Economics 202A: Section 3

Dynamic Optimization & the Ramsey Model

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1 Outline

Recall from last week analyzing dynamic optimization problems usually takes the following two steps:

1. use of the Hamiltonian (in continuous time) to obtain the first-order conditions
2. analyze this dynamic system (i.e. system of difference equations plus boundary conditions) to discover the properties of the solution

We will review the Hamiltonian (ie. step 1) briefly but today the focus is on step 2.

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

2 Analyzing Difference Equations

The typical output of the Kuhn-Tucker conditions for dynamic optimization problems:

- a set of difference equations governing the evolution of the system over time
- one initial condition for each predetermined variable (e.g. capital or assets)
- one terminal condition for each non-predetermined or "jumping" variable (such as consumption)

The terminal condition is required in order to know what value the jumping variables should take in the first period. Once we know all the first period values, the difference equations govern the subsequent evolution of the system.

We have essentially three tools for analyzing systems of *non-linear* difference or differential equations:

1. graphical analysis of nonlinear system
2. study *linear* approximations (around a steady state) instead
3. numerical solutions (not covered today)

2.1 Linearization

Using the Solow model with cobb douglass production as an example, let $x = \ln(k)$ and rewrite the dynamic equation

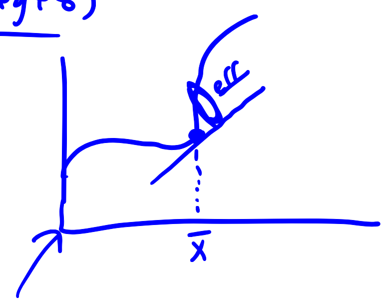
$$\boxed{\dot{k} = sk^{\alpha} - (n + g + \delta)k} \tag{1}$$

$$\frac{\dot k}{k} = sk^{\alpha-1} - (n+g+\delta) = s(e^{x})^{\alpha-1} - (\cdot)$$

$$\frac{\dot k}{k} = s(e^{(\alpha-1)x}) - (n+g+\delta)$$

Finally, note: $\dot x = \frac{\partial}{\partial t}\ln k = \frac{\dot k}{k} \Rightarrow \dot x = s(e^{(\alpha-1)x}) - (n+g+\delta)$

Taylor Thm: $f(x) = f'(\bar x)(x-\bar x) + f(\bar x) + err(x)$



note: $\dot x = f(x)$, let $\bar x = \bar k$ or ss k

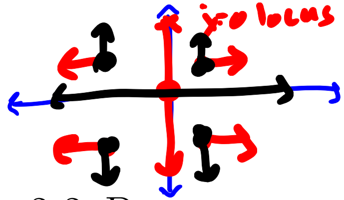
$$\dot x \approx (\alpha-1)se^{(\alpha-1)\bar x}(x-\bar x) + \underbrace{f(\bar x)}_{0}$$

note: $f(x) = se^{(\alpha-1)x} - (n+g+\delta)$

$\Rightarrow se^{(\alpha-1)x} = f(x) + n + g + \delta$

$$\dot x = (\alpha-1)[\underbrace{f(\bar x)}_{0} + n + g + \delta](x-\bar x) + 0 \Rightarrow \boxed{\dot x = (\alpha-1)(n+g+\delta)(x-\bar x)}$$

if $\bar x = 0$, $\frac{\dot x}{x} = $ constant



2.2 Diagonalization and the Phase Diagram

Let's start by looking at a diagonal system of linear equations. Solving a diagonal system is easy because each equation can be solved independently of the others.

$\boxed{\text{X locus is: the set of points s.th. } \dot{x}(t)=0}$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{2}$$

$\boxed{\dot{x}(t) = a_1 (x(t))}$

$\frac{\dot{x}(t)}{x(t)} = a_1$

We know the general solution to such a system takes the form

$\boxed{\dot{y}(t) = a_2 y(t)}$

$y(t)=0$

$\frac{\dot{L}(t)}{L(t)} = n$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \tag{3}$$

for arbitrary initial conditions $x(0), y(0)$ and a_1 and a_2 are non-zero real numbers. Now we will determine the stability properties of the system by drawing a phase diagram for 3 possible cases.

