_K(t) = [r(t) - \tau i(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t).$

Substituting $i(t) = r(t) + \pi(t)$ into equation (2) gives us

(3) $r_K(t) = [r(t) - \tau r(t) - \tau\pi(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t),$

which implies

(4) $r_K(t) = [(1 - \tau)r(t) - \tau\pi(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t).$

To see how an increase in inflation for a given real interest rate affects $r_K(t)$, take the derivative of $r_K(t)$ with respect to $\pi(t)$:

(5) $\partial r_K(t)/\partial\pi(t) = -\tau p_K(t) < 0.$

An increase in inflation reduces the user cost of owner-occupied housing since it increases tax-deductible nominal interest payments. Thus an increase in inflation increases the desired stock of owner-occupied housing.

Problem 9.4

(a) The planner's problem is to maximize the discounted value of lifetime utility for the representative household, which is given by

$$(1)\quad U = \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \qquad \beta \equiv \rho - n - (1-\theta)g,$$

subject to the capital-accumulation equation given by

$$(2)\quad \dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

The control variable is the variable that can be controlled freely by the planner, consumption per unit of effective labor, c(t). The state variable is the variable with the property that its value at any time is determined by past decisions of the planner. Here, that is capital per unit of effective labor, k(t). Finally, the shadow value of the state variable is the costate variable, denoted $\mu(t)$.

The current-value Hamiltonian is thus

$$(3)\quad H\big(k(t), c(t)\big) = \frac{c(t)^{1-\theta}}{1-\theta} + \mu(t)[f(k(t)) - c(t) - (n+g)k(t)].$$

(b) The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point is zero, or

$$(4)\quad \frac{\partial H\big(k(t), c(t)\big)}{\partial c(t)} = c(t)^{-\theta} - \mu(t) = 0.$$

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time, or

$$(5)\quad \frac{\partial H\big(k(t), c(t)\big)}{\partial k(t)} = \mu(t) f'(k(t)) - \mu(t)(n+g) = \beta\mu(t) - \dot{\mu}(t).$$

The final condition is the transversality condition. The limit as t goes to infinity of the discounted value of the costate variable times the state variable equals zero, or

$$(6)\quad \lim_{t\to\infty} e^{-\beta t}\mu(t)k(t) = 0.$$

(c) From equation (4) we have

$$(7)\quad \mu(t) = c(t)^{-\theta}.$$

Taking the time derivative of both sides of equation (7) yields

$$(8)\quad \dot{\mu}(t) = -\theta c(t)^{-\theta-1}\dot{c}(t) = -\theta c(t)^{-\theta}\frac{\dot{c}(t)}{c(t)}.$$

From equation (5), we have

$$(9)\quad \dot{\mu}(t) = \mu(t)[\beta - f'(k(t)) + (n+g)].$$

Equating these two expressions for $\dot{\mu}(t)$ gives us

$$(10)\quad -\theta c(t)^{-\theta}\frac{\dot{c}(t)}{c(t)} = \mu(t)[\beta - f'(k(t)) + (n+g)].$$

Substituting equation (7) for $\mu(t)$ and $f'(k(t)) = r(t)$ into equation (10) yields

$$(11)\quad -\theta c(t)^{-\theta}\frac{\dot{c}(t)}{c(t)} = c(t)^{-\theta}[\beta - r(t) + (n+g)].$$

Canceling the $c(t)^{-\theta}$, dividing both sides by $-\theta$, and substituting for $\beta \equiv \rho - n - (1-\theta)g$ gives us

(12) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{r(t) - (n+g) - \rho + n + (1-\theta)g}{\theta}$,

which simplifies to

(13) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{r(t) - \rho - \theta g}{\theta}$.

Equation (13) is identical to the Euler equation in the decentralized equilibrium. See equation (2.20) in the text.

(d) Dividing both sides of equation (9) by $\mu(t)$ leaves us with

(14) $\dfrac{\dot{\mu}(t)}{\mu(t)} = \beta + (n+g) - r(t)$,

where we have substituted for $f'(k(t)) = r(t)$. Note that equation (14) can be written as

(15) $\partial \ln\mu(t)/\partial t = \beta + (n + g) - r(t)$.

Integrating both sides of equation (15) from time $\tau = 0$ to time $\tau = t$ gives us

(16) $\ln\mu(t) - \ln\mu(0) = [\beta + (n+g)]\tau\Big|_{\tau=0}^{\tau=t} - \displaystyle\int_{\tau=0}^{t} r(\tau)d\tau$.

Using the definition of R(t) and simplifying gives us

(17) $\ln\mu(t) = \ln\mu(0) + \beta t + (n + g)t - R(t)$.

Taking the exponential function of both sides of equation (17) yields

(18) $\mu(t) = \mu(0)e^{\beta t}\, e^{(n+g)t}\, e^{-R(t)}$

Thus $e^{-\beta t}\,\mu(t)$ is proportional to $e^{-R(t)}\, e^{(n+g)t}$.

This implies that the transversality condition, equation (6), is equivalent to

(19) $\lim\limits_{t\to\infty} e^{-R(t)} e^{(n+g)t} k(t) = 0$.

From equation (2.15) in the text, the household's budget constraint, expressed in terms of limiting behavior, is given by

(20) $\lim\limits_{t\to\infty} e^{-R(t)} e^{(n+g)t} k(t) \geq 0$.

Comparing equations (19) and (20), we can see that the transversality condition will hold if and only if the budget constraint is met with equality. Thus we have shown that the solution to the social planner's problem in the Ramsey model is the same as the decentralized equilibrium. Hence that decentralized equilibrium must be Pareto efficient.

Problem 9.5

The planner's problem is to choose the path of $L_A(t)$ to maximize the lifetime utility of the representative individual, given by

(1) $U = \displaystyle\int_{t=0}^{\infty} e^{-\rho t} \ln C(t)dt, \quad \rho > 0$.

Output in the Romer model is given by

(2) $Y(t) = A(t)^{(1-\phi)/\phi} L_Y(t)$.

The production function for new ideas is given by

(3) $\dot{A}(t) = BL_A(t)A(t)$.

Equilibrium in the labor market requires the sum of the number of workers engaged in R&D and the number of workers producing inputs to equal the fixed population, or

(4) $L_A(t) + L_Y(t) = \overline{L}.$

We must now set-up the current-value Hamiltonian. The control variable is the variable that can be controlled freely by the planner, which is the path of $L_A(t)$. The state variable is the variable with the property that its value at any time is determined by past decisions of the planner. Here, that is the existing stock of knowledge, A(t). Finally, the shadow value of the state variable is the costate variable, denoted $\lambda(t)$.

The current-value Hamiltonian is therefore given by

(5) $H(A(t), L_A(t)) = \ln C(t) + \lambda(t)BL_A(t)A(t)$,

Because all output is consumed in this model, the representative individual's consumption is given by $1/\overline{L}$ times output. Thus we can write the Hamiltonian as

(6) $H(A(t), L_A(t)) = \ln(Y(t)/\overline{L}) + \lambda(t)BL_A(t)A(t)$.

Substituting equations (2) and (4) into equation (6) gives us

(7) $H(A(t), L_A(t)) = \dfrac{1-\phi}{\phi}\ln A(t) + \ln\left(\dfrac{\overline{L} - L_A(t)}{\overline{L}}\right) + \lambda(t)BL_A(t)A(t)$.

To find the optimum we must use the three Hamiltonian conditions. The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point in time is zero, or $\partial H/\partial L_A = 0$. Taking the partial derivative using equation (7) gives us

(8) $\lambda(t)BA(t) = \dfrac{1}{\overline{L} - L_A(t)}$.

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time, or $\partial H/\partial A = \rho\lambda(t) - \dot{\lambda}(t)$. Taking the partial derivative using (7), we get

(9) $\dfrac{1-\phi}{\phi}\dfrac{1}{A(t)} + \lambda(t)BL_A(t) = \rho\lambda(t) - \dot{\lambda}(t)$.

The final condition is the transversality condition. The limit as t goes to infinity of the discounted value of the costate variable times the state variable equals zero, or

(10) $\lim\limits_{t\to\infty} e^{-\rho t}\lambda(t)A(t) = 0$.

If we restrict ourselves to a solution in which L_A is constant this condition will be satisfied, since $\lambda(t)A(t)$ will be constant.

Dividing both sides of equation (9) by $\lambda(t)$ and rearranging gives us

(11) $\dfrac{\dot{\lambda}(t)}{\lambda(t)} = \rho - \dfrac{1-\phi}{\phi}\dfrac{1}{\lambda(t)A(t)} - BL_A(t)$

Rearranging equation (8) yields

(12) $\dfrac{1}{\lambda(t)A(t)} = B(\overline{L} - L_A(t))$.

Substituting equation (12) into equation (11) gives us

(13) $\dfrac{\dot{\lambda}(t)}{\lambda(t)} = \rho - \dfrac{1-\phi}{\phi}B(\overline{L} - L_A(t)) - BL_A(t)$.

If $L_A(t)$ is constant, then from equation (8), so is $\lambda(t)A(t)$ and thus $\dot{\lambda}(t)/\lambda(t) = -\dot{A}(t)/A(t)$. Therefore, equation (13) becomes

(14) $\dfrac{\dot{A}(t)}{A(t)} = -\rho + \dfrac{1-\phi}{\phi} B(\bar{L} - L_A(t)) + BL_A(t)$.

From equation (3), the production function for new ideas, we can write $\dot{A}(t)/A(t) = BL_A(t)$ so equation (14) becomes

(15) $BL_A(t) = -\rho + \dfrac{1-\phi}{\phi} B(\bar{L} - L_A(t)) + BL_A(t)$.

Simplifying gives us

(16) $\dfrac{1-\phi}{\phi} BL_A(t) = \dfrac{1-\phi}{\phi} B\bar{L} - \rho$.

Solving for the socially optimal constant value of L_A then gives us

(17) $L_A{}^{OPT} = \bar{L} - \dfrac{\phi}{1-\phi}\dfrac{\rho}{B}$.

This is essentially equivalent to the socially optimal level of L_A we solved for in chapter 3 as given by equation (3.49).

In chapter 3 we showed that the equilibrium value of L_A (ignoring the possibility of $L_A < 0$) was given by equation 3.43, or

(3.43) $L_A{}^{EQ} = (1-\phi)\bar{L} - \dfrac{\phi\rho}{B}$.

Thus, the equilibrium and optimal levels of L_A are related in the following way:

(18) $L_A{}^{EQ} = (1-\phi)L_A{}^{OPT}$.

Problem 9.6

The equation of motion for the market value of capital, q, is

(1) $\dot{q}(t) = rq(t) - \pi(K(t))$,

where $\pi'(\bullet) < 0$. The condition required for $\dot{q} = 0$ is given by

(2) $q = \pi(K)/r$.

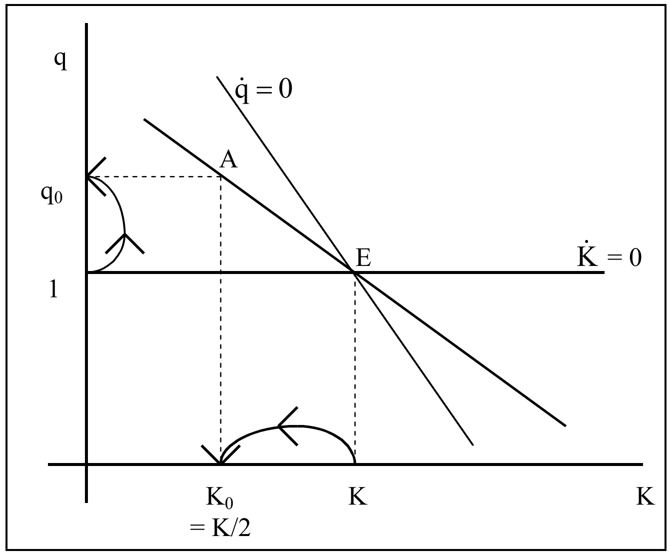
The equation of motion for capital, K, is

(3) $\dot{K}(t) = f(q(t))$,

where $f(q) = NC'^{-1}(q - 1)$ with $f(1) = 0$ and $f'(\bullet) > 0$. The condition required for $\dot{K} = 0$ is given by

(4) $q = 1$.

(a) The destruction of half of the capital stock does not cause either the $\dot{K} = 0$ or the $\dot{q} = 0$ loci to shift. Both of these are already drawn allowing for K to vary. At the time of the destruction, K falls to $K_0 = K/2$.



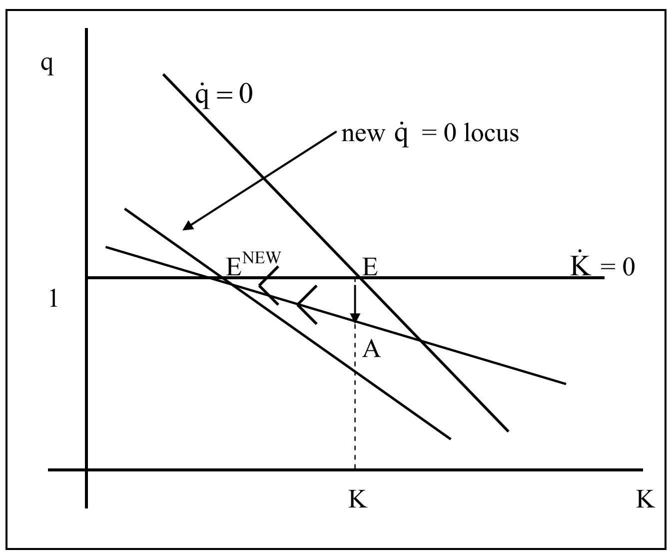
For the economy to return to a stable equilibrium, q must adjust so that the economy is on the saddle path. Thus q must jump up to q_0 , putting the economy at point A in the figure at right. Intuitively, since profits are higher at the lower K, the capital that is left is more valuable and so the market value of capital is now higher.

The economy then moves down the saddle path with q falling and K rising. Intuitively, the higher market value of capital attracts investment and so the capital stock begins to build back up. As it does so, profits begin to fall and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is back at its original level. Hence the economy eventually returns to point E.

(b) Profits at a given K are now $(1 - \tau)\pi(K)$ rather than $\pi(K)$. The condition required for $\dot{q} = 0$ is now given by

(5) $q = (1 - \tau)\pi(K)/r$.

At a given K, the value of q that makes $\dot{q} = 0$ is now lower so the new $\dot{q} = 0$ locus lies below the old one. In addition, the slope of the $\dot{q} = 0$ locus is $\partial q/\partial K = (1 - \tau)\pi'(K)/r$ rather than $\pi'(K)/r$. With $(1 - \tau) < 1$, this new slope is less negative and so the $\dot{q} = 0$ locus becomes flatter. The $\dot{K} = 0$ locus is unaffected. See the figure at right.



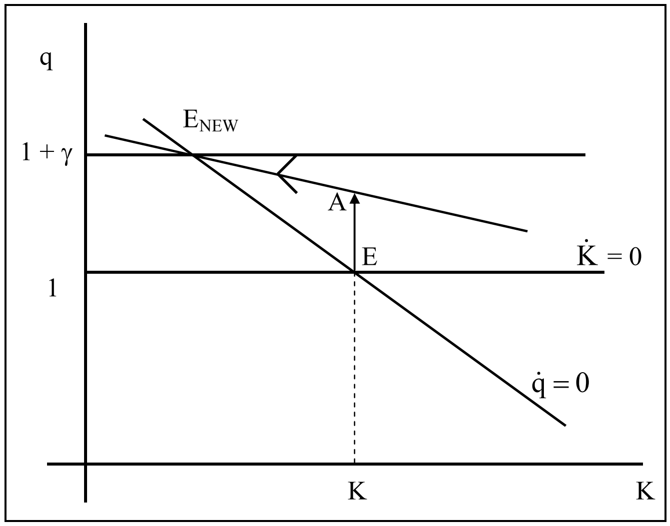
K, the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump down so that the economy is on the new saddle path at point A. Intuitively, since the government is now taking a fraction of profits, existing capital is less valuable and so the market value of capital falls. The economy then moves up the new saddle path with K falling and q rising. Intuitively, the lower market value of capital discourages investment and so the capital stock begins falling. As it does so, profits begin to rise and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is at a permanently lower level. The economy winds up at point E_{NEW} in the diagram. The lower capital and thus higher pretax profits offset the fact that the government takes a fraction of those profits.

(c) One of the conditions required for optimization is that the firm invests to the point at which the cost of acquiring capital equals the value of that capital, q. With this tax on investment, the cost of acquiring a unit of capital is the purchase price (which is fixed at one) plus the tax, γ, plus the marginal adjustment cost, C '(I). Thus analogous to equation (8.21) in the text, we now have

(6) $1 + \gamma + C'(I(t)) = q(t)$.

Since C '(0) is zero, equation (6) implies that I(t) is zero (and thus $\dot{K} = 0$) when $q(t) = 1 + \gamma$. So the equation of the $\dot{K} = 0$ locus is now

(7) $q = 1 + \gamma$.



Thus an investment tax of γ shifts the $\dot{K} = 0$ locus up by γ. The $\dot{q} = 0$ locus is unaffected. See the figure.

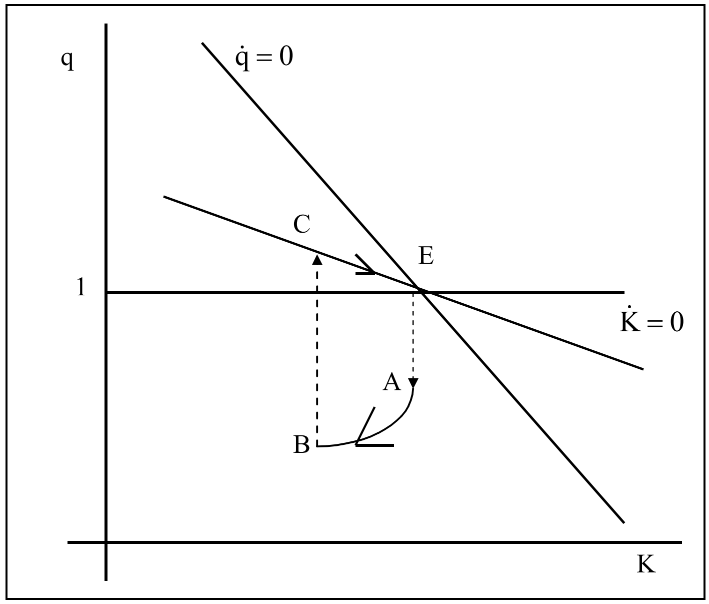
K, the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump up so that the economy is on the new saddle path at point A. Intuitively, because the tax will reduce investment, it means that the industry's profits (neglecting the tax) will eventually be higher, and thus that existing capital is more valuable. The economy then moves up the new saddle path until it reaches point E_{NEW}. The capital stock is permanently lower and the pretax market value of capital is equal to $1 + \gamma$; the after-tax market value is again equal to one.

Problem 9.7

The important point is that q is anticipated to jump up discontinuously at the time of the capital levy, time T. Consider what is required, if there is a market for shares in firms, for individuals to be willing to hold those shares through the interval where the one-time tax on capital holdings is imposed. Consider the market value of capital an instant, ε, before the levy and an instant after the levy and then look at what happens as ε goes to zero. The key point is that the market value of capital an instant before the levy, $q(T - \varepsilon)$, must equal (1 - f) times the market value of capital an instant after the levy. If it did not – in light of the levy – holders of shares in firms would be expecting capital losses that they could avoid. Therefore, $q(T - \varepsilon)$ must equal $(1 - f)q(T + \varepsilon)$ or

(1) $q(T - \varepsilon)/q(T + \varepsilon) = (1 - f)$.

For example, if f = 0.10 or ten percent, then the value of q an instant before the levy must equal 90 percent of its value an instant after the levy. Thus at time T, q jumps up to close that 10 percent gap. In addition, that jump must put the economy somewhere on the saddle path in order for the economy to return to a stable equilibrium.



Thus at the time of the news, q must jump down, putting the economy at a point such as A in the figure at right. The economy is then in a region where both q and K are falling. Thus between the time of the news and the time the levy is imposed, the market value of capital and the capital stock are falling. Intuitively, firms begin decumulating capital in anticipation of the one-time levy.

Point A must be chosen so that at the time of the levy, q can jump up by the required amount discussed above and that required jump must put the economy right on the saddle path. The stock of capital does not jump at the time of the levy. Thus at time T, the economy jumps from a point such as B to a point such as C where $q_B / q_C = (1 - f)$.

After the time of the levy, the economy moves down the saddle path, eventually returning to the original equilibrium at point E. Intuitively, once the one-time tax is over with, since K is lower, profits are higher and so investment is attractive once again. Thus the capital stock begins rising back to its initial level.

Problem 9.8

(a) The evolution of the stock of housing is given by

(1) $\dot{H} = I(p_H) - \delta H$.

Thus the condition required for $\dot{H} = 0$ is given by $I(p_H) = \delta H$. That is, in order for the stock of housing to remain constant, new investment in housing – which is an increasing function of the real price of housing – must exactly offset depreciation of the existing housing stock. Differentiating both sides of this expression with respect to H gives us the following slope of the $\dot{H} = 0$ locus:

(2) $I'(p_H) dp_H / dH = \delta$,

or

(3) $dp_H / dH = \delta / I'(p_H) > 0$.

Since $I'(p_H) > 0$, the $\dot{H} = 0$ locus is upward-sloping in (H, p_H) space.

Rental income plus capital gains must equal the exogenous rate of return, r, or

(4) $\dfrac{R(H) + \dot{p}_H}{p_H} = r$.

Solving equation (4) for $\dot{p}_H$ yields

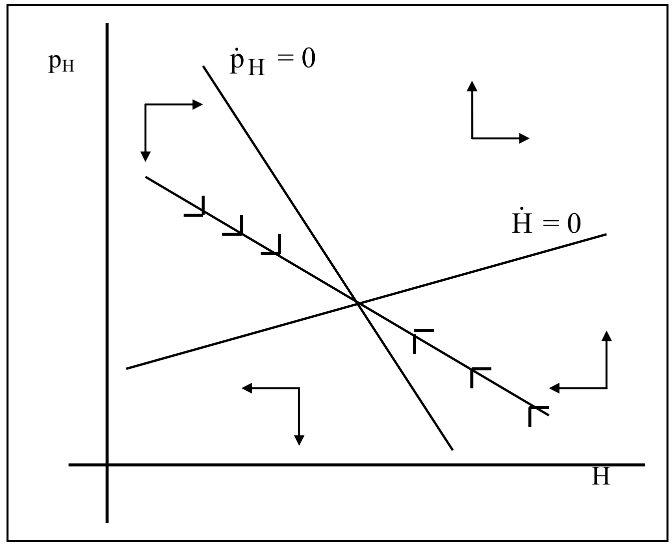
(5) $\dot{p}_H = rp_H - R(H)$.

Therefore the condition required for $\dot{p}_H = 0$ is $rp_H - R(H) = 0$ or $p_H = R(H)/r$. Differentiating both sides of this expression with respect to H gives us the following slope of the $\dot{p}_H = 0$ locus:

(6) $dp_H / dH = R'(H)/r$.

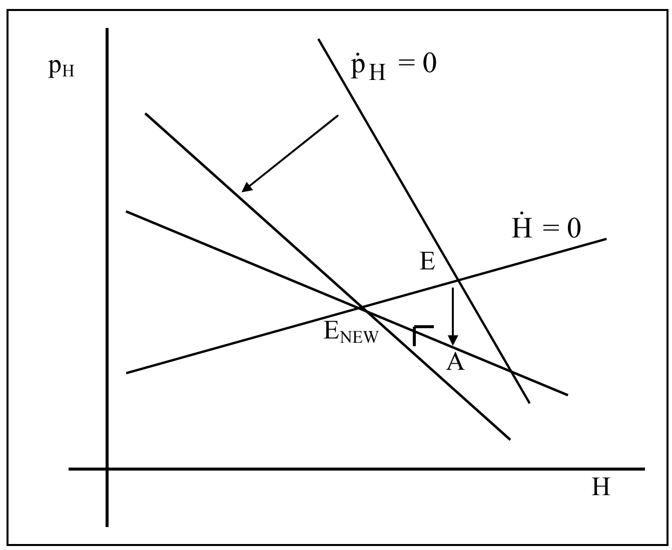
Since $R'(H) < 0$ – rent is a decreasing function of the stock of housing – the $\dot{p}_H = 0$ locus is downward-sloping in (H, p_H) space.

(b) Since $I'(p_H) > 0$, then from equation (1), $\dot{H}$ is increasing in p_H. This means that above the $\dot{H} = 0$ locus, $\dot{H} > 0$ and so H is rising. Intuitively, at a given H, if p_H is higher than the price necessary to keep the stock of housing constant, investment – which is an increasing function of p_H – is higher than necessary to offset depreciation. Thus the stock of housing is rising above the $\dot{H} = 0$ locus. Similarly, below the $\dot{H} = 0$ locus, $\dot{H} < 0$ and so H is falling. Intuitively, p_H and thus investment are too low to offset depreciation and keep the stock of housing constant at a given H. Thus the stock of housing is falling below the $\dot{H} = 0$ locus.



Since $R'(H) < 0$, then from equation (5), $\dot{p}_H$ is increasing in H. This means that to the right of the $\dot{p}_H = 0$ locus, $\dot{p}_H > 0$ and so p_H is rising. Intuitively, at a given p_H, if H is higher – and thus rent lower – than the level necessary to keep the price of housing constant, this lower rent must be offset by capital gains – a rising p_H – if investors are to earn the required exogenous return of r. Similarly, to the left of the $\dot{p}_H = 0$ locus, $\dot{p}_H < 0$ and so p_H is falling. If H is lower – and thus rent higher – than the level necessary to keep the price of housing constant, this higher rent must be offset by capital losses in order for investors to earn the rate of return r.

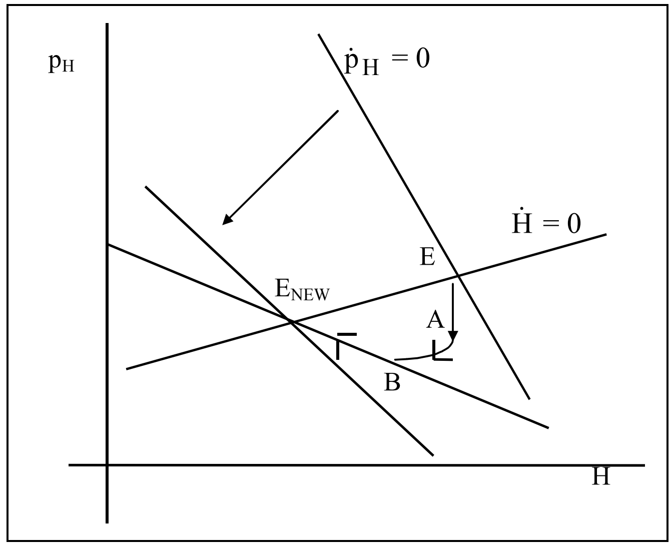
(c) The $\dot{p}_H = 0$ locus is defined by $p_H = R(H)/r$. A rise in r means that the p_H that makes $\dot{p}_H = 0$ is now lower at a given H. Thus the new $\dot{p}_H$ locus lies below the old one. In addition, the slope of the $\dot{p}_H = 0$ locus is $R'(H)/r$ and so the rise in r makes the slope less negative. Thus the new $\dot{p}_H = 0$ locus is flatter than the old one. The $\dot{H} = 0$ locus is defined by $I(p_H) = \delta H$. Since r does not appear in this equation, the $\dot{H} = 0$ locus is unaffected.



At the time of the increase in r, H – the stock of existing housing – cannot jump discontinuously. The real price of housing, p_H, must jump down to put the economy on the new saddle path. In the figure, at the time of the rise in r, the economy jumps from point E to point A.

The discontinuous downward jump in p_H causes the amount of investment to jump down. Thus investment is no longer enough to offset depreciation at the initial value of H – we are below the $\dot{H} = 0$ locus – and so the stock of housing begins to fall. As H begins to fall, rent begins to rise since $R'(H) < 0$. As the economy moves up the new saddle path, the real price of housing is rising. This means that investment is rising back up since $I'(p_H) > 0$. The economy eventually reaches point E_{NEW} where p_H is constant at a new lower level and thus investment is constant at a new lower level. In addition, the stock of housing is constant at a new lower level. Finally, rent is higher.

(d) An important point is that the dynamics of the system are still governed by the original $\dot{p}_H = 0$ and $\dot{H} = 0$ loci until the actual time of the increase in r. At the time of the change, H cannot jump and, importantly, neither can p_H. If p_H did jump, an instant before the rise in r, people would be expecting capital gains or losses that could be arbitraged away. Thus right at the time of the increase in r, the economy must be somewhere on the new saddle path.



At the time of the news, since the stock of housing, H, cannot jump, p_H must jump down so that the economy is at a point such as A in the figure at right. The economy is then below the $\dot{H} = 0$ locus and so H is falling. Intuitively, the price of housing is now lower and thus so is investment. It is no longer high enough to offset depreciation at the initial level of H and so H begins to fall. The economy is also to the left of the $\dot{p}_H = 0$ locus – this locus does not shift until the rise in r actually occurs – and so p_H is falling. Thus between the time of the news and the time of the increase in r, H is falling and so rent, R, is increasing. In addition, p_H is falling and thus investment, which jumped down initially, continues to fall.

Right at the time of the increase in r, the $\dot{p}_H = 0$ locus shifts to the left and becomes flatter. The economy is at a point such as B on the new saddle path. The economy is still below the $\dot{H} = 0$ locus and thus the stock of housing continues to fall while rent, R, continues to rise. The economy is now to the right of the relevant $\dot{p}_H = 0$ locus, however, and so the real price of housing begins to rise and thus so does investment. The economy moves up the new saddle path until it reaches a new long-run equilibrium at point E_{NEW}. At this new long-run equilibrium, H is lower, R is higher, p_H is lower and I is lower.

(e) Adjustment costs are not internal in this model. There are no direct costs of building housing in this setup. Internal costs are, as the text explains, the actual cost of building the new capital (here, housing) which can include the cost of training workers to do it and so on. This model exhibits external adjustment costs. As firms undertake more investment in housing, the real price of housing adjusts so that individuals do not wish to invest or disinvest at infinite rates.

(f) The $\dot{H} = 0$ locus is not horizontal because investment depends on the real price of housing, p_H. Depreciation is proportional to the stock of housing and thus is higher at higher levels of H. Therefore, to keep H constant at higher levels of H requires more investment. But in order to have more investment, the price of housing must be higher since $I'(p_H) > 0$. This means that the $\dot{H} = 0$ locus is upward-sloping.

Problem 9.9

(a) The costs of adjustment are now given by $C(\dot{\kappa}/\kappa)\kappa$. Since the capital-accumulation equation is given by $\dot{\kappa} = I - \delta\kappa$, this can be written as $C((I/\kappa) - \delta)\kappa$. The firm's profits at time t are $\pi(K(t))\kappa(t) - I(t) - C([I(t)/\kappa(t)] - \delta)\kappa(t)$. Thus the present-value Hamiltonian is

$$(1)\quad H(\kappa(t), I(t)) = \pi(K(t))\kappa(t) - I(t) - C\left(\frac{I(t)}{\kappa(t)} - \delta\right)\kappa(t) + q(t)[I(t) - \delta\kappa(t)].$$

(b) The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point is zero. Here, the control variable is investment and thus

$$(2)\quad \frac{\partial H(\kappa(t), I(t))}{\partial I(t)} = -1 - C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\frac{1}{\kappa(t)}\kappa(t) + q(t) = 0.$$

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time. Since the state variable is the capital stock, we have

$$(3)\quad \frac{\partial H(\kappa(t), I(t))}{\partial \kappa(t)} = \pi(K(t)) - C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\left(\frac{-I(t)}{\kappa(t)^2}\right)\kappa(t) - C\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right) - \delta q(t) = rq(t) - \dot{q}(t).$$

The final condition is the transversality condition. The limit as t goes to infinity of the present value of the costate variable times the state variable must be zero. Thus, we have

$$(4)\quad \lim_{t\to\infty} e^{-rt} q(t)\kappa(t) = 0.$$

(c) Equation (2) states that each firm invests to the point at which the purchase price of capital plus the marginal adjustment cost equals the value of capital: $1 + C'(\dot{\kappa}/\kappa) = q$. Since $C'(\dot{\kappa}/\kappa)$ is increasing in $\dot{\kappa}/\kappa$, this condition implies that $\dot{\kappa}/\kappa$ is increasing in q. And since $C'(0)$ is zero, it also implies that $\dot{\kappa}/\kappa$ is zero when q equals one. Finally, note that since q is the same for all firms, all firms choose the same value of $\dot{\kappa}/\kappa$. Thus the growth rate of the aggregate capital stock, $\dot{K}/K$, is given by the value of $\dot{\kappa}/\kappa$ that satisfies (2). Putting this information together, we can write

(5) $\dot{K}(t)/K(t) = f(q(t)) \qquad f(1) = 0,\ f'(\bullet) > 0,$

where f(q) is the value of $\dot{K}/K$ that satisfies $C'(\dot{K}/K) = q - 1$: $f(q) = C'^{-1}(q - 1)$. Equation (5) implies that K is increasing when q > 1, decreasing when q < 1 and constant when q = 1. Thus the $\dot{K} = 0$ locus is a horizontal line at q = 1 when drawn in (K, q) space.

(d) Rearranging equation (3) to solve for $\dot{q}(t)$ yields

$$(6)\quad \dot{q}(t) = (r+\delta)q(t) - \left[\pi(K(t)) + C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\left(\frac{I(t)}{\kappa(t)}\right) - C\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\right].$$

To simplify this expression, note first that I/κ equals $(\dot{\kappa} + \delta\kappa)/\kappa$ or $(\dot{\kappa}/\kappa) + \delta$. In addition, as we just showed, the growth rate of the representative firm's capital stock, $\dot{\kappa}/\kappa$, is the same as the growth rate of the industry-wide capital stock, $\dot{K}/K$. Thus we can rewrite equation (6) as

$$(7)\quad \dot{q}(t) = (r+\delta)q(t) - \left[\pi(K(t)) + C'\left(\frac{\dot{K}(t)}{K(t)}\right)\left(\frac{\dot{K}(t)}{K(t)} + \delta\right) - C\left(\frac{\dot{K}(t)}{K(t)}\right)\right].$$

We can now use equation (5), $\dot{K}/K = f(q)$, to substitute for $\dot{K}/K$, and then use the fact that the definition of $f(\bullet)$ implies $C'(f(q)) = q - 1$. This yields

$$(8)\quad \dot{q}(t) = (r+\delta)q(t) - \left[\pi(K(t)) + [q(t) - 1][f(q(t)) + \delta] - C\big(f(q(t))\big)\right] \equiv G\big(K(t), q(t)\big).$$

(e) The condition G(K, q) = 0 implicitly defines the locus of points in (K, q) space for which $\dot{q}$ is zero. To see how q varies with K along this locus, we therefore implicitly differentiate this condition with respect to K. This yields

$$(9)\quad G_K(K,q) + G_q(K,q)\left.\frac{dq}{dK}\right|_{\dot{q}=0} = 0,$$

or

(10) $$\left.\frac{dq}{dK}\right|_{\dot{q}=0} = \frac{-G_K(K,q)}{G_q(K,q)},$$

where subscripts denote partial derivatives and $\left.\frac{dq}{dK}\right|_{\dot{q}=0}$ denotes the derivative of q with respect to K along the $\dot{q} = 0$ locus.

Using equation (8) to compute the derivatives in (10) yields

(11) $G_K(K, q) = -\pi'(K)$,

and

(12) $G_q(K, q) = (r + \delta) - [(q - 1)f'(q) + (f(q) + \delta) - C'(f(q))f'(q)] = r - f(q)$,

where we again use the fact that $C'(f(q)) = q - 1$.

Substituting equations (11) and (12) into equation (10) gives us

(13) $$\left.\frac{dq}{dK}\right|_{\dot{q}=0} = \frac{\pi'(K)}{r - f(q)}.$$

Note that f(q) is zero when q equals one. Thus the slope of the $\dot{q} = 0$ locus at the point where q = 1 is simply $\pi'(K)/r$. Note that at this particular point, this slope is exactly the same as the slope of the $\dot{q} = 0$ locus in the version of the model in the text where adjustment costs took the form $C(\dot{\kappa})$.

Problem 9.10

(a) One of the conditions for optimization is that the marginal revenue product of capital, $\pi(K(t))$, equals its user cost, $rq(t) - \dot{q}(t)$. Rearranging this condition gives us the following equation of motion for q:

(1) $\dot{q}(t) = rq(t) - \pi(K(t))$.

Substituting the profit function, $\pi(K) = a - bK$, into equation (1) gives us

(2) $\dot{q}(t) = rq(t) - a + bK(t)$.

The $\dot{q} = 0$ locus is therefore given by

(3) $rq - a + bK = 0$,

or solving for q as a function of K, we have

(4) $q = (a - bK)/r$.

So the $\dot{q} = 0$ locus has a constant slope of $-b/r$.

To find the long-run-equilibrium value of K, we need to find the intersection of the $\dot{q} = 0$ locus – as given by equation (4) – and the $\dot{K} = 0$ locus. The $\dot{K} = 0$ locus is given by q = 1, which means that we already know that the long-run-equilibrium value of q, q*, is one. Substituting q = 1 into equation (4) and solving for K* gives us

(5) $K^* = (a - r)/b$.

We can now use the method of Section 2.6 to find the slope of the saddle path. We first need to solve for the equation of motion of K(t). One of the conditions for optimization is that each firm invests to the point at which the purchase price of capital (which is fixed at one), plus the marginal adjustment cost, equals the value of capital, q. We are assuming quadratic costs of adjustment, $C(\dot{\kappa}) = \alpha\dot{\kappa}^2/2$, and thus the marginal adjustment cost is

(6) $\partial C(\dot{\kappa})/\partial\dot{\kappa} = \alpha\dot{\kappa}$.

Thus we have $1 + \alpha\dot{\kappa} = q$, which implies

(7) $\dot{\kappa} = (q - 1)/\alpha$.

Since q is the same for all firms, all firms choose the same value of investment, $\dot{\kappa}$. Thus the rate of change of the aggregate capital stock, $\dot{K}$, is given by

(8) $\dot{K} = N(q - 1)/\alpha$,

where N is the number of firms.

Define $\tilde{q} \equiv q - q^*$ and $\tilde{K} \equiv K - K^*$. Since q* and K* are constants, $\dot{q}$ and $\dot{K}$ are equivalent to $\dot{\tilde{q}}$ and $\dot{\tilde{K}}$ respectively. Thus we can rewrite the equations of motion, equations (2) and (8), as

(9) $\dot{\tilde{q}} = rq - a + bK$, and (10) $\dot{\tilde{K}} = N(q - 1)/\alpha$.

Dividing both sides of equation (9) by $\tilde{q}$ gives us

(11) $\dfrac{\dot{\tilde{q}}}{\tilde{q}} = \dfrac{rq - a + bK}{\tilde{q}}$.

From equation (5), we can write

(12) $\tilde{K} = (bK - a + r)/b$,

or rearranging to solve for bK:

(13) $bK = b\tilde{K} + a - r$.

Substituting equation (13) into equation (10) gives us

(14) $\dfrac{\dot{\tilde{q}}}{\tilde{q}} = \dfrac{rq - a + b\tilde{K} + a - r}{\tilde{q}} = \dfrac{r(q-1)}{\tilde{q}} + \dfrac{b\tilde{K}}{\tilde{q}} = r + b\dfrac{\tilde{K}}{\tilde{q}}$,

where we have used the fact that q* = 1 so that $\tilde{q} \equiv q - q^* = q - 1$.

Dividing both sides of equation (10) by $\tilde{K}$ and noting that q* = 1 we have

(15) $\dfrac{\dot{\tilde{K}}}{\tilde{K}} = \dfrac{N}{\alpha}\dfrac{\tilde{q}}{\tilde{K}}$.

Equations (14) and (15) imply that the growth rates of $\tilde{q}$ and $\tilde{K}$ depend only on the ratio of $\tilde{q}$ to $\tilde{K}$. Given this, consider what happens if the values of q and K are such that $\tilde{q}$ and $\tilde{K}$ are falling at the same rate. This implies that the ratio of $\tilde{q}$ to $\tilde{K}$ is not changing, and thus that their growth rates are not changing. Thus $\tilde{q}$ and $\tilde{K}$ continue to fall at equal rates. In terms of a phase diagram, from a point at which $\tilde{q}$ and $\tilde{K}$ are falling at equal rates, the economy simply moves along a straight-line saddle path to (K*, q*) with the distance from (K*, q*) falling at a constant rate.

Let μ denote $\dot{\tilde{K}}/\tilde{K}$. Then equation (15) implies

(16) $\mu = \dfrac{N}{\alpha}\dfrac{\tilde{q}}{\tilde{K}}$,

or solving for the ratio of $\tilde{q}$ to $\tilde{K}$:

(17) $\dfrac{\tilde{q}}{\tilde{K}} = \dfrac{\alpha\mu}{N}$.

From equation (14), the condition that $\dot{\tilde{q}}/\tilde{q}$ always equals $\dot{\tilde{K}}/\tilde{K}$ is thus

(18) $\mu = r + (bN/\alpha\mu)$,

or

(19) $\alpha\mu^2 - \alpha r\mu - bN = 0$.

