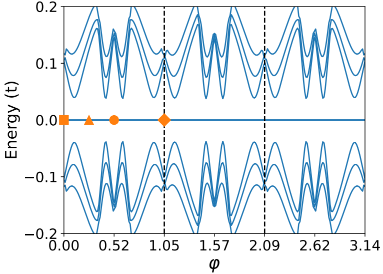
Emergent Topological Phenomena in Low-Dimensional Systems Induced by Gauge Potentials



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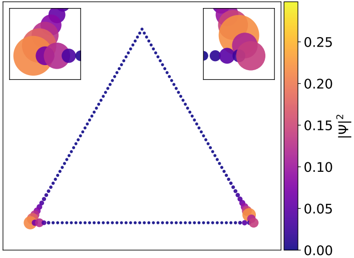
March 14, 2025



Built in LATEX

Presentation objectives

Part I: Topological superconducting triangular islands

- Construct a new platform for topological quantum logic gates via:
  - Triangular geometry
  - Applied gauge potentials

Part II: Floquet quantum Hall effect

- Induce quantum Hall effect with inhomogeneous, circularly polarized light via:
  - Floquet theorem
  - High-frequency approximation

Part I: Topological superconducting triangular islands outline

- Background:
  - What does topology offer for quantum computing
  - What are Majorana fermions and where to find them
  - Kitaev chain and Majorana number
- Motivation:
  - Braiding and topological quantum computing
  - T-junctions to triangular structures
- Results:
  - Minimal Kitaev triangle
  - Hollow triangular islands
  - Minimal braiding example
- Summary

What does topology offer for quantum computing?

- Quantum computers have two prominent "local" errors (noise, perturbations).
  - Classical error: flips a qubit from empty to occupied, $|0\rangle \to |1\rangle$, or vice versa.
  - Phase error: changes the sign of an occupied qubit, $|1\rangle \to -|1\rangle$.
  - Normal qubits have various means for reducing error and achieving fault-tolerance.

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  - Their qubit information is stored “non-locally”, making the above errors unlikely to occur.
  - What system allows for topological qubits?

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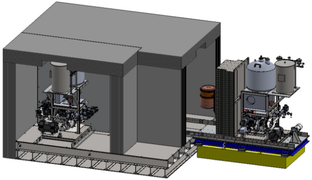
Majorana fermions are candidates for building topological quantum computers.

What are Majorana fermions?

Ettore Majorana



MAJORANA project

- Fermions
  - Fermi-Dirac statistics
  - Pauli exclusion principle
  - Half-odd-integer spin
- Majorana Fermions (MFs)
  - Particle = Antiparticle
  - Neutral, Massive
  - Negative results for Majorana particles as elementary particles

Where to find Majorana fermions?

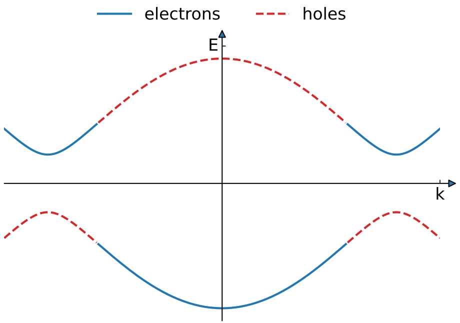
- Superconductors (SCs)
  - Cooper pairs
    - Electron-phonon interaction pairs two electrons with opposite spin and momenta.

Where to find Majorana fermions?

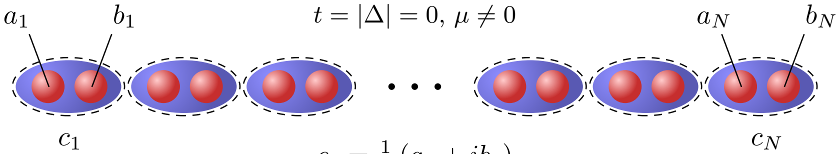
- Superconductors (SCs)
  - Cooper pairs
    - Electron-phonon interaction pairs two electrons with opposite spin and momenta.
  - Bogoliubov quasiparticles
    - Excitation, pairs an electron to a hole with opposite momenta.

$$H_{BdG} = \begin{bmatrix} \epsilon(k) & \Delta(k) \\ \Delta^*(k) & -\epsilon(-k) \end{bmatrix}$$

    - Zero-energy excitations in non-tivial *p*-wave SC may be MFs.
    - Come in pairs.



Kitaev chain - 1D p-wave superconducting wire



$t = |\Delta| = 0,\ \mu \neq 0$

$c_j = \frac{1}{2}\left(a_j + ib_j\right)$

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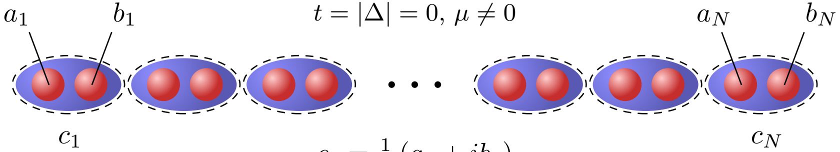
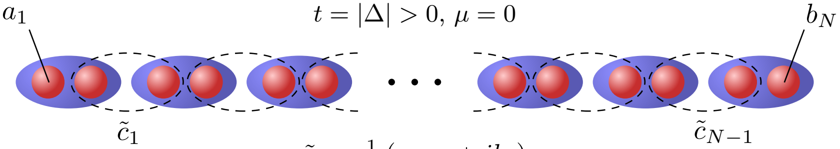


$$c_j = \tfrac{1}{2}\left(a_j + i b_j\right)$$

$t = |\Delta| > 0,\ \mu = 0$



$$\tilde{c}_j = \tfrac{1}{2}\left(a_{j+1} + i b_j\right)$$

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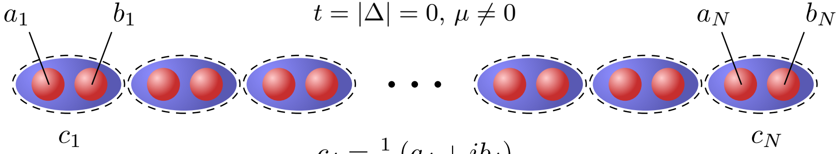

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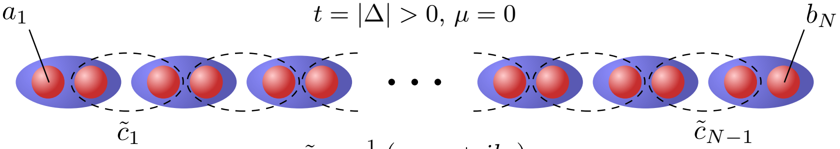
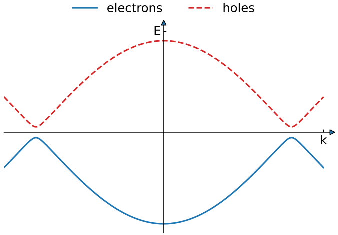


$$\tilde{c}_j = \frac{1}{2}\left(a_{j+1} + i b_j\right)$$

$$\boxed{f = \frac{1}{2}\left(a_1 + i b_N\right)}$$

Band gap and topological phase

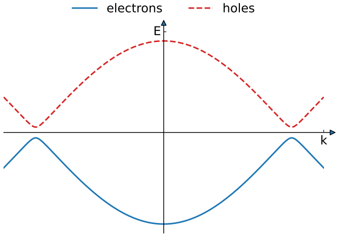


$$\mu = 2.25t$$

$$\mathcal{M} = 1$$

Trivial

Band gap and topological phase



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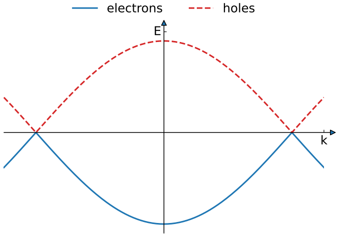
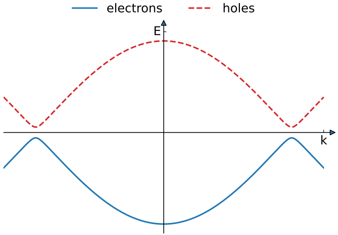