Exercise 1 *Draw the phase diagram in x, y space for the cases when $a_1, a_2 > 0$, $a_1, a_2 < 0$, and $a_1 > 0, a_2 < 0$ and classify each case as stable, unstable or saddlepath stable.*

To do this requires completing the following steps:

1. Find the set of points – also called the locus – at which $\dot{x}(t) = 0$, and the set at which $\dot{y}(t) = 0$.

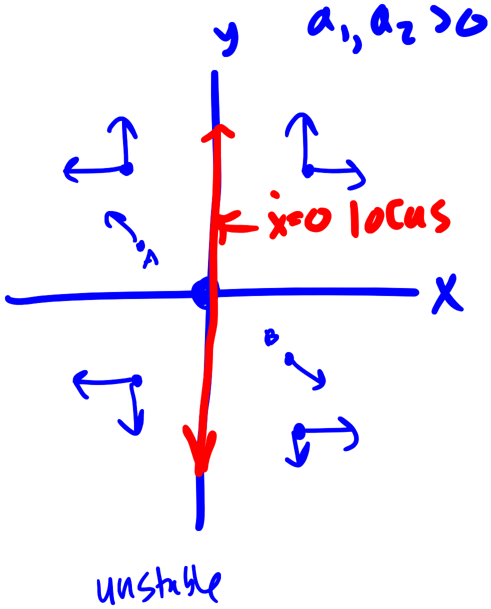
2. Determine dynamics of $x(t)$ and $y(t)$ in each region created by these loci. For example, if $x(t)$ is to the right (i.e., greater) than the $\dot{x}(t) = 0$ locus, is $x(t)$ increasing or decreasing.

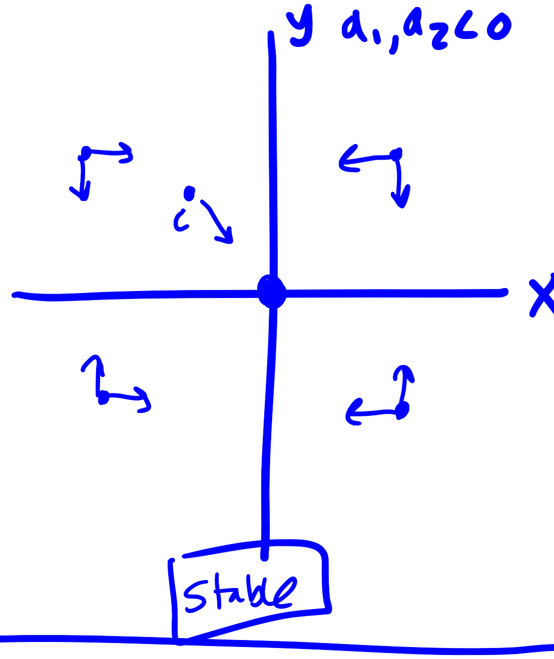
3. Use the dynamics you find above to determine the joint behavior of both variables in the four regions created by the $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$ loci.

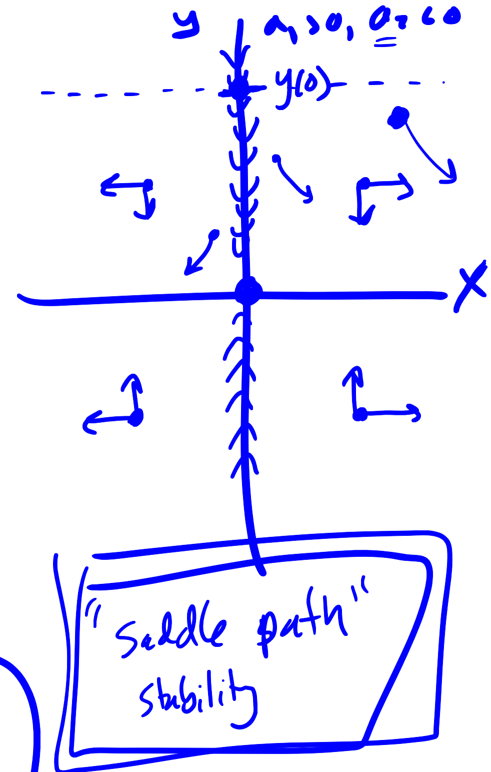
4. Use any boundary / initial conditions to determine which path constitutes a solution.

