Problem 1

Basic Properties of Growth Rates: Romer 1.1

Use the fact that the growth rate of a variable equals the time derivative of its log to show:

(a) The growth rate of the product of two variables equals the sum of their growth rates. That is, if $Z(t) = X(t)Y(t)$, then $\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] + [\dot{Y}(t)/Y(t)]$.

Assuming $Z = XY$ and X, Y are both univariate functions of t, then

$$\begin{aligned}
\frac{\dot{Z}}{Z} &= \frac{d}{dt}\ln Z \\
&= \frac{d}{dt}\ln(XY) \\
&= \frac{d}{dt}(\ln X + \ln Y) \\
&= \frac{d}{dt}\ln X + \frac{d}{dt}\ln Y \\
&= \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}
\end{aligned}$$

(b) The growth rate of the ratio of two variables equals the difference of their growth rates. That is, if $Z(t) = X(t)/Y(t)$, then $\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] - [\dot{Y}(t)/Y(t)]$.

Assuming $Z = X/Y$ and X, Y are both univariate functions of t, then

$$\begin{aligned}
\frac{\dot{Z}}{Z} &= \frac{d}{dt}\ln Z \\
&= \frac{d}{dt}\ln(X/Y) \\
&= \frac{d}{dt}(\ln X - \ln Y) \\
&= \frac{d}{dt}\ln X - \frac{d}{dt}\ln Y \\
&= \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}
\end{aligned}$$

(c) If $Z(t) = \alpha X(t)^\alpha$, then $\dot{Z}(t)/Z(t) = \alpha \dot{X}(t)/X(t)$.

Assuming $Z = \alpha X^\alpha$ and X is a univariate function of t, then

$$\begin{aligned}
\frac{\dot{Z}}{Z} &= \frac{d}{dt} \ln Z \\
&= \frac{d}{dt} \ln(\alpha X^\alpha) \\
&= \frac{d}{dt}(\ln \alpha + \alpha \ln X) \\
&= \frac{d}{dt} \ln \alpha + \alpha \frac{d}{dt} \ln X \\
&= 0 + \alpha \frac{\dot{X}}{X}
\end{aligned}$$

Problem 2

Constant-Elasticity-of-Substitution Production Functions

Consider the production function

$$Y = F(K, L) = A[aK^{\phi} + (1-a)L^{\phi}]^{1/\phi} \tag{1}$$

where $0 < a < 1$ and $\phi < 1$.

Recall that the elasticity of substitution between capital and labor is the curvature of the isoquants of the production function. The slope of an isoquant is

$$\text{Slope} = \frac{\partial F(\cdot)/\partial K}{\partial F(\cdot)/\partial L}$$

The elasticity is then

$$\left[\frac{\partial(\text{Slope})}{\partial(L/K)} \frac{L/K}{\text{Slope}}\right]^{-1}$$

(a) Show that the elasticity of subtitution of the production function (1) is constant and equal to $1/(1-\phi)$.

First, let's find both partial derivatives of (1)

$$\frac{\partial F}{\partial K} = A\frac{1}{\phi}[...]^{1/\phi-1}aK^{\phi-1}$$
$$\frac{\partial F}{\partial L} = A\frac{1}{\phi}[...]^{1/\phi-1}(1-a)L^{\phi-1}$$

So, with some cancellation, the slope is then

$$\begin{aligned}\text{Slope} &= \frac{\frac{\partial F}{\partial K}}{\frac{\partial F}{\partial L}} \\ &= \frac{a}{1-a}\left(\frac{K}{L}\right)^{\phi-1} \\ &= \frac{a}{1-a}\left(\frac{L}{K}\right)^{1-\phi}\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial(\text{Slope})}{\partial(L/K)} &= \frac{a}{1-a}(1-\phi)\left(\frac{L}{K}\right)^{-\phi} \\ &= (1-\phi)\left(\frac{L}{K}\right)^{-1}\text{Slope}\end{aligned}$$