Using the quadratic formula to solve for μ yields

(20) $\mu = \dfrac{\alpha r \pm \sqrt{\alpha^2 r^2 + 4\alpha bN}}{2\alpha} = \dfrac{r \pm \sqrt{r^2 + (4bN/\alpha)}}{2}$.

If μ is positive, then $\tilde{q}(t) \equiv q(t) - q^*$ and $\tilde{K}(t) \equiv K(t) - K^*$ are growing. That is, instead of moving along a straight line toward (K^*, q^*), the economy is moving on a straight line away from (K^*, q^*). Thus μ must be negative and hence

$$(21)\quad \mu_1 = \frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2}.$$

Thus equation (17) with $\mu = \mu_1$ tells us how q and K must be related on the saddle path. Substituting equation (21) into equation (17) gives us

$$(22)\quad \frac{\tilde{q}}{\tilde{K}} \equiv \frac{q - q^*}{K - K^*} = \frac{\alpha\left[r - \sqrt{r^2 + (4bN/\alpha)}\right]}{2N},$$

or solving for q as a function of K:

$$(23)\quad q = q^* + \alpha\left[\frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2N}\right](K - K^*).$$

Thus, the slope of the saddle path is

$$(24)\quad \left.\frac{\partial q}{\partial K}\right|_{sp} = \alpha\left[\frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2N}\right] < 0.$$

Problem 9.11

(a) Consider the situation in which $a(t + \tau)$ is certain to equal $E_t[a(t + \tau)]$ for all $\tau \geq 0$ so that there is no uncertainty. The value of q at some date $t + \tau$ can then be written as the value of q in period t plus the "sum" of all the changes in q from time t to time $t + \tau$. More formally:

$$(1)\quad \hat{q}(t+\tau, t) = q(t) + \int_{s=t}^{\tau} \dot{\hat{q}}(t+s, t)ds,$$

where $\hat{q}(t+\tau, t)$ denotes the path of q when a is certain to equal its expected value. Since $\dot{q} = rq - \pi(K)$, then with this particular profit function we have $\dot{q} = rq - a + bK$. Thus equation (1) can be written as

$$(2)\quad \hat{q}(t+\tau, t) = q(t) + \int_{s=t}^{\tau} \left[r\hat{q}(t+s, t) - E_t[a(t+s)] + b\hat{K}(t+s, t)\right]ds,$$

where we have substituted in for $a(t + s) = E_t[a(t + s)]$ and where $\hat{K}(\bullet)$ denotes the path of K given that a is certain to equal its expected value.

Now consider the situation in which $a(t + \tau)$ is uncertain. Then the expected value, as of time t, of q at some future date $t + \tau$ can be written as the value of q in period t plus the "sum" of all the expected changes in q from time t to time $t + \tau$. More formally:

$$(3)\quad E_t[q(t+\tau)] = q(t) + \int_{s=t}^{\tau} E_t[\dot{q}(t+s)]ds.$$

Since equation (9.34) in the text, $E_t[\dot{q}(t)] = rq(t) - \pi(K(t))$, holds in all periods and since $\pi(K(t)) = a - bK(t)$, we can write

(4) $E_{t+s}[\dot{q}(t + s)] = rq(t + s) - a(t + s) + bK(t + s)$.

Taking the expected value, as of information available at time t, of both sides of equation (4) yields

(5) $E_t[\dot{q}(t + s)] = rE_t[q(t + s)] - E_t[a(t + s)] + bE_t[K(t + s)]$,

where we have used the law of iterated projections so that $E_t E_{t+s} [\dot{q}(t+s)] = E_t [\dot{q}(t+s)]$. Substituting equation (5) into equation (3) gives us

(6) $E_t[q(t+\tau)] = q(t) + \int_{s=t}^{\tau} \left[rE_t[q(t+s)] - E_t[a(t+s)] + bE_t[K(t+s)] \right] ds.$

If $E_t [q(t+\tau)] = \hat{q}(t+\tau, t)$ for all $\tau \geq 0$, then the right-hand sides of equations (2) and (6) must be equal for all $\tau \geq 0$. Thus

(7) $q(t) + \int_{s=t}^{\tau} \left[r\hat{q}(t+s,t) - E_t[a(t+s)] + b\hat{K}(t+s,t) \right] ds =$

$$q(t) + \int_{s=t}^{\tau} \left[rE_t[q(t+s)] - E_t[a(t+s)] + bE_t[K(t+s)] \right] ds.$$

Again, using $E_t [q(t+\tau)] = \hat{q}(t+\tau, t)$ for all $\tau \geq 0$, this simplifies to

(8) $b \int_{s=t}^{\tau} \hat{K}(t+s,t) ds = b \int_{s=t}^{\tau} E_t[K(t+s)] ds.$

Canceling the b's and using Leibniz's rule to take the derivative of both sides of equation (8) with respect to τ gives us

(9) $\hat{K}(t+\tau,t) = E_t [K(t+\tau)]$.

Equation (9) holds for all $\tau \geq 0$.

(b) Consider the situation in which $a(t+\tau)$ is certain to equal $E_t [a(t+\tau)]$ for all $\tau \geq 0$ so that there is no uncertainty. Then from equation (9.24) in the text, we can write the market value of capital at time t as the present value of its future marginal revenue products and so

(10) $\hat{q}(t,t) = \int_{\tau=0}^{\infty} e^{-r\tau} \left[E_t[a(t+\tau)] - b\hat{K}(t+\tau,t) \right] d\tau,$

where we have used the facts that $\pi(K) = a - bK$ and $a(t+\tau) = E_t [a(t+\tau)]$. Now $\hat{q}(t,t)$ denotes the value of q given that a always equals its expected value and $\hat{K}(t+\tau,t)$ has the same meaning as in part (a). It is the path of K given that a is always certain to equal its expected value.

Now consider the situation where $a(t+\tau)$ is uncertain. Then, using $\pi(K) = a - bK$, equation (9.32) in the text becomes

(11) $q(t) = \int_{\tau=0}^{\infty} e^{-r\tau} \left[E_t[a(t+\tau)] - bE_t[K(t+\tau)] \right] d\tau.$

As shown in part (a), if $E_t [q(t+\tau)] = \hat{q}(t+\tau,t)$ for all $\tau \geq 0$, then $E_t [K(t+\tau)] = \hat{K}(t+\tau,t)$ for all $\tau \geq 0$. This means that the right-hand sides of equations (10) and (11) are equal and so $q(t) = \hat{q}(t,t)$. That is, for the case in which π is linear and the uncertainty concerns the intercept of the π function, the market value of capital is the same with the uncertainty as it is if the future values of the π function are certain to equal their expected values.

Even with uncertainty, each firm invests to the point at which the cost of acquiring a unit of new capital equals the market value of capital. That is, investment satisfies

(12) $1 + C'(I(t)) = q(t)$.

Since $C = \alpha I^2/2$, $C'(I) = \alpha I$. In addition, as we have just shown, $q(t) = \hat{q}(t,t)$. Thus equation (12) can be rewritten as

(13) $1 + \alpha I(t) = \hat{q}(t,t)$.

By definition, the change in each firm's capital stock is equal to I(t). Since each firm faces the same $\hat{q}$ (t,t), they choose the same level of investment. Thus the change in the aggregate capital stock is given by $\dot{K}(t)$ = NI(t), where N is the number of firms. Substituting this expression into equation (13) yields

(14) $1 + \alpha\dot{K}(t)/N = \hat{q}(t,t)$.

Solving (14) for $\dot{K}(t)$ gives us

(15) $\dot{K}(t) = N[\hat{q}(t,t) - 1]/\alpha$

Under these special circumstances – when π is linear, the uncertainty concerns the intercept of the π function, and adjustment costs are quadratic – investment is the same with the uncertainty as it is if the future values of the π function are equal to their expected values with certainty.

Problem 9.12

(a) The equation of motion for the market value of capital, q, is given by

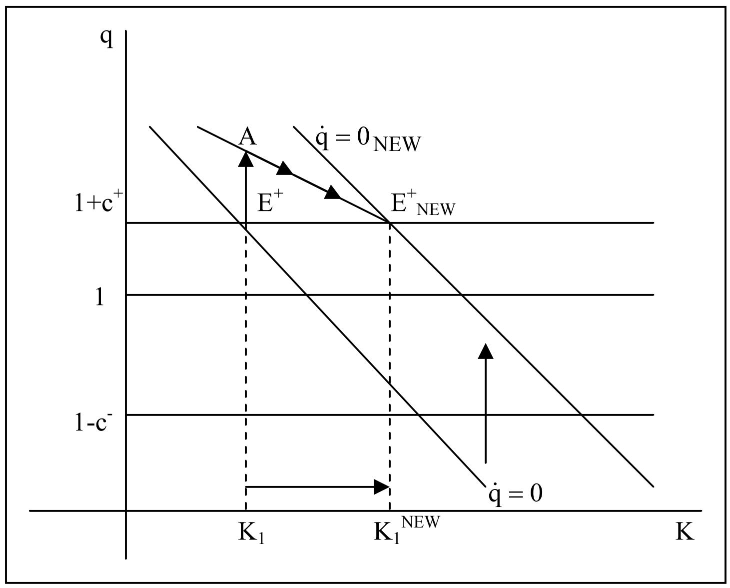
(1) $\dot{q}(t) = rq(t) - \pi(K(t))$,

where $\pi'(\bullet) < 0$. The condition required for $\dot{q} = 0$ is given by

(2) $q = \pi(K)/r$.

A permanent upward shift of the $\pi(\bullet)$ function means that at any given K, the value of q that makes $\dot{q} = 0$ is higher. Assuming $\pi'(K)$ is still the same for any given K, the $\dot{q} = 0$ locus shifts up without a change in slope. Since neither c^+ nor c^- changes, the range of q for which $\dot{K} = 0$ is unchanged.

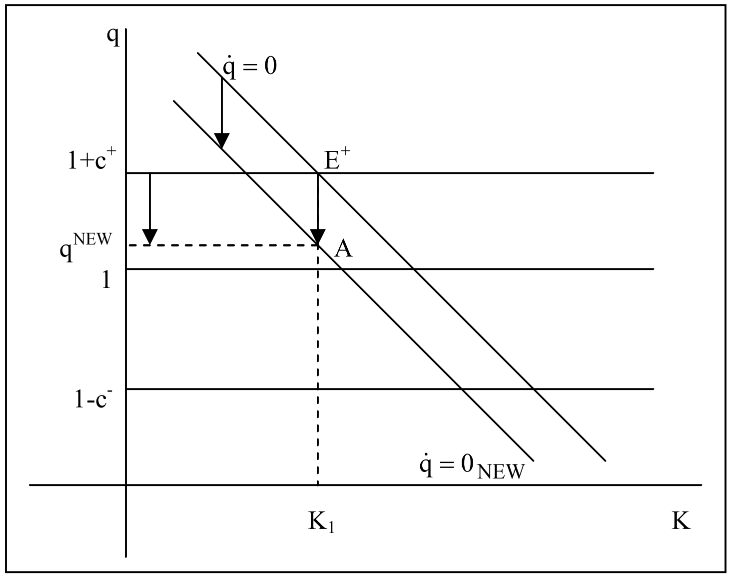
The economy starts at point E^+ in the figure at right, with a capital stock equal to K_1 and $q = 1 + c^+$. The $\dot{q} = 0$ locus then shifts up. The stock of capital, K, cannot jump discontinuously at the time of the upward shift of the $\pi(\bullet)$ function.



Thus q must jump to point A in the figure so that the economy is on the new saddle path. Intuitively, since the profit function has shifted up, existing capital is more valuable and so the market value of capital rises.

The economy then moves down the new saddle path with K rising and q falling. Intuitively, the higher market value of capital encourages investment and so the capital stock begins rising. As it does so, profits fall and thus so does the market value of capital. This process continues until we reach E^+_{NEW}, at which point q has returned to $1 + c^+$ and the capital stock is permanently higher at K_1^{NEW}.

(b) A permanent rise in the interest rate means that at any given K, the value of q that makes $\dot{q} = 0$ is lower; see equation (2). Thus the $\dot{q} = 0$ locus shifts down. Since neither c^+ nor c^- changes, the range of q for which $\dot{K} = 0$ is unchanged.

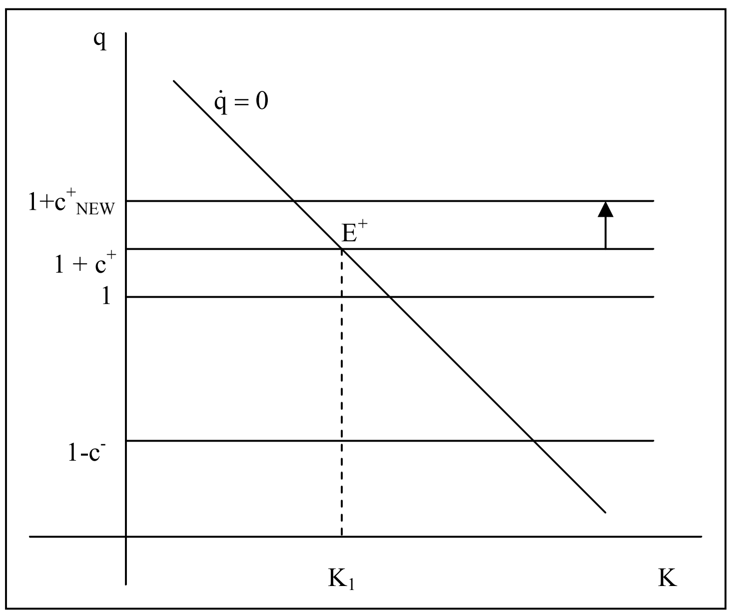


The economy starts at point E^+ in the figure at right, with $q = 1 + c^+$ capital equal to K_1.

At the time of the rise in r, K cannot jump discontinuously. Thus q must fall to the point on the $\dot{q} = 0$ locus at the initial capital stock K_1. The economy moves to point A in the figure.

We will assume the fact that the rise in r is "small" to imply that the new $\dot{q} = 0$ locus does not shift down so much that the new value of q at K_1 is less than $1 - c^-$. Thus, as long as the shift of the $\dot{q} = 0$ locus is small enough, q is still in the range $[1 - c^-, 1 + c^+]$ and so investment does not change. The capital stock remains at K_1. There are no further dynamics.

(c) A rise in the cost of the first unit of positive investment, c^+, shifts the $1+c^+$ line up to $1+c^+_{NEW}$. The range of values of q for which $\dot{K} = 0$ is now larger. Since c^+ does not appear in equation (2), the $\dot{q} = 0$ locus is unaffected.

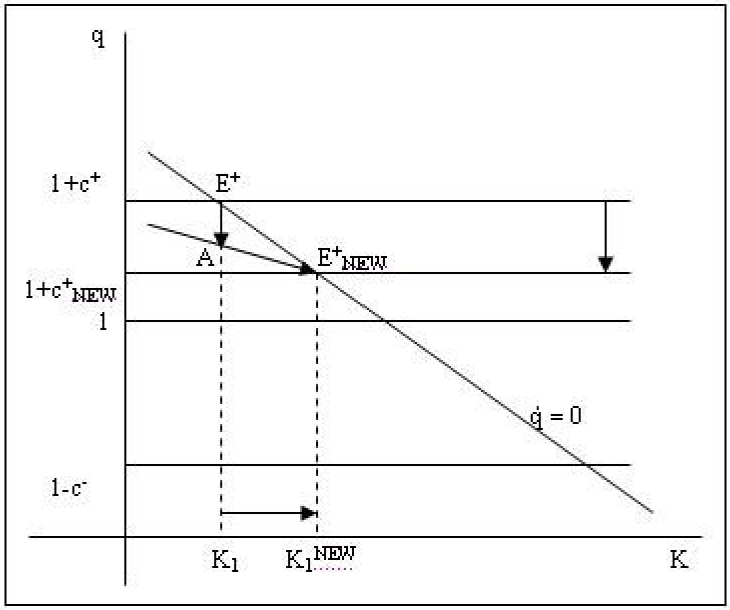


The economy starts at point E^+ in the figure at right with $q = 1+c^+$ and the capital stock equal to K_1.

Since K does not jump discontinuously at the time of the rise in c^+ and since the $\dot{q} = 0$ locus is unaffected, q is in the range $[1 - c^-, 1 + c^+_{NEW}]$ and so investment does not change. The capital stock remains at K_1, the market value of capital remains at $1 + c^+$ and there are no dynamics.

(d) A fall in the cost of the first unit of positive investment, c^+, shifts the $1+c^+$ line down to $1+c^+_{NEW.}$ The range of values of q for which $\dot{K} = 0$ is now smaller. Since c^+ does not appear in equation (2), the $\dot{q} = 0$ locus is unaffected.
The economy starts at point E^+ in the figure at right with $K = K_1$ and the market value of capital equal to $1+c^+$. After the fall in c^+ , the initial value of q is outside the range $[1 - c^-, 1 + c^+_{NEW}]$. Since K_1 does not

jump discontinuously, q must fall so that the economy is on the new saddle path at point A in the figure. After q jumps down, it still exceeds $1 + c^{+}_{NEW}$ and so investment rises; additional investment is now profitable at the initial level of the capital stock because of the drop in c^{+}. The economy moves down the new saddle path with K rising and q falling until we reach point E^{+}_{NEW}. At that point, the capital stock is permanently higher at K_1^{NEW} and the market value of capital is permanently lower at $q = 1 + c^{+}_{NEW}$.



Problem 9.13

(a) If the firm does not undertake the investment, its expected profits are zero. Thus we have

(1) $E[\pi^{NO}] = 0$.

If the firm does undertake the investment, its expected profits are the certain payoff in period 1 plus the expected payoff in period 2 less the cost of undertaking the investment. Thus we have

(2) $E[\pi^{YES}] = \pi_1 + E[\pi_2] - I$.

The firm will undertake the investment if its expected profits from doing so are greater than its expected profits from not investing, or when

(3) $E[\pi^{YES}] > E[\pi^{NO}]$,

or simply when

(4) $\pi_1 + E[\pi_2] - I > 0$.

(b) Suppose the firm does not invest in period 1. Then in period 2, if $\pi_2 > I$, it will invest and earn $\pi_2 - I$. If $\pi_2 < I$, it will not invest in period 2 and will earn zero. Thus the expected profits from not investing in period 1 are

(5) $E[\pi^{NO\ IN\ 1}] = \text{Prob}(\pi_2 > I)E[\pi_2 - I \mid \pi_2 > I]$.

From equation (2), the expected profits from investing in period 1 are

(6) $E[\pi^{YES\ IN\ 1}] = \pi_1 + E[\pi_2] - I$.

Thus the difference in the firm's expected profits between not investing in period 1 and investing in period 1 are

(7) $E[\pi^{NO\ IN\ 1}] - E[\pi^{YES\ IN\ 1}] = \text{Prob}(\pi_2 > I)E[\pi_2 - I \mid \pi_2 > I] - (\pi_1 + E[\pi_2] - I)$.

Even if $\pi_1 + E[\pi_2] - I > 0$, as long as $\text{Prob}(\pi_2 > I)E[\pi_2 - I \mid \pi_2 > I]$ is greater than $\pi_1 + E[\pi_2] - I$, the firm's expected profits are higher if it does not invest in period 1 than if it does.

(c) The cost of waiting is that the firm foregoes any payoff in period 1. That is, it foregoes π_1 and hence

(8) cost of waiting $= \pi_1$.

The benefit of waiting is that the firm can observe π_2, see if it is less than I, and decide not to invest and avoid a loss if this is the case. The expected loss that the firm avoids by waiting is equal to the probability that π_2 is less than I, multiplied by the expected loss given that $\pi_2 < I$, which is $E[I - \pi_2 \mid \pi_2 < I]$. Hence

(9) benefit of waiting $= \text{Prob}(\pi_2 < I)\ E[I - \pi_2 \mid \pi_2 < I]$.

Note that by the definition of conditional expected values, we can write

(10) $E[\pi_2 - I] = \text{Prob}(\pi_2 > I)\, E[\pi_2 - I \mid \pi_2 > I] + \text{Prob}(\pi_2 < I)\, E[\pi_2 - I \mid \pi_2 < I]$.

Substituting this into equation (7) yields

(11) $E[\pi^{\text{NO IN 1}}] - E[\pi^{\text{YES IN 1}}] = \text{Prob}(\pi_2 > I)\, E[\pi_2 - I \mid \pi_2 > I] - \pi_1 - \text{Prob}(\pi_2 > I)\, E[\pi_2 - I \mid \pi_2 > I] - \text{Prob}(\pi_2 < I)\, E[\pi_2 - I \mid \pi_2 < I]$.

Note that we can write $\text{Prob}(\pi_2 < I)E[\pi_2 - I \mid \pi_2 < I] = -\text{Prob}(\pi_2 < I)\, E[I - \pi_2 \mid \pi_2 < I]$. Using this fact, equation (11) becomes

(12) $E[\pi^{\text{NO IN 1}}] - E[\pi^{\text{YES IN 1}}] = -\pi_1 + \text{Prob}(\pi_2 < I)\, E[I - \pi_2 \mid \pi_2 < I]$.

Since π_1 is the cost of waiting and $\text{Prob}(\pi_2 < I)\, E[I - \pi_2 \mid \pi_2 < I]$ is the benefit of waiting, we do have

(13) $E[\pi^{\text{NO IN 1}}] - E[\pi^{\text{YES IN 1}}]$ = benefit of waiting - cost of waiting.

Problem 9.14

(a) Consider the value of a unit of debt. It pays off one unit of output at time $t + \tau$, for all $\tau \geq 0$. The consumer values this payoff according to the marginal utility of consumption at each time $t + \tau$. Thus the value of having one unit of output at time $t + \tau$ rather than at t is equal to the discounted marginal utility of consumption at time $t + \tau$ relative to the marginal utility of consumption at time t, which is given by $e^{-\rho t} u'(C(t + \tau))/u'(C(t))$. Thus the value of a unit of debt at time t is simply the appropriately discounted "sum" of all the future payoffs, or

$$(1)\quad P(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\right] d\tau.$$

Equity holders are the residual claimant and thus at time $t + \tau$, $\tau \geq 0$, they receive the additional profit generated by the marginal unit of capital, $\pi(K(t + \tau))$, minus the total amount paid to bond holders, which is b (the total number of outstanding bonds). Again, individuals value this payoff at time $t + \tau$ according to the discounted marginal utility of consumption at time $t + \tau$ relative to the marginal utility of consumption at time t. Thus the value of the equity in the marginal unit of capital is

$$(2)\quad V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\big(\pi(K(t+\tau)) - b\big)\right] d\tau.$$

(b) Adding equation (2) to b times equation (1) gives us the following market value of the claim on the marginal unit of capital:

$$(3)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} bE_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\big(\pi(K(t+\tau)) - b\big)\right] d\tau.$$

Combining the integrals yields

$$(4)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\big(b + \pi(K(t+\tau)) - b\big)\right] d\tau,$$

and thus

$$(5)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'(C(t+\tau))}{u'(C(t))}\pi(K(t+\tau))\right] d\tau.$$

The division of financing between bonds and equity as captured by b, the number of outstanding bonds, does not affect the market value of the claims on the marginal unit of capital. The present discounted value of that unit of capital is determined by its expected effect on the path of profits. Since the division of $\pi(K(t + \tau))$ between bonds and equity does not affect the size of $\pi(K(t + \tau))$, it does not affect the market value of the claim on the unit of capital.

(c) The market value of each of the n assets is given by

$$(6)\quad V_i(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)} d_i(t+\tau)\right] d\tau.$$

There will be n equations of the form of (6). Adding these n equations together gives us the following total value of the n financial instruments:

$$(7)\quad V_1(t)+\ldots+V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)} \big(d_1(t+\tau)+\ldots+d_n(t+\tau)\big)\right] d\tau.$$

Since $d_1(t+\tau) + \ldots + d_n(t+\tau) = \pi(K(t+\tau))$, we can rewrite equation (7) as

$$(8)\quad V_1(t)+\ldots+V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)} \pi\big(K(t+\tau)\big)\right] d\tau.$$

The total market value of the n financial instruments is determined by the expected effect on the path of profits of the marginal unit of capital. It does not depend upon the individual payoffs to the assets.

(d) The value of a unit of debt continues to be given by equation (1). The value of a unit of equity is now

$$(9)\quad V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)} \big\{(1-\theta)\big[\pi\big(K(t+\tau)\big)-b\big]\big\}\right] d\tau.$$

Adding equation (9) to b times equation (1) gives the following market value of the claims on the marginal unit of capital:

$$(10)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} bE_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)}\right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)} \big\{(1-\theta)\big[\pi\big(K(t+\tau)\big)-b\big]\big\}\right] d\tau.$$

Combining the integrals yields

$$(11)\quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t\left[\frac{u'\big(C(t+\tau)\big)}{u'\big(C(t)\big)} \big[(1-\theta)\pi\big(K(t+\tau)\big)+\theta b\big]\right] d\tau.$$

Now the division of the financing between bonds and equity does matter. The number of bonds issued, b, does affect the market value of the claim on the marginal unit of capital. The division of the additional profits between bonds and capital does affect the size of those profits. Specifically, a switch toward debt financing increases profits since interest payments are tax deductible.

SOLUTIONS TO CHAPTER 10

Problem 10.1

(a) Substituting the expression for the average wage, $w_a = fw_u + (1 - f)w_n$, into the expression for the nonunion wage, $w_n = (1 - bu)w_a/(1 - \beta)$, yields

(1) $w_n = \dfrac{(1-bu)}{(1-\beta)}\left[fw_u + (1-f)w_n\right]$.

Substituting the union wage, $w_u = (1 + \mu)w_n$, into equation (1) yields

(2) $w_n = \dfrac{(1-bu)}{(1-\beta)}\left[f(1+\mu)w_n + (1-f)w_n\right] = \dfrac{(1-bu)}{(1-\beta)}\left[(1+\mu f)w_n\right]$.

Simplifying gives us

(3) $(1 - bu)(1 + \mu f) = (1 - \beta)$.

Since $(1 - bu)(1 + \mu f) = 1 + \mu f - bu - b\mu fu$, equation (3) can be rewritten as

(4) $-u(b + b\mu f) = -\beta - \mu f$,

and thus the equilibrium unemployment rate is

(5) $u = \dfrac{\beta + \mu f}{b(1+\mu f)}$.

(b) (i) Substituting $\mu = f = 0.15$, $\beta = 0.06$ and $b = 1$ into equation (5) gives us

(6) $u = \dfrac{(0.06) + (0.15)(0.15)}{1 + (0.15)(0.15)} = \dfrac{0.0825}{1.0225} = 0.081$.

Equilibrium unemployment is approximately 8.1 percent, which is higher than the 6 percent obtained with $\beta = 0.06$ and $b = 1$ in the standard version of this model without a union sector.

In order to determine the proportion by which the cost of effective labor in the union sector exceeds that in the nonunion sector, we need to calculate the equilibrium effort level in each sector. The union wage as a function of the average wage is

(7) $w_u = (1 + \mu)w_n = (1 + \mu)(1 - bu)w_a/(1 - \beta)$.

Substituting equation (7) and the definition of the index of labor-market conditions, $x = (1 - bu)w_a$, into the expression for effort, $e = [(w - x)/x]^\beta$, gives us

(8) $e_u = \left[\dfrac{\left[(1+\mu)(1-bu)w_a/(1-\beta)\right] - (1-bu)w_a}{(1-bu)w_a}\right]^\beta = \left[\dfrac{(1+\mu)(1-bu) - (1-\beta)(1-bu)}{(1-\beta)(1-bu)}\right]^\beta$,

or simply

(9) $e_u = \left[\left(\dfrac{1+\mu}{1-\beta}\right) - 1\right]^\beta = \left(\dfrac{\mu+\beta}{1-\beta}\right)^\beta$.

Substituting $w_n = (1 - bu)w_a/(1 - \beta)$ into the expression for effort yields

(10) $e_n = \left[\dfrac{\left[(1-bu)w_a/(1-\beta)\right] - (1-bu)w_a}{(1-bu)w_a}\right]^\beta = \left[\dfrac{(1-bu) - (1-\beta)(1-bu)}{(1-\beta)(1-bu)}\right]^\beta$,

or simply

(11) $e_n = \left[\dfrac{1}{1-\beta} - 1\right]^\beta = \left(\dfrac{\beta}{1-\beta}\right)^\beta$.

In the union sector, it costs a firm w_u to buy one unit of labor that provides e_u units of effective labor. Thus it costs a firm w_u/e_u to buy one unit of effective labor. Using the fact that $w_u = (1 + \mu)w_n$ and equation (9), we can write

$$(12)\quad \frac{w_u}{e_u} = \frac{(1+\mu)w_n}{\left[(\mu+\beta)/(1-\beta)\right]^{\beta}}.$$

Similarly, the cost to a nonunion firm of obtaining one unit of effective labor is w_n/e_n. Using equation (11), we can write

$$(13)\quad \frac{w_n}{e_n} = \frac{w_n}{\left[\beta/(1-\beta)\right]^{\beta}}.$$

Dividing equation (12) by equation (13) gives us the following ratio of the cost of effective labor in the union to the nonunion sector:

$$(14)\quad \frac{w_u/e_u}{w_n/e_n} = \frac{(1+\mu)w_n}{\left[(\mu+\beta)/(1-\beta)\right]^{\beta}}\frac{\left[\beta/(1-\beta)\right]^{\beta}}{w_n} = (1+\mu)\left(\frac{\beta}{\mu+\beta}\right)^{\beta}.$$

Substituting $\mu = 0.15$ and $\beta = 0.06$ into equation (14) gives us

$$(15)\quad \frac{w_u/e_u}{w_n/e_n} = (1.15)\left(\frac{0.06}{0.21}\right)^{0.06} = 1.0667.$$

Note that although the cost of labor in the union sector exceeds the cost of labor in the nonunion sector by a factor of $(1 + \mu) = 1.15$, the cost of effective labor is only higher by a factor of about 1.07. This is because union workers exert more effort since they are paid a higher wage.

(b) (ii) Substituting $\mu = f = 0.15$, $\beta = 0.03$ and $b = 0.5$ into equation (5) yields

$$(16)\quad u = \frac{(0.03)+(0.15)(0.15)}{0.5[1+(0.15)(0.15)]} = \frac{0.0525}{0.51125} = 0.103.$$

Equilibrium unemployment is now higher at about 10.3%. Substituting $\mu = 0.15$ and $\beta = 0.03$ into equation (14) gives us

$$(17)\quad \frac{w_u/e_u}{w_n/e_n} = (1.15)\left(\frac{0.03}{0.18}\right)^{0.03} = 1.0898.$$

With the elasticity of effort with respect to the wage lower at $\beta = 0.03$ and less weight on unemployment in the index of labor-market conditions, the ratio of the cost of effective labor in the union sector to that in the nonunion sector is now higher.

Problem 10.2

(a) (i) With e fixed at 1 and taking w as given, the firm's problem is to choose L in order to maximize profits as given by

(1) $\pi = L^{\alpha}/\alpha - wL$.

The first-order condition is

(2) $\partial\pi/\partial L = L^{\alpha-1} - w = 0$,

and thus the firm's choice of employment is

(3) $L = w^{-1/(1-\alpha)}$.

Substituting equation (3) into the expression for profits yields

(4) $\pi = w^{-1/(1-\alpha)}/\alpha - w^{[(1-\alpha)-1]/(1-\alpha)} = w^{-\alpha/(1-\alpha)}\,[(1/\alpha) - 1]$,

and thus the level of profits is

(5) $\pi = [(1 - \alpha)/\alpha]w^{-\alpha/(1-\alpha)}$.

(a) (ii) Substituting equation (3) for L into the union's objective function, U = (w - x)L, gives us
(6) $U = (w - x)x^{-1/(1-\alpha)}$.
Using equation (6) and equation (5) for profits gives us the following bargaining problem:

$$(7)\quad \max_{w} \quad (w-x)^{\gamma} w^{-\gamma/(1-\alpha)} \left[\left(\frac{1-\alpha}{\alpha} \right) w^{-\alpha/(1-\alpha)} \right]^{1-\gamma}.$$

It simplifies the algebra to maximize the log of $U^{\gamma} \pi^{1-\gamma}$; doing so transforms the problem into

$$(8)\quad \max_{w} \quad \gamma \ln(w-x) - \frac{\gamma}{1-\alpha} \ln w + (1-\gamma) \ln\left(\frac{1-\alpha}{\alpha} \right) - \frac{\alpha(1-\gamma)}{1-\alpha} \ln w.$$

The first-order condition is

$$(9)\quad \frac{\partial\left[\ln\left(U^{\gamma}\pi^{1-\gamma}\right)\right]}{\partial w} = \gamma \frac{1}{w-x} - \frac{\gamma}{1-\alpha}\frac{1}{w} - \frac{\alpha(1-\gamma)}{(1-\alpha)}\frac{1}{w} = 0.$$

Equation (9) can be rewritten as

$$(10)\quad \gamma \frac{1}{w-x} = \frac{\gamma+\alpha-\alpha\gamma}{1-\alpha}\frac{1}{w}.$$

Cross-multiplying yields
(11) $(1 - \alpha)\gamma w = [\alpha + \gamma(1 - \alpha)](w - x)$.
Subtracting $(1 - \alpha)\gamma w$ from both sides of equation (11) and rearranging yields
(12) $\alpha w = [\alpha + (1 - \alpha)\gamma]x$,
and thus finally, the wage chosen in the bargaining process is

$$(13)\quad w = \frac{\alpha + (1-\alpha)\gamma}{\alpha} x.$$

(b) (i) Substituting $e = [(w - x)/x]^{\beta}$ into the expression for profits allows us to write the firm's problem as

$$(14)\quad \max_{L} \quad \pi = \frac{1}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta} L^{\alpha} - wL.$$

The first-order condition is

$$(15)\quad \frac{\partial \pi}{\partial L} = \left(\frac{w-x}{x}\right)^{\alpha\beta} L^{\alpha-1} - w = 0,$$

and thus the firm's choice of employment is

$$(16)\quad L = \left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-1/(1-\alpha)}.$$

Substituting equation (16) into the expression for profits yields

$$(17)\quad \pi = \frac{1}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta}\left(\frac{w-x}{x}\right)^{\alpha^{2}\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)} - w^{1-[1/(1-\alpha)]}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)}.$$

Since $\alpha\beta + [\alpha^2 \beta/(1 - \alpha)] = [\alpha\beta - \alpha^2 \beta + \alpha^2 \beta]/(1 - \alpha) = \alpha\beta/(1 - \alpha)$ and $1 - [1/(1 - \alpha)] = [(1 - \alpha - 1)/(1 - \alpha)] = -\alpha/(1 - \alpha)$, equation (17) can be written as

$$(18)\quad \pi = \frac{1}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)} - w^{-\alpha/(1-\alpha)}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)}.$$

Collecting terms and simplifying yields

$$(19)\quad \pi = \frac{1-\alpha}{\alpha}\left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)}.$$

(b) (ii) Substituting equation (16) for L into the union's objective function, U = (w - x)L, gives us

(20) $U = (w - x)(w - x)^{\frac{\alpha\beta}{1-\alpha}}(1/x)^{\frac{\alpha\beta}{1-\alpha}}w^{\frac{-1}{1-\alpha}}$,

which simplifies to

(21) $U = (w - x)^{\frac{1-\alpha(1-\beta)}{1-\alpha}}(1/x)^{\frac{\alpha\beta}{1-\alpha}}w^{\frac{-1}{1-\alpha}}$.

Using equation (21) and equation (19) for profits gives us the following bargaining problem:

(22) $\max\limits_{w}\ (w - x)^{\frac{[1-\alpha(1-\beta)]\gamma}{1-\alpha}}(1/x)^{\frac{\alpha\beta\gamma}{1-\alpha}}w^{\frac{-\gamma}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}(w - x)^{\frac{\alpha\beta(1-\gamma)}{1-\alpha}}(1/x)^{\frac{\alpha\beta(1-\gamma)}{1-\alpha}}w^{\frac{-\alpha(1-\gamma)}{1-\alpha}}$.

We will again maximize the log of $U^{\gamma}\pi^{1-\gamma}$. Ignoring the terms not involving w, we have the following bargaining problem:

(23) $\max\limits_{w}\ \frac{[1-\alpha(1-\beta)]\gamma}{1-\alpha}\ln(w - x) - \frac{\gamma}{1-\alpha}\ln w + \frac{\alpha\beta(1-\gamma)}{1-\alpha}\ln(w - x) - \frac{\alpha(1-\gamma)}{1-\alpha}\ln w$.

The first-order condition is

(24) $\frac{\partial\left[\ln\left(U^{\gamma}\pi^{1-\gamma}\right)\right]}{\partial w} = \frac{[1-\alpha(1-\beta)]\gamma}{1-\alpha}\frac{1}{w - x} - \frac{\gamma}{1-\alpha}\frac{1}{w} + \frac{\alpha\beta(1-\gamma)}{1-\alpha}\frac{1}{w - x} - \frac{\alpha(1-\gamma)}{1-\alpha}\frac{1}{w} = 0$,

which can be rewritten as

(25) $\frac{1}{1-\alpha}[\gamma - \alpha\gamma(1-\beta) + \alpha\beta - \alpha\beta\gamma]\frac{1}{w - x} = \frac{1}{1-\alpha}[\gamma + \alpha - \alpha\gamma]\frac{1}{w}$.

Multiplying both sides of (25) by (1 - α) and simplifying yields

(26) $[\gamma - \alpha(\gamma - \beta)]\frac{1}{w - x} = [\alpha + (1-\alpha)\gamma]\frac{1}{w}$.

Cross-multiplying gives us

(27) $[\gamma - \alpha(\gamma - \beta)]w = [\alpha + (1 - \alpha)\gamma](w - x)$.

Subtracting γw from both sides of (27) and simplifying yields

(28) $-\alpha(\gamma - \beta)w = \alpha(1 - \gamma)w - [\alpha + (1 - \alpha)\gamma]x$.

Collecting the terms in w gives us

(29) $[-\alpha\gamma + \alpha\beta - \alpha + \alpha\gamma]w = - [\alpha + (1 - \alpha)\gamma]x$,

which simplifies to

(30) $- [\alpha(1 - \beta)]w = - [\alpha + (1 - \alpha)\gamma]x$.

Thus the wage chosen in the bargaining process is

(31) $w = \frac{\alpha + (1-\alpha)\gamma}{\alpha(1-\beta)}x$.

Note that in the case of $\beta = 0$, equation (31) does simplify to equation (13).

(b) (iii) The proportional impact of workers' bargaining power on wages can be measured by the elasticity $\partial[\ln w]/\partial\gamma$. In the absence of efficiency wages, the wage chosen in the bargaining process is given by equation (13). Comparing equation (13) to equation (31) we can see that efficiency-wage considerations simply raise the wage by a multiplicative factor of $1/(1 - \beta)$. Thus the presence of efficiency wages does not affect the elasticity given by $\partial[\ln w]/\partial\gamma$. Thus in this model, the proportional impact on wages of workers' bargaining power is not greater with efficiency wages than without and is not greater when efficiency-wage effects are greater.

Problem 10.3

The no-shirking condition (NSC) is given by

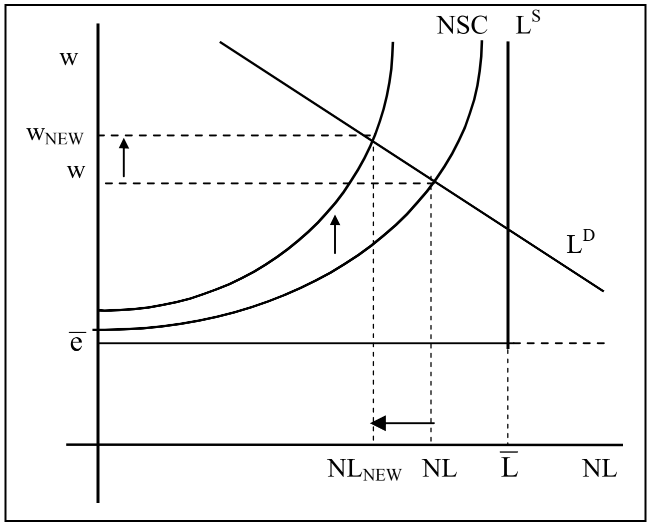
(1) $w = \bar{e} + \left(\rho + \dfrac{\bar{L}}{\bar{L} - NL} b\right)\dfrac{\bar{e}}{q}$,

and the labor demand curve is given by

(2) $F'(\bar{e}L) = w/\bar{e}$.

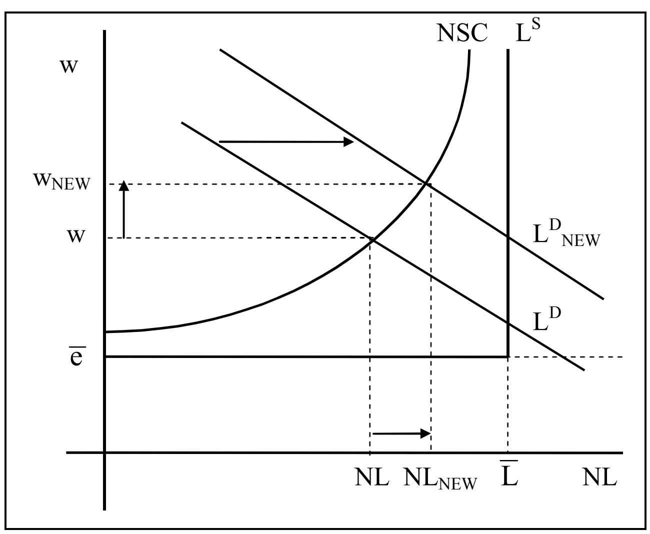
Equation (2) states that firms choose L so that the marginal product of effective labor equals the marginal cost of effective labor, where the wage, w, is set to satisfy (1).

