The Neoclassical Growth Model: aka Ramsey-Cass-Koopmans Model

Jón Steinsson

University of California, Berkeley

Fall 2022

Neoclassical Growth Model

- Solow model makes many simplifying assumptions
- Now we relax exogenous savings rate assumption
- Households maximize utility / firms maximize profits
- Households and firms interact in competitive market
- Ramsey-Cass-Koopmans Model (or Ramsy model for short)

Technology

- The same as in the Solow model
- Production function:

$$Y = F(K, AL)$$

- A grows exogenously at rate g
- L grows exogenously at rate n
- Evolution of capital stock:

$$\dot{K}(t) = Y(t) - C(t)L(t)$$

- No depreciation of capital (for simplicity)
- $C(t)$ denotes per capita consumption (but $Y(t)$ is total output)

Change of Variables

- It is useful to rewrite

$$\dot{K}(t) = Y(t) - C(t)L(t)$$

in terms of capital, output, and consumption per effective unit of labor

- A few lines of algebra yield

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

Details

Firms

- Large number of identical firms seek to maximize profits
- Hire workers and capital in competitive factor markets
- Can use A for free
- Profit maximization implies that firms hire labor and capital to the point where their marginal product is equal to their price
- First order condition for capital:

$$\frac{\partial F(K, AL)}{\partial K} = r(t)$$

which implies

$$f'(k(t)) = r(t)$$

Details

Firms

- First order condition for labor:

$$\frac{\partial F(K, AL)}{\partial L} = W(t)$$

- We can rewrite this as

$$f(k(t)) - k(t)f'(k(t)) = w(t)$$

where $w(t) = W(t)/A(t)$ (wage per effective unit of labor)

Details

Representative Firm

- One "representative" firm same as many small firms if
  - Common production function
  - Constant returns to scale
  - Common factor prices
- Easy to derive for Cobb-Douglas case:

$$Y = AK^{\alpha}L^{1-\alpha}$$

- Maximizing behavior implies Details

$$Y = A\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left(\frac{W}{r}\right)^{\alpha}L$$

- Since production is linear in L, firm has no optimal size
- Might as well be one large firm!

HOUSEHOLDS

- Large number H of households
- Size of each household grows at rate n
- Each member supplies one unit of labor
- Initial capital holdings $K(0)/H$
- Household rents capital to firms
- Key household choice:
  - How to divide income between consumption and savings

Household Utility

- Households maximize utility

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$$

where

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}$$

- ρ is discount rate
- $u(\cdot)$ gives each member's utility
- $L(t)/H$ is the number of household members
- $1/\theta$ is elasticity of intertemporal substitution

Household Budget Constraint

- Household's intertemporal budget constraint:
  - Present value of consumption less than or equal to present value of income plus initial wealth
- Mathematically:

$$\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} W(t) \frac{L(t)}{H} dt$$

- Discounting with time-varying interest rate:

$$R(t) = \int_{\tau=0}^{t} r(\tau) d\tau$$

Household Budget Constraint

- Why is this the budget constraint?
- This condition rules out Ponzi-schemes:
  - Borrow today and consume
  - Borrow more tomorrow to pay back principle and interest
  - Do this forever
- Intertemporal budget constraint implies that

$$\lim_{s \to \infty} e^{-R(s)} \frac{K(s)}{H} \geq 0$$

(See Romer (2019, p. 54) for derivation)

- Ponzi schemes violate this condition

Change of Variables

- Let's write household objective function in terms of consumption and labor per effective worker

$$\frac{C(t)^{1-\theta}}{1-\theta} = \frac{[A(t)c(t)]^{1-\theta}}{1-\theta}$$
$$= \frac{[A(0)e^{gt}]^{1-\theta}c(t)^{1-\theta}}{1-\theta}$$
$$= A(0)^{1-\theta}e^{(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta}$$

Change of Variables

- Overall utility function becomes:

$$
\begin{aligned}
U &= \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \left[A(0)^{1-\theta} e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} \right] \frac{L(0)e^{nt}}{H} dt \\
&= B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt
\end{aligned}
$$

where

$$
B = A(0)^{1-\theta} \frac{L(0)}{H} \qquad \beta = \rho - n - (1-\theta)g
$$

- A similar change of variables for the budget constraint yields:

$$
\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt
$$

Household's Problem

- Maximize:

$$U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

- Subject to:

