# Problem 1

**No-Capital Model: Romer 3.1**

Consider the model of Section 3.2 with $\theta < 1$.

(a) On the balanced growth path, $\dot{A} = g_A^* A$ where $g_A^*$ is the balanced-growth-path value of $g_A$. Use this fact and equation (3.6) to derive an expression for $A(t)$ on the balanced growth path in terms of $B, a_L, \gamma, \theta$, and $L(t)$.

First, we can begin by finding an expression for $g_A$.

$$\begin{aligned} \dot{A} &= g_A^* A \\ \dot{A} &= B(a_L L)^\gamma A^\theta \\ \implies g_A^* &= B(a_L L)^\gamma A^{\theta-1} \end{aligned}$$

On the balanced growth path, the growth rate $g_A$ is constant. So $\dot{g}_A = 0$. We can find $\dot{g}_A$ by taking the time derivative of $\ln g_A^*$

$$\begin{aligned} \ln g_A^* &= \ln(B a_L^\gamma) + \gamma \ln L + (\theta - 1)\ln A \\ \frac{\dot{g}_A^*}{g_A^*} &= \gamma \frac{\dot{L}}{L} + (\theta - 1)\frac{\dot{A}}{A} \\ &= \gamma n + (\theta - 1) g_A \end{aligned}$$

Since $\dot{g}_A = 0$ on the BGP,

$$g_A^* = \frac{\gamma n}{(1-\theta)}$$

From earlier, this implies

$$\frac{\gamma n}{(1-\theta)} = B(a_L L)^\gamma A^{\theta-1}$$

And solving for $A$ yields

$$A = \left[ B(a_L L)^\gamma \ \frac{(1-\theta)}{\gamma n} \right]^{\dfrac{1}{1-\theta}}$$

(b) Use your answer to part (a) and the production function, (3.5), to obtain an expression for $Y(t)$ on the balanced growth path. Find the value of $a_L$ that maximizes output on the balanced growth path.

$$Y = A(1 - a_L)L$$

Plugging in our $A$ function from part (a), we get

$$Y = \left[B(a_L L)^\gamma \ \frac{(1-\theta)}{\gamma n}\right]^{\dfrac{1}{1-\theta}} (1 - a_L)L$$

Since our goal is to maximize $Y$, we can simplify by maximizing $\ln Y$ instead:

$$\ln Y = \frac{1}{(1-\theta)} \ln\left(BL^\gamma \frac{(1-\theta)}{\gamma n}\right) + \frac{\gamma}{(1-\theta)} \ln a_L + \ln(1 - a_L) + \ln L$$

Then the FOC of $\ln Y$ is

$$\begin{aligned}
\frac{\partial \ln Y}{\partial a_L} &= \frac{\gamma}{(1-\theta)}\frac{1}{a_L} - \frac{1}{1-a_L} = 0 \\
\implies a_L &= \frac{\gamma}{1 - \theta + \gamma}
\end{aligned}$$

# Problem 2

**A model with Capital: Romer 3.3**

Consider the economy analyzed in Section 3.3. Assume that $\theta + \beta < 1$ and $n > 0$, and that the economy is on its balanced growth path. Describe how each of the following changes affect the $\dot{g}_A = 0$ and $\dot{g}_K = 0$ lines and the position of the economy in $(g_A, g_K)$ space at the moment of the change:

(a) An increase in $n$.

The $\dot{g}_A = 0$ line is defined by

$$\begin{aligned} \dot{g}_A = 0 &= \beta g_K + \gamma n + (\theta - 1) g_A \\ \implies g_K &= \frac{1-\theta}{\beta} g_A - \frac{\gamma}{\beta} n \end{aligned}$$

Which means the $\dot{g}_A = 0$ line shifts vertically downward by $\frac{\gamma}{\beta}$ for every 1-unit increase in $n$.

Similarly, the $\dot{g}_K = 0$ line is defined by

$$\begin{aligned} \dot{g}_K = 0 &= (1-\alpha)(g_A + n - g_K) \\ \implies g_K &= g_A + n \end{aligned}$$

So the $\dot{g}_K = 0$ line shifts vertically up by the amount $n$ has increased.

In the case that $\theta + \beta < 1$, we know that

$$g_A^* = \frac{\beta + \gamma}{1 - (\theta + \beta)} n$$

So $g_A^*$ has shifted out by $\frac{\beta + \gamma}{1 - (\theta + \beta)}$ for every 1-unit increase in $n$.

We know that $g_K = g_A + n$, so

$$g_K^* = \left[ \frac{\beta + \gamma}{1 - (\theta + \beta)} + 1 \right] n$$

And $g_K^*$ shifts up. Thus the new BGP position of the economy in $(g_A, g_K)$ space at the moment of the change is out and up compared to the old position in $(g_A, g_K)$ space.

