Economics 202A: Section 1

Introduction to Solow

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Housekeeping and Welcome:

- Covid: please do your best to follow (changing) university guidelines!
  - Will try to offer online section option
- What is section:
  - Review and practice problems!
  - Sometimes more technical and covers material beyond course (not on test but generally useful)
  - We may not cover everything but the leftover is useful as practice problems (both for midterm now and possibly field exams later). I will publish answer keys.
- First Year Advice: help each other and don't stress about grades (especially now given everything else going on)

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will refer to Acemoglu's textbook *Introduction to Modern Economic Growth* which provides a slightly more technical discussion than these notes.

The outline of these notes are as follows:

1. Properties of the Solow Aggregate Production Function (APF)

2. Intensive Form

3. Reviewing the Solow Setup

4. Solving Solow Through Graphs

5. Analyzing the Solow APF

1 Properties of the Solow APF

Definition 1 *A production function $F(X_1,\ X_2, ...)$ is a technology for transforming a vector of inputs $(X_1,\ X_2, ...)$ into output F. The most common production function considers output as a function of capital and labor: F(K,L).*

- Inputs are also referred to as *factors of production*.
- The inputs and output should be interpreted as quantities, not values.

The Solow model's APF is *neoclassical*. For concreteness, in this section and the rest of these notes I'll focus on production functions of the form $F(K,\ L,\ A)$, and omit the dependence on A when it is inessential.

Definition 2 *Consider a production function $F(K,\ L,\ A) : \mathbf{R}_+^3 \to \mathbf{R}_+$ which is twice continuously differentiable in K and L. The production function is* ***neoclassical*** *if it satisfies*[1]

1. *Constant returns to scale (CRS):*

$$F(\lambda K,\ \lambda L,\ A) = \lambda F(K,\ L,\ A) \text{ for all } \lambda > 0. \tag{1}$$

2. *Positive and diminishing marginal products with respect to each input:*

$$\frac{\partial F}{\partial K} > 0,\ \frac{\partial^2 F}{\partial K^2} < 0, \tag{2}$$

$$\frac{\partial F}{\partial L} > 0,\ \frac{\partial^2 F}{\partial L^2} < 0,\ \text{for all } K > 0,\ L > 0. \tag{3}$$

3. *Inada conditions:*

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty, \tag{4}$$

$$\lim_{K\to \infty} \frac{\partial F}{\partial K} = \lim_{L\to \infty} \frac{\partial F}{\partial L} = 0. \tag{5}$$

[1] These conditions are taken from Acemoglu 2009, Ch.2

- Condition 1 is also known as *homogeneity of degree 1*: A function $F(X_1,\ X_2,...)$ is homogeneous of degree n if, for all $\lambda > 0$, $F(\lambda X_1,\ \lambda X_2,...) = \lambda^n F(X_1,\ X_2,...)$. It means that a proportional increase in all inputs increases output by the same proportion. While intuitively plausible, this condition does rule out fixed costs of production, externalities in production and other potentially interesting features.

- Condition 2 is similarly straightforward: holding all other inputs fixed, increasing any single input always increases output, but at a diminishing rate as more of the input is used. Notice that there is no contradiction with the assumption of constant returns to scale, which involves increasing all inputs proportionally.

- Condition 3 is less intuitive. Plays a technical role in ensuring the existence, uniqueness and stability of the steady state with positive output in growth models

Exercise 1 *Show that Cobb-Douglas production function $Y = K^\alpha L^{1-\alpha}$ and $\alpha \in (0,1)$ satisfies neoclassical assumptions.*

1. $F(\lambda K,\ \lambda L) = (\lambda K)^a (\lambda L)^{1-a} = \lambda^a \lambda^{1-a} K^\alpha L^{1-a} = \lambda F(K,\ L)$
2. $\frac{\partial F}{\partial K} = a\left(\frac{L}{K}\right)^{1-a} > 0$; $\frac{\partial^2 F}{\partial K^2} = -a(1-a)\frac{L^{1-a}}{K^{2-a}} < 0$ (analogous for L)
3. $\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{K\to 0} a\left(\frac{L}{K}\right)^{1-a} = \infty$; $\lim_{K\to\infty} \frac{\partial F}{\partial K} = \lim_{K\to\infty} a\left(\frac{L}{K}\right)^{1-a} = 0$ (analogous for L)