$\mu_c = 2t$

Band gap and topological phase



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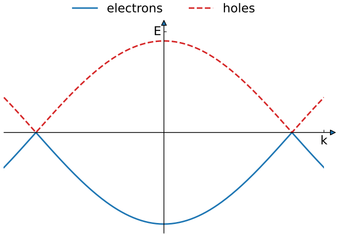

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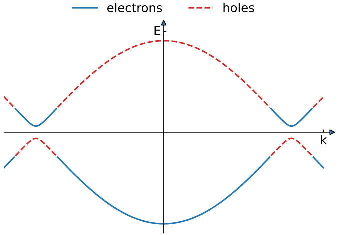
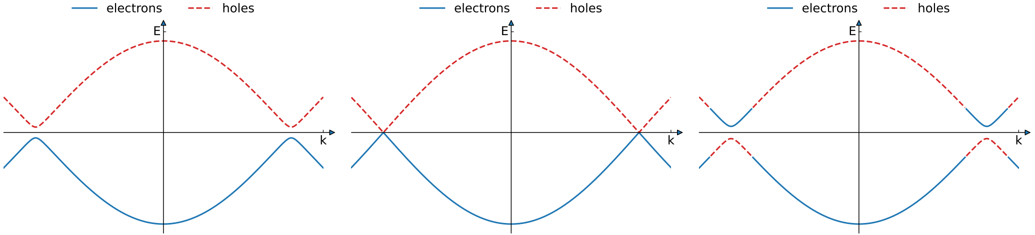


$\mu = 1.75t$

$\mathcal{M} = -1$

Non-trivial



$\mu = 2.25t$	$\mu_c = 2t$	$\mu = 1.75t$
$\mathcal{M} = 1$		$\mathcal{M} = -1$
Trivial		Non-trivial

<u>Non-trivial phase</u>

larger band gap $\rightarrow$ more robust against local errors $\rightarrow$ stronger topological protection

Transform to Majorana basis

$$A = -iU\mathcal{H}U^{\dagger}$$

Majorana number

$$\mathcal{M} = \mathsf{sgn}[\mathsf{Pf}(A)]$$

- If $|\mu| > 2t$, $\mathcal{M} = +1$, trivial topology
- If $|\mu| < 2t$, $\mathcal{M} = -1$, non-trivial topology

Majorana number & bulk-edge correspondence

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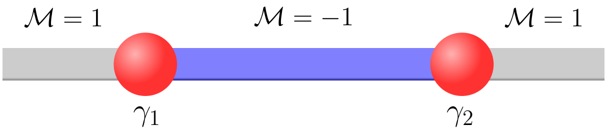

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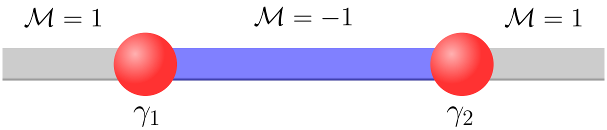
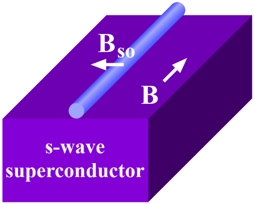
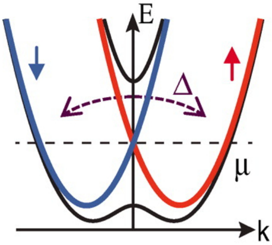
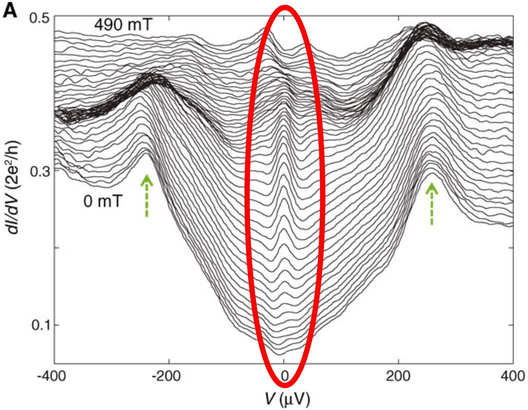


Warning! There are no *p*-wave SCs currently, but we can build effective *p*-wave SCs!

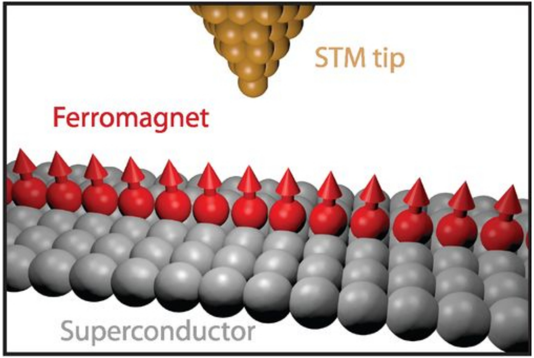


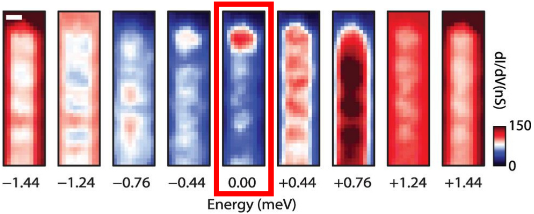




Mourik et al., *Science* **336**, 1003 (2012)

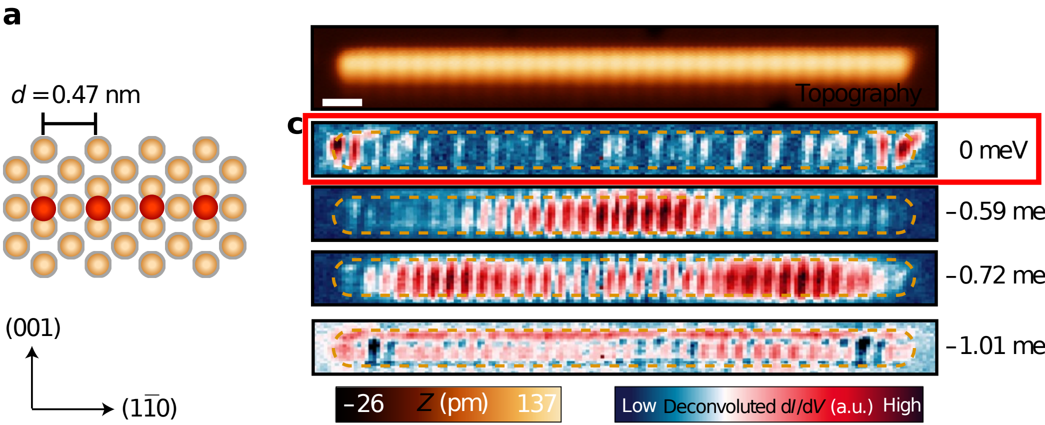
Experimental evidence of Majorana fermions II





