Economics 202A: Section 6

Endogenous Growth

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1. Romer Model: Planner's Problem
2. Extensions to the Romer Model
3. Midterm Practice Problems

*I thank Todd Messer, Nick Sander, Evan Rose, and many other past 202A GSIs for sharing their notes. Occasionally I will make reference to Acemoglu's textbook *Introduction to Modern Economic Growth* which has been used in this class in the past and is recommended reading for those wanting a slightly more technical discussion than we provide here.

1 Review of the Romer Model: Planner's Problem

Key equations:

$$
\begin{aligned}
Y(t) &= A(t)^{\frac{1-\phi}{\phi}} L_Y(t) && (1)\\
L &= L_Y(t) + L_A(t) && (2)\\
\dot{A}(t) &= B L_A(t) A(t) && (3)
\end{aligned}
$$

Here we have already imposed the result that the planner will optimally allocate $L_y(t)$ equally across all inputs i (as in the competetive equilibrium). Note we no longer have conditions for private optimality (profit maximization, etc.). Instead, we will have new optimality conditions which result from the *planner* maximizing household consumption $C(t) = Y(t)/L$,

$$
\max_{\{C(t)\}} \int_0^\infty e^{-\rho t} \ln C(t) dt
$$

This will yield a different result than when the household solves this problem because the planner internalizes all the externalities that result from the household's decision to e.g. start a new firm/invent a new variety. What are the externalities here?

1. Consumer Surplus Externality: every new firm creates some consumer surplus that they don't capture $\Rightarrow$ too little entry in the competetive equilibrium.

2. Business Stealing Externality: every new firm lowers profits for existing firms $\Rightarrow$ too much entry in the competetive equilibrium.

3. R&D Externality: new firms improve the efficiency of the R&D sector $\Rightarrow$ too little entry in the competetive equilibrium.

Question: does the presence of externalities in a model *always* imply that the market equilibrium is inefficient?

Answer: No! For example, there exist versions of the Dixit Stiglitz model with free entry (and without the R&D externality) where the market equilibrium can be efficient because externalities can cancel out.

Fundamentally, the planner's problem is really just to allocate a fixed L between the two sectors, R&D and Goods. I recommend writing the problem in terms of

$$\theta(t) \equiv \frac{L_Y(t)}{L}$$

Plugging in our constraints, the problem boils down to

$$\max_{\{\theta(t)\}} \int_0^\infty e^{-\rho t} \left(\ln \theta(t) + \frac{1-\phi}{\phi} \ln A(t) \right) dt$$

such that

$$\frac{\dot{A}(t)}{A(t)} = BL(1 - \theta(t))$$

- $\theta(t)$ is the control variable and $A(t)$ is the state variable (why?)
- We ignore the inequality constraints on $L_A(t)$ here (so $\theta(t) \in (0, 1)$) and search for an interior solution that is not in a "corner" (is this restrictive?)
- We can use the second theorem in the Hamiltonian reference sheet. Using the FONCs, we can show that $\dot{\theta}(t)$ is characterized by a quadratic eqaution in $\theta(t)$ which makes solving for the long run dynamics relatively easy.[1]

[1]This part was a bit trickier than intended; see the HW solution key for details.

As in the book, you should find $\theta(0) = \theta^* \equiv \frac{\phi\rho}{(1-\phi)BL}$ is the unique, constant solution to the planner's problem.

Exercise 1 *Consider the solution to the planners problem above*

1. *If $\theta^* > 1$, does this mean there is no solution to the planner's problem?*
2. *Recall that the optimal choice of L_A on the BGP in the Competetive Equilibrium (CE) was given by*

$$L_A^{CE} = (1-\phi)L - \frac{\phi\rho}{B}$$

is this higher or lower than the planners choice of L_A? (Show this formally.) What does your answer say about the relative importance of the externalities we listed above?

(1) if $\theta^* > 1$, this implies that no *interior* solution exists. We are on a corner with $L_Y(t) = L$, and the reason for this is that R&D is not productive enough to be "worth it." Note that $\theta^* \to \infty$ if $B \to 0$.