(a) An increase in ρ shifts up the no-shirking locus. From equation (1), for a given NL, the wage needed to get workers to exert effort is now higher. Intuitively, since workers discount the future more, it matters less to them if they are caught shirking, are fired and have to go through a period of unemployment. Thus at a given level of employment, firms must pay a higher wage to deter shirking. The labor demand curve is unaffected. As shown in the figure at right, equilibrium employment falls and the equilibrium wage rises.

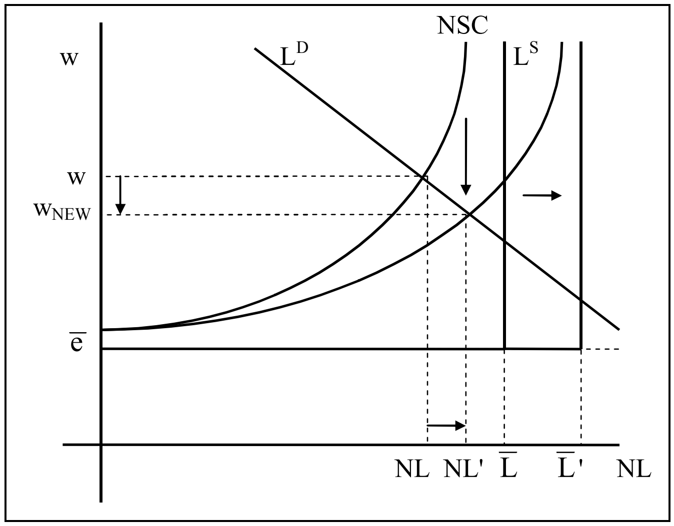


(b) An increase in the job breakup rate, b, shifts up the no-shirking locus. From equation (1), for a given NL, the wage required to get workers to exert effort is now higher. Intuitively, since workers are more likely to lose their job anyway, the value of being employed is lower. Thus workers are not as concerned about being caught shirking and fired. So at a given level of employment, firms must pay a higher wage to deter shirking. The labor demand curve is unaffected. Equilibrium employment falls and the wage rises.

(c) The rise in A shifts the labor demand curve to the right. The no-shirking locus is unaffected. As shown in the figure at right, the equilibrium wage rises as does the level of employment. Note that if efficiency wages were not present, inelastic labor supply would mean that increases in technology would lead only to increases in the wage, not to increases in employment.



(d) The vertical portion of the labor supply curve shifts to the right. The labor demand curve is unaffected. The no-shirking locus shifts down. Intuitively, at a given NL, $\overline{L}$ - NL is now higher. Thus at a given level of employment, if workers become unemployed, they are likely to stay unemployed longer. Thus at a given level of employment, the cost of shirking is greater for a worker and thus firms can get away with paying a lower wage to deter shirking. From the figure at right, the equilibrium wage falls and employment rises.



<u>Problem 10.4</u>

(a) The total number of unemployed workers is $\overline{L}$ - NL. If there is no shirking, the number of workers becoming unemployed per unit time is the number of firms, N, times the number of workers per firm, L, times the rate of job breakup, b. In a steady state, this is also the number of workers becoming employed per unit time. If people who have been unemployed the longest are hired first, the length of time it takes to get a job, which we can denote t*, is equal to the total number of unemployed workers divided by the number of people who get hired per unit time. For example, if there are 1000 unemployed workers and 100 workers become employed per unit time, then the number of units of time it takes to get a job is 1000/100 = 10. Thus in general

(1) $t^* = \dfrac{\overline{L} - NL}{NLb}.$

(b) There is no uncertainty involved when calculating the value of becoming newly unemployed as a function of the value of being employed. When a worker loses her job, she knows that she will be unemployed for $t^* = (\overline{L} - NL)/NLb$ units of time, at which point she will become employed again. Thus the value of becoming newly unemployed is that t* units of time into the future, the individual will have the value of being employed. The discounted value of being employed t* units of time into the future is given by $e^{-\rho t^*} V_E$. Thus

(2) $V_U = e^{-\rho(\overline{L} - NL)/NLb} V_E.$

(c) As in the usual version of the Shapiro-Stiglitz model, the firm chooses a wage so that the value of being employed, V_E, just equals the value of shirking, V_S. From equation (10.33) in the text, this implies

(3) $V_E - V_U = \dfrac{\overline{e}}{q}.$

Substituting equation (2) into equation (3) yields

(4) $V_E - e^{-\rho t^*} V_E = \dfrac{\overline{e}}{q}.$

Solving for V_E gives us

(5) $V_E = \dfrac{\bar{e}}{(1-e^{-\rho t^*})q}$.

The next step is to determine what the wage must be in order for the value of employment to be given by equation (5). Equation (10.28) in the text gives the return from being employed, $\rho V_E = (w - \bar{e}) - b(V_E - V_U)$. Solving for w gives us

(6) $w = \bar{e} + \rho V_E + b(V_E - V_U)$.

Substituting equations (3) and (5) into equation (6) yields

(7) $w = \bar{e} + \dfrac{\rho\bar{e}}{(1-e^{-\rho t^*})q} + \dfrac{b\bar{e}}{q}$.

Substituting $t^* = (\bar{L} - NL)/NLb$ into equation (7) gives us

(8) $w = \bar{e} + \left[\dfrac{\rho}{1-e^{-\rho(\bar{L}-NL)/NLb}} + b\right]\dfrac{\bar{e}}{q}$.

Equation (8) is the no-shirking condition. Note that as unemployment goes to zero (as $NL \to \bar{L}$), there is no wage that will deter shirking. As the number of unemployed workers goes to zero, there is no line to stand in and wait for a job. An individual who is caught shirking and is fired will be at the front of the line and will be rehired instantly. Note also that as $NL \to 0$, the wage needed to deter shirking goes to $\bar{e} + (\rho + b)\bar{e}/q$. This is exactly the same wage needed to deter shirking as $NL \to 0$ in the standard Shapiro-Stiglitz model.

(d) In order to compare the equilibrium unemployment rate in this model with the equilibrium unemployment rate in the Shapiro-Stiglitz model, we need to compare the two no-shirking loci. If the wage needed to deter shirking for a given level of employment is higher in one of the two models, equilibrium unemployment will be higher in that model.

Intuitively, in both models, the value of being newly unemployed comes from the possibility of becoming employed. For a given level of employment, the expected time to becoming employed is the same in the two models. Here it is certain; in the Shapiro-Stiglitz model it is uncertain. That is, in this model, the newly unemployed worker knows that she will be rehired in t* units of time; in the Shapiro-Stiglitz model, she has probability 1/t* per unit time of becoming employed again and thus on average will be employed again in t* units of time.

Now, since $e^{-\rho t}$ is convex in t, the uncertainty about the time it takes to get employed again in the Shapiro-Stiglitz model raises V_U for a given V_E relative to this model. This means that firms must pay a higher wage in the Shapiro-Stiglitz model, for a given level of employment, to deter shirking. Thus equilibrium unemployment is higher in the Shapiro-Stiglitz model.

More formally, our claim is that

(9) NSC wage for a given NL in Shapiro-Stiglitz > NSC wage for a given NL in this model.

From equation (10.37) in the text and equation (8) here, the claim is

(10) $\bar{e} + \left[\rho + \dfrac{\bar{L}}{\bar{L}-NL}b\right]\dfrac{\bar{e}}{q} > \bar{e} + \left[\dfrac{\rho}{1-e^{-\rho t^*}} + b\right]\dfrac{\bar{e}}{q}$.

Subtracting $\bar{e}$ from both sides and dividing both sides of the resulting expression by $\bar{e}/q$ leaves us with

(11) $\rho + \dfrac{\bar{L}}{\bar{L}-NL}b > \dfrac{\rho}{1-e^{-\rho t^*}} + b$.

Now, using the definition of $t^* = (\bar{L} - NL)/NLb$, we can write $NLbt^* = \bar{L} - NL$ or

(12) $NL = \overline{L}/(1 + bt^*)$.
Substituting equation (12) into $[\overline{L}/(\overline{L} - NL)]b$ gives us

(13) $$\frac{\overline{L}}{\overline{L} - NL}b = \frac{\overline{L}}{\overline{L} - [\overline{L}/(1 + bt^*)]}b = \frac{1 + bt^*}{(1 + bt^*) - 1}b = \frac{1 + bt^*}{t^*}.$$

Substituting equation (13) into our claim gives us

(14) $$\rho + \frac{1 + bt^*}{t^*} > \frac{\rho}{1 - e^{-\rho t^*}} + b.$$

Multiplying both sides of (14) by t* gives us the following equivalent expression:

(15) $$\rho t^* + 1 + bt^* > \frac{\rho t^*}{1 - e^{-\rho t^*}} + bt^*.$$

Subtracting bt* from both sides of (15) and then multiplying both sides of the resulting expression by $(1 - e^{-\rho t^*})$ yields

(16) $$\rho t^* - \rho t^* e^{-\rho t^*} + 1 - e^{-\rho t^*} > \rho t^*.$$

Thus, finally, our original claim is equivalent to

(17) $$1 - e^{-\rho t^*} - \rho t^* e^{-\rho t^*} > 0.$$

We need to show that (17) actually holds. Note that it takes the form $1 - e^{-x} - xe^{-x}$ where $x \equiv \rho t^*$. This expression is greater than zero for $x > 0$. To formally see this, let f(x) denote the left-hand side of equation (17). Then $f(0) = 0$ and

(18) $f'(x) = e^{-x} + xe^{-x} - e^{-x} = xe^{-x} > 0$ for $x > 0$.

Thus, since f is equal to zero at zero and is increasing for all x greater than zero, it must be positive for all $x > 0$. Therefore our claim holds and equilibrium unemployment is higher in the Shapiro-Stiglitz model than it is in this model.

Problem 10.5

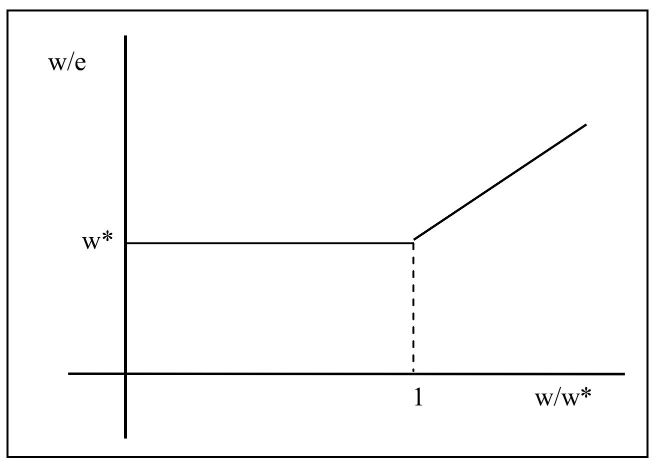
(a) The firm obtains e units of effective labor for a wage cost of w. Thus the cost to the firm of one unit of effective labor is w/e.

For $w/w^* < 1$, $e = w/w^*$ and so we have

(1) $$\frac{w}{e} = \frac{w}{w/w^*} = w^*.$$

For $w/w^* \geq 1$, $e = 1$ and so we have

(2) $$\frac{w}{e} = \frac{w}{1} = w.$$



See the figure at right, which plots the cost of a unit of effective labor, w/e, as a function of the firm's wage relative to the fair wage, w/w*. We can see that any wage such that $w/w^* \leq 1$ or $w \leq w^*$, minimizes the cost of effective labor.

(b) (i) Given the assumption that the firm pays the highest wage in the range described in part (a), it chooses $w = w^*$; that is, in order to minimize the cost per unit of effective labor, the firm chooses to pay the fair wage. Thus

(3) $w = \overline{w} + a - bu$.

(b) (ii) Assume that there is positive unemployment. If this is the case, the firm is unconstrained in its choice of the wage and therefore pays the fair wage, nothing more. Since this is true for all firms, the average wage, $\overline{w}$, must equal w. Thus we have
(4) $w = w + a - bu$.
Solving equation (4) for the unemployment rate yields
(5) $u = a/b$.
Note that the higher is a – the higher above the average wage is the perceived fair wage – the higher is the equilibrium unemployment rate. Also, the lower is b – the less responsive is the fair wage to unemployment – the higher is the equilibrium unemployment rate.

Since we have derived this under the assumption of positive unemployment, we need to verify that this is actually the case. If $u = 0$, then the fair wage is $w^* = w + a$, where we have used the fact that $\overline{w} = w$ (all firms pay the same wage). But if $a > 0$, this means that $w < w^*$ and so firms are paying less than the fair wage. But this violates our assumption that firms do not choose to pay below the fair wage. So as long as $a > 0$, there will be positive unemployment in equilibrium.

(b) (iii) From part (b), (ii), we can see that if $a = 0$, equilibrium unemployment will be zero. The fair wage will equal the actual wage. If $a < 0$, the perceived fair wage is always less than the average wage for any value of unemployment. Thus the representative firm, taking the average wage as given, wants to pay less than the average wage. Since workers are willing to work at any positive wage, firms need only pay ε above zero to get workers to be willing to work, even with zero unemployment.

(c) (i) Analysis such as that in part (a) continues to hold for each type of worker. The representative firm attempts to minimize the cost of effective labor for each type of worker. If the firm is unconstrained in its choice of w, this can be accomplished by paying any wage such that $w_1 \leq w_1^*$ for the high-productivity workers and $w_2 \leq w_2^*$ for the low-productivity workers. Assuming the firm pays the highest wage in these ranges, neither type of worker will be paid less than its fair wage.

(c) (ii) Firms will hire each type of worker until the cost of one unit of effective labor is the same for each type of worker. If this were not the case, firms could reduce their costs by hiring more of the workers with lower effective labor costs and fewer of the workers with higher effective labor costs.

For a low-productivity worker, the firm obtains e_2 units of effective labor at a wage cost of w_2. Thus the cost to the firm of one unit of low-productivity effective labor is w_2/e_2. For a high-productivity worker, the firm obtains Ae_1 units of effective labor at a wage cost of w_1. Thus the cost to the firm of one unit of high-productivity effective labor is w_1/Ae_1. Equating these effective labor costs gives us
(6) $\dfrac{w_1}{Ae_1} = \dfrac{w_2}{e_2}$.
Since both types of workers are paid at least the fair wage, $e_1 = e_2 = 1$ and so equation (6) can be rewritten as
(7) $w_1 = Aw_2$.
The wage for the high-productivity workers exceeds that of the low-productivity workers by a factor of A.

(c) (iii) In equilibrium, there will be no unemployment among the high-productivity workers. Suppose instead that there was. We have just shown that high-productivity workers have a higher wage than low-productivity workers and so w_1 is higher than the average wage or
(8) $w_1 > (w_1 + w_2)/2$,

where we have used the fact that all firms pay the same wage so that $\overline{w}_1 = w_1$ and $\overline{w}_2 = w_2$. Thus the fair wage for high-productivity workers is

(9) $w_1^* = (w_1 + w_2)/2 - bu_1 < w_1$.

But inequality (9) says that the firm is paying a wage higher than the fair wage to a group that has unemployment. But the firm does not need to do this; it is unconstrained in its choice of w_1 if $u_1 > 0$. It could cut the wage down to the fair wage level. Thus there cannot be unemployment among the high-productivity workers.

(c) (iv) In equilibrium, there will be unemployment among low-productivity workers. In part (c), (ii), we explained that the low-productivity workers receive a lower wage than the high-productivity workers. This means that their wage is less than the average wage, or

(10) $w_2 < (w_1 + w_2)/2$.

Now suppose that there was no unemployment among the low-productivity workers. Then the fair wage for them would be

(11) $w_2^* = (w_1 + w_2)/2 > w_2$.

But inequality (11) violates our assumption that firms will not pay a wage below the fair wage. Thus there must be some positive unemployment rate, $u_2 > 0$, such that $w_2 = w_2^*$.

Problem 10.6

(a) Suppose there are N states of the world. Then the firm's expected profits are

$$(1)\quad E(\pi) = \sum_{i=1}^{N} p_i\left[A_i F(L_i) - C_i^E L_i - C_i^U(\overline{L} - L_i)\right].$$

The expected utility of a representative worker is

$$(2)\quad E(u) = \sum_{i=1}^{N} p_i\left\{\left(\frac{L_i}{\overline{L}}\right)\left[U(C_i^E) - K\right] + \left(\frac{\overline{L} - L_i}{\overline{L}}\right)U(C_i^U)\right\}.$$

The firm's problem is to choose the L_i 's, C_i^E 's and C_i^U 's in order to maximize equation (1) subject to equation (2). Thus the Lagrangian is

$$(3)\quad \mathcal{L} = \sum_{i=1}^{N} p_i\left[A_i F(L_i) - C_i^E L_i - C_i^U(\overline{L} - L_i)\right] + \lambda\left[\sum_{i=1}^{N} p_i\left\{\left(\frac{L_i}{\overline{L}}\right)\left[U(C_i^E) - K\right] + \left(\frac{\overline{L} - L_i}{\overline{L}}\right)U(C_i^U)\right\} - u_0\right].$$

(b) The first-order conditions are

$$(4)\quad \frac{\partial \mathcal{L}}{\partial L_i} = p_i A_i F'(L_i) - p_i C_i^E + p_i C_i^U + \lambda p_i\left(\frac{1}{\overline{L}}\right)\left[U(C_i^E) - K\right] - \lambda p_i\left(\frac{1}{\overline{L}}\right)U(C_i^U) = 0,$$

$$(5)\quad \frac{\partial \mathcal{L}}{\partial C_i^E} = -p_i L_i + \lambda p_i\left(\frac{L_i}{\overline{L}}\right)U'(C_i^E) = 0, \text{ and}$$

$$(6)\quad \frac{\partial \mathcal{L}}{\partial C_i^U} = -p_i(\overline{L} - L_i) + \lambda p_i\left(\frac{\overline{L} - L_i}{\overline{L}}\right)U'(C_i^U) = 0.$$

Solving equation (5) for $U'(C_i^E)$ gives us

(7) $U'(C_i^E) = \overline{L}/\lambda$.

Equation (7) implies the marginal utility of consumption for the employed workers is constant across states and thus, with $U''(\bullet) < 0$, consumption of the employed workers is constant across states.

Solving equation (6) for $U'(C_i^U)$ gives us
(8) $U'(C_i^U) = \overline{L}/\lambda$.
Thus consumption of the unemployed workers is also constant across states. Comparing equations (7) and (8), it is also true that the marginal utility of consumption is the same for both employed and unemployed workers. This implies that the level of consumption of both types of workers is the same. That is,
(9) $C^E = C^U$.
So C^E and C^U do not depend on the state and are always equal to each other.

(c) The unemployed workers are actually better off. They consume the same amount as the employed workers and do not suffer the disutility of work, K.

Problem 10.7
(a) With efficient contracts, as shown in Section 10.5 of the text, $C = wL$ is constant across states. In addition, employment is increasing in A so that $L_G > L_B$. Given A and given the fact that wL is constant, profit, $\pi = AF(L) - wL$, is increasing in employment. Thus when the true state is A_B, the firm is better off announcing that A is actually A_G and employing L_G. When the true state is A_G, the firm is again better off announcing that A is A_G. Thus when the state is A_G, it is in the firm's interest to announce the true state. In the bad state, however, it is not in the firm's interest to announce the true state and so the efficient contract is not incentive-compatible.

(b) The incentive-compatibility constraint that is binding is that the firm not prefer to claim that $A = A_G$ when in fact $A = A_B$. Assuming that this constraint holds with equality, this requires
(1) $A_B F(L_B) - C_B = A_B F(L_G) - C_G$.
The left-hand side of equation (1) is the firm's profit in the bad state if it announces the bad state whereas the right-hand side of equation (1) is the firm's profit in the bad state if it announces the good state. The other constraint is that workers' expected utility be equal to u_0 or
(2) $[U(C_B) - V(L_B)]/2 + [U(C_G) - V(L_G)]/2 = u_0$.
The firm's expected profit is
(3) $E[\pi] = [A_B F(L_B) - C_B]/2 + [A_G F(L_G) - C_G]/2$.

The problem facing the firm is to choose L_B, C_B, L_G and C_G to maximize expected profits subject to equations (1) and (2). The Lagrangian is
(4) $\mathcal{L} = [A_B F(L_B) - C_B]/2 + [A_G F(L_G) - C_G]/2 + \lambda_1 \{[U(C_B) - V(L_B)]/2 + [U(C_G) - V(L_G)]/2 - u_0\}$
$+\lambda_2 \{[A_B F(L_B) - C_B] - [A_B F(L_G) - C_G]\}$.

The first-order conditions are
(5) $\partial\mathcal{L}/\partial C_B = (-1/2) + (1/2)\lambda_1 U'(C_B) - \lambda_2 = 0$,
(6) $\partial\mathcal{L}/\partial C_G = (-1/2) + (1/2)\lambda_1 U'(C_G) + \lambda_2 = 0$,
(7) $\partial\mathcal{L}/\partial L_B = (1/2)A_B F'(L_B) - (1/2)\lambda_1 V'(L_B) + \lambda_2 A_B F'(L_B) = 0$, and
(8) $\partial\mathcal{L}/\partial L_G = (1/2)A_G F'(L_G) - (1/2)\lambda_1 V'(L_G) - \lambda_2 A_B F'(L_G) = 0$.

(c) Adding equations (5) and (6) yields
(9) $-1 + (1/2)\lambda_1 U'(C_B) + (1/2)\lambda_1 U'(C_G) = 0$.
Solving for λ_1 gives us
(10) $\lambda_1 = \dfrac{2}{U'(C_B) + U'(C_G)}$.
Substituting equation (10) into equation (5) yields

(11) $2\lambda_2 = -1 + \dfrac{2U'(C_B)}{U'(C_B) + U'(C_G)} = \dfrac{-U'(C_B) - U'(C_G) + 2U'(C_B)}{U'(C_B) + U'(C_G)}$,

and thus

(12) $\lambda_2 = \dfrac{U'(C_B) - U'(C_G)}{2\left[U'(C_B) + U'(C_G)\right]}$.

Substituting equations (10) and (12) into equation (7) yields

(13) $\dfrac{1}{2}A_B F'(L_B) - \dfrac{V'(L_B)}{U'(C_B) + U'(C_G)} + \dfrac{A_B F'(L_B)\left[U'(C_B) - U'(C_G)\right]}{2\left[U'(C_B) + U'(C_G)\right]} = 0.$

Multiplying both sides of equation (13) by $2[U'(C_B) + U'(C_G)]$ gives us

(14) $A_B F'(L_B)[U'(C_B) + U'(C_G)] - 2V'(L_B) + A_B F'(L_B)[U'(C_B) - U'(C_G)] = 0.$

Simplifying yields

(15) $2A_B F'(L_B) U'(C_B) = 2V'(L_B)$,

and finally

(16) $A_B F'(L_B) = \dfrac{V'(L_B)}{U'(C_B)}$.

Equation (16) states that, in the bad state, the marginal product and the marginal disutility of labor are equated.

(d) Substituting equations (10) and (12) into equation (8) yields

(17) $\dfrac{1}{2}A_G F'(L_G) - \dfrac{V'(L_G)}{U'(C_B) + U'(C_G)} - \dfrac{A_B F'(L_G)\left[U'(C_B) - U'(C_G)\right]}{2\left[U'(C_B) + U'(C_G)\right]} = 0.$

Multiplying both sides of equation (17) by $[U'(C_B) + U'(C_G)]$ and rearranging yields

(18) $\dfrac{A_G F'(L_G)\left[U'(C_B) + U'(C_G)\right]}{2} - \dfrac{A_B F'(L_G)\left[U'(C_B) - U'(C_G)\right]}{2} = V'(L_G).$

Dividing both sides of equation (18) by $U'(C_G)$ and factoring out an $F'(L_G)$ from the left-hand side leaves us with

(19) $\left[\dfrac{A_G}{2} + \dfrac{A_B}{2} + \dfrac{A_G U'(C_B) - A_B U'(C_B)}{2U'(C_G)}\right] F'(L_G) = \dfrac{V'(L_G)}{U'(C_G)}$.

This is equivalent to

(20) $\left[A_G - \dfrac{A_G}{2} + \dfrac{A_B}{2} + \dfrac{A_G U'(C_B) - A_B U'(C_B)}{2U'(C_G)}\right] F'(L_G) = \dfrac{V'(L_G)}{U'(C_G)}$.

Collecting terms yields

(21) $\left\{A_G + \dfrac{A_G}{2}\left[\dfrac{U'(C_B)}{U'(C_G)} - 1\right] - \dfrac{A_B}{2}\left[\dfrac{U'(C_B)}{U'(C_G)} - 1\right]\right\} F'(L_G) = \dfrac{V'(L_G)}{U'(C_G)}$,

and thus finally

(22) $\left\{A_G + \left(\dfrac{A_G - A_B}{2}\right)\left[\dfrac{U'(C_B) - U'(C_G)}{U'(C_G)}\right]\right\} F'(L_G) = \dfrac{V'(L_G)}{U'(C_G)}$.

The contract must clearly involve $L_G > L_B$ and must therefore have $C_G > C_B$ to be incentive-compatible. [Note that if $L_G = L_B$, then C_G must equal C_B for incentive-compatibility. But then equation (12) implies $\lambda_2 = 0$, which implies that equations (7) and (8) cannot both be satisfied.] Since $C_G > C_B$, it follows that

$U'(C_B) > U'(C_G)$ and so the second term in brackets is positive. Thus $V'(L_G)/U'(C_G)$ exceeds $A_G F'(L_G)$. The marginal disutility of work exceeds the marginal product of labor in the good state. In other words, there is overemployment in the good state.

(e) Given the fact that there is no unemployment in the bad state and overemployment in the good state, this model does not appear to be helpful in understanding the high level of average unemployment. But since it is in the good state that the overemployment occurs, the model does suggest a reason that employment might be procyclical and more responsive to shocks than under symmetric information.

Problem 10.8

(a) The firm's profit function is given by

(1) $\pi = AF(L_I + L_O) - w_I L_I - w_O L_O$.

Since A is random, we can take the expected value of profits to yield

(2) $E[\pi] = \sum_{i=1}^{K} p_i [A_i F(\overline{L}_I + L_{Oi}) - w_{Ii}\overline{L}_I - Rw_{Ii}L_{Oi}]$,

where we have used the facts that L_I always equals $\overline{L}_I$ and $w_{Oi} = Rw_{Ii}$. The firm must offer insiders at least the minimal amount of expected utility u_0. We are told that expected utility for the representative insider in each state i is given by $u_I = U(w_I)$. Thus, the firm chooses w_{Ii} and L_{Oi} to maximize expected profits as given by equation (2) subject to the constraint that

(3) $E[u] = \left[\sum_{i=1}^{K} p_i U(w_{Ii})\right] - V(\overline{L}_I)$.

The Lagrangian is therefore given by

(4) $\mathcal{L} = \sum_{i=1}^{K} p_i [A_i F(\overline{L}_I + L_{Oi}) - w_{Ii}\overline{L}_I - Rw_{Ii}L_{Oi}] + \lambda\left(\left[\sum_{i=1}^{K} p_i U(w_{Ii})\right] - V(\overline{L}_I) - u_0\right)$.

(b) The first-order condition for the choice of L_{Oi} is given by

(5) $0 = p_i[A_i F'(\overline{L}_I + L_{Oi}) - Rw_{Ii}]$.

Solving for the real wage paid to insiders gives us

(6) $w_{Ii} = \dfrac{A_i F'(\overline{L}_I + L_{Oi})}{R}$.

Since R is less than 1, firms pay insiders a real wage that exceeds their marginal product of labor. Outsiders receive a real wage given by $w_{Oi} = Rw_{Ii}$ and are thus paid their marginal product.

(c) The first-order condition for the choice of w_{Ii} is given by

(7) $0 = p_i(-\overline{L}_I - RL_{Oi}) + \lambda p_i U'(w_{Ii})$.

Solving for $U'(w_{Ii})$ gives us

(8) $U'(w_{Ii}) = \dfrac{\overline{L}_I + RL_{Oi}}{\lambda}$.

The first-order condition therefore implies that the higher is L_{Oi}, the higher will be marginal utility. Since $U''(\bullet) < 0$ this implies that the real wage paid to insiders would be lower for higher values of L_{Oi}.

Problem 10.9

(a) In equilibrium, the number of people in the primary sector equals the number of employed people. This in turn equals the number of primary sector jobs, N_p, plus the number of unemployed people in the economy, U. In equilibrium—since individuals are picked at random for jobs—the probability of obtaining a primary-sector job, q, is equal to the total number of jobs, N_p, divided by the equilibrium pool of primary-sector workers, $N_p + U$. Thus in equilibrium

(1) $q = N_p/(N_p + U)$.

In addition, in equilibrium, the expected utility of choosing the primary sector, $qw_p + (1 - q)b$, must equal the expected utility of choosing the secondary sector, w_s. Thus in equilibrium

(2) $qw_p + (1 - q)b = w_s$.

Solving equation (2) for q gives us

(3) $q = (w_s - b)/(w_p - b)$.

We have two conditions that q must satisfy in equilibrium. Setting the right-hand sides of equations (1) and (3) equal, we have

(4) $N_p/(N_p + U) = (w_s - b)/(w_p - b)$,

which can be rewritten as

(5) $N_p(w_p - b) = N_p(w_s - b) + (w_s - b)U$.

Solving for equilibrium unemployment we have

(6) $(w_s - b)U = (w_p - w_s)N_p$,

or simply

(7) $$U = \left(\frac{w_p - w_s}{w_s - b}\right)N_p.$$

(b) To see the way in which an increase in the number of primary-sector jobs affects unemployment, take the derivative of U with respect to N_p:

(8) $$\frac{\partial U}{\partial N_p} = \left(\frac{w_p - w_s}{w_s - b}\right) > 0.$$

This derivative is positive because we are assuming $b < w_s < w_p$. Equation (8) implies that a rise in the number of primary-sector jobs actually increases equilibrium unemployment. The fact that there are more of these jobs does, for a given number of primary-sector workers, increase the likelihood of getting a job. But that very fact encourages more individuals to choose the primary sector over the secondary sector. And indeed, so many more choose the primary sector that the number of primary-sector workers who do not get jobs actually rises.

(c) To see the effects of an increase in the level of unemployment benefits, take the derivative of U with respect to b:

(9) $$\frac{\partial U}{\partial b} = \frac{(w_p - w_s)}{(w_s - b)^2}N_p > 0.$$

Thus unemployment rises when b rises. Again, the intuition is that higher unemployment benefits make the primary sector more attractive. Thus more individuals choose the primary sector. Since there are a fixed number of jobs in the primary sector, more individuals will end up unemployed.

Problem 10.10

(a) Note that $V = E[w - Cn'] = E[w] - CE[n']$ and is the expected value of the wage the worker will eventually accept if she searches more minus the expected cost of further searching. The expected cost of further searching is the expected number of jobs to be sampled multiplied by the (known) cost of sampling each job. Thus V can be interpreted as the expected value of further searching. Clearly, if the worker is offered a job that pays a wage of $\hat{w}$, where $\hat{w}$ exceeds the expected value of further searching, it is optimal to stop searching and take that job. However, if the wage offered is less than the expected value of further searching, it is optimal to reject that job and continue searching.

(b) Note that

(1) $V = F(V)V + \int_{w=V}^{\infty} wf(w)dw - C$

can be rewritten as

(2) $V = \dfrac{\int_{w=V}^{\infty} wf(w)dw}{1-F(V)} - \dfrac{C}{1-F(V)}.$

Consider the first term on the right-hand side of equation (2). The denominator is the probability that a wage drawn from the distribution will be greater than the cutoff, V. Thus this first term represents the expected wage conditional on that wage being greater than the reservation wage of V.

Now consider the second term on the right-hand side of equation (2). Since 1 - F(V) is the probability of drawing a wage greater than V, 1/[1 - F(V)] is the expected number of jobs that will need to be sampled in order to draw a job with a wage greater than V. For example, if the probability of drawing a wage greater than V is 1/2, then on average it will take two draws in order to obtain a wage greater than V. Therefore, C/[1 - F(V)] is the expected cost of sampling jobs. Thus V = E[w] - CE[n'] must satisfy equation (2).

(c) By Leibniz's rule and the chain rule, we have

(3) $\dfrac{\partial\left[\int_{w=V(C)}^{\infty} wf(w)dw\right]}{\partial C} = \dfrac{\partial\left[\int_{w=V(C)}^{\infty} wf(w)dw\right]}{\partial V}\dfrac{\partial V}{\partial C} = -Vf(V)\dfrac{\partial V}{\partial C}.$

Differentiating both sides of equation (1) with respect to C and using the result in equation (3) yields

(4) $\dfrac{\partial V}{\partial C} = \left[f(V)\dfrac{\partial V}{\partial C}\right]V + F(V)\dfrac{\partial V}{\partial C} - Vf(V)\dfrac{\partial V}{\partial C} - 1.$

Collecting terms in equation (4) gives us

(5) $[1 - F(V)]\partial V/\partial C = -1$,

or simply

(6) $\dfrac{\partial V}{\partial C} = \dfrac{-1}{1-F(V)}.$

With $F(V) < 1$, $\partial V/\partial C < 0$ so that an increase in the cost of sampling jobs reduces the value of the reservation wage.

(d) A searcher will never accept a job that she has previously rejected. From equation (2), V is a constant. If a searcher rejects a job paying some wage $\hat{w}$, it must mean that $\hat{w}$ is less than V and will always be less than V. Thus the worker will never accept the job paying $\hat{w}$.

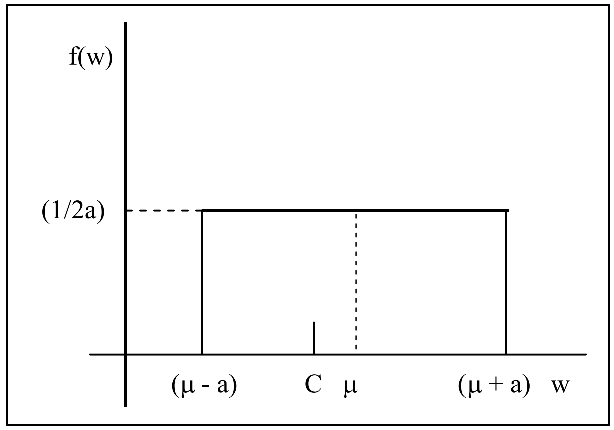
Problem 10.11

(a) The distribution of wages is depicted at right. The cost of sampling a job, C, is also depicted under the assumption that $C < \mu$. Over the interval from $(\mu - a)$ to $(\mu + a)$, this uniform distribution has a probability density function given by

(1) $f(w) = 1/2a$,

and an associated cumulative distribution function given by

(2) $F(w) = \dfrac{w - (\mu - a)}{2a}$.



As explained in the solution to part (b) of Problem 10.10, the cutoff or reservation wage, V, must satisfy

(3) $V = F(V)V + \int_{w=V}^{\mu+a} wf(w)dw - C$.

Note that we can write the upper bound of the integral as $(\mu + a)$ since $f(w) = 0$ for all $w > (\mu + a)$.
Substituting equations (1) and (2) into equation (3) yields

(4) $V = \left[\dfrac{V - (\mu - a)}{2a}\right]V + \int_{w=V}^{\mu+a} (w/2a)\,dw - C$.

The value of the integral in equation (4) is

(5) $\int_{w=V}^{\mu+a} (w/2a)\,dw = \dfrac{1}{2a}\left[\dfrac{1}{2}w^2 \Big|_{w=V}^{w=\mu+a}\right] = \dfrac{1}{4a}\left[(\mu + a)^2 - V^2\right]$.

Substituting equation (5) into equation (4) and multiplying both sides of the resulting expression by 4a gives us

(6) $4aV = 2V^2 - 2(\mu - a)V + (\mu + a)^2 - V^2 - 4aC$.

Collecting terms yields

(7) $V^2 - 2(\mu + a)V + (\mu + a)^2 - 4aC = 0$.

Using the quadratic formula, we have

(8) $V = \dfrac{2(\mu + a) \pm \sqrt{4(\mu + a)^2 - 4(\mu + a)^2 + 16aC}}{2} = \dfrac{2(\mu + a) \pm 4\sqrt{aC}}{2}$.

We can ignore the solution with $V > (\mu + a)$ since $(\mu + a)$ is the highest possible wage. Thus V is given by

(9) $V = (\mu + a) - 2a^{1/2}C^{1/2}$.

Note that if there is no cost to sampling a job so that $C = 0$, then $V = (\mu + a)$ meaning that the worker simply keeps searching until she is offered the highest wage in the distribution. In addition, if $C = a$, then $V = (\mu + a) - 2a$ or $V = (\mu - a)$ meaning that the worker will accept any wage. Finally, if $C > a$, then $V < (\mu + a)$ and so again, the worker accepts any wage that is offered.

To see the way in which V varies with a (which measures the dispersion of wages), use equation (9) to find the derivative of V with respect to a:

(10) $\partial V/\partial a = 1 - a^{-1/2}C^{1/2} = 1 - (C/a)^{1/2}$

With $C < a$, a rise in a increases the reservation wage, V. Intuitively, the fact that there are now more higher paying jobs increases the value of further searching and thus increases the cutoff wage.

Problem 10.12

(a) From equation (10.66) in the text, the rV_V locus is given by

(1) $$rV_V = -c + \frac{(1-\phi)\alpha}{\phi a + (1-\phi)\alpha + \lambda + r}(y-b).$$

Thus an increase in λ directly reduces rV_V for a given level of employment. However, α and a also depend on λ and so we must examine how a rise in λ affects them for a given level of employment. From equation (10.67) in the text, a, the rate at which unemployed workers find jobs, is

(2) $$a = \frac{\lambda E}{1-E}.$$

Thus a rise in λ increases a for a given level of employment. From equation (1), a rise in a reduces rV_V for a given level of employment. From equation (10.69) in the text, α, the rate at which vacancies are filled, is

(3) $\alpha = k^{1/\gamma} (\lambda E)^{(\gamma - 1)/\gamma} (1 - E)^{(1-\gamma)/\gamma}$.

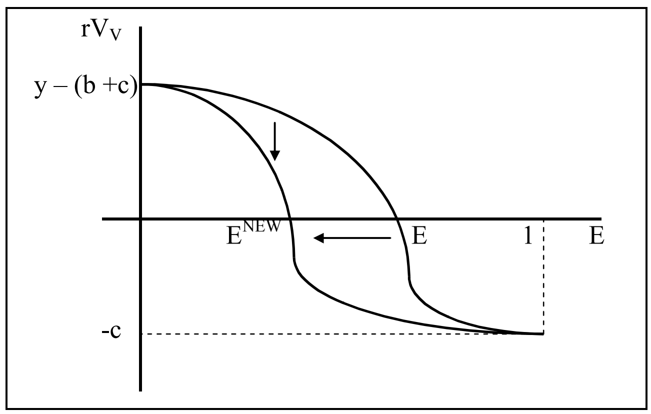
With $\gamma < 1$, a rise in λ reduces α for a given level of employment. From equation (1),

(4) $$\frac{\partial [rV_V]}{\partial \alpha} = (y-b)(1-\phi)\left[\frac{-(1-\phi)\alpha}{[\phi a + (1-\phi)\alpha + \lambda + r]^2} + \frac{1}{\phi a + (1-\phi)\alpha + \lambda + r}\right],$$

which simplifies to

(5) $$\frac{\partial [rV_V]}{\partial \alpha} = (y-b)(1-\phi)\left[\frac{\phi a + \lambda + r}{[\phi a + (1-\phi)\alpha + \lambda + r]^2}\right] > 0.$$

Thus a fall in α reduces rV_V for a given level of employment. In summary, all of these effects work in the same direction. The rise in the job breakup rate, λ, reduces rV_V for a given level of employment. Thus the rV_V locus shifts down as shown in the figure at right.



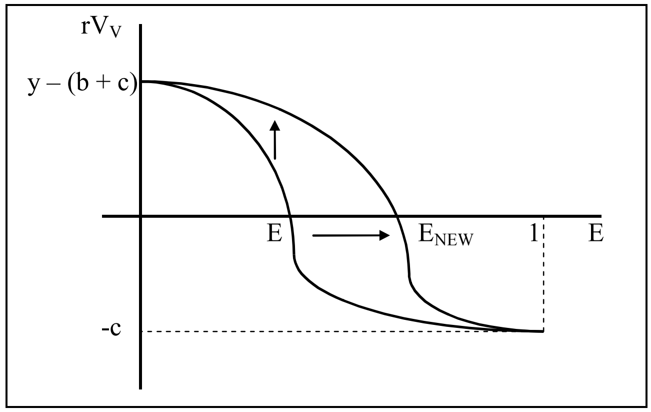
The equilibrium level of employment, which is given by the intersection of the rV_V locus with the free-entry condition that implies $rV_V = 0$, falls from E to E^{NEW}.

(b) We need to determine if the rV_V locus shifts up or down as a result of the increase in r. At a given level of employment, since a and α do not depend on r, we have

(6) $$\frac{\partial [rV_V]}{\partial r} = \frac{-(1-\phi)\alpha}{[\phi a + (1-\phi)\alpha + \lambda + r]^2}(y-b) < 0.$$

Thus the rV_V locus shifts down; the equilibrium level of employment falls as a result of the increase in the interest rate.

