Economics 202A: Section 1

Introduction to Solow

Jacob Weber*

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Housekeeping and Welcome:

- Covid: please do your best to follow (changing) university guidelines!
  - Will try to offer online section option
- What is section:
  - Review and practice problems!
  - Sometimes more technical and covers material beyond course (not on test but generally useful)
  - We may not cover everything but the leftover is useful as practice problems (both for midterm now and possibly field exams later). I will publish answer keys.
- First Year Advice: help each other and don't stress about grades (especially now given everything else going on)

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will refer to Acemoglu's textbook *Introduction to Modern Economic Growth* which provides a slightly more technical discussion than these notes.

The outline of these notes are as follows:

1. Properties of the Solow Aggregate Production Function (APF)
2. Intensive Form
3. Reviewing the Solow Setup
4. Solving Solow Through Graphs
5. Analyzing the Solow APF

1 Properties of the Solow APF

Definition 1 *A production function $F(X_1,\ X_2,...)$ is a technology for transforming a vector of inputs $(X_1,\ X_2,...)$ into output F. The most common production function considers output as a function of capital and labor: F(K,L).*

- Inputs are also referred to as *factors of production.*
- The inputs and output should be interpreted as quantities, not values.

The Solow model's APF is *neoclassical*. For concreteness, in this section and the rest of these notes I'll focus on production functions of the form $F(K,\ L,\ A)$, and omit the dependence on A when it is inessential.

Definition 2 *Consider a production function $F(K,\ L,\ A) : \mathbf{R}^3_+ \to \mathbf{R}_+$ which is twice continuously differentiable in K and L. The production function is **neoclassical** if it satisfies*[1]

1. *Constant returns to scale (CRS):*

$$F(\lambda K,\ \lambda L,\ A) = \lambda F(K,\ L,\ A) \text{ for all } \lambda > 0. \tag{1}$$

2. *Positive and diminishing marginal products with respect to each input:*

$$\frac{\partial F}{\partial K} > 0,\ \frac{\partial^2 F}{\partial K^2} < 0, \tag{2}$$

$$\frac{\partial F}{\partial L} > 0,\ \frac{\partial^2 F}{\partial L^2} < 0, \text{ for all } K > 0,\ L > 0. \tag{3}$$

3. *Inada conditions:*

$$\boxed{F(K,0) = F(0,L) = 0}$$

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty, \tag{4}$$

$$\lim_{K\to\infty} \frac{\partial F}{\partial K} = \lim_{L\to\infty} \frac{\partial F}{\partial L} = 0. \tag{5}$$

[1] These conditions are taken from Acemoglu 2009, Ch.2

- Condition 1 is also known as *homogeneity of degree 1*: A function $F(X_1, X_2, ...)$ is homogeneous of degree n if, for all $\lambda > 0$, $F(\lambda X_1, \lambda X_2, ...) = \lambda^n F(X_1, X_2, ...)$. It means that a proportional increase in all inputs increases output by the same proportion. While intuitively plausible, this condition does rule out fixed costs of production, externalities in production and other potentially interesting features.

- Condition 2 is similarly straightforward: holding all other inputs fixed, increasing any single input always increases output, but at a diminishing rate as more of the input is used. Notice that there is no contradiction with the assumption of constant returns to scale, which involves increasing all inputs proportionally.

- Condition 3 is less intuitive. Plays a technical role in ensuring the existence, uniqueness and stability of the steady state with positive output in growth models

Exercise 1 *Show that Cobb-Douglas production function $Y = K^{\alpha} L^{1-\alpha}$ and $\alpha \in (0, 1)$ satisfies neoclassical assumptions.*

① $F(\lambda K, \lambda L) = (\lambda K)^{\alpha} (\lambda L)^{1-\alpha} = \lambda^{\alpha} \lambda^{(1-\alpha)} K^{\alpha} L^{1-\alpha} = \lambda f(K, L)$

② $\frac{\partial f}{\partial K} = \alpha K^{\alpha-1} L^{(1-\alpha)} = \alpha \left(\frac{L}{K}\right)^{1-\alpha} > 0 \quad \Big\| \quad \frac{\partial^2 F}{\partial^2 K} \quad \alpha(\alpha-1) K^{\alpha-2} L^{(1-\alpha)}$