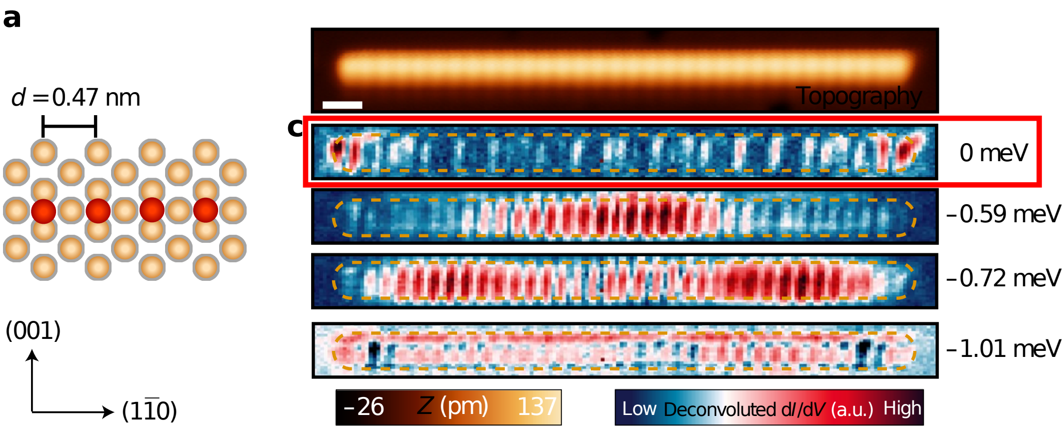
Nadj-Perge et al., *Science* **346**, 602 (2014)

Experimental evidence of Majorana fermions III



Mn atoms (red spheres) on top of superconducting Nb (brown spheres).

Schneider et al., *Nature Nanotechnology* **17**, 384 (2022)



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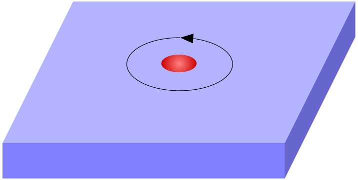
Schneider et al., *Nature Nanotechnology* **17**, 384 (2022)

Zero-energy states could be Andreev bound states. "Braiding" can distinguish them.

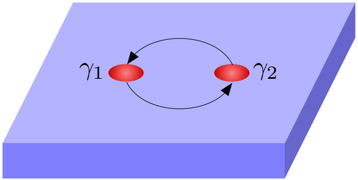
- 2D p-wave (triplet pairing) SC exhibit half-quantum vortices.



What makes Majorana fermions so cool?

- 2D *p*-wave (triplet pairing) SC exhibit half-quantum vortices.
- MF state accumulate $e^{i\theta/2}$ phase.

$$\gamma_1 \rightarrow -\gamma_2$$
$$\gamma_2 \rightarrow \gamma_1$$



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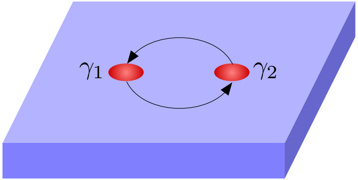

$$U_{12} = \exp\left(\pm\frac{\pi}{4}\gamma_1\gamma_2\right)$$

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- Combine with MFs topological protection,
  - $\rightarrow$ Topological quantum computer.
  - $\rightarrow$ Benefits: fewer repeated operations compared to normal qubits.

$$\gamma_1 \rightarrow -\gamma_2$$
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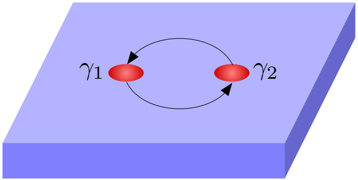

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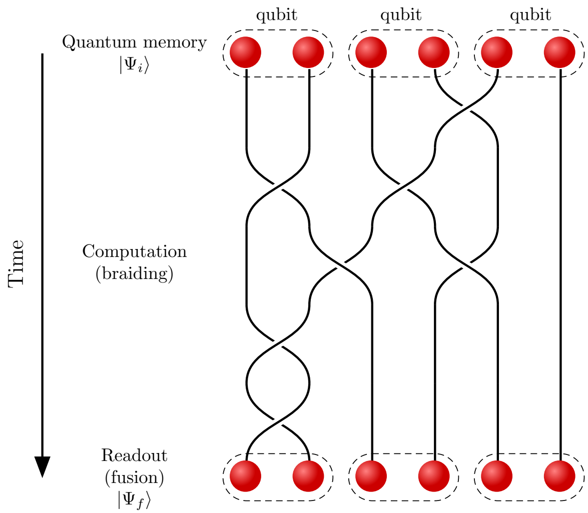
Braiding in a topological quantum computer

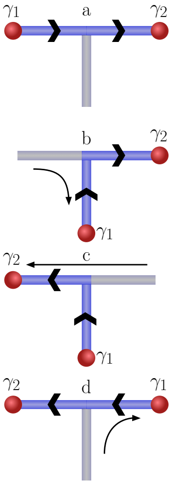


Generalized braiding of two neighboring MFs

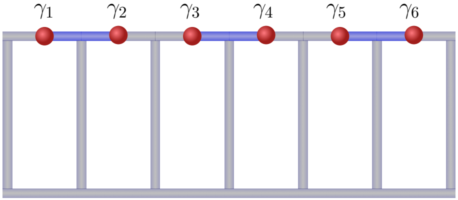
$$U_{nm} = \exp\left(\pm\frac{\pi}{4}\gamma_n\gamma_m\right)$$

Readout after braiding

$$|\Psi_f\rangle = U_{n'm'}\cdots U_{nm}|\Psi_i\rangle$$

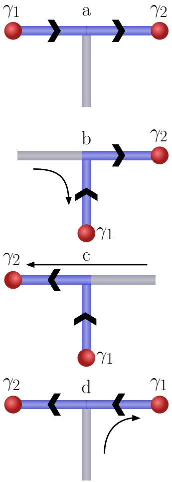


- Take pairing term $|\Delta|e^{i\phi}c_jc_{j+1}$ such that the site indices:
- Increase moving $\rightarrow$ / $\uparrow$ in the horizontal/vertical wires: $\phi = 0$,
- Decrease moving $\leftarrow$ / $\downarrow$ in the horizontal/vertical wires: $\phi = \pi$.



Alicea et al., *Nature Phys.* **7**, 412 (2011)

T-junction as a quantum logic gate



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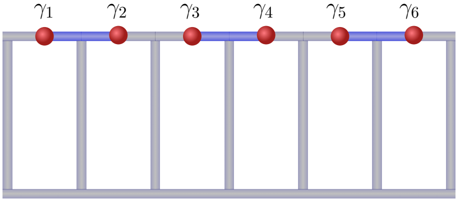


Alicea et al., *Nature Phys.* **7**, 412 (2011)

Difficult to build, manipulate, and read.

Not a smooth connection between 1D and 2D SCs.