(c) At a given level of employment, a, which is given by equation (2), does not depend upon k. At a given level of employment, α, which is given by $\alpha = k^{1/\gamma} (\lambda E)^{(\gamma-1)/\gamma} (1- E)^{(1-\gamma)/\gamma}$, is increasing in k. As shown in equation (5), a rise in α causes rV_V to rise for a given level of employment. Thus an increase in the effectiveness of matching, k, shifts the rV_V locus up as depicted in the figure at right. The increase in the effectiveness of matching causes the equilibrium level of employment to rise from E to E_{NEW} .



(d) At a given level of employment, a and α, which are given by equations (2) and (3) do not depend upon b. From equation (1),

$$(7)\quad \frac{\partial[rV_V]}{\partial b} = \frac{-(1-\phi)\alpha}{\phi a + (1-\phi)\alpha + \lambda + r} < 0 .$$

Thus, an increase in income when unemployed causes a fall in the equilibrium level of employment.

(e) At a given level of employment, a and α, which are given by equations (2) and (3) do not depend upon φ. We again use equation (1) to obtain

$$(8)\quad \frac{\partial[rV_V]}{\partial \phi} = \alpha(y-b)\left[\frac{-(1-\phi)(a-\alpha)}{[\phi a + (1-\phi)\alpha + \lambda + r]^2} - \frac{1}{\phi a + (1-\phi)\alpha + \lambda + r}\right].$$

Obtaining a common denominator for the term in brackets gives us

$$(9)\quad \frac{\partial[rV_V]}{\partial \phi} = \alpha(y-b)\left[\frac{-(1-\phi)(a-\alpha) - [\phi a + (1-\phi)\alpha + \lambda + r]}{[\phi a + (1-\phi)\alpha + \lambda + r]^2}\right],$$

which simplifies to

$$(10)\quad \frac{\partial[rV_V]}{\partial \phi} = \alpha(y-b)\left[\frac{-(a+\lambda+r)}{[\phi a + (1-\phi)\alpha + \lambda + r]^2}\right] < 0 .$$

Thus, an increase in worker's bargaining power leads to a fall in the equilibrium level of employment.

Problem 10.13

(a) From equation (10.64), the wage is given by

$$(1)\quad w = b + \frac{(a+\lambda+r)\phi}{\phi a + (1-\phi)\alpha + \lambda + r}(y-b).$$

With the condition that φ = 0, this simplifies to

(2) w = b,

and so the wage is equal to the value of leisure.

From equation (10.70), the equilibrium condition is

$$(3)\quad -c + \frac{(1-\phi)\alpha(E)}{\phi a(E) + (1-\phi)\alpha(E) + \lambda + r}(y-b) = 0.$$

With φ = 0 this simplifies to

$$(4)\quad -c + \frac{\alpha(E)}{\alpha(E) + \lambda + r}(y-b) = 0.$$

(b) If $\varphi = 1$, the wage given by equation (1) simplifies to
(5) $w = y$,
so that the wage of a given worker equals that worker's output.

The equilibrium condition in this model is that $rV_V = 0$. From equation (10.65), rV_V is given by

(6) $$rV_V = -c + \alpha \frac{y - w}{\alpha + \lambda + r}.$$

With $w = y$, this implies that

(7) $$rV_V = -c.$$

Regardless of whether the job is filled or not, the firm's profits in this case are always – c, where c is the exogenous cost per unit time of maintaining a job. Since free entry and exit implies that the value of a vacancy must be 0 in equilibrium, no firms will want to create jobs. Thus, the equilibrium in this situation is that employment will be 0.

<u>Problem 10.14</u>

After the fall in y, there is no reason for firms whose positions are filled to discharge their workers. Thus employment and unemployment do not change discontinuously at the time of the shock. The reduced attractiveness of hiring does cause the value of a vacancy, V_V, to fall. However, since exiting is not allowed, we do not require $V_V = 0$ and so vacancies do not change. Since employment, unemployment and vacancies are not affected at the time of the fall in y, the number of new matches continues to equal the flows into unemployment. In summary, if we rule out entry and exit, unemployment and vacancies do not respond at all to the fall in y.

<u>Problem 10.15</u>

(a) Equation (10.65) in the text gives the value of a vacancy as a function of the wage, w, the rate per unit time that vacant jobs are filled, α, and exogenous parameters of the model. Using the fact that $V_V =$ 0 in equilibrium, (10.65) implies

(1) $$\alpha \frac{(y - w)}{\alpha + \lambda + r} = c.$$

Solving equation (1) for α yields

(2) $\alpha(y - w - c) = (\lambda + r)c$,

or simply

(3) $$\alpha = \frac{(\lambda + r)c}{(y - w - c)}.$$

Equation (10.69) in the text gives us an expression for α in terms of E and exogenous parameters of the model:

(4) $$\alpha = k^{1/\gamma}(\lambda E)^{(\gamma-1)/\gamma}(1 - E)^{(1-\gamma)/\gamma},$$

which simplifies to

(5) $$\alpha = k^{1/\gamma}\left(\frac{1 - E}{\lambda E}\right)^{(1-\gamma)/\gamma}.$$

Substituting equation (5) into equation (3) yields

(6) $$k^{1/\gamma}\left(\frac{1 - E}{\lambda E}\right)^{(1-\gamma)/\gamma} = \frac{(\lambda + r)c}{(y - w - c)}.$$

We need to solve equation (6) for E. Dividing both sides of (6) by $k^{1/\gamma}$ and then taking both sides of the resulting expression to the exponent $\gamma/(1 - \gamma)$ gives us

$$(7)\quad \frac{1-E}{\lambda E} = k^{-1/(1-\gamma)}[(\lambda + r)c]^{\gamma/(1-\gamma)}(y - w - c)^{-\gamma/(1-\gamma)}.$$

Multiplying both sides of equation (7) by λE and collecting the terms in E leaves us with

$$(8)\quad E[1 + \lambda k^{-1/(1-\gamma)}[(\lambda + r)c]^{\gamma/(1-\gamma)}(y - w - c)^{-\gamma/(1-\gamma)}] = 1,$$

or simply

$$(9)\quad E = \frac{1}{[1 + \lambda k^{-1/(1-\gamma)}[(\lambda + r)c]^{\gamma/(1-\gamma)}(y - w - c)^{-\gamma/(1-\gamma)}]}.$$

Equation (9) expresses employment as a function of the wage, w, and exogenous parameters of the model.

(b) The change in employment due to a change in y is given by

$$(10)\quad \frac{\partial E}{\partial y} = \frac{-1}{B^2}\left(\frac{-\gamma}{1-\gamma}\right)\lambda k^{-1/(1-\gamma)}[(\lambda + r)c]^{\gamma/(1-\gamma)}(y - w - c)^{-1/(1-\gamma)}\left(1 - \frac{\partial w}{\partial y}\right),$$

where $B \equiv [1 + \lambda k^{-1/(1-\gamma)}[(\lambda + r)c]^{\gamma/(1-\gamma)}(y - w - c)^{-\gamma/(1-\gamma)}]$. Note that $E = 1/B$ and thus equation (10) can be rewritten as

$$(11)\quad \frac{\partial E}{\partial y} = \frac{E}{B}\left(\frac{\gamma}{1-\gamma}\right)\lambda k^{-1/(1-\gamma)}[(\lambda + r)c]^{\gamma/(1-\gamma)}(y - w - c)^{-1/(1-\gamma)}\left(1 - \frac{\partial w}{\partial y}\right).$$

If the wage is fixed, then $\partial w/\partial y = 0$. As explained in the text, if w adjusts so that $V_E - V_U$ remains equal to $V_F - V_V$ then $\partial w/\partial y > 0$. Thus the change in employment due to a change in y is smaller when w adjusts.

Problem 10.16

At equilibrium, equation (10.71) says

$$(1)\quad \frac{M(U,V)}{V}(y - w) - c = 0.$$

Since $\varphi = w/y$, where φ denotes the fraction of the surplus from forming the match that goes to the worker, then $w = \varphi y$ and we can rewrite the term $y - w$ as $y - \varphi y = (1 - \varphi)y$. Then (1) becomes

$$(2)\quad \frac{M(U,V)}{V}(1 - \phi)y - c = 0.$$

After rearranging the terms, we obtain

$$(3)\quad \frac{M(U,V)}{V}(1 - \phi) = \frac{c}{y}.$$

Next, we consider the optimal allocation of V. A social planner would choose V to maximize $Ey - Vc = M(U, V)y - Vc$. The first-order condition for V is

$$(4)\quad \frac{\partial M(U,V)}{\partial V}y - c = 0.$$

After rearranging the terms, we are left with

$$(5)\quad \frac{\partial M(U,V)}{\partial V} = \frac{c}{y}.$$

Since we have assumed that $M(\cdot)$ is smooth and well-behaved, and V is strictly positive at both the equilibrium and optimal levels, then we notice that when we compare (3) and (5), c/y can be expressed as two different and smooth functions of V. Equation (3) gives the equilibrium state; (5) gives the optimal state. For the decentralized equilibrium to be efficient, we will need the equilibrium and optimal levels of vacancies, V, to be equal, or

(6) $\dfrac{M(U,V)}{V}(1-\phi) = \dfrac{\partial M(U,V)}{\partial V}$.

After rearranging the terms, we obtain

(7) $\dfrac{\partial M(U,V)}{\partial V}\dfrac{V}{M(U,V)} = (1-\phi)$.

Thus, the answer to the question is yes. The condition for the decentralized equilibrium to be efficient is still that the elasticity of matches with respect to vacancies, $[\partial M(U,V)/\partial V][V/M(U,V)]$, equals the share of the surplus that goes to the firm, $1 - \varphi$.

Problem 10.17

(a) The number of unemployed workers is given by $U = 1 - E(N)$. The number of vacancies is given by $V = N - E(N)$. Equation (10.50) implies that $M(U,V) = Um(V/U)$. The matching function, (10.53), implies that $m(V/U) = k(V/U)^{\gamma}$, where $k > 0$ and $o < \gamma < 1$. Thus, we can write

(1) $M(U,V) = Uk(V/U)^{\gamma}$,

or

(2) $M(U,V) = kV^{\gamma}U^{1-\gamma}$.

Now substitute the expressions for V and U to give us

(3) $M(U,V) = k[N - E(N)]^{\gamma}[1 - E(N)]^{1-\gamma}$.

The steady-state condition, $M(U,V) = \lambda E$, implies that

(4) $\lambda E(N) = k[N - E(N)]^{\gamma}[1 - E(N)]^{1-\gamma}$.

Differentiating both sides of equation (4) with respect to N gives us

(5) $\lambda E'(N) = k\{\gamma[N - E(N)]^{\gamma-1}[1 - E'(N)]\,[1 - E(N)]^{1-\gamma} + (1-\gamma)\,[1 - E(N)]^{-\gamma}[-E'(N)]\,[N - E(N)]^{\gamma}\}$.

Then we collect the common factors of $[N - E(N)]^{\gamma-1}$ and $[1 - E(N)]^{-\gamma}$, yielding

(6) $\lambda E'(N) = k[N - E(N)]^{\gamma-1}[1 - E(N)]^{-\gamma}\,\{\gamma[1 - E'(N)]\,[1 - E'(N)] - (1-\gamma)E'(N)[N - E(N)]\}$.

To simplify the algebra, we return to the expressions using U and V, where $U = 1 - E(N)$ and $V = N - E(N)$. This allows us to rewrite equation (6) as

(7) $\lambda E'(N) = kV^{\gamma-1}U^{-\gamma}\,\{\gamma U[1 - E'(N)] - (1-\gamma)E'(N)V\}$,

or simply

(8) $\lambda E'(N) = kV^{\gamma-1}U^{-\gamma}\,\{\gamma U - [\gamma U + (1-\gamma)V]E'(N)\}$.

Collecting the terms in E'(N) gives us

(9) $E'(N)[\lambda + \gamma kV^{\gamma-1}U^{1-\gamma} + (1-\gamma)kV^{\gamma}U^{-\gamma}] = \gamma kV^{\gamma-1}U^{1-\gamma}$.

Finally, we arrive at an expression for the impact of a change in the number of jobs on employment, E'(N), in terms of E(N) and the parameters of the model:

(10) $E'(N) = \dfrac{\gamma kV^{\gamma-1}U^{1-\gamma}}{\lambda + \gamma kV^{\gamma-1}U^{1-\gamma} + (1-\gamma)kV^{\gamma}U^{-\gamma}}$.

(b) Differentiating both sides of $W(N) = (y - b)E(N) - Nc + b$ with respect to N gives us

(11) $W'(N) = (y - b)E'(N) - c$.

Substitute the expression for E'(N) from part (a) into equation (11) to obtain

(12) $W'(N) = \dfrac{\gamma kV^{\gamma-1}U^{1-\gamma}}{\lambda + \gamma kV^{\gamma-1}U^{1-\gamma} + (1-\gamma)kV^{\gamma}U^{-\gamma}}(y-b) - c$.

We can rewrite equation (12) as

(13) $W'(N) = \dfrac{\gamma kV^{\gamma}U^{1-\gamma}V^{-1}}{\lambda + \gamma kV^{\gamma}U^{1-\gamma}V^{-1} + (1-\gamma)kV^{\gamma}U^{1-\gamma}U^{-1}}(y-b) - c$.

Since $M(U,V) = \lambda E(N) = kV^{\gamma}U^{1-\gamma}$, we may further simplify (13) to

$$(14)\quad W'(N) = \frac{\gamma\lambda E(N)V^{-1}}{\lambda + \gamma\lambda E(N)V^{-1} + (1-\gamma)\lambda E(N)U^{-1}}(y-b) - c\,.$$

Dividing the top and bottom of (14) by λ and simplifying leaves us with

$$(15)\quad W'(N) = \frac{\gamma E(N)V^{-1}}{1 + E(N)[\gamma V^{-1} + (1-\gamma)U^{-1}]}(y-b) - c\,.$$

(c) To simplify the notation, we can drop the "EQ" subscripts on N, recalling that N is the number of jobs in equilibrium. In the case of r = 0, equation (10.66) simplifies to

$$(16)\quad c = \frac{(1-\phi)\alpha}{\phi a + (1-\phi)\alpha + \lambda}(y-b)\,.$$

Substituting $a = \lambda E/(1 - E)$ and $\alpha = \lambda E/V$ into equation (16) gives us

$$(17)\quad c = \frac{(1-\phi)(\lambda E/V)}{\phi[\lambda E/(1-E)] + (1-\phi)(\lambda E/V) + \lambda}(y-b)\,.$$

Simplifying yields

$$(18)\quad c = \frac{(1-\phi)E(1-E)}{\phi EV + (1-\phi)E(1-E) + (1-E)V}(y-b)\,.$$

Substituting $V = N - E$ into (18) gives us

$$(19)\quad c = \frac{(1-\phi)E(N)[1-E(N)]}{\phi E(N)[N-E(N)] + (1-\phi)E(N)[1-E(N)] + [N-E(N)][1-E(N)]}(y-b)\,,$$

where N is the number of jobs in equilibrium.

(d) We will first simplify equation (19) by using the definitions of $U = 1 - E(N)$ and $V = N - E(N)$ to obtain

$$(20)\quad c = \frac{(1-\phi)EU}{\phi EV + (1-\phi)EU + VU}(y-b)\,.$$

Now substitute equation (20) into equation (15), yielding

$$(21)\quad W'(N) = \frac{\gamma EV^{-1}}{1 + E[\gamma V^{-1} + (1-\gamma)U^{-1}]}(y-b) - \frac{(1-\phi)EU}{\phi EV + (1-\phi)EU + VU}(y-b)\,.$$

Obtaining a common denominator gives us

$$(22)\quad W'(N) = \frac{\gamma EV^{-1}[\phi EV + (1-\phi)EU + VU] - (1-\phi)EU\{1 + E[\gamma V^{-1} + (1-\gamma)U^{-1}]\}}{\{1 + E[\gamma V^{-1} + (1-\gamma)U^{-1}]\}[\phi EV + (1-\phi)EU + VU]}(y-b)\,.$$

Simplifying the terms in the numerator gives us

$$(23)\quad W'(N) = \frac{\gamma\phi E^2 + \gamma(1-\phi)E^2V^{-1}U + \gamma EU - (1-\phi)EU - \gamma(1-\phi)E^2V^{-1}U - (1-\phi)(1-\gamma)E^2}{\{1 + E[\gamma V^{-1} + (1-\gamma)U^{-1}]\}[\phi EV + (1-\phi)EU + VU]}(y-b)\,.$$

Note that in the numerator of (23), the second and fifth terms sum to 0. We can then simplify further by combining the first and sixth terms, and the third and fourth terms, to yield

$$(24)\quad W'(N) = \frac{E^2[\gamma\phi - (1-\phi)(1-\gamma)] + EU[\gamma - (1-\phi)]}{\{1 + E[\gamma V^{-1} + (1-\gamma)U^{-1}]\}[\phi EV + (1-\phi)EU + VU]}(y-b)\,.$$

Since $\gamma\phi - (1-\phi)(1-\gamma) = \gamma - (1-\phi)$, we can rewrite equation (24) as

$$(25)\quad W'(N) = \frac{(E^2 + EU)[\gamma - (1-\phi)]}{\{1 + E[\gamma V^{-1} + (1-\gamma)U^{-1}]\}[\phi EV + (1-\phi)EU + VU]}(y-b)\,,$$

where N is the number of jobs in equilibrium. Since $(y - b)$, $E^2 + EU$, $1 + E[\gamma V^{-1} + (1 - \gamma)U^{-1}]$, and $\varphi EV + (1 - \varphi)EU + VU$ are all positive, the sign of $W'(N_{EQ})$ will be determined by the sign of $\gamma - (1 - \varphi)$. If $\gamma > (1 - \varphi)$ then $W'(N_{EQ}) > 0$. If $\gamma < (1 - \varphi)$ then $W'(N_{EQ}) < 0$.

SOLUTIONS TO CHAPTER 11

Problem 11.1

(a) From $m_t - p_t = c - b(E_t\, p_{t+1} - p_t)$, collecting the terms in p_t yields

(1) $p_t\,(1 + b) = m_t - c + bE_t\, p_{t+1}$,

and so p_t is given by

$$(2)\quad p_t = \left(\frac{b}{1+b}\right)E_t p_{t+1} + \left(\frac{1}{1+b}\right)\left(m_t - c\right).$$

(b) Equation (2) holds in all periods so that we can write p_{t+1} as

$$(3)\quad p_{t+1} = \left(\frac{b}{1+b}\right)E_{t+1} p_{t+2} + \left(\frac{1}{1+b}\right)\left(m_{t+1} - c\right).$$

Taking the expected value, as of time t, of both sides of equation (3) yields

$$(4)\quad E_t p_{t+1} = \left(\frac{b}{1+b}\right)E_t p_{t+2} + \left(\frac{1}{1+b}\right)\left(E_t m_{t+1} - c\right),$$

where we have used the law of iterated projections, which states that $E_t\, E_{t+1}\, p_{t+2} = E_t\, p_{t+2}$ .If this did not hold, individuals would be expecting to revise their estimate of p_{t+2} either up or down, which would imply that their original estimate was not rational.

(c) Substituting equation (4) into equation (2) yields

$$(5)\quad p_t = \left(\frac{b}{1+b}\right)^2 E_t p_{t+2} + \left(\frac{1}{1+b}\right)\left[\left(m_t - c\right) + \left(\frac{b}{1+b}\right)\left(E_t m_{t+1} - c\right)\right].$$

Again using the fact that equation (2) holds in all periods, we can write p_{t+2} as

$$(6)\quad p_{t+2} = \left(\frac{b}{1+b}\right)E_{t+2} p_{t+3} + \left(\frac{1}{1+b}\right)\left(m_{t+2} - c\right).$$

Taking the expected value, as of time t, of both sides of equation (6) gives us

$$(7)\quad E_t p_{t+2} = \left(\frac{b}{1+b}\right)E_t p_{t+3} + \left(\frac{1}{1+b}\right)\left(E_t m_{t+2} - c\right),$$

where we have again used the law of iterated projections so that $E_t\, E_{t+2}\, p_{t+3} = E_t\, p_{t+3}$. Substituting equation (7) into equation (5) leaves us with

$$(8)\quad p_t = \left(\frac{b}{1+b}\right)^3 E_t p_{t+3} + \left(\frac{1}{1+b}\right)\left[\left(m_t - c\right) + \left(\frac{b}{1+b}\right)\left(E_t m_{t+1} - c\right) + \left(\frac{b}{1+b}\right)^2\left(E_t m_{t+2} - c\right)\right].$$

The pattern should now be clear. We can write p_t as the following infinite sum:

$$(9)\quad p_t = \left(\frac{1}{1+b}\right)\left[\left(m_t - c\right) + \left(\frac{b}{1+b}\right)\left(E_t m_{t+1} - c\right) + \left(\frac{b}{1+b}\right)^2\left(E_t m_{t+2} - c\right) + \left(\frac{b}{1+b}\right)^3\left(E_t m_{t+3} - c\right) + \ldots\right].$$

(d) With output and the real interest rate constant, the price level must adjust to clear the money market. If m_{t+i} is higher, p_{t+i} will need to be higher to clear the money market. Thus if individuals expect, in period m_{t+i-1} , that m_{t+i} will be higher they will also expect p_{t+i} to be higher. Thus in period t + i - 1, expected inflation will be higher. This reduces real money demand in period t + i - 1. For a given value of m_{t+i-1} , this means that p_{t+i-1} will need to rise to clear the money market. Now go back one more period. Suppose that individuals expect, in period t + i - 2, that m_{t+i} will be higher. Then they expect, through the reasoning above, that p_{t+i-1} will be higher. Thus expected inflation in t + i - 2 will be higher, real money demand will be lower and thus p_{t+i-2} will be need to be higher to clear the money market. Reasoning backward, as soon as people expect the nominal money supply to rise in some future period, the price level will rise in the current period.

(e) Equation (9) can be written using summation notation as

(10) $p_t = \frac{1}{1+b}\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i\left(E_t m_{t+i} - c\right).$

Substituting the assumption that $E_t\, m_{t+i} = m_t + gi$ into equation (10) yields

(11) $p_t = \frac{1}{1+b}\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i(m_t + gi - c) = \frac{1}{1+b}\left[(m_t - c)\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i + g\sum_{i=0}^{\infty} i\left(\frac{b}{1+b}\right)^i\right].$

Now we can use the facts that

(12) $\sum_{i=0}^{\infty}\left(\frac{b}{1+b}\right)^i = 1 + \left(\frac{b}{1+b}\right) + \left(\frac{b}{1+b}\right)^2 + \ldots = \frac{1}{1-[b/(1+b)]} = 1+b,$

and

(13) $\sum_{i=0}^{\infty} i\left(\frac{b}{1+b}\right)^i = \frac{b/(1+b)}{\{1-[b/(1+b)]\}^2} = \frac{b/(1+b)}{1/(1+b)^2} = b(1+b).$

Equation (13) uses the result that

(14) $\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}.$

A (not entirely rigorous) way to see why (14) and thus (13) hold is to note that with $x < 1$, we have

(15) $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$

Differentiating both sides of equation (15) with respect to x (which means differentiating term by term on the left-hand side) gives us

(16) $1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2}.$

Multiplying both sides of equation (16) by x yields

(17) $x + 2x^2 + 3x^3 + \ldots = \frac{x}{(1-x)^2}.$

Note that (17) and (14) are equivalent; the left-hand side of equation (14) is simply the left-hand side of (17) written in summation notation.

Substituting equations (12) and (13) into equation (11) yields

(18) $p_t = \frac{1}{1+b}\left[(m_t - c)(1+b) + gb(1+b)\right].$

Thus the price level is given by

(19) $p_t = (m_t - c) + bg.$

To see how the price level changes when money growth changes, use equation (19) to take the derivative of p_t with respect to g:

(20) $\frac{\partial p_t}{\partial g} = b > 0.$

Thus a rise in money growth, even without a rise in the level of the current period's money supply, causes an upward jump in the current price level.

Problem 11.2

(a) Substituting the normalized, flexible-price level of output, $y_0 = 0$, into the IS equation, $y_0 = c - ar_0$, gives us $0 = c - ar_0$. Solving for the real interest rate in period 0 yields

(1) $r_0 = c/a$.

Since the nominal money stock is expected to be constant, the price level is expected to be constant and thus expected inflation from period 0 to period 1 is

(2) $E_0[p_1] - p_0 = 0$.

The nominal interest rate in period 0, $i_0 = r_0 + [E_0[p_1] - p_0]$, is simply equal to the real interest rate:

(3) $i_0 = c/a$.

Finally, substituting the assumptions that $m_0 = 0$ and $y_0 = 0$ as well as equation (3) into the condition for equilibrium in the money market, $m_0 - p_0 = b + hy_0 - ki_0$, yields $- p_0 = b - (ck/a)$ or simply

(4) $p_0 = (ck/a) - b$.

(b) In period 2, the economy is once again at the flexible-price equilibrium level of output, which is 0. Substituting this fact into the IS equation allows us to solve for the real interest rate in period 2:

(5) $r_2 = c/a$.

Since expected inflation from period 2 to period 3 is equal to g – the price level is expected to rise by the same amount as the nominal money supply each period – the nominal interest rate in period 2 is given by

(6) $i_2 = (c/a) + g$.

Since m was equal to 0 in period 0 and then increases by g in each following period, the nominal money supply in period 2 is $m_2 = 2g$. Substituting this fact as well as $y_2 = 0$ and $i_2 = (c/a) + g$ into the condition for equilibrium in the money market leaves us with

(7) $2g - p_2 = b - (ck/a) - kg$.

Solving for p_2 gives us

(8) $p_2 = - b + (ck/a) + (2 + k)g$.

(c) The price level is completely unresponsive to unanticipated monetary shocks for one period. Thus the price level in period 1 does not change from its period 0 value and hence

(9) $p_1 = (ck/a) - b$.

The expectation of inflation from period 1 to period 2, $E_1[p_2] - p_1$, is therefore

(10) $E_1[p_2] - p_1 = - b + (ck/a) + (2 + k)g - (ck/a) + b = (2 + k)g$,

where we have used equations (8) and (9) to substitute for p_2 and p_1.

Now substitute the IS equation, $y_1 = c - ar_1$, into the money-market-equilibrium condition, $m_1 - p_1 = b + hy_1 - ki_1$, to obtain

(11) $m_1 - p_1 = b + hc - ahr_1 - ki_1$.

By assumption, the nominal money supply in period 1 is g. In addition, $i_1 = r_1 + [E_1[p_2] - p_1]$, which, using equation (10), is equivalent to $i_1 = r_1 + (2 + k)g$. Substituting these facts as well as equation (9) for the price level into equation (11) gives us

(12) $g - (ck/a) + b = b + hc - ahr_1 - kr_1 - (2 + k)kg$.

Simplifying and collecting the terms in r_1 yields

(13) $r_1[ah + k] = hc + (ck/a) - g - (2 + k)kg$.

Thus the real interest rate in period 1 is given by

(14) $$r_1 = \frac{hc + (ck/a) - g - (2 + k)kg}{ah + k}.$$

Finally, substituting equations (10) and (14) into $i_1 = r_1 + [E_1[p_2] - p_1]$ gives us

(15) $$i_1 = \frac{hc + (ck/a) - g - (2 + k)kg}{ah + k} + (2 + k)g = \frac{hc + (ck/a) - g - (2 + k)kg + (2 + k)ahg + (2 + k)kg}{ah + k}.$$

Thus the nominal interest rate in period 1 is given by

$$(16)\ \ i_1 = \frac{hc + (ck/a) - g + (2 + k)ahg}{ah + k}.$$

(d) Using equations (16) and (3), the change in the nominal interest rate from period 0 to period 1 is

$$(17)\ \ i_1 - i_0 = \frac{hc + (ck/a) - g + (2 + k)ahg}{ah + k} - (c/a) = \frac{hc + (ck/a) - g + (2 + k)ahg - hc - (ck/a)}{ah + k}.$$

Simplifying yields

$$(18)\ \ i_1 - i_0 = \frac{(2 + k)ahg - g}{ah + k}.$$

We can determine the condition required of the parameters in order for the nominal interest rate to fall from period 0 to period 1; that is, for $i_1 - i_0 < 0$. From equation (18), this condition is

$$(19)\ \ \frac{g\left[(2 + k)ah - 1\right]}{ah + k} < 0,$$

or simply

$$(20)\ \ (2 + k)ah < 1.$$

The smaller is a (the elasticity of output with respect to changes in the real interest rate), the smaller is h (the income elasticity of real money demand) and the smaller is k (the interest semi-elasticity of real money demand), the more likely it is for the condition in (20) to be satisfied and thus the more likely it is for the nominal interest rate to fall in response to the monetary expansion.

For the nominal interest rate, $i = r + \pi^e$, to fall, we need the liquidity effect to outweigh the expected inflation effect. That is, we need the real interest rate to fall by more than expected inflation rises. With the price level fixed by assumption in period 1, y and i must adjust to ensure money market equilibrium. If k is small, changes in i will not affect real money demand very much. We need y to rise to increase real money demand and get it equal to the new higher real money stock. If h is small, we need y to rise a lot in order to accomplish this. If y is to rise a lot, we need – from the IS equation – the real interest rate to fall a lot. If furthermore, a is small, we need r to fall a lot just to generate an increase in output. Thus small values of k, h, and a all work to make the drop in r larger and thus make it more likely that i will fall.

Problem 11.3

(a) Any shock to the nominal money supply in period t + 1 is fully reflected in the price level by period t + 2. That is, the only reason the price level will change from period t + 1 to period t + 2 is if there is a non-zero realization of u in period t + 1. From the law of iterated projections, we have

$$(1)\ \ E_t\,[E_{t+1}\,[p_{t+2}] - p_{t+1}] = E_t\,[p_{t+2} - p_{t+1}].$$

Since the expected value, as of period t, of u_{t+1} is zero, the price level is not expected to change from period t + 1 to period t + 2. Thus

$$(2)\ \ E_t\,[E_{t+1}\,[p_{t+2}] - p_{t+1}] = 0.$$

Since the condition for money market equilibrium must hold each period, we can write

$$(3)\ \ m_{t+1} - p_{t+1} = b + hy_{t+1} - kr_{t+1} - k(E_{t+1}\,[p_{t+2}] - p_{t+1}),$$

where we have substituted in for $i_{t+1} = r_{t+1} + (E_{t+1}\,[p_{t+2}] - p_{t+1})$. Taking the expected value of both sides of equation (3) yields

$$(4)\ \ E_t m_{t+1} - E_t p_{t+1} = b + hy^n - kr^n,$$

where we have used the result from equation (2) that $E_t\,[E_{t+1}\,[p_{t+2}] - p_{t+1}] = 0$. In addition, since y_{t+1} and r_{t+1} will only depend on the u_{t+1} shock, which is expected to be zero, they are expected to be equal to their flexible-price values.

(b) Rearranging equation (4), we have

(5) $E_t p_{t+1} = E_t m_{t+1} - b - hy^n + kr^n$.

Since $m_{t+1} = m_t + u_{t+1}$, $E_t\, m_{t+1} = m_t$. Using this fact and subtracting p_t from both sides of equation (5) yields

(6) $E_t p_{t+1} - p_t = (m_t - p_t) - b - hy^n + kr^n$.

As explained in part (a), expected inflation is equal to u_t and thus we can write

(7) $u_t = (m_t - p_t) - b - hy^n + kr^n$.

Substituting $m_t = m_{t-1} + u_t$ into equation (7) and rearranging to solve for p_t yields

(8) $p_t = m_{t-1} - b - hy^n + kr^n$.

The next step is to solve for output in period t. Rearranging the condition for money market equilibrium to solve for i_t yields

(9) $i_t = [b + hy_t - (m_t - p_t)] / k$.

From equation (7), we have

(10) $(m_t - p_t) = u_t + b + hy^n - kr^n$.

Substituting equation (10) into equation (9) gives us

(11) $i_t = \dfrac{b + hy_t - u_t - b - hy^n + kr^n}{k} = \dfrac{h(y_t - y^n) + kr^n - u_t}{k}$.

Substituting equation (11) for i_t and using the fact that $\pi_t^e = u_t$, the IS equation becomes

(12) $y_t = c - a\left[\dfrac{h(y_t - y^n) + kr^n - u_t}{k}\right] + au_t$.

Collecting the terms in y_t , we have

(13) $\left[\dfrac{k + ah}{k}\right] y_t = c + \dfrac{ahy^n - akr^n + au_t}{k} + au_t$,

which implies

(14) $y_t = \dfrac{kc + ahy^n - akr^n + au_t + kau_t}{k + ah}$,

and thus output in period t is given by

(15) $y_t = \dfrac{kc + a[hy^n - kr^n + (1 + k)u_t]}{k + ah}$.

In order to determine the real interest rate, rearrange the IS equation to obtain

(16) $r_t = (c/a) - (y_t /a)$.

Substituting equation (15) into equation (16) yields

(17) $r_t = \dfrac{c}{a} - \dfrac{kc + a[hy^n - kr^n + (1 + k)u_t]}{a(k + ah)}$,

which implies

(18) $r_t = \dfrac{ck + cah - kc - a[hy^n - kr^n + (1 + k)u_t]}{a(k + ah)} = \dfrac{ch - [hy^n - kr^n + (1 + k)u_t]}{k + ah}$.

Thus the real interest rate in period t is

(19) $r_t = \dfrac{h(c - y^n) + kr^n - (1+k)u_t}{k + ah}$.

The nominal interest rate is $i_t = r_t + \pi_t^e$, where $\pi_t^e = u_t$:
(20) $i_t = r_t + u_t$.
Substituting equation (19) into equation (20) gives us

(21) $i_t = \dfrac{h(c - y^n) + kr^n - (1+k)u_t + (k+ah)u_t}{k + ah} = \dfrac{h(c - y^n) + kr^n + (ah - 1)u_t}{k + ah}$.

(c) From equation (21), with $\pi_t^e = u_t$, we have

(22) $i_t = \dfrac{h(c - y^n) + kr^n}{k + ah} + \dfrac{ah - 1}{k + ah}\pi_t^e$.

From equation (22), we can see that changes in expected inflation are not reflected one-for-one in the nominal rate. This is due to the fact that prices are completely unresponsive to the monetary disturbance for one period. This means that, in general, output and the nominal interest rate will adjust to clear the money market. In order for output to change, the real interest rate must change and therefore, in general, the nominal interest rate will not move one-for-one with inflation.

Problem 11.4
(a) Under rational expectations,
(1) $\pi_{t+1} = E_t\, \pi_{t+1} + \varepsilon_{t+1}$,
where ε_{t+1} is a disturbance that is uncorrelated with anything known at t. Now consider the regression:
(2) $i_t = a + b\pi_{t+1} + e_t$.
Using the hint in the question, the OLS estimator of b is given by

(3) $\hat{b} = \dfrac{\text{cov}(i_t, \pi_{t+1})}{\text{var}(\pi_{t+1})}$.

Using $i_t = r_t + E_t\, \pi_{t+1}$ and equation (1), we can write the covariance in the numerator as
(4) $\text{cov}(i_t , \pi_{t+1}) = \text{cov}(r_t + E_t\, \pi_{t+1} , E_t\, \pi_{t+1} + \varepsilon_{t+1})$.
Since r_t and $E_t\, \pi_{t+1}$ are uncorrelated and ε_{t+1} is uncorrelated with anything known at t, this implies
(5) $\text{cov}(i_t , \pi_{t+1}) = \text{var}(E_t\, \pi_{t+1})$.
Again using equation (1), the variance in the denominator of equation (3) can be written as
(6) $\text{var}(\pi_{t+1}) = \text{var}(E_t\, \pi_{t+1} + \varepsilon_{t+1}) = \text{var}(E_t\, \pi_{t+1}) + \text{var}(\varepsilon_{t+1})$,
where we have used the fact that $\text{cov}(E_t\, \pi_{t+1} , \varepsilon_{t+1}) = 0$. Substituting equations (5) and (6) into equation (3) allows us to write the OLS estimator as

(7) $\hat{b} = \dfrac{\text{var}(E_t \pi_{t+1})}{\text{var}(E_t \pi_{t+1}) + \text{var}(\varepsilon_{t+1})} < 1$.

The hypothesis that the real interest rate is constant, so that changes in expected inflation cause one-for-one movements in the nominal interest rate, only predicts that the coefficient on π_{t+1} should be positive and less than one, not that it will take on any specific value.

(b) Now consider a regression of the form
(8) $\pi_{t+1} = a' + b'\, i_t + e_t'$.
The OLS estimator of b' is of the form

(9) $\hat{b}' = \dfrac{\text{cov}(i_t, \pi_{t+1})}{\text{var}(i_t)}$.

The covariance in the numerator of equation (9) is still given by equation (5). Since $i_t = r_t + E_t\,\pi_{t+1}$, we can write the denominator of equation (9) as

(10) $\text{var}(i_t) = \text{var}(r) + \text{var}(E_t\,\pi_{t+1})$,

where we have used the fact that $\text{cov}(r, E_t\,\pi_{t+1}) = 0$. Substituting equations (5) and (10) into equation (9) gives us the following OLS estimator:

(11) $$\hat{b}' = \frac{\text{var}(E_t \pi_{t+1})}{\text{var}(r) + \text{var}(E_t \pi_{t+1})}.$$

The hypothesis that the real interest rate is constant, so that var(r) = 0, predicts a coefficient of one on i_t.

(c) Consider the following regression:

(12) $i_t = a + b_0\,\pi_t + b_1\,\pi_{t-1} + \ldots + b_n\,\pi_{t-n} + \varepsilon_t$.

So, for example, the coefficient b_0 represents the direct impact on i_t of a change in π_t, holding the other π's constant.

Now suppose that the behavior of actual inflation is given by

(13) $\pi_t = \rho\pi_{t-1} + e_t$.

If $i_t = r + E_t\,\pi_{t+1}$, with r constant, changes in expected inflation should cause one-for-one movements in i_t. Thus since $\pi_{t+1} = \rho\pi_t + e_{t+1}$, a change in π_t of $\Delta\pi_t$ will cause $E_t\,\pi_{t+1}$, and thus i_t, to change by $\rho\Delta\pi_t$. So we would expect $b_0 = \rho$ in the above regression.

But now, controlling for π_t, the other variables – π_{t-1}, ..., π_{t-n} – provide no new information about π_{t+1}. Any effect that π_{t-1}, say, has on π_{t+1} is already captured indirectly by π_{t-1}'s impact on π_t. Thus we would expect $b_1 = \ldots = b_n = 0$ in the above regression. Thus the claim is incorrect since we would have $b_0 + b_1 + \ldots + b_n = \rho$, not $b_0 + b_1 + \ldots + b_n = 1$.

Problem 11.5

(a) We have $\pi_t = p_t - p_{t-1}$ and $\pi_t^e = p_t^e - p_{t-1}$. Thus $\pi_t - \pi_t^e = (p_t - p_{t-1}) - (p_t^e - p_{t-1}) = p_t - p_t^e$. We can therefore write the Lucas supply function as

(1) $y_t = y^n + b(p_t - p_t^e)$.

Setting aggregate supply equal to aggregate demand (which is given by $y_t = m_t - p_t$) gives us

(2) $m_t - p_t = y^n + b(p_t - p_t^e)$.

Solving equation (2) for p_t yields

(3) $$p_t = \frac{1}{1+b} m_t + \frac{b}{1+b} p_t^e - \frac{1}{1+b} y^n.$$

With rational expectations, the expected value of both sides of equation (3) must be equal. Hence

(4) $$p_t^e = \frac{1}{1+b}(m_{t-1} + a) + \frac{b}{1+b} p_t^e - \frac{1}{1+b} y^n,$$

where we have used the fact that the expected value of $m_t = m_{t-1} + a + \varepsilon_t$ is equal to $m_{t-1} + a$ since ε is white noise. Subtracting equation (4) from equation (3) yields

(5) $$p_t - p_t^e = \frac{1}{1+b} m_t - \frac{1}{1+b}(m_{t-1} + a) = \frac{1}{1+b}(m_t - m_{t-1} - a).$$

Substituting equation (5) into equation (1) gives us

(6) $$y_t = y^n + \frac{b}{1+b}(m_t - m_{t-1} - a).$$

(b) From equation (6), we can see that we also need to know a, as well as m_t and m_{t-1}, in order to determine the current level of output. Intuitively, equation (6) says that only unexpected money affects output since the difference between m_t and $(m_{t-1} + a)$ is the random shock, ε_t . However, if we don't know a, we cannot determine how much of the change in the nominal money stock from period t - 1 to period t was due to a (and thus was expected) and how much was due to ε (and thus was unexpected).

(c) Again, it must be true that with rational expectations, the expected value of both sides of equation (3) must be equal. However, the expected value of m_t is now $m_{t-1} + \rho(0) + (1 - \rho)a = m_{t-1} + (1 - \rho)a$ since private agents believe that the probability that a = 0 is ρ. Thus

(7) $p_t^e = \frac{1}{1+b}\left[m_{t-1} + (1-\rho)a\right] + \frac{b}{1+b}p_t^e - \frac{1}{1+b}y^n$.

Subtracting equation (7) from equation (3) yields

(8) $p_t - p_t^e = \frac{1}{1+b}\left[m_t - m_{t-1} - (1-\rho)a\right]$.

Substituting equation (8) into equation (1) gives us

(9) $y_t = y^n + \frac{b}{1+b}\left[m_t - m_{t-1} - (1-\rho)a\right]$.

(d) Equation (6) holds in any period in which there is no regime shift. Thus if there is no regime shift in period t - 1, we can write

(10) $y_{t-1} = y^n + \frac{b}{1+b}\left(m_{t-1} - m_{t-2} - a\right)$.