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$$

- Household's choice variable is the path for $c(t)$

- We form a Lagrangian:

$$\mathcal{L} = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt + \lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}dt - \int_{t=0}^{\infty} e^{-R(t)}c(t)e^{(n+g)t}dt\right]$$

- Differentiate this with respect to $c(t)$:

$$Be^{-\beta t}c(t)^{-\theta} = \lambda e^{-R(t)}e^{(n+g)t}$$

Household Optimization

$$Be^{-\beta t}c(t)^{-\theta} = \lambda e^{-R(t)}e^{(n+g)t}$$

- Let's take logs:

$$\log B - \beta t - \theta \log c(t) = \log \lambda - \int_{\tau=0}^{t} r(\tau)d\tau + (n+g)t$$

- Differentiate both sides with respect to time

$$-\beta - \theta\frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g)$$

- Rearrange and use definition of β:

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

Consumption Euler Equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

- And for per capita consumption:

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{c}(t)}{c(t)} \\ &= g + \frac{r(t) - \rho - \theta g}{\theta} \\ &= \frac{r(t) - \rho}{\theta}\end{aligned}$$

- Consumption growth depends on $r(t) - \rho$ and nothing else

Consumption Euler Equation

$$\frac{\dot{C}(t)}{C(t)} == \frac{r(t) - \rho}{\theta}$$

- Consumption growth depends on $r(t) - \rho$ and nothing else.
- Why?
  - Why nothing else?
  - Why higher growth with higher $r(t)$?

Consumption Euler Equation

$$\frac{\dot{C}(t)}{C(t)} == \frac{r(t) - \rho}{\theta}$$

- Consumption growth depends on $r(t) - \rho$ and nothing else.
- Why?
  - Why nothing else?
  - Why higher growth with higher $r(t)$?
- Embodies desire to smooth consumption

  (due to diminishing marginal utility)

Dynamic Optimization in Continuous Time

- Our derivation was somewhat heuristic
- More rigorous to use Hamiltonians / Maximum Principle
- Barro R.J. and X. Sala-I-Martin (2004): *Economic Growth* (Appendix A.3)
- Obstfeld, M. (1992): "Dynamic Optimization in Continuous-Time Economic Models (A Guide for the Perplexed)
- Acemoglu, D. (2009): *Introduction to Modern Economic Growth* (Chapter 7)

Interest Rate in General Equilibrium

- Household takes $r(t)$ as given:

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

- But in general equilibrium:

$$r(t) = f'(k(t))$$

- So we have that

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

Dynamic System

- Two endogenous variables $c(t)$, $k(t)$
- Two dynamic equations:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

- Two boundary conditions:
  - k(0) given (initial condition)
  - Intertemporal budget constraint with equality (terminal condition)
- It is the fact that dynamic system has a terminal condition (rather than full set of initial conditions) that makes the system “forward looking”.

Dynamics of $c(t)$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

- $\dot{c}(t) = 0$ if $f'(k(t)) = \rho + \theta g$
  - Recall that $f'(k)$ is decreasing in k
  - Let k^* denote k for which $f'(k) = \rho + \theta g$
  - Then $\dot{c}(t) = 0$ if $k(t) = k^*$
- $\dot{c}(t) > 0$ if $f'(k(t)) > \rho + \theta g$ or equivalently $k(t) < k^*$
- $\dot{c}(t) < 0$ if $f'(k(t)) < \rho + \theta g$ or equivalently $k(t) > k^*$

Dynamics of $c(t)$

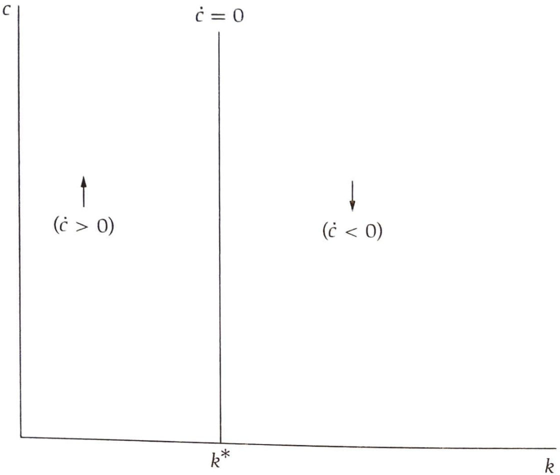

FIGURE 2.1 The dynamics of c

Source: Romer (2019)