From section 3.3, we know that

$$g_K = s(1 - a_K)^\alpha (1 - a_L)^{1-\alpha} \left[ \frac{AL}{K} \right]^{1-\alpha}$$

and

$$g_A = Ba_K^{\beta} a_L^{\gamma} K^{\beta} L^{\gamma} A^{\theta - 1}$$

Since $n$ isn't in either of these equations, and $K, L, A$ all need to change continuously, there is no jump in the location of the economy in $(g_A, g_K)$ space at the moment of the change. The economy smoothly moves outward in $(g_A, g_K)$ space toward the new $g_A^*, g_K^*$ location.

(b) An increase in $a_K$.

From part (a), we have that

$$g_A^* = \frac{\beta + \gamma}{1 - (\theta + \beta)} n$$
$$g_K^* = \left[\frac{\beta + \gamma}{1 - (\theta + \beta)} + 1\right] n$$

Thus there is no change in the BGP location in $(g_A, g_K)$ space.

From section 3.3, we know that

$$g_K = s(1 - a_K)^{\alpha}(1 - a_L)^{1-\alpha} \left[\frac{AL}{K}\right]^{1-\alpha}$$

and

$$g_A = B a_K^{\beta} a_L^{\gamma} K^{\beta} L^{\gamma} A^{\theta - 1}$$

So immediately after the increase in $a_K$, the economy jumps to a point down and to the right of the old BGP location. And then the economy will decrease $g_A$ and increase $g_K$ in order to get back to the BGP location.

(c) An increase in $\theta$.

From section 3.3, we know that

$$g_K = s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}\left[\frac{AL}{K}\right]^{1-\alpha}$$
$$g_A = Ba_K^{\beta}a_L^{\gamma}K^{\beta}L^{\gamma}A^{\theta-1}$$

So the economy does not jump in $(g_A, g_K)$ space. We also know that

$$g_A^* = \frac{\beta+\gamma}{1-(\theta+\beta)}n$$
$$g_K^* = \left[\frac{\beta+\gamma}{1-(\theta+\beta)}+1\right]n$$

So $g_A^*$ shifts to inward (to the left) and $g_K^*$ shifts downward. Thus the economy moves smoothly down and to the left in $(g_A, g_K)$ space.

# Problem 3

**Identical capital and knowledge growth rates: Romer 3.5**

Consider the model of Section 3.3 with $\beta + \theta = 1$ and $n = 0$.

(a) Using (3.14) and (3.16), find the value that $A/K$ must have for $g_K$ and $g_A$ to be equal.

Equations (3.14) and (3.16) give

$$\begin{aligned} g_K &= s(1-a_K)^\alpha (1-a_L)^{1-\alpha} \left[\frac{AL}{K}\right]^{1-\alpha} \\ g_A &= Ba_K^\beta a_L^\gamma K^\beta L^\gamma A^{\theta-1} \end{aligned}$$

Setting these equal gives

$$\begin{aligned} g_K &= g_A \\ s(1-a_K)^\alpha (1-a_L)^{1-\alpha} \left(\frac{AL}{K}\right)^{1-\alpha} &= Ba_K^\beta a_L^\gamma K^\beta L^\gamma A^{\theta-1} \end{aligned}$$

Since $\beta + \theta = 1$, the right hand side becomes

$$\begin{aligned} s(1-a_K)^\alpha (1-a_L)^{1-\alpha} \left(\frac{AL}{K}\right)^{1-\alpha} &= Ba_K^\beta a_L^\gamma \left(\frac{A}{K}\right)^{-\beta} L^\gamma \\ s(1-a_K)^\alpha (1-a_L)^{1-\alpha} \left(\frac{A}{K}\right)^{1-\alpha} L^{1-\alpha} &= Ba_K^\beta a_L^\gamma \left(\frac{A}{K}\right)^{-\beta} L^\gamma \\ \left(\frac{A}{K}\right)^{1-\alpha+\beta} &= \frac{Ba_K^\beta a_L^\gamma L^{\gamma+\alpha-1}}{s(1-a_K)^\alpha (1-a_L)^{1-\alpha}} \\ \frac{A}{K} &= \left[\frac{Ba_K^\beta a_L^\gamma L^{\gamma+\alpha-1}}{s(1-a_K)^\alpha (1-a_L)^{1-\alpha}}\right]^{\frac{1}{1-\alpha+\beta}} \end{aligned}$$

(b) Using your result in part (a), find the growth rate of $A$ and $K$ when $g_K = g_A$.