Subtracting equation (10) from equation (6) yields

(11) $y_t - y_{t-1} = \frac{b}{1+b}\left[\left(m_t - m_{t-1}\right) - \left(m_{t-1} - m_{t-2}\right)\right]$.

Defining $\Delta y_t \equiv y_t - y_{t-1}$ and $\Delta m_t \equiv m_t - m_{t-1}$, we have

(12) $\Delta y_t = \frac{b}{1+b}\left[\Delta m_t - \Delta m_{t-1}\right]$.

Equation (12) states that in the absence of regime shifts, output growth is determined by the change in money growth.

If there is a regime shift in period t, equation (9) holds. Subtracting equation (10) from equation (9) yields

(13) $y_t - y_{t-1} = \frac{b}{1+b}\left[\left(m_t - m_{t-1}\right) - \left(m_{t-1} - m_{t-2}\right)\right] + \frac{b}{1+b}\left[a - (1-\rho)a\right]$,

or simply

(14) $\Delta y_t = \frac{\rho ab}{1+b} + \frac{b}{1+b}\left[\Delta m_t - \Delta m_{t-1}\right]$.

Under the null hypothesis of no credibility of the announcement of the regime shift, ρ = 0, the first term on the right-hand side of equation (14) is equal to zero. Thus if the announcement is not believed, equations (14) and (12) are identical. Thus we can run a regression of Δy_t on $[\Delta m_t - \Delta m_{t-1}]$ and a dummy variable that equals one in the period of a regime shift. The coefficient on that dummy variable will reflect the amount of credibility of the policymaker's announcement. In fact, since we will have an estimate of b/(1 + b) and can determine a (the average change in the money stock before the regime shift), we can calculate an estimate of ρ from the coefficient on the dummy variable.

Problem 11.6

(a) (i) The one-period nominal interest rate is given by $i_t^1 = E_t\,\pi_{t+1}$ since the real interest rate is assumed constant at zero. Since $\pi_{t+1} = \Delta m_{t+1}$, we have

(1) $i_t^1 = E_t\,\Delta m_{t+1}$.

Since money growth is given by

(2) $\Delta m_t = k\Delta m_{t-1} + \varepsilon_t$,

and since equation (2) holds in all periods, we can write

(3) $\Delta m_{t+1} = k\Delta m_t + \varepsilon_{t+1}$.

Substituting equation (3) into equation (1), we have

(4) $i_t^1 = E_t\,[k\Delta m_t + \varepsilon_{t+1}\,] = k\Delta m_t$,

where we have used the fact that Δm_t is known as of time t and $E_t\,[\varepsilon_{t+1}\,] = 0$.

(a) (ii) The expectation, as of time t, of the nominal interest rate from period t + 1 to t + 2 is

(5) $E_t\,i_{t+1}^1 = E_t\,\pi_{t+2} = E_t\,\Delta m_{t+2}$.

Since equation (2) holds every period, we can write

(6) $\Delta m_{t+2} = k\Delta m_{t+1} + \varepsilon_{t+2}$.

Substituting equation (3) into equation (6) gives us Δm_{t+2} as a function of Δm_t :

(7) $\Delta m_{t+2} = k^2\,\Delta m_t + k\varepsilon_{t+1} + \varepsilon_{t+2}$.

Substituting equation (7) into equation (5) gives us

(8) $E_t\,i_{t+1}^1 = E_t\,[k^2\,\Delta m_t + k\varepsilon_{t+1} + \varepsilon_{t+2}\,] = k^2\,\Delta m_t$,

where we have used the fact that Δm_t is known at t and the ε's are mean-zero disturbances.

(a) (iii) Under the rational-expectations theory of the term structure, the two-period interest rate is

(9) $i_t^2 = [i^1 + E_t\,i_{t+1}^1\,]/2$.

Substituting equation (8) into equation (9), we have

(10) $i_t^2 = [i_t^1 + k^2\,\Delta m_t\,]/2$.

From equation (4), $k\Delta m_t = i_t^1$ and so equation (10) can be rewritten as

(11) $i_t^2 = [i_t^1 + ki_t^1\,]/2 = i_t^1\,(1 + k)/2$.

(a) (iv) From equation (11), a rise in k will increase the two-period interest rate, i_t^2 , for any given one-period rate. For a given level of inflation in period t, expected inflation for period t + 1 will now be higher. Thus for a given one-period interest rate in t, the one-period rate in t + 1 is expected to be higher. Therefore i_t^2 , which is the average of the one-period rate in t and the expected one-period rate in t + 1, will now be higher for a given i_t^1 .

Note that as k goes to one, so that money growth and thus inflation approach a random walk, the two-period interest rate becomes equal to the one-period interest rate. That is because with inflation a random walk, next period's inflation (and thus next period's one-period nominal rate) is expected to be equal to this period's inflation (and thus this period's one-period nominal rate).

(b) (i) Equation (4) holds in all periods and thus the actual one-period interest rate in t + 1 is

(12) $i_{t+1}^1 = k\Delta m_{t+1}$.

Substituting equation (3) into equation (12) yields

(13) $i_{t+1}^1 = k^2\,\Delta m_t + k\varepsilon_{t+1}$.

Thus

(14) $i_{t+1}^1 - i_t^1 = k^2\Delta m_t + k\varepsilon_{t+1} - k\Delta m_t = k(k - 1)\Delta m_t + k\varepsilon_{t+1}$.

From equation (11), we can write

(15) $i_t^2 - i_t^1 = [i_t^1\,(1 + k)/2] - i_t^1 = [i_t^1\,(1 + k - 2)/2]$,

or substituting in for $i_t^1 = k\Delta m_t$, we have

(16) $i_t^2 - i_t^1 = \dfrac{k(k-1)\Delta m_t}{2}$.

Using the hint in the question, the OLS estimator of b in the following regression:

(17) $i_{t+1}^1 - i_t^1 = a + b[i_t^2 - i_t^1\,] + e_{t+1}$,

is given by

(18) $\hat{b} = \dfrac{\mathrm{cov}[(i_{t+1}^1 - i_t^1),(i_t^2 - i_t^1)]}{\mathrm{var}(i_t^2 - i_t^1)}$.

Using equations (14) and (16), the covariance in the numerator of (18) can be written as

(19) $\mathrm{cov}[(i_{t+1}^1 - i_t^1),(i_t^2 - i_t^1)] = \mathrm{cov}\left[k(k-1)\Delta m_t + k\varepsilon_{t+1}, \dfrac{k(k-1)\Delta m_t}{2}\right]$.

Since ε is white noise and $\mathrm{var}(\Delta m_t) = \sigma_\varepsilon^2$, we have

(20) $\mathrm{cov}[(i_{t+1}^1 - i_t^1),(i_t^2 - i_t^1)] = \dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2$.

Using equation (16), the variance in the denominator of equation (18) can be written as

(21) $\mathrm{var}(i_t^2 - i_t^1) = \dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2$.

Substituting equations (20) and (21) into equation (18) gives us

(22) $\hat{b} = \dfrac{\dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2}{\dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2} = 2$.

(b) (ii) With the time-varying term premium, equation (16) becomes

(23) $i_t^2 - i_t^1 = \dfrac{k(k-1)\Delta m_t}{2} + \theta_t$.

Using equations (14) and (23), the covariance in the numerator of equation (18) is now given by

(24) $\mathrm{cov}[(i_{t+1}^1 - i_t^1),(i_t^2 - i_t^1)] = \mathrm{cov}\left[k(k-1)\Delta m_t + k\varepsilon_{t+1}, \dfrac{k(k-1)\Delta m_t}{2} + \theta_t\right]$.

Since ε and θ are white noise, this is simply

(25) $\mathrm{cov}[(i_{t+1}^1 - i_t^1),(i_t^2 - i_t^1)] = \dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2$.

This covariance is the same as it was without the time-varying term premium. The variance of $(i_t^2 - i_t^1\,)$ will change, however. It is now given by

(26) $\mathrm{var}(i_t^2 - i_t^1) = \dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2 + \sigma_\theta{}^2$,

where we have used the fact that the covariance between ε and θ is zero.

Substituting equations (25) and (26) into equation (18) gives us

$$(27)\quad \hat{b} = \frac{\dfrac{k^2(k-1)^2}{2}\sigma_\varepsilon{}^2}{\left(\dfrac{k^2(k-1)^2}{4}\sigma_\varepsilon{}^2\right) + \sigma_\theta{}^2} = \frac{2}{1 + \left(\dfrac{4\sigma_\theta{}^2}{k^2(k-1)^2}\right)}.$$

(b) (iii) Since $k^2(k-1)^2$ reaches a maximum at k = 1/2, the OLS estimator is highest when k = 1/2. For k > 1/2, an increase in k (more persistent money growth and inflation), reduces the value of the OLS estimator. As k approaches one, so that money growth, inflation and thus the one-period nominal interest rate all approach random walks, the OLS estimator goes to zero.

Problem 11.7

The new aggregate supply equation can be written as

(1) $\pi_t = \pi_{t-1} + \alpha\tilde{y}_{t-1} + \varepsilon_t^\pi$.

To find q*, we need to find $E[(y - y^*)^2] + \lambda E[\pi^2]$ as a function of q. The AS expression applied to period t + 1 implies

(2) $E_t[\pi_{t+1}] = \pi_t + \alpha\tilde{y}_t$,

since $E_t[\varepsilon_{t+1}^\pi] = 0$. Equation (11.20) in the text, $\tilde{y}_t = -\beta r_{t-1} + u_t^{IS} - y_t^n$, implies that

(3) $\tilde{y}_t = E_{t-1}[\tilde{y}_t] + \varepsilon_t^{IS} - \varepsilon_t^Y$.

In addition, the AS expression implies

(4) $\pi_t = E_{t-1}[\pi_t] + \varepsilon_t^\pi$.

Substituting equations (3) and (4) into (2) yields

(5) $E_t[\pi_{t+1}] = E_{t-1}[\pi_t] + \varepsilon_t^\pi + \alpha\left(E_{t-1}[\tilde{y}_t] + \varepsilon_t^{IS} - \varepsilon_t^Y\right)$.

In this version of the model, optimal policy takes the form

(6) $E_t[\tilde{y}_{t+1}] = -qE_t[\pi_{t+1}]$.

Equation (6) applied to period t implies that

(7) $E_{t-1}[\tilde{y}_t] = -qE_{t-1}[\pi_t]$.

Substituting equation (7) into (5) gives us

(8) $E_t[\pi_{t+1}] = E_{t-1}[\pi_t] + \varepsilon_t^\pi + \alpha\left(-qE_{t-1}[\pi_t] + \varepsilon_t^{IS} - \varepsilon_t^Y\right)$,

which simplifies to

(9) $E_t[\pi_{t+1}] = (1 - \alpha q)E_{t-1}[\pi_t] + \varepsilon_t^\pi + \alpha\varepsilon_t^{IS} - \alpha\varepsilon_t^Y$.

The shocks are all uncorrelated with each other and with $E_{t-1}[\pi_t]$. Thus, taking expectations of both sides of the square of (9) yields

(10) $E[(E_t[\pi_{t+1}])^2] = (1 - \alpha q)^2 E[(E_{t-1}[\pi_t])^2] + \sigma_\pi^2 + \alpha^2\sigma_{IS}^2 + \alpha^2\sigma_Y^2$,

where σ_π^2 is the variance of ε^π. In the long run, the expectations of $(E_t[\pi_{t+1}])^2$ and $(E_{t-1}[\pi_t])^2$ are equal and so we can rewrite equation (10) as

$$(11)\quad \begin{aligned} E[(E_{t-1}[\pi_t])^2] &= \frac{\sigma_\pi^2 + \alpha^2\sigma_{IS}^2 + \alpha^2\sigma_Y^2}{1 - (1 - \alpha q)^2} \\ &= \frac{\sigma_\pi^2 + \alpha^2\sigma_{IS}^2 + \alpha^2\sigma_Y^2}{\alpha q(2 - \alpha q)} \end{aligned} .$$

Again, the AS expression implies $\pi_t = E_{t-1}[\pi_t] + \varepsilon_t^\pi$ and squaring both sides gives us

(12) $\pi_t{}^2 = (E_{t-1}[\pi_t])^2 + 2\varepsilon_t^{\pi} E_{t-1}[\pi_t] + (\varepsilon_t^{\pi})^2$.

Taking expectations of (12) then gives us

(13) $E[\pi^2] = E[(E_{t-1}[\pi_t])^2] + \sigma_{\pi}^2$.

Substituting equation (11) into equation (13) leaves us with

(14) $E[\pi^2] = \dfrac{\sigma_{\pi}^2 + \alpha^2\sigma_{IS}^2 + \alpha^2\sigma_{Y}^2}{\alpha q(2 - \alpha q)} + \sigma_{\pi}^2$.

Having found $E[\pi^2]$ in terms of q, we turn our attention to finding $E[(y - y^*)^2]$. First, note that $y - y^*$ equals $(y - y^n) - (y^* - y^n)$. This, in turn, equals $\tilde{y} - \Delta$ since $(y - y^n) \equiv \tilde{y}$ and $(y^* - y^n) = \Delta$ [see equation (11.19)]. From equation (3), we have $\tilde{y}_t = E_{t-1}[\tilde{y}_t] + \varepsilon_t^{IS} - \varepsilon_t^{Y}$, and from equation (7), $E_{t-1}[\tilde{y}_t] = -qE_{t-1}[\pi_t]$. Finally, as explained in the text, the mean of $\tilde{y}$ must be 0 in order for inflation to be bounded. Thus, we can write

(15) $E[(y - y^*)^2] = q^2 E[(E_{t-1}[\pi_t])^2] + \sigma_{IS}^2 + \sigma_{Y}^2 + \Delta^2$.

Substituting equation (11) into (15) yields

(16) $E[(y - y^*)^2] = \dfrac{q[\sigma_{\pi}^2 + \alpha^2\sigma_{IS}^2 + \alpha^2\sigma_{Y}^2]}{2\alpha - \alpha^2 q} + \sigma_{IS}^2 + \sigma_{Y}^2 + \Delta^2$.

The central bank will choose q to minimize $E[(y - y^*)^2] + \lambda E[\pi^2]$. The first-order condition is given by

(17) $\dfrac{\partial E[(y - y^*)^2]}{\partial q} + \lambda \dfrac{\partial E[\pi^2]}{\partial q} = 0$,

or simply

(18) $\dfrac{\partial E[\pi^2]/\partial q}{\partial E[(y - y^*)^2]/\partial q} = -\dfrac{1}{\lambda}$.

We must now solve for the derivatives in equation (18). We can rewrite equation (14) as

(19) $E[\pi^2] = [\alpha q(2 - \alpha q)]^{-1} K + \sigma_{\pi}^2$,

where we have defined $K \equiv \sigma_{\pi}^2 + \alpha^2\sigma_{IS}^2 + \alpha^2\sigma_{Y}^2$. Taking the derivative with respect to q gives us

(20) $\dfrac{\partial E[\pi^2]}{\partial q} = -[\alpha q(2 - \alpha q)]^{-2}(2\alpha - 2\alpha^2 q)K$,

or simply

(21) $\dfrac{\partial E[\pi^2]}{\partial q} = \dfrac{-[2\alpha(1 - \alpha q)]K}{[\alpha q(2 - \alpha q)]^2}$.

We can rewrite equation (16) as

(22) $E[(y - y^*)^2] = q(2\alpha - \alpha^2 q)^{-1} K + \sigma_{IS}^2 + \sigma_{Y}^2 + \Delta^2$.

The derivative with respect to q is given by

(23) $\dfrac{\partial E[(y - y^*)^2]}{\partial q} = [(2\alpha - \alpha^2 q)^{-1} + q(-1)(2\alpha - \alpha^2 q)^{-2}(-\alpha^2)]K$,

which we can rewrite as

(24) $\dfrac{\partial E[(y-y^*)^2]}{\partial q} = \left[\dfrac{1}{2\alpha - \alpha^2 q} + \dfrac{\alpha^2 q}{(2\alpha - \alpha^2 q)^2}\right] K$.

Equation (24) simplifies to

(25) $\dfrac{\partial E[(y-y^*)^2]}{\partial q} = \dfrac{2\alpha K}{(2\alpha - \alpha^2 q)^2}$.

Substituting equations (21) and (25) into the first-order condition given by (18) yields

(26) $\dfrac{-[2\alpha(1-\alpha q)]K}{[\alpha q(2-\alpha q)]^2}\dfrac{(2\alpha - \alpha^2 q)^2}{2\alpha K} = -\dfrac{1}{\lambda}$.

Noting that $(2\alpha - \alpha^2 q)^2$ can be written as $\alpha^2(2 - \alpha q)^2$ and canceling terms leaves us with

(27) $\dfrac{1-\alpha q}{q^2} = \dfrac{1}{\lambda}$.

Thus, we have the following quadratic equation:

(28) $q^2 + \lambda\alpha q - \lambda = 0$.

Solving (28) using the quadratic formula yields

(29) $q^* = \dfrac{-\lambda\alpha + \sqrt{\alpha^2\lambda^2 + 4\lambda}}{2}$.

We find that q* is unchanged from the one described by (11.27). Note that once again, we can ignore the negative root since a negative q* causes the variances of y and π to be infinite.

Problem 11.8

(a) When $\varphi_\pi = 1$, the matrix A simplifies to

(1) $A = \begin{bmatrix} 1 & 0 \\ \kappa & \beta \end{bmatrix}$.

The eigenvalues of this matrix are given by

(2) $\begin{vmatrix} 1-t & 0 \\ \kappa & \beta - t \end{vmatrix} = (1-t)(\beta - t) = 0$

The characteristic equation implies that the eigenvalues are β and 1.

To analyze the self-fulfilling movements, we first make the observations that

(3) $E_t\tilde{y}_{t+1} = \lambda\tilde{y}_t$ and $E_t\pi_{t+1} = \lambda\pi_t$.

Then, we substitute this information into equation (11.41) to obtain

(4) $\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \kappa & \beta \end{bmatrix} \begin{bmatrix} \lambda\tilde{y}_t \\ \lambda\pi_t \end{bmatrix} = \begin{bmatrix} \lambda\tilde{y}_t \\ \kappa\lambda\tilde{y}_t + \beta\lambda\pi_t \end{bmatrix}$.

From (4), we have the equality:

(5) $\tilde{y}_t = \lambda\tilde{y}_t$.

There are two possibilities to satisfy (5)—either $\tilde{y}_t = 0$ or λ = 1. Now, we analyze the second equality that we get from (4), which is

(6) $\pi_t = \kappa\lambda\tilde{y}_t + \beta\lambda\pi_t$.

If we consider the first possibility by substituting $\tilde{y}_t = 0$ into (6), we find $\lambda = 1/\beta$. However, $\lambda = 1/\beta$ implies $\lambda > 1$ which contradicts the assumption of the problem. Therefore, we eliminate the possibility that $\tilde{y}_t = 0$ and conclude $\lambda = 1$. We can now substitute $\lambda = 1, \pi_t = \lambda^t Z$, and $\tilde{y}_t = c\lambda^t Z$ into (6) to get

(7) $Z = \kappa c Z + \beta Z$.

Equation (7) implies that

(8) $c = (1 - \beta)/\kappa$.

Therefore, we can conclude that the self-fulfilling movements of $\tilde{y}$ and π result in both variables staying at the constant values Z and $[(1 - \beta)/\kappa]Z$, respectively.

(b) In part (a), we found that when $\varphi_\pi = 1$, one of the eigenvalues is 1 and the other is strictly inside the unit circle. According to equation (11.42), the eigenvalues are continuous functions of φ_π. Consequently, we will examine the derivative of γ with respect to φ, evaluated at $\varphi_\pi = 1$, using equation (11.42). This will allow us to determine whether the eigenvalues will increase or decrease for values of φ_π that are infinitesimally larger or smaller than 1. We will look only at the value of γ given by

$$(9)\quad \gamma = \frac{1+\beta+\alpha+\sqrt{(1+\beta+\alpha)^2 - 4\beta}}{2},$$

where $\alpha \equiv \kappa(1-\phi_\pi)/\theta$, since this implies $\gamma = 1$ when $\varphi_\pi = 1$. Now, we take the derivative of γ with respect to φ_π to yield

$$(10)\quad \frac{\partial \gamma}{\partial \phi_\pi} = \frac{1}{2}\frac{\partial \alpha}{\partial \phi_\pi} + \frac{1}{4}[(1+\beta+\alpha)^2 - 4\beta]^{-1/2} \cdot 2(1+\beta+\alpha)\frac{\partial \alpha}{\partial \phi_\pi},$$

where

$$(11)\quad \frac{\partial \alpha}{\partial \phi_\pi} = \frac{-\kappa}{\theta}.$$

Now we substitute $\varphi_\pi = 1$ in order to determine the sign of the derivative. Since $\alpha = 0$ when $\varphi_\pi = 1$, we have

$$(12)\quad \frac{\partial \gamma}{\partial \phi_\pi} = -\frac{\kappa}{2\theta} - \frac{1}{2}[(1+\beta)^2 - 4\beta]^{-1/2}(1+\beta)\frac{\kappa}{\theta}.$$

Since $(1 + \beta)^2 - 4\beta$ equals $(1 - \beta)^2$, the derivative is negative when $\varphi_\pi = 1$. Consequently, for values infinitesimally larger than 1, the root is inside the unit circle, and for values infinitesimally smaller than 1, the root is outside the unit circle.

(c) Using the given information, the matrix A simplifies to

$$(13)\quad A = \begin{bmatrix} 1 & -[2(1+\beta)]/\kappa \\ \kappa & \beta - 2(1+\beta) \end{bmatrix}.$$

To find the eigenvalues, first find the characteristic equation, which is given by

$$(14)\quad \begin{vmatrix} 1-t & -[2(1+\beta)]/\kappa \\ \kappa & \beta - 2(1+\beta) - t \end{vmatrix} = t^2 + t(1+\beta) + \beta.$$

Next, use the the quadratic formula to find the roots of the polynomial equation:

$$(15)\quad t = \frac{-(1+\beta) \pm \sqrt{(1+\beta)^2 - 4\beta}}{2},$$

which simplifies to

$$(16)\quad t = \frac{-(1+\beta) \pm (1-\beta)}{2}.$$

We conclude that the eigenvalues are -1 and $-\beta$.

We find the values of λ and c that satisfy (11.41) the same way we found them in (a). First we substitute our known values into (11.41) to obtain

(17) $\begin{bmatrix}\tilde{y}_t\\ \pi_t\end{bmatrix}=\begin{bmatrix}c\lambda^t Z\\ \lambda^t Z\end{bmatrix}=\begin{bmatrix}1 & -[2(1+\beta)]/\kappa\\ \kappa & \beta-2(1+\beta)\end{bmatrix}\begin{bmatrix}c\lambda^{t+1}Z\\ \lambda^{t+1}Z\end{bmatrix}=\begin{bmatrix}c\lambda^{t+1}Z-[2\lambda^{t+1}Z(1+\beta)]/\kappa\\ \kappa c\lambda^{t+1}Z+\beta\lambda^{t+1}Z-2(1+\beta)\lambda^{t+1}Z\end{bmatrix}.$

Using the equation described by the second row of the system described by (17), we can solve for c to get an expression in terms of λ, which is

(18) $c=\dfrac{1+(2+\beta)\lambda}{k\lambda}.$

Substituting (18) into the equation described by the first row of (17), we can simplify to obtain

(19) $0 = \beta\lambda^2 + (1 + \beta)\lambda + 1.$

Then, using the quadratic formula to solve for λ gives us

(20) $\lambda=\dfrac{-(1+\beta)\pm\sqrt{(1+\beta)^2-4\beta}}{2\beta},$

which simplifies to

(21) $\lambda=\dfrac{-(1+\beta)\pm(1-\beta)}{2\beta}.$

Thus, the possible solutions are $\lambda = -(1/\beta)$ or $\lambda = -1$. Since $-(1/\beta) < -1$, this solution violates the condition that $|\lambda|\le 1$, so we will eliminate it. Thus, $\lambda = -1$ is our solution. We then substitute this value of λ into equation (15) to obtain

(22) $c=\dfrac{1+\beta}{\kappa}.$

We can now conclude that the self-fulfilling movements take the form:

(23) $\pi_t=(-1)^t Z,$

and

(24) $\tilde{y}_t=\left(\dfrac{1+\beta}{\kappa}\right)(-1)^t Z.$

Thus, π and $\tilde{y}$ oscillate between the values $\pm Z$ and $\pm[(1+\beta)/\kappa]Z$, respectively.

Problem 11.9

(a) When the policymaker fixes i, the money-market equilibrium condition is irrelevant. Equilibrium output is determined by the IS curve and the fixed nominal interest rate, $\bar{i}$. Substituting $\bar{i}$ into the IS curve yields

(1) $y = c - a\bar{i} + \varepsilon_1.$

The variance of y is simply

(2) $\text{var}(y) = \text{var}(\varepsilon_1) = \sigma_1^2.$

(b) When the policymaker fixes m, the equilibrium level of output is determined by the intersection of the IS and money-market equilibrium equations. Rearranging the IS curve to solve for i gives us

(3) $i = (c + \varepsilon_1 - y)/a.$

Substituting equation (3) and the assumption that $m = \bar{m}$ into the money-market equilibrium equation, $m - p = hy - ki + \varepsilon_2$, gives us

(4) $\bar{m} - p = hy - [k(c + \varepsilon_1 - y)/a] + \varepsilon_2 = [h + (k/a)]y - (kc/a) - (k/a)\varepsilon_1 + \varepsilon_2.$

Solving for y yields

(5) $y = \dfrac{\overline{m} - p + (kc/a) + (k/a)\varepsilon_1 + \varepsilon_2}{h + (k/a)} = \dfrac{a(\overline{m} - p) + kc + k\varepsilon_1 + a\varepsilon_2}{ah + k}$.

The variance of y is

(6) $\text{var}(y) = \left(\dfrac{k}{ah + k}\right)^2 \sigma_1{}^2 + \left(\dfrac{a}{ah + k}\right)^2 \sigma_2{}^2$.

(c) If $\sigma_1{}^2 = 0$ – if there are only monetary shocks – then from equations (2) and (6):

(7) $\text{var}(y)\big|_{i=\bar{i}} = 0$,

and

(8) $\text{var}(y)\big|_{m=\overline{m}} = \left(\dfrac{a}{ah + k}\right)^2 \sigma_2{}^2 > 0$.

Thus interest-rate targeting leads to a lower variance of output than money-stock targeting. In fact, output is constant under interest targeting.

(d) If $\sigma_2{}^2 = 0$ – if there are only IS shocks – then from equations (2) and (6):

(9) $\text{var}(y)\big|_{i=\bar{i}} = \sigma_1{}^2$,

and

(10) $\text{var}(y)\big|_{m=\overline{m}} = \left(\dfrac{k}{ah + k}\right)^2 \sigma_1{}^2 < \sigma_1{}^2$.

Thus money-stock targeting leads to a lower variance of output than interest-rate targeting.

(e) Consider the situation in part (c) in which there are only monetary shocks. If the policymaker targets the nominal interest rate, it ensures that i remains constant in the face of any monetary shock. Since i is not allowed to change, planned expenditure does not change and thus the level of output that equates planned and actual expenditure does not change. If the policymaker targets the nominal money stock, the monetary shocks do require a change in the interest rate to restore money-market equilibrium. The change in the interest rate then changes planned expenditure and thus the level of output that equates planned and actual expenditure.

Consider the situation in part (d) in which there are only IS shocks. A positive IS shock shifts the IS curve to the right. If the policymaker keeps m fixed, then as Y rises to equate planned and actual expenditure, i rises as well in order for the money market to remain in equilibrium. This rise in i reduces planned expenditure and thus mitigates some of the positive shock. If, instead, the policymaker targets the nominal interest rate, equilibrium output changes by the full extent of the shock to the IS curve.

(f) If there are only IS shocks, it is possible to keep y constant at some target level $\hat{y}$. By rearranging the money-market equilibrium equation with y set to $\hat{y}$, the nominal money supply must be such that

(11) $m = p + h\hat{y} - ki$.

The policymaker knows the fixed p, has picked $\hat{y}$ herself and can observe i. Thus when i changes – and since there are only IS shocks, we know this must be due to a shift of the IS curve – the policymaker must change m accordingly. As i rises, for example, the policymaker must reduce m. The policymaker can stop reducing m when m and i are such that equation (11) is satisfied.

Problem 11.10

(a) Using the fact that for a random variable X, $var(X) = E[X^2] - (E[X])^2$ or $E[X^2] = var(X) + (E[X])^2$, we have

(1) $E[(y - y^*)^2] = var(y - y^*) + (E[y - y^*])^2$.

Substituting the expression for output, $y = x + (k + \varepsilon_k)z + u$, into $var(y - y^*)$ and simplifying yields

(2) $var(y - y^*) = var(x + kz + \varepsilon_k z + u - y^*) = z^2 \sigma_k^2 + \sigma_u^2$.

Substituting for output in $(E[y - y^*])^2$ and simplifying yields

(3) $(E[y - y^*])^2 = (E[x + kz + \varepsilon_k z + u - y^*])^2 = (x + kz - y^*)^2$,

where we have used the fact that ε_k and u both have mean zero. Substituting equations (2) and (3) into equation (1) gives us

(4) $E[(y - y^*)^2] = z^2 \sigma_k^2 + \sigma_u^2 + (x + kz - y^*)^2$.

(b) The policymaker chooses z in order to minimize $E[(y - y^*)^2]$. Using equation (4), the first-order condition is

(5) $\partial(E[(y - y^*)^2])/\partial z = 2z\sigma_k^2 + 2k(x + kz - y^*) = 0$.

Collecting the terms in z yields

(6) $z(\sigma_k^2 + k^2) = (y^* - x)k$,

and thus the optimal choice of z is

(7) $$z = \frac{(y^* - x)k}{\sigma_k^2 + k^2}.$$

(c) To see the way in which policy should respond to shocks (i.e. changes in x), to take the derivative of z, as given by equation (7), with respect to x:

(8) $$\frac{\partial z}{\partial x} = -\frac{k}{\sigma_k^2 + k^2} < 0.$$

The fact that the derivative in (8) is negative implies that higher values of x should be offset with lower values of z in order to keep output from varying as much.

Note that $\partial z/\partial x$ does not depend on σ_u^2, the parameter that represents uncertainty about the state of the economy. Thus in this model, the optimal degree of "fine-tuning" does not depend on the amount of uncertainty about the state of the economy.

(d) In contrast, $\partial z/\partial x$ does depend on σ_k^2, the parameter that represents uncertainty about the effects of the policy instrument. In fact, we have

(9) $$\frac{\partial[\partial z/\partial x]}{\partial \sigma_k^2} = \frac{k}{(\sigma_k^2 + k^2)^2} > 0.$$

Since $\partial z/\partial x$ is negative to begin with, a rise in σ_k^2 makes it less negative. That is, higher values of σ_k^2 – more uncertainty about the effects of the policy instrument – reduces the amount that policy should respond to shocks or in other words, reduces the amount of "fine-tuning" that should be done.

Problem 11.11

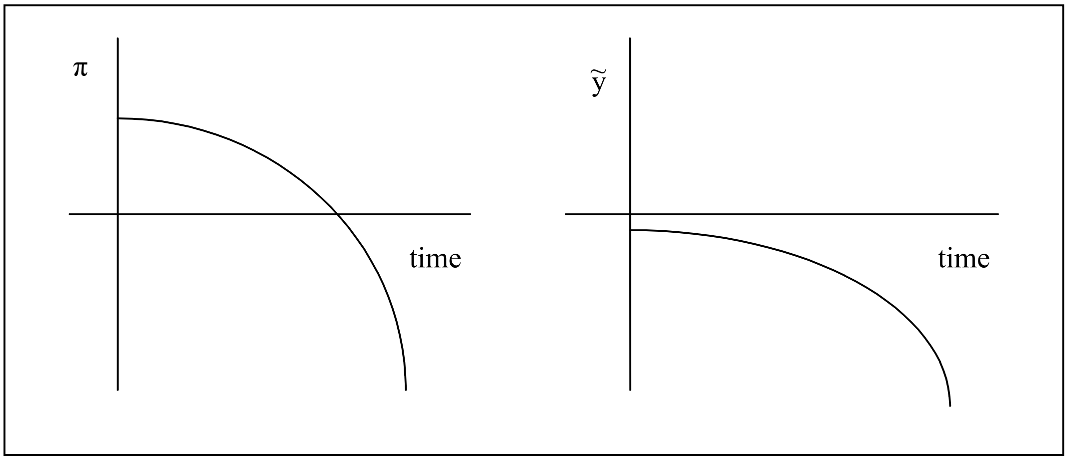
(a) From the assumption of the problem, we first conclude

(1) $\tilde{y}(0) = -b[i(0) - \pi(0)] < 0$.

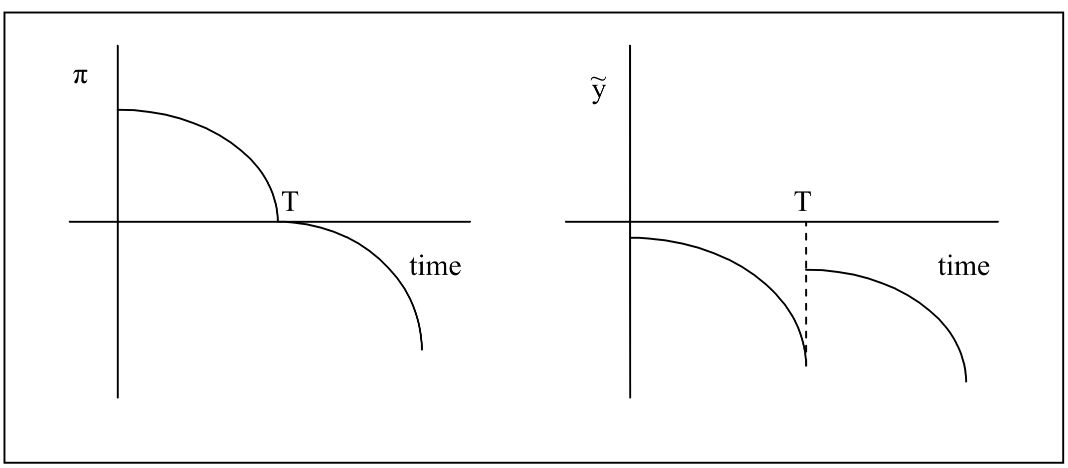
The inequality expressed by (1) also implies

(2) $\dot{\pi}(0) = \lambda\tilde{y}(0) < 0$,

since λ is positive. Therefore, π is decreasing at time zero. We also know that $\pi(0)$ is a positive quantity since we are told to assume
(3) $0 < b\pi(0), \quad b > 0.$
Inflation decreases over time since its time derivative is negative. Because inflation decreases, output also decreases since
(4) $\tilde{y}(t) = -b[i(t) - \pi(t)] = b\pi(t) - i(0)$.
As $\tilde{y}(t)$ decreases, $\dot{\pi}(t)$ decreases causing $\pi(t)$ to decrease at a faster rate. Because the interest rate is constant, this cycle continues indefinitely, causing $\tilde{y}(t)$ and $\pi(t)$ to diverge to $-\infty$ as t goes to ∞, as reflected in the following figures.



(b) Our figures will look identical to those in part (a) until $\pi(t)$ hits 0, which occurs at a time we will call time T. At time T, $\tilde{y}(t)$ will immediately increase to a less negative value since
(5) $\tilde{y}(T) = -b[i(T) - \pi(T)] = b\pi(T) > b\pi(T) - i(0)$.
This implies that the derivative of inflation also increases, since
(6) $\dot{\pi}(T) = \lambda\tilde{y}(T)$.
But since the time derivative is still negative, $\pi(t)$ continues to decrease after time T, causing $\tilde{y}(t)$ to also decrease after time T. This behavior is illustrated in the figures below.



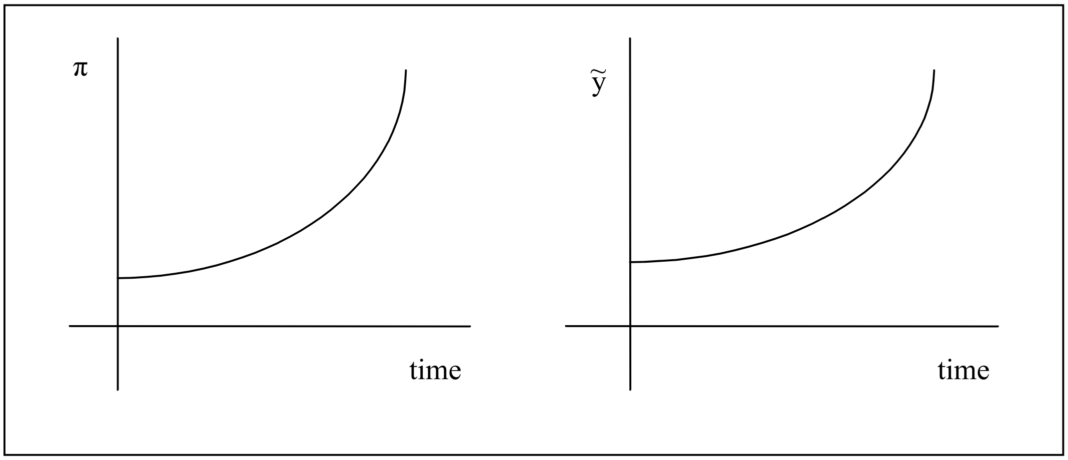
(c) Since i(0) = 0, we conclude

(7) $\tilde{y}(0) = -b[i(0) - \pi(0)] = b\pi(0) > 0$.

Since $\tilde{y}(0) > 0$, we also deduce

(8) $\dot{\pi}(t) = \lambda\tilde{y}(t) > 0$,

since $\lambda > 0$. Thus, $\pi(t)$ initially increases since its time derivative is positive. As inflation increases, $\tilde{y}(t)$ also increases, causing $\dot{\pi}(t)$ to increase even more. This cycle continues and causes both $\tilde{y}(t)$ and $\pi(t)$ to diverge to ∞ as t goes to ∞, as reflected in the figures below.



(d) According to this model, waiting to reduce the interest rate until the economy experiences deflation can result in a brief bump in output followed by a relapse back into recession. On the other hand, acting quickly can immediately boost output and avoid a recession, while also causing inflation in the near future. Having successfully avoided the deflationary collapse, the central bank presumably could raise i at some point to prevent inflation from exploding.

Problem 11.12

As described in the text, in equilibrium, output equals y^n and inflation equals $\pi^* + (b/a)(y^* - y^n)$. Substituting these values into the loss function given by equation (11.54) in the text, which is given by $L = (1/2)(y - y^*)^2 + (1/2)a(\pi - \pi^*)^2$, yields the following value of the loss function in equilibrium:

(1) $$L^{EQ} = \frac{1}{2}\left(y^n - y^*\right)^2 + \frac{1}{2}a\left[\frac{b}{a}\left(y^* - y^n\right)\right]^2 = \frac{1}{2}\left(y^* - y^n\right)^2 + \frac{1}{2}\frac{b^2}{a}\left(y^* - y^n\right)^2,$$

or simply

(2) $$L^{EQ} = \frac{1}{2}\left(y^* - y^n\right)^2\left[1 + \frac{b^2}{a}\right].$$

Output equals y^n in equilibrium, regardless of the value of a. Thus to see how the equilibrium loss varies with a, use equation (2) to take the derivative of L^{EQ} with respect to a:

(3) $$\frac{\partial L^{EQ}}{\partial a} = \frac{-b^2}{2a^2}\left(y^* - y^n\right)^2 < 0.$$

Equation (3) states that a fall in a increases L^{EQ}. That is, a reduction in the cost of inflation increases the loss to society. It is true that any given deviation in inflation from its optimal level, π^*, has a lower cost to society. However, the problem is that this causes the equilibrium level of inflation itself to be higher. Intuitively, at a given π^e, the marginal cost of additional inflation is now lower for the policymaker. For a given π^e, it then becomes optimal to set a higher inflation rate. But the public knows this and thus the level of π for which $\pi^e = \pi$ is now higher. It turns out that the fact that π^{EQ} exceeds π^* by more than it used to, outweighs the fact that any given deviation in π^{EQ} from π^* has a lower cost to society.

Problem 11.13

(a) Let S be the amount of social welfare from a given policy. Thus, we have

(1) $S = \left(y_1 - \frac{a}{2}\pi_1^2\right) + \left(y_2 - \frac{a}{2}\pi_2^2\right)$.

Substituting equation (11.53), we get

(2) $S = (y^n + b\pi_1 - \frac{a}{2}\pi_1^2 - b\pi_1^e) + (y^n + b\pi_2 - \frac{a}{2}\pi_2^2 - b\pi_2^e)$.

Taking the derivative of (2) with respect to π_2 and setting the result equal to zero gives the solution $\pi_2 = b/a$.

(b) Since the type 1 policymaker never chooses $\pi_1 = 0$, there is no doubt in the second period that the policymaker is of type 1, since a type 2 policymaker would have picked $\pi_1 = 0$. Therefore, people will expect the policymaker to maximize social welfare in the second period and so $\pi_2^e = b/a$.

Taking the derivative of (2) with respect to π_1 and setting the result equal to zero, we find that the policymaker selects $\pi_1 = b/a$ in order to maximize social welfare. Setting $\pi_1 = \pi_2 = \pi_2^e = b/a$ in equation (2), we get

(3) $S = 2y^n - b\pi_1^e$.