And finally, the elasticity of substitution between capital and labor is

$$\left[\frac{\partial(\text{Slope})}{\partial(L/K)}\frac{L/K}{\text{Slope}}\right]^{-1} = \left[(1-\phi)\left(\frac{L}{K}\right)^{-1}\text{Slope}\frac{L/K}{\text{Slope}}\right]^{-1}$$
$$= (1-\phi)^{-1}$$

> (b) Show that when the elasticity of substitution approaches 1 ($\phi \to 0$), this production function approaches the Cobb-Douglas form $Y = (\text{constant})K^{\alpha}L^{1-\alpha}$.
> (Hint: You will need to use l'Hopital's rule.)

Because $\ln(\cdot)$ is monotone and we expect the output to always be positive, we just need to show that $\ln(F)$ is equal to the log form of a Cobb-Douglas in the limit:

$$\text{Let } \gamma = aK^{\phi} + (1-a)L^{\phi}$$
$$\text{Then } \frac{\partial \gamma}{\partial \phi} = aK^{\phi}\ln K + (1-a)L^{\phi}\ln L$$
$$\text{and } \ln F = \ln A + \frac{1}{\phi}\ln \gamma$$

$$\lim_{\phi\to 0}\ln\frac{F}{A} = \frac{\lim_{\phi\to 0}\frac{\partial}{\partial\phi}\ln\gamma}{\lim_{\phi\to 0}\frac{\partial}{\partial\phi}\phi}$$

$$= \frac{\lim_{\phi\to 0}\frac{1}{\gamma}\frac{\partial}{\partial\phi}\gamma}{1}$$

$$= \lim_{\phi\to 0}\frac{aK^{\phi}\ln K + (1-a)L^{\phi}\ln L}{aK^{\phi} + (1-a)L^{\phi}}$$

$$= \frac{a\ln K + (1-a)\ln L}{a + 1 - a}$$

$$= \ln[K^{a}L^{1-a}]$$

Which implies

$$\lim_{\phi\to 0}F = AK^{a}L^{1-a}$$

(c) Divide through equation (1) by L and rewrite the production function in terms of output per person y and capital per person k. Let's refer to this production function as $y = f(k)$.

$$
\begin{aligned}
y &= f(k) \\
&= \frac{F}{L} \\
&= A\left[\frac{1}{L^\phi}\Big(aK^\phi + (1-a)L^\phi\Big)\right]^{1/\phi} \\
&= A[ak^\phi + (1-a)]^{1/\phi}
\end{aligned}
$$

(d) Derive expressions for $f'(k)$ and $f(k)/k$.

$$\begin{aligned}
\frac{f(k)}{k} &= \frac{A}{k}[ak^{\phi} + (1-a)]^{1/\phi} \\
&= A[a + (1-a)k^{-\phi}]^{1/\phi}
\end{aligned}$$

$$\begin{aligned}
f'(k) &= A\frac{1}{\phi}[ak^{\phi} + (1-a)]^{1/\phi - 1} a\phi k^{\phi-1} \\
&= \frac{Aa[ak^{\phi} + (1-a)]^{1/\phi - 1}}{k^{1-\phi}} \\
&= Aa[a + (1-a)k^{-\phi}]^{1/\phi - 1}
\end{aligned}$$

Note that

$$f'(k) = A^{\phi} a \left(\frac{f(k)}{k}\right)^{1-\phi}$$

(e) Consider the case where capital and labor are gross substitutes ($0 < \phi < 1$). Does the production function satisfy the Inada conditions?

