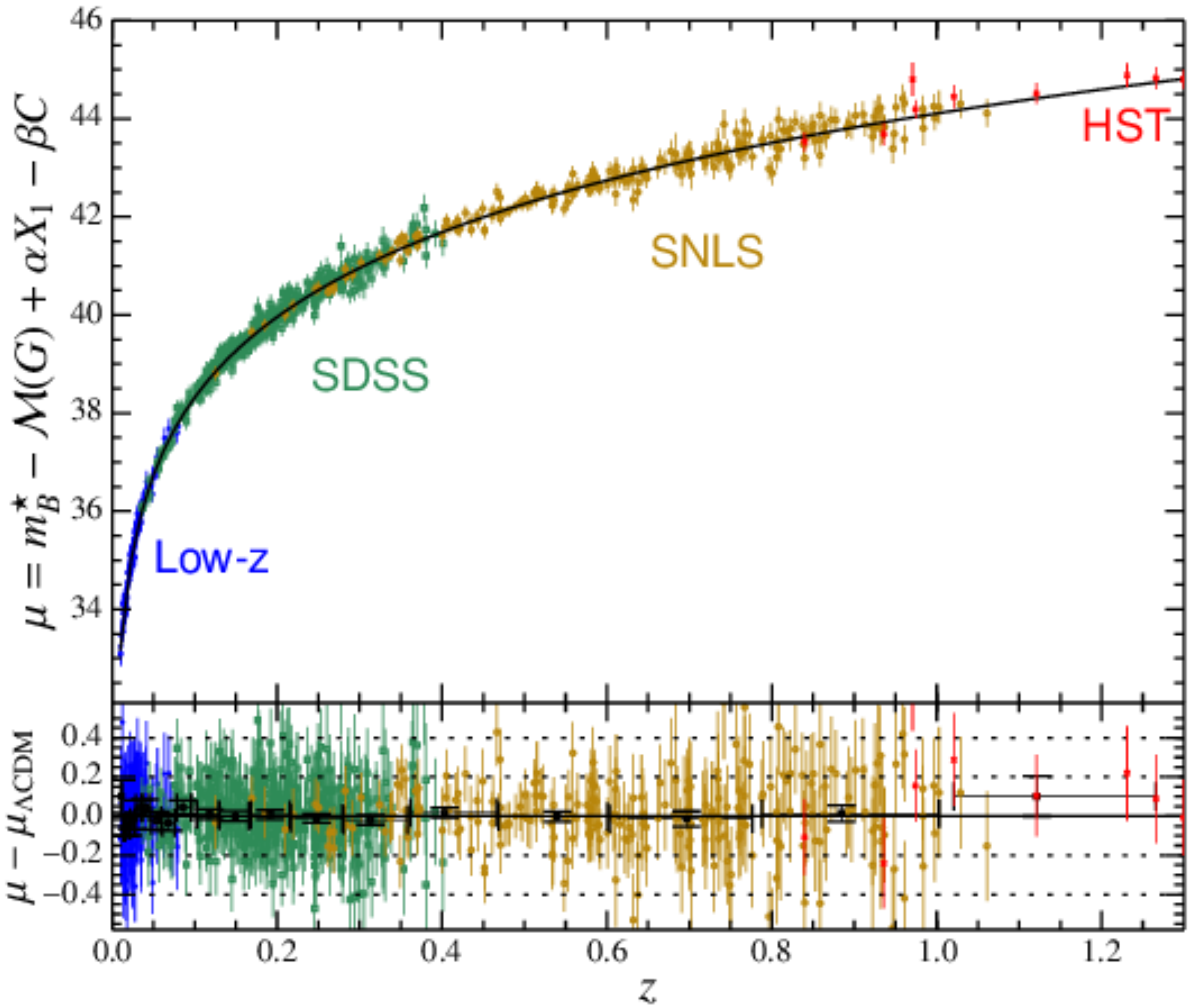
Astrostatistics: Wed 26 Feb 2020

https://github.com/CambridgeAstroStat/PartIII-Astrostatistics-2020

- Today: continue Bayesian computation / MCMC
  - MacKay: Ch 29-30; Bishop: Ch 11; Gelman BDA
  - Givens & Hoeting. "Computational Statistics"
    (Free online through Cambridge Library iDiscover)
- Example Sheet 1 solutions & code posted
- Lecture MCMC code examples online: lecture_codes/
- Example Class 2, Thu Feb 27, 3:30pm MR13

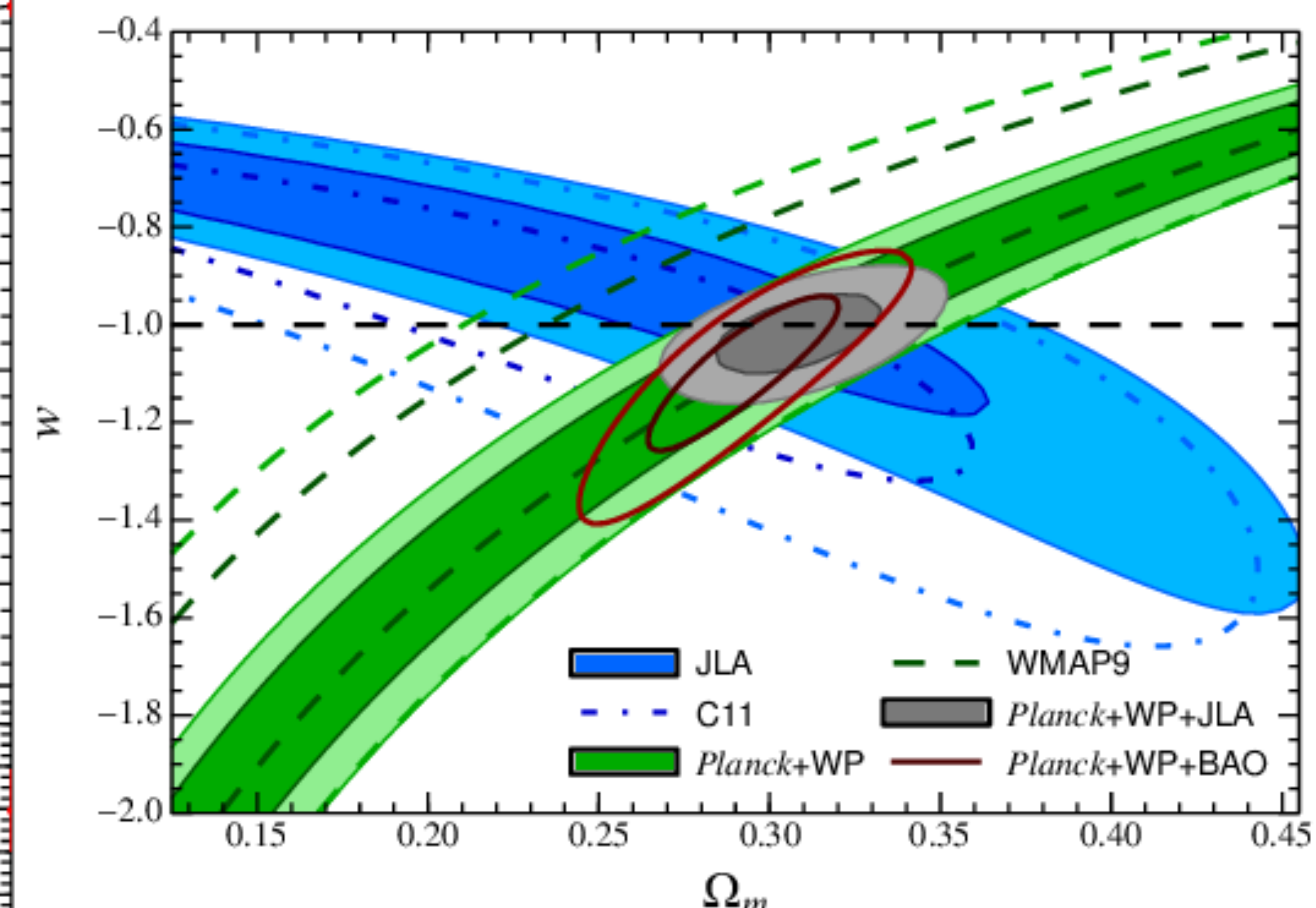
Type Ia SN Cosmology

Hubble Diagram
Modern SN Ia Surveys



Joint Lightcurve Analysis
(JLA, Betoule et al. 2014)

Cosmological Constraints



$w = -1.027 \pm 0.055$ (stat+sys)

Cosmology with standard candles

For us: Simplify:
Let's say Supernovae are *standard* candles

$$M_s \sim N(M_0, \sigma^2_{\mathrm{int}})$$ SN Population Distribution

$$m_s = M_s + \mu(z_s; H_0, \mathbf{\Omega})$$ (Log) Inverse Square Law

$$\mathbf{\Omega} = (\Omega_M, \Omega_\Lambda, w)$$

Cosmological Parameters:

- Hubble Constant H_0
- Matter Density Ω_M
- Dark Energy Density Ω_Λ
- DE EoS parameter w

Assume apparent magnitudes and redshifts
$\{m_s, z_s\}$ measured perfectly

Theoretical Distances

1 Comoving Distance

The dimensionless comoving distance $\tilde{d}$ to an object with redshift z is:

$$\tilde{d}(z;\Omega_M,\Omega_\Lambda,w) = \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3+\Omega_k(1+z')^2+\Omega_\Lambda(1+z')^{3(w+1)}}} \tag{1}$$

where Ω_M is the matter density, Ω_Λ is the dark energy density, the curvature is $\Omega_k \equiv 1-\Omega_M-\Omega_\Lambda$, and w is the equation-of-state parameter of dark energy.

In terms of the scale factor $a \equiv (1+z)^{-1}$, the integral can be expressed as:

$$\tilde{d}(z;\Omega_M,\Omega_\Lambda,w) = \int_{(1+z)^{-1}}^1 \frac{da'/a'}{\sqrt{\Omega_M/a'+\Omega_k+\Omega_\Lambda/a'^{(1+3w)}}} \tag{2}$$

(derived from standard FLRW metric)

Theoretical Luminosity Distances (Arbitrary Curvature)

2 Luminosity Distance

The theoretical cosmological distance to an object with redshift z inferred from the standard candle method are given by the luminosity distance d_L. The dimensionless distance is:

$$\tilde{d}_L(z;\Omega_M,\Omega_\Lambda,w) = (1+z)\begin{cases} |\Omega_k|^{-\frac{1}{2}}\sinh[\sqrt{|\Omega_k|}\tilde{d}(z;\Omega_M,\Omega_\Lambda,w), & \Omega_k > 0 \\ \tilde{d}(z;\Omega_M,\Omega_\Lambda,w), & \Omega_k = 0 \\ |\Omega_k|^{-\frac{1}{2}}\sin[\sqrt{|\Omega_k|}\tilde{d}(z;\Omega_M,\Omega_\Lambda,w), & \Omega_k < 0 \end{cases} \tag{3}$$

For a given value of the Hubble constant, the dimensionful luminosity distance is

