Economics 202A: Section 6

Endogenous Growth

Jacob Weber*

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1. Romer Model: Planner's Problem
2. Extensions to the Romer Model
3. Midterm Practice Problems

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

1 Review of the Romer Model: Planner's Problem

Key equations:

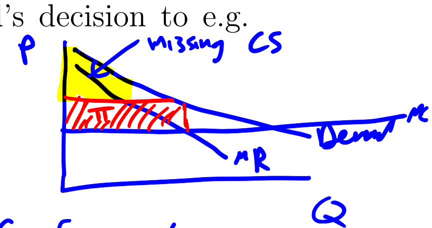
$Y(t) = \left[\int y_i(t)^{\phi} di\right]^{\frac{1}{\phi}}$ $\quad y_i(t) = l_i(t) = \frac{L_Y(t)}{A(t)}$

$$
\begin{aligned}
Y(t) &= A(t)^{\frac{1-\phi}{\phi}} L_Y(t) & (1)\\
L &= L_Y(t) + L_A(t) & (2)\\
\dot{A}(t) &= B L_A(t) A(t) & (3)
\end{aligned}
$$

Here we have already imposed the result that the planner will optimally allocate $L_y(t)$ equally across all inputs i (as in the competetive equilibrium). Note we no longer have conditions for private optimality (profit maximization, etc.). Instead, we will have new optimality conditions which result from the *planner* maximizing household consumption $C(t) = Y(t)/L$,

$$\max_{\{C(t)\}} \int_0^{\infty} e^{-\rho t} \ln C(t) dt$$

This will yield a different result than when the household solves this problem because the planner internalizes all the externalities that result from the household's decision to e.g. start a new firm/invent a new variety. What are the externalities here?



1. Consumer Surplus: firms don't capture all the CS from entry ⇒ too little entry!

2. Business Stealing: entry reduces profits for other firms ⇒ too much entry

$\dot{A}(t) = B A(t) L_A(t)$

3. "standing on shoulders of Giants" ⇒ too little entry!

Question: does the presence of externalities in a model *always* imply that the market equilibrium is inefficient?

Fundamentally, the planner's problem is really just to allocate a fixed L between the two sectors, R&D and Goods. I recommend writing the problem in terms of

$$\boxed{\theta(t) \equiv \frac{L_Y(t)}{L}}$$

Plugging in our constraints, the problem boils down to

$$\max_{\{\theta(t)\}} \int_0^\infty e^{-\rho t} \left(\ln \theta(t) + \frac{1-\phi}{\phi} \ln A(t) \right) dt$$

such that

$$\frac{\dot{A}(t)}{A(t)} = BL(1-\theta(t))$$

- $\theta(t)$ is the control variable and $A(t)$ is the state variable (why?)
- We ignore the inequality constraints on $L_A(t)$ here (so $\theta(t) \in (0,1)$) and search for an interior solution that is not in a "corner" (is this restrictive?)
- We can use the second theorem in the Hamiltonian reference sheet. Using the FONCs, we can show that $\dot{\theta}(t)$ is characterized by a quadratic eqaution in $\theta(t)$ which makes solving for the long run dynamics relatively easy.[1]

[1] This part was a bit trickier than intended; see the HW solution key for details.

As in the book, you should find $\theta(0) = \theta^* \equiv \frac{\phi\rho}{(1-\phi)BL}$ is the unique, constant solution to the planner's problem.

$$\boxed{L_y^* = \frac{\phi\rho}{1-\phi B}}$$

Exercise 1 *Consider the solution to the planners problem above*

1. *If $\theta^* > 1$, does this mean there is no solution to the planner's problem?*

2. *Recall that the optimal choice of L_A on the BGP in the Competetive Equilibrium (CE) was given by*

$$L_A^{CE} = (1-\phi)L - \frac{\phi\rho}{B}$$

is this higher or lower than the planners choice of L_A? (Show this formally.) What does your answer say about the relative importance of the externalities we listed above?