Dynamics of $k(t)$

$$\dot{k}(t) = \underbrace{f(k(t)) - c(t)}_{\text{Investment}} - \underbrace{(n+g)k(t)}_{\text{breakeven investment}}$$

- $\dot{k}(t) = 0$ if investment is equal to breakeven investment
- Rearranging, we get that $\dot{k}(t) = 0$ when

$$c = f(k) - (n+g)k$$

- Let's think about this as a function of k
  - $f(0) = 0$ and $(n+g)0 = 0$
  - $f'(k)$ is very high for small k
  - $f(k)$ is concave (slope goes to $-(n+g)$ when $k \to \infty$)

Dynamics of $k(t)$

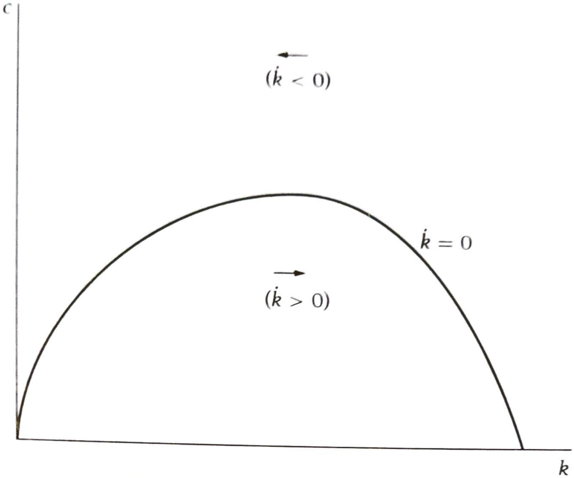

FIGURE 2.2 The dynamics of k

Source: Romer (2019)

Dynamics of $c(t)$ and $k(t)$

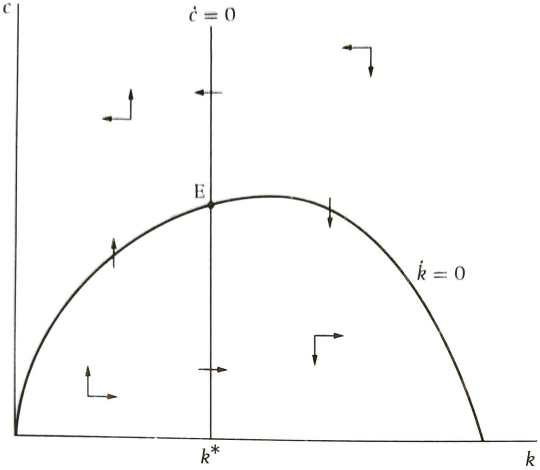

FIGURE 2.3 The dynamics of c and k

Source: Romer (2019)

k^* is below Golden Rule k

- Steady state k: $f'(k^*) = \rho + \theta g$
- Golden Rule k: $f'(k_{GR}) = n + g$
- Since $f''(k) < 0$, $k^* < k_{GR}$ if

$$\rho + \theta g > n + g$$

$$\rho - n - (1 - \theta) g > 0$$

$$\beta > 0$$

- We assume $\beta > 0$ since otherwise household utility is infinite

Which Path Will Economy Take?

- Phase diagram has many paths
- All of them satisfy the two dynamic equations
- Which one of these paths will the economy take?

Which Path Will Economy Take?

- Phase diagram has many paths
- All of them satisfy the two dynamic equations
- Which one of these paths will the economy take?

- Answer determined by boundary conditions
  - Boundary condition #1: Initial condition for k
  - But there is no initial condition for c!!!
  - $c(0)$ is a choice of the household

Which Path Will Economy Take?

- Phase diagram has many paths
- All of them satisfy the two dynamic equations
- Which one of these paths will the economy take?

- Answer determined by boundary conditions
  - Boundary condition #1: Initial condition for k
  - But there is no initial condition for c!!!
  - $c(0)$ is a choice of the household

- So, how do we determine $c(0)$?

Which Path Will Economy Take?

- Phase diagram has many paths
- All of them satisfy the two dynamic equations
- Which one of these paths will the economy take?