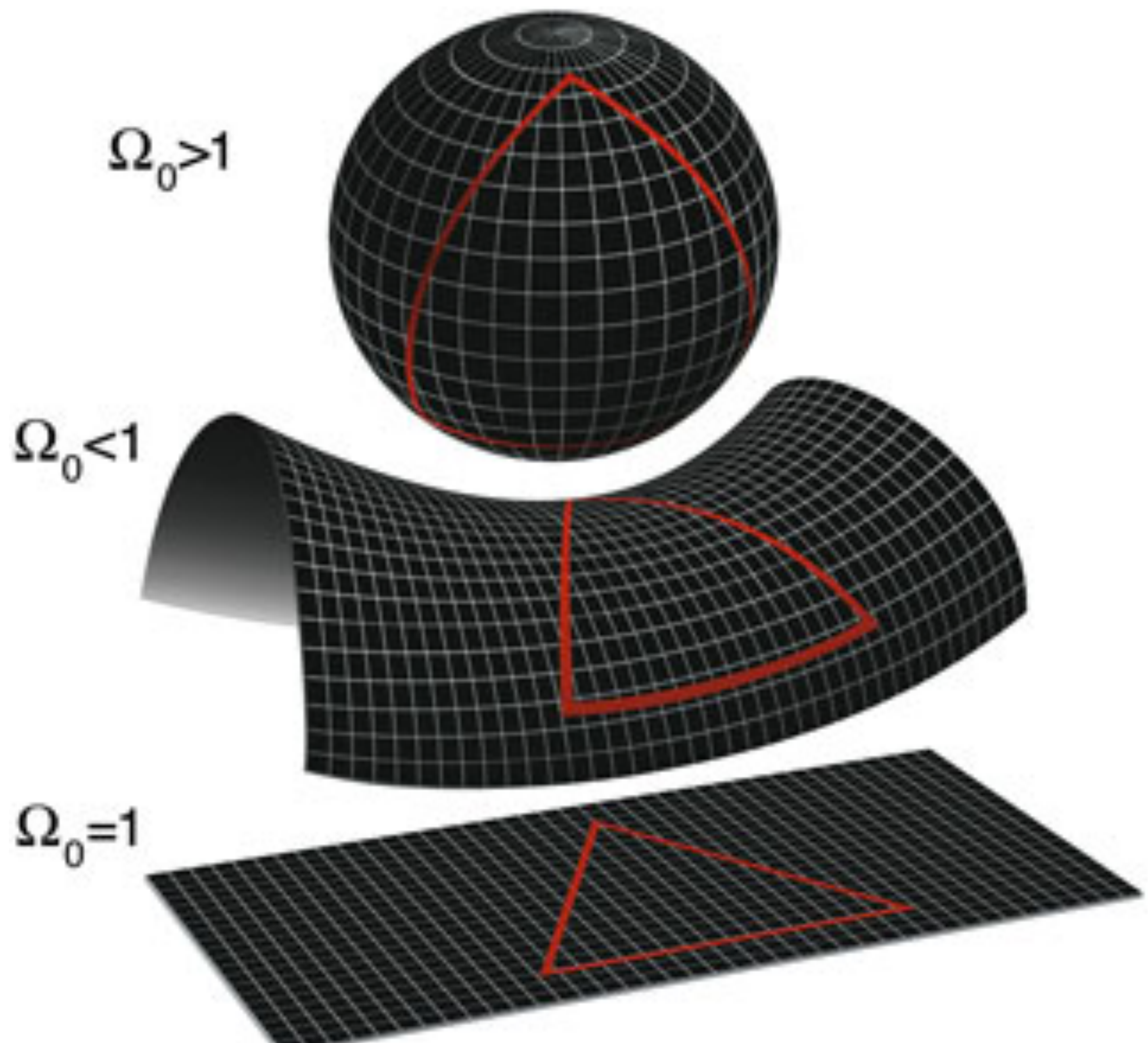
$$d_L(z;\Omega_M,\Omega_\Lambda,w,H_0) = \frac{c}{H_0}\,\tilde{d}_L(z;\Omega_M,\Omega_\Lambda,w) \tag{4}$$

Distance Modulus:

$$\mu(z;H_0,\mathbf{\Omega}) = 25 + 5\log_{10}\left(\frac{c}{H_0}\tilde{d}_L(z;\mathbf{\Omega})\right)$$

