Economics 202A: Section 4

OLG Models

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Outline for Today:

1. Review unanticipated/anticipated shocks in the Ramsey model's phase diagram (continuous time)

2. Review of Diamond OLG and Derivation of Dynamic Inefficiency (discrete time)

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

1 Returning to Ramsey

For reference, the equations of the Ramsey model are

$$\frac{\dot{c}}{c} = \frac{(f'(k) - \delta - \rho - \theta g)}{\theta} \tag{1}$$

$$\dot{k} = f(k) - c - (\delta + n + g)k \tag{2}$$

along with the initial and transversality conditions.

Exercise from last section: Suppose initially the Ramsey economy is in the steady state. Using phase diagrams, show how the economy responds to (i.e. how c, $\dot{c}$, k, and $\dot{k}$ react, do any loci shift, etc.)

1. A one-time unanticipated reduction in capital stock.
2. A one-time anticipated reduction in capital stock.
3. A one-time, unanticipated reduction in the depreciation rate.
4. A one-time, anticipated reduction in the depreciation rate.

   For the next two questions suppose that you start from a point below the BGP

5. An unanticipated permanent change in consumption-smoothing preferences (θ goes down).
6. An anticipated permanent change in consumption-smoothing preferences (θ goes down).

Answers go here (will draw graphs in section)

2 The Intertemporal budget constraint and Ricardian Equivalence

Here we review discrete time optimization and digress on "Ricardian Equivalence" in the Ramsey model So far we have formulated our optimization problems in terms of the *flow* budget constraint that maps current period income and expenditures into the change in assets carried into the next period. For our savings problem in discrete time, the budget constraint was

$$a_{t+1} - a_t = ra_t + \bar{y} - c_t \tag{3}$$

where a denotes savings (assets), r is the *net* real interest rate, y denotes income, and c denotes consumption. In continuous time, this can be written as

$$\dot{a} = ra(t) + \bar{y} - c(t) \tag{4}$$

When we solved this problem in the discrete time case, we iteratively substituted for a_{t+1} in this equation and applied the transversality condition to get

$$0 = \sum_{t=o}^{\infty}(1+r)^{-t}(\bar{y} - c_t) + (1+r)a_0 \tag{5}$$

which we can rearrange to get

$$(1+r)a_0 = \sum_{t=o}^{\infty}(1+r)^{-t}(c_t - \bar{y}) \tag{6}$$

This is the consumer's *intertemporal budget constraint* equating current assets to the present discounted value of the excess of consumption over income. Note that any path for c_t that satisfies this constraint is a feasible path, since every sequential budget constraint and the No-Ponzi scheme are satisfied. Thus we have replaced an infinite series of budget constraints with a single budget constraint involving an infinite discounted sum of consumption. We could now go on to solve the reformulated problem

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{7}$$

$$s.t.\ (1+r)a_0 = \sum_{t=o}^{\infty} (1+r)^{-t}(c_t - \bar{y}) \tag{8}$$

using the usual Lagrangian methods, and you can check that we would get the same answer as before.[1]

The assumption behind this formulation is that the *only* restriction on the pattern of consumer borrowing and spending is that, *in the long run*, they do not consume more than their income plus initial assets. There is no restriction on how much debt the consumer can run up in the short term. If the consumer did face such short term borrowing constraints we would have to include those as side constraints and they could potentially affect the solution.

The idea of Ricardian equivalence is that any transfer that affects both sides of equation (8) equally will leave the consumer facing the same constraint as before, and therefore will have no effect on the solution to the maximization problem. The canonical example is when the government gives you d_0 in bonds at time 0, which pay interest rate r, but will hit you with a sequence of taxes τ_t such that the bond is eventually paid off. How does this change the budget constraint (8)? We add d_0 to the LHS and $\sum_{t=0}^{\infty}(1+r)^{-t}\tau_t$ to the RHS to get

$$d_0 + (1+r)a_0 = \sum_{t=o}^{\infty} (1+r)^{-t}(c_t - \bar{y}) + \sum_{t=0}^{\infty}(1+r)^{-t}\tau_t \tag{10}$$

But since $d_0 = \sum_{t=0}^{\infty}(1+r)^{-t}\tau_t$ we could just subtract d_0 from both sides and we would just have our original constraint. Since the optimization problem has not changed, the optimal consumption path has not changed.