Exercise 2 *Give an example of production function that exhibits diminishing returns in each input, yet satisfies the property of homogeneity of degree $n > 1$.*

$F(K,\ L) = K^a L^b; 1 >\ a,\ b > 0;\ a + b > 1$ would do.

$$\begin{aligned} F(\lambda K,\ \lambda L) &= (\lambda K)^a (\lambda L)^b = \lambda^a \lambda^b K^\alpha L^{1-a} = \lambda^{a+b} F(K,\ L) > \lambda F(K,\ L) \\ \frac{\partial F}{\partial K} &= aK^{a-1}L^b > 0;\ \frac{\partial^2 F}{\partial K^2} = -a(1-a)\frac{L^b}{K^{2-a}} < 0 \text{ (analogous for } L) \end{aligned}$$

Exercise 3 *For a neoclassical production function, prove that as the amount of one input goes to infinity, so does output (for strictly positive amounts of other inputs), i.e.* $F(\infty,\ L) = F(K, \infty) = \infty^2$. *(If this is too hard, try showing it for the Cobb-Douglass case and I will show the proof for the general case)*

First, observe that[3]

$$\begin{aligned}
\lim_{K\to\infty} F(K,\ L) &= \lim_{K\to\infty} K \cdot F\left(1,\ \frac{L}{K}\right) = \text{(by CRS; now rewrite it to apply l'Hopital's rule)} \\
&= \lim_{K\to\infty} \frac{F\left(1,\ \frac{L}{K}\right)}{1/K} = \left[\frac{0}{0}\right] = \text{(apply l'Hopital's rule)} \\
&= \lim_{K\to\infty} \frac{-\frac{L}{K^2} F_2\left(1,\ \frac{L}{K}\right)}{-\frac{1}{K^2}} = \text{(by Inada conditions)} \\
&= \lim_{K\to\infty} L \cdot F_2\left(1,\ \frac{L}{K}\right) = \infty.
\end{aligned}$$

Analogously, $\lim_{L\to\infty} F(K,\ L) = \infty$. Then

$$\begin{aligned}
F(1,\ 0) &= \lim_{K\to\infty} F\left(1,\ \frac{L}{K}\right) = \text{(by CRS)} \\
&= \lim_{K\to\infty} \frac{F(K,\ L)}{K} = \left[\frac{\infty}{\infty}\right] = \text{(can apply l'Hopital's rule given our result above)} \\
&= \lim_{K\to\infty} F_1(K,\ L) = 0 \text{ (by Inada conditions)}, \\
F(K,\ 0) &= K \cdot F(1,\ 0) = K \cdot 0 = 0 \text{ (using CRS one more time)}.
\end{aligned}$$

Analogously, $F(0,\ L) = 0$.

[2] $F(\infty, \cdot) \equiv \lim_{x\to\infty} F(x, \cdot)$.

[3] The first line of the derivation is a bit tricky. We do not yet know what $F(1,\ \frac{L}{K})$ is: since F was assumed continuous and differentiable at zero we know that the limit exists and is weakly positive given the assumptions on F. Obviously, if it is anything greater than 0, the result follows immediately. So, here we "assume" it is 0 and show that the limit is still infinite. Later in this proof we will see that our guess of zero is correct.

Exercise 4 *(Romer 2.1) Consider N firms each with constant-returns-to-scale production function $Y = F(K, AL)$, or (using the intensive form) $Y = ALf(k)$. Assume $f'(\cdot) > 0, f''(\cdot) < 0$. Assume that all firms can hire labor at wage wA and rent capital at cost r, and that all firms have the same value of A. Ignore nonnegativity constraints on K and L.*

(a) Consider the problem of a firm trying to produce Y units of output at minimum cost. Show that the cost-minimizing level of k is uniquely defined and is independent of Y, and that all firms therefore choose the same value of k.

(b) Show that the total output of the N cost-minimizing firms equals the output that a single firm with the same production function has if it uses all the labor and capital used by the N firms.

Solution to (a): Set-up the firms Lagrangian.

$$\mathcal{L} = wAL + rK + \lambda\left(Y - ALf(k)\right)$$

The first order conditions require that:

$$\begin{aligned} r &= \lambda f'(k) \\ w &= \lambda\left(f(k) - kf'(k)\right) \end{aligned}$$

k is thus uniquely and implicitly defined by these equations and is independent of Y.