Curvature of the Universe

$$\Omega_K + \Omega_M + \Omega_\Lambda = 1$$

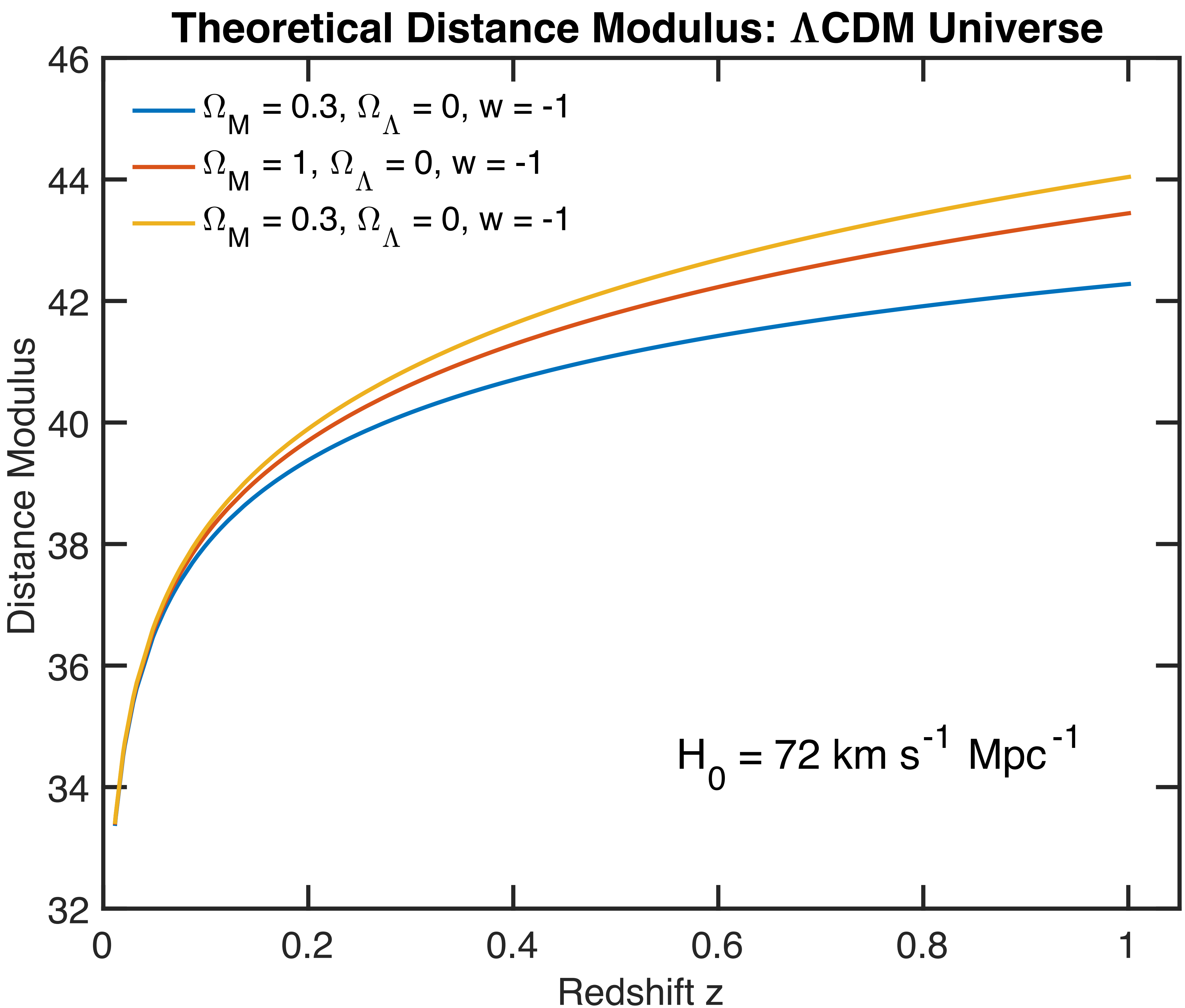


$$\Omega_k > 0$$

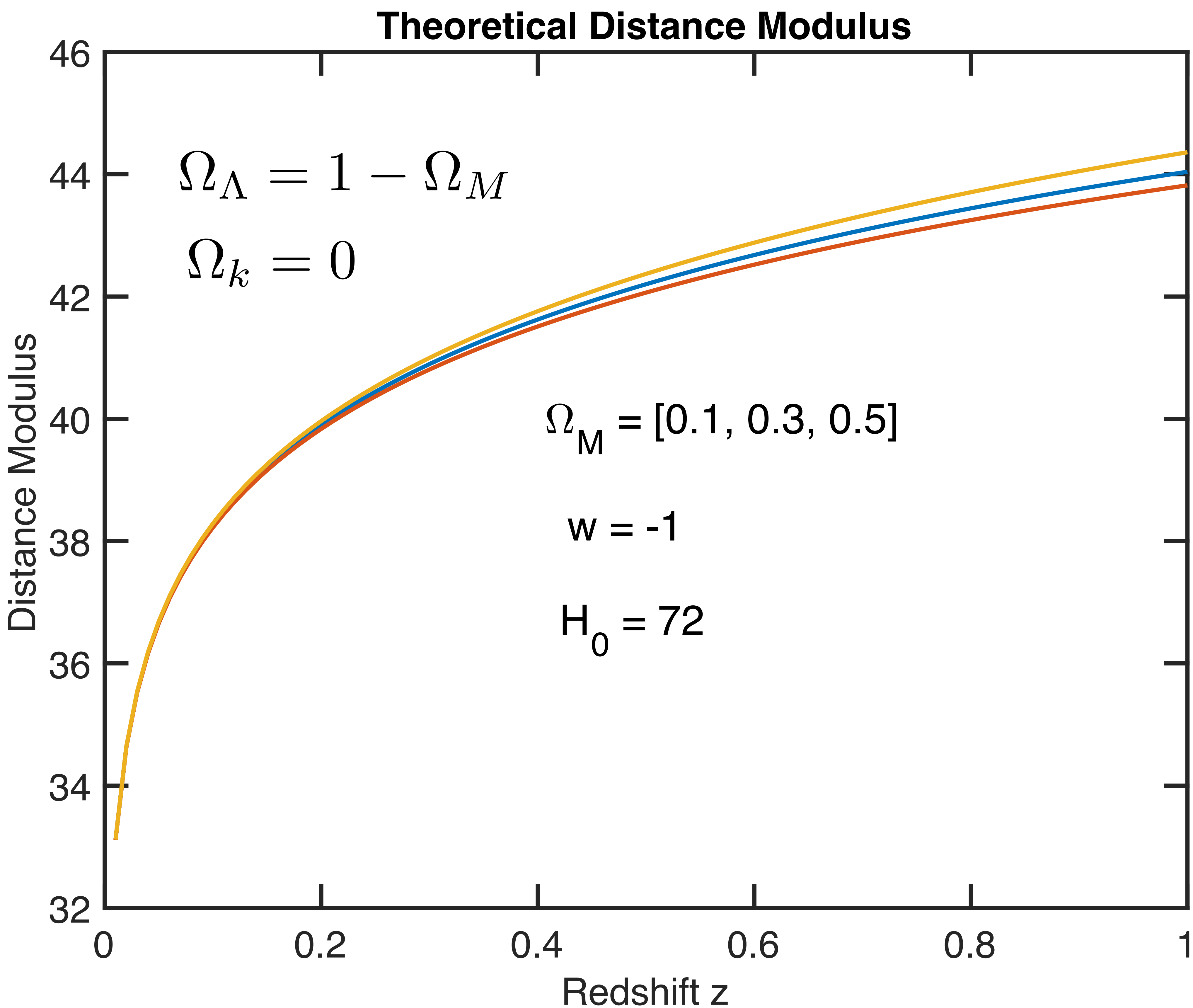
$$\Omega_k < 0$$

$$\Omega_k = 0$$

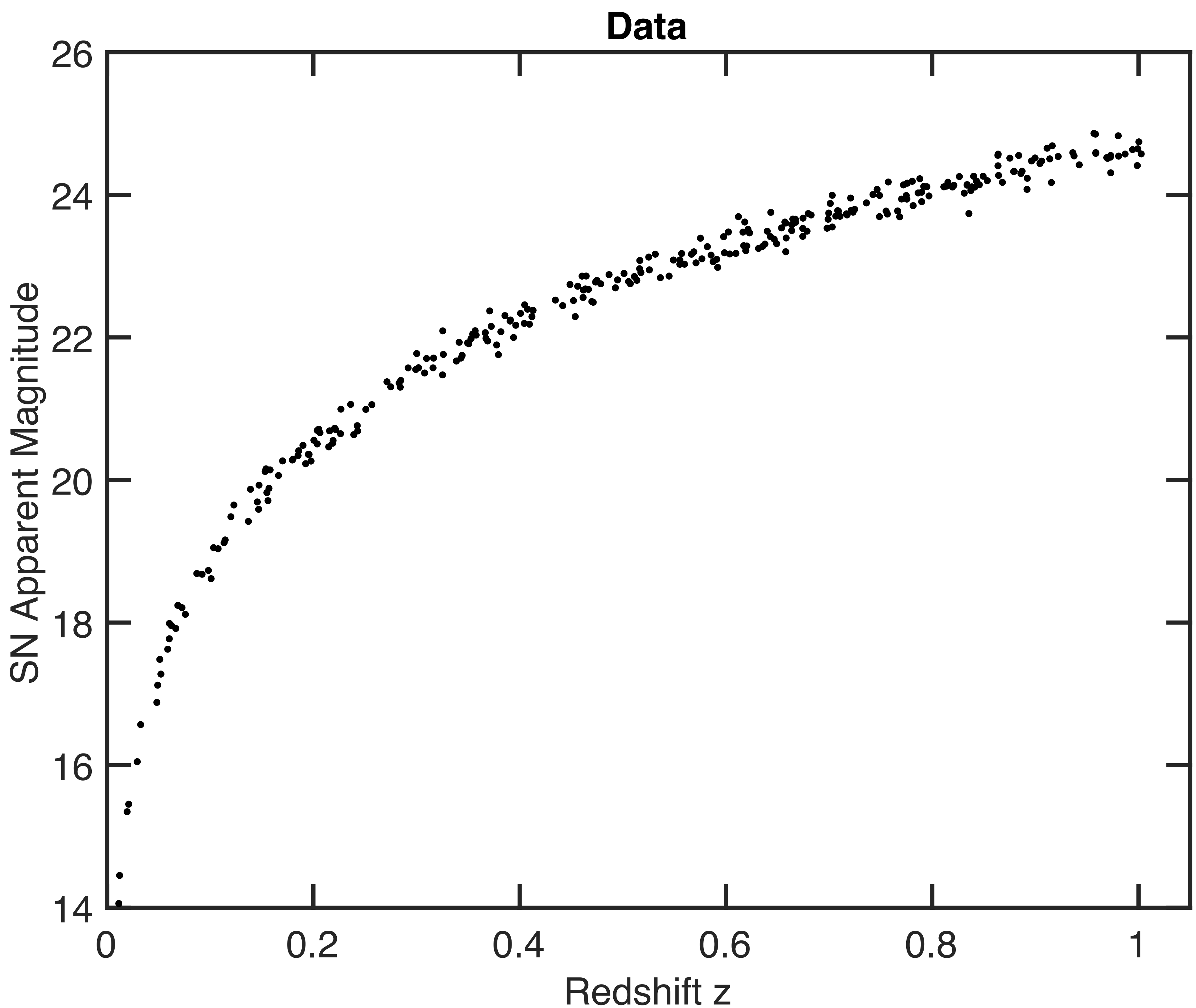
Theoretical Distance Modulus
(Arbitrary Curvature)



