Economics 202A: Section 5

Towards Endogenous Technological Change

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1. Dixit-Stiglitz model of Monopolistic Competition
   a) Input Demand Functions
   b) "Love of Variety"
2. Romer Model

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

1 Dixit-Stiglitz

Two equivalent ways to structure models of monopolistic competition with free entry:

1. a continuum of intermediate goods producers who sell their output to a final goods sector that combines these intermediate goods into a final consumption good with CES technology (Romer model from class)

2. a monopolistically competitive sector sells consumption goods directly to a consumer with CES preferences over those goods

You see both frequently. The second is a little easier conceptually. Will establish their equivalence.

1.1 Input Demand Functions

The consumer's problem takes the structure:

$$\max_{C} U(C) \tag{1}$$

$$C = \left(\int_0^M C(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \tag{2}$$

$$s.t. \quad \int_0^M p(i)C(i)di \le w \tag{3}$$

The Lagrangian can be written as

$$\mathcal{L} = U\left(\left(\int_0^M C(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}\right) + \lambda(w - \int_0^M p(i)C(i))$$

Taking first-order conditions with respect to an arbitrary good $C(i)$ gives

$$0 = U'(C)\frac{\sigma}{\sigma - 1}\left(\int_0^M C(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} \frac{\sigma - 1}{\sigma}C(i)^{-1/\sigma} - \lambda$$

Taking the ratio of this FONC for any good i and j gives

$$\frac{C(i)}{C(j)} = \left(\frac{p(i)}{p(j)}\right)^{-\sigma} \tag{4}$$

which explains why σ is the elasticity of substitution between any two goods. It can also be shown that demand can be written as

$$C(i) = \left(\frac{p(i)}{P}\right)^{-\sigma} C \tag{5}$$

$$P \equiv \left(\int_0^M p(i)^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} \tag{6}$$

The first equation can be found by integrating Equation 4 with respect to j.

Why care about P? In essence, P is a kind of “sufficient statistic” for utility. To see this, note that:

$$\int_0^M p(i)C(i)di \leq w \tag{7}$$

$$P^\sigma C \int_0^M p(i)^{1-\sigma} di \leq w \tag{8}$$

$$P^\sigma C P^{1-\sigma} \leq w \tag{9}$$

$$C \leq \frac{w}{P} \tag{10}$$

Notes:

- w/P measures a real wage that converts numeraire unit w into util-units of the final consumption aggregate. "P" is an ideal price index for the final consumption good.
- Since only relative prices matter, we are free to make a normalization. A particularly convenient choice in many settings is $P = 1$.

The next exercise asks you to show that interpreting this problem through the lens of a firm's cost minimization problem hardly changes anything in the context of the Romer model.

Exercise 1 *Consider the problem of a competetive final goods producer, who takes wages and prices as given, and produces final good Y by combining inputs $Y(i)$ according to*

$$Y = \left[\int_0^A Y(i)^\phi di \right]^{\frac{1}{\phi}}$$

where $p(i)$ is the price of each variety $Y(i)$.

A. *Show that the input demand takes the same form as in the consumers problem above: $Y(i) = \left(\frac{p(i)}{P}\right)^{\frac{1}{\phi-1}} Y$ where $P \equiv \left[\int_0^A p(i)^{\frac{\phi}{\phi-1}} di \right]^{\frac{\phi-1}{\phi}}$ is the marginal cost for a final goods producer (and hence the price of final good Y).*

B. *Assume the supplier of variety i can set the price $p(i)$ and produces according to $Y(i) = L(i)$. Show that if they take aggregate output Y and other prices (here, $p(j)$ and the wage w) as given, we obtain the price setting rule $p(i) = \frac{1}{\phi}w$ from lecture.*