(c) Since the public expects the policymaker to select $\pi_2 = b/a$ with probability p and $\pi_2 = 0$ with probability $(1 - p)$, we have

(4) $\pi_2^e = p(b/a) + (1-p)0 = p(b/a)$.

Therefore, substituting $\pi_1 = 0$, $\pi_2 = b/a$, and $\pi_2^e = p(b/a)$ into equation (2), we get

(5) $S = 2y^n - b\pi_1^e + \frac{b^2}{a}\left(\frac{1}{2} - p\right)$.

(d) Since $0 < p < 1/2$, the value of S given by equation (5) is larger than the value of S given by equation (3) because the term $(b^2/a)[(1/2) - p]$ is positive. Thus, a type 1 policymaker would select $\pi_1 = 0$. Notice also that as p gets smaller, S gets larger. This implies that a strong reputation as a policymaker who is tough on inflation can allow a type 1 policymaker to achieve higher social welfare by first selecting $\pi_1 = 0$.

(e) Now consider the case in which $1/2 < p < 1$. If the public believes the type 1 policymaker will pick $\pi_1 = 0$, then $\pi_2^e = p(b/a)$ as in part (c). But $[(1/2) - p]$ is now negative and consequently the term $(b^2/a)[(1/2) - p]$ reduces the value of S in equation (5) relative to its value in equation (3). Thus, if the public believes that a type 1 policymaker will pick $\pi_1 = 0$, he or she will choose $\pi_1 = b/a$.

Suppose instead the public believes the type 1 policymaker will pick $\pi_1 = b/a$. If he or she picked $\pi_1 = b/a$, the public would know that the policymaker was of type 1 and would adjust their expectation so that $\pi_2^e = (b/a)$. Substituting $\pi_1 = \pi_2 = \pi_2^e = b/a$ into equation (2) gives us

(6) $S = 2y^n - b\pi_1^e$.

If the policymaker chose $\pi_1 = 0$, the public would believe for certain that she is a type 2 policymaker (because the public believes that type 1 policymakers always choose $\pi_1 = b/a$), and so $\pi_2^e = 0$. Choosing $\pi_2 = b/a$ would then achieve a level of social welfare given by

(7) $S = 2y^n - b\pi_1^e + \dfrac{b^2}{2a}$.

Since (7) yields a higher level of social welfare than (6), the policymaker of type 1 will choose $\pi_1 = 0$ if the public believes that a type 1 policymaker will pick $\pi_1 = b/a$.

Therefore, neither the public always believing a type 1 policymaker will pick $\pi_1 = 0$ nor the public always believing a type 1 policymaker will pick $\pi_1 = b/a$ can be an equilibrium. The equilibrium will instead consist of the type 1 policymaker playing a mixed strategy for period 1 in which he or she selects $\pi_1 = 0$ with some probability and $\pi_1 = b/a$ with some probability.

Problem 11.14

(a) The policymaker chooses inflation in order to maximize her objective function, which is given by $W = c\gamma y - (a\pi^2/2)$, subject to the constraint that output is given by the Lucas Supply function, $y = \overline{y} + b(\pi - \pi^e)$. Thus the policymaker's problem is

(1) $\max\limits_{\pi} \; W = c\gamma[y^n + b(\pi - \pi^e)] - (a\pi^2/2)$.

The first-order condition is

(2) $\partial W/\partial\pi = bc\gamma - a\pi = 0$.

Thus the policymaker's choice of π is

(3) $\pi = bc\gamma/a$.

(b) The public knows the policymaker sets inflation according to equation (3). Thus with rational expectations, expected inflation must equal the expectation of the right-hand side of equation (3):

(4) $\pi^e = E[bc\gamma/a] = bcE[\gamma]/a = bc\,y^n/a$.

(c) The true social welfare function is given by $W^{SOC} = \gamma y - (a\pi^2/2)$. Taking the expectation of both sides of this expression with respect to the public's information set, so that γ is random, gives us

(5) $E[W^{SOC}] = E[\gamma(y^n + b(\pi - \pi^e)) - (a\pi^2/2)]$,

where we have substituted for $y = y^n + b(\pi - \pi^e)$. Now substitute the policymaker's choice of π, equation (3), and the public's expectation of inflation, equation (4), into equation (5):

(6) $E[W^{SOC}] = E\left[\gamma\left[y^n + b\left(\dfrac{bc\gamma}{a} - \dfrac{bc\overline{\gamma}}{a}\right)\right] - \dfrac{ab^2c^2\gamma^2}{2a^2}\right]$.

Simplifying yields

(7) $E[W^{SOC}] = y^n E[\gamma] + \dfrac{b^2cE[\gamma^2]}{a} - \dfrac{b^2c\overline{\gamma}E[\gamma]}{a} - \dfrac{b^2c^2E[\gamma^2]}{2a}$.

Since $E[\gamma] = \overline{\gamma}$, equation (7) becomes

(8) $E[W^{SOC}] = y^n\overline{\gamma} + \dfrac{b^2c}{a}\left[E[\gamma^2] - \overline{\gamma}^2\right] - \dfrac{b^2c^2E[\gamma^2]}{2a}$.

Now use the facts that for a random variable X:

(9) $\mathrm{var}(X) = E[X^2] - (E[X])^2$,

and

(10) $E[X^2] = \text{var}(X) + (E[X])^2$.
Here, this means that we can write
(11) $\sigma_\gamma^2 = E[\gamma^2] - \bar{\gamma}^2$,
and
(12) $E[\gamma^2] = \sigma_\gamma^2 + \bar{\gamma}^2$.
Substituting equations (11) and (12) into equation (8) gives us the following expected value of the true social welfare function:

(13) $$E[W^{SOC}] = y^n\bar{\gamma} + \frac{b^2c}{a}\sigma_\gamma^2 - \frac{b^2c^2}{2a}\left(\sigma_\gamma^2 + \bar{\gamma}^2\right).$$

(d) To find the first-order condition for the maximization, use equation (13) to set the derivative of the expected value of the social welfare function with respect to c equal to zero:

(14) $$\frac{\partial E[W^{SOC}]}{\partial c} = \frac{b^2}{a}\sigma_\gamma^2 - \frac{b^2c}{a}\left(\sigma_\gamma^2 + \bar{\gamma}^2\right) = 0.$$

Solving for c yields

(15) $$c = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \bar{\gamma}^2}.$$

There is a tradeoff here. From equation (3), we can see that choosing a more "conservative" policymaker—one with a low value of c—produces a better performance in terms of average inflation. Such a policymaker would not respond well to the shocks, however. Thus there is some optimal level of "conservatism" that balances these two forces.

The value of c that maximizes the expected value of true social welfare is decreasing in the mean of γ. Since we know that π^e equals π on average (since γ equals $\bar{\gamma}$ on average), output equals full-employment output on average, regardless of the values of c or $\bar{\gamma}$. From equation (3), we can see that if γ is higher on average, inflation will also be higher on average, for a given c. Thus it will be welfare-improving to offset this higher average γ and keep inflation lower on average by having a policymaker with a lower c; that is, having a more "conservative" policymaker.

However, the value of c that maximizes expected social welfare is increasing in the variance of the γ shock. The more variable is the shock, the less "conservative" the central banker should be. Since the policymaker can act after γ is realized, she can choose to offset any deviation in γ from its expected value, which raises welfare. The policymaker will do this only to the extent that she cares about the shock's effect. Thus the more that γ varies, the better it is to have a policymaker who cares about the shock's effect and will act to offset it.

Problem 11.15
(a) Social welfare is higher when the policymaker turns out to be a Type-1, the type that shares the public's preferences concerning output and inflation. The choice of setting $\pi = 0$ in both periods – as the Type-2 policymaker does – is a choice available to the Type-1 policymaker. She chooses not to do this; in order to maximize social welfare, she decides to choose another pair of inflation rates. Since she is attempting to maximize social welfare, welfare must be higher under the choices made by the Type-1 policymaker. For example, as explained in the text, if $\beta < 1/2$, it is optimal for the Type-1 policymaker to choose $\pi_1 = b/a$ and $\pi_2 = b/a$. That must be because it achieves higher welfare than choosing $\pi_1 = 0$, $\pi_2 = 0$.

(b) Expected inflation, π^e, is determined by the public's beliefs. So both the " a' " policymaker and the "a" policymaker face the same π^e, since in either case, the public believes it is facing an " a' " policymaker. Thus both policymakers have the same choice set. The "a" policymaker makes her choice to maximize true social welfare, whereas the " a' " policymaker makes her choice to maximize something else. Thus social welfare must be higher with the "a" policymaker.

Problem 11.16

(a) Suppose that π differs from $\hat{\pi}$ in some period t_0. Then $\pi^e = b/a$ for all periods after t_0. Substituting this expression for expected inflation into the Lucas supply function, $y_t = y^n + b(\pi_t - \pi_t^e)$, gives us output in each subsequent period:

(1) $y_t = y^n + b(\pi_t - b/a)$ for all $t > t_0$.

With expected inflation now constant into the future, the equilibrium in each period is independent of the policymaker's action in the previous period. Thus we can concentrate on a representative period, t; the equilibrium in all periods will be the same. Substituting equation (1) into the policymaker's objective function for period t, $w_t = y_t - (a\pi^2/2)$, yields

(2) $w_t = y^n + b(\pi_t - b/a) - a\pi_t^2/2$ for all $t > t_0$.

The first-order condition for the choice of inflation is

(3) $\partial w_t/\partial \pi_t = b - a\pi_t = 0$,

and thus the policymaker chooses

(4) $\pi_t = b/a$ for all $t > t_0$.

Since $\pi_t = \pi_t^e = b/a$, then from the Lucas supply function we have

(5) $y_t = y^n$ for all $t > t_0$.

(b) To keep things simple, we can assume that the monetary authority chooses to depart from $\pi = \hat{\pi}$ in period 0. This does not alter the message. Since π has always been equal to $\hat{\pi}$, $\pi_0^e = \hat{\pi}$. Substituting this into the Lucas supply function gives us

(6) $y_0 = y^n + b(\pi_0 - b/a)$.

Given the fact that the policymaker is choosing to depart from $\pi = \hat{\pi}$, the particular choice of π_0 does not affect π^e and thus the equilibrium in future periods. Thus only the current period's objective function matters to the policymaker. She will choose π_0 to maximize

(7) $w_0 = y^n + b(\pi_0 - \hat{\pi}) - (a\pi_0^2/2)$.

The first-order condition for the choice of π_0 is

(8) $\partial w_0/\partial \pi_0 = b - a\pi_0 = 0$,

and thus the policymaker chooses

(9) $\pi_0 = b/a$.

With this choice of inflation, using the Lucas supply function, output in period 0 is given by

(10) $y_0 = y^n + b[(b/a) - \hat{\pi}]$.

Substituting equations (9) and (10) into the policymaker's objective function, $w_0 = y_0 - (a\pi_0^2/2)$, yields

(11) $w_0 = y^n + (b^2/a) - b\hat{\pi} - (b^2/2a)$,

or simply

(12) $w_0 = y^n + (b^2/2a) - b\hat{\pi}$.

As shown in part (a), in all subsequent periods after the policymaker has deviated, $\pi_t = b/a$ and $y_t = y^n$. Substituting these values of output and inflation into the objective function, $w_t = y_t - (a\pi^2/2)$, gives us

(13) $w_t = y^n - (b^2/2a)$ for all $t > 0$.

Thus the policymaker's lifetime objective function if she deviates is given by

(14) $W^D = y^n + (b^2/2a) - b\hat{\pi} + \sum_{t=1}^{\infty}\beta^t\left[y^n - (b^2/2a)\right].$

Pulling the $[y^n - (b^2/2a)]$ out of the summation sign and using the fact that, since $\beta < 1$, we have

(15) $\sum_{t=1}^{\infty}\beta^t = \beta + \beta^2 + \beta^3 + \ldots = \beta\left(1 + \beta + \beta^2 + \ldots\right) = \beta/(1-\beta),$

we can write the lifetime objective function as

(16) $W^D = y^n + \dfrac{b^2}{2a} - b\hat{\pi} + \left(\dfrac{\beta}{1-\beta}\right)\left[y^n - \dfrac{b^2}{2a}\right] = \left(1 + \dfrac{\beta}{1-\beta}\right)y^n - b\hat{\pi} + \left(1 - \dfrac{\beta}{1-\beta}\right)\dfrac{b^2}{2a},$

or simply

(17) $W^D = \left(\dfrac{1}{1-\beta}\right)y^n - b\hat{\pi} + \left(\dfrac{1-2\beta}{1-\beta}\right)\dfrac{b^2}{2a}.$

If the policymaker chooses $\pi = \hat{\pi}$ every period, output will be equal to y^n every period. The policymaker's objective function in each period is therefore given by

(18) $w_t = y^n - (a\hat{\pi}^2/2).$

Thus the policymaker's lifetime objective function if she does not deviate is given by

(19) $W^{ND} = \sum_{t=0}^{\infty}\beta^t\left[y^n - (a\hat{\pi}^2/2)\right].$

Pulling the $[y^n - (a\hat{\pi}^2/2)]$ out of the summation sign and using the fact that, since $\beta < 1$, we have

(20) $\sum_{t=0}^{\infty}\beta^t = 1 + \beta + \beta^2 + \ldots = 1/(1-\beta),$

we can write the lifetime objective function as

(21) $W^{ND} = \left(\dfrac{1}{1-\beta}\right)\left[y^n - \dfrac{a\hat{\pi}^2}{2}\right].$

(c) One way of solving the problem is to calculate the benefit and cost of deviating as a function of $\hat{\pi}$ and the other parameters. We can then examine the range of $\hat{\pi}$'s over which the cost exceeds the benefit and thus the range of $\hat{\pi}$'s over which the policymaker will choose not to deviate from $\pi = \hat{\pi}$.

The benefit of departing from $\pi = \hat{\pi}$ in some period t_0 is that welfare in period t_0 is $y^n + (b^2/2a) - b\hat{\pi}$ [see equation (12)] rather than $y^n - (a\hat{\pi}^2/2)$ [see equation (18)]. Thus the benefit of deviating, B, is

(22) $B = y^n + (b^2/2a) - b\hat{\pi} - y^n + (a\hat{\pi}^2/2),$

or simply

(23) $B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}.$

The cost of deviating is that in all periods subsequent to t_0, welfare will be $y^n - (b^2/2a)$ [see equation (13)] rather than $y^n - (a\hat{\pi}^2/2)$. Thus the cost of deviating in each future period is $y^n - (a\hat{\pi}^2/2) - y^n + (b^2/2a)$ or simply $(b^2/2a) - (a\hat{\pi}^2/2)$. The total cost of deviating, discounted to time t_0 is

(24) $C = \sum_{t=t_0+1}^{\infty}\beta^{t-t_0}\left[(b^2/2a) - (a\hat{\pi}^2/2)\right] = \left[(b^2/2a) - (a\hat{\pi}^2/2)\right]\left(\beta + \beta^2 + \beta^3 + \ldots\right).$

Substituting the result in equation (15) into equation (24) gives us the following cost of deviating:

(25) $C = \left(\dfrac{\beta}{1-\beta}\right)\left[\dfrac{b^2}{2a} - \dfrac{a\hat{\pi}^2}{2}\right].$

We can plot the benefit and cost from deviating as a function of $\hat{\pi}$. First, we will deal with the benefit from deviating. From equation (23), we have

(26) $\partial B/\partial\hat{\pi} = a\hat{\pi} - b$,

and

(27) $\partial^2 B/\partial\hat{\pi}^2 = a > 0$.

Thus B is a parabola that reaches a minimum at $\hat{\pi} = b/a$. From equation (23), at $\hat{\pi} = 0$, $B = b^2/2a$. Finally, at its minimum at $\hat{\pi} = b/a$, $B = (b^2/2a) + (b^2/2a) - (b^2/a) = 0$. B, the benefit from deviating as a function of $\hat{\pi}$, is plotted in both figures below.

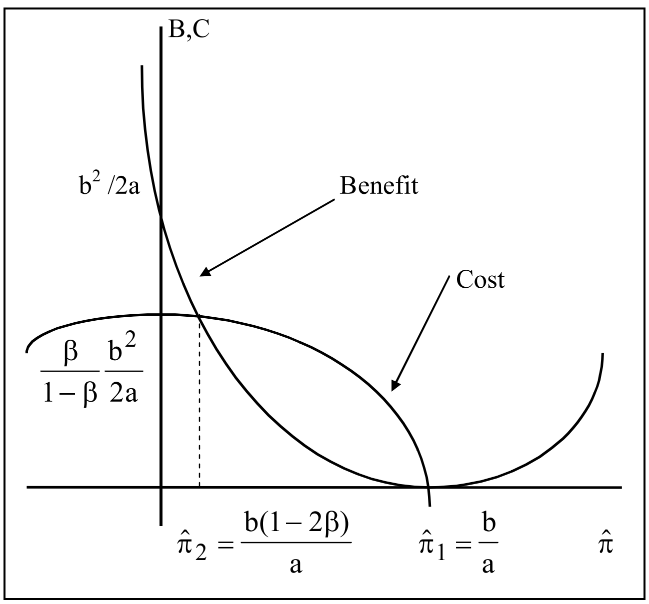
Now dealing with the cost of deviation, we have from equation (25)

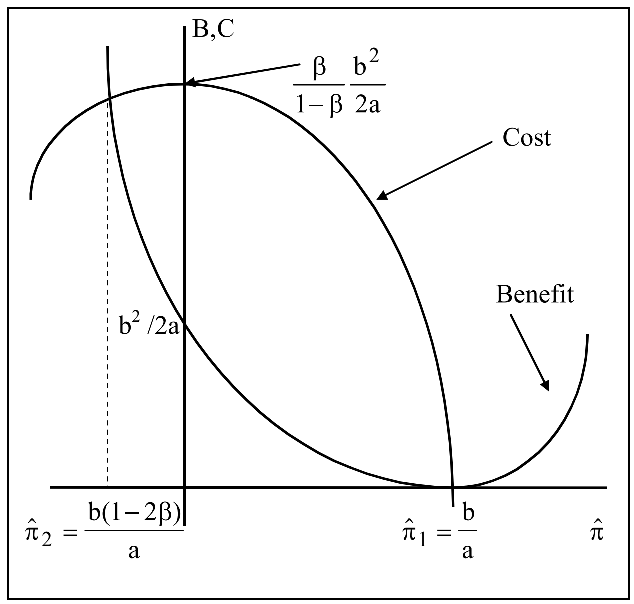
(28) $\partial C/\partial\hat{\pi} = -[\beta/(1-\beta)]a\hat{\pi}$,

and

(29) $\partial^2 C/\partial\hat{\pi}^2 = -[\beta/(1-\beta)]a < 0$.

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. From equation (25), the value of the cost of deviating at $\hat{\pi} = 0$ is given by $C = [\beta/(1-\beta)](b^2/2a)$. In addition, at $\hat{\pi} = b/a$, $C = 0$.





The case of $\beta < 1/2$ – so that $\beta/(1-\beta) < 1$ – is depicted in the left figure. The case of $\beta > 1/2$ – so that $\beta/(1-\beta) > 1$ – is depicted in the right figure.

We need to solve for the values of $\hat{\pi}$ where the benefit of deviating equals the cost of deviating. Setting the right-hand sides of equations (23) and (25) equal to each other yields

(30) $$\frac{b^2}{2a} + \frac{a\hat{\pi}^2}{2} - b\hat{\pi} = \frac{\beta}{1-\beta}\left[\frac{b^2}{2a} - \frac{a\hat{\pi}^2}{2}\right],$$

which implies that

(31) $$\left(1 + \frac{\beta}{1-\beta}\right)\frac{a\hat{\pi}^2}{2} - b\hat{\pi} + \left(1 - \frac{\beta}{1-\beta}\right)\frac{b^2}{2a} = 0,$$

or simply

(32) $\left(\dfrac{1}{1-\beta}\right)\dfrac{a\hat{\pi}^2}{2} - b\hat{\pi} + \left(\dfrac{1-2\beta}{1-\beta}\right)\dfrac{b^2}{2a} = 0.$

Multiplying both sides of equation (32) by (1 - β)2a gives us an equivalent condition for B = C :

(33) $a^2\,\hat{\pi}^2 - 2a(1 - \beta)b\hat{\pi} + (1 - 2\beta)b^2 = 0.$

Using the quadratic formula, we have

(34) $\hat{\pi} = \dfrac{2ab(1-\beta) \pm \sqrt{4a^2b^2(1-\beta)^2 - 4a^2b^2(1-2\beta)}}{2a^2} = \dfrac{2ab(1-\beta) \pm \sqrt{4a^2b^2\left[1-2\beta+\beta^2-1+2\beta\right]}}{2a^2}.$

Some further algebra yields

(35) $\hat{\pi} = \dfrac{2ab(1-\beta) \pm 2ab\beta}{2a^2} = \dfrac{b(1-\beta) \pm b\beta}{a},$

and thus finally

(36) $\hat{\pi}_1 = \dfrac{b(1-\beta)+b\beta}{a} = \dfrac{b}{a},$

and

(37) $\hat{\pi}_2 = \dfrac{b(1-\beta)-b\beta}{a} = \dfrac{b(1-2\beta)}{a}.$

These two values of $\hat{\pi}$ for which the benefit of deviating just equals the cost of deviating are depicted in the figures above. Note that for the case of β > 1/2 – the figure on the right – $\hat{\pi}_2$ is negative and is thus not relevant. We can now interpret the figures.

For the case of β > 1/2 – depicted in the figure on the right – the cost of deviating exceeds the benefit of deviating for any $\hat{\pi}$ such that $0 \le \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate from $\pi = \hat{\pi}$. Right at $\hat{\pi} = b/a$, the policymaker is indifferent and in fact at $\hat{\pi} = b/a$, deviating is actually the same as producing $\pi = \hat{\pi}$. Finally, for any value of $\hat{\pi} > b/a$, the benefit of deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

For the case of β < 1/2 – depicted in the figure on the left – the cost of deviating exceeds the benefit of deviating for any value of $\hat{\pi}$ such that $[b(1 - 2\beta)]/a < \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate from $\pi = \hat{\pi}$. Right at $\hat{\pi} = b/a$ and $\hat{\pi} = [b(1 - 2\beta)]/a$, the policymaker is indifferent. Finally, for any value of $\hat{\pi} < [b(1 - 2\beta)]/a$ or $\hat{\pi} > b/a$, the benefit of deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

For the policymaker to actually set π = 0 if $\hat{\pi} = 0$, we would need the cost of deviating to exceed the benefit of deviating, evaluated at $\hat{\pi} = 0$. From our earlier discussion, we know this will be true as long as β > 1/2. Thus regardless of the values of a and b, the policymaker will choose to set inflation to zero if $\hat{\pi} = 0$ as long as the discount rate is greater than 1/2.

Problem 11.17

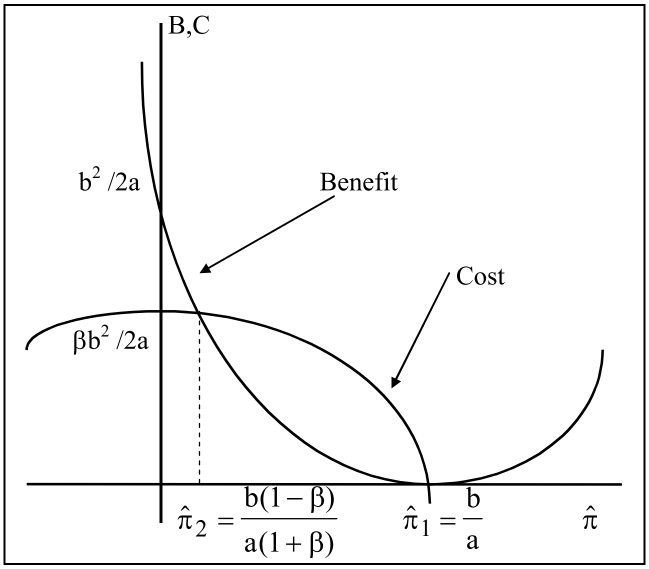
(a) We can use the same technique as in part (c) of the solution to Problem 11.16. We can examine the range of $\hat{\pi}$'s over which the cost of deviating from setting $\pi = \hat{\pi}$ exceeds the benefit of deviating. This gives the range of $\hat{\pi}$'s over which the policymaker chooses $\pi = \hat{\pi}$ each period. The benefit from deviating, B, is the same as it was in Problem 11.16. Thus we have

(1) $B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}.$

The cost of deviating in some period is that in the following period, $\pi^e = b/a$ rather than $\pi^e = \hat{\pi}$. As shown in part (a) of the solution to Problem 11.16, when $\pi^e = b/a$, the policymaker chooses π = b/a. Thus output is equal to y^n in the following period. Since the policymaker chooses $\pi = \pi^e$ in the period after

deviating, expected inflation reverts to $\pi^e = \hat{\pi}$ in all subsequent periods. Thus there is only a one-period cost to deviating. Specifically, the cost is that in the period after deviating, the value of the policymaker's objective function is $y^n - (b^2/2a)$ rather than $y^n - (a\hat{\pi}^2/2)$. Discounting that back to the period in which the deviation occurs yields the following cost, C:
(2) $C = \beta[y^n - (a\hat{\pi}^2/2) - y^n + (b^2/2a)] = \beta[(b^2/2a) - (a\hat{\pi}^2/2)]$.

We can now plot the benefit and cost of deviating as a function of $\hat{\pi}$. The benefit from deviating is the same as in Problem 11.16 and so we can concentrate on the cost. From equation (2):
(3) $\partial C/\partial\hat{\pi} = -\beta a\hat{\pi}$,
and
(4) $\partial^2 C/\partial\hat{\pi}^2 = -\beta a < 0$.
Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. From equation (2), the value of the cost of deviating at $\hat{\pi} = 0$ is given by $C = \beta(b^2/2a) < (b^2/2a)$ since $\beta < 1$. The next step is to solve for the values of $\hat{\pi}$ where the benefit of deviating equals the cost of deviating. Setting the right-hand sides of equations (1) and (2) equal to each other yields
(5) $\dfrac{b^2}{2a} + \dfrac{a\hat{\pi}^2}{2} - b\hat{\pi} = \dfrac{\beta b^2}{2a} - \dfrac{\beta a\hat{\pi}^2}{2}$,
which simplifies to
(6) $\dfrac{(1+\beta)a}{2}\hat{\pi}^2 - b\hat{\pi} + \dfrac{(1-\beta)b^2}{2a} = 0$.
Multiplying both sides of equation (6) by 2a gives us the following equivalent condition for B = C:
(7) $(1+\beta)a^2\hat{\pi}^2 - 2ab\hat{\pi} + (1-\beta)b^2 = 0$.
Using the quadratic formula gives us
(8) $\hat{\pi} = \dfrac{2ab \pm \sqrt{4a^2b^2 - 4a^2b^2(1+\beta)(1-\beta)}}{2a^2(1+\beta)} = \dfrac{2ab \pm \sqrt{4a^2b^2\left[1 - 1 + \beta^2\right]}}{2a^2(1+\beta)}$.
Some further algebra yields
(9) $\hat{\pi} = \dfrac{2ab \pm 2ab\beta}{2a^2(1+\beta)} = \dfrac{b(1\pm\beta)}{a(1+\beta)}$,
and thus finally
(10) $\hat{\pi}_1 = \dfrac{b(1+\beta)}{a(1+\beta)} = \dfrac{b}{a}$, and (11) $\hat{\pi}_2 = \dfrac{b(1-\beta)}{a(1+\beta)}$.



From the figure at left, we can see that the cost of deviating from $\pi = \hat{\pi}$ exceeds the benefit from deviating for any $\hat{\pi}$ such that

(12) $b(1 - \beta)/a(1 + \beta) < \hat{\pi} < b/a$.

With these values of the parameters, the policymaker will choose not to deviate. For any value of $\hat{\pi}$ greater than b/a or less than $b(1 - \beta)/a(1 + \beta)$, the benefit from deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

(b) Again, we will employ the same technique. The benefit from deviating remains the same; it is given by equation (1), $B = (b^2 / 2a) + (a\hat{\pi}^2 / 2) - b\hat{\pi}$.

We need to determine the cost of deviating for the policymaker. Suppose the policymaker deviates in some period t. Then in period t + 1, $\pi_{t+1}^e = \pi_0 > b/a$. We can also write this as $\pi_{t+1}^e = b/a + x$, $x > 0$. The variable x captures the extent of the punishment for deviating. When the policymaker takes expected inflation as given, she chooses to set inflation equal to b/a. Thus, using the Lucas supply function, output in period t + 1, the period after deviating, is

(13) $y_{t+1} = y^n + b[(b/a) - (b/a) - x] = y^n - bx$.

Thus output is below the natural rate the period after deviating. The value of the policymaker's objective function in period t + 1 is

(14) $w_{t+1} = y_{t+1} - (a\pi^2 / 2) = y^n - bx - (b^2 / 2a)$.

Thus the cost of deviating in period t + 1 is that welfare is given by (14) rather than $y^n - (a\hat{\pi}^2 / 2)$. Discounting this back to period t, we have the cost in period t + 1:

(15) $C_{t+1} = \beta[y^n - (a\hat{\pi}^2 / 2) - y^n + bx + (b^2 / 2a)]$,

or simply

(16) $C_{t+1} = \beta[bx - (a\hat{\pi}^2 / 2) + (b^2 / 2a)]$.

Now consider the situation in period t + 2, two periods after a deviation. Expected inflation equals b/a. Taking expected inflation as given, the policymaker chooses to set inflation equal to b/a. Thus output is at the natural rate. The value of the policymaker's objective function in t + 2 is

(17) $w_{t+2} = y_{t+2} - (a\pi^2 / 2) = y^n - (b^2 / 2a)$.

Thus the cost of deviating in period t + 2 is that welfare is equal to $y^n - (b^2 / 2a)$ rather than $y^n - (a\hat{\pi}^2 / 2)$. Discounting this back to period t, we have the cost in period t + 2:

(18) $C_{t+2} = \beta^2[y^n - (a\hat{\pi}^2 / 2) - y^n + (b^2 / 2a)] = \beta^2[(b^2 / 2a) - (a\hat{\pi}^2 / 2)]$.

In period t + 3, since actual inflation last period was equal to expected inflation last period, expected inflation reverts to $\hat{\pi}$ and there is no further cost to the deviation in period t.

Thus the total cost of the deviation is

(19) $C = \beta[bx - (a\hat{\pi}^2/2) + (b^2/2a)] + \beta^2[(b^2/2a) - (a\hat{\pi}^2/2)]$,

or simply

(20) $C = \beta bx + \beta(1+\beta)[(b^2/2a) - (a\hat{\pi}^2/2)]$.

From equation (20),

(21) $\partial C/\partial\hat{\pi} = \beta(1+\beta)[-2a\hat{\pi}/2] = -\beta(1+\beta)a\hat{\pi}$,

and

(22) $\partial^2 C/\partial\hat{\pi}^2 = -\beta(1+\beta)a < 0$.

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. The value of the cost of deviating at $\hat{\pi} = 0$ is given by $\beta bx + \beta(1+\beta)(b^2/2a)$. From earlier analysis, we know that the benefit of deviating at $\hat{\pi} = 0$ is $b^2/2a$. Thus if the value of x, the excess punishment, is high enough, the cost of deviating at $\hat{\pi} = 0$ will exceed the benefit and there can be an equilibrium with zero inflation. Specifically, we need the following condition to hold:

(23) $\beta bx + \beta(1+\beta)(b^2/2a) > b^2/2a$,

or

(24) $\beta bx > (b^2/2a)[1 - \beta(1+\beta)]$,

or simply

(25) $x > (b^2/2a)[1 - \beta(1+\beta)]/\beta b$.

We can determine the value of $\hat{\pi}$ at which C = 0. From equation (20), C = 0 when

(26) $\beta(1+\beta)[(a\hat{\pi}^2/2) - (b^2/2a)] = \beta bx$,

which implies

(27) $(a\hat{\pi}^2/2) - (b^2/2a) = bx/(1+\beta)$,

and thus

(28) $\hat{\pi}^2 = [2bx/a(1+\beta)] + (b^2/a^2)$.

Therefore C = 0 when

(29) $\hat{\pi} = \sqrt{\dfrac{2bx}{a(1+\beta)} + \dfrac{b^2}{a^2}} > \dfrac{b}{a}$.

Thus the cost of deviating is equal to zero at a value of $\hat{\pi}$ greater than the one for which the benefit of deviating is equal to zero (which is $\hat{\pi} = b/a$). The values of $\hat{\pi}$ for which it is an equilibrium for the policymaker not to deviate are those—just as in part (a)—where the cost of deviating exceeds the benefit. The basic idea is that higher values of x lead to a wider range of $\hat{\pi}$'s for which the cost exceeds the benefit and thus a wider range of $\hat{\pi}$'s for which the policymaker does not deviate.

(c) As we have shown previously, if the policymaker takes expected inflation as given, she chooses inflation equal to b/a. Thus if $\pi^e = b/a$, the policymaker chooses $\pi = b/a$, so that the public's expectation is fulfilled and output is at the natural rate. There is no incentive for the policymaker to choose a different inflation rate and there is no incentive for the public to change its expectation of inflation and thus $\pi = \pi^e = b/a$ will be an equilibrium for any $a > 0$, $b > 0$.

Problem 11.18

Consider the situation in the last period, denoted T. The policymaker's choice of π has no effect on next period's expected inflation; there is no next period. Thus the policymaker's problem in the final period is to take expected inflation as given and choose π in order to maximize the period T objective function. From previous analysis in the solution to Problem 11.16, we know that the policymaker's choice of inflation in this type of situation is $\pi_T = b/a$. Since the public knows how the policymaker behaves, expected inflation also equals b/a and thus output equals y^n.

Now consider the situation in period T - 1. The important point is that the policymaker knows her choice of π_{T-1} will have no bearing on what happens the next and final period. Regardless of the level of π she chooses in period T - 1, expected inflation next period will be b/a, as described above. Since the policymaker's problem has no impact on the future, she chooses π, taking π^e as given, in order to maximize the period T - 1 objective function. Again, the optimal choice is $\pi_{T-1} = b/a$. The public knows this and so $\pi_{T-1}^e = b/a$ and thus output in period T - 1 equals y^n.

Working backward, the same thing happens each period. The policymaker knows that expected inflation the following period will be b/a regardless of what she does this period. Thus she acts to maximize the one-period objective function and chooses $\pi = b/a$, which results in output equal to the natural rate. Therefore the unique equilibrium for all periods is $\pi_t^e = \pi_t = b/a$ and $y_t = y^n$.

Problem 11.19

The politician faces the following problem, where E is defined as the probability of being reelected:

(1) $\max\limits_{u_1,u_2} E = \Pr[\pi_2 + \beta u_2 < K]$,

subject to

(2) $\pi_t = \pi_{t-1} - \alpha(u_t - u^n) + \varepsilon_t^s$,

and

(3) $u_L \le u_t \le u_H$,

for t = 1, 2. Substituting equation (2) evaluated at t = 2 into the probability that the politician is reelected yields

(4) $E = \Pr[\pi_1 - \alpha(u_2 - u^n) + \varepsilon_2^s + \beta u_2 < K]$.

We can rewrite equation (4) as

(5) $E = \Pr[\varepsilon_2^s < K - \pi_1 - \alpha u^n + (\alpha - \beta)u_2]$.

The probability on the right-hand side of equation (5) is simply the cumulative distribution function of ε_2^S evaluated at $K - \pi_1 - \alpha u^n + (\alpha - \beta)u_2$ and so we can write the probability of being reelected as

(6) $E = F(K - \pi_1 - \alpha u^n + (\alpha - \beta)u_2)$.

Note that the choice of u_2 does not depend on π_1, which in turn is a function of u_1. To see the way in which the probability of being reelected varies with the choice of u_1, take the derivative of E with respect to u_1:

(7) $\dfrac{\partial E}{\partial u_1} = f(K - \pi_1 - \alpha u^n + (\alpha - \beta)u_2)(-\dfrac{\partial \pi_1}{\partial u_1})$,

where $f(\bullet)$ is the probability density function of ε_2^S. Since $\pi_1 = \pi_0 - \alpha(u_1 - u^n) + \varepsilon_1^s$, where π_0 is given, $\partial \pi_1 / \partial u_1 = -\alpha$. Thus we can write

(8) $\dfrac{\partial E}{\partial u_1} = \alpha f(K - \pi_1 - \alpha u^n + (\alpha - \beta)u_2)$

Since $f(\bullet) \geq 0$, the derivative given in equation (8) is greater than or equal to zero. Thus picking a higher value for first-period unemployment can never reduce the probability of being reelected and might increase it. Thus it is optimal to pick the highest feasible level of unemployment in the first period, u_H.

Intuitively, since only second-period inflation and unemployment determine the probability of being reelected, the politician wants to face the best possible inflation-unemployment tradeoff in period 2. From equation (2), we can see that is accomplished by having the lowest possible inflation rate in the previous period, period 1. That, in turn, is accomplished by having the highest possible unemployment rate in period 1.

Problem 11.20

(a) Rearranging the relationship between output and inflation given by

(1) $y_t = y^n + b(\pi_t - E_{t-1}\pi_t)$,

to solve for π_t gives us

(2) $\pi_t = (1/b)(y_t - y^n) + E_{t-1}\pi_t$.

The liberal leader chooses y_t to maximize $a_L y_t - \pi_t^2/2$ subject to inflation being determined by equation (2). The first-order condition is

(3) $a_L - \pi_t \dfrac{\partial \pi_t}{\partial y_t} = 0$.

From equation (2), taking $E_{t-1}\pi_t$ as given, the derivative of π_t with respect to y_t is (1/b). Substituting that fact, as well as equation (2), into the first-order condition gives us

(4) $a_L - [(1/b)(y_t - y^n) + E_{t-1}\pi_t](1/b) = 0$.

Equation (4) can be rewritten as

(5) $(1/b)(y_t - y^n) + E_{t-1}\pi_t = ba_L$.

Solving equation (5) for y_t gives us

(6) $(1/b)y_t = (1/b)y^n + ba_L - E_{t-1}\pi_t$,

and multiplying both sides of equation (6) by b yields

(7) $y_t^L = y^n + b^2 a_L - bE_{t-1}\pi_t$.

Equation (7) gives the value of output that a liberal leader will choose.

An analogous exercise would allow us to derive the following expression for the level of output chosen by a conservative leader:

(8) $y_t^C = y^n + b^2 a_C - bE_{t-1}\pi_t$.

(b) Substitute the liberal's choice for output into equation (2) to obtain

(9) $\pi_t = (1/b)(y^n + b^2 a_L - bE_{t-1}\pi_t - y^n) + E_{t-1}\pi_t$

Thus the inflation rate with a liberal leader is given by

(10) $\pi_t^L = ba_L$.

Similarly, the inflation rate with a conservative leader is

(11) $\pi_t^C = ba_C$.

Individuals know this is how the leaders will behave in period 1. Since the public knows that the probability of a liberal leader is p, and the probability of a conservative leader is (1 – p), the expected

value of inflation in period 1 is a weighted average of the two possible inflation rates given by (10) and (11), where the probabilities serve as weights. That is,
(12) $E_0\pi_1 = pba_L + (1-p)ba_C = b[pa_L + (1-p)a_C]$.

To determine output in period 1 under a liberal leader, substitute equation (12) into equation (7):
(13) $y_1^L = y^n + b^2 a_L - bE_0\pi_1 = y^n + b^2 a_L - b^2[pa_L + (1-p)a_C]$.
Equation (13) can be rearranged to obtain
(14) $y_1^L = y^n + b^2 a_L(1-p) - b^2(1-p)a_C$,
or simply
(15) $y_1^L = y^n + b^2(1-p)[a_L - a_C]$.

To determine output in period 1 under a conservative leader, substitute equation (12) into equation (8):
(16) $y_1^C = y^n + b^2 a_C - bE_0\pi_1 = y^n + b^2 a_C - b^2[pa_L + (1-p)a_C]$.
Equation (16) can be rearranged to obtain
(17) $y_1^C = y^n - b^2[pa_L - pa_C]$,
or simply
(18) $y_1^C = y^n - b^2 p[a_L - a_C]$.

Since $a_L > a_C > 0$, output and inflation are higher in period 1 under a liberal leader than they are under a conservative leader.

(c) In period 1, the public knows for certain who the leader will be in period 2. Thus if a liberal is in power, the public expects inflation in period 2 to equal ba_L; similarly, if a conservative is in power, the public expects inflation in period 2 to equal ba_C.

The leaders continue to maximize their objective function, taking expected inflation as given. Thus, from equation (7), the liberal leader's choice of y_2 is given by
(19) $y_2^L = y^n + b^2 a_L - bE_1\pi_2$.
Substituting the fact that $E_1\pi_2 = ba_L$ if there is a liberal leader gives us
(20) $y_2^L = y^n + b^2 a_L - b^2 a_L$.
And thus period-2 output under a liberal leader is simply
(21) $y_2^L = y^n$.

Similarly, from equation (8), the conservative leader's choice of y_2 is given by
(22) $y_2^C = y^n + b^2 a_C - bE_1\pi_2$.
Substituting the fact that $E_1\pi_2 = ba_C$ if there is a conservative leader gives us
(23) $y_2^C = y^n + b^2 a_C - b^2 a_C$.
And thus period-2 output under a conservative leader is also given by
(24) $y_2^C = y^n$.
Without the uncertainty about who the leader will be, output will not deviate from potential.