Flat Universe: $\Omega_k = 0$



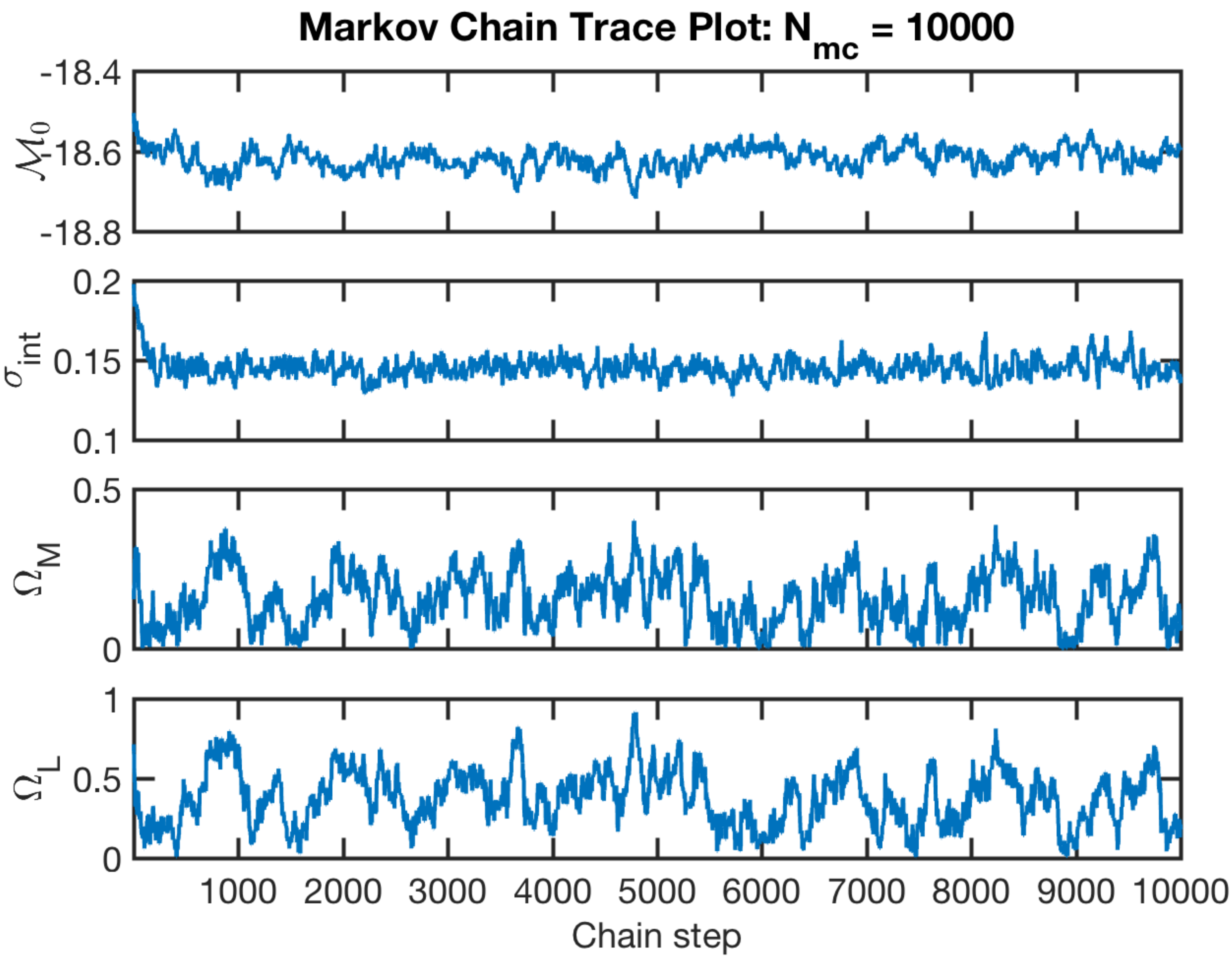
Idealised supernova dataset



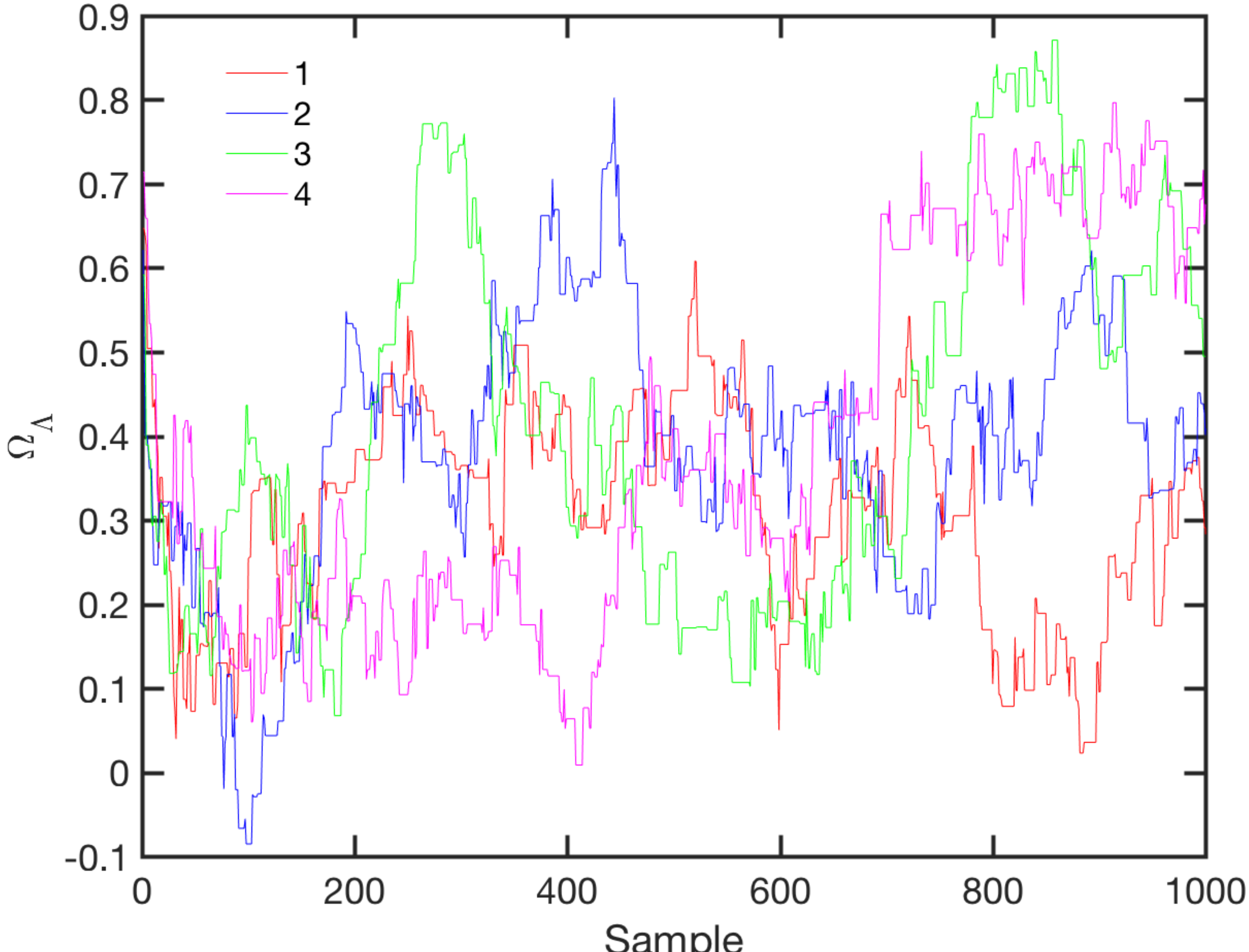
First assume $w = -1$

Write down model & posterior

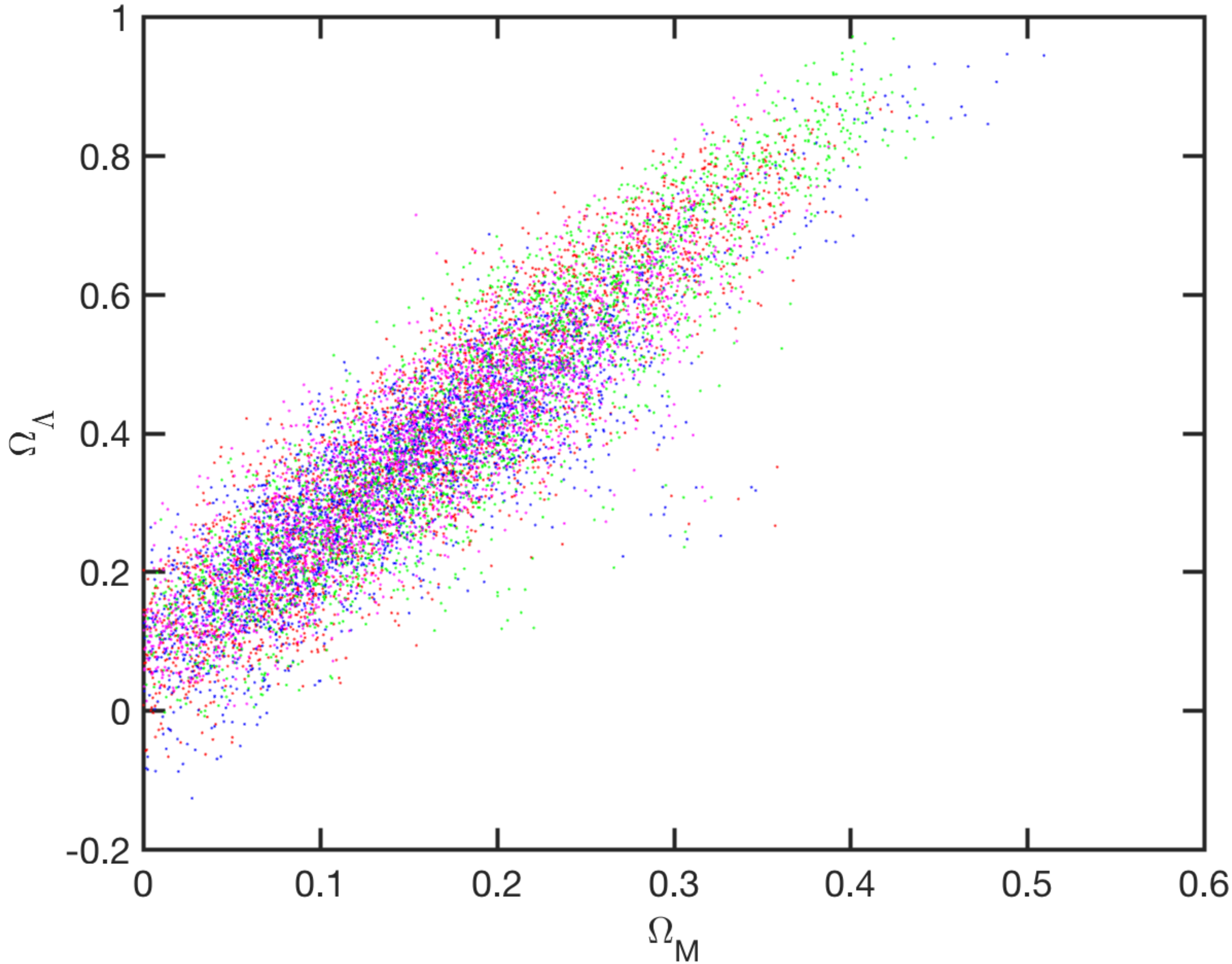
4D Metropolis: one chain



Multiple Independent Chains



Multiple Independent Chains



Assessing Convergence with multiple chains: Gelman-Rubin (G-R) ratio

Monitoring convergence of each scalar estimand

Suppose we have simulated m parallel sequences, each of length n (after discarding the first half of the simulations). For each scalar estimand ψ, we label the simulation draws as ψ_{ij} $(i = 1, \ldots, n; j = 1, \ldots, m)$, and we compute B and W, the between- and within-sequence variances:

$$B = \frac{n}{m-1}\sum_{j=1}^{m}(\overline{\psi}_{.j} - \overline{\psi}_{..})^2, \quad \text{where} \quad \overline{\psi}_{.j} = \frac{1}{n}\sum_{i=1}^{n}\psi_{ij}, \quad \overline{\psi}_{..} = \frac{1}{m}\sum_{j=1}^{m}\overline{\psi}_{.j}$$

$$W = \frac{1}{m}\sum_{j=1}^{m}s_j^2, \quad \text{where} \quad s_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\psi_{ij} - \overline{\psi}_{.j})^2.$$

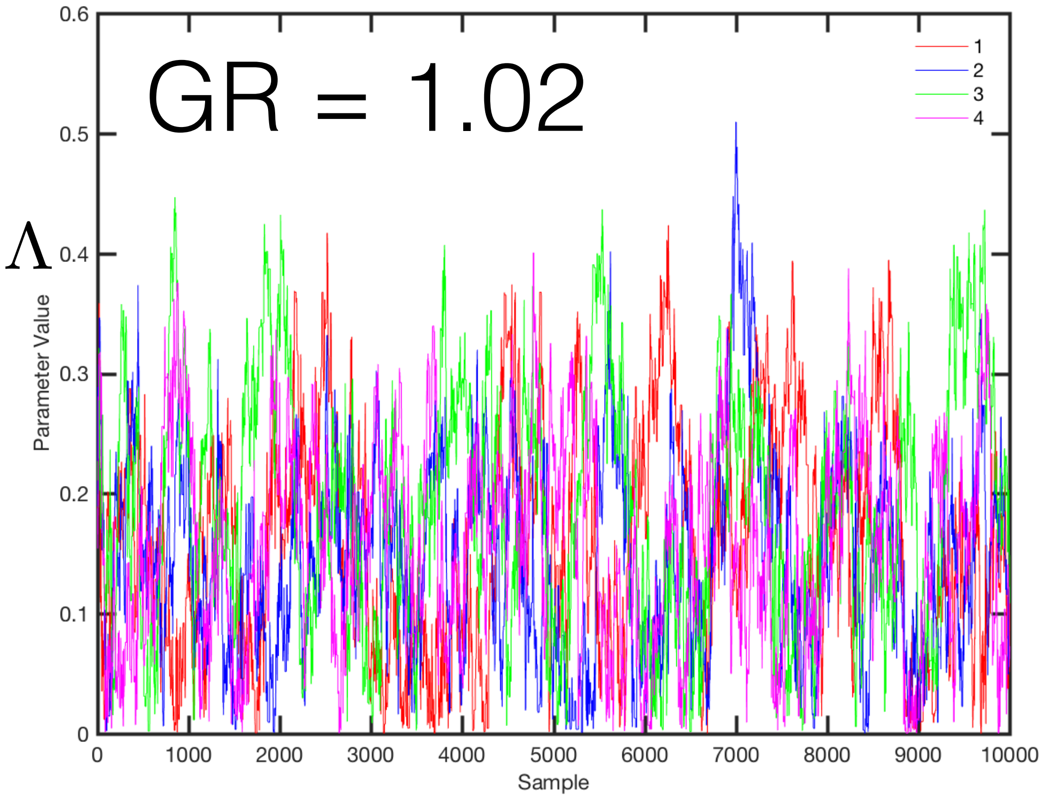
We can estimate $\text{var}(\psi|y)$, the marginal posterior variance of the estimand, by a weighted average of W and B, namely

$$\widehat{\text{var}}^+(\psi|y) = \frac{n-1}{n}W + \frac{1}{n}B. \tag{11.3}$$

G-R ratio: $\hat{R} = \sqrt{\frac{\widehat{\text{var}}^+(\psi|y)}{W}}, \approx 1$

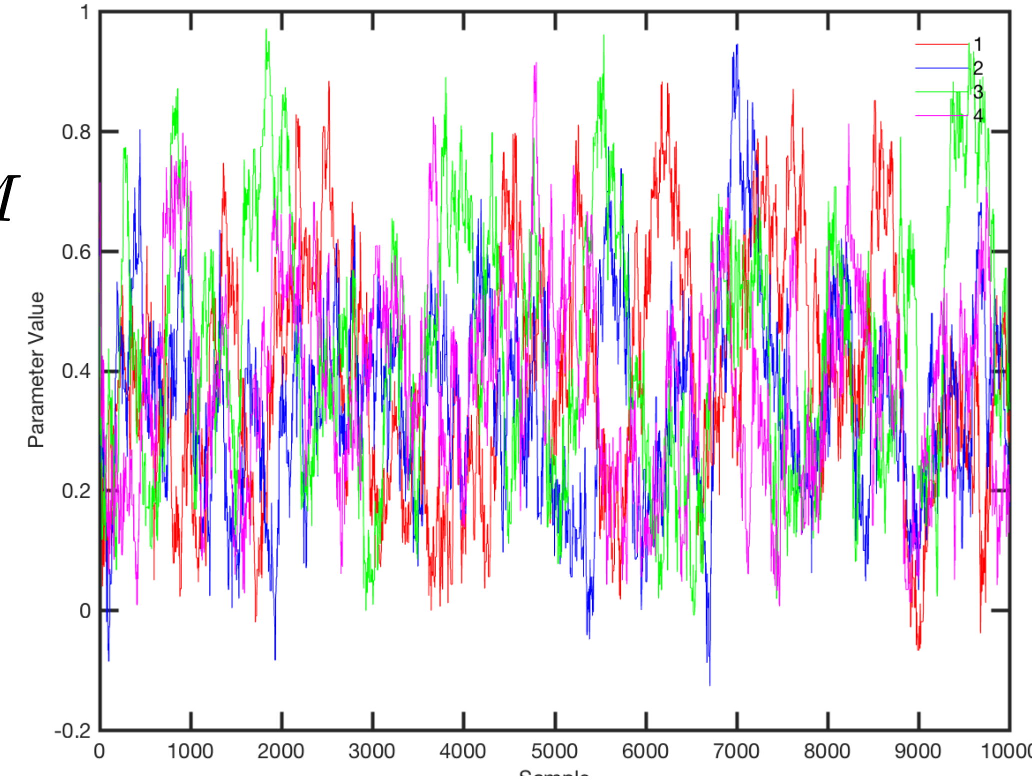
4 Independent Chains (10000 iters)