What if the consumer faced a non-negativity constraint on assets, i.e. no borrowing, and

[1] In continuous time the intertemporal budget constraint takes the form

$$a(0) = \int_0^{\infty} e^{-rt}(c(t) - \bar{y})dt \tag{9}$$

Make sure you know how to solve a linear differential equation so you can derive this on an exam.

suppose further that $y_0 < y_t = \bar{y} \; \forall t > 0$ and that $a_0 = 0$? We know that if the non-negativity constraints on assets do not bind, the Euler equation tells us that $c_t = \bar{c}$. If we went ahead and solved the problem assuming this constraint did not bind, we would find

$$\bar{c} = \frac{ry_0 + \bar{y}}{1 + r} \tag{11}$$

But this would require borrowing in period 0 because $\bar{c} > y_0$, so this is not feasible. So the optimal policy is to consume y_0 in the first period and $\bar{y}$ thereafter.

Now suppose the government implements its scheme and that $d_0 \geq \bar{y} - y_0$. The consumer may now implement her unconstrained optimal plan of consuming $\bar{c}$ from equation (11) because she has enough assets in period 0 to finance her extra consumption in that period. Thus the timing of government debt and taxes can potentially affect whether or not borrowing constraints bind, and thus affect consumption behavior through this channel. Government debts provide "liquidity" in this model. The role of government debt in providing market liquidity is an active area of research in macroeconomics.

3 Basics of the OLG model

Here we review the Diamond Overlapping Generations (OLG) model. Each period, L_t young are born, each endowed with one unit of labor income. L_t grows at $L_t = (1+n)L_{t-1}$. People live for two periods, but only supply labor when young. Thus young consumers in every period solve

$$\max_{c_t^y, c_{t+1}^o} = u(c_t^y) + \frac{1}{1+\rho} u(c_{t+1}^o) \tag{12}$$

$$s.t. \quad c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} = w_t A_t \tag{13}$$

whereas the old simply consume their interest income and their capital stock in every period (after the capital has been used to produce output).[2] Output is produced using the production function $f(k_t)$ and factors earn their marginal product.

How can we analyze this model? Does a BGP exist, like in the Solow model?

In every period but the first, capital is equal to last period's savings,

$$k_t = \frac{s_{t-1}^y}{(1+n)(1+g)} = \frac{w_{t-1} - c_{t-1}^y}{(1+n)(1+g)} \tag{14}$$

while in period 0 it is given as k_0 and owned entirely by the period-0 old.[3]

That's it. Solving the model involves solving the very easy maximization problem (12) for consumption in both periods as a function of w_t and r_{t+1}. Once you know c_t^y you know k_{t+1} as a function of w_t and r_{t+1} using (14). Since

$$w_t = f(k_t) - f'(k_t)k_t \tag{15}$$

$$r_{t+1} = f'(k_{t+1}) \tag{16}$$

[2] Notice also the assumption on interest rates. Here we assume interest is acrued on savings at the start of the period, which is obvious if you write the budget constraint as: $c_{t+1}^o = (w_t - c_t^y)(1 + r_{t+1})$.

[3] If the terms in the denominator confuse you, recall that $k_t = \frac{K_t}{A_t L_t}$. Aggregate capital is equal to the total savings of the young in the last period: $K_t = (W_{t-1} - C_{t-1}^y)L_{t-1}$. Dividing by $A_t L_t$ and converting to "intensive form" yet again gives the desired result.

we have 3 equations in 4 unknowns (r_{t+1}, k_t, k_{t+1} and w_t) so we can solve for the equation of motion for capital,

$$k_{t+1} = g(k_t) \tag{17}$$

Solving this single differential equation completes the solution to the model. In general this will be a non-linear differential equation that has no explicit solution and its behavior can be quite complicated depending on the form of utility and production function.[4] If we assume CRRA utility, we can make some progress using the strategy of considering a small change in consumption. If the agent is optimizing, we must have:

$$\triangle_c (c_t^y)^{-\theta} = \frac{1}{1+\rho}(c_{t+1}^o)^{-\theta}(1+r_{t+1})\triangle_c \tag{18}$$

$$\frac{c_{t+1}^o}{c_t^y} = (\frac{1+r_{t+1}}{1+\rho})^{\frac{1}{\theta}} \tag{19}$$

Substituting back into the budget constraint, we have can write the savings *rate* as a function of r_{t+1} only:

$$s(r_{t+1}) = \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \tag{20}$$

Recall from our previous sections that the very special log utility is limiting form of CRRA utility as $\theta \to 1$. It should be clear from equation 20 that the savings rate in this case is $\frac{1}{2+\rho}$. Log utility has the special feature that income and substitution effects cancel, so that savings behavior does not depend on the interest rate. If we also assume Cobb-Douglas production, we can get a concise expression for k_{t+1} as a function of k_t.