$$L_A^{CE} = (1-\phi)\left(L - \frac{\phi\rho}{(1-\phi)B}\right) = (1-\phi)\left(L - L_y^*\right) = (1-\phi)L_A^* \Leftrightarrow \boxed{L_A^* = \frac{L_A^{CE}}{1-\phi}}$$

since $\phi < 1$

$$\left(P_i = \frac{w}{\phi}\right)$$

Note: we just showed $L_A^* = (1-\phi) L_A^{CE}$

Exercise 2 *Assume we implement the Planners solution to the Romer model by "commanding" every household to supply their labor accordingly to both sectors and then gifting them an equal share of output of the final good each period. Further assume that households are able to borrow/lend to each other in bonds at some rate $r(t)$. For each household i, the budget constraint is then*

$$\boxed{\dot{B}_i(t) = r(t)B_i(t) + Y_i(t) - C_i(t)}$$

where $Y_i(t) = Y(t)/L$. Note households are not allowed to start firms (since we assumed the planner chooses $L_A(t)$). As in the competetive equilibrium, household optimality requires:

$$\boxed{\frac{\dot{C}_i(t)}{C_i(t)} = r(t) - \rho} \qquad (y = c)$$

1. *Show that $r(t)$ in the Planner's equilibrium is constant and a function of the growth rate of A, g_A, as in the BGP of the competetive equilibrium (hint: since all households are the same, can we ever have $B_i(t) \neq 0$ in equilibrium?)*
2. *Is this interest rate higher or lower than on the BGP of the competetive equilibrium? What does this mean about the optimal quantity of saving in the market equilibrium of the Romer model?*

$$g_c = g_y = \frac{1-\phi}{\rho} g_A = BL_A^*$$

$$g_A = BL_A^*$$

(1) $r(t) = \dfrac{\dot{C}(t)}{C(t)} + \rho$

$$r = \frac{1-\phi}{\phi} g_A + \rho \Rightarrow r = \frac{1-\phi}{\rho} BL_A^* + \rho$$

(2) $L_A^* > L_A^{CE}$ since $L_A^* = (1-\phi)^{-1} L_A^{CE}$

2 Beyond the Textbook Romer Model: Other Knowledge Production Functions

What implications does the knowledge production function have for the BGP in an endogenous growth model? A more general form of equation (3) is

$$\boxed{\dot{A}(t) = F\left[A(t),\ L_A(t)\right]} \tag{4}$$

where $L_A(t)$ is the labor input devoted to research. We often assume $F(\bullet)$ is Cobb-Douglas,

$$\boxed{\dot{A}(t) = BA(t)^{\lambda} L_A(t)^{\gamma}} \tag{5}$$

$$\frac{\dot{A}(t)}{A(t)} = BA(t)^{\lambda - 1} L_A(t)^{\gamma 1}$$

Above, we assumed $\lambda = \gamma = 1$, but there are other possibilities:

- $\lambda < 1$: Easier discoveries made first ("low hanging fruit" hypothesis)
- $\lambda > 1$: Standing on the shoulders of giants
- $\gamma > 1$: Peer effects
- $\gamma < 1$: Too many cooks spoil the broth

We will consider the case where $\gamma = 1$, and allow λ to vary.

As in our previous models, the knowledge production function alone will already tell us the growth rates on a BGP, if it exists. Divide the production function (5) by $A(t)$,

$$\boxed{g(t) = BA(t)^{\lambda-1} L_A(t)} \tag{6}$$

On a BGP, g_t must be constant. Taking logs and differentiating w.r.t. time gives

$$\frac{\partial}{\partial t} \ln g(t) = \frac{\partial}{\partial t}\left[\cancel{\ln B} + (\lambda - 1)\ln A(t) + \ln L_A(t)\right]$$

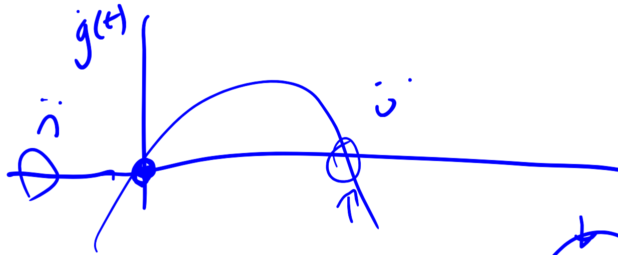
$$\left[\frac{1}{g(t)} \cdot \dot{g}(t)\right] = (\lambda - 1)\underbrace{\frac{1}{A(t)} \cdot \dot{A}(t)}_{g(t)} + \underbrace{\frac{1}{L_A(t)} \cdot \dot{L}_A(t)}_{n_A(t)}$$

$$\frac{\dot{g}(t)}{g(t)} = (\lambda - 1)\, g(t) + n_A(t)$$

on a BGP: $0 = (\lambda-1)g + n_A$

is it stable?