Ω_Λ

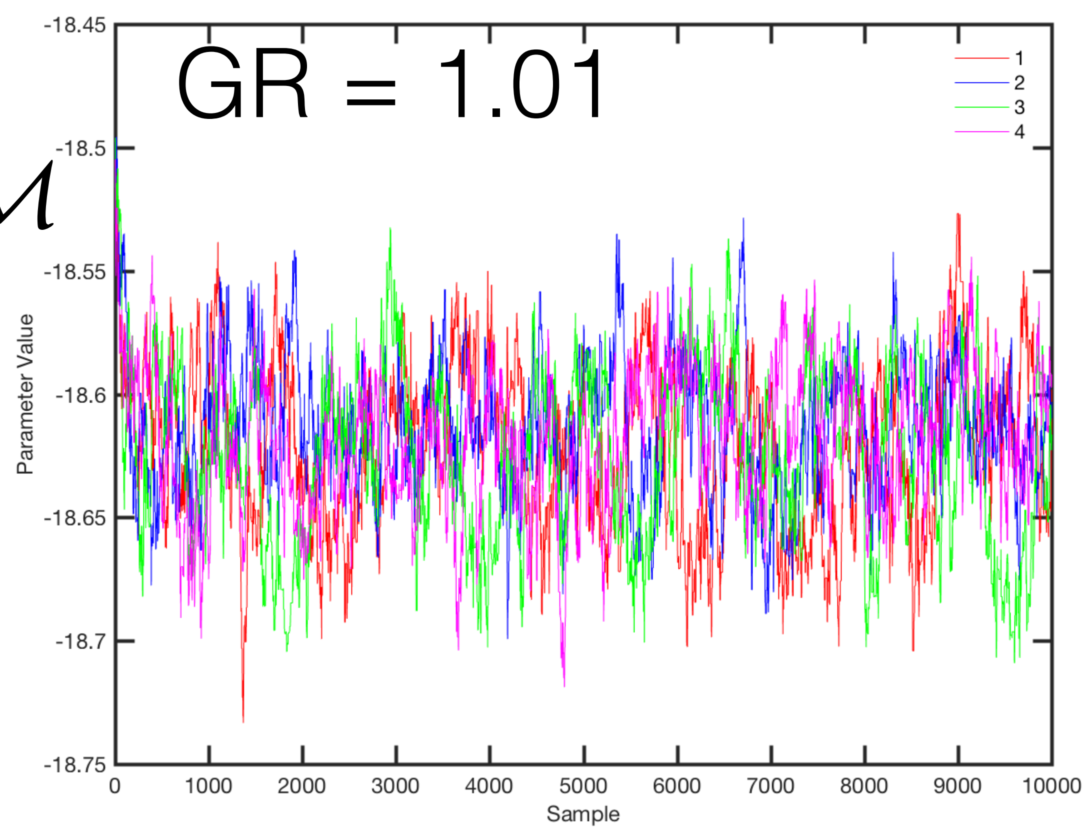


GR = 1.02

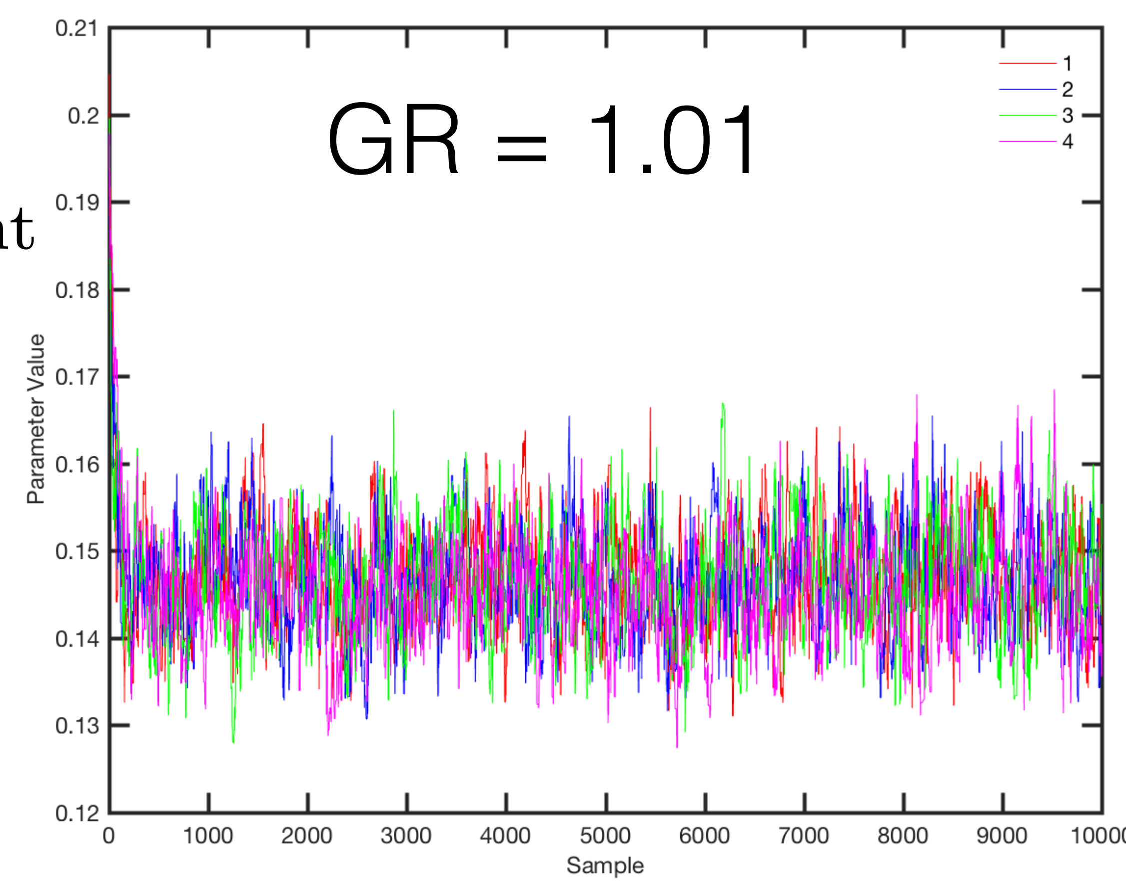
Ω_M



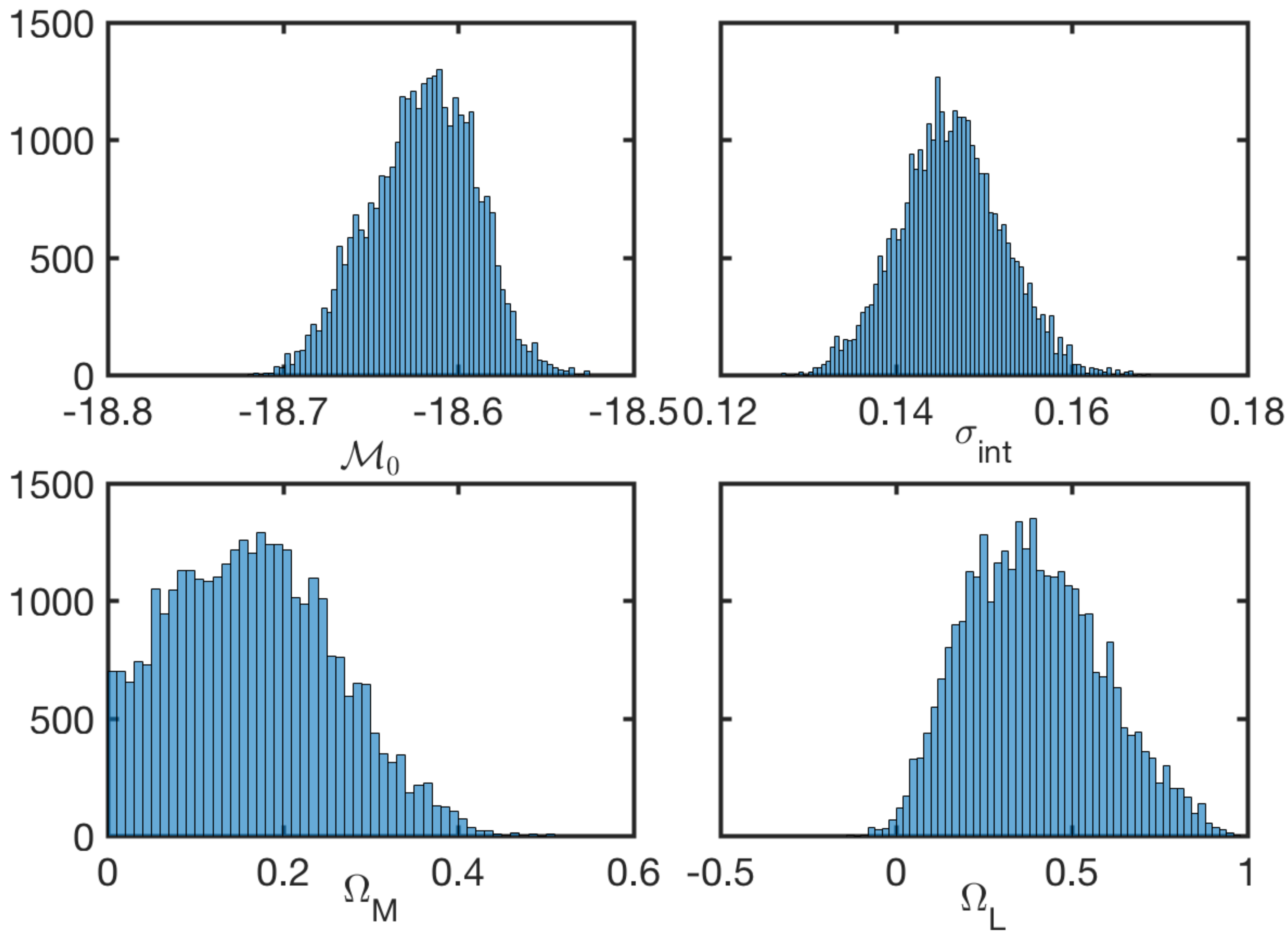
$\mathcal{M}$



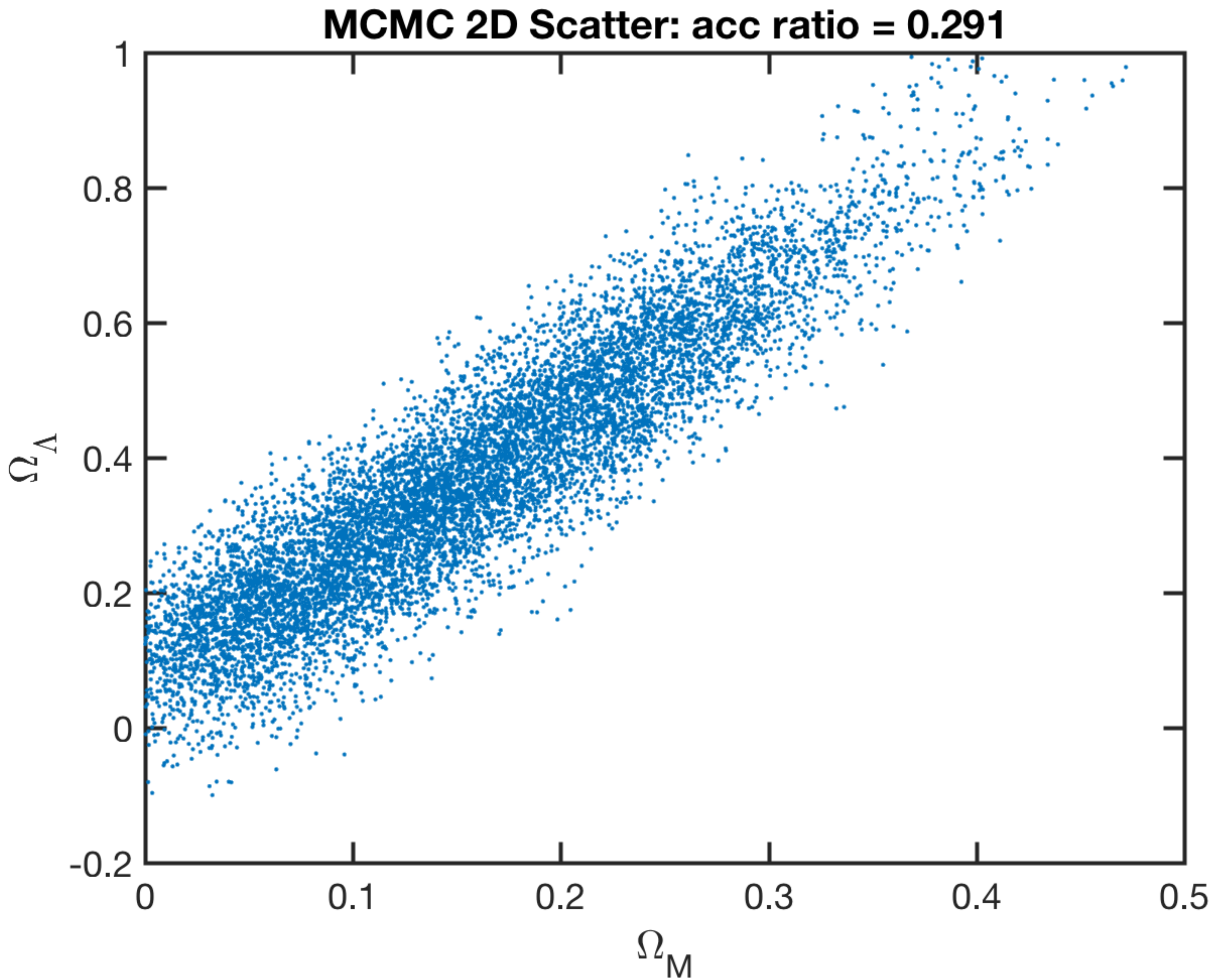
$\sigma_{\rm int}$