Assume $\phi \in (0,1)$. First consider the limit as $k \to 0$:

$$\lim_{k\to 0} f'(k) = \lim_{k\to 0} \frac{Aa[ak^{\phi} + (1-a)]^{1/\phi - 1}}{k^{1-\phi}} = \infty$$

so the first Inada conditions is met. Let's now consider the limit as $k \to \infty$. Using the relationship between f' and f/k from the last problem, we have:

$$\begin{aligned}
\lim_{k\to\infty} f'(k) &= \lim_{k\to\infty} A^{\phi} a \left(\frac{f(k)}{k}\right)^{1-\phi} \\
&= A^{\phi} a \left(\lim_{k\to\infty} \frac{f(k)}{k}\right)^{1-\phi}
\end{aligned}$$

$$\begin{aligned}
\lim_{k\to\infty} \frac{f(k)}{k} &= \lim_{k\to\infty} A[a + (1-a)k^{-\phi}]^{1/\phi} \\
&= A[a + (1-a) \lim_{k\to\infty} k^{-\phi}]^{1/\phi} \\
&= Aa^{1/\phi} \\
&\neq 0
\end{aligned}$$

And because $\phi \in (0,1)$,

$$\implies \lim_{k\to\infty} f'(k) \neq 0$$

So this production function does not satisfy the Inada conditions.

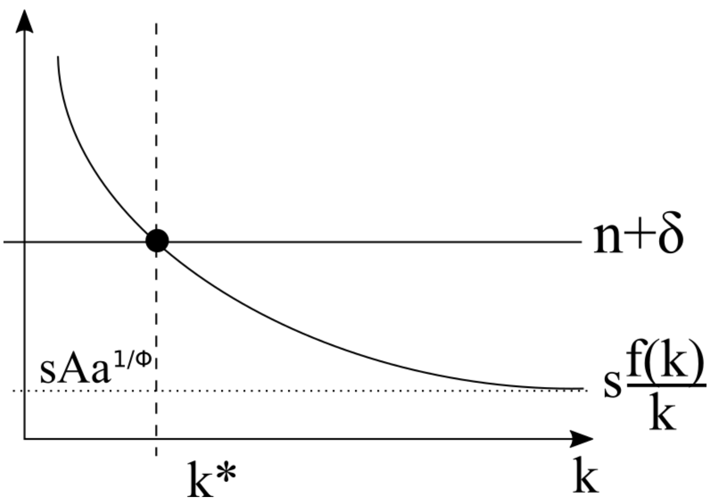
(f) Recall that

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (n+\delta)$$

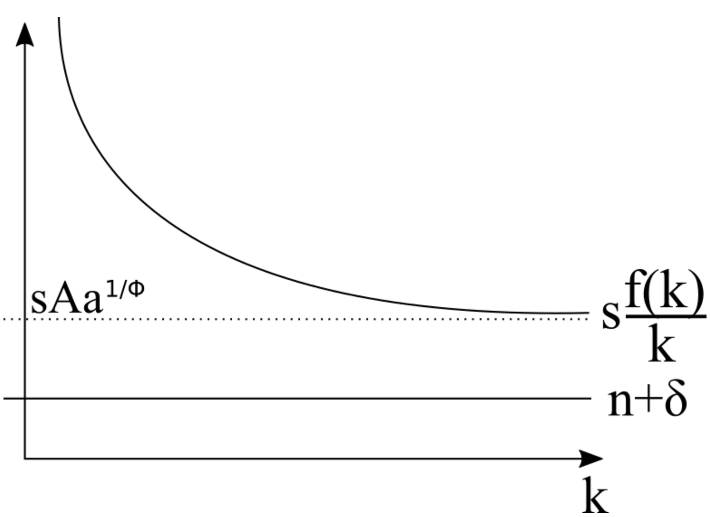
where s is the savings rate, n is the population growth rate, and δ is the depreciation rate of capital.

Can capital accumulation result in sustained growth in this economy given a constant savings rate? (Hint: Plot $sf(k)/k$ and $(n+\delta)$ as a function of k on the same figure.)