③ $\lim_{K \to 0} \frac{\partial f}{\partial K} = \lim_{K \to 0} \alpha \left(\frac{L}{K}\right)^{1-\alpha} = \infty$

$\lim_{K \to \infty} \frac{\partial f}{\partial K} = \lim_{K \to \infty} \alpha \left(\frac{L}{K}\right)^{\alpha} = 0$

Exercise 2 *Give an example of production function that exhibits diminishing returns in each input, yet satisfies the property of homogeneity of degree $n > 1$.*

$Y = K^{\alpha} L^{\beta}$ where $\alpha, \beta \in (0,1)$ and $\alpha + \beta > 1$

$F(\lambda X) = \lambda^n F(X)$

$F(K,0) = F(0,L) = 0$

Exercise 3 *For a neoclassical production function, prove that as the amount of one input goes to infinity, so does output (for strictly positive amounts of other inputs), i.e.* $F(\infty,\ L) = F(K,\infty) = \infty$[2]. *(If this is too hard, try showing it for the Cobb-Douglass case and I will show the proof for the general case)*

$\lim_{K\to\infty} F(K,L) = \lim_{K\to\infty} K \cdot \underline{F(1,\tfrac{L}{K})}$ → assume $\lim_{K\to\infty} F(1,\tfrac{L}{K}) = 0$

$= \lim_{K\to\infty} \dfrac{F(1,\frac{L}{K})}{\frac{1}{K}} = \left[\frac{0}{0}\right]$ (use L'hopitals)

$= \lim_{K\to\infty} \dfrac{-\frac{L}{K^2} \cdot f_2(1,\frac{L}{K})}{-\frac{1}{K^2}}$

$\boxed{= \lim_{K\to\infty} L \cdot f_2(1,\tfrac{L}{K}) = \infty}$ ///

$F(x)$

WTS: $F(1,0) = 0$

($\Rightarrow F(K,0) = 0\ \forall K$)

note: $F(1,0) = \lim_{K\to\infty} F(1,\tfrac{L}{K})$

$= \lim_{K\to\infty} \dfrac{F(K,L)}{K} = \left[\frac{\infty}{\infty}\right]$

$= \lim_{K\to\infty} \dfrac{F_1(K,L)}{1} = 0$

[2] $F(\infty,\cdot) \equiv \lim_{x\to\infty} F(x,\cdot)$.

$k \equiv \frac{K}{AL}$

Exercise 4 *(Romer 2.1) Consider N firms each with constant-returns-to-scale production function $Y = F(K, AL)$, or (using the intensive form) $\boxed{Y = ALf(k).}$ Assume $f'(\cdot) > 0$, $f''(\cdot) < 0$. Assume that all firms can hire labor at wage wA and rent capital at cost r, and that all firms have the same value of A. Ignore nonnegativity constraints on K and L.*

Review Static opt.

(a) Consider the problem of a firm trying to produce Y units of output at minimum cost. Show that the cost-minimizing level of k is uniquely defined and is independent of Y, and that all firms therefore choose the same value of k.

(b) Show that the total output of the N cost-minimizing firms equals the output that a single firm with the same production function has if it uses all the labor and capital used by the N firms.

$$\mathcal{L} = -((wA)L + rK) - \lambda(Y - ALf(k))$$

$\max_{L,K,\lambda} \mathcal{L}$

$$\boxed{\frac{r}{w} = \frac{f'(k)}{f(k) - f'(k)k}}$$

CRS

$$Y = \sum_i AL_i f(k^*) = Af(k^*)\sum_i L_i \equiv Af(k^*)\bar{L} = F(\bar{K}, \bar{L})$$

Exercise 5 *Consider a cost minimizing firm with a neoclassical production function $F(K, L)$ and facing prices r and w for inputs K and L, respectively. Using λ to denote the costate variable, show that the optimal choices of K and L satisfy:*[3]

$$F_K(K, L)K + F_L(K, L)L = \lambda F(K, L)$$

What does this imply about the profits earned by a cost minimizing, price taking firm with a neoclassical production function?

