Economics 202A
Macroeconomics
Fall 2021

Problem Set 6
Due on November 1, 2022
Please upload on Gradescope

You are allowed to work in groups. But please write up your own solutions. Please note that copying from old solutions constitutes plagiarism. Please write up your solutions as clearly as possible. Your grade will be reduced if your solution is unreasonably difficult to follow.

1. Eat-the-Pie Problem: Consider a household that must live forever off of an initial stock of wealth A_0 that pays a return R. The household seeks to maximize the utility function

$$\sum_{t=0}^{\infty} \beta^t u(C_t).$$

The household's wealth evolves according to

$$A_{t+1} = R(A_t - C_t).$$

The Bellman equation for the household's problem is

$$V(A) = \max_{C \in [0,A]} \{u(C) + \beta V(R(A - C))\}$$

A. Using Blackwell's sufficiency conditions, prove that the Bellman operator T:

$$(TV)(A) = \max_{C \in [0,A]} \{u(C) + \beta V(R(A - C))$$

is a contraction mapping. For simplicity, you can assume that $u(C)$ is bounded for $C \in [0, A]$. See chapter 3 of Stokey and Lucas (1989) for a discussion of the notion of a contraction mapping and Blackwell's sufficiency conditions.

B. Assume that

$$u(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0, \infty) \text{ and } \gamma \neq 1 \\ \log C & \text{if } \gamma = 1. \end{cases}$$

Let's guess that the value function takes the form

$$V(A) = \begin{cases} \psi \frac{A^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0, \infty) \text{ and } \gamma \neq 1 \\ \phi + \psi \log A & \text{if } \gamma = 1. \end{cases}$$

Confirm that this is in fact a solution to the Bellman equation.

C. Derive the optimal policy rule

$$C = \psi^{-\gamma^{-1}} A$$

where

$$\psi^{-\gamma^{-1}} = 1 - (\beta R^{1-\gamma})^{\gamma^{-1}}$$

D. When $\gamma = 1$, the consumption rule becomes $C = (1-\beta)A$. Why does consumption not depend on the value of the interest rate in this case? (Hint: Think about income and substitution effects)

2. Consider a household that lives for T+1 periods from period 0 to period T and faces a consumption-savings decision. The household seeks to maximize

$$\sum_{t=0}^{T} \beta^t u(C_t)$$

where $u'(C_t) > 0$ and $u''(C_t) < 0$. The household starts off with wealth A_0 and receives a constant income stream of Y per period. The interest rate in the economy is R. The household's budget constraint is therefore

$$C_t + A_{t+1} = Y + (1+R)A_t.$$

The household is constrained to die without debt: $A_{T+1} \geq 0$. Since the problem is non-stationary (time to death varies with t), the value function will be different for different periods. The value function will therefore have a time subscript, i.e. $V_t(A)$.

A. What is the value function for the household in period T?

B. Write a Bellman equation for the household for $t < T$.

For the remainder to this problem, we make the simplifying assumption that $\beta(1+R) = 1$. We want to show that the value function takes the following form

$$V_t(A) = \frac{1-\beta^{T-t+1}}{1-\beta} u\left(Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A\right)$$

and the optimal policy rule for the household is

$$C_t(A) = Y + \frac{1-\beta}{1-\beta^{T-t+1}}(1+R)A.$$

C. Show that the value function and policy rule above are correct for t=T.

D. Use an inductive argument to show that the value function and policy rule above are correct for $t < T$. I.e., assume they are correct for $t+1$ and show that conditional on this they are correct for t.

E. What happens as $T \to \infty$?

F. (Optional) I have posted matlab code (adapted from code originally written by Peter Maxted[1]) on bCourses that numerically solves the problem described above using a similar backward-induction procedure. The PS2-partf.m program solves 4 different versions of this problem, which can each be run by simply changing the variable $step = 1, 2, 3, 4$. Descriptions of these cases and how to switch between them can be found in the comments of the file. These different versions vary the numbers of periods T, interest rates R and utility functions $u(\cdot)$. The program then graphs the numeric approximation and analytic solution found in parts A through E of this problem.

Please use this code to answer the following questions: Qualitatively, under what wealth and time (A, t) conditions does the numerical solution approximate the analytic solution well ($step = 1$)? Are the conditions the same for the value function and the consumption policy function? Does increasing the number of periods T improve the numerical approximation ($step = 2$)? Does changing the utility function from log-utility $\log(c)$ to isoelastic utility $\dfrac{c^{1-\rho} - 1}{1 - \rho}$ with $\rho = 0.5$ improve or worsen the fit ($step = 3$)? What effect does lowering the interest rate R, while still maintaining that $(1+R)\beta = 1$, have on the shape of the value function and consumption policy function ($step = 4$)?

3. Optimal Stopping Problem: Each period a worker draws a job offer from a uniform distribution with support in the unit interval: $x \sim \mathrm{U}[0, 1]$. The worker can either accept the offer and realize a net present value of x or wait for another period and draw again. The problem ends when the worker accepts an offer. The worker discounts the future at a rate β per period.

A. Write down a Bellman equation for this problem.

B. Using Blackwell's conditions, show that the Bellman operator is a contraction mapping.

C. Starting with a guess $V_0(x) = 1$, analytically iterate on the Bellman operator to show that

$$V(x) = \begin{cases} x^* & \text{if } x \leq x^* \\ x & \text{if } x > x^* \end{cases}$$

where

$$x^* = \beta^{-1}(1 - \sqrt{1 - \beta^2})$$

Hint: Each iteration will give rise to a cutoff value for x. Let's denote the cutoff in iteration n as x_n^*. Derive a condition that relates the cutoff value x_n^* to the cutoff value in the previous iteration x_{n-1}^*. Solve for a fixed point of this dynamic equation.

References

STOKEY, N. L., AND R. E. LUCAS (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, Mass.

[1] https://scholar.harvard.edu/maxted/research_code