Note from the last problem that $\lim_{k\to\infty} \frac{sf(k)}{k} = sAa^{1/\phi>0}$. So in the case where $n+\delta > sAa^{1/\phi}$, the plot would look like



We can see in the above picture that there is a k^* where $n+\delta$ meets $\frac{sf(k)}{k}$, indicating that there is a steady state in capital per unit of effective labor and that capital accumulation alone cannot lead to sustained growth. If we consider the case where $n+\delta < sAa^{1/\phi}$, the plot would look like this:



In this version, there is no single crossing point because $\frac{sf(k)}{k}$ asymptotically approaches $sAa^{1/\phi}$ and never crosses $n+\delta$. Thus there is no steady state and capital will continue to accumulate and grow the economy.

Problem 3

The Harrod-Domar Model

Prior to the development of the Solow model, work on development and growth often used a Leontief production function

$$Y = F(K, L) = min(AK, BL), \tag{2}$$

where $A = 0$ and $B = 0$. As we discussed above, this corresponds to a CES production function with $\phi \to -\infty$. Well-known papers to use this type of production function are Harrod (1939) and Domar (1946). Growth models that use this production function have since usually been referred to as Harrod-Domar models. Harrod and Domar predicted that capitalist economies would suffer from persistent problems of unemployment of either labor or machines. We now explore how this conclusion follows from the production function (2).

(a) Rewrite the production function (2) in terms of per capita output y and capital per capita k. Plot the resulting production function as a function of k. Also plot the marginal product of capital as a function of k.

$$Y = \min(AK, BL) = \begin{cases} AK & \text{if } AK < BL \\ BL & \text{if } AK > BL \end{cases}$$

$$\frac{Y}{L} = \begin{cases} A\frac{K}{L} & \text{if } AK < BL \\ B & \text{if } AK > BL \end{cases}$$

$$y = \begin{cases} Ak & \text{if } Ak < B \\ B & \text{if } Ak > B \end{cases}$$

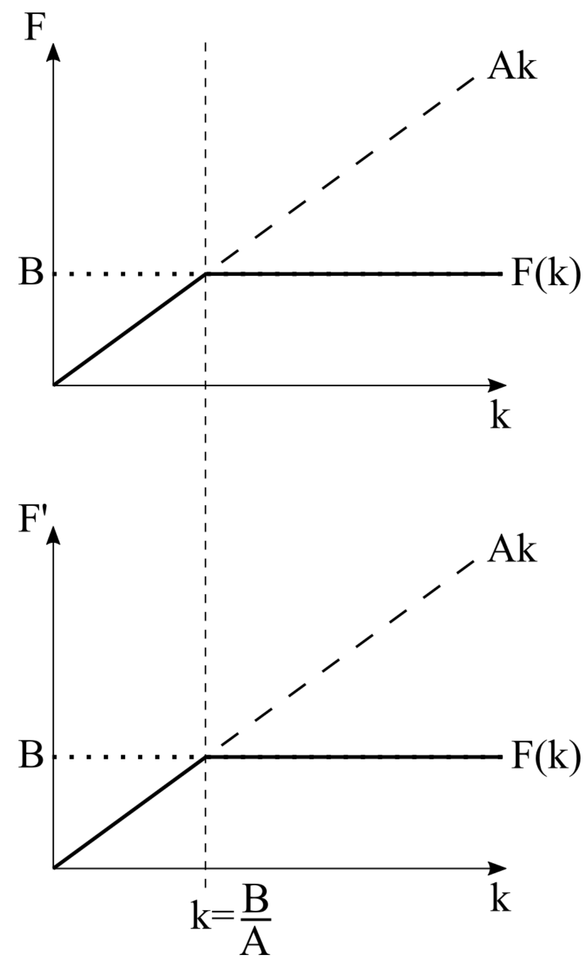


Figure 1: Production function (top) and marginal product of capital (bottom).