A: The firm's cost minimization problem can be rewritten as the maximization problem:

$$\max_{\{Y(i)\}} - \int_0^A p(i)Y(i) - \lambda \left(\bar{Y} - \left[\int_0^A Y(i)^\phi di \right]^{\frac{1}{\phi}} \right)$$

which yields the first order condition

$$p(i) = \lambda \bar{Y}^{1-\phi} y(i)^{\phi-1}$$

Dividing the FOC for i by the FOC for j yields $\left(\frac{p(i)}{p(j}\right)^{\frac{1}{\phi-1}} = \frac{y(i)}{y(j)}$. Manipulate this,

$$\left(\frac{p(i)}{p(j}\right)^{\frac{\phi}{\phi-1}} = \frac{y(i)^\phi}{y(j)^\phi}$$

Integrating w.r.t i,

$$\frac{\int p(i)^{\frac{\phi}{\phi-1}} di}{p(j)^{\frac{\phi}{\phi-1}}} = \frac{\int y(i)^\phi di}{y(j)^\phi} = \frac{Y^\phi}{y(j)^\phi}$$

Defining $P \equiv \left(\int p(i)^{\frac{\phi}{\phi-1}} di\right)^{\frac{\phi-1}{\phi}}$ yields

$$\begin{aligned}
\left(\frac{P}{p(j)}\right)^{\frac{\phi}{\phi-1}} &= \frac{Y^{\phi}}{y(j)^{\phi}} \\
\left(\frac{p(j)}{P}\right)^{\frac{1}{\phi-1}} Y &= y(j)
\end{aligned}$$

Finally, plug the demand function here back into the FOC for good j. Cancellations reveal $P = \lambda$, so that our definition for P implies that this is equivalent to the marginal cost and will be the price if final goods production is competetive.

B: a monopolist for variety j maximizes

$$\max_{p(j)} p(j)Y(j) - wL(j)$$

subject to the demand for their product $Y(j)$ solved for above, and production function $Y(j) = L(j)$. We can solve the unconstrained problem by writing:

$$\max_{p(j)} \left(p(j) - w\right) Y\left(p(j)\right)$$

The FOC is

$$\begin{aligned}
Y\left(p(j)\right) + \left(p(j) - w\right) \frac{\partial}{\partial p(j)} Y\left(p(j)\right) &= 0 \\
Y\left(p(j)\right) + \left(p(j) - w\right) \frac{p(j)^{-1}}{\phi - 1} Y\left(p(j)\right) &= 0 \\
1 + \left(p(j) - w\right) \frac{p(j)^{-1}}{\phi - 1} &= 0 \\
\frac{p(j) - w}{p(j)} &= 1 - \phi \\
-\frac{w}{p(j)} &= -\phi \\
p(j) &= \frac{w}{\phi}
\end{aligned}$$

1.2 Love of Variety

A key feature of this utility structure is that consumers "love" variety. To show this, suppose that the prices of all goods are the same and $p(i) = p = 1$. Then:

$$U(C) = U(w/P) \tag{11}$$

$$P = \left(\int_0^M p(i)^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} \tag{12}$$

$$P = M^{\frac{1}{1-\sigma}} \tag{13}$$

$$U(C) = U(w/M^{\frac{1}{1-\sigma}}) \tag{14}$$

Since we assume $\sigma > 1$, the denominator inside the utility function is increasing in M. Thus for any fixed wage and price level, the more goods the consumer enjoys the happier she becomes.

Exercise 2 *Consider the final goods producer in the previous exercise. What interpretation does "love of variety" have in this setting? (Hint: use the expression for the firms marginal cost when all prices are equal).*

Mechanically, the math is the same as in the consumers problem above. Recall $P \equiv \left[\int_0^A p(i)^{\frac{\phi}{\phi-1}} di\right]^{\frac{\phi-1}{\phi}}$, which is equal to marginal cost if the final goods firm is cost-minimizing. For tractability, pretend it faces $p(i) = p$ for all i (note this will be the case in equilibrium in e.g. the Romer model). We obtain

$$P = \lambda \left(\int_0^A p^{\frac{\phi}{\phi-1}} di\right)^{\frac{\phi-1}{\phi}} = pA^{\frac{\phi-1}{\phi}}$$

where P (marginal cost) is decreasing with A, since $\phi < 1$. Thus, holding wages fixed, increasing the number of varieties (or chemical compounds) the firm can choose from reduces the price of the final good and raises real wages for households. (You should also be able to show as David did in lecture that if you assume $Y(i) = L(i)$ and allocate some fixed stock of labor equally across all i that output in the final good sector is increasing in A).