$$\mathcal{L} = -(wL + rK) - \lambda(Y - F(K,L))$$

FOC L: $\boxed{\begin{array}{l} w = \lambda F_L \\ r = \lambda F_K \end{array}} \Rightarrow \begin{array}{l} wL = \lambda F_L \cdot L \\ rK = \lambda F_K \cdot K \end{array} \Rightarrow \begin{array}{l} wL + rK = \lambda(F_L \cdot L + F_K \cdot K) \\ wL + rK = \lambda F(K,L) \end{array}$

[3]The following theorem, known as **Euler's Theorem**, may be helpful (taken from Acemoglu 2009, Chapter 2):

Theorem 1 *Suppose that $g : \mathcal{R}^{K+2} \to \mathcal{R}$ is differentiable in $x \in \mathcal{R}$ and $y \in \mathcal{R}$, with partial derivatives denoted by g_x and g_y, and is homogenous of degree m in x and y. Then:*

$$mg(x, y, z) = g_x(x, y, z)x + g_y(x, y, z)$$

e.g. $F(K,L) = F_K \cdot K + F_L \cdot L$

for all $x \in \mathcal{R}$, $y \in \mathcal{R}$, and $z \in \mathcal{R}^K$. Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogenous of degree $m - 1$ in x and y.

A second important choice is the role of technology in this production function. In the Solow model, technology is taken to be labor augmenting or Harrod Neutral (this is not innocuous)

Definition 3 *Technology is labor augmenting (Harrod-neutral) if it enters in the form: $F(K, AL)$. Technology is capital augmenting if it enters in the form: $F(AK, L)$. And it is Hicks-neutral if it enters in the form $AF(K, L)$.*

$\underline{A} \cdot K^{\alpha} L^{(1-\alpha)}$

2 Intensive Form

It will often be convenient to express the production function in *intensive form* by dividing through by one of its arguments. Consider the Solow production function $F(K, AL)$. Using the constant returns to scale property, we can write

$$Y = F(K,\ AL) = AL \cdot F(K/AL,\ 1) = AL \cdot f(k) \tag{6}$$

where $k = K/AL$ is capital per "effective worker" and $f(k) \equiv F(k, 1)$. If we divide both sides by AL we get an expression for output per effective worker as a function of k, $y = f(k)$.

Why do this?

- A judiciously chosen normalization can reduces the number of variables (you will see a similar trick probably with consumption-savings models in part two)
- If you ever need to switch back and forth, just substitute the definition (i.e., $y = Y/AL$) and apply the normal rules of algebra / calculus.

3 Reviewing the Solow Setup

Ignoring the distribution of income among factors, the Solow model can be summarized by the following equations:

$$Y(t) = F(K(t),\ A(t)L(t)) \tag{7}$$

$\dot{X}(t) \equiv \frac{\partial}{\partial t} X(t)$

$$\boxed{\dot{K}(t) = sY(t) - \delta K(t)} \tag{8}$$

$$L(t) = L(0)e^{nt} \tag{9}$$

$\dot{L}(t) = n \cdot L(t) \iff \frac{\dot{L}(t)}{L(t)} = n$

$$A(t) = A(0)e^{gt} \tag{10}$$

$\frac{\dot{A}(t)}{A(t)} = g$

There are two exogenous variables, A and L, and two endogenous variables K and Y. However, Y is a static function of the rest of the variables, so once we know $K(t)$ we can determine $Y(t)$ easily through (7).

- A solution to this system of differential equations would deliver the value of $K(t)$ for all t. In general we cannot find an explicit solution even if we know the functional form of F, as is common with nonlinear differential equations.

- A key feature of the Solow model that allows us to learn so much about the solution is that it eventually converges to a *balanced growth path* (BGP), a trajectory in which the endogenous variables (here, K and Y) grow at constant rates. This means that we can completely characterize the solution for the time period which the economy spends of the balanced growth path. We can use graphical techniques and approximations to study the behavior of the economy away from the steady state.