(b) Recall that with a constant savings rate s we have

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (n+\delta)$$

Consider first the case where $sA < (n+\delta)$. Plot $sf(k)/k$ and $n+\delta$ as a function of k. Discuss the dynamics of this economy starting from some initial capital stock k_0. Comment in particular on the degree of unemployment of labor in the long run in such an economy.

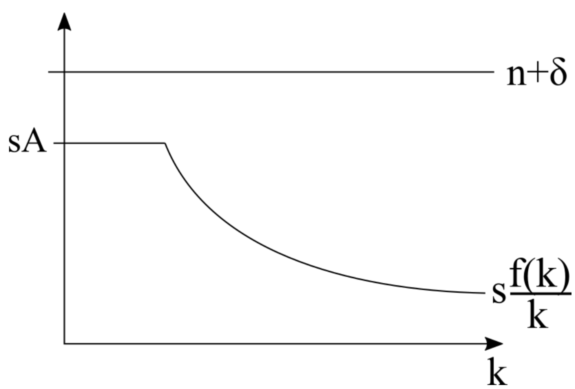
Since

$$f(k) = \begin{cases} Ak & \text{if } Ak < B \\ B & \text{if } Ak > B \end{cases}$$

Then

$$\frac{sf(k)}{k} = \begin{cases} sA & \text{if } Ak < B \\ \dfrac{sB}{k} & \text{if } Ak > B \end{cases}$$

So the plot looks like



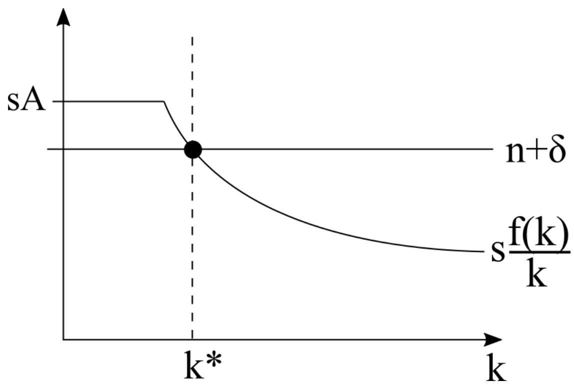
So if $sA < (n+\delta)$, there is no crossing point and no steady state in capital per capita.

$$\begin{aligned} n+\delta &> \frac{sf(k)}{k}\ \forall k \\ &\implies \frac{\dot{k}}{k} < 0\ \forall k \\ &\implies k \to 0 \text{ in the long run, from any } k_0 \end{aligned}$$

and $\frac{\dot{k}}{k}$ is larger for larger k, so the rate of shrinkage of the capital stock starts off larger and slows down until $\frac{\dot{k}}{k} = sA - (n+\delta) < 0$ and stays constant until $\dot{k} = 0$.

$\implies$ we have long run total unemployment (since there will be no capital to use and labor and capital are compliments in production).

(c) Now consider the case were $sA > (n+\delta)$. Again, plot $sf(k)/k$ and $n+\delta$ as a function of k. Discuss the dynamics of this economy starting from some initial capital stock k_0. Comment on the degree of unemployment of labor in the long run as well as the possible idleness of capital in the long run.



Since $s, \delta, n \in (0,1)$,

$$sA > n\delta \implies \frac{sA}{n+\delta} > 1$$

and since

$$k^* = \frac{sB}{n+\delta}$$

Then

$$f(k^*) = \min\left(A\frac{sB}{n+\delta}, B\right) = B\frac{sA}{n+\delta} > B$$

So there is a greater amount of capital employed than labor in the long run, meaning there will be full employment of labor, but some unused captial (or underutilized).

(d) Is it possible for both capital and labor to be fully employed in a Harrod-Domar economy in the long run?

It would only be possible to fully employ both labor and capital if $k^* = \frac{B}{A}$ exactly, but since k^* depends on continuous fundemental parameters n, δ, s, $k^* = \frac{B}{A}$ would happen with probability zero (meaning no, it is not possible to have fully employed labor and captial in the Harrod-Domar model).