- Answer determined by boundary conditions
  - Boundary condition #1: Initial condition for k
  - But there is no initial condition for c!!!
  - $c(0)$ is a choice of the household

- So, how do we determine $c(0)$?
  - Boundary condition #2: Intertemporal budget constrain holds with equality

DETERMINATION OF $c(0)$

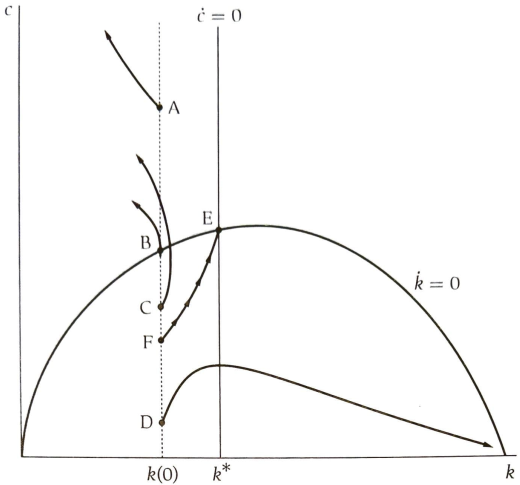

FIGURE 2.4 The behavior of c and k for various initial values of c

Source: Romer (2019)

Determination of $c(0)$

- Paths starting at A, B, and C eventually hit $k = 0$. For consumption Euler equation to continue to hold, k must keep falling, which it can't.
- Path starting at D, eventually rises above Golden Rule level of consumption. At this point $f'(k) < n + g$, so $e^{-R(s)}e^{(n+g)s}$ is rising in s and $k(s)$ is rising in s. So:

$$\lim_{s \to \infty} e^{-R(s)}e^{(n+g)s}k(s) = \infty$$

  In other words, net present value of future wealth is exploding, which means household should consumer more. (Violates intertemporal budget constraint.)

Saddle Path

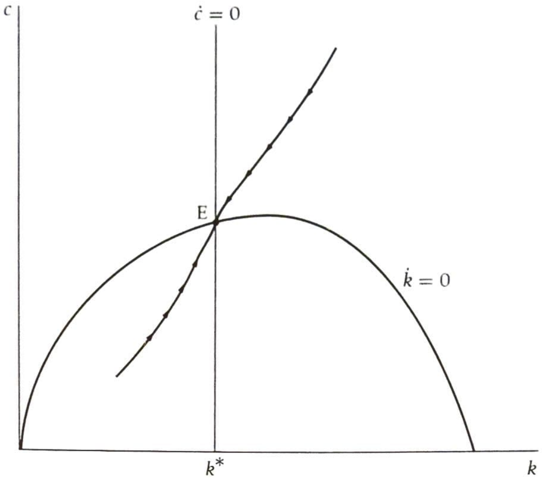

FIGURE 2.5 The saddle path

Source: Romer (2019)

Long-Run Growth in Ramsey Model

- In the long run, k^*, y^*, c^* are constant
- Level variables grow at rate $n + g$
- Per capita variables grow at rate g
- Long-run growth rate independent of many model parameters.
  E.g., Production function parameters, preference parameters (ρ and θ)

- Dynamic equations:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

- To calculate steady state, we set $\dot{c}(t) = 0$ and $\dot{k}(t) = 0$:

$$f'(k^*) = \rho + \theta g$$

$$c^* = f(k^*) - (n+g)k^*$$

- Level of steady state **is** a function of ρ, θ, and production function

Unanticipated Fall in Discount Rate

- Consider an unanticipated fall in the discount rate ρ
- Conceptually similar to an increase in the savings rate in the Solow model
- Long-run effect depends on shift in steady state:

$$f'(k^*) = \rho + \theta g$$

$$c^* = f(k^*) - (n+g)k^*$$

- Shift in ρ only affects first of these equation
- Since $f''(k) < 0$, $k^*_{\text{NEW}} > k^*_{\text{OLD}}$