4 Solving Solow Through Graphs

4.1 Finding the BGP

Exercise 6 *Show that the Solow model converges to a BGP. What are the growth rates of endogenous variables K and Y on the BGP?*

$$\frac{\dot K}{AL} = s\frac{F(K,L)}{AL} - \delta\frac{K}{AL}$$

note: $\dot k = \frac{\partial}{\partial t}\frac{K}{AL} = \frac{\dot K}{AL} - \frac{K}{A^2L}\dot A - \frac{K}{AL^2}\dot L$

$\dot k = \frac{\dot K}{AL} - k\cdot g - kn \Rightarrow \frac{\dot K}{AL} = \dot k + k\cdot g + kn$

$$\frac{\dot K}{AL} = sf(k) - \delta k$$

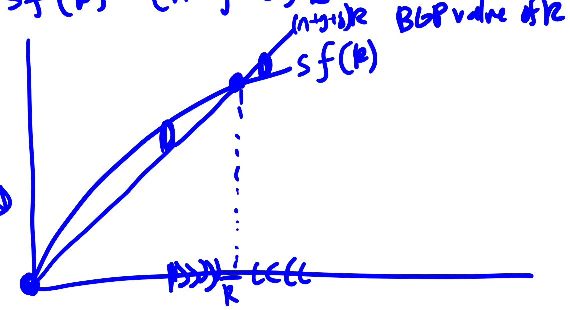
$$\dot k + k\cdot g + k\cdot n = sf(k) - \delta k$$

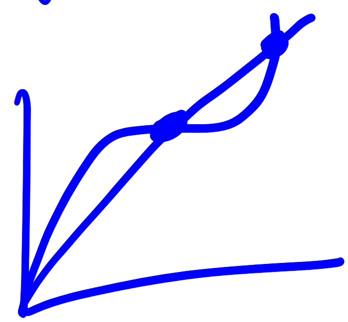
$$\boxed{\dot k = sf(k) - (n+g+\delta)k}$$

$0 = sf(k) - (n+g+\delta)k$ → call $\bar k$ the BGP value of k

if little "k" constant, $\boxed{\frac{K}{AL} \text{ is constant} \Rightarrow g_K = n+g}$

similarly, $y = f(k)$ is constant. $\Rightarrow \frac{Y}{AL}$ constant $\Rightarrow g_Y = n+g$





Exercise 7 *Assume the Solow economy is on its BGP. Show the impact of an change in the parameters s, δ, g, n on capital accumulation ($\dot k$ and k) and output per worker (Y/L) during the transition to the new BGP.*

4.2 "Golden Rules"

Suppressing time subscripts and letting consumption be defined as everything not saved, or $C = (1-s)Y$, we have $c \equiv \frac{C}{AL} = (1-s)y$. Note that for given values of g, n and δ, the level of s uniquely determines the steady state level of capital in the economy, k^*. The steady state value of consumption is likewise pinned down by s as:

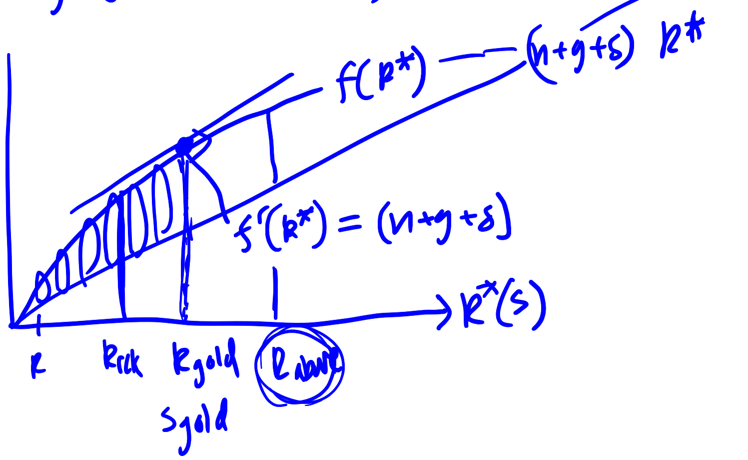
$$\boxed{c^* = (1-s)f(k^*)} \tag{11}$$

This implies that for different levels of s, we will get different levels of c. It turns out that there is a unique $s \in (0,1)$, and corresponding k^*, which maximizes c^* in the Solow model. We call the capital stock consistent with this value of s the "Golden Rule" level of the capital stock as it results in the maxiumum possible consumption along a BGP.