(2) Note the expression for L_A^{CE} implies that

$$L_A^{CE} = (1-\phi)\left(L - \frac{\phi\rho}{(1-\phi)B}\right) = (1-\phi)(L - \theta^* L) = (1-\phi)L_A^*$$

where L_A^* is the labor allocated to R&D in the solution to the planner's problem. Since $\phi \in (0,1)$ this shows that the planner allocates more labor to R&D, and that the externalities above do not balance out, and that there is too little entry/knowledge creation in the competetive equilibrium.

Exercise 2 *Assume we implement the Planners solution to the Romer model by "commanding" every household to supply their labor accordingly to both sectors and then gifting them an equal share of output of the final good each period. Further assume that households are able to borrow/lend to each other in bonds at some rate $r(t)$. For each household i, the budget constraint is then*

$$\dot{B}_i(t) = r(t)B_i(t) + Y_i(t) - C_i(t)$$

where $Y_i(t) = Y(t)/L$. Note households are not allowed to start firms (since we assumed the planner chooses $L_A(t)$). As in the competetive equilibrium, household optimality requires:

$$\frac{\dot{C}_i(t)}{C_i(t)} = r(t) - \rho$$

1. *Show that $r(t)$ in the Planner's equilibrium is constant and a function of the growth rate of A, g_A , as in the BGP of the competetive equilibrium (hint: since all households are the same, can we ever have $B_i(t) \neq 0$ in equilibrium?)*
2. *Is this interest rate higher or lower than on the BGP of the competetive equilibrium? What does this mean about the optimal quantity of saving in the market equilibrium of the Romer model?*

(1): since all households are identical in their willingness to demand and supply bonds, $B_i(t)$ is zero always, implying $Y_i(t) = C_i(t) = Y(t)/L$. Thus from the Euler Equation,

$$r(t) = \frac{\dot{C}_i(t)}{C_i(t)} + \rho = \frac{\dot{Y}(t)}{Y(t)} + \rho = \frac{1-\phi}{\phi}g_A(t) + \rho = \frac{1-\phi}{\phi}g_A^* + \rho$$

where the last two equalities follow from the solution to the planner sproblem above, in which $g_A(t)$ is constant. This is the same as in the textbook for the competetive equilibrium.

(2) r is higher than in the competetive equilibrium because g_A is higher. Since the planner manufactures higher income growth, every household wants to borrow against their future income more, which bids up the interest rate on the untraded bond relative to the case in the competetive equilibrium.

This suggests that one way of looking at the inefficiency in the market equilibrium is that there is "undersaving" because the private value of saving (namely, foregoing output today to start a new firm) is lower than the social value. This should suggest to you that there is likely a way to implement the planner's outcome by lump-sum taxing households and subsidizing savings.

2 Beyond the Textbook Romer Model: Other Knowledge Production Functions

What implications does the knowledge production function have for the BGP in an endogenous growth model? A more general form of equation (3) is

$$\dot{A}(t) = F\left[A(t),\ L_A(t)\right] \tag{4}$$

where $L_A(t)$ is the labor input devoted to research. We often assume $F(\bullet)$ is Cobb-Douglas,

$$\dot{A}(t) = BA(t)^{\lambda} L_A(t)^{\gamma} \tag{5}$$

Above, we assumed $\lambda = \gamma = 1$, but there are other possibilities:

- $\lambda < 1$: Easier discoveries made first ("low hanging fruit" hypothesis)
- $\lambda > 1$: Standing on the shoulders of giants
- $\gamma > 1$: Peer effects
- $\gamma < 1$: Too many cooks spoil the broth

We will consider the case where $\gamma = 1$, and allow λ to vary.