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} k_t^{\alpha} \tag{21}$$

which we can analyze in a similar fashion as the Solow model using a phase diagram in k_{t+1} and k_t. Notice that this equation is also log-linear, so we can explicitly solve for the entire dynamic path as well.

[4]Have a look at the textbook for an analysis of some of the more interesting possibilities.

[Extra space if needed]

Exercise 1 *Draw the phase diagram for the OLG model with log utility and Cobb-Douglass production, find the steady state and characterize the dynamics of the system.*

We can use this diagram to analyze the impact of changes in parameters, similar to the Ramsey phase diagram.

See Figure 2.10 in the textbook. We can use this diagram to analyze the impact of changes in parameters, similar to the Ramsey phase diagram.

Exercise 2 (Midterm 07) *Consider an economy described by the Diamond overlapping generations model in the special case of logarithmic utility and Cobb-Douglas production. Assume that initially k is above its balanced-growth-path level. Now suppose there is an unexpected permanent increase in agents' discount factor ρ. Sketch the resulting paths of k, and what that path would have been if ρ had not changed. Explain your answers.*

We will answer this in class using the phase diagram. The capital accumulation equation will rotate inward, as the higher ρ makes households more impatient and want to save less. This lowers the steady-state capital stock. The transition path will converge to a lower value of k^* at a faster rate at first, as the difference between savings and break-even investment is now larger at each point in time.

Exercise 3 (Midterm 06) *In the Diamond OLG model with log utility and Cobb-Douglas production, analyze how changes in* n,g,α *and* β *affect the likelihood of an economy becoming dynamically inefficient. Give the intuition for the impact of each change.*

Hint: proceed in two steps. First, calculate the Golden Rule capital stock in this model (try working from $K_{t+1} = F(K_t, A_tL_t) - C_t + K_t$ where C is aggregate consumption across both generations). You should get a condition for the marginal product of capital. Then, compare this condition to the marginal product of capital on the BGP of this model.

The steady state value of capital is found by solving the accumulation equation (21) for $k_{t+1} = k_t$ to get

$$\bar{k} = \left(\frac{(1-\alpha)}{(2+\rho)(1+g)(1+n)}\right)^{\frac{1}{1-\alpha}} \tag{22}$$

$$\tag{23}$$

and the steady state marginal product of capital is:

$$f'(\bar{k}) = \frac{\alpha}{1-\alpha}(2+\rho)(1+n)(1+g) \tag{24}$$

Note the contrast with the Golden Rule marginal product of capital $f'(k^{GR}) = (1+n)(1+g) - 1$. In the OLG model we will overshoot the Golden Rule level of steady state capital if

$$\frac{\alpha}{1-\alpha}(2+\rho) < 1 - \frac{1}{(1+n)(1+g)} \tag{25}$$

There is nothing inherent in the model that prevents this from occurring.

Economic intuition is as follows: the source of dynamic inefficiency in the OLG model is that capital is the only savings vehicle, and saving is the only way to consume in old age. Thus anything that lowers the productivity of capital will increase the likelihood of being dynamically inefficient, because capital will become a less effective savings vehicle. A higher capital share implies a higher productivity of capital and lowers the likelihood of dynamic inefficiency. More patience (higher β) increases the desire to save, increasing the changes of dynamic inefficiency. Higher n and g also increase the chance of dynamic inefficiency, although the intuition is a bit harder in this case. Increases in both lower the golden rule level of capital because they imply faster capital accumulation just to keep the same steady state capital per effective worker. It has the same effect on the steady state capital stock in the OLG model, but the effect is smaller because people have additional motives to save in the OLG model.[5]

[5] This depends on α, as can be seen in the equation for the marginal product of capital at k^*. For reasonable values of α, though, this will be the case.

Exercise 4 *In the OLG model with no depreciation, which of the following is not correct:*

1. *If the growth rate of population and exogenous technical change are zero, the standard competitive equilibrium allocation will not be dynamically inefficient.*
2. *If the production function is Cobb-Douglas, a lower capital elasticity makes it more likely that the competitive equilibrium will overaccumulate capital.*
3. *If individuals do not discount over time, the standard competitive equilibrium allocation will not be dynamically inefficient.*

(3): When individual generations do not discount, $\rho = 0$. Check our condition for dynamic inefficiency equation (25) above and note that we can definitely satisfy this condition with $\rho = 0$: just pick α arbitrarily close to zero. Note (1) is only "technically" true because in this case, there is no golden rule capital stock (since capital never decays or "dilutes" when $n = g = \delta = 0$ we can obtain arbitrarily high consumption). So the economy is never dynamically inefficient. But the planner could still improve on the competetive equilibrium!