Problem 11.21

We can focus on a situation in which g_M , π, i, and r are constant and in which $\pi^e = \pi$. Although not technically correct – since Y and thus M/P are growing – such a situation will be referred to as a steady state in what follows. Under these assumptions, it is therefore reasonable to assume that output, and the real interest rate are unaffected by the rate of money growth and that actual and expected inflation are equal. Taking the exponential function of both sides of the money demand function, which is given by ln(M(t)/P(t)) = a - bi + lnY(t), yields

(1) $M(t)/P(t) = e^{a-bi}Y(t)$.

The nominal interest rate is given by $i = r + \pi^e$. In steady state, π^e and r are constant and thus so is the nominal interest rate. Thus in steady state, the quantity of real balances must grow at the same rate as Y(t). In other words, $\dot{M}(t)/M(t) - \dot{P}(t)/P(t) = g_Y$. Solving for inflation yields

(2) $\pi = g_M - g_Y$,

where g_M is the growth rate of the nominal money stock. This means that the nominal interest rate in steady state is given by

(3) $i = \bar{r} + \pi = \bar{r} + g_M - g_Y$,

where we have used the fact that actual and expected inflation are equal. Substituting equation (3) into equation (1) gives steady-state real balances:

(4) $M(t)/P(t) = e^a e^{-b(\bar{r}+g_M-g_Y)}Y(t)$.

Seignorage is given by

(5) $S(t) = \dfrac{\dot{M}(t)}{P(t)} = \dfrac{\dot{M}(t)}{M(t)}\dfrac{M(t)}{P(t)} = g_M \dfrac{M(t)}{P(t)}$.

Substituting equation (4) into equation (5) gives steady-state seignorage:

(6) $S(t) = g_M e^a e^{-b(\bar{r}+g_M-g_Y)}Y(t) = Cg_M e^{-bg_M}Y(t)$,

where $C \equiv e^a e^{-b(\bar{r}-g_Y)}$. We need to find the choice of nominal money growth, g_M , that maximizes steady-state seignorage. Again, we are assuming that output is unaffected by money growth. The first-order condition is

(7) $\partial S(t)/\partial g_M = Ce^{-bg_M}Y(t) - bCg_M e^{-bg_M}Y(t) = 0$,

which simplifies to

(8) $1 - bg_M = 0$.

Thus seignorage is maximized when money growth is given by

(9) $g_M = 1/b$.

From equation (2), we know that $\pi = g_M - g_Y$ and thus the rate of inflation that maximizes seignorage is

(10) $\pi = (1/b) - g_Y$.

Equation (10) implies that the higher is the growth rate of real output, the lower is the rate of inflation that maximizes steady-state seignorage.

Problem 11.22

(a) Desired real money holdings are given by

(1) $m(t) = Ce^{-b\pi^e(t)}$.

The assumption is that expected inflation adjusts gradually toward actual inflation. Specifically, our assumption is

(2) $\dot{\pi}^e(t) = \beta[\pi(t) - \pi^e(t)]$.

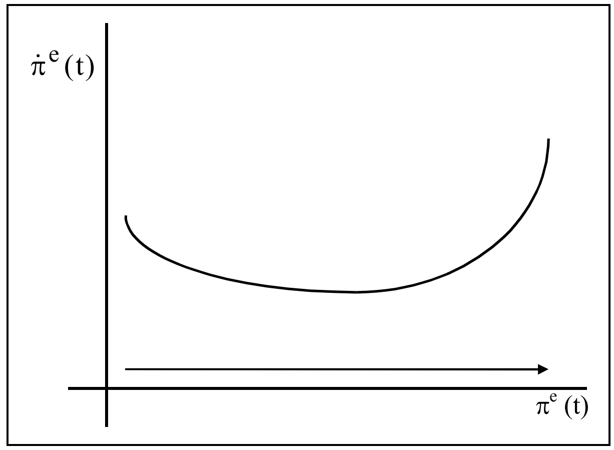
As usual, seignorage is given by $\dot{M}(t)/P(t)$ or equivalently $[\dot{M}(t)/M(t)][M(t)/P(t)]$. Assuming that the nominal money supply is growing at rate $g_M(t)$, we can write seignorage as

(3) $S(t) = g_M(t)m(t)$.

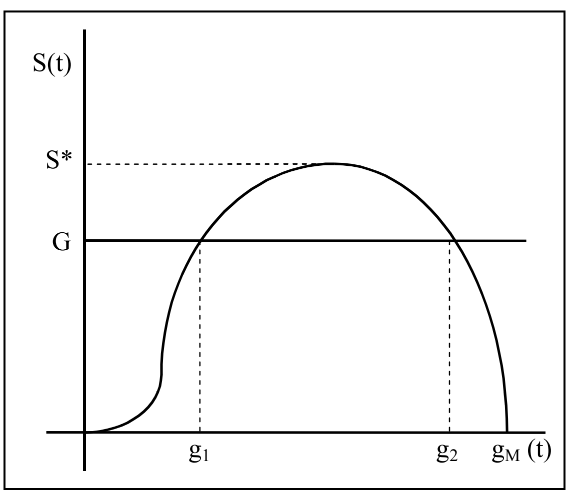
To see the dynamics of inflation and money holdings formally, note that the growth rate of real money, $\dot{m}(t)/m(t)$, equals the growth rate of nominal money, $g_M(t)$, minus the rate of inflation, $\pi(t)$. Rewriting this as an equation for inflation gives us
(4) $\pi(t) = g_M(t) - [\dot{m}(t)/m(t)]$.
Define G as the amount of real purchases that the government needs to finance with seignorage. Thus from equation (3), we have
(5) $g_M(t) = G/m(t)$.
Taking the time derivative of both sides of equation (1) yields
(6) $\dot{m}(t) = -b\dot{\pi}^e(t)Ce^{-b\pi^e(t)}$.
Dividing both sides of equation (6) by m(t) gives us
(7) $\dot{m}(t)/m(t) = -b\dot{\pi}^e(t)$.
Substituting equations (5) and (7) into equation (4) yields
(8) $\pi(t) = \dfrac{G}{m(t)} + b\dot{\pi}^e(t)$.
Substituting equation (8) into equation (2) gives us
(9) $\dot{\pi}^e(t) = \beta\left[\dfrac{G}{m(t)} + b\dot{\pi}^e(t) - \pi^e(t)\right]$.
Collecting the terms in $\dot{\pi}^e(t)$ yields
(10) $\dot{\pi}^e(t)\left[1 - \beta b\right] = \beta\left[\dfrac{G}{m(t)} - \pi^e(t)\right]$,
and thus
(11) $\dot{\pi}^e(t) = \dfrac{\beta}{1-\beta b}\left[\dfrac{G - \pi^e(t)m(t)}{m(t)}\right]$.

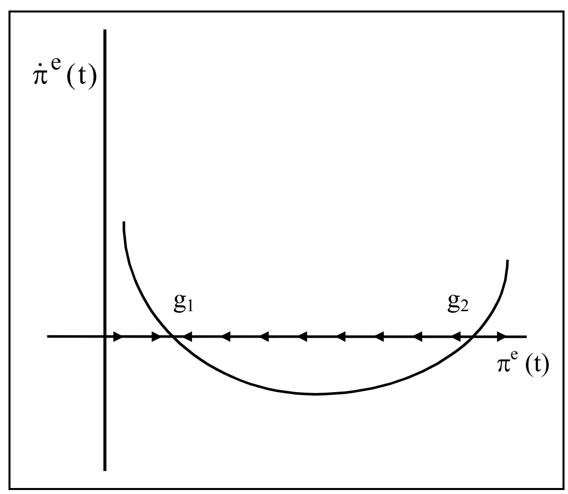
(b) The assumption that G > S* (where S* represents the maximum steady-state value of seignorage) is equivalent to $G > \pi^e m$ for all possible values of π^e. Thus since $\beta b < 1$, the right-hand side of equation (11) is everywhere positive: regardless of where it starts, expected inflation grows without bound. To examine the nature of the phase diagram, substitute $m(t) = Ce^{-b\pi^e(t)}$ into equation (11):
(12) $\dot{\pi}^e(t) = \dfrac{\beta G}{(1-\beta b)Ce^{-b\pi^e(t)}} - \dfrac{\beta}{1-\beta b}\pi^e(t)$.
The following derivatives will be useful:
(13) $\dfrac{d\dot{\pi}^e(t)}{d\pi^e(t)} = \dfrac{\beta b G e^{b\pi^e(t)}}{(1-\beta b)C} - \dfrac{\beta}{1-\beta b}$,
and
(14) $\dfrac{d^2\dot{\pi}^e(t)}{d\pi^e(t)^2} = \dfrac{\beta b^2 G e^{b\pi^e(t)}}{(1-\beta b)C} > 0$.

By setting the right-hand side of equation (13) equal to zero, it is straightforward to show that $\dot{\pi}^e(t)$ reaches a minimum at $\pi^e(t) = [\ln(C/bG)]/b$. Thus the phase diagram has the shape depicted in the figure at right. From equation (1), and since π^e rises without bound, the real money stock is continually falling. If m(t) is continually falling, then from equation (3), it must be the case that the growth rate of the nominal money supply, $g_M(t)$, is continually rising if the government is to obtain G in seignorage.



(c) Now consider the case of $G < S^*$. The left-hand figure below reproduces Figure 11.8 from the text. It depicts the amount of seignorage the government can obtain in steady state as a function of the growth rate of the nominal money supply. In the case of $G < S^*$, there are two possible growth rates of the nominal money supply, labeled g_1 and g_2 in the figure, consistent with raising the amount G in seignorage. Recall that in a steady state, expected inflation equals actual inflation which in turn equals the constant growth rate of the nominal money supply. Thus, by assumption, $\pi^e(t)m(t) = G$ at $\pi^e(t) = g_1$ and $\pi^e(t) = g_2$. From equation (11) then, $\dot{\pi}^e(t) = 0$ at $\pi^e(t) = g_1$ and $\pi^e(t) = g_2$. From the figure on the left, when $g_1 < \pi^e(t) < g_2$, we have $\pi^e(t)m(t) > G$ and thus $\dot{\pi}^e(t) < 0$. Otherwise, $\pi^e(t)m(t) < G$ and thus $\dot{\pi}^e(t) < 0$. Putting all of this information together gives us the phase diagram depicted on the right. The low-inflation steady state with $\pi^e(t) = \pi(t) = g_1$ is stable and the high-inflation steady state with $\pi^e(t) = \pi(t) = g_2$ is unstable.





SOLUTIONS TO CHAPTER 12

Problem 12.1

(a) (i) Taking the time derivative of d(t) = D(t)/Y(t) gives us

(1) $\dot{d}(t) = \dfrac{\dot{D}(t)Y(t) - D(t)\dot{Y}(t)}{Y(t)^2}$.

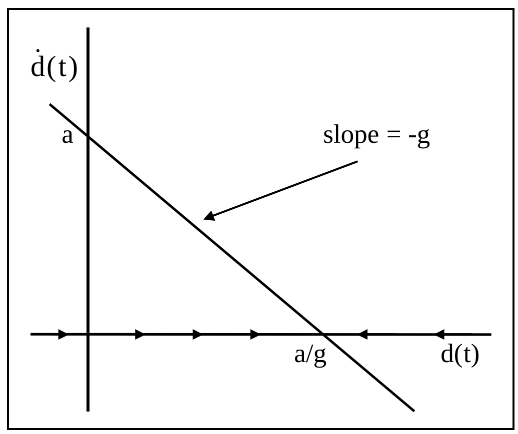
Substituting $\dot{D}(t) = \delta(t)$ – the rate of change of the amount of debt outstanding equals the budget deficit – and the fact that $\dot{Y}(t) = Y(t)g$, which follows from the fact that output grows at rate g, and simplifying gives us

(2) $\dot{d}(t) = \dfrac{\delta(t)}{Y(t)} - \dfrac{D(t)g}{Y(t)}$.

Substituting the assumption that the deficit-to-output ratio is constant – δ(t)/Y(t) = a – and using the definition of d(t) = D(t)/Y(t), yields

(3) $\dot{d}(t) = a - gd(t)$.

(a) (ii) The phase diagram for the ratio of debt to output is depicted in the figure at right.



In (d, $\dot{d}$) space, equation (3) is a line with slope equal to -g. We can see that the system is stable. If the debt-to-output ratio is less than a/g, $\dot{d}(t) > 0$ and so d(t) rises toward a/g.

Similarly, if the debt-to-output ratio is greater than a/g, $\dot{d}(t) < 0$ and so d(t) falls toward a/g.

Note that the value of the debt-to-output ratio to which the economy converges is increasing in the deficit-to-output ratio, a, and decreasing in the growth rate of output, g.

(b) (i) Once again, taking the time derivative of d(t) = D(t)/Y(t) gives us

(4) $\dot{d}(t) = \dfrac{\dot{D}(t)Y(t) - D(t)\dot{Y}(t)}{Y(t)^2}$.

Substituting $\dot{D}(t) = \delta(t) = aY(t) + r(d(t))D(t)$ and the fact that $\dot{Y}(t) = Y(t)g$ into equation (4) and simplifying gives us

(5) $\dot{d}(t) = a + \dfrac{r(d(t))D(t)}{Y(t)} - \dfrac{D(t)g}{Y(t)}$.

Using the definition of d(t) = D(t)/Y(t), yields

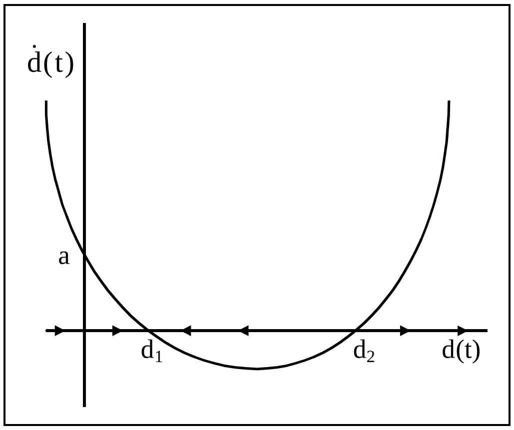
(6) $\dot{d}(t) = a + [r(d(t)) - g]d(t)$.

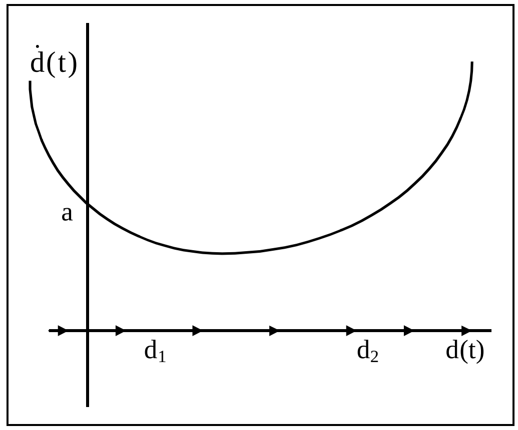
(b) (ii) When plotted in (d, $\dot{d}$) space, the slope of the locus given by equation (6) equals r(d(t)) - g. Given the assumptions about the behavior of r, this slope is negative for large negative values of d(t) and increases as d increases and so equation (6) defines a convex function as depicted in the figures below.

Once again, $\dot{d}(t) = a > 0$ when d(t) equals zero. And $\dot{d}(t) = 0$ if $d(t) = \dfrac{a}{g - r(d(t))}$.

The case in which a is sufficiently small that $\dot{d}$ is negative for some values of d is depicted on the left-hand side. In this case, d_1 is a stable equilibrium whereas d_2 is not. If the debt-to-output ratio starts off less than d_2, it converges to d_1. If the debt-to-output ratio starts off greater than d_2, it rises without bound.

The case in which a is sufficiently large that $\dot{d}$ is positive for all values of d is depicted in the figure on the right-hand side below. In this case, d will always be rising and there is no stable equilibrium.





Problem 12.2

Throughout, we will assume $U'(\bullet) > 0$ and $U''(\bullet) < 0$. In addition, since the expected value of Y_2 is equal to Y_1, we can write $Y_2 = Y_1 + \varepsilon$ with $E[\varepsilon] = 0$.

(a) The individual's problem is to choose C_1 and C_2 in order to maximize $U(C_1) + E[U(C_2)]$ subject to

(1) $C_2 = (1 - \tau_1)Y_1 - C_1 + (1 - \tau_2)(Y_1 + \varepsilon)$.

We can substitute for C_2 and solve the following unconstrained problem of choosing C_1:

(2) max $U(C_1) + E\Big[U\big((1-\tau_1)Y_1 - C_1 + (1-\tau_2)(Y_1+\varepsilon)\big)\Big]$.

The first-order condition is given by

(3) $U'(C_1) + E\big[U'(C_2)(-1)\big] = 0$,

or simply

(4) $U'(C_1) = E\big[U'(C_2)\big]$.

If the individual is optimizing, the marginal utility of consumption in period one must equal the expected marginal utility of consumption in period two.

(b) If Y_2 is not random, the first-order condition reduces to $U'(C_1) = U'(C_2)$. With $U''(\bullet) < 0$ everywhere, this implies that $C_1 = C_2$. If utility is quadratic then $U'(C_2)$ is a linear function of C_2 and so $E[U'(C_2)] = U'(E[C_2])$. Thus the first-order condition given by equation (2) can be rewritten as $U'(C_1) = U'(E[C_2])$. Since $U''(\bullet) < 0$ everywhere, this implies that $C_1 = E[C_2]$.

(c) We are now told that $U'(\bullet) > 0$, $U''(\bullet) < 0$ and $U'''(\bullet) > 0$. In this case, marginal utility is a convex function of consumption and so by Jensen's inequality $E[U'(C_2)] > U'(E[C_2])$. Combining this with the first-order condition, $U'(C_1) = E\big[U'(C_2)\big]$, yields $U'(C_1) > U'(E[C_2])$. Since $U'(\bullet) > 0$ and $U''(\bullet) < 0$ we have $C_1 < E[C_2]$. The individual plans in such a way that if second-period income turns out to be equal to its expected value, C_2 would turn out to be higher than C_1. Thus, in the face of uncertainty and with U ''' (•) > 0, the individual undertakes "precautionary saving".

(d) The government cuts first-period taxes, τ_1, and raises second-period taxes, τ_2, in such a way that expected tax revenue remains unchanged. Expected tax revenue, $\overline{R}$, can be expressed as $\tau_1 Y_1 + \tau_2 E[Y_1 + \varepsilon] = \overline{R}$. Using the fact that E[ε] = 0, we can solve for τ_2:

(5) $\tau_1 Y_1 + \tau_2 Y_1 = \overline{R}$,

which implies that

(6) $\tau_2 = \overline{R}/Y_1 - \tau_1$.

In order to keep $\overline{R}$ constant, the change in taxes must satisfy

(7) $\partial\tau_2/\partial\tau_1 = -1$.

The question is whether or not this change in the timing of taxes alters the individual's consumption behavior. Substitute equation (1) into the first-order condition, equation (4), to obtain

(8) $U'(C_1) = E\left[U'\left((1-\tau_1)Y_1 - C_1 + (1-\tau_2)(Y_1+\varepsilon)\right)\right]$.

Differentiating both sides of this equation with respect to τ_1 yields

(9) $U''(C_1)\,\partial C_1/\partial\tau_1 = E\left[U''(C_2)\left\{-Y_1 - \partial C_1/\partial\tau_1 - (Y_1+\varepsilon)\,\partial\tau_2/\partial\tau_1\right\}\right]$.

Substituting equation (7), $\partial\tau_2/\partial\tau_1 = -1$, into equation (9) gives us

(10) $U''(C_1)\,\partial C_1/\partial\tau_1 = E\left[U''(C_2)\left(-Y_1 - \partial C_1/\partial\tau_1 + Y_1 + \varepsilon\right)\right]$,

which simplifies to

(11) $U''(C_1)\,\partial C_1/\partial\tau_1 = E\left[U''(C_2)\left(-\partial C_1/\partial\tau_1\right)\right] + E\left[U''(C_2)\,\varepsilon\right]$,

or

(12) $\left[U''(C_1) + E\left[U''(C_2)\right]\right]\partial C_1/\partial\tau_1 = E\left[U''(C_2)\,\varepsilon\right]$.

Now use the fact that for any two random variables X and Y, E[XY] = E[X]E[Y] + cov[X,Y]:

(13) $\left[U''(C_1) + E\left[U''(C_2)\right]\right]\partial C_1/\partial\tau_1 = E\left[U''(C_2)\right]E\left[\varepsilon\right] + \mathrm{cov}\left[U''(C_2),\varepsilon\right]$.

Finally, E[ε] = 0 and thus the change in first-period consumption due to this change in the timing of taxes is given by

(14) $\dfrac{\partial C_1}{\partial\tau_1} = \dfrac{\mathrm{cov}\left[U''(C_2),\varepsilon\right]}{U''(C_1) + E\left[U''(C_2)\right]}$.

(e) If Y_2 is not random then $\varepsilon = 0$ always and thus the covariance in the numerator of equation (14) is 0. In addition, if utility is quadratic, then U '' (•) is a constant and again the covariance is 0. In both of these cases, $\partial C_1/\partial\tau_1 = 0$ and thus first-period consumption does not change in response to the tax cut.

(f) In the case of U ''' (•) > 0 we need to show that $\partial C_1/\partial\tau_1 < 0$. That is, we need to show that C_1 rises in response to the reduction in τ_1. Intuitively, the higher is ε, the higher will be C_2. The individual simply consumes any extra random income in the second period. If U ''' (•) > 0, then as C_2 rises so will U '' (C_2), and thus it will be the case that cov[U '' (C_2),ε] > 0. The denominator of equation (14) will be negative since U '' (•) < 0 and thus $\partial C_1/\partial\tau_1 < 0$ as required. The intuition is that the change in the timing of taxes leaves the individual with the same expected after-tax lifetime income, but more of it comes with certainty in the first period. If the individual is undertaking precautionary saving – and if U ''' (•) > 0 she is – the amount of such saving will be reduced and she will consume more in the first period.

Problem 12.3

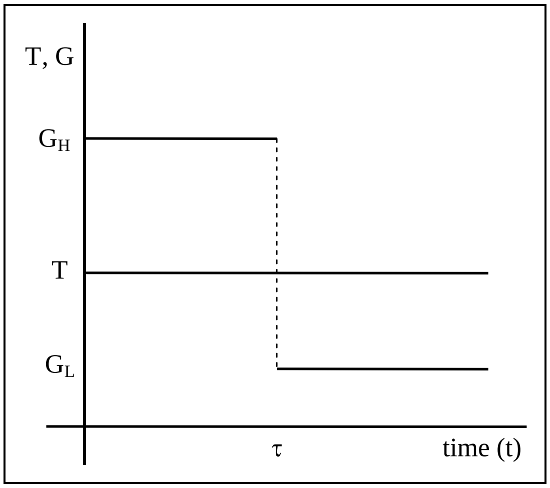
In the Barro tax-smoothing model in which output and the real interest rate are constant, the government finds it optimal to set taxes equal to a constant such that its budget constraint is satisfied with equality.

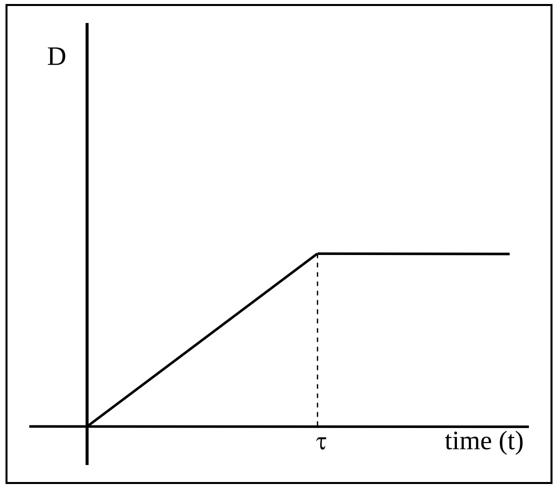
Thus the government will find it optimal to set taxes such that $T < G_H$ .That is, during the war from time $0 \le t \le \tau$, $T < G_H$ and so the government runs a deficit and its debt will be growing over time. The deficit, which equals the rate of change of debt, will be

(1) $\dot{D}(t) = G_H - T + rD(t) > 0$.

Even though the primary deficit, G_H - T, is constant, the total deficit will be rising over time since the government debt outstanding and thus interest payments on that debt, rD(t), are rising.

At time τ, government debt will be positive: $D(\tau) > 0$. To satisfy its budget constraint at that time, taxes must have been set so that $T = G_L + rD(\tau) > G_L$. Thus for $t > \tau$, the budget will balance so that the deficit is zero and the debt will then be constant at its level as of time τ. See the figures below.





Problem 12.4

(a) First, we will use dynamic programming (see Section 10.4, the Shapiro-Stiglitz model, for more information regarding this technique) to find an expression for the expected present value of the revenue the government must raise when $G = G_H$, denoted $V_H(\Delta t)$. This expression is given by

(1) $$V_H(\Delta t) = \int_{t=0}^{\Delta t} e^{-rt} e^{-at}(G_H + rD)dt + e^{-r\Delta t}[e^{-a\Delta t}V_H(\Delta t) + (1 - e^{-a\Delta t})V_L(\Delta t)].$$

The first term on the right-hand side of (1) reflects the revenue the government must raise during the interval $(0, \Delta t)$. The probability that government spending is still high at time t is e^{-at}, in which case the government must raise $G_H + rD$. The e^{-rt} term discounts this using the constant interest rate, r. The second term reflects revenue needs after Δt. At time Δt, government purchases are still high with probability $e^{-a\Delta t}$, and have switched to being low with probability $(1 - e^{-a\Delta t})$. V_H and V_L denote the expected present value of the revenue the government must raise in each case. And this is then discounted by the $e^{-r\Delta t}$ term.

The integral in (1) can be solved as follows:

$$(2)\quad \int_{t=0}^{\Delta t} e^{-(a+r)t}(G_H + rD)dt = (G_H + rD)\left[\frac{-1}{(a+r)} e^{-(a+r)t}\Bigg|_{t=0}^{\Delta t}\right],$$

which simplifies to

$$(3)\quad \int_{t=0}^{\Delta t} e^{-(a+r)t}(G_H + rD)dt = \frac{G_H + rD}{a+r}\left[1 - e^{-(a+r)\Delta t}\right].$$

Substituting equation (3) into (1) yields

$$(4)\quad V_H(\Delta t) = \frac{G_H + rD}{a+r}[1 - e^{-(a+r)\Delta t}] + e^{-(a+r)\Delta t}V_H(\Delta t) + e^{-r\Delta t}(1 - e^{-a\Delta t})V_L(\Delta t),$$

and collecting terms in $V_H(\Delta t)$ gives us

$$(5)\quad V_H(\Delta t)[1 - e^{-(a+r)\Delta t}] = \frac{G_H + rD}{a+r}[1 - e^{-(a+r)\Delta t}] + e^{-r\Delta t}(1 - e^{-a\Delta t})V_L(\Delta t),$$

or simply

$$(6)\quad V_H(\Delta t) = \frac{G_H + rD}{a+r} + \frac{e^{-r\Delta t}(1 - e^{-a\Delta t})}{1 - e^{-(a+r)\Delta t}}V_L(\Delta t).$$

As described in Section 10.4, we now take the limit of the expression in (6) as the interval of time goes to zero. This requires using l'Hopital's rule. The derivative with respect to Δt of the numerator of the second term on the right-hand side of (6) is $-re^{-r\Delta t} + (a+r)e^{-(a+r)\Delta t}$. The limit of this as $\Delta t \to 0$ is a. The derivative of the denominator of that same term is $(a+r)e^{-(a+r)\Delta t}$ which goes to (a + r) as $\Delta t \to 0$. Thus, as $\Delta t \to 0$, we have

$$(7)\quad V_H = \frac{G_H + rD}{a+r} + \frac{a}{a+r}V_L = \frac{G_H + rD + aV_L}{a+r}.$$

Rearranging (7) gives an expression that can be interpreted as an asset-pricing condition:

(8) $rV_H = (G_H + rD) - a(V_H - V_L)$.

Similar analysis to the above would yield the following expression for V_L, the expected present value of the revenue the government must raise when $G = G_L$:

(9) $rV_L = (G_L + rD) - b(V_L - V_H)$,

or

$$(10)\quad V_L = \frac{G_L + rD + bV_H}{b+r}.$$

We can now solve (7) and (10) for V_H and V_L. Substituting equation (7) into equation (9) yields

$$(11)\quad rV_L = G_L + rD - b\left[V_L - \left(\frac{G_H + rD + aV_L}{a+r}\right)\right] = G_L + rD - b\left(\frac{rV_L - G_H - rD}{a+r}\right).$$

Collecting the terms in V_L yields

$$(12)\quad \left(r + \frac{br}{a+r}\right)V_L = G_L + rD + \frac{b}{a+r}(G_H + rD),$$

which simplifies to

$$(13)\quad \left[\frac{r(a+b+r)}{a+r}\right]V_L = G_L + \frac{b}{a+r}G_H + \frac{r(a+b+r)}{a+r}D,$$

and thus we have

$$(14)\quad V_L = \frac{(a+r)G_L + bG_H}{r(a+b+r)} + D.$$

Substituting (14) into (7) yields

(15) $$V_H = \frac{G_H + rD + a\left[\dfrac{(a+r)G_L + bG_H}{r(a+b+r)} + D\right]}{a+r},$$

which implies

(16) $$V_H = \frac{aG_L}{r(a+b+r)} + \frac{\left[r(a+b+r)+ab\right]G_H}{(a+r)r(a+b+r)} + D.$$

Note that r(a + b + r) + ab = br + r(a + r) + ab = b(a + r) + r(a + r) = (a + r)(b + r), and so

(17) $$V_H = \frac{aG_L + (b+r)G_H}{r(a+b+r)} + D.$$

Equations (14) and (17) give the expected present value of the revenue the government must raise as a function of its expenditures, the amount of debt outstanding, and the parameters of the model. With quadratic distortion costs and constant output, the optimal policy is for taxes to be expected to be constant also. Thus, when government spending is high, the government expects to impose a tax, T_H, such that

(18) $$\int_{t=0}^{\infty} e^{-rt} T_H \, dt = V_H.$$

Solving the integral and using equation (17) for V_H yields

(19) $$\frac{1}{r}T_H = \frac{aG_L + (b+r)G_H}{r(a+b+r)} + D,$$

or simply

(20) $$T_H = \frac{aG_L + (b+r)G_H}{(a+b+r)} + rD.$$

Similar analysis would show that when $G = G_L$, the government sets taxes, T_L, equal to

(21) $$T_L = \frac{(a+r)G_L + bG_H}{(a+b+r)} + rD.$$

(b) From equations (20) and (21) we can see that the path of taxes during an interval in which G is constant is driven by the path of outstanding debt, D. In general, the change in debt – or the budget deficit – is given by

(22) $$\dot{D} = G - T + rD.$$

From equation (20), the path of taxes during an interval in which G equals G_H is given by

(23) $$\dot{T}_H = r\dot{D} = r(G_H - T_H + rD).$$

Substituting equation (20) for T_H into equation (23) yields

(24) $$\dot{T}_H = r\left[G_H - \left(\frac{aG_L + (b+r)G_H}{a+b+r} + rD\right) + rD\right],$$

which simplifies to

(25) $$\dot{T}_H = r\left[\frac{(a+b+r)G_H - aG_L - (b+r)G_H}{a+b+r}\right],$$

or

(26) $$\dot{T}_H = \frac{ar(G_H - G_L)}{a+b+r} > 0.$$

As long as G equals G_H , the government runs a deficit and taxes are thus increasing over time because of the increased interest on the outstanding debt. Intuitively, the government knows there is a probability that its expenditures will fall in the future and so it runs a deficit in order to smooth taxes over time.

At the moment that G falls to G_L , taxes will drop from T_H to T_L . The path of taxes when G equals G_L is again driven by the path of outstanding debt, so that

(27) $\dot{T}_L = r\dot{D} = r(G_L - T_L + rD)$.

Substituting equation (21) for T_L yields

(28) $\dot{T}_L = r\left[G_L - \left(\dfrac{(a+r)G_L + bG_H}{a+b+r} + rD\right) + rD\right]$,

which simplifies to

(29) $\dot{T}_L = r\left[\dfrac{(a+b+r)G_L - bG_H - (a+r)G_L}{a+b+r}\right]$,

or

(30) $\dot{T}_L = \dfrac{br(G_L - G_H)}{a+b+r} < 0$.

As long as G equals G_L , the government runs a surplus and taxes are thus decreasing over time because of the decreased interest on the outstanding debt. Intuitively, the government knows there is a probability that its expenditures will rise in the future and so it runs a surplus in order to smooth taxes over time.

Problem 12.5

It is true that the model has an implication about the long run that is clearly incorrect and undesirable. But as with all models, we need to look at whether this is important for the issues the model was meant to address. This implication about the long run does not mean that the model does not provide a good approximation to actual and/or optimal fiscal policy in the short and medium terms.

The motive for studying tax smoothing is to examine its implications for the behavior of deficits over short and moderate time frames. And in fact, the model does provide interesting implications for the behavior of deficits during such short-run phenomena as wars and recessions. The simplifying assumptions that give rise to the result that the tax rate is a random walk – and thus that the tax rate would eventually exceed 100 percent or become negative – should only be considered problematic if they cause the model to give incorrect answers to the questions it was meant to address.

Problem 12.6

Assuming that everyone votes truthfully in each two-way contest, policy A would beat policy B by a vote of two to one and policy B would beat policy C by a vote of two to one. If society's preferences as a whole exhibited transitivity we would then expect policy A should beat policy C. But instead policy C would defeat policy A by a vote of two to one.

Thus the order of the pairwise voting would determine the outcome. If policies A and B were voted on first, C would be the eventual winner whereas, for example, if policies B and C were voted on first, A would be the eventual winner. Thus the voting choice essentially reverts back to the choice of agenda.

Also note that this provides for the incentive not to vote truthfully. For example, consider Voter 2. If the first vote is between policies B and C and Voter 2 votes truthfully (as do the other voters), then A – Voter 2's least-desired outcome – would be the eventual winner. If, however, Voter 2 casts her ballot for policy C in the first vote then C wins that vote and goes on to beat A in the second vote. Thus Voter 2 winds up better off with her second-best alternative by misrepresenting her preferences in that first vote.

Problem 12.7

Since the real interest rate is assumed to be zero, the period-1 policymaker has no interest payments on the initial debt, D_0. Thus the period-1 budget constraint remains

(1) $M_1 + N_1 = W + D$,

where W is the economy's endowment and D is the amount of debt the period-1 policymaker issues. In period 2, the policymaker must now pay off the initial debt, D_0, plus whatever was borrowed in the first period. Thus the period-2 constraint is

(2) $M_2 + N_2 = W - (D + D_0)$.

As explained in the text, the period-2 policymaker simply devotes all available resources, which are now given by $W - (D + D_0)$, to the type of government purchases preferred by the period-2 median voter.

Consider the first period and assume the period-1 median voter has $\alpha = 1$. Her expected utility, denoted E[V], as a function of D is given by

(3) $E[V] = U(W + D) + \pi U(W - (D + D_0)) + (1 - \pi)U(0)$.

The first term on the right-hand side of (3) reflects the fact that with $\alpha = 1$ for the median voter, the period-1 policymaker chooses $M_1 = W + D$ and $N_1 = 0$ and thus receives utility $U(W + D)$. With probability π, the period-2 median voter has $\alpha = 1$ and devotes all available resources, $W - (D + D_0)$, to military goods giving utility $U(W - (D + D_0))$ to the period-1 policymaker. Finally, with probability $(1 - \pi)$, the period-2 median voter has $\alpha = 0$ and so all available resources are devoted to non-military goods giving $U(0)$ to the period-1 policymaker.

The first-order condition for the period-1 policymaker's choice of D is

(4) $U'(W + D) - \pi U'(W - (D + D_0)) = 0$.

To see how the first-period deficit, $D = M_1 + N_1 - W$, responds to a change in D_0, implicitly differentiate equation (4) with respect to D_0 to obtain

(5) $$U''(W+D)\frac{\partial D}{\partial D_0} - \pi U''(W-(D+D_0))\left(-\frac{\partial D}{\partial D_0} - 1\right) = 0.$$

Collecting the terms in $\partial D/\partial D_0$ gives us

(6) $$[U''(W+D) + \pi U''(W-(D+D_0))]\frac{\partial D}{\partial D_0} = -\pi U''(W-(D+D_0)),$$

and thus

(7) $$\frac{\partial D}{\partial D_0} = \frac{-\pi U''(W-(D+D_0))}{U''(W+D) + \pi U''(W-(D+D_0))}.$$

Since $U''(\bullet) < 0$ and π is between zero and one, we can see that $-1 < \partial D/\partial D_0 < 0$.

Similar analysis for the case in which the period-1 median voter has $\alpha = 0$ would yield the following expression for the change in the first-period deficit due to a change in D_0:

(8) $$\frac{\partial D}{\partial D_0} = \frac{-(1-\pi)U''(W-(D+D_0))}{U''(W+D) + (1-\pi)U''(W-(D+D_0))},$$

and so again we have $-1 < \partial D/\partial D_0 < 0$.

Thus an increase in initial debt reduces the period-1 deficit; that is, it reduces borrowing by the first-period policymaker. An increase in debt, all else equal, reduces the resources available to the period-2 policymaker since she is the one that has to pay off this initial debt. In this model, the reason there are deficits is that there is a positive probability that the period-2 policymaker will devote the economy's resources to an activity that, in the view of the period-1 policymaker, simply wastes resources. The period-1 policymaker therefore has an incentive to reduce resources available in the second period by transferring resources from the second period to the first period by borrowing.

There is, however, also a chance that the period-2 policymaker will share the same preferences as the period-1 policymaker and devote all resources to the same type of purchases. But since an increase in initial debt reduces the resources available in period two, it reduces the amount the period-2 policymaker can purchase and thus increases the marginal utility of purchases in the second period. Since it is optimal to smooth purchases over time this would mean that the period-1 policymaker would actually have incentive to transfer resources to the second period to the extent that it is possible that the period-2 policymaker shares the same preferences. And this competing incentive increases as initial debt increases. Thus the period-1 policymaker borrows less the higher is the initial level of debt.

Problem 12.8

(a) Consider an individual with $\alpha = 1$; that is, someone who prefers military goods. In period one, with probability π the median voter also has $\alpha = 1$ and so the policymaker purchases all military goods giving the individual utility of $U(W + D)$. With probability $(1 - \pi)$, the median voter has $\alpha = 0$ resulting in the purchase of all non-military goods giving the individual $U(0)$.

Similarly in period two, with probability π the median voter has $\alpha = 1$ and so the policymaker devotes all available resources, $W - D$, to military goods giving utility of $U(W - D)$ to the $\alpha = 1$ individual. With probability $(1 - \pi)$, the median voter has $\alpha = 0$ resulting in the purchase of all non-military goods giving the individual $U(0)$.

Thus the individual with $\alpha = 1$ has expected utility, denoted $E[V]$, given by

(1) $E[V] = \pi U(W + D) + (1 - \pi)U(0) + \pi U(W - D) + (1 - \pi)U(0)$.

(b) The first-order condition for this individual's most preferred value of D is

(2) $\dfrac{\partial E[V]}{\partial D} = \pi U'(W + D) + \pi U'(W - D)(-1) = 0,$

or

(3) $U'(W + D) = U'(W - D)$.

With a well-behaved utility function, for example with $U''(\bullet) < 0$ everywhere, this implies

(4) $W + D = W - D$,

and thus implies

(5) $D = 0$.

The individual prefers a balanced budget so that no debt is issued.

(c) Similarly for someone with $\alpha = 0$ -- someone who prefers all non-military goods -- expected utility is given by

(6) $E[V] = \pi U(0) + (1 - \pi)U(W + D) + \pi U(0) + (1 - \pi)U(W - D)$.

The first-order condition is given by

(7) $\dfrac{\partial E[V]}{\partial D} = (1 - \pi)U'(W + D) + (1 - \pi)U'(W - D)(-1) = 0,$

or

(8) $U'(W + D) = U'(W - D)$,

and thus, again, this implies

(9) $D = 0$.

(d) Since all voters prefer D = 0, so does the median voter and thus the policymaker will pursue a balanced budget policy and not issue any debt.

(e) A balanced-budget requirement forces D = 0 for everyone. Without a requirement, the period-1 policymaker would choose D freely. Thus, it is possible that an $\alpha = 0$ individual would choose a different D than an $\alpha = 1$ individual; in fact, unless $\pi = 1/2$, they definitely would choose different values of D. Thus, answering part (d) does not answer the question of whether individuals will support a balanced-budget requirement.

Problem 12.9

(a) Since $\alpha = 1$, the period-1 median voter – who controls policy in both periods one and two – purchases all military goods in those two periods giving utility of $U(W + D_1)$ in the first period and $U(W + D_2)$ in the second period, where D_i represents the amount of debt issued in period i. In the third period, with probability π the period-3 median voter has $\alpha = 1$ and devotes all available resources, $W - D_1 - D_2$, to military purchases giving utility of $U(W - D_1 - D_2)$ to the $\alpha = 1$ individual. With probability $(1 - \pi)$, the period-3 median voter purchases all non-military goods giving utility of U(0) to the $\alpha = 1$ individual. Thus expected utility for someone with $\alpha = 1$, denoted E[V], is

(1) $E[V] = U(W + D_1) + U(W + D_2) + \pi U(W - D_1 - D_2) + (1 - \pi)U(0)$.