Triangular structures for braiding

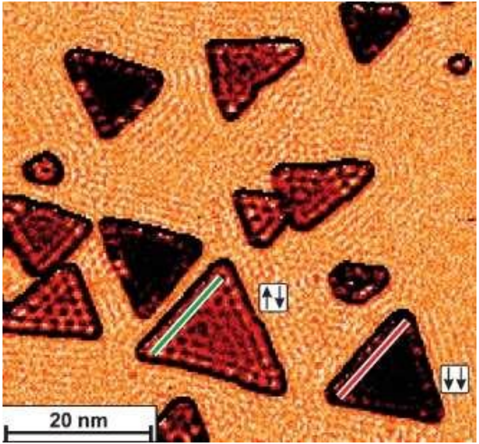
- T-junction $\rightarrow$ Triangle

  3 endpoints $\rightarrow$ 3 vertices

Triangular structures for braiding

- T-junction → Triangle

  3 endpoints → 3 vertices
- Islands of three-fold rotational symmetry occur naturally in epitaxial growth on close-packed metal surfaces.



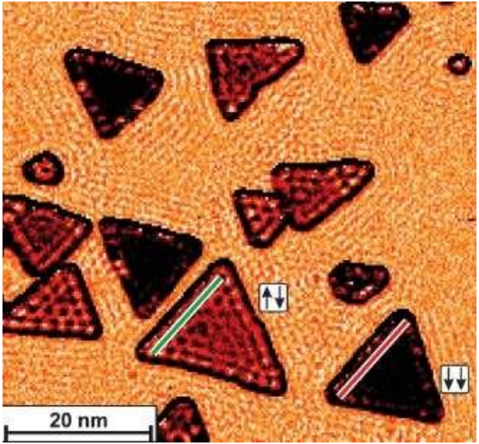
Triangular Co islands on Cu(111).

Pietzsch et al., *PRL* **96**, 237203 (2006)

Triangular structures for braiding

- T-junction $\rightarrow$ Triangle

  3 endpoints $\rightarrow$ 3 vertices
- Islands of three-fold rotational symmetry occur naturally in epitaxial growth on close-packed metal surfaces.
- In plane gauge potentials can break a triangle's 3-fold symmetry.
- Make a smoother connection between 1D and 2D superconductors.



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Two proposals

- Exactly solvable minimal "Kitaev Triangle"
  - Three fermion sites compared to minimal four in T-junctions
  - Three edges controlled by Peierls phase

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  - Three fermion sites compared to minimal four in T-junctions
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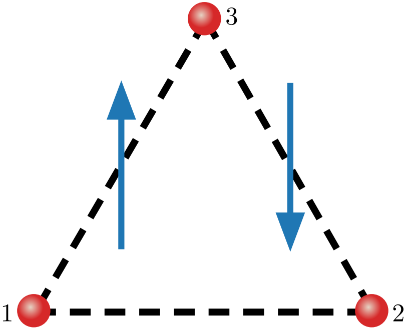
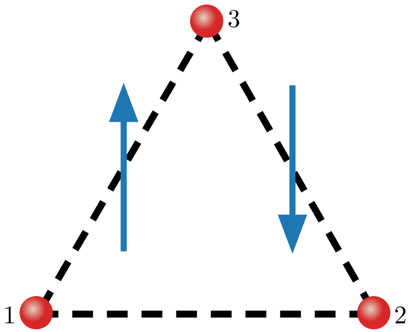
- Finite-size triangle with hollow interior
  - Uniform gauge potential
  - Bulk-edge correspondence

Kitaev triangle



$$\mathbf{A} = [1 - 2\Theta(x)]\,\frac{2\pi}{3\sqrt{3}}\hat{\mathbf{y}}$$

Kitaev triangle



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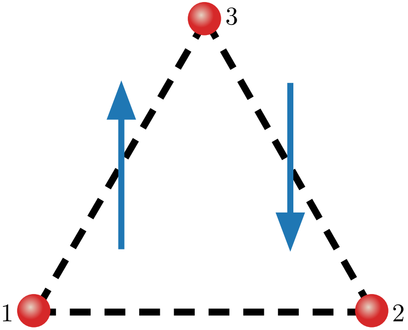
$$\phi_{jl} = \frac{ie}{\hbar}\int_{r_j}^{r_l} \mathbf{A}\cdot d\mathbf{l}$$

$$\mathcal{H} = \sum_{\langle j,l\rangle} (-te^{i\phi_{jl}} c_j^\dagger c_l + \Delta c_j^\dagger c_l^\dagger + h.c.) - \mu c_j^\dagger c_j$$



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In Kitaev limit, $t = \Delta \neq 0$ and $\mu = 0$,

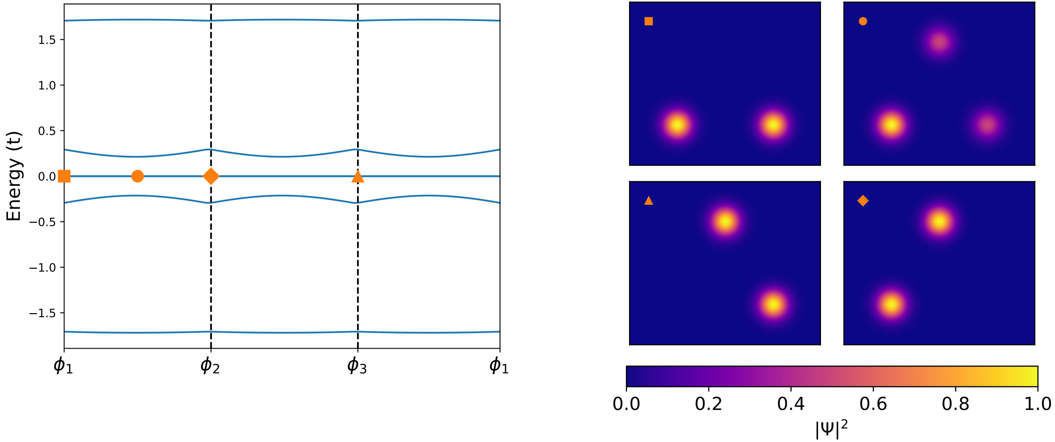
$$(\phi_{12}, \phi_{23}, \phi_{31}) = (0, -\tfrac{\pi}{3}, -\tfrac{\pi}{3}) = \boldsymbol{\phi}_1$$

MZMs localized at sites 1 and 2

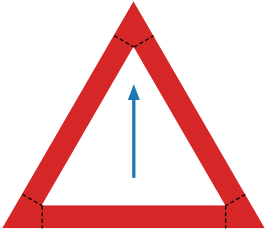
Kitaev triangle braiding

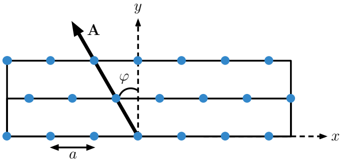
A closed parameter path linearly interpolated between the following sets of ϕ_{jk}:

$$(\phi_{12}, \phi_{23}, \phi_{31}) : \boldsymbol{\phi}_1 \rightarrow \boldsymbol{\phi}_2 \rightarrow \boldsymbol{\phi}_3 \rightarrow \boldsymbol{\phi}_1 \tag{1}$$



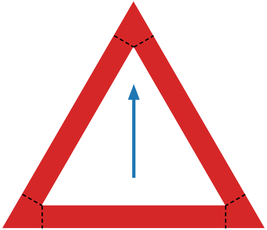
$$\phi_{jl} = \frac{e}{\hbar}\int_{\mathbf{r}_j}^{\mathbf{r}_l} \mathbf{A}\cdot d\mathbf{l} = \mathbf{A}\cdot\mathbf{r}_{jl} = -\phi_{lj}$$