Unanticipated Fall in ρ

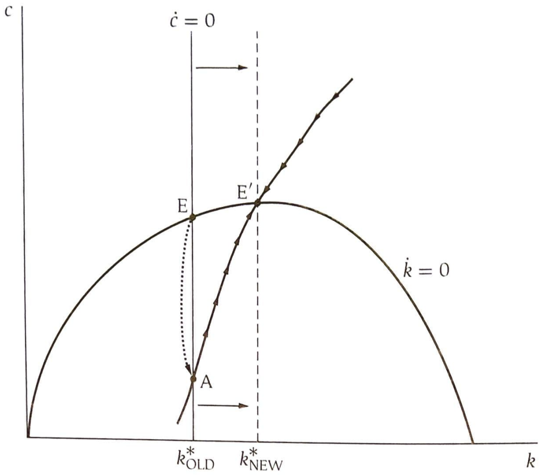

FIGURE 2.6 The effects of a fall in the discount rate

Source: Romer (2019)

Shape of the Saddle Path

- The saddle path gives c(k) (called the policy function)
- What is the shape of this path?
- Consider different values of θ
- Recall that $1/\theta$ is the intertemporal elasticity of substitution
- High θ (low $1/\theta$) implies strong desire to smooth consumption
  Household will try to shift consumption from the future
  Saddle path will be close to $\dot{k}(t) = 0$ locus

Unanticipated Fall in ρ

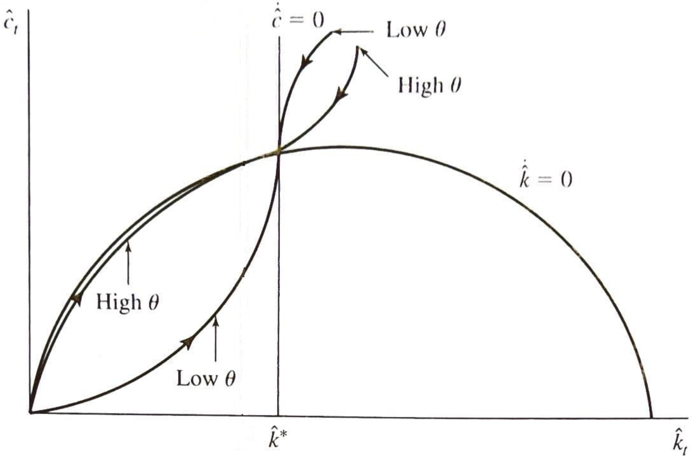

Figure 2.2

Source: Barro and Sala-I-Martin (2004)

SAVINGS RATE IN THE RAMSEY MODEL

- Savings rate constant in Solow model. What about Ramsey model?
- In general it is complicated
- Intuitively two forces:
  - Substitution effect: As k rises, $f'(k)$ falls, and r falls. Incentive to save fall.
  - Income effect: When households are below steady state, their income is below future income. They would like to smooth consumption by consuming a lot relative to income. As income rises, this force weakens and savings rise.
- Which force wins depends on parameters.

Savings Rate in the Ramsey Model

- Cobb-Douglas case ($y = k^\alpha$): savings rate is monotonic
- If $\alpha(g + n) = (\rho + \theta g)/\theta$, savings rate is constant
- If $\alpha(g + n) > (\rho + \theta g)/\theta$, savings rate is increasing

  (e.g., small value of $1/\theta$ – weak substitution effect)
- If $\alpha(g + n) < (\rho + \theta g)/\theta$, savings rate is decreasing

  (e.g., large value of $1/\theta$ – strong substitution effect)
- See Barro and Sala-I-Martin (2004, sec. 2.6.4) for details
  - They argue that $\alpha(g + n) \approx (\rho + \theta g)/\theta$ is not unreasonable.
  - Empirically they argue savings rate rise a slight bit with development