As in our previous models, the knowledge production function alone will already tell us the growth rates on a BGP, if it exists. Divide the production function (5) by $A(t)$,

$$g(t) = BA(t)^{\lambda-1} L_A(t) \tag{6}$$

On a BGP, g_t must be constant. Taking logs and differentiating w.r.t. time gives

$$0 = (\lambda - 1)g + n_A \tag{7}$$

where n_A is the growth rate of researchers, which must equal the growth rate of population on the BGP (why?). Let's consider a few cases.

Knowledge Production with $\gamma = 1$:

$$\dot{A}(t) = BA(t)^{\lambda} L_A(t)$$

Characterize the BGP: does it exist?

1. In lecture and the textbook's version of the Romer model, $\lambda = 1$ and $n_A = 0$.

   In this case equation (17) does not imply a value for g because $\lambda - 1 = 0$ so we have to go back to the production function (5) to see that $g = BL_{A,0}$ Note that the initial number of researchers determines the growth rate. This is known as a *scale effect* because the level of one variable (researchers) determines the growth rate of another (output). This has strong empirical implications which are not generally borne out in the data.[2]

2. Suppose $\lambda \geq 1$ and $n_A \geq 0$ and either $\lambda > 1$ or $n_A > 0$.

   There is no BGP; instead, $\frac{\dot{g}(t)}{g(t)} = (\lambda - 1)g(t) + n_A$. Growth grows more than exponentially. This is also not borne out in the data.

3. Suppose $\lambda < 1$ and $n_A \geq 0$.

   Then research production runs into diminishing returns and (17) implies that $g = n_a/(1 - \lambda)$. The steady state growth rate depends on the growth rate of researchers, with the growth rate approaching infinity as $\lambda \to 1$. There are no scale effects. This is known as *semi-endogenous* growth because growth ultimately still depends on growth in an exogenous variable. This case therefore conforms more with the fact that the rising number of researchers in the last 50 years has not led to a significant increase in the growth rate. Observe that the policy implications are suddenly quite negative: to raise growth, we need to increase the *growth rate* of researchers, which would ultimately require an increase in population growth. However, given the small fraction of labor

[2]If we think of the scale effect happening at the level of countries it is clearly empirically false. But if we think of it occurring at the world level then it may be consistent with growth in the very long run.

devoted to research in even advanced economies, the growth rate of researchers could rise substantially for quite some time without increasing population growth.

In many endogenous growth models you will see similar production functions buried within, for example, human capital accumulation. Usually if these generate scale effects, it is because knowledge is not subject to decreasing returns, i.e. $\lambda = 1$. It is something worth looking out for since this particular assumption is often not spelled out clearly in papers.

Exercise 3 *Explore the implications of $\gamma \neq 1$ for long-run growth along a BGP using the Cobb-Douglass form of the production function for ideas* (5)*:*

$$\dot{A}(t) = BA(t)^{\lambda} L_A(t)^{\gamma}$$

and assuming $\lambda \neq 1$.

When $\gamma \neq 1$, we have that

$$g_t = BA_t^{\lambda-1} L_{A,t}^{\gamma}$$

Which, when taking logs and differentiating along the steady-state, gives

$$0 = (\lambda - 1)g + \gamma n_A$$

Rearranging gives

$$g = \frac{\gamma n_A}{1 - \lambda}$$

We have the same as before when $\gamma = 1$. Assume for the following that $\lambda \neq 1$.

For the most part, we have the same as before. However, we can see that γ scales up and down the size of the effect of population growth. Suppose $0 < \gamma < 1$. This lowers growth in the semi-endogenous model $\lambda < 1$. We still have no BGP if $\lambda > 1$.