Plugging in $A/K$ from part (a) into equation (3.14), we get

$$g \equiv g_K = s(1-a_K)^\alpha(1-a_L)^{1-\alpha}\left[\frac{AL}{K}\right]^{1-\alpha}$$

$$= s(1-a_K)^\alpha(1-a_L)^{1-\alpha}\left[\frac{Ba_K^\beta a_L^\gamma L^{\gamma+\alpha-1}}{s(1-a_K)^\alpha(1-a_L)^{1-\alpha}}\right]^{\dfrac{1-\alpha}{1-\alpha+\beta}} L^{1-\alpha}$$

And consolidating the $L$ terms, we have

$$= s(1-a_K)^\alpha(1-a_L)^{1-\alpha}\left[\frac{Ba_K^\beta a_L^\gamma}{s(1-a_K)^\alpha(1-a_L)^{1-\alpha}}\right]^{\dfrac{1-\alpha}{1-\alpha+\beta}} L^{\frac{(\gamma+\alpha-1)(1-\alpha)}{1-\alpha+\beta}} L^{1-\alpha}$$

$$= s(1-a_K)^\alpha(1-a_L)^{1-\alpha}\left[\frac{Ba_K^\beta a_L^\gamma}{s(1-a_K)^\alpha(1-a_L)^{1-\alpha}}\right]^{\dfrac{1-\alpha}{1-\alpha+\beta}} L^{\dfrac{(\gamma+\beta)(1-\alpha)}{1-\alpha+\beta}}$$

And simplifying the $s, a_K, a_L$ terms, we have

$$= \left[s(1-a_K)^\alpha(1-a_L)^{1-\alpha}\right]^{\dfrac{\beta}{1-\alpha+\beta}} \left[Ba_K^\beta a_L^\gamma\right]^{\dfrac{1-\alpha}{1-\alpha+\beta}} L^{\dfrac{(\gamma+\beta)(1-\alpha)}{1-\alpha+\beta}}$$

(c) How does an increase in $s$ affect the long-run growth rate of the economy?

Since $n = 0$, the growth rate of the economy is

$$\frac{\dot{Y}}{Y} = \alpha g_K + (1-\alpha) g_A$$

and since $g_K = g_A = g$

$$\begin{aligned} &= g \\ &= \left[s(1-a_K)^\alpha (1-a_L)^{1-\alpha}\right]^{\frac{\beta}{1-\alpha+\beta}} \left[B a_K^\beta a_L^\gamma\right]^{\frac{1-\alpha}{1-\alpha+\beta}} L^{\frac{(\gamma+\beta)(1-\alpha)}{1-\alpha+\beta}} \end{aligned}$$

So an increase in $s$ by 1% will increase the long-run growth rate of the economy by approximately a factor of $1.01^{\frac{\beta}{1-\alpha+\beta}}$. Since $\frac{\beta}{1-\alpha+\beta} < 1$, this factor is between 1 and 1.01.

(d) What value of $a_K$ maximizes the long-run growth rate of the economy? Intuitively, why is this value not increasing in $\theta$, the importance of capital in the R&D sector?

To maximize the long-run growth, we want to maximize $g$. We can also maximize $\ln g$:

$$\begin{aligned}
\ln g &= \frac{\beta}{1-\alpha+\beta} \ln\left[s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}\right] + \frac{1-\alpha}{1-\alpha+\beta} \ln\left[B a_K^{\beta} a_L^{\gamma}\right] + \frac{(\gamma+\beta)(1-\alpha)}{1-\alpha+\beta} \ln L \\
&= \frac{\beta\alpha}{1-\alpha+\beta} \ln(1-a_K) + \frac{(1-\alpha)\beta}{1-\alpha+\beta} \ln a_K + \text{non-}a_K \text{ terms}
\end{aligned}$$

Then the FOC for $a_K$ is

$$\begin{aligned}
\frac{(1-\alpha)\beta}{(1-\alpha+\beta)a_K} &= \frac{\beta\alpha}{(1-\alpha+\beta)(1-a_K)} \\
a_K\alpha &= (1-a_K)(1-\alpha) \\
&= 1-\alpha-a_K(1-\alpha) \\
(\alpha+1-\alpha)a_K &= 1-\alpha \\
a_K &= 1-\alpha
\end{aligned}$$

In this case of the model, $\beta = 1-\theta$. So, if $\theta$ increased, the effectiveness of knowledge would increase in the sense that an additional unit of $A$ would have larger impacts on $g_A$. But an increase in $\theta \implies$ an decrease in $\beta$, which would decrease the effectiveness of $K$ on $g_A$. So in this case of the model, we can think of the effects of $\theta$ as cancelling out with the effects of $\beta$ on the growth-rate maximizing value of $a_K$.