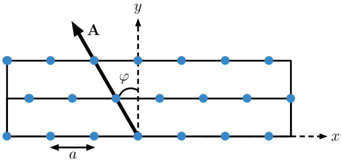


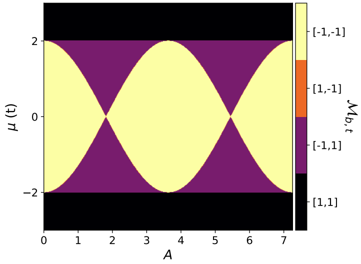


Triangular ribbon and topological phases

$$\phi_{jl} = \frac{e}{\hbar}\int_{\mathbf{r}_j}^{\mathbf{r}_l} \mathbf{A}\cdot d\mathbf{l} = \mathbf{A}\cdot\mathbf{r}_{jl} = -\phi_{lj}$$

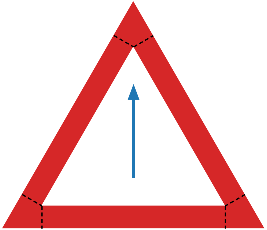


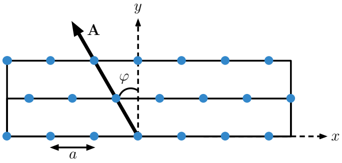


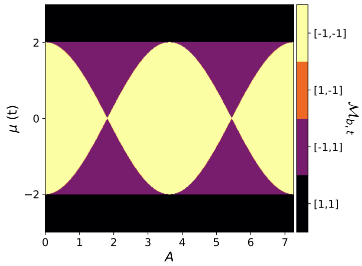


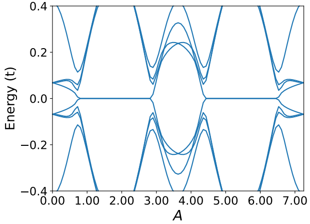
Triangular ribbon and topological phases

$$\phi_{jl} = \frac{e}{\hbar}\int_{\mathbf{r}_j}^{\mathbf{r}_l} \mathbf{A}\cdot d\mathbf{l} = \mathbf{A}\cdot\mathbf{r}_{jl} = -\phi_{lj}$$

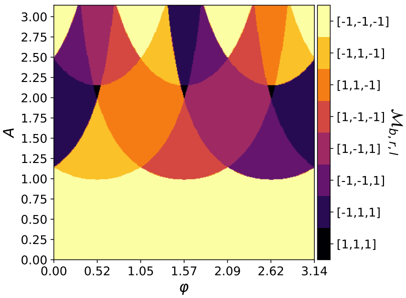








Braiding Majorana zero modes triangular chain (W=1)



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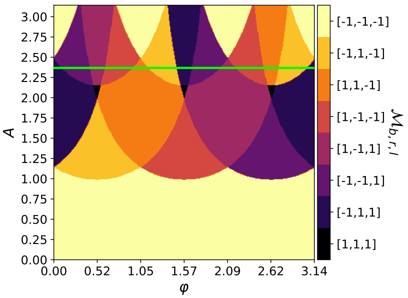

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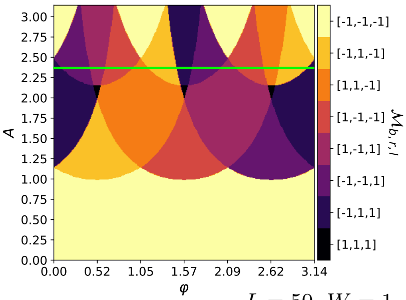

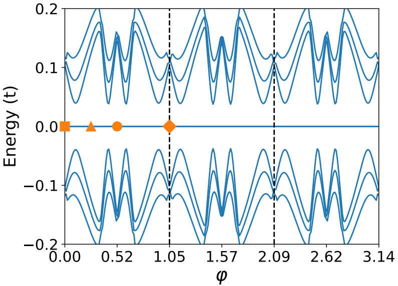

$L = 50,\ W = 1,\ \mu = 1.1,\ A = 2.35$

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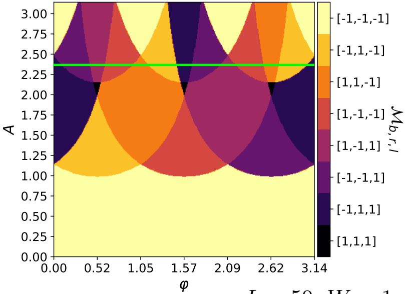

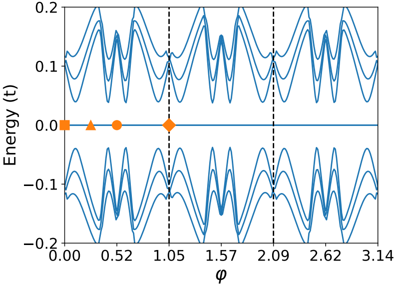

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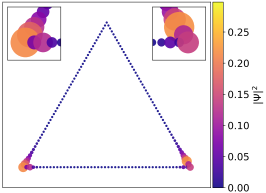

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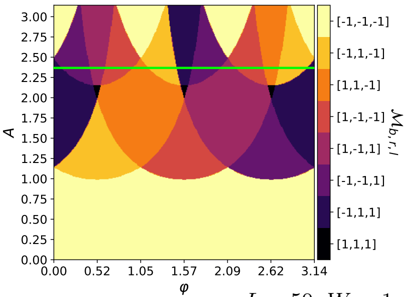

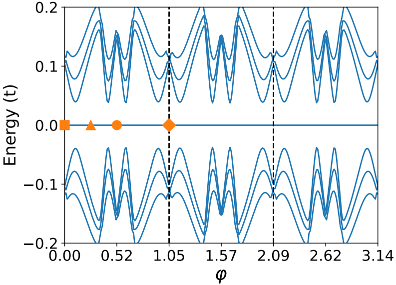

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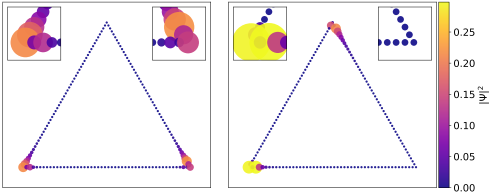

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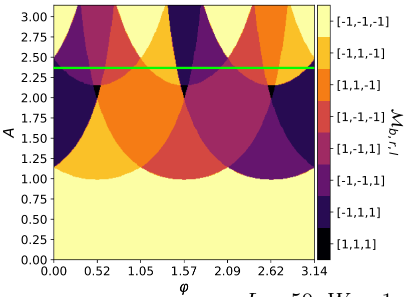

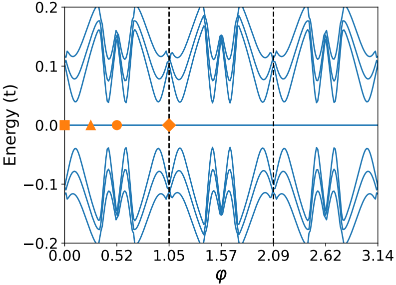

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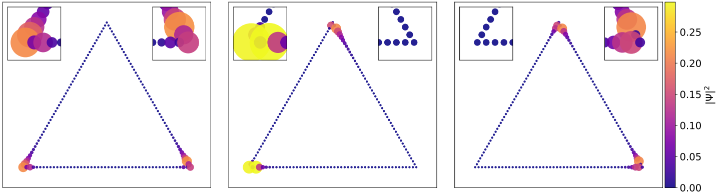