2 Reviewing the Romer Model

Armed with tools to handle monopolistic competition, we return to (David's version) of the Romer model. In it, a fixed stock of labor L is allocated across two sectors: the production of goods, and the R&D sector which produces ideas.

Goods Sector

The goods sector produces output $Y(t)$ according to the production function

$$Y(t) = \left[\int_0^{A(t)} L_i(t)^{\phi} di\right]^{\frac{1}{\phi}} \tag{15}$$

Where $\phi > 0$. Note:

- This is the same as above except David has skipped ahead and plugged in the equilibrium condition $Y_i(t) = L_i(t)$ which holds for all i and t
- We know the expression for marginal cost/price $P(t)$ but we will normalize this to one and use the final consumption good as the numeraire

Finally, David implicitly defines total labor in the goods market as

$$L_y(t) = \int_0^{A(t)} L_i(t) di \tag{16}$$

Cost minimization implies that the final goods producer has the following demand for input i as we showed in exercise one:

$$Y_i(t) = L_i(t) = p_i(t)^{\frac{1}{\phi-1}} Y(t) \tag{17}$$

And the monopolist's profit maximization implies that

$$p_i(t) = \frac{1}{\phi} w(t) \tag{18}$$

At this point, note that we have shown that $p_i(t) = p(t)$: all monopolists will charge the same price, because they face the same profit maximization problem. Thus $L_i(t) = L_y(t)/A(t)$ as David stated in lecture. This is a result which follows from our other assumptions.

R&D Sector

- Anyone can hire labor and generate new ideas for "chemical formulas" or inputs i.
- Their reward for doing so are monopoly rents that flow from the resulting firm i.
- The amount of new ideas that you recieve from the labor you hire is governed by the following law of motion:

$$\dot{A}(t) = BL_A(t)A(t) \tag{19}$$

This implies that the marginal cost of producing an idea is $\frac{w(t)}{BA(t)}$.

Households and Market Clearing

There are a fixed number of L households each supplying one unit of labor. We assume they maximize $U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t)$. This implies that the euler equation holds:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho \tag{20}$$

where all final goods are consumed:

$$LC(t) = Y(t) \tag{21}$$

and labor markets clear, so that

$$L = L_A(t) + L_y(t) \tag{22}$$

Finally, equilibrium in the R&D "startup" market (if you like) requires that the cost of entry (i.e. cost of making a patent) equals the present discounted value of the profits that will accrue to the new firm over it's lifetime:

$$\frac{w(t)}{BA(t)} = \int_{\tau=t}^{\infty} e^{-R(\tau)} \Pi_i(\tau) d\tau \tag{23}$$

where $R(t)$ is some possibly time varying interest rate – though as we are about to show, this will be constant along a balanced growth path.

The Balanced Growth Path

- The textbook establishes existence, but not uniqueness, of an equilibrium in this version of the Romer model.
- David posits the existence of a solution in which L_A is *constant* and then proceed to show that it exists, and results in a solution which is a balanced growth path.
- We can then study the properties of this solution to the model (we claim but will not show that it is unique).

The first step to showing that an equilibrium in this model with L_A constant is a BGP is to note that if L_A is some constant (to be determined in equilibrium), this implies that $L_y = L - L_A$ is also constant. Then, we analyze the goods market.