SAVINGS RATE IN RAMSEY MODEL

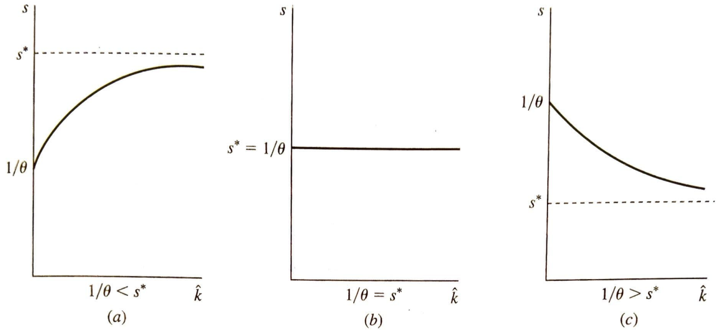

Figure 2.3
Phase diagram for the behavior of the saving rate (in the Cobb–Douglas case). In the Cobb–Douglas case, the savings rate behaves monotonically. Panel *a* shows the phase diagram for $\hat{c}/\hat{y}$ and $\hat{k}$ when the parameters are such that $(\delta + \rho + \theta x)/\theta > \alpha \cdot (x + n + \delta)$. Since the stable arm is upward sloping, the consumption ratio increases as the economy grows toward the steady state. Hence, in this case, the saving rate (one minus the consumption rate) declines monotonically during the transition. Panel *b* considers the case in which $(\delta + \rho + \theta x)/\theta < \alpha \cdot (x + n + \delta)$. The stable arm is now downward sloping and, therefore, the saving rate increases monotonically during the transition. Panel *c* considers the case $(\delta + \rho + \theta x)/\theta + \alpha \cdot (x + n + \delta)$. The stable arm is now horizonal, which means that the saving rate is constant during the transition.

Source: Barro and Sala-I-Martin (2004)

WELFARE IN THE RAMSEY MODEL

- Is the market outcome in the Ramsey model a ”good” outcome?
- First Welfare Theorem holds in the Ramsay model:
  - Markets are competitive and complete
  - Households are rational and optimize
  - Property rights exist over all objects of value
- This means decentralized outcome is Pareto efficient

  (Actually there are some technicalities which we will come back to in OLG lecture)
- Easy to verify by solving the “planner’s problem”

Ramsey versus Solow

- Basic message of Solow model carries over to Ramsey model
  - Balanced growth in steady state
  - Savings rate $(y - c)/y$ constant in steady state
  - Level variables grow at rate $n + g$
  - Per capita growth due to exogenous growth in A
  - Growth higher below steady state, lower above steady state
- Differences in Ramsey:
  - Savings rate not necessarily constant (but constant not unreasonable)
  - n does not affect k^*. ($f'(k^*) = \rho - \theta g$)
    Higher n lowers c^*. ($c^* = f(k) - (n + g)k^*$)

Appendix

Capital Accumulation – Change of Variables

Notice that:

$$\dot{k}(t) = \frac{d}{dt}\left(\frac{K(t)}{A(t)L(t)}\right) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}[A(t)\dot{L}(t) + \dot{A}(t)L(t)]$$

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - (n+g)k$$

Dividing capital accumulation equation by $A(t)L(t)$ we get

$$\frac{\dot{K}(t)}{A(t)L(t)} = \frac{Y(t)}{A(t)L(t)} - \frac{C(t)L(t)}{A(t)L(t)}$$

$$\dot{k}(t) + (n+g)k = y(t) - c(t)$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

Back

Capital Demand – Details

- Recall that $f(k) \equiv F(k, 1)$.
- $F(\cdot, \cdot)$ is homogeneous of degree one
- This implies that $F_1(\cdot, \cdot)$ is homogeneous of degree zero

  (where the subscript denotes a derivative with respect to the first argument)

$$f'(k) = F_1(k, 1) = F_1(K/AL, 1) = F_1(K, AL) = F_K(K, AL)$$

Back

Labor Demand – Details

- Notice that

$$\frac{\partial F(K, AL)}{\partial L} = A\frac{\partial F(K, AL)}{\partial AL} = AF_2(K, AL)$$

- Since $F(K, AL)$ is homogeneous of degree 1 we have that

$$\begin{aligned} F(K, AL) &= F_1(K, AL)K + F_2(K, AL)AL \\ ALF(k, 1) &= F_1(K, AL)K + F_2(K, AL)AL \\ f(k) &= f'(k)k + F_2(K, AL) \\ F_2(K, AL) &= f(k) - f'(k)k \end{aligned}$$

Back

Cobb-Douglas Firm

- Labor demand:

$$W = (1-\alpha) A K^{\alpha} L^{1-\alpha}$$

- Capital demand:

$$r = \alpha A K^{\alpha-1} L^{1-\alpha}$$

- Divide one by the other

$$\frac{K}{L} = \frac{\alpha}{1-\alpha} \frac{W}{r}$$

- Plug this into production function for K:

$$Y = A K^{\alpha} L^{1-\alpha} = A \left(\frac{\alpha}{1-\alpha} \right)^{\alpha} \left(\frac{W}{r} \right)^{\alpha} L^{\alpha} L^{1-\alpha}$$

Back