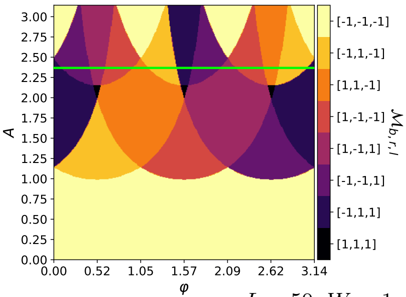

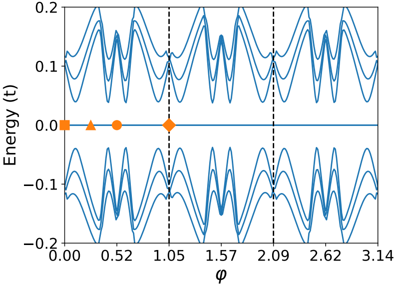

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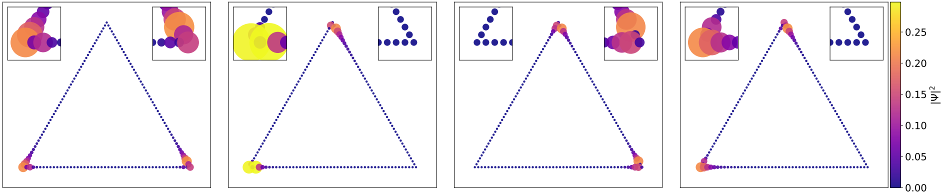
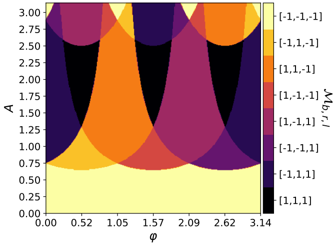
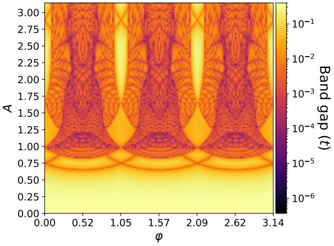


Braiding Majorana zero modes hollow triangle (W=3)





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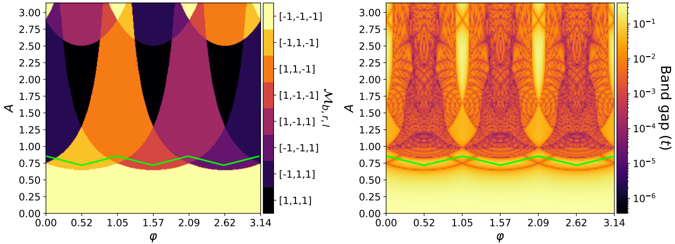

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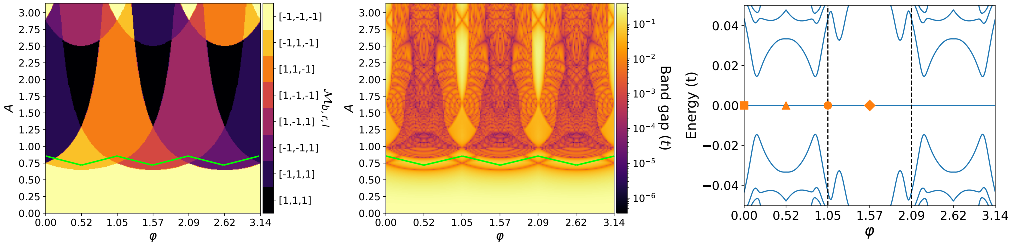
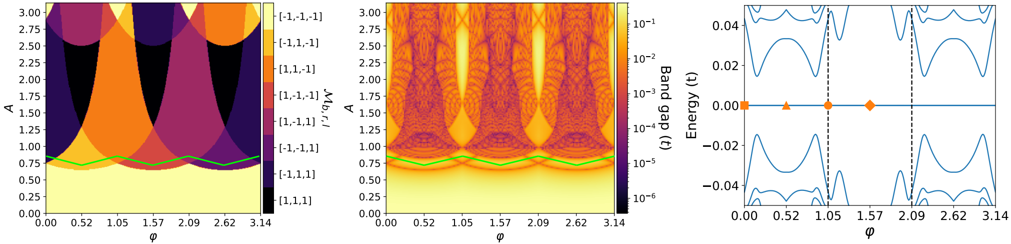


$$L = 80,\ W = 3,\ \mu = 1.6,\ (A, \varphi) = (0.83, 0) \to (0.77, \tfrac{\pi}{6}) \to (0.83, \tfrac{\pi}{3}) \to (0.77, \tfrac{\pi}{2}) \dots$$

Braiding Majorana zero modes hollow triangle (W=3)



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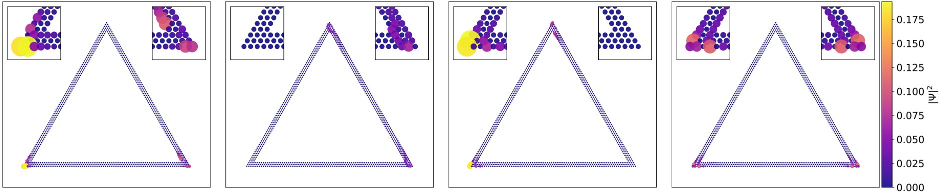
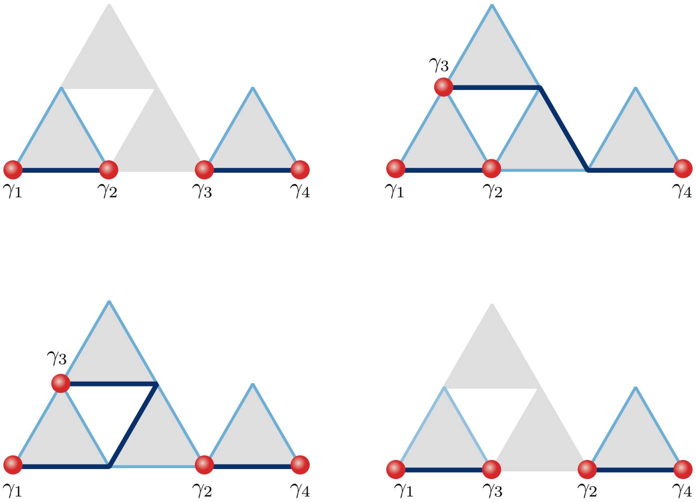


Braiding Two of Four Majorana zero modes



Part I: Summary

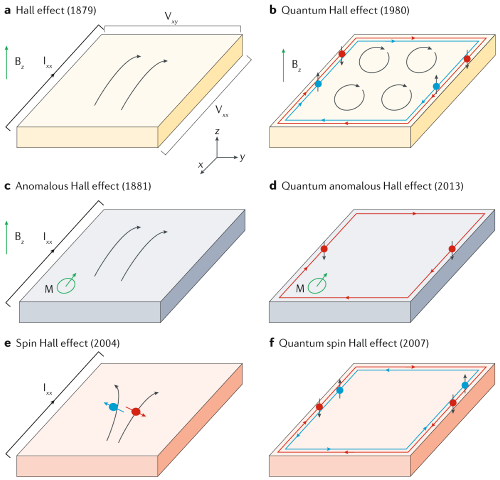
- A minimal Kitaev triangle can emerge as an effective theory for three fermionic sites and is sufficient for braiding.
- Hollow triangles maintain 1D bulk-edge correspondence, gauge potential strength and rotation allows for additional topological tunability, and demonstrates braiding.
- MZMs can be hosted and braided on a network of triangular islands.