After completing these steps, you should find that the system is unstable when both coefficients are greater than 0, stable when both roots are less than 0, and saddlepath stable when $a_1 > 0, a_2 < 0$.

So saddlepath stability requires that one coefficient of the diagonal system be greater than 0, and one should be less than 0. But the system will be on the stable arm *only* if $x(0) = 0$; otherwise the solution will be an explosive trajectory.



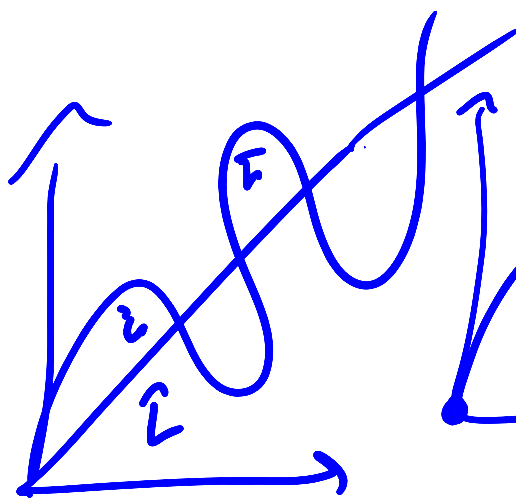


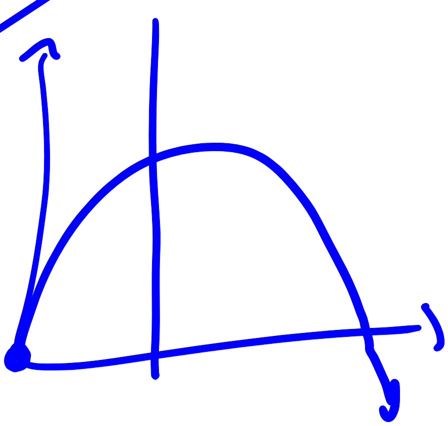


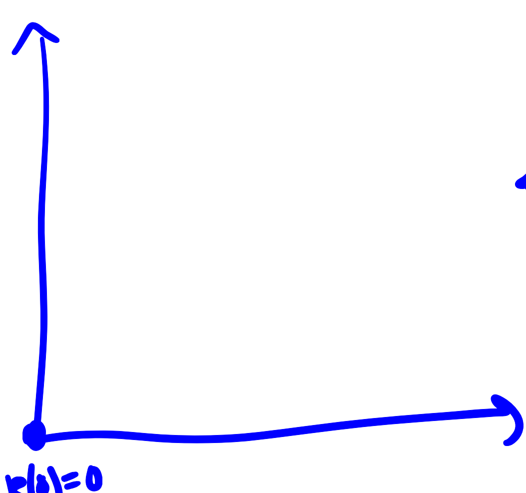
Locus for a variable $X(t) \equiv$ the set of points such that $\dot{x}(t) = 0$

$\dot{x}(t) = a_1 \cdot x(t)$

$\dot{x}(t) = 0 \Leftrightarrow x(t) = 0$







$\beta > 0$

ρ, θ, g

What about non-diagonal systems, such as

$$\boxed{\dot{\mathbf{x}} = A\mathbf{x}} \tag{4}$$

We can transform a non-diagonal system into a diagonal system by finding the eigenvalues λ_1, λ_2 of the matrix A and a matrix of eigenvectors Φ, and multiplying both sides by Φ^{-1}.[1]

$$\dot{\mathbf{z}} = \Lambda \mathbf{z} \tag{7}$$

where $\mathbf{z} = \Phi^{-1}\mathbf{x}$ and

$$\Lambda = \Phi^{-1} A \Phi = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{8}$$

We can solve the diagonal system for the transformed variables, then reverse the transformation to recover the solution to the original system.[2] The key property of the transformation Φ^{-1} is that it preserves the stability properties of the original system; thus *we can analyze the diagonal system to draw conclusions about the stability of the original non-diagonal system.*

To recap: to learn about the stability properties of a complicated nonlinear system around some steady state,

1. Log linearize the system around that steady state (so that we can use linear algebra)
2. Diagonalize the system as above (so that we can analyze it's stability properties more easily)

[1] Recall that the eigenvalues λ and eigenvectors z of a matrix A satisfy

$$Az_i = \lambda_i z_i \tag{5}$$

The interpretation is that, along the eigenvector z_i, the action of A simply represents a dilation or contraction along the same direction. We can find the eigenvalues by solving the characteristic equation

$$\det(A - \lambda I) \tag{6}$$

[2] I am ignoring the possibility of complex eigenvalues here, which could give rise to oscillatory behavior. The models that we work with in this class will all generate real eigenvalues.