Solution to (b): Since each firm uses the same technology and has the same level of A, the will employ the same k implicitly defined above. Total output is therefore:

$$Y = \sum_{i=1}^{n} AL_i f(k^*) = A\bar{L}f(k^*) = F(\bar{K}, A\bar{L})$$

Exercise 5 *Consider a cost minimizing firm with a neoclassical production function $F(K, L)$*

and facing prices r and w for inputs K and L, respectively. Using λ to denote the costate variable, show that the optimal choices of K and L satisfy:[4]

$$rK + wL = \lambda F(K, L)$$

What does this imply about the profits earned by a cost minimizing, price taking firm with a neoclassical production function?

Answer: The cost-minimizing firm's problem is to minimize costs subject to the constraint that it produces at least some Y. Set up the following lagrangian:

$$\mathcal{L} = -(wL + rK) + \lambda\left(F(K, L) - Y\right)$$

The first order conditions for this problem imply that:

$$r = \lambda F_K(K, L)$$
$$w = \lambda F_L(K, L)$$

Using Eulers theorem for the CRS ($m = 1$) case yields:

$$rK^* + wL^* = \lambda F_K(K^*, L^*)K^* + \lambda F_L(K^*, L^*)L^* = \lambda F(K^*, L^*)$$

where $*$ indicates the optimal choices for K, L. What does this imply about the profits earned by a cost minimizing, price taking firm with a neoclassical production function? Recall that λ has the interpretation here as marginal cost, so unless the firm has market power it will be making zero profits.

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[4] The following theorem, known as **Euler's Theorem**, may be helpful (taken from Acemoglu 2009, Chapter 2):

Theorem 1 *Suppose that $g : \mathcal{R}^{K+2} \to \mathcal{R}$ is differentiable in $x \in \mathcal{R}$ and $y \in \mathcal{R}$, with partial derivatives denoted by g_x and g_y, and is homogenous of degree m in x and y. Then:*

$$mg(x, y, z) = g_x(x, y, z)x + g_y(x, y, z)$$

for all $x \in \mathcal{R}$, $y \in \mathcal{R}$, and $z \in \mathcal{R}^K$. Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogenous of degree $m - 1$ in x and y.

A second important choice is the role of technology in this production function. In the Solow model, technology is taken to be labor augmenting or Harrod Neutral (this is not innocuous)

Definition 3 *Technology is labor augmenting (Harrod-neutral) if it enters in the form:* $F(K, AL)$*. Technology is capital augmenting if it enters in the form:* $F(AK, L)$*. And it is Hicks-neutral if it enters in the form* $AF(K, L)$*.*

2 Intensive Form

It will often be convenient to express the production function in *intensive form* by dividing through by one of its arguments. Consider the Solow production function $F(K, AL)$. Using the constant returns to scale property, we can write

$$Y = F(K,\ AL) = AL \cdot F(K/AL,\ 1) = AL \cdot f(k) \tag{6}$$

where $k = K/AL$ is capital per "effective worker" and $f(k) \equiv F(k, 1)$. If we divide both sides by AL we get an expression for output per effective worker as a function of k, $y = f(k)$.

Why do this?

- A judiciously chosen normalization can reduces the number of variables (you will see a similar trick probably with consumption-savings models in part two)
- If you ever need to switch back and forth, just substitute the definition (i.e., $y = Y/AL$) and apply the normal rules of algebra / calculus.

3 Reviewing the Solow Setup

Ignoring the distribution of income among factors, the Solow model can be summarized by the following equations:

$$Y(t) = F(K(t),\ A(t)L(t)) \tag{7}$$

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{8}$$

$$L(t) = L(0)e^{nt} \tag{9}$$

$$A(t) = A(0)e^{gt} \tag{10}$$

There are two exogenous variables, A and L, and two endogenous variables K and Y. However, Y is a static function of the rest of the variables, so once we know $K(t)$ we can determine $Y(t)$ easily through (7).

- A solution to this system of differential equations would deliver the value of $K(t)$ for all t. In general we cannot find an explicit solution even if we know the functional form of F, as is common with nonlinear differential equations.