Aidan Winblad and Hua Chen, *PRB* **109**, 205158 (2024).

Part II: Floquet quantum Hall effect outline

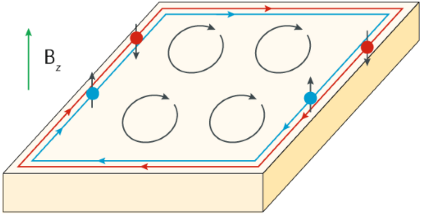
- Background & Motivation:
  - Quantum Hall effect
  - Floquet engineering and nonequilibrium physics
- Formulation:
  - Floquet theorem and high-frequency approximation
  - Inhomogeneous, circularly polarized light
- Results:
  - Dirac and 2DEG systems
- Summary

Quantum Hall effect at the Hall effect Zoo



von Klitzing et al. *Nat. Rev. Phys.* **2**, 397-401 (2020)

Quantum Hall effect at the Hall effect Zoo



Dirac: $$\begin{cases} \mathcal{H}^D = v_F \boldsymbol{\sigma} \cdot (\mathbf{p} + eBx\hat{\mathbf{y}}) \\ \epsilon_n^D = \pm v_F \sqrt{2n\hbar eB} \end{cases}$$

2DEG: $$\begin{cases} \mathcal{H}^{\text{2DEG}} = \frac{1}{2m^*}(\mathbf{p} + eBx\hat{\mathbf{y}})^2 \\ \epsilon_n^{\text{2DEG}} = \frac{\hbar eB}{m^*}\left(n + \frac{1}{2}\right) \end{cases}$$

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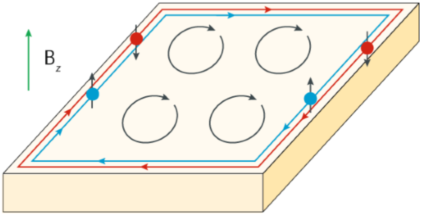

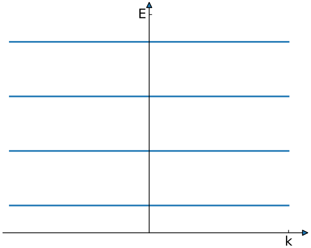

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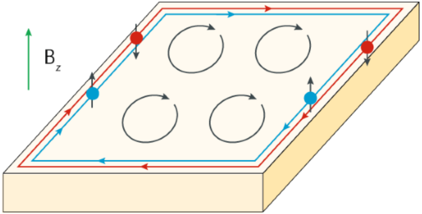

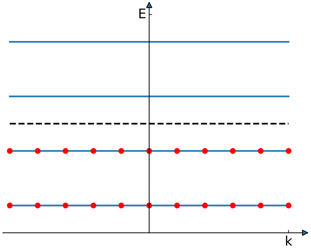

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QHE has quantized Hall conductance:

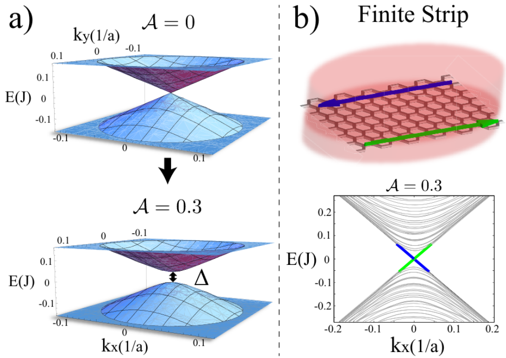
$$\sigma_H = -\frac{Ce^2}{h}$$

Floquet exhibit at the Hall effect Zoo

Pioneer research on nonequilibrium systems exhibiting equilibrium physics. Circularly polarized light provides time periodicity to use Floquet theorem and high-frequency approximation.

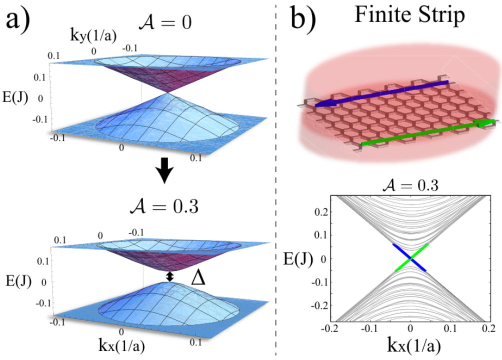
- Photovoltaic Hall effect in graphene
  - Oka et al., *PRB* **79**, 0814406(R) (2009)
- Floquet topological insulator in semiconductor quantum wells
  - Lindner et al., *Nat. Phys.* **7**, 490-495 (2011)
- Observations of Floquet-Bloch states on the surface of a topological insulator
  - Wang et al., *Science* **342**, 453-457 (2013)
- Floquet Fractional Chern Insulators
  - Grushin et al., *PRL* **112**, 156801 (2014)

Floquet quantum anomalous Hall effect

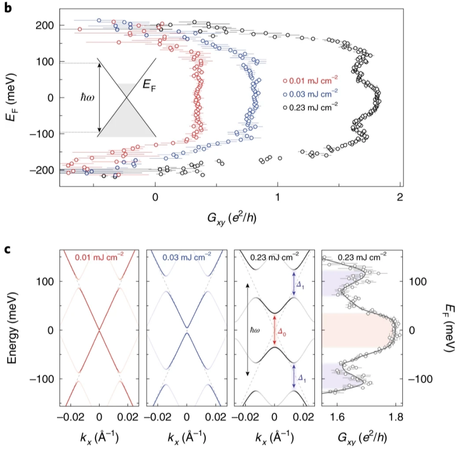


Kitagawa et. al., *PRB* **84**, 235108 (2011)

Floquet quantum anomalous Hall effect



Kitagawa et. al., *PRB* **84**, 235108 (2011)



McIver et. al., *Nature Phys.* **16**, 38 (2020)

Analogous to Bloch's theorem

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Exhibits discrete time translation symmetry

$$H(t+T) = H(t)$$

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Exhibits discrete time translation symmetry

$$H(t+T) = H(t)$$

For such a system, Floquet theorem must hold

$$\psi_\epsilon(t) = e^{-i\epsilon t/\hbar} u_\epsilon(t)$$
$$u_\epsilon(t+T) = u_\epsilon(t)$$

Let wavefunction be an arbitrary wave approximation

$$\psi(t) = \sum_\epsilon u_\epsilon e^{-i\epsilon t/\hbar}$$

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Discrete time Fourier series of Hamiltonian

$$H(t) = \sum_n H_n e^{in\omega t}$$

Where

$$H_n = \frac{1}{T}\int_0^T H(t) e^{-in\omega t}$$

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Then, $i\hbar\partial_t\psi(t) = H(t)\psi(t)$ becomes

$$\bar{Q}_{m,m+n} = H_n - m\hbar\omega\delta_{n0}$$

Floquet Theorem

Analogous to Bloch's theorem

Exhibits discrete time translation symmetry

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For such a system, Floquet theorem must hold

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Then, $i\hbar\partial_t \psi(t) = H(t)\psi(t)$ becomes

$$\bar{Q}_{m,m+n} = H_n - m\hbar\omega\delta_{n0}$$

$\bar{Q}$ is an infinite quasienergy matrix! How do we handle such a matrix?