Cut burn-in (20%) and combine chains
Posterior Histograms



Posterior Scatter plot



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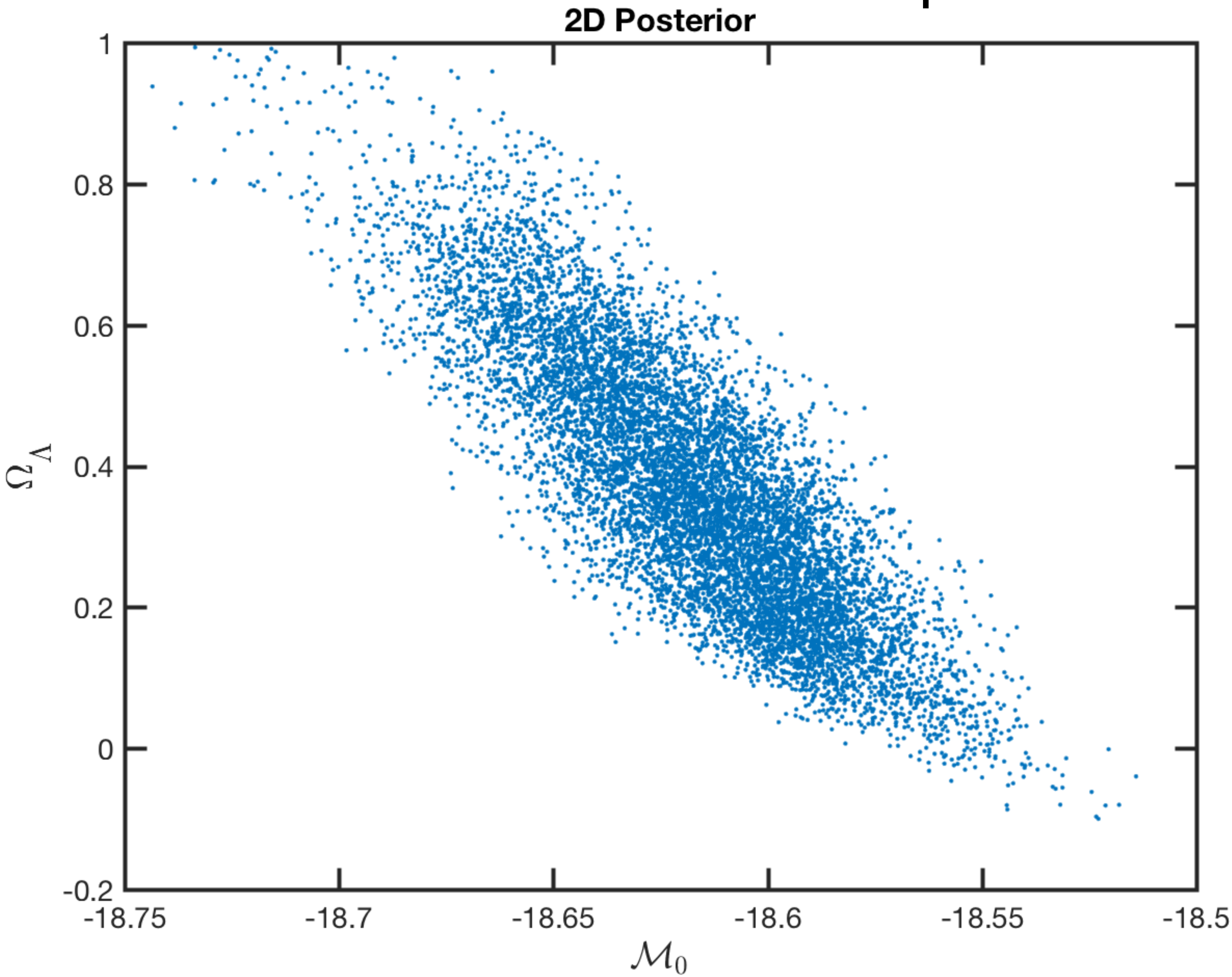
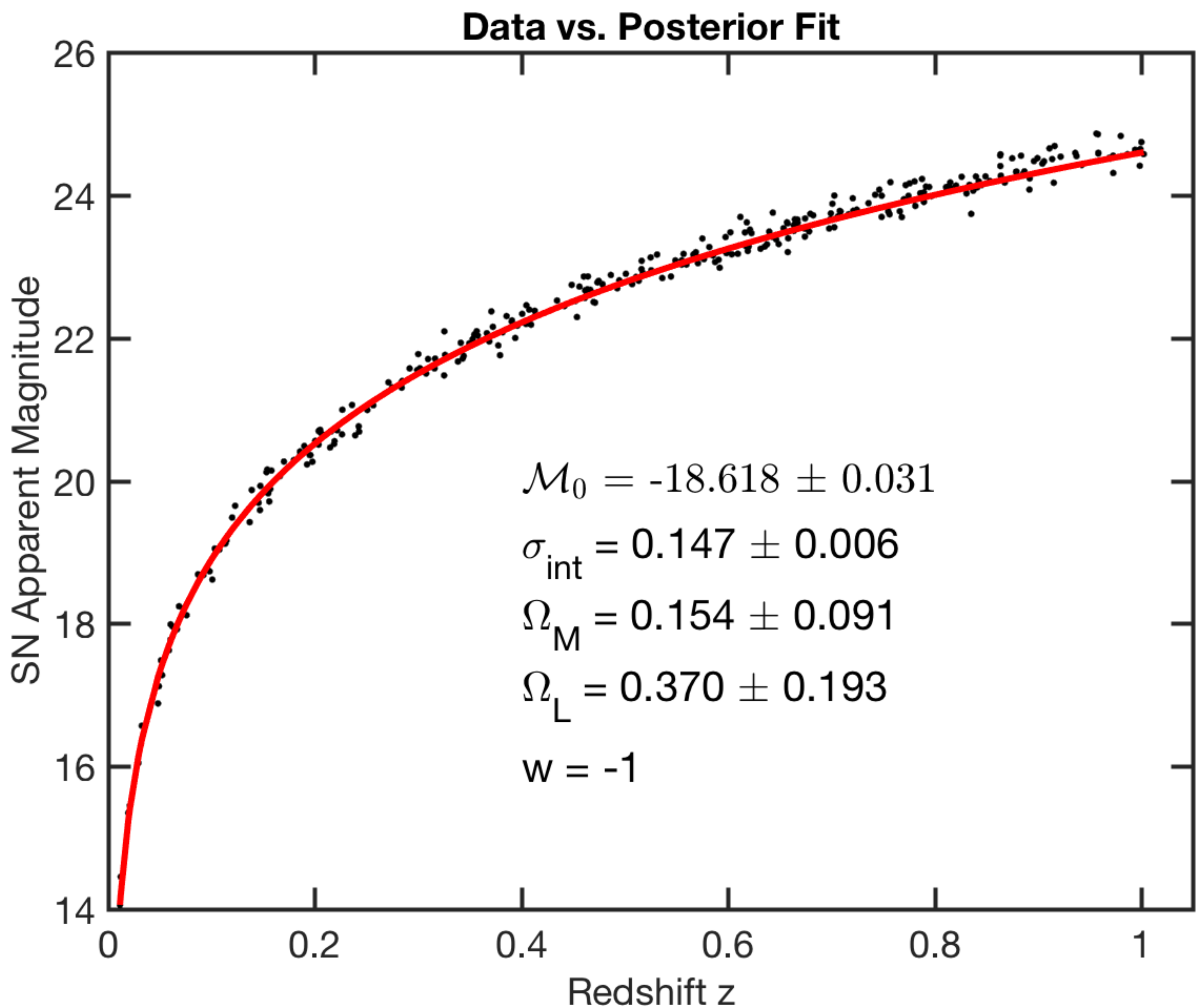
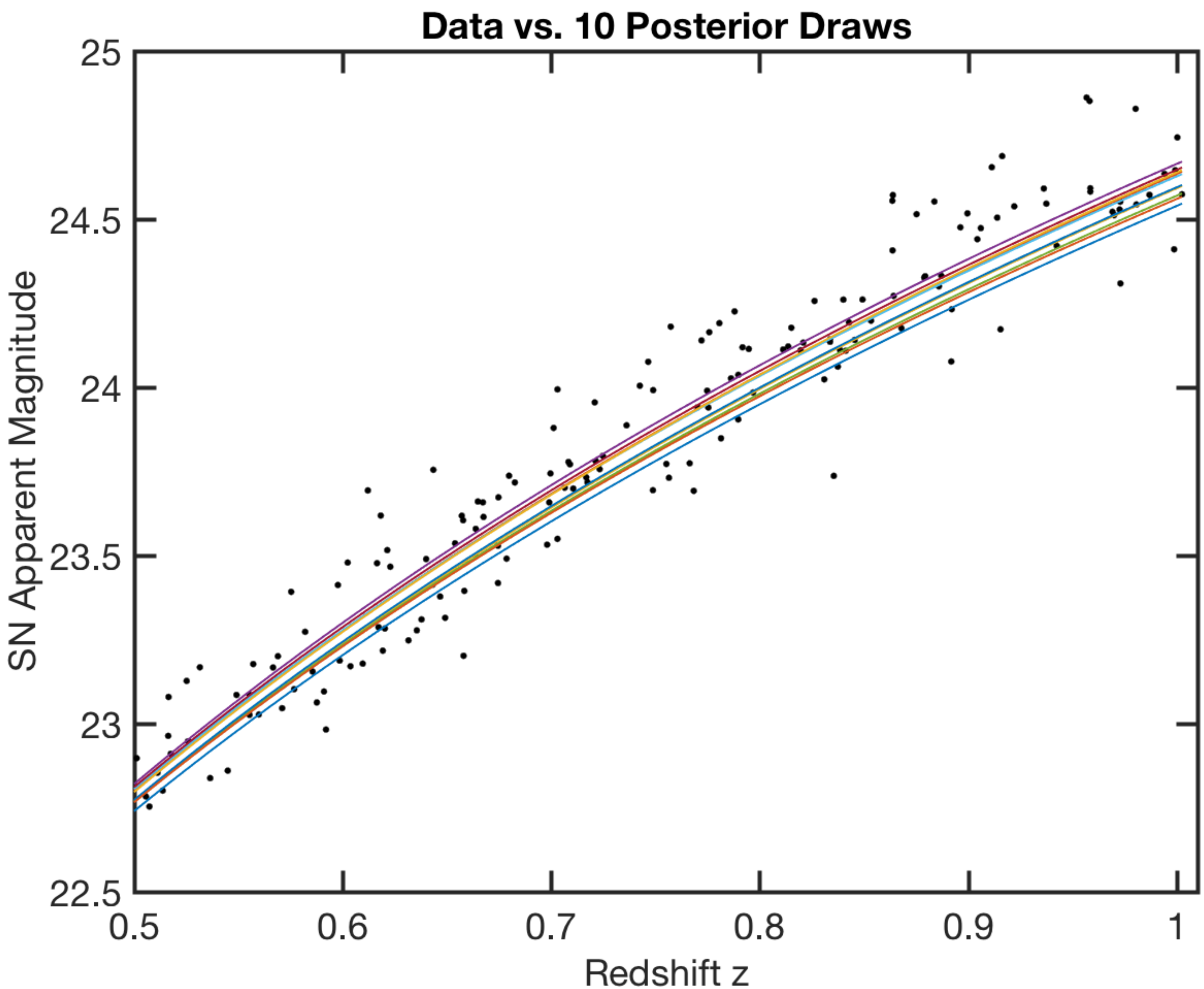