Exercise 8 *Solve for the golden rule capital stock in the Solow model analytically, and then graphically. What is the intuition?*

$$c^* = f(k^*) - \underbrace{sf(k^*)}_{(n+g+\delta)k} = f(k^*) - (n+g+\delta)k^*$$

FOC: $f'(k^*) = (n+g+\delta)$



4.3 Do all models have BGPs? Uzawa's Theorem

The choice of labor-augmenting technological growth is not arbitrary. Uzawa's theorem shows that if the APF exhibits constant returns to scale and there exists a steady-state with constant growth rates for Y, K, and C, then the APF can be represented with labor-augmenting technological progress. While this theorem does not imply that other forms of technological progress are impossible, it suggests that labor-augmenting technological progress (or at least an equivalent representation of the APF) and BGPs are co-dependent, given CRS production.

Theorem 2 *Consider a growth model with aggregate production function*

$$Y = F(K, L, A)$$

where $F : \mathcal{R}^3_+ \to \mathcal{R}_+$, and variables are defined as above. Suppose that F exhibits constant returns to scale in K and L. The aggregate resource constraint is

$$\dot{K} = Y - C - \delta K$$

Suppose that there is a constant growth rate of population n and that there exists $T < \infty$ such that for all $t \geq T$, $\dot{Y}/Y = g_Y > 0$, $\dot{K}/K = g_K > 0$, and $\dot{C}/C = g_C > 0$. Then

1. $g_Y = g_K = g_C$
2. *For any $t \geq T$, there exists a function $F : \mathcal{R}^2_+ \to \mathcal{R}_+$, homogenous of degree 1 in its two arguments, such that the aggregate production function can be represented as*

$$Y = F(K, AL)$$

where $A \in \mathcal{R}_+$ and

$$\dot{A}/A = g = g_Y - n$$

For a proof, see Acemoglu (2009), Chapter 2.

5 Analyzing the Solow APF

The production function encodes a lot of information about the relationship between output and relative input use. This information can often be summarized in a parameter like the *elasticity of substitution*. This turns out to have important implications for a broad spectrum of growth models (e.g. the stability of factor shares).

5.1 Elasticity of Substitution

Definition 4 *The elasticity of substitution (ES) of capital with labor (ε) is the percentage change in (L/K) with respect to a percent change in the ratio of marginal products (MPK/MPL) such that output is unchanged.*[4]

$$\varepsilon(K,L) \equiv \frac{d\ln L/K}{d\ln MPK/MPL}|_{F(K,\ L)=c} = \frac{dL/K}{dMPK/MPL}|_{F(K,\ L)=c} \cdot \frac{MPK/MPL}{L/K} \tag{12}$$

Since in a competitive environment firms hire factors until their marginal product equals their cost, $\epsilon(K,L)$ also provides a measure of the change in relative demand for capital and labor in response to change in relative prices.

Exercise 9 *Using the definition, find the elasticity of substitution for the Cobb-Douglas production function for any given L/K.*

[4]This verbal definition is not completely precise because we are dealing with infinitesimal changes, not discrete percentage changes. The mathematical definition is precise.

In general a neoclassical production function can have a different elasticity of substitution for different values of L/K. In most macroeconomic applications we use *isoelastic* or *Constant Elasticity of Substitution* (CES) production functions, which have constant ε at all points.

Definition 5 *The CES production function takes the form*

$$F(K,\ L) = \left(aK^{\frac{\rho-1}{\rho}} + (1-a)\,L^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}},\ \rho > 0,\ a \in (0,1). \tag{13}$$

The Cobb-Douglas, Leontief, and linear production functions are all special cases of this more general production function, as the following exercises will show.

Exercise 10 *Prove that the CES production function exhibits a constant elasticity of substitution.*

Exercise 11 *How does an increase in K/L affect the labor share of output when $\rho > 1$ and $\rho < 1$? Explain intuitively.*

Exercise 12 *What is the limiting value of the CES production function when $\rho = 1$? When $\rho \to \infty$? When $\rho \to 0$? (Important but tedious: will only do if we have extra time at the end)*