- These stability properties are also essential to finding *particular* solutions to linear systems. Recall that we need one boundary condition for each endogenous variable in the system in order to find the correct initial values of the variables.

- A *predetermined* variable is one whose initial condition is given in the problem. A *nonpredetermined* or *jumping* variable has no given initial condition; instead, it may take any initial value consistent with the remaining conditions of the problem. These conditions typically include a condition on the limiting behavior of the system; for example, the transversality condition in the Ramsey model.

- Recall our 3 types of diagonal systems. Suppose y is jumping and x is predetermined.

- Suppose further (as is common in economic models) that the terminal condition implies that in the long run $x = 0$.[3]

  1. If the system is unstable, then the only point consistent with the terminal condition is $(0, 0)$. If $x(0) \neq 0$ then there is no value of $y(0)$ that we can choose that will get us to the steady state, so the system has no solution.

  2. If the system is stable, then since any starting value leads towards the steady state, any choice of $y(0)$ is consistent with the terminal condition. In this case there are an infinity of solutions.

  3. When the system is saddlepath stable, there is only one initial value for $y(0)$ consistent with convergence to the steady state: $y(0) = 0$. Thus there is a unique solution to the system.

- In higher-dimensional systems, we need the number of jumping variables to equal the number of positive eigenvalues to ensure a unique solution.

- Not all models you will encounter will have unique solutions. Sometimes, multiple solutions are a fact of life!

[3] Or more generally, takes on some steady state value.

3 The Ramsey Model

3.1 Ramsey Solution: A Review

The purpose of the Ramsey model is to endogenize savings choices, which you will recall was exogenous and fixed in the basic Solow model. Let H be the number of households in the economy and $L(t)$ be the number of workers/people, while $C(t)$ is consumption per person. Each household maximizes:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u[C(t)] \frac{L(t)}{H} dt \tag{9}$$

subject to a budget constraint where:

$$\dot{B}(t) = r(t)B(t) + W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H} \tag{10}$$

We also impose a No Ponzi condition, so that $\lim_{t\to\infty} B(t)e^{-R(t)} \geq 0$ where $R(t) = \int_0^t r(s)ds$.

We assume that firms hire capital up to the point that its marginal product equals the interest rate, and likewise for labor and firms. Note that in equilibrium the economy's level of capital will equal total family wealth, since families own the firms and this is a closed economy.

First, rewrite the variables in "effective" terms, as in the Solow model. Technology has a constant growth rate g, so $A(t) = A(0)e^{gt}$. Likewise, $L(t)$ grows at rate n. Define $c(t)A(t) = C(t), w(t)A(t) = W(t), b(t)A(t)L(t)/H = B(t)$. With some work we can write the household budget constraint and No Ponzi condition as:

$\boxed{\dot{B}} \Rightarrow \dot{b}$

$B \neq \dot{b}$

$$\dot{b}(t) = (r(t) - g - n)\, b(t) + w(t) - c(t) \tag{11}$$

$$\lim_{t\to\infty} b(t)e^{-R(t)+(n+g)t} \geq 0 \tag{12}$$

$\lim_{t\to\infty} B(t)\, e^{-R(t)} \geq 0$

$\lim_{t\to\infty} \frac{B(t)}{A(t)L(t)} A(t)L(t)\, e^{-R(t)} \geq 0$

$\lim_{t\to\infty} b(t)\cdot A(0)e^{gt} \cdot L(0)\, e^{nt}\, e^{-R(t)} \geq 0$

If we use CRRA utility, $u[C(t)] = \frac{C(t)^{1-\theta}}{1-\theta}$, we can also simplify the objective function to:

$$U = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt \tag{13}$$

$$B = A(0)^{1-\theta}\frac{L(0)}{H}$$

$$\beta = \rho - n - (1-\theta)g$$

We assume $\beta > 0$ so utility is bounded. You can ignore B, since as a scalar shifter it won't affect the choice of the optimal path of consumption. You should see we have everything we need to use the Hamiltonian approach. The (present-value) Hamiltonian is:

$$H = e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta} + \mu(t)\left[(r(t) - g - n)\, b(t) + w(t) - c(t)\right] \tag{14}$$

Given this, what are the necessary conditions for an optimum? Use the *first* theorem in the Hamiltonian handout and write them (you should have two "infinity constraints" plus the NPG, and the original budget constraint – I'll discuss the TVC in a moment)

1.