- A key feature of the Solow model that allows us to learn so much about the solution is that it eventually converges to a *balanced growth path* (BGP), a trajectory in which the endogenous variables (here, K and Y) grow at constant rates. This means that we can completely characterize the solution for the time period which the economy spends of the balanced growth path. We can use graphical techniques and approximations to study the behavior of the economy away from the steady state.

4 Solving Solow Through Graphs

4.1 Finding the BGP

Exercise 6 *Show that the Solow model converges to a BGP. What are the growth rates of endogenous variables K and Y on the BGP?*

There are two ways to find the balanced growth path of the Solow model. It is easiest to begin with the intensive form of the capital accumulation equation (8):

$$\begin{aligned}
\dot{K} &= sY(t) - \delta K(t) \\
\frac{\dot{K}}{AL} &= s\frac{Y(t)}{AL} - \delta\frac{K(t)}{AL} \\
\frac{\dot{K}}{K}\frac{K}{AL} &= sf(k) - \delta k \\
k\left(\frac{\dot{k}}{k} + g + n\right) &= sf(k) - \delta k \\
\dot{k} &= sf(k) - (n+g+\delta)k \qquad (11)
\end{aligned}$$

where we used the fact that $\dot{K}/K = \dot{k}/k + \dot{A}/A + \dot{L}/L$. We know that for small values of k, $sf(k) > (n+g+\delta)k$ by the Inada conditions. Similarly, we know that for sufficiently large k we have k, $sf(k) < (n+g+\delta)k$, again by the Inada conditions. This implies that there is a value of k, call it $\bar{k}$, for which $\dot{k} = 0$. This value of k is the economy's balanced growth path, since capital accumulation through savings is just enough to offset depreciation, population growth and technological progress. If there were more capital, the gains to production would not be large enough to offset the forces of de-accumulation. If there were less, capital's contribution to savings through production would do more than offset these forces.

We can solve for this value of $\dot{k}$ by setting $\dot{k} = 0$ in (11) to get

$$\frac{f(\bar{k})}{\bar{k}} = \frac{n+g+\delta}{s} \qquad (12)$$

Furthermore our analysis above shows that this steady state is *stable*, meaning that if $k \neq \bar{k}$ it will tend to converge to this steady state value. This also implies that $y = f(k)$ will be constant on the BGP. These properties can also be illustrated in the famous Solow diagram. Once we know that $\dot{k} = 0$ we can solve for the growth rates of K and Y:

$$0 = \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{\dot{A}}{A} \;\Rightarrow\; g_K = n + g \tag{13}$$

$$\frac{\dot{Y}}{Y} = \alpha(t)\frac{\dot{K}}{K} + (1 - \alpha(t))\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right) \;\Rightarrow\; g_Y = n + g \tag{14}$$

One limitation of this method is that we had to guess the appropriate normalization to ensure a steady state in advance. A perhaps more widely applicable method is to start with the accumulation equation (8) to figure out what variables grow at the same rate. Dividing equation (8) through by K, we get

$$g_K = s\frac{Y}{K} - \delta \tag{15}$$

Since g_K is constant on the BGP, Y/K must be constant on the BGP as well. Then

$$\frac{Y}{K} = constant \;\Rightarrow\; g_Y - g_K = 0 \Rightarrow g_Y = g_K \tag{16}$$

Now we use this equality into our growth decomposition 14 to get

$$g_Y = \alpha(t)g_K + (1 - \alpha(t))(g_L + g_A) \;\Rightarrow\; g_K = g_L + g_A \tag{17}$$

Now we have shown that on the balanced growth path $\dot{k} = 0$ and we can derive equation (11) and use it to find the steady state value of k and y as before. The point is that this procedure tells you what the appropriate normalization of K will be to ensure a steady state, whereas before we simply guessed that it would be K/AL.

Exercise 7 *Assume the Solow economy is on its BGP. Show the impact of an change in the parameters* s, δ, g, n *on capital accumulation* ($\dot{k}$ *and* k) *and output per worker* (Y/L) *during the transition to the new BGP.*

The easiest way to answer these questions is using the Solow diagram. We will do the case of the depreciation rate in class, and you can do the others on your own. The case of s was covered in lecture and in the textbook.

Differences in the Solow BGP as a result of s, δ, g, n can results in long-run differences in capital per effective worker, output per worker, and consumption per worker. To the extent that two economies share the same rate of technological progress, population growth, and savings behaviors, they should converge to similar levels of income. This is the notion of *conditional convergence*, which helps explain why although there is no relationship between 1960 GDP and GDP growth over the next decade among all countries, there appears to be a strong *negative* relationship among similar countries (i.e., the OECD).