Exercise 4 *Extra Problem 13 in PS4 Fall 2021: (From an old midterm). Consider the model of endogenous knowledge accumulation presented in the book and in lecture for the case of* $\theta < 1$*:*

$$Y(t) = (1 - \alpha_L)L(t)A(t), \quad 0 < a_L < 1$$
$$\dot{A}(t) = B[a_L L(t)]^{\gamma} A(t)^{\theta}, \quad B > 0, \gamma > 0, \theta < 1$$
$$\dot{L}(t) = nL(t)$$

Assume $L(0) > 0$*,* $A(0) > 0$*. As in the usual model,* a_L *is exogenous and constant. In contrast to the baseline version of the model, assume that the rate of population growth is a decreasing function of the fraction of workers who are engaged in R&D:* $n = n(a_L)$*,* $n'(\cdot) < 0$*,* $n(\cdot) > 0$*. (The idea is that, for some reason, scientists on average have fewer children than other workers.) Suppose the economy is on a balanced growth path, and that there is a permanent increase in* a_L*. Sketch the resulting path of* $\ln A$ *and what that path would have been without the increase in* a_L*. Explain your answer.*

First, solve for g:

$$g = \frac{\dot{A}_t}{A_t} = B[a_L L_t]^{\gamma} A_t^{\theta - 1} \tag{8}$$

Now take logs and derivatives and consider the growth rate of g:

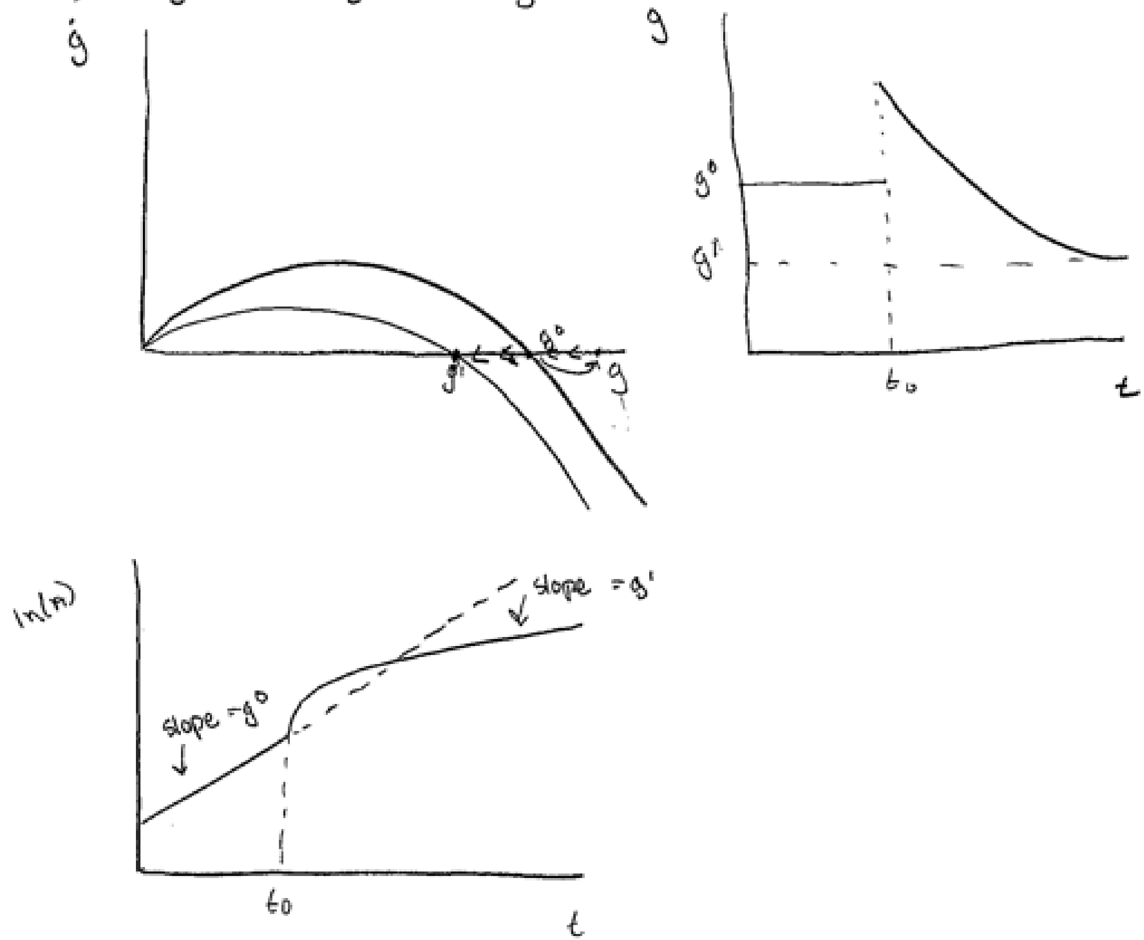
$$\frac{\dot{g}}{g} = \gamma n(a_L) + (\theta - 1)g \tag{9}$$