4 Money and Taxes in the OLG Model

The original OLG models of Allais and Samuelson were even simpler than the Diamond version of the model, and can also be used to illustrate the potential inefficiency of equilibria and the scope for monetary and fiscal policy to improve the allocation.

- Assume log utility, no discounting ($\rho = 0$), and no production: each young person is endowed with A units of the consumption good in period t, which they may either consume or store. Initial old are endowed with Z units of the good.
- storing S of the good yields xS of the good the next period.
- Population grows at rate n, and assume $x < 1 + n$.[6]

Exercise 5 *Solve the optimization problem solved by the young, and answer the following:*

1. *Describe the decentralized equilibrium of the economy. Is it efficient?*
2. *Describe a tax and transfer scheme carried out by an infinitely lived government that improves welfare.*

1. In the first period, the old simply consume their income. The young solve

$$\max \;\; \log(c_1^y) + \log(c_2^o) \tag{26}$$

$$s.t. \; c_1^y + \frac{c_2^o}{x} = A \tag{27}$$

which has solution $c_1^y = A/2$, $c_2^o = xA/2$. Note that next period the situation is exactly the same; the old have an endowment that they consume, and the young solve the same problem. This describes the equilibrium in every period. This equilibrium is not efficient. Suppose that no one ever saved any output. Instead, in every period (besides perhaps the first) the young transfer half of their endowment to the old and consume the rest. Then everyone's consumption when young is the same, but every old person' consumption is now $(1 + n)A/2$, which is greater: a Pareto improvement.

2. The government simply taxes the young by half their output and gives it to the old.

[6]This corresponds to the Diamond model with $g = 0$ and $F(K, L) = AL + xK$.

Exercise 6 *(Extra) For this problem, assume that the initial old are also endowed with a quantity M of a storable good called money that does not yield utility.*

1. *Suppose money can be exchanged for goods at price P_t per good. What is an individual's behavior as a function of P_t/P_{t+1}?*
2. *Show there is an equilibrium with $P_{t+1} = P_t/(1+n)$ for all t. How much is stored? Is this equilibrium efficient?*
3. *Show there is an equilibrium with $P_{t+1} = P_t/x$ for all t. How much is stored? Is this equilibrium efficient?*
4. *Show there is an equilibrium with $P_t = \infty$, that is, money is worthless.*
5. *Which of these equilibria will actually occur?*

1. The young person's problem is now to maximize utility subject to

$$c_t^y = A - sA - mA \tag{28}$$

$$c_{t+1}^o = sxA + \frac{P_t}{P_{t+1}}mA \tag{29}$$

where s is the fraction stored in period t, m is the fraction of goods traded for money in period t, and $s + m < 1$, $s, m > 0$. If $x > P_t/P_{t+1}$ then the rate of return on storage dominates the rate of return on money, and $m = 0$. If $x < P_t/P_{t+1}$ then the rate of return on money dominates the rate of return on storage, and $s = 0$. When $x = P_t/P_{t+1}$ the consumer is indifferent between holding money and storage. However, with log utility savings behavior does not depend on the rate of return, so

$$c_t^y = A/2 \tag{30}$$

$$c_t^o = xA/2 \;\text{ if }\; x \geq P_t/P_{t+1} \tag{31}$$

$$= \frac{P_t}{P_{t+1}}A/2 \;\text{ if }\; x < P_t/P_{t+1} \tag{32}$$

2. The implied consumption behavior is

$$c_t^y = A/2 \tag{33}$$

$$c_{t+1}^o = (1+n)A/2 \tag{34}$$

from the first part of the problem. This is certainly feasible; it replicates the equilibrium from the last exercise. Obviously, there is no storage. The equilibrium is pareto efficient.

3. The implied consumption behavior is

$$c_t^y = A/2 \tag{35}$$

$$c_{t+1}^o = xA/2 \tag{36}$$

which replicates consumption in the autarkic equilibrium without money in the first exercise. Since money has value, it must all be held in equilibrium so at least some portion of savings must be held in monetary form (and therefore not stored). Nonetheless, consumption is as if all savings consisted of stored output. Can you figure out how this works?

4. This is an equilibrium in which there is complete storage and no one holds money. It is an equilibrium because if no one will accept money for goods, no one has any incentive to hold it.

5. This is a trick question. We have no criteria for selecting between multiple potential equilibria in this model. Government policies could potentially eliminate some of the equilibria; for example, the equilibrium with $P_t = \infty$ will not exist if the government levies taxes and accepts payment in money.