4.2 "Golden Rules"

Suppressing time subscripts and letting consumption be defined as everything not saved, or $C = (1-s)Y$, we have $c \equiv \frac{C}{AL} = (1-s)y$. Note that for given values of g, n and δ, the level of s uniquely determines the steady state level of capital in the economy, k^*. The steady state value of consumption is likewise pinned down by s as:

$$c^* = (1-s)f(k^*) \tag{18}$$

This implies that for different levels of s, we will get different levels of c. It turns out that there is a unique $s \in (0,1)$, and corresponding k^*, which maximizes c^* in the Solow model. We call the capital stock consistent with this value of s the "Golden Rule" level of the capital stock as it results in the maxiumum possible consumption along a BGP.

Exercise 8 *Solve for the golden rule capital stock in the Solow model analytically, and then graphically. What is the intuition?*

Using the fact that in the steady state, capital accumulation exactly offsets accrued depreciation, we can also write c^* as:

$$c^* = f(k^*) - (n+g+\delta)k^* \tag{19}$$

If a policy maker could chose s, she may want to do so in order to maximize consumption, assuming that utility is derived from consumption alone. It's clear from the representation in equation 19 that the value of k^* that maximizes c^* satisfies:

$$f'(k^*) = n+g+\delta \tag{20}$$

This level of capital is commonly revered to as the "golden rule" level of capital. We can find it on our standard Solow graph by including $f(k)$ in addition to $sf(k)$. Consumption is simply the distance between these two curves, which is maximized at the point where $f'(k) = n+g+\delta$.

What's the intuition here? If we are below the golden rule level of capital, we have $f'(k^*) > n + g + \delta$. This means that if we increase capital by a small amount, output more than increases enough to account for the incremental depreciation caused by this extra capital accumulation. So if we increased savings on the margin, output would increase more than enough to cover our extra depreciation costs. This implies that consumption can increase as well.

An economy with savings that are "too high," in the sense that s is larger than the value that delivers the golden rule level of capital, is said to be *dynamically inefficient*. This means that if s were lowered to the golden rule level, consumption could be increased both in the steady state and in transition. The steady state equilibrium in a dynamically inefficient is therefore not pareto optimal. Everyone can be made better off without making anyone worse off.

We can prove this by studying the Solow graph again. If $s > s_{GOLD}$ initially and is dropped to s_{GOLD}, consumption jumps initially and decreases monotonically to a level of consumption $c_{GOLD} > c$, where c is the initial level of consumption. Consumption is thus higher in transition and the steady state, meaning the economy does not have to pay a price (in terms of temporarily lower consumption) in order to transition to the optimal level of k. While here dynamic inefficiency only arises because our exogenous level of savings is set too high, it can also crop up in more realistic models of overlapping generations with endogenous savings decisions, which has brought the idea substantial empirical and theoretical attention.

4.3 Do all models have BGPs? Uzawa's Theorem

The choice of labor-augmenting technological growth is not arbitrary. Uzawa's theorem shows that if the APF exhibits constant returns to scale and there exists a steady-state with constant growth rates for Y, K, and C, then the APF can be represented with labor-augmenting technological progress. While this theorem does not imply that other forms of technological progress are impossible, it suggests that labor-augmenting technological progress (or at least an equivalent representation of the APF) and BGPs are co-dependent, given CRS production.

Theorem 2 *Consider a growth model with aggregate production function*

$$Y = F(K, L, A)$$

where $F : \mathcal{R}^3_+ \to \mathcal{R}_+$, and variables are defined as above. Suppose that F exhibits constant returns to scale in K and L. The aggregate resource constraint is

$$\dot{K} = Y - C - \delta K$$

Suppose that there is a constant growth rate of population n and that there exists $T < \infty$ such that for all $t \geq T$, $\dot{Y}/Y = g_Y > 0$, $\dot{K}/K = g_K > 0$, and $\dot{C}/C = g_C > 0$. Then

1. $g_Y = g_K = g_C$
2. *For any $t \geq T$, there exists a function $F : \mathcal{R}^2_+ \to \mathcal{R}_+$, homogenous of degree 1 in its two arguments, such that the aggregate production function can be represented as*

$$Y = F(K, AL)$$

where $A \in \mathcal{R}_+$ and

$$\dot{A}/A = g = g_Y - n$$

For a proof, see Acemoglu (2009), Chapter 2.