The steady state level of g, g^* is given by:

$$g^* = \frac{\gamma n(a_L)}{1 - \theta} \tag{10}$$

So the long run impact of an increase in a_L will be to lower growth rates. However, at any point in time g is also given by $g = B[a_L L(t)]^{\gamma} A(t)^{\theta - 1}$. So an increase in a_L will also increase growth in the short run.

5) $\dot{g} = \gamma n(a_L) g + (\theta - 1) g^2$



The final result is that $ln(A)$ will be increasing at a constant rate, kink upwards at the time of the a_L shock, and then gradually bend back down to a slope that is smaller than its slope before the change. See the attached graphs for an illustration.

Exercise 5 (Learning-by-doing) *This exercise is based on the first endogenous growth paper by Paul Romer. Suppose a representative firm solves*

$$\max_{\{K(t),\ L(t)\}} A(t)K(t)^{\alpha}L(t)^{1-\alpha} - r(t)K(t) - w(t)L(t) \tag{11}$$

Firms take $A(t)$ as given. However, in the aggregate, $A(t) = K(t)^{\phi}$ where $\phi < 1$, so there are positive peer effects in production. Capital evolves according to

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{12}$$

where we assume that agents save a constant fraction of their income as in the Solow model. Finally, assume $\dot{L}(t) = nL(t)$ with $n > 0$.

a. *Find a condition for ϕ such that a BGP exists, and show that Y/L is growing along it.*

b. *Establish that this BGP is stable.*

This is essentially the Solow model with increasing returns to scale. Let's first look for a balanced growth path by taking logs of the production function, differentiating with respect to time and recognizing that $g_Y = g_K$ to get

$$g_Y = (\phi + \alpha)g_K + (1-\alpha)n \tag{13}$$

$$\Rightarrow \quad g_K = \frac{1-\alpha}{1-\alpha-\phi} \cdot n \tag{14}$$

There are several things to note here. One is that a balanced growth path exists if and only if $\phi < 1 - \alpha$; otherwise, growth increases exponentially. Note that this also implies that output is growing faster than population, so that Y/L is growing on the BGP. To see this last point, note that $g_k > n$ along a bgp. Thus

$$g_y > (\phi + \alpha)n + (1-\alpha)n > n$$

To show that the balanced growth path is stable, let $Y/K \equiv x = K^{\alpha+\phi-1}L^{1-\alpha}$. Then

$$\frac{\dot{x}}{x} = (\alpha + \phi - 1)sx + \delta(1 - \alpha - \phi) + (1 - \alpha)n \tag{15}$$

This equation has a steady state when

$$x = \frac{(1-\alpha)n}{(1-\alpha-\phi)s} + \frac{\delta}{s} \tag{16}$$

When x is above this steady state value, $\dot{x}$ is negative. When x is below this steady state value, $\dot{x}$ is positive. Therefore the BGP is stable.

Exercise 6 *Extra Problem 6 in PS5 Fall 2021: From the 2019 Midterm (Problem 7):*

Motivated by the Solow model, a researcher is interested in studying the impact of the fraction of output devoted to investment on output per worker. They therefore consider the model:

$$\ln \frac{Y_i}{L_i} = a + b \ln s_i + e_i$$

where i indexes countries, Y_i/L_i is output per worker in country i in 2015, and s_i is the average ratio of investment to GDP in country i over the period 1985–2014.

a. In no more than two sentences, give one reason that the estimate of b that results from estimating the equation above by OLS might suffer from omitted variable bias.

b. Suppose the researcher has found a way of estimating b that you are confident does not suffer from omitted variable bias. Suppose the resulting point estimate of b is 1.0, with a standard error of 0.7. Explain briefly how you would interpret that finding.