2.

3.

4.

There is a small technical point I would like to make here regarding the two theorems in the Hamiltonian handout and the Transversality Condition (TVC)

- Strictly speaking, you have to use theorem one here: theorem two only holds in the case when the Hamiltonian only depends on time through discounting (and the state/control variables). But here we have time varying interest rates and income as well
- But theorem one doesn't let you show the transversality condition is a result! So we add it as a separate assumption – this is what David does in lecture.

Thus we also assume the TVC,

$$\boxed{\lim_{t\to\infty} \mu(t)b(t) \leq 0} \tag{15}$$

Recall David's language: the NPG says "don't do impossible things" (we rule out borrowing and achieving infinite consumption) while the TVC says "don't do globally stupid things" (i.e. postpone consumption forever).

Given the NPG and TVC, we can show what Acemoglu calls a "stronger form of the TVC":

$$\boxed{\lim_{t\to\infty} \mu(t)b(t) = 0} \tag{16}$$

Take logs and time derivatives to obtain the "Euler Equation (or LOM for consumption):

This is about as far as we can go given the non-linearities. To "close" the model, assume $r(t) = f'(k(t)) - \delta$ and $w(t) = f(k) - k(t)f'(k(t))$. We then have two difference equations in two variables $k(t)$ and $c(t)$, and can draw a phase diagram.

3.2 Analyzing the Ramsey Model Using the Phase Diagram

For simple non-linear systems like the Ramsey model, we can directly draw the phase diagram without first linearizing the system.

$$\frac{\dot{c}}{c} = \frac{(f'(k) - \tilde{\delta} - \rho - \theta g)}{\theta} \tag{17}$$

$$\dot{k} = f(k) - c - (\tilde{\delta} + n + g)k \tag{18}$$

along with the initial and transversality conditions.

Exercise 2 *Draw the phase diagram for the Ramsey model (putting c on the vertical axis and k on the horizontal axis as before).*

$\beta \equiv \rho - n - (1-\theta)g$

$\dot{c}=0 \Leftrightarrow \boxed{f'(k) = \delta + \rho + \theta g}$

$= \delta + \rho - n + n - g + \theta g + g$

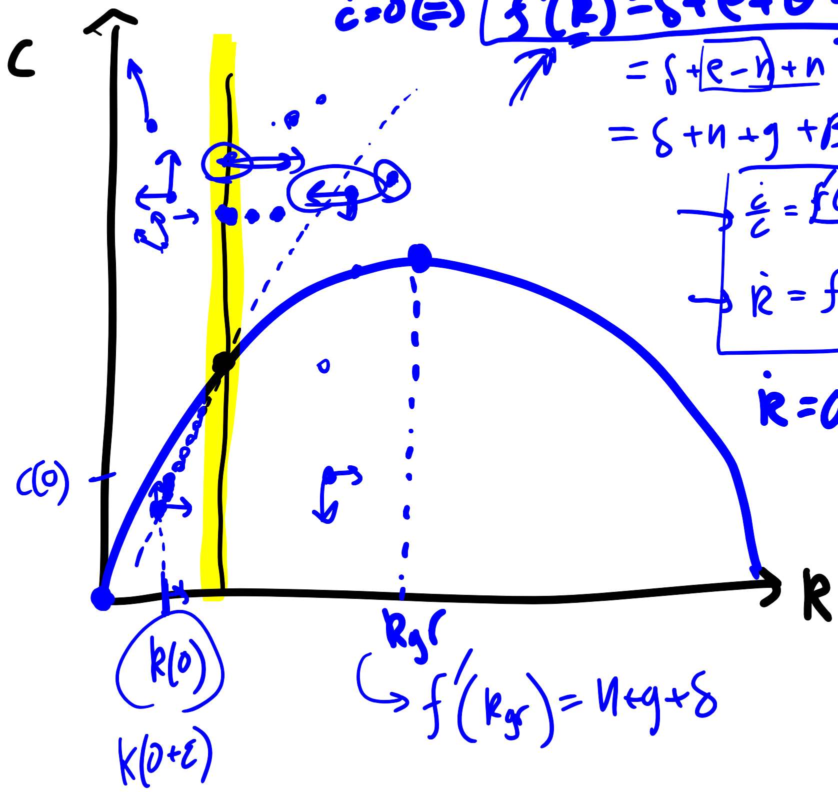
$= \delta + n + g + \beta > n + g + \delta$