The period-1 median voter chooses D_1 and D_2. The first-order conditions are

(2) $$\frac{\partial E[V]}{\partial D_1} = U'(W + D_1) - \pi U'(W - D_1 - D_2) = 0,$$

and

(3) $$\frac{\partial E[V]}{\partial D_2} = U'(W + D_2) - \pi U'(W - D_1 - D_2) = 0.$$

Equations (2) and (3) imply

(4) $U'(W + D_1) = U'(W - D_2)$.

With $U''(\bullet) < 0$ everywhere, this implies

(5) $W + D_1 = W + D_2$,

and so

(6) $D_1 = D_2$.

Thus the policymaker issues the same amount of debt in each of the first two periods and so purchases in each of the first two periods, $M_1 = W + D_1$ and $M_2 = W + D_2$, must also be equal.

(b) To see the way in which the amount of debt issued in period two, D_2, varies with π we can implicitly differentiate the first-order condition given by equation (3) with respect to π. Note that we are treating D_1 as given since we are assuming that the change in π occurs after period one and thus after D_1 has been chosen. We have

(7) $$U''(W + D_2)\frac{\partial D_2}{\partial \pi} + (-1)U'(W - D_1 - D_2) + (-\pi)U''(W - D_1 - D_2)\left(-\frac{\partial D_2}{\partial \pi}\right) = 0.$$

Collecting the terms in $\partial D_2 / \partial \pi$ gives us

(8) $$[U''(W + D_2) + \pi U''(W - D_1 - D_2)]\frac{\partial D_2}{\partial \pi} = U'(W - D_1 - D_2),$$

and thus

(9) $$\frac{\partial D_2}{\partial \pi} = \frac{U'(W - D_1 - D_2)}{U''(W + D_2) + \pi U''(W - D_1 - D_2)} < 0,$$

since U '(•) > 0 and U "(•) < 0. Thus a fall in π increases D_2 . Thus the policymaker issues more debt and increases purchases in period two after the news that it is less likely that the period-3 median voter also prefers military goods. Intuitively, since it is now more likely that the period-3 median voter will prefer non-military goods, which the period-1 median voter deems wasteful, the period-1 median voter transfers more resources from the third period to the second period by borrowing more and devotes the additional resources with certainty to the type of good she prefers.

Problem 12.10

(a) The period-2 policymaker's objective function is

(1) $F_2 = U + \alpha_2 [V(G_1) + V(G_2)]$.

Substituting for private utility, $U = W - C(T_1) - C(T_2)$, and using the fact that taxes in period two must equal government consumption plus debt, $T_2 = G_2 + D$, gives us

(2) $F_2 = W - C(T_1) - C(G_2 + D) + \alpha_2 [V(G_1) + V(G_2)]$.

The period-2 policymaker takes W, T_1 , and D as given and thus the first-order condition is

(3) $$\frac{\partial F_2}{\partial G_2} = -C'(G_2 + D) + \alpha_2 V'(G_2) = 0.$$

(b) Implicitly differentiating equation (3) with respect to D yields

(4) $$-C''(G_2 + D)\left[\frac{\partial G_2}{\partial D} + 1\right] + \alpha_2 V''(G_2)\frac{\partial G_2}{\partial D} = 0,$$

or

(5) $$\left[\alpha_2 V''(G_2) - C''(G_2 + D)\right]\frac{\partial G_2}{\partial D} = C''(G_2 + D).$$

This implies

(6) $$\frac{\partial G_2}{\partial D} = \frac{C''(G_2 + D)}{\alpha_2 V''(G_2) - C''(G_2 + D)} < 0,$$

since C"(•) > 0 and V"(•) < 0. Thus an increase in debt reduces the period-2 policymaker's choice of government consumption.

(c) The period-1 policymaker's objective function, substituting for private utility, is

(7) $F_1 = W - C(T_1) - C(T_2) + \alpha_1 [V(G_1) + V(G_2)]$.

Note that G_2 is a function of D or $G_2 = G_2(D)$, and that since $D = G_1 - T_1$ we can write $T_1 = G_1 - D$. In addition, $T_2 = G_2 + D$. Thus (7) becomes

(8) $F_1 = W - C(G_1 - D) - C(G_2(D) + D) + \alpha_1 [V(G_1) + V(G_2(D))]$.

The first-order conditions for the choices of G_1 and D are

(9) $$\frac{\partial F_1}{\partial G_1} = -C'(G_1 - D) + \alpha_1 V'(G_1) = 0,$$

and

(10) $$\frac{\partial F_1}{\partial D} = -C'(G_1 - D)(-1) - C'(G_2(D) + D)[G_2'(D) + 1] + \alpha_1 V'(G_2(D))G_2'(D) = 0.$$

(d) Solving equation (3) for $V'(G_2(D))$ gives us

(11) $$V'(G_2(D)) = \frac{C'(G_2(D) + D)}{\alpha_2}.$$

Substituting equation (11) into equation (10) yields

(12) $C'(G_1 - D) - C'(G_2(D) + D)[G_2'(D) + 1] + \dfrac{\alpha_1}{\alpha_2} C'(G_2(D) + D)G_2'(D) = 0,$

which can be rewritten as

(13) $C'(G_1 - D) - C'(G_2(D) + D) = C'(G_2(D) + D)G_2'(D) - \dfrac{\alpha_1}{\alpha_2} C'(G_2(D) + D)G_2'(D).$

Collecting terms on the right-hand side of (13) gives us

(14) $C'(G_1 - D) - C'(G_2(D) + D) = C'(G_2(D) + D)G_2'(D)\left(1 - \dfrac{\alpha_1}{\alpha_2}\right).$

As shown in part (b), $G_2'(D) < 0$ and since $C'(\bullet) > 0$, then if $\alpha_1 < \alpha_2$, the right-hand side of (14) is negative. Thus

(15) $C'(G_1 - D) - C'(G_2(D) + D) < 0,$

or

(16) $C'(G_1 - D) < C'(G_2(D) + D).$

Since $C''(\bullet) > 0$ this implies

(17) $G_1 - D < G_2(D) + D.$

Since $D = G_1 - T_1$ or $T_1 = G_1 - D$ and $T_2 = G_2(D) + D$, this is equivalent to

(18) $T_1 < T_2$.

Intuitively, if $\alpha_1 < \alpha_2$ this means the period-1 policymaker values government consumption less than the period-2 policymaker. Thus the period-1 policymaker attempts to "enforce discipline" on the period-2 policymaker. The lower-α policymaker in period one keeps taxes low and thus passes along a relatively higher level of D in order to force the period-2 policymaker to choose a lower level of government consumption.

(e) Not necessarily. If $\alpha_1 < \alpha_2$, the period-1 policymaker will choose a lower level of government purchases than the period-2 policymaker. To see this, substitute equations (3) and (9) into the first-order condition given by (10):

(19) $\alpha_1 V'(G_1) - \alpha_2 V'(G_2(D))[G_2'(D) + 1] + \alpha_1 V'(G_2(D))G_2'(D) = 0,$

which can be rewritten as

(20) $\alpha_1 V'(G_1) = V'(G_2(D))[-\alpha_1 G_2'(D) + \alpha_2 (G_2'(D) + 1)],$

which implies

(21) $\dfrac{V'(G_1)}{V'(G_2(D))} = \dfrac{\alpha_2}{\alpha_1} + \left(\dfrac{\alpha_2 - \alpha_1}{\alpha_1}\right)G_2'(D).$

Adding and subtracting $(\alpha_2 - \alpha_1)/\alpha_1$ from the right-hand side of (21) yields

(22) $\dfrac{V'(G_1)}{V'(G_2(D))} = \dfrac{\alpha_2}{\alpha_1} - \dfrac{\alpha_2 - \alpha_1}{\alpha_1} + \left(\dfrac{\alpha_2 - \alpha_1}{\alpha_1}\right)[G_2'(D) + 1],$

or simply

(23) $\dfrac{V'(G_1)}{V'(G_2(D))} = 1 + \left(\dfrac{\alpha_2 - \alpha_1}{\alpha_1}\right)[G_2'(D) + 1].$

From equation (6), we can see that $G_2'(D) > -1$ or $G_2'(D) + 1 > 0$. In addition, our assumption is that $\alpha_2 - \alpha_1 > 0$. Thus

(24) $\dfrac{V'(G_1)}{V'(G_2(D))} > 1,$

or $V'(G_1) > V'(G_2(D))$. Since $V'(\bullet) < 0$, this implies $G_1 < G_2(D)$. Thus, not only does the period-1 policymaker choose a lower level of taxes, she also chooses a lower level of government consumption than the period-2 policymaker. Thus $D = G_1 - T_1 = T_2 - G_2$ can be either positive or negative.

Problem 12.11

(a) As T, the amount of taxes that reform requires, falls then V '(X) at X = A also falls since

(1) $$V'(X = A) = \frac{[B - (W - T)] - 2A}{B - A},$$

and we have

(2) $$\frac{\partial V'(X = A)}{\partial T} = \frac{1}{B - A} > 0.$$

In the case in which V '(X) at X = A was already negative, it is now more negative and there is no effect on workers' offer or the probability of reform. Workers continue to offer X* = A and the probability of reform, P(X*), continues to equal one.

If initially V '(X) at X = A was positive and the change in T is small enough, V '(X = A) will still be positive. In this case, from equation (12.37) in the text, workers' offer is

(3) $$X^* = \frac{B - (W - T)}{2},$$

and so

(4) $$\frac{\partial X^*}{\partial T} = \frac{1}{2} > 0.$$

Thus a fall in T reduces workers' offer. From equation (12.38) in the text, the probability of reform is

(5) $$P(X^*) = \frac{B + (W - T)}{2(B - A)},$$

and so

(6) $$\frac{\partial P(X^*)}{\partial T} = \frac{-1}{2(B - A)} < 0.$$

The fall in T increases the probability of reform in this case.

Finally, if V '(X) at X = A was initially positive and the change in T is large enough, it will become negative. In this case, workers' offer will now be X* = A and reform will now occur with certainty.

(b) An increase in B, the upper bound on capitalists' pre-tax payoff from reform, means that V '(X) at X = A also increases since

(7) $$\frac{\partial V'(X = A)}{\partial B} = \frac{(B - A) - [B - (W - T)] + 2A}{(B - A)^2} = \frac{(W - T) + A}{(B - A)^2} > 0.$$

Thus, if initially V '(X) at X = A was positive, it still will be. Workers' offer continues to be given by equation (3) and

(8) $$\frac{\partial X^*}{\partial B} = \frac{1}{2} > 0.$$

Thus workers' offer increases. That is, with an increase in the upper bound on capitalists' payoff, workers ask capitalists to pay a greater share of the costs of reform. Using equation (5) we have

(9) $$\frac{\partial P(X^*)}{\partial B} = \frac{2(B - A) - [B + (W - T)]2}{4(B - A)^2} = \frac{-[A + (W - T)]}{2(B - A)^2} < 0.$$

The increase in B causes the probability of reform to fall.

If initially V '(X) at X = A was negative and the rise in B is small enough, it will continue to be negative. There will be no effect on workers' offer, which continues to be X* = A, or on the probability of reform, which continues to be one. If, however, the rise in B is large enough, V '(X) at X = A becomes positive in which case workers' offer will now be greater than A and the probability of reform will fall below one.

(c) An upward shift in the distribution of capitalists' payoff – an equal increase in A and B – means that V '(X) at the new X = A' will be lower than V '(X) at the original X = A. We can see this since

$$(10)\ \ V'(X = A) = \frac{[B - (W - T)] - 2A}{B - A}.$$

An equal increase in A and B leaves the denominator unchanged and reduces the numerator.

In the case in which V '(X) at X = A was negative, it is now more negative at the new X = A'. Thus workers' offer rises and equals the new A' and the probability of reform continues to be one.

If initially V '(X) at X = A was positive and the change in A and B is small enough, V ' (X) will still be positive at the new X = A'. Note that A does not enter workers' offer here and so we need only examine the derivative of X* with respect to B:

$$(11)\ \ \frac{\partial X^*}{\partial B} = \frac{1}{2} > 0.$$

Hence workers' offer rises proportionately less than B (or A). From equation (5) which gives the probability of reform, we can see that the probability of reform increases since the numerator rises whereas the denominator is unchanged.

Finally, if V '(X) at X = A was positive and the change in A and B is large enough, V '(X) at the new X = A' will be negative. Thus workers' offer will equal the new A' and the probability of reform becomes one.

<u>Problem 12.12</u>

(a) If the capitalists accept the workers' proposal and reform occurs, their payoff is π - X. If they reject the proposal, their payoff is now -C, C $\geq$ 0, rather than zero. They therefore accept when π - X > -C, or π > X - C. Since π is distributed uniformly on [A, B] this probability is

$$(1)\ \ P(X) = \begin{cases} 1 & \text{if} \quad X - C \le A \quad \text{or} \quad X \le A + C \\ \dfrac{B - (X - C)}{B - A} & \text{if} \quad A < X - C < B \quad \text{or} \quad A + C < X < B + C \\ 0 & \text{if} \quad X - C \ge B \quad \text{or} \quad X \ge B + C, \end{cases}$$

where we have used the fact that for A + C < X < B + C, P(X) = P(π > X - C) = 1 - P(π < X - C) which in turns equals 1 - [(X - C) - A]/(B - A) or simply [B - (X - C)]/(B - A).

The workers receive (W - T) + X if their proposal is accepted and -C if it is rejected. Their expected payoff, V(X), therefore equals P(X)[(W - T) + X] + [1 - P(X)](-C). Using equation (1), this equals

$$(2)\ \ V(X) = \begin{cases} (W - T) + X & \text{if} \quad X \le A + C \\ \dfrac{[B - (X - C)][(W - T) + X + F]}{B - A} + \left[\dfrac{1 - (B - (X - C))}{B - A}\right](-C) & \text{if} \quad A + C < X < B + C \\ -C & \text{if} \quad X \ge B + C. \end{cases}$$

As in the model in the text, there are two possibilities. First, the workers may choose a value of X in the interior of [A + C, B + C] so that the probability of the capitalists accepting the proposal is strictly between zero and one. Second, the workers may make the least-generous proposal that they know will be accepted for sure, which is X = A + C.

Using equation (2) to find the derivative of V(X) with respect to X for A + C < X < B + C yields

$$(3)\quad V'(X) = \frac{B-(W-T)-2X-C+C}{B-A} = \frac{[B-(W-T)]-2X}{B-A}.$$

Note that V "(X) is negative over the whole range we are considering. Thus if V '(X) is negative at X = A + C, it is negative over all of [A + C, B + C]. In this case, workers propose X = A + C, the least-generous proposal they know will be accepted for sure. This occurs when V '(X = A + C) < 0 or when [B - (W - T)] - 2(A + C) < 0.

The alternative is for V '(X) to be positive at X = A + C. In this case, the optimum is interior to the interval [A + C, B + C] and is defined by V '(X) = 0. From equation (3), this occurs when [B - (W - T)] - 2X = 0. Thus, analogous to equation (12.37) in the text, we have

$$(4)\quad X^* = \begin{cases} A+C & \text{if} \quad [B-(W-T)]-2(A+C) \le 0 \\ \dfrac{B-(W-T)}{2} & \text{if} \quad [B-(W-T)]-2(A+C) > 0. \end{cases}$$

Thus, using equation (1) and substituting for X*, we have the following expression for the equilibrium probability that the proposal is accepted:

$$(5)\quad P(X^*) = \begin{cases} 1 & \text{if} \quad [B-(W-T)]-2(A+C) \le 0 \\ \dfrac{B+(W-T)+2C}{B-A} & \text{if} \quad [B-(W-T)]-2(A+C) > 0. \end{cases}$$

Equation (5) is analogous to equation (12.38) in the text.

(b) If, in equilibrium, V '(X) at X = A + C is less than or equal to zero, then workers offer X* = A + C and P(X*) = 1. In this case, workers get (W - T) + (A + C) and capitalists expected payoff is E[π] - (A + C). Thus social welfare, SW(X*), is given by
(6) SW(X*) = (W - T) + (A + C) + E[π] - (A + C) = (W - T) + E[π].
Since π is distributed uniformly on [A, B], E[π] = (A + B)/2 and thus
(7) SW(X*) = (W - T) + (A + B)/2.

From equation (3), we can see that V '(X) evaluated at X = A + C is decreasing in C. Thus if V '(X) is negative initially, it still will be after an increase in C and social welfare will remain unchanged as reform still occurs with probability one. Social welfare is higher with reform than without and so initially if V '(X) at X = A + C is positive and the increase in C is large enough, it becomes negative at the new X = A + C'. The reform now occurs with certainty and social welfare is therefore higher.

Finally, if V '(X) at X = C was initially positive and the rise in C is small enough, V '(X) at the new X = A + C' will still be positive. We need to determine equilibrium social welfare in this case and the change in equilibrium social welfare due to a change in C.

For workers, the expected payoff, denoted V(X*), equals the probability of acceptance times the payoff from acceptance plus the probability of rejection – which is one minus the probability of acceptance – times the payoff from rejection, or from equation (2),

(8) $$V(X^*) = \frac{[B-(X^*-C)][(W-T)+X^*]}{B-A} + \left[1 - \frac{B-(X^*-C)}{B-A}\right](-C),$$

which can be rewritten as

(9) $$V(X^*) = \frac{[B-(X^*-C)](W-T)}{B-A} + \frac{[B-(X^*-C)]X^*}{B-A} + \frac{[A-(X^*-C)]C}{B-A}.$$

For capitalists, if π turns out to be less than X* - C, they reject the proposal and receive -C. If π turns out to be greater than X* - C, they accept the proposal and receive π - X*. Since π is distributed uniformly on [A, B], the probability density function of π over that interval is $f(\pi) = 1/(B - A)$. Thus, capitalists' expected payoff, denoted K(X*), is given by

(10) $$K(X^*) = \int_{\pi=A}^{X^*-C} \frac{-C}{B-A}\,d\pi + \int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A}\,d\pi.$$

The first integral on the right-hand side of equation (10) is given by

(11) $$\int_{\pi=A}^{X^*-C} \frac{-C}{B-A}\,d\pi = \frac{-C[(X^*-C)-A]}{B-A} = \frac{[A-(X^*-C)]C}{B-A}.$$

The second integral on the right-hand side of (10) is given by

(12) $$\int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A}\,d\pi = \frac{1}{B-A}\left[\left(\frac{1}{2}\pi^2 - X^*\pi\right)\Bigg|_{\pi=X^*-C}^{B}\right],$$

or

(13) $$\int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A}\,d\pi = \frac{1}{B-A}\left[\frac{1}{2}B^2 - BX^* - \frac{1}{2}(X^*-C)^2 + (X^*-C)X^*\right],$$

which can be factored as follows:

(14) $$\int_{\pi=X^*-C}^{B} \frac{\pi - X^*}{B-A}\,d\pi = \frac{1}{2(B-A)}\left[B^2 - (X^*-C)^2\right] - \frac{1}{B-A}\left[B-(X^*-C)\right]X^*.$$

Social welfare, which is the sum of the expected payoffs of workers and capitalists, can be obtained by adding equations (9), (11), and (14):

(15) $$SW(X^*) = \frac{[B-(X^*-C)](W-T)}{B-A} + 2\frac{[A-(X^*-C)]C}{B-A} + \frac{[B^2-(X^*-C)^2]}{2(B-A)}.$$

Since X* does not depend on C – see equation (4) – the change in equilibrium social welfare due to a change in the cost of a crisis, C, is

(16) $$\frac{\partial SW(X^*)}{\partial C} = \frac{(W-T)+2A-2X^*+4C}{B-A} + \frac{2X^*-2C}{2(B-A)} = \frac{2(W-T)+4A+6C-2X^*}{2(B-A)}.$$

Substituting for X* = [B - (W - T)]/2 gives us

(17) $$\frac{\partial SW(X^*)}{\partial C} = \frac{2(W-T)+4A+6C-B+(W-T)}{2(B-A)} = \frac{3(W-T)+4A+6C-B}{2(B-A)}.$$

Thus, depending on the magnitude of B relative to 3(W - T) + 4A + 6C, an increase in the cost of a crisis can, but does not necessarily, increase social welfare. So, for example, a high value of B – the upper bound on capitalists' pre-tax payoff from reform – makes it less likely that an increase in the cost of a crisis will increase social welfare.

Problem 12.13

If the capitalists accept the workers' proposal and reform occurs, their payoff is $\pi + F - X$, where $F > 0$ is the amount of aid they receive from the international agency. If they reject the proposal, they receive zero. They therefore accept when $\pi + F - X > 0$, or $\pi > X - F$. Since π is distributed uniformly on [A, B] this probability is

$$(1)\quad P(X) = \begin{cases} 1 & \text{if} \quad X - F \le A \quad \text{or} \quad X \le A + F \\ \dfrac{B - (X - F)}{B - A} & \text{if} \quad A < X - F < B \quad \text{or} \quad A + F < X < B + F \\ 0 & \text{if} \quad X - F \ge B \quad \text{or} \quad X \ge B + F, \end{cases}$$

where we have used the fact that for $A + F < X < B + F$, $P(X) = P(\pi > X - F) = 1 - P(\pi < X - F)$ which in turns equals $1 - [(X - F) - A]/(B - A)$ or simply $[B - (X - F)]/(B - A)$.

The workers receive $(W - T) + X + F$ if their proposal is accepted and zero if it is rejected. Their expected payoff, V(X), therefore equals $P(X)[(W - T) + X + F]$. Using equation (1), this equals

$$(2)\quad V(X) = \begin{cases} (W - T) + X + F & \text{if} \quad X \le A + F \\ \dfrac{[B - (X - F)][(W - T) + X + F]}{B - A} & \text{if} \quad A + F < X < B + F \\ 0 & \text{if} \quad X \ge B + F. \end{cases}$$

As in the model in the text, there are two possibilities. First, the workers may choose a value of X in the interior of [A + F, B + F] so that the probability of the capitalists accepting the proposal is strictly between zero and one. Second, the workers may make the least-generous proposal that they know will be accepted for sure, which is $X = A + F$.

Using equation (2) to find the derivative of V(X) with respect to X for $A + F < X < B + F$ yields

$$(3)\quad V'(X) = \frac{B - (W - T) - 2X - F + F}{B - A} = \frac{[B - (W - T)] - 2X}{B - A}.$$

Note that V ''(X) is negative over the whole range we are considering. Thus if V '(X) is negative at $X = A + F$, it is negative over all of [A + F, B + F]. In this case, workers propose $X = A + F$, the least-generous proposal they know will be accepted for sure. This occurs when $V'(X = A + F) < 0$ or when $[B - (W - T)] - 2(A + F) < 0$.

The alternative is for V '(X) to be positive at $X = A + F$. In this case, the optimum is interior to the interval [A + F, B + F] and is defined by $V'(X) = 0$. From equation (3), this occurs when $[B - (W - T)] - 2X = 0$. Thus, analogous to equation (12.37) in the text, we have

$$(4)\quad X^* = \begin{cases} A + F & \text{if} \quad [B - (W - T)] - 2(A + F) \le 0 \\ \dfrac{B - (W - T)}{2} & \text{if} \quad [B - (W - T)] - 2(A + F) > 0. \end{cases}$$

Thus, using equation (1) and substituting for X*, we have the following expression for the equilibrium probability that the proposal is accepted:

$$(5)\quad P(X^*) = \begin{cases} 1 & \text{if} \quad [B - (W - T)] - 2(A + F) \le 0 \\ \dfrac{B + (W - T) + 2F}{B - A} & \text{if} \quad [B - (W - T)] - 2(A + F) > 0. \end{cases}$$

Comparing equation (5) to equation (12.38) in the text, we can see that the presence of $F > 0$, the positive amount of aid, increases the probability of reform, if reform did not already occur with certainty. If F is large enough, reform now occurs with probability one since, as discussed above, reform occurs with certainty if [B - (W - T)] - 2(A + F) < 0.

Otherwise, from equation (5), we can see that P(X*) rises as F rises since

(6) $$\frac{\partial P(X^*)}{\partial F} = \frac{2}{B - A} > 0.$$

We now need to determine the impact of the international aid on social welfare, defined as the sum of the expected payoffs of workers and capitalists. If, in equilibrium, V '(X) at X = A + F is less than or equal to zero, then workers offer X* = A + F and P(X*) = 1. In this case, workers get (W - T) + X* + F or simply (W - T) + A + 2F. Capitalists expected payoff is E[π] - X* + F or simply E[π] - A. Thus social welfare, SW(X*), is given by

(7) SW(X*) = (W - T) + A + 2F + E[π] - A = (W - T) + 2F + E[π].

Since π is distributed uniformly on [A, B], E[π] = (A + B)/2 and thus

(8) SW(X*) = (W - T) + 2F + (A + B)/2.

From equation (3), we can see that V '(X) evaluated at X = A + F is decreasing in F. Thus if V '(X) is negative initially, it still will be. Here, since reform would have occurred anyway, social welfare simply increases by the total payoff from the international agency, which is 2F, and the entire amount of aid is extracted by workers.

If initially, V '(X) at X = A was positive and F is large enough, the aid causes V '(X) at X = A + F to be negative so that reform now occurs with certainty. Since social welfare is higher with reform, social welfare is higher in this case also.

Finally, if V '(X) at X = A was initially positive and F is small enough, V '(X) at X = A + F will still be positive. We need to determine equilibrium social welfare in this case. Equation (2) describes workers' expected payoff. It equals the probability of acceptance times the payoff from acceptance; the payoff from rejection is zero. Thus, from equation (2),

(9) $$V(X^*) = \frac{[B - (X^* - F)][(W - T) + X^* + F]}{B - A}.$$

Substituting X* = [B - (W - T)]/2 into equation (9) gives us

(10) $$V(X^*) = \frac{\left[B - \left(\frac{B - (W - T)}{2}\right) + F\right]\left[(W - T) + \left(\frac{B - (W - T)}{2}\right) + F\right]}{B - A},$$

which simplifies to

(11) $$V(X^*) = \frac{[2B - B + (W - T) + 2F][2(W - T) + B - (W - T) + 2F]}{4(B - A)},$$

and thus workers' expected payoff is given by

(12) $$V(X^*) = \frac{[B + (W - T) + 2F]^2}{4(B - A)}.$$

For capitalists, if π turns out to be less than X* - F, they reject the proposal and receive zero. If π turns out to be greater than X* - F, they accept the proposal and receive π - X* + F or π - (X* - F). Since π is distributed uniformly on [A, B], the probability density function of π over that interval is f(π) = 1/(B - A). Thus, capitalists' expected payoff, denoted K(X*), is given by

(13) $K(X^*) = \int_{\pi = X^* - F}^{B} \frac{\pi - (X^* - F)}{B - A} d\pi$.

Solving the integral in equation (13) gives us

(14) $K(X^*) = \frac{1}{B - A}\left[\left(\frac{1}{2}\pi^2 - (X^* - F)\pi\right)\Big|_{\pi = X^* - F}^{B}\right]$,

which simplifies to

(15) $K(X^*) = \frac{1}{B - A}\left[\frac{1}{2}B^2 - (X^* - F)B - \frac{1}{2}(X^* - F)^2 + (X^* - F)^2\right]$,

which can be factored as

(16) $K(X^*) = \frac{1}{2(B - A)}\left[B^2 - 2B(X^* - F) + (X^* - F)^2\right] = \frac{1}{2(B - A)}\left[B - (X^* - F)\right]^2$.

Substituting X* = [B - (W - T)]/2 into equation (16) gives us

(17) $K(X^*) = \frac{1}{2(B - A)}\left[B - \left(\frac{B - (W - T)}{2}\right) + F)\right]^2 = \frac{1}{8(B - A)}\left[B + (W - T) + 2F\right]^2$.

Total social welfare is the sum of the expected payoffs for workers and capitalists. Adding equations (12) and (17) gives us

(18) $SW(X^*) = V(X^*) + K(X^*) = \frac{[B + (W - T) + 2F]^2}{4(B - A)} + \frac{[B + (W - T) + 2F]^2}{8(B - A)}$,

or simply

(19) $SW(X^*) = \frac{3[B + (W - T) + 2F]^2}{8(B - A)}$.

From equation (19), we can see that social welfare is increasing in F and so the aid package from the international agency does raise social welfare unambiguously.

Problem 12.14

(a) Of the fraction f of the population that knows its welfare under both policies, fraction α is better off with Policy A. Thus fraction αf of those who know their welfare prefer Policy A.

Ex ante, the individuals in the fraction (1 - f) of the population that does not know its welfare are all identical. Each of these individuals will prefer Policy A if their expected utility from A exceeds that from B. The expected utility from Policy A, relative to that from Policy B, denoted $E[U^A]$, is given by

(1) $E[U^A] = \beta(+1) + (1 - \beta)(-1) = 2\beta - 1$,

since with probability β they will be one unit of utility better off and with probability $(1 - \beta)$ they will be one unit of utility worse off. These individuals will all prefer Policy A if $2\beta - 1 > 0$ or $\beta > 1/2$. If $\beta < 1/2$, all of these individuals prefer Policy B and if $\beta = 1/2$, they are indifferent.

Thus the fraction of the population that prefers Policy A under uncertainty, denoted X_u^A, is given by

(2) $X_u^A = \begin{cases} \alpha f + (1 - f) & \text{if} \quad \beta > 1/2 \\ \alpha f & \text{if} \quad \beta < 1/2. \end{cases}$

(Note that if $\beta = 1/2$, the fraction of the population that prefers Policy A would be αf plus (1 - f) times the fraction of those who are indifferent who decide to choose A.)

(b) Of the fraction f that always knows its welfare under both policies, fraction α prefer A. Now, of the fraction (1 - f) that previously did not know its welfare, fraction β find out that they are definitely better off under Policy A. Thus β(1 - f) now prefer A. Thus the fraction of the population who prefer Policy A under certainty, denoted X_c^A, is given by

(3) $X_c^A = \alpha f + \beta(1 - f)$.

(c) There are cases when whichever policy is initially in effect is retained. Suppose Policy A is in effect. From equation (2) we can see that a proposal to switch to Policy B will be defeated if, for example, $\beta > 1/2$ and $\alpha f + (1 - f) \geq 1/2$. The sum of the people who know their welfare and are better off with A, αf, plus the entire fraction of the population that is uncertain, (1 - f), vote to retain A in this case. If they constitute at least half of the population, the proposal is defeated.

Suppose Policy B is in effect. We are assuming that no one votes for a switch to Policy A if they know that once everyone's welfare is revealed, the majority would vote to revert back to Policy B. From equation (3), once welfare is revealed, fraction $\alpha f + \beta(1 - f)$ prefer A. If this is less than 1/2, the majority would vote to return to Policy B.

Thus whichever policy is in effect would be retained if $\beta > 1/2$, $\alpha f + (1 - f) \geq 1/2$, and $\alpha f + \beta(1 - f) < 1/2$. Because $\alpha f + (1 - f)$ is greater than $\alpha f + \beta(1 - f)$, it is easy to find parameter values that satisfy these conditions. One example is f = 0.5, α = 0.2, and β = 0.6. In this particular example, everyone knows that ex post, Policy B is preferred by the majority. Yet if Policy A is in effect, it is retained. This is driven by the fact that the entire portion of the population that is uncertain about its welfare maximizes its expected utility by voting for A but once that fraction of the population learns its welfare, not enough of them are better under A to constitute a majority when joined with the others who always preferred A.

Problem 12.15

(a) The representative from district j will maximize the utility of the representative person in that district, which is given by

(1) $U_j = E + V(G_j) - C(T)$,

subject to the budget constraint given by

(2) $\sum_{i=1}^{M} G_i = MT$,

which can be rewritten as

(3) $T = \dfrac{\sum_{i=1}^{M} G_i}{M}$.

Substituting equation (3) into equation (1) gives us

(4) $U_j = E + V(G_j) - C\left(\dfrac{\sum_{i=1}^{M} G_i}{M}\right)$.

The first-order condition is given by

(5) $\dfrac{\partial U_j}{\partial G_j} = V'(G_j) - C'\left(\dfrac{\sum_{i=1}^{M} G_i}{M}\right)\dfrac{1}{M} = 0$,

or

(6) $V'(G_j) = \dfrac{1}{M} C'\left(\dfrac{\sum_{i=1}^{M} G_i}{M}\right)$.

(b) We want a value of G, denoted G^N, that is optimal for a representative to choose given that all other representatives are choosing that level. Substituting that common choice of G^N for G_j and all the G_i's in the condition defining the optimal choice of G, equation (6), gives us

$$(7)\quad V'(G^N) = \frac{1}{M} C'\left(\frac{\sum_{i=1}^{M} G^N}{M}\right) = \frac{1}{M} C'\left(\frac{MG^N}{M}\right),$$

or simply

$$(8)\quad V'(G^N) = \frac{1}{M} C'(G^N).$$

Representatives choose a level of the local public good, G^N, such that the marginal utility of G^N equals only their district's share of the marginal distortion costs of the taxes needed to finance that good.

(c) To see if the Nash equilibrium is Pareto efficient, we can examine the social planner's problem. A social planner would maximize the sum of the utilities of the representative person in each district, which is given by

$$(9)\quad \sum_{j=1}^{M} U_j = \sum_{j=1}^{M}\left[E + V(G_j) - C\left(\frac{\sum_{i=1}^{M} G_i}{M}\right)\right],$$

where we have already substituted for the budget constraint using equation (3). Equation (9) simplifies to

$$(10)\quad \sum_{j=1}^{M} U_j = ME + \sum_{j=1}^{M} V(G_j) - MC\left(\frac{\sum_{i=1}^{M} G_i}{M}\right).$$

The social planner chooses the same level of the public good in each district, which we can denote $\overline{G}$. Thus equation (10) becomes

$$(11)\quad \sum_{j=1}^{M} U_j = ME + \sum_{j=1}^{M} V(\overline{G}) - MC\left(\frac{M\overline{G}}{M}\right) = ME + MV(\overline{G}) - MC(\overline{G}).$$

The first-order condition for the choice of $\overline{G}$ is

$$(12)\quad \frac{\partial \sum_{j=1}^{M} U_j}{\partial \overline{G}} = MV'(\overline{G}) - MC'(\overline{G}) = 0,$$

or simply

$$(13)\quad V'(\overline{G}) = C'(\overline{G}).$$

The social planner equates the marginal utility of the level of the local public good with the total marginal distortion costs of the taxes required to finance that good. Comparing equations (6) and (13), then since $V''(\bullet) < 0$, we can see that the social planner chooses a lower level of G for each district than the representatives do in the Nash equilibrium. That is, the Nash equilibrium involves an inefficiently high level of local public goods.

Intuitively, in the decentralized equilibrium, an increase in the level of a local public good in any given district gives the representative person in that district marginal utility of $V'(G)$. But the marginal cost of the distortion caused by the extra taxation needed to finance that good is borne by all individuals in all districts. Essentially, there is a negative externality from higher government purchases. Since individuals in any given district do not bear all the costs of extra purchases in that district, purchases are inefficiently high.

Problem 12.16

(a) The representative from district j will maximize the utility of the representative person in that district, which is given by

(1) $U_j = E + V(G_j) - C(T)$,

subject to the budget constraint given by

(2) $D + \sum_{i=1}^{M} G_i = MT$,

which can be rewritten as

(3) $T = \dfrac{D}{M} + \dfrac{\sum_{i=1}^{M} G_i}{M}$.

Substituting equation (3) into equation (1) gives us

(4) $U_j = E + V(G_j) - C\left(\dfrac{D}{M} + \dfrac{\sum_{i=1}^{M} G_i}{M}\right)$.

The first-order condition is given by

(5) $\dfrac{\partial U_j}{\partial G_j} = V'(G_j) - C'\left(\dfrac{D}{M} + \dfrac{\sum_{i=1}^{M} G_i}{M}\right)\dfrac{1}{M} = 0$.

The Nash equilibrium value of G, denoted G^N, is the one that is optimal for a representative to choose given that all other representatives are choosing that level. Substituting that common choice of G^N for G_j and all the G_i's into equation (5), the condition defining the optimal choice of G, gives us

(6) $V'(G^N) - \dfrac{1}{M}C'\left(\dfrac{D}{M} + G^N\right) = 0$.

To see how G^N is affected by changes in the initial amount of debt, we can implicitly differentiate equation (6) with respect to D, which yields

(7) $V''(G^N)\dfrac{\partial G^N}{\partial D} - \dfrac{1}{M}C''\left(\dfrac{D}{M} + G^N\right)\left[\dfrac{\partial G^N}{\partial D} + \dfrac{1}{M}\right] = 0$.

Collecting the terms in $\partial G^N/\partial D$ gives us

(8) $\left[V''(G^N) - \dfrac{1}{M}C''\left(\dfrac{D}{M} + G^N\right)\right]\dfrac{\partial G^N}{\partial D} = \dfrac{1}{M^2}C''\left(G^N + \dfrac{D}{M}\right)$,

and thus

(9) $\dfrac{\partial G^N}{\partial D} = \dfrac{\dfrac{1}{M^2}C''\left(G^N + \dfrac{D}{M}\right)}{V''(G^N) - \dfrac{1}{M}C''\left(\dfrac{D}{M} + G^N\right)} < 0$,

since $C''(\bullet) > 0$ and $V''(\bullet) < 0$. Thus an increase in initial debt reduces the Nash equilibrium level of the local public good.

(b) As explained in the solution to Problem 12.15, the representatives would choose an inefficiently high level of local public goods in the first period; the distortion costs of the taxation needed to finance those goods would be inefficiently high. As shown in part (a), the representatives know that by having $D > 0$, they can reduce the purchases of public goods and thus the distortion costs of the taxes because a positive value of debt will reduce the inefficiently high level of government purchases in the second period.

(c) If representatives were to choose D before the first-period value of G is determined, the representatives would choose not to issue any debt. It is true that with $D = 0$, there will be distortion in the choice of local public goods each period as shown in Problem 12.5; the level of local public goods will be inefficiently high. Choosing $D > 0$ reduces the choice of local public goods in the second period, as shown in part (a), which at the margin is desirable. But analogous reasoning would show that it would raise the choice of local public goods in the first period, which at the margin is undesirable. Thus having $D > 0$ does not clearly counteract the "common-pool" distortion. In addition, it introduces departures from tax-smoothing and expenditure-smoothing, and thus it appears that representatives would not choose to issue any debt.

Problem 12.17

The probability density function of T is given by

$$(1)\quad f(T) = \begin{cases} \dfrac{1}{2X} & \text{if} \quad \mu - x \le T \le \mu + x \\ 0 & \text{otherwise.} \end{cases}$$

The associated cumulative distribution function is given by

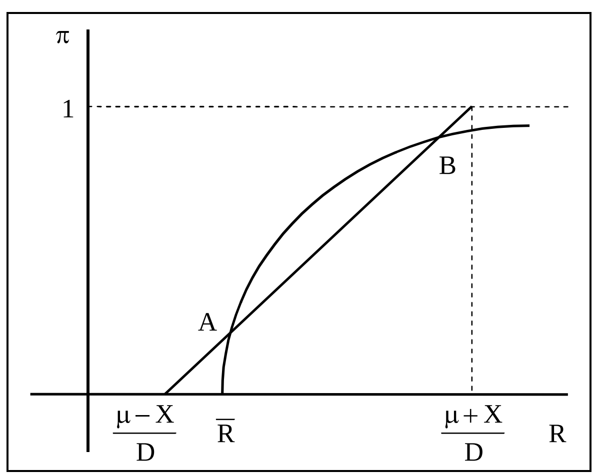
$$(2)\quad F(T) = \begin{cases} 0 & \text{if} \quad T < \mu - X \\ \dfrac{T - (\mu - X)}{2X} & \text{if} \quad \mu - x \le T \le \mu + x \\ 1 & \text{if} \quad T > \mu - X. \end{cases}$$

The probability of a default equals the probability that tax revenue, T, is less than the amount due on the debt, RD, and thus equals F(RD). So from equation (2), we can see that the probability of default, π, is given by

$$(3)\quad \pi = F(RD) = \begin{cases} 0 & \text{if} \quad RD < \mu - X \quad \text{or} \quad R < (\mu - X)/D \\ \dfrac{RD - (\mu - X)}{2X} & \text{if} \quad \mu - x \le RD \le \mu + x \quad \text{or} \quad (\mu - x)/D \le R \le (\mu + x)/D \\ 1 & \text{if} \quad RD > \mu + X \quad \text{or} \quad R > (\mu + X)/D. \end{cases}$$

The other equilibrium condition describing combinations of R and π for which investors are willing to hold the economy's debt is still given by

$$(4)\quad \pi = \frac{R - \overline{R}}{R}.$$



Equations (3) and (4) are depicted in the figure at right. This shows the possible situation of multiple equilibria. Under the plausible dynamics described in the text, the equilibrium at A is stable whereas the equilibrium at B is not. Another stable equilibrium occurs when investors are unwilling to hold the economy's debt at any interest rate.

(a) A rise in μ represents an upward shift in the distribution of possible tax revenue without a change in its dispersion. The probability of default line shifts to the right by the change in μ. The locus given by equation (4) is unaffected. The stable equilibrium would now involve a lower interest factor and a lower probability of default.

(b) A fall in X represents a decrease in the dispersion of possible tax revenue without a change in its expected value. The locus given by equation (4) is unaffected. The slope of the probability of default line is given by $\partial\pi/\partial R = D/2X$. Thus this line essentially rotates; it becomes steeper over a smaller range and still goes through the point ($\pi = 1/2$, $R = \mu/D$). If the original intersection between the two equilibrium conditions was at $RD < \mu$ or $R < \mu/D$ (as in the case depicted in the figure above), the new stable equilibrium would involve a lower interest factor and a lower probability of default.