(a) some potential candidates for OVB include:

- Insitutional or cultural factors that lead to both more saving and higher effort/output per worker (e.g. imagine more educated workers are both more productive and tend to save more; this would cause us to overestimate b).
- If country specfic idioscyncratic productivity is correlated with the saving rate, then this would lead to OVB

(b) I think there are two things David may look for immediately here. The first is to note that this means that you may interpret b as the true marginal effect of increasing the savings rate on current output per worker, while the second point is to note also that e.g. zero is not ruled out.

To say more, let us "microfound" this regression equation as David suggested in lecture. Viewed from the lens of the Solow model, what is the prediction for b?

Along a balanced growth path in the Solow model, for country i we should have

$$s_i f(k_i^*) = (n_i + g + \delta)k_i^* \tag{17}$$

Note that I have assumed all countries share the same technology (so that g, $f(\cdot)$ and δ are the same across countries) but that they likely vary in population growth i. This may make sense if we assume that the regression was estimated over countries with similar institutions and levels of technology. Note that this means k_i^* varies across countries due to the different savings rates s_i and population growth rates n_i. This is a promising place to start because we have a relationship between s and output per effective unit of labor. Let's now work towards something that looks more like the regression equation:

$$\begin{aligned}
\frac{f(k_i^*)}{k_i^*} &= \frac{n_i + g + \delta}{s_i} \\
\frac{Y_i(t)}{K_i(t)} &= \frac{n_i + g + \delta}{s_i} \\
\frac{Y_i(t)}{L_i(t)} &= \frac{n_i + g + \delta}{s_i}\frac{K_i(t)}{L_i(t)}
\end{aligned}$$

We should not stop here, because the capital labor ratio along a BGP is also a function of s. To make progress (and make life easy on ourselves) let's assume production is Cobb-Douglass, so $f(k) = k^\alpha$. Then equation (17) implies that along a BGP,

$$\begin{aligned}
(k^*)^{\alpha-1} &= \frac{n_i + g + \delta}{s_i} \\
\frac{K_i(t)}{A_i(t)L_i(t)} &= \left(\frac{n_i + g + \delta}{s_i}\right)^{\frac{1}{\alpha-1}} \\
\frac{K_i(t)}{L_i(t)} &= \left(\frac{n_i + g + \delta}{s_i}\right)^{\frac{1}{\alpha-1}} A_i(t)
\end{aligned}$$

Plugging this back in yields

$$
\begin{aligned}
\frac{Y_i(t)}{L_i(t)} &= \left(\frac{n_i+g+\delta}{s_i}\right)^{\left(1+\frac{1}{\alpha-1}\right)} A_i(t) \\
\ln\frac{Y_i(t)}{L_i(t)} &= \frac{\alpha}{\alpha-1}\ln(n_i+g+\delta) - \frac{\alpha}{\alpha-1}\ln s_i + \ln A_i(t) \\
\ln\frac{Y_i(t)}{L_i(t)} &= -\frac{\alpha}{1-\alpha}\ln(n_i+g+\delta) + \frac{\alpha}{1-\alpha}\ln s_i + \ln A_i(t)
\end{aligned}
$$

This makes it clear that in the canonical Solow model with cobb-douglass production, $b = \alpha/(1-\alpha) > 0$. So the fact that we cannot reject zero means that the data are also consistent with other models. The point estimate suggests that $\alpha = 1/2$ which is on the high side (usually we calibrate $\alpha = 1/3$ to reflect that labor's share of income is historically around 2/3). However, the standard error means that we cannot rule out $\alpha = 1/3$ (or much higher/lower values of α). Thus the regression does not tell us much about the deep parameters of the solow model either, even if we assume that this is the true model of the world.