$\dot{g}(t) = (\lambda-1)g(t)^2 + n_A g(t)$



Knowledge Production with $\gamma = 1$:

$$\dot{A}(t) = BA(t)^{\lambda}L_A(t) \rightarrow g_A(t) = BA(t)^{\lambda-1}L_A(t)$$

assume $\gamma=1$

Characterize the BGP: does it exist?

1. In lecture and the textbook's version of the Romer model, $\lambda = 1$ and $n_A = 0$.

$\boxed{g(t) = BL_A}$

BGP exists!

Also: there are scale effects. Level of population (researchers) impacts the growth.

2. Suppose $\lambda \geq 1$ and $n_A \geq 0$ and either $\lambda > 1$ or $n_A > 0$.

$\rightarrow \lambda > 1 \Rightarrow$ we know $\boxed{g = \frac{n_A}{1-\lambda}} \leq 0$

$\lambda \geq 1,\ n_A > 0 \Rightarrow \boxed{g = \frac{n_A}{1-\lambda}} < 0$

No BGP — faster than constant growth! (generally rejected by data)

3. Suppose $\lambda < 1$ and $n_A \geq 0$.

$g = \frac{n_A}{1-\lambda} > 0 \Rightarrow$ we do have a BGP

$\boxed{\rightarrow \text{Semi-endogenous Growth}}$

n_A

$\rightarrow$ need more and more

Exercise 3 *Explore the implications of $\gamma \neq 1$ for long-run growth along a BGP using the Cobb-Douglass form of the production function for ideas (5):*

$$\dot{A}(t) = BA(t)^{\lambda} L_A(t)^{\gamma}$$

$$\frac{\dot{A}(t)}{A(t)} = BA(t)^{\lambda-1} L_A(t)^{\gamma}$$

~~*and assuming*~~ *and assuming $\lambda \neq 1$.*

$\boxed{\text{and } \gamma > 0}$

$g = ?$

$$\boxed{g(t) = BA(t)^{\lambda-1} L_A(t)^{\gamma}}$$

$$\Rightarrow \frac{\dot{g}(t)}{g(t)} = (\lambda-1)g(t) + (\gamma)g_{L_A}(t)$$

on a BGP:

$$0 = (\lambda-1)g + (\gamma)n_A$$

$$g = \frac{\gamma n_A}{1-\lambda} \quad \left(g = \frac{n_A}{1-\lambda} \text{ before}\right)$$

→ still no BGP if $\lambda > 1$ (growth too fast!)

→ otherwise, pretty similar!

Exercise 4 *Extra Problem 13 in PS4 Fall 2021: (From an old midterm). Consider the model of endogenous knowledge accumulation presented in the book and in lecture for the case of $\theta < 1$:*

$$Y(t) = (1 - \alpha_L)L(t)A(t), \quad 0 < a_L < 1$$
$$\dot{A}(t) = B[a_L L(t)]^\gamma A(t)^\theta, \quad B > 0, \gamma > 0, \theta < 1$$
$$\dot{L}(t) = nL(t)$$

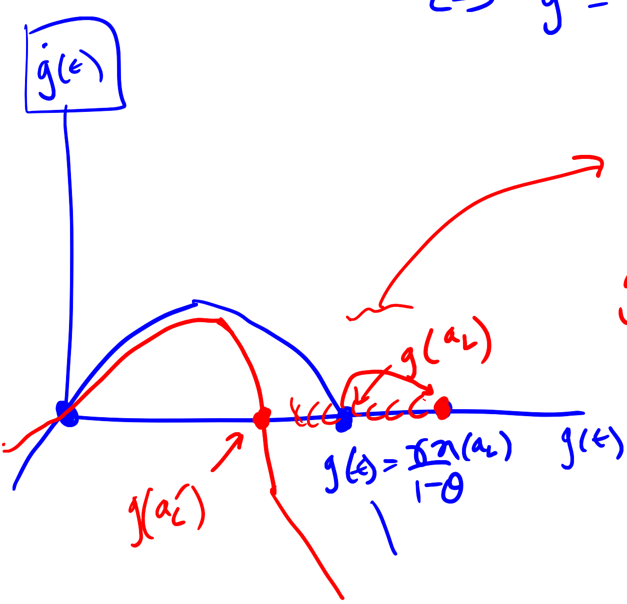
Assume $L(0) > 0$, $A(0) > 0$. As in the usual model, a_L is exogenous and constant. In contrast to the baseline version of the model, assume that the rate of population growth is a decreasing function of the fraction of workers who are engaged in R&D: $n = n(a_L)$, $n'(\cdot) < 0$, $n(\cdot) > 0$. (The idea is that, for some reason, scientists on average have fewer children than other workers.) Suppose the economy is on a balanced growth path, and that there is a permanent increase in a_L. Sketch the resulting path of $\ln A$ and what that path would have been without the increase in a_L. Explain your answer.