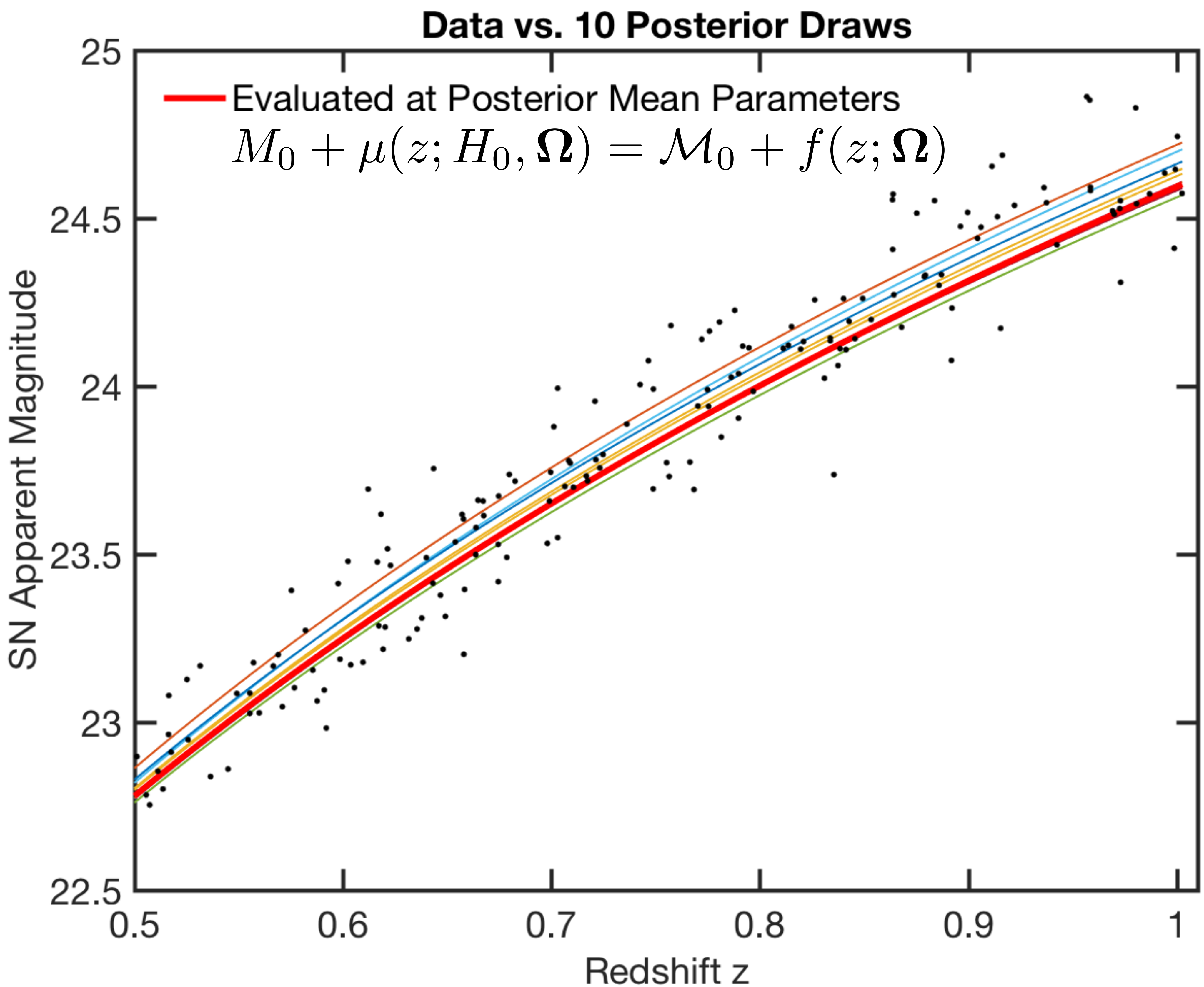
Fit with posterior mean



Random Posterior Draws of parameters from chain



Random Posterior Draws of parameters from chain



The deceleration parameter q_0

of the cosmological models. "It is a striking and slightly puzzling fact that almost all current cosmological observations can be summarized by the simple statement: The jerk of the Universe equals one" [25].

Let us make Taylor expansion of the scale factor in time using the above introduced parameters

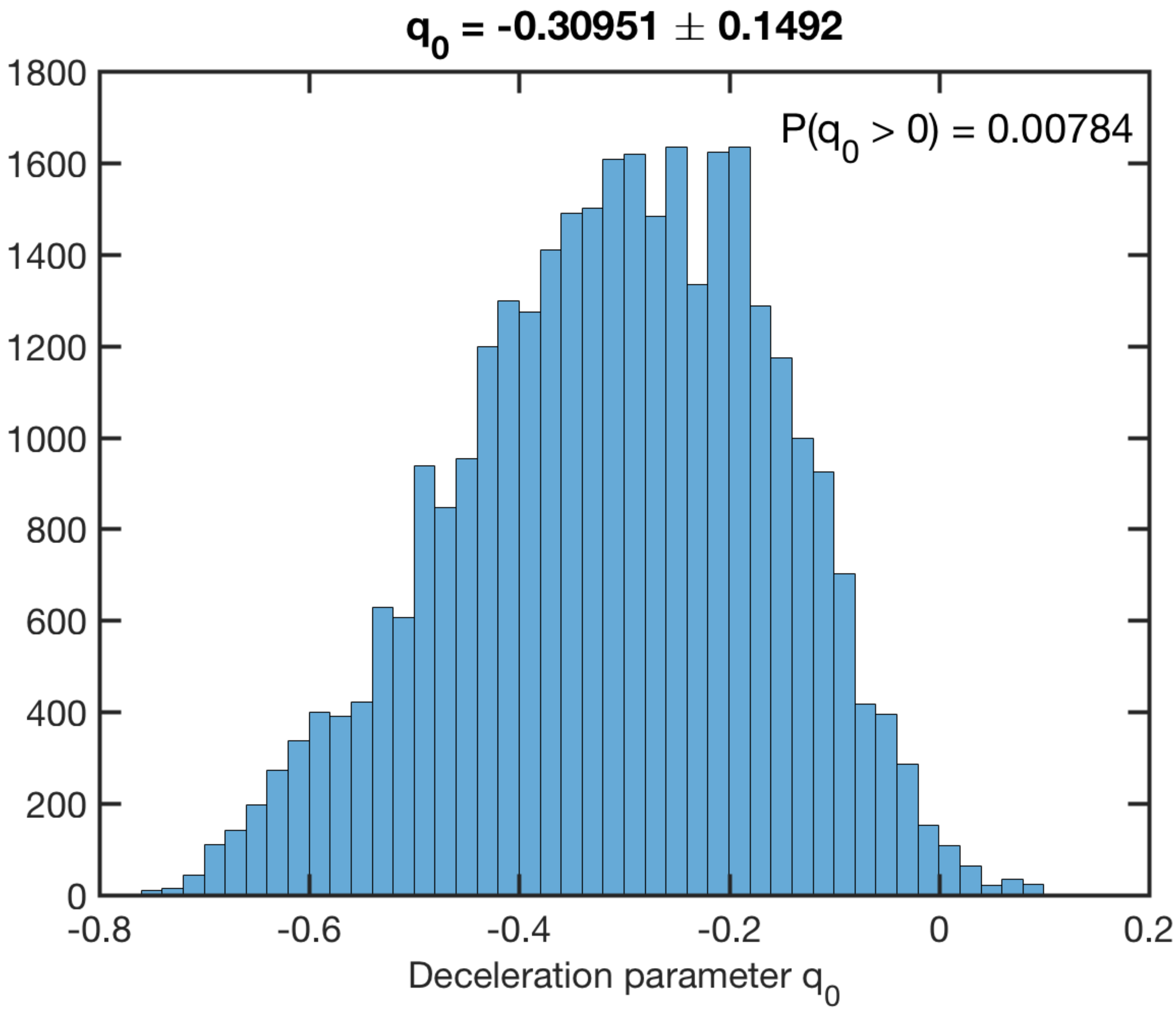
$$\begin{aligned} a(t) =& a_0 \Bigg[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t-t_0)^3 \\ &+ \frac{1}{4!} s_0 H_0^4 (t-t_0)^4 + \frac{1}{5!} l_0 H_0^5 (t-t_0)^5 + O\left((t-t_0)^6\right) \Bigg]. \end{aligned} \tag{2.49}$$

a(0) = 0 (Big Bang)
a(1) = 1 (Today)

$$q_0 = \Omega_M/2 - \Omega_\Lambda$$

The posterior of derived quantities

Deceleration Parameter: $q_0 = \Omega_M/2 - \Omega_\Lambda$



$q_0 < 0$
Expansion of Universe is accelerating!

Metropolis-Hastings Algorithm: More General Jumping Rule: $J(\theta^*|\theta_i)$

[Need not be symmetric: $J(\theta_a \mid \theta_b) \neq J(\theta_b \mid \theta_a)$]

1. Choose a random starting point θ_0

2. At step i = 1…N, propose a new parameter value: $\theta^* \sim J(\theta^*|\theta_{i-1})$

3. Evaluate M-H ratio of posteriors at proposed vs current values.
   $r = [P(\theta^* \mid \mathbf{y}) / J(\theta^*|\theta_{i-1})] / [P(\theta_{i-1} \mid \mathbf{y}) / J(\theta_{i-1}|\theta^*)]$

4. Accept θ^* with probability min(r,1): $\theta_i = \theta^*$. If not accept, stay at same value $\theta_i = \theta_{i-1}$ & include in chain.

