Summary of the Hamiltonian Approach

Consider the (summarized) maximization problem stated in Acemoglu section 7.3 Thm. 7.9:

$$\max_{x(t),y(t)} W\left(x(t),y(t)\right) \equiv \int_0^\infty f(t,x(t),y(t))dt$$

subject to:

$$\dot{x} = g(t,x(t),y(t))$$

and $y(t) \in \mathbf{R}$ for all t, where $x(0) = x_0$, and $\lim_{t\to\infty} x_t \geq x_1$ The Hamiltonian is

$$H\left(t,x(t),y(t),\lambda(t)\right) = f(t,x(t),y(t)) + \lambda(t)g(t,x(t),y(t))$$

where $y(t)$ is the control variable, and $x(t)$ is the state variable and $\lambda(t)$ is a costate variable to be characterized. Under some restrictions/assumptions, the First Order Necessary Conditions (FONCs) for a potential solution to be optimal are the following: letting H_i be the partial derivative of H w.r.t. variable i:

$$\begin{aligned} H_y &= 0 \\ H_x &= -\dot{\lambda}(t) \\ \dot{x} &= H_\lambda \\ \lim_{t\to\infty} x(t) &\geq x_1 \end{aligned}$$

and $x(0) = x_0$.The multivariate case extends trivially.

Hamiltonian with Discounted Utility Maximization Problems

Consider the (summarized) maximization problem stated in Acemoglu (2009) section 7.5, which is a special case of the above:

$$\max_{x(t),y(t)} W\left(x(t), y(t)\right) \equiv \int_0^{\infty} \exp(-\rho t) f(x(t), y(t)) dt$$

where $\rho > 0$ and $x(0) = x_0$, subject to:

$$\dot{x} = g(x(t), y(t))$$

and $\lim_{t\to\infty} x(t) \geq x_1$. The present/currrent value Hamiltonians are, respectively:

$$\begin{aligned} H &= \exp(-\rho t) f(x(t), y(t)) + \lambda(t) g(x(t), y(t)) \\ \hat{H} &= f(x(t), y(t)) + \bar{\mu}(t) g(x(t), y(t)) \end{aligned}$$

where $y(t)$ is the control variable, and $x(t)$ is the state variable. Under some restrictions/assumptions,[1] the First Order Necessary Conditions (FONCs) for a potential solution to be optimal are the following:

$$\begin{aligned} \hat{H}_y &= 0 \\ \hat{H}_x &= \rho\bar{\mu}(t) - \dot{\bar{\mu}}(t) \\ \dot{x} &= g(x(t), y(t)) \\ \lim_{t\to\infty} \exp(-\rho t)\bar{\mu}(t) x(t) &= 0 \end{aligned}$$

The FONCs for the present value formulation result from substituting in $\bar{\mu}(t) = \exp(\rho t)\lambda(t)$ to the above:[2] In that case the FONCs are:

$$\begin{aligned} H_y &= 0 \\ H_x &= -\dot{\lambda}(t) \\ \dot{x} &= g(x(t), y(t)) \\ \lim_{t\to\infty} \lambda(t) x(t) &= 0 \end{aligned}$$

[1]These include assumptions necessary to write the transversality condition as an equality (e.g. monotonicity of f and g), which will generally hold in the examples we give in class; I have tried here to give the simplest possible exposition necessary for our course, but you should see Acemoglu for details/proofs.

[2]You should verify this for yourself for practice!