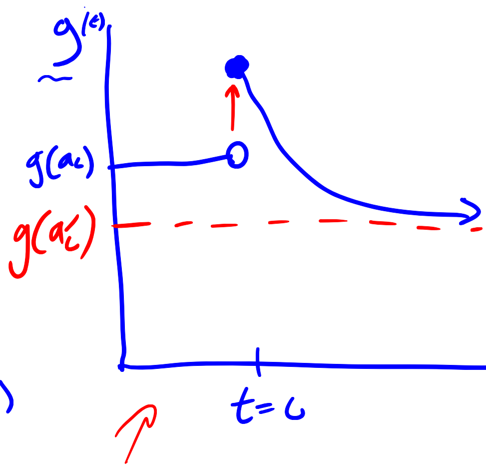
$$\boxed{(*)\ \frac{\dot A(t)}{A(t)} = g(t) = B[a_L L(t)]^\gamma A(t)^{\theta-1}} \Rightarrow \boxed{\frac{\dot g(t)}{g(t)} = \gamma n + (\theta-1) g(t)}$$

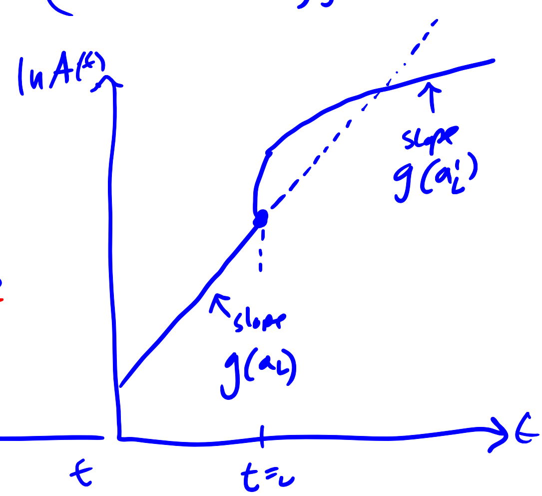
on a BGP: $0 = \gamma n + (\theta - 1) g$

$\iff g = \frac{\gamma n}{1-\theta}$

$\Rightarrow \dot g(t) = (\gamma n + (\theta-1) g(t)) g(t)$







Exercise 5 (Learning-by-doing) *This exercise is based on the first endogenous growth paper by Paul Romer. Suppose a representative firm solves*

$$\max_{\{K(t),\ L(t)\}} A(t)K(t)^{\alpha}L(t)^{1-\alpha} - r(t)K(t) - w(t)L(t) \tag{7}$$

Firms take $A(t)$ as given. However, in the aggregate, $A(t) = K(t)^{\phi}$ where $\phi < 1$, so there are positive peer effects in production. Capital evolves according to

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{8}$$

where we assume that agents save a constant fraction of their income as in the Solow model. Finally, assume $\dot{L}(t) = nL(t)$ with $n > 0$.

a. *Find a condition for ϕ such that a BGP exists, and show that Y/L is growing along it.*

b. *Establish that this BGP is stable.*

Exercise 6 *Extra Problem 6 in PS5 Fall 2021: From the 2019 Midterm (Problem 7):*

Motivated by the Solow model, a researcher is interested in studying the impact of the fraction of output devoted to investment on output per worker. They therefore consider the model:

$$\ln \frac{Y_i}{L_i} = a + b \ln s_i + e_i$$

where i indexes countries, Y_i/L_i is output per worker in country i in 2015, and s_i is the average ratio of investment to GDP in country i over the period 1985–2014.

a. *In no more than two sentences, give one reason that the estimate of b that results from estimating the equation above by OLS might suffer from omitted variable bias.*

b. *Suppose the researcher has found a way of estimating b that you are confident does not suffer from omitted variable bias. Suppose the resulting point estimate of b is 1.0, with a standard error of 0.7. Explain briefly how you would interpret that finding.*