5 Analyzing the Solow APF

The production function encodes a lot of information about the relationship between output and relative input use. This information can often be summarized in a parameter like the *elasticity of substitution*. This turns out to have important implications for a broad spectrum of growth models (e.g. the stability of factor shares).

5.1 Elasticity of Substitution

Definition 4 *The elasticity of substitution (ES) of capital with labor (ε) is the percentage change in (L/K) with respect to a percent change in the ratio of marginal products (MPK/MPL) such that output is unchanged.*[5]

$$\varepsilon(K,L) \equiv \frac{d\ln L/K}{d\ln MPK/MPL}|_{F(K,\ L)=c} = \frac{dL/K}{dMPK/MPL}|_{F(K,\ L)=c} \cdot \frac{MPK/MPL}{L/K} \tag{21}$$

Since in a competitive environment firms hire factors until their marginal product equals their cost, $\epsilon(K,L)$ also provides a measure of the change in relative demand for capital and labor in response to change in relative prices.

Exercise 9 *Using the definition, find the elasticity of substitution for the Cobb-Douglas production function for any given L/K.*

$MPK = aK^{a-1}L^{1-a} = aY/K$; $MPL = (1-a)K^aL^{-a} = (1-a)Y/L \Rightarrow MPK/MPL = \frac{a}{1-a}\frac{L}{K} \Rightarrow \ln MPK/MPL = \ln\frac{a}{1-a} + \ln\frac{L}{K} \Rightarrow \ln\frac{L}{K} = \ln MPK/MPL - \ln\frac{a}{1-a} \Rightarrow \frac{d\ln L/K}{d\ln MPK/MPL} = 1$

[5] This verbal definition is not completely precise because we are dealing with infinitesimal changes, not discrete percentage changes. The mathematical definition is precise.

In general a neoclassical production function can have a different elasticity of substitution for different values of L/K. In most macroeconomic applications we use *isoelastic* or *Constant Elasticity of Substitution* (CES) production functions, which have constant ε at all points.

Definition 5 *The CES production function takes the form*

$$F(K,\ L) = \left(aK^{\frac{\rho-1}{\rho}} + (1-a)\,L^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}},\ \rho > 0,\ a \in (0,1). \tag{22}$$

The Cobb-Douglas, Leontief, and linear production functions are all special cases of this more general production function, as the following exercises will show.

Exercise 10 *Prove that the CES production function exhibits a constant elasticity of substitution.*

$$\begin{aligned}
MPK &= \frac{\partial F}{\partial K} = \frac{\rho}{\rho-1}F^{\frac{1}{\rho}}a\left(\frac{\rho-1}{\rho}\right)K^{-\frac{1}{\rho}} = aF^{\frac{1}{\rho}}K^{-\frac{1}{\rho}}, \\
MPL &= \frac{\partial F}{\partial L} = \frac{\rho}{\rho-1}F^{\frac{1}{\rho}}(1-a)\left(\frac{\rho-1}{\rho}\right)L^{-\frac{1}{\rho}} = (1-a)F^{\frac{1}{\rho}}L^{-\frac{1}{\rho}} \Rightarrow \\
\frac{MPK}{MPL} &= \frac{a}{1-a}\left(\frac{K}{L}\right)^{-\frac{1}{\rho}} \Rightarrow \\
\frac{L}{K} &= \left(\frac{1-a}{a}\right)^{\rho}\left(\frac{MPK}{MPL}\right)^{\rho} \Rightarrow \\
\ln\frac{L}{K} &= \rho\ln(1-a) - \rho\ln a + \rho\ln\frac{MPK}{MPL} \Rightarrow \\
\varepsilon^s_{KL} &\equiv \frac{d\ln L/K}{d\ln MPK/MPL}|_{F(K,\ L)=c} = \rho
\end{aligned}$$

Exercise 11 *How does an increase in K/L affect the labor share of output when $\rho > 1$ and $\rho < 1$? Explain intuitively.*

When $\rho = 1$, labor's share of output is constant. This is because the special form of Cobb-Douglas production implies changes in the quantity of each factor used is exactly proportional to changes in their prices, so that they offset.