$$\frac{\dot{c}}{c} = \frac{f(k) - \theta g - \rho}{\theta}$$

$$\dot{k} = f(k) - c - (n+g+\delta)$$

$$\frac{\dot{c}}{c} = \frac{f'(k) - \theta g - \rho}{\theta}$$

$\dot{k}=0 \Leftrightarrow c = f(k) - (n+g+\delta)k$



Exercise 3 *Suppose initially the Ramsey economy is in the steady state. Using phase diagrams, show how the economy responds to (i.e. how c, $\dot{c}$, k, and $\dot{k}$ react, do any loci shift, etc.)*

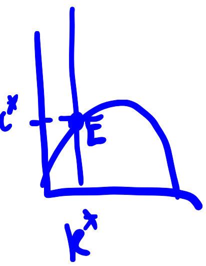
$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \theta g - \rho}{\theta}$$

$$\dot{k} = f(k) - c - (n+g+\delta)k$$

1. *A one-time unanticipated reduction in capital stock.*
2. *A one-time anticipated reduction in capital stock.*
3. *A one-time, unanticipated reduction in the depreciation rate.*
4. *A one-time, anticipated reduction in the depreciation rate.*

   For the next two questions suppose that you start from a k below steady state:

5. *An unanticipated permanent change in consumption-smoothing preferences (θ goes down).*
6. *An anticipated permanent change in consumption-smoothing preferences (θ goes down).*



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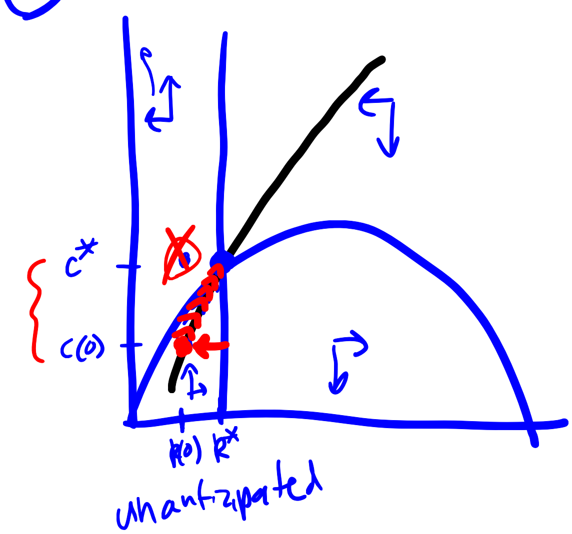

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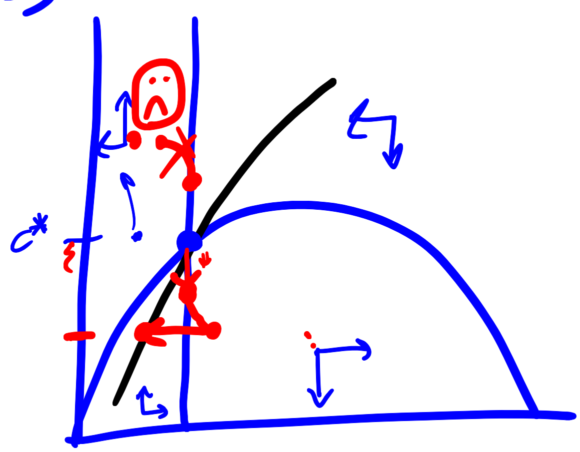

@ t=0, learn about shock to k
@ time T>0