5. Repeat steps 2-4 until reach some measure of convergence and gather enough samples to compute your inference

Metropolis-Hastings Algorithm: More General Jumping Rule: $J(\theta^*|\theta_i)$

[Need not be symmetric: $J(\theta_a \mid \theta_b) \neq J(\theta_b \mid \theta_a)$]

- d-dim Metropolis is just a special case, where $J(\theta^* \mid \theta_i) = N(\theta^* \mid \theta_i, \mathbf{\Sigma_p}) = N(\theta_i \mid \theta^*, \mathbf{\Sigma_p}) = J(\theta_i \mid \theta^*)$ is a symmetric proposal distribution

- More general asymmetric proposals, allow “biased” proposals —> more probable to propose towards a certain direction

- With some knowledge of structure of the posterior, can sometims engineer a clever proposal $J(\theta^*|\theta_i)$

Gibbs Sampling

- Multi-dimensional sampling, when you can utilise the set of conditional posterior distributions.

- If joint posterior is $P(\theta, \phi | \boldsymbol{D})$

- And you can solve for tractable conditionals:

$$P(\theta | \, \phi, \boldsymbol{D})$$

$$P(\phi | \, \theta, \boldsymbol{D})$$

- Jump along each parameter-dimension one at a time

2-dim Gibbs Sampler

1. Choose a random starting point (θ^0, ϕ^0)

2. At cycle t, update $\theta^t \sim P(\theta | \, \phi^{t-1}, \boldsymbol{D})$

3. Then update $\phi^t \sim P(\phi | \, \theta^t, \boldsymbol{D})$

   (Each complete set (pair) of updates is called a Gibbs cycle)

4. Record current values of chain (θ^t, ϕ^t)

5. Repeat steps 2-4, until reach some measure of convergence (G-R) and gather enough independent samples to compute your inference (reduce Monte Carlo error)

d-dim Gibbs Sampler

Parameter vector $\boldsymbol{\theta} = (\theta_1 \dots \theta_d)$

Current state at j-th update within cycle t: $(\boldsymbol{\theta}^t_j, \boldsymbol{\theta}^t_{-j})$

$$\boldsymbol{\theta}^t_{-j} \equiv (\theta_1^{t+1}, \dots, \theta_{j-1}^{t+1}, \theta_{j+1}^t, \dots, \theta_d^t)$$

1. Choose a random starting point θ_0.

2. In each cycle t, update through the d-parameters:
   For each j = 1… d, move jth parameter to

$$\theta_j \sim P(\theta_j | \boldsymbol{\theta}^t_{-j}, \boldsymbol{D})$$

   (update θ_j conditional on current values of all other parameters)

3. After updating all d parameters, record current state: $\boldsymbol{\theta}^{t+1}$

4. Repeat steps 2-3 until reach convergence and enough samples

Gibbs Sampling: Example (Gelman BDA Section 11.1)

Likelihood: $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$ $\quad \rho$ known

Priors: $P(\theta_1) = P(\theta_2) \propto 1$

Posterior: $\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \Big| \boldsymbol{y} \sim N\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$

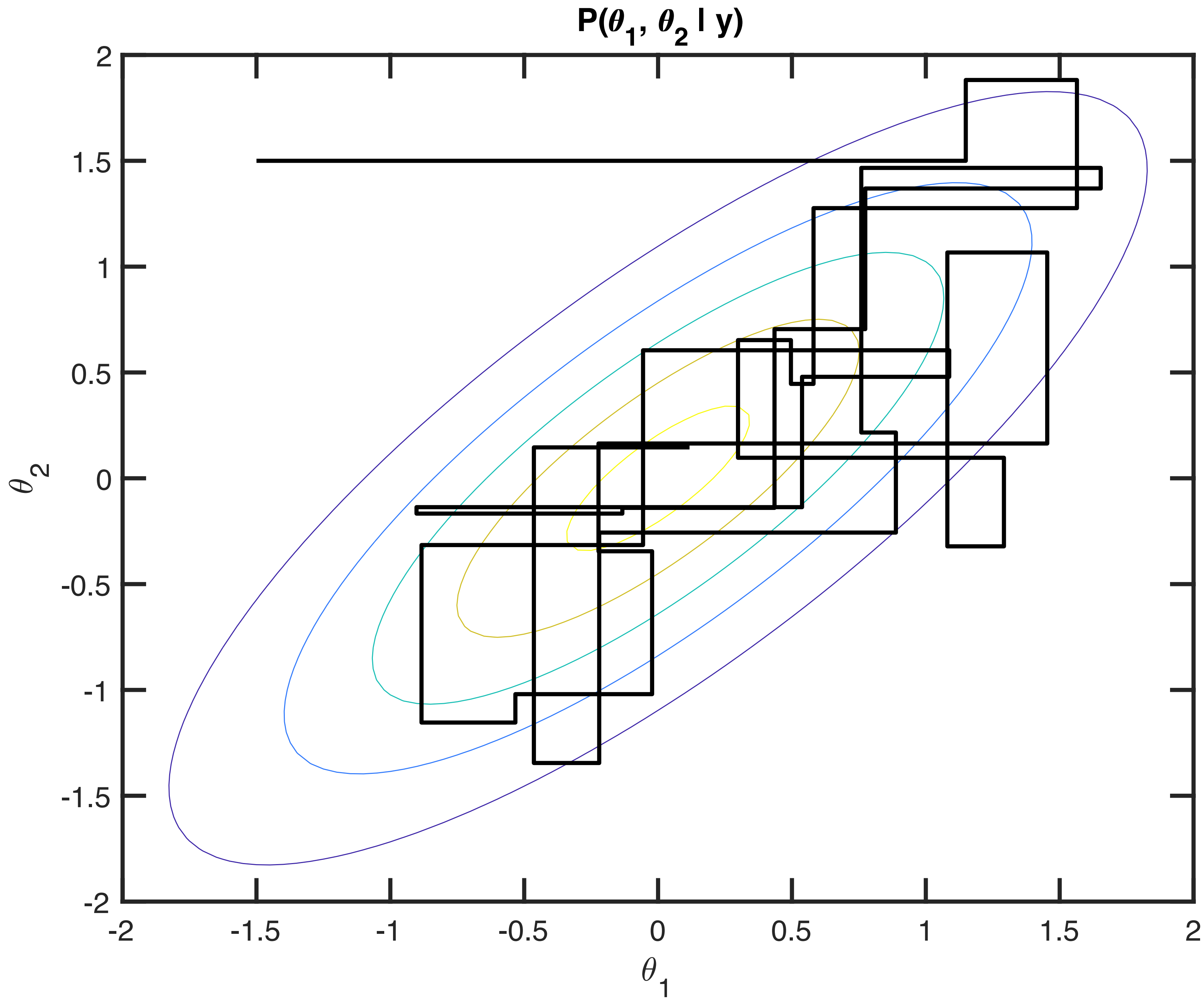
$$P(\boldsymbol{\theta}|y) = P(\theta_1|\theta_2, y)P(\theta_2|y) = P(\theta_2|\theta_1, y)P(\theta_1|y)$$

Conditional Posteriors:

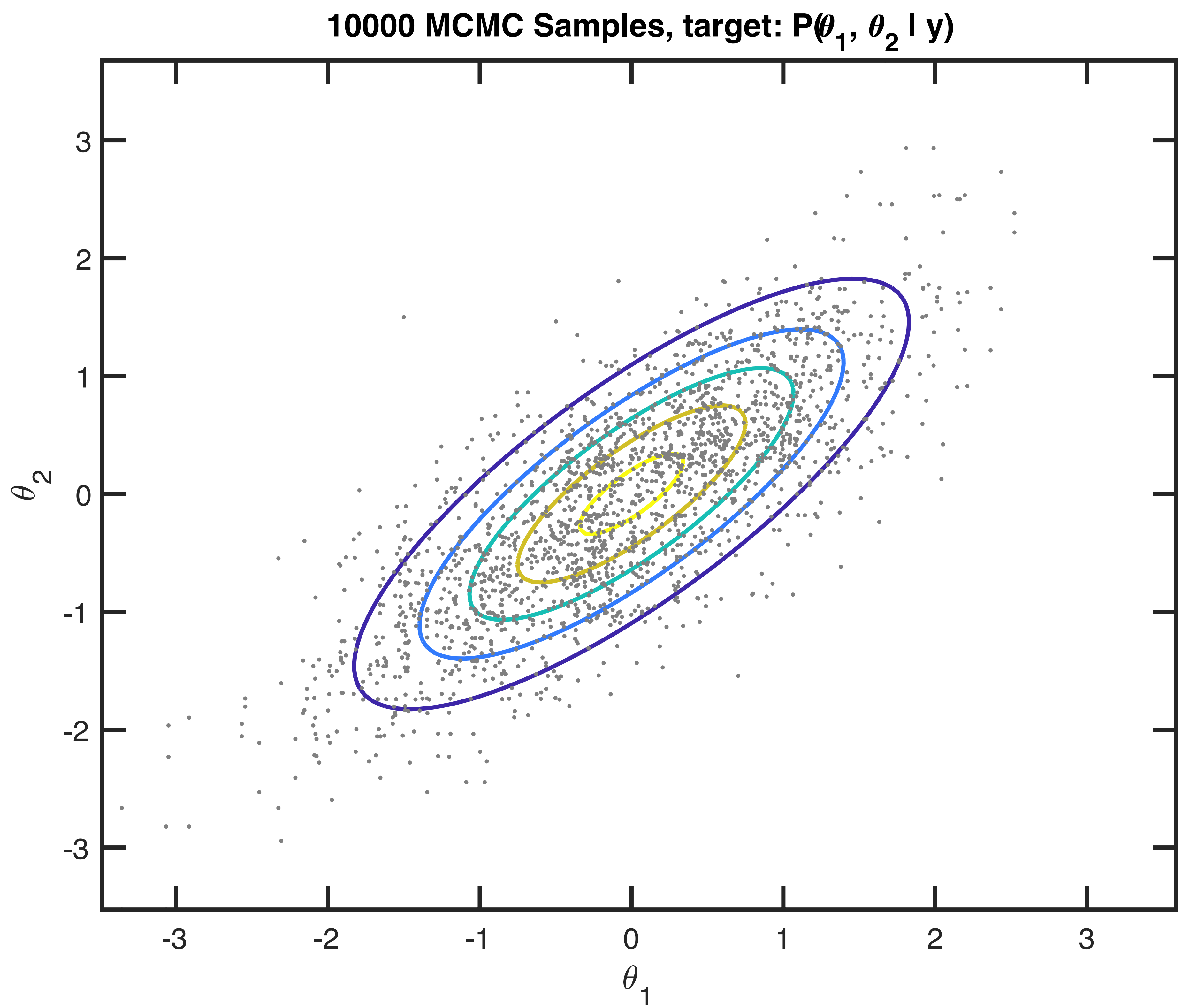
$$\theta_1|\theta_2, \boldsymbol{y} \sim N(y_1 + \rho(\theta_2 - y_2), 1 - \rho^2)$$

$$\theta_2|\theta_1, \boldsymbol{y} \sim N(y_2 + \rho(\theta_1 - y_1), 1 - \rho^2)$$

Gibbs Sampling: demo
gibbs_example.m
2D Trace Paths for 50 iterations



Gibbs Sampling: demo
gibbs_example.m (demo different ρ's)
Joint Posterior Densities



Gibbs Sampling: demo
gibbs_example.m
Marginal Posterior Densities