Exercise 3 *Show that $\Pi_i(t)$, the profits of firm i, are identical across firms by rewriting it in terms of the wage $w(t)$ and the number of input producers/stock of ideas, $A(t)$.*

Defining the flow of profits to a monopolist as $\Pi_i(t) = p_i(t)L_i(t) - w(t)L_i(t)$, and recalling that the price is just a constant markup over the wage given in equation (18)

$$\begin{aligned}
\Pi_i(t) &= \left(\frac{1}{\phi}w(t) - w(t)\right) L_i(t) \\
\Pi_i(t) &= \left(\frac{1}{\phi} - 1\right) w(t)L_i(t) \\
\Pi_i(t) &= \frac{1-\phi}{\phi} w(t) \left(\frac{L_Y(t)}{A(t)}\right)
\end{aligned}$$

where the last step follows from $L_i(t) = L_Y(t)/A(t)$ since all firms have the same production function, price and demand from final good firms. Then plugging in for $L_Y(t) = L - L_A$ yields

$$\Pi_i(t) = \frac{1-\phi}{\phi} w(t) \left(\frac{L - L_A}{A(t)} \right)$$

To make progress, we can simplify the market clearing condition for R&D, equation (23), by noting that the interest rate will be constant. This requires the following steps:

1. First, note that equation (19) tells us that with L_A constant, the growth rate of ideas A will be constant and equal to $g_A = BL_A$.
2. Then establish that $Y(t) = A(t)^{\frac{1-\phi}{\phi}} L_Y$ by plugging $L_i(t) = L_Y/A(t)$ into the production function, which implies the growth rate of $Y(t)$ is constant and $g_Y = \frac{1-\phi}{\phi} g_A$
3. Goods market clearing (21) makes it clear that the growth rate of $C(t)$ is then constant and equal to the growth rate of Y.
4. Finally, the Euler Equation (20) links the growth rate of $C(t)$ to the interest rate and establishes that this is constant and equal to

$$r(t) = \rho + g_Y = \rho + \frac{1-\phi}{\phi} BL_A$$

So we can write $r(t) = r$ and rewrite (23) as:

$$\frac{w(t)}{B} = (L - L_A)\frac{1-\phi}{\phi} \int_{\tau=t}^{\infty} e^{-(r+g_A)(\tau-t)} w(\tau) d\tau \tag{24}$$

Where we have used the fact that $A(\tau) = A(t)e^{g_A \tau}$ grows at a constant rate. Now the only thing keeping the integral "messy" is the dependence of the wage on time. Fortunately, it grows at a constant rate too.

Exercise 4 *Show that $w(t)$ grows at the same rate as output (hint: use our result in exercise 2 above!) and use this fact to solve for $L_A = (1-\phi)L - \frac{\phi\rho}{B}$ by simplifying equation (24):*

Above, we showed that $P = pA^{\frac{\phi}{1-\phi}}$, where p was the price of a variety and P the marginal cost of the final consumption good. Here, we've normalized $P(t) = 1$ and can use the fact that $p_i(t) = \frac{1}{\phi} w(t)$ to see that

$$w(t) = \phi A(t)^{\frac{1-\phi}{\phi}}$$

which implies that $w(t)$ grows at $g_W = \frac{1-\phi}{\phi} g_A = g_Y$.

Given this fact, we can rewrite (24):

$$\begin{aligned}\frac{w(t)}{B} &= (L-L_A)\frac{1-\phi}{\phi}\int_{\tau=t}^{\infty} e^{-(r+g_A)(\tau-t)}w(t)e^{g_y(\tau-t)}d\tau \\ \frac{\phi}{(1-\phi)B} &= (L-L_A)\int_{\tau=t}^{\infty} e^{-(r+g_A-g_y)(\tau-t)}d\tau\end{aligned}$$

Noting that $r+g_A-g_y=\rho+g_A=\rho+BL_A$

$$\frac{\phi}{(1-\phi)B} = \frac{L-L_A}{\rho+BL_A}$$

Some algebra will yield

$$L_A=(1-\phi)L-\frac{\phi\rho}{B}$$

as in the text.

[More room for work here]

Discussion

Note that we have found a solution to the model consistent with L_A constant which is a BGP. This is not necessarily unique. Additionally, we assumed implicitly (i.e. whenever we divided by L_A that L_A) was nonzero. Technically, we have

$$L_A = max\left[(1-\phi)L - \frac{\phi\rho}{B}, 0\right]$$

Where L_A may be zero if it is not profitable to create new inputs / do R&D in equilibrium.