If $\rho > 1$, the elasticity of substitution between capital and labor is high. So when there is relatively more capital, labor's share of income decreases since its relative scarcity is less of a problem.

When $\rho < 1$, the elasticity of substitution between capital and labor is low. So when there is relatively more capital, labor earns a premium because capital is not a close substitute (and hence it is very productive). In the extreme case of perfect complements, for example, incremental labor is essentially invaluable because of the oversupply of capital.

Exercise 12 *What is the limiting value of the CES production function when $\rho = 1$? When $\rho \to \infty$? When $\rho \to 0$? (Important but tedious: will only do if we have extra time at the end)*

The limiting value of the CES function as $\rho \to 1$ can be determined using l'Hopital's rule.

$$\begin{aligned}
\lim_{\rho\to1} F(K,\ L) &= \lim_{\rho\to1} \exp\left\{\ln F(K,\ L)\right\} = \lim_{\rho\to1} \exp\left\{\frac{\rho \ln\left(aK^{1-\frac{1}{\rho}} + (1-a)\,L^{1-\frac{1}{\rho}}\right)}{\rho-1}\right\} \\
&= \exp\left\{\lim_{\rho\to1} \frac{\rho \ln\left(aK^{1-\frac{1}{\rho}} + (1-a)\,L^{1-\frac{1}{\rho}}\right)}{\rho-1}\right\} = \left[\frac{0}{0}\right] \\
&= \exp\left\{\lim_{\rho\to1} \left[\begin{array}{c} \ln\left(aK^{1-\frac{1}{\rho}} + (1-a)\,L^{1-\frac{1}{\rho}}\right) \\ +\rho\frac{aK^{1-\frac{1}{\rho}}\ln K + (1-a)L^{1-\frac{1}{\rho}}\ln L}{aK^{1-\frac{1}{\rho}} + (1-a)L^{1-\frac{1}{\rho}}}\frac{1}{\rho^2} \end{array}\right]\right\} \\
&= \exp\left\{a\ln K + (1-a)\ln L\right\} \\
&= \exp\left\{\ln K^a L^{1-a}\right\} = K^a L^{1-a}
\end{aligned}$$

Thus we recover the Cobb-Douglass production function as the limiting value of the CES *production function as the elasticity of substitution goes to* 1.

$$\begin{aligned}
\lim_{\rho\to\infty} F(K,\ L) &= \lim_{\rho\to\infty} \exp\ln F(K,\ L) = \lim_{\rho\to\infty} \exp \frac{\rho \ln\left(aK^{1-\frac{1}{\rho}} + (1-a)\, L^{1-\frac{1}{\rho}}\right)}{\rho - 1} = \\
&= \exp \lim_{\rho\to\infty} \frac{\rho \ln\left(aK^{1-\frac{1}{\rho}} + (1-a)\, L^{1-\frac{1}{\rho}}\right)}{\rho - 1} = \left[\frac{\infty}{\infty}\right] = \\
&= \exp \lim_{\rho\to\infty} \left[\begin{array}{c} \ln\left(aK^{1-\frac{1}{\rho}} + (1-a)\, L^{1-\frac{1}{\rho}}\right) + \\ +\rho \frac{aK^{1-\frac{1}{\rho}} \ln K + (1-a) L^{1-\frac{1}{\rho}} \ln L}{aK^{1-\frac{1}{\rho}} + (1-a) L^{1-\frac{1}{\rho}}} \frac{1}{\rho^2} \end{array} \right] = \\
&= \exp\ln[aK + (1-a)\, L] = aK + (1-a)\, L
\end{aligned}$$

Consider the case when $K > L$.

$$\begin{aligned}
\lim_{\rho\to 0+} F(K,\ L) &= \lim_{\rho\to 0+} L \cdot F\left(\frac{K}{L},\ 1\right) = \text{ (by CRS)} \\
&= L \cdot \lim_{\rho\to 0+} \left(a \left(\frac{K}{L}\right)^{1-\frac{1}{\rho}} + (1-a) \right)^{\frac{\rho}{\rho-1}} = L
\end{aligned}$$

Analogously, $\lim_{\rho\to 0+} F(K,\ L) = K$ *if* $K \leq L$. *Hence,* $\lim_{\rho\to 0+} F(K,\ L) = \min\{K,\ L\}$