$$\bar{Q} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & H_0 - \hbar\omega & H_{-1} & H_{-2} & \cdots \\ \cdots & H_1 & H_0 & H_{-1} & \cdots \\ \cdots & H_2 & H_1 & H_0 + \hbar\omega & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Quasienergy operator and high-frequency approximation

$$\bar{Q} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & H_0 - \hbar\omega & H_{-1} & H_{-2} & \cdots \\ \cdots & H_1 & H_0 & H_{-1} & \cdots \\ \cdots & H_2 & H_1 & H_0 + \hbar\omega & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \to \boxed{H_{\pm n} \ll \hbar\omega} \to$$

Quasienergy operator and high-frequency approximation

$$\bar{Q} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & H_0 - \hbar\omega & H_{-1} & H_{-2} & \cdots \\ \cdots & H_1 & H_0 & H_{-1} & \cdots \\ \cdots & H_2 & H_1 & H_0 + \hbar\omega & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \rightarrow \boxed{H_{\pm n} \ll \hbar\omega} \rightarrow \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & H_{\text{eff}} - \hbar\omega & 0 & 0 & \cdots \\ \cdots & 0 & H_{\text{eff}} & 0 & \cdots \\ \cdots & 0 & 0 & H_{\text{eff}} + \hbar\omega & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Quasienergy operator and high-frequency approximation

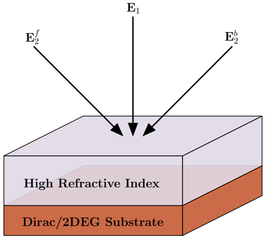
$$\bar{Q} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & H_0 - \hbar\omega & H_{-1} & H_{-2} & \cdots \\ \cdots & H_1 & H_0 & H_{-1} & \cdots \\ \cdots & H_2 & H_1 & H_0 + \hbar\omega & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \rightarrow \boxed{H_{\pm n} \ll \hbar\omega} \rightarrow \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & H_{\text{eff}} - \hbar\omega & 0 & 0 & \cdots \\ \cdots & 0 & H_{\text{eff}} & 0 & \cdots \\ \cdots & 0 & 0 & H_{\text{eff}} + \hbar\omega & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$H_{\text{eff}} = H^{F(1)} + H^{F(2)} + H^{F(3)} + \ldots$$

$$H^{F(1)} = H_0$$

$$H^{F(2)} = \sum_{m>0} \frac{2[H_m, H_{-m}]}{m\hbar\omega}$$

$$H^{F(3)} = \sum_{m>0} \left(\frac{[H_{-m}, [H_0, H_m]] + h.c.}{2(m\hbar\omega)^2} + \sum_{m' \neq m} \frac{[H_{-m'}, [H_{m'-m}, H_m]] + h.c.}{3mm'(\hbar\omega)^2} \right)$$



Generalized electric field at substrate surface

$$\mathbf{E} = E\langle \cos{(a\omega t)}, -\cos{(Kx)}\sin{(b\omega t)}\rangle$$

where

$$K = \frac{\omega \sin{(\theta_i)}}{v_p}$$

Dirac systems

$$\mathcal{H}(t) = v_F \boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}(t))$$
$$\mathbf{E} = E\langle \cos{(\omega t)}, \sin{(Kx)} \sin{(2\omega t)} \rangle$$

Dirac systems

$$\mathcal{H}(t) = v_F \boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}(t))$$
$$\mathbf{E} = E\langle \cos(\omega t), \sin(Kx)\sin(2\omega t)\rangle$$

Use floquet theorem and high-frequency approximation, and in limit $Kx \ll 1$

$$\mathcal{H}_{\text{eff}}^{D} = v_F \sigma_x p_x + v_F \sigma_y \left(C p_y + eB^D x\right)$$
$$B^D = \frac{K v_F^2 e^2 E^3}{4\hbar^2 \omega^5}$$

2DEG systems

$$\mathcal{H}(t) = \frac{1}{2m^*}\left(\mathbf{p} + e\mathbf{A}(t)\right)^2$$
$$\mathbf{E} = E\langle\cos{(\omega t)}, -\cos{(Kx)}\sin{(\omega t)}\rangle$$

2DEG systems

$$\mathcal{H}(t) = \frac{1}{2m^*}(\mathbf{p} + e\mathbf{A}(t))^2$$

$$\mathbf{E} = E\langle \cos(\omega t), -\cos(Kx)\sin(\omega t)\rangle$$

Use floquet theorem and high-frequency approximation, and in limit $Kx \ll 1$

$$\mathcal{H}_{\text{eff}}^{\text{2DEG}} = \frac{1}{2m^*}\left[p_x^2 + \left(p_y - eB^{\text{2DEG}}x\right)^2\right]$$

$$B^{\text{2DEG}} = \frac{K^2 eE^2}{m^*\omega^3}$$

$$\mathcal{H}(t) = \frac{1}{2m^*}(\mathbf{p} + e\mathbf{A}(t))^2$$
$$\mathbf{E} = E\langle\cos(\omega t), -\cos(Kx)\sin(\omega t)\rangle$$

Use floquet theorem and high-frequency approximation, and in limit $Kx \ll 1$

$$\mathcal{H}_{\text{eff}}^{\text{2DEG}} = \frac{1}{2m^*}\left[p_x^2 + \left(p_y - eB^{\text{2DEG}}x\right)^2\right]$$
$$B^{\text{2DEG}} = \frac{K^2 e E^2}{m^* \omega^3}$$

Effective Hamiltonians are approximations, energies and states are Landau level like.

Expected analytical values and tight-binding models

- Parameters from previous literature can produce effective magnetic fields on the order of mT to hundreds of mT in Dirac and 2DEG.
- Tight-binding models were performed for both systems
  - Strong agreement with high-frequency approximation.
  - Interpreting data of lower frequencies is difficult and inconclusive when modes overlap.

Part II: Summary

- Inhomogeneous, circularly polarized light induces QHE in Dirac and 2DEG systems.
- Showed nonequilibrium systems exhibit equilibrium physics.
- Effective magnetic field can be enhanced by several parameters.

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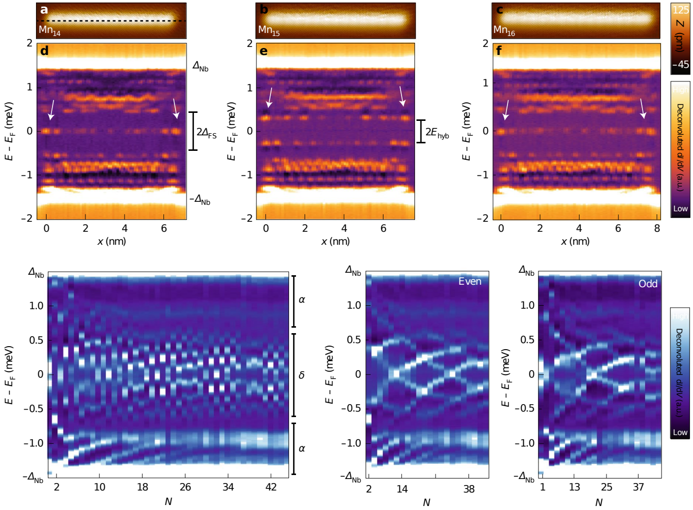


a
Mn14
b
Mn15
c
Mn16
125
Z (pm)
-45
d
e
f
ΔNb
2ΔFS
-ΔNb
2Ehyb
E - EF (meV)
x (nm)
Deconvoluted dI/dV (a.u.)
High
Low
α
δ
Even
Odd
N

Majorana fermion notation and coupling isolation

The complex fermion operator can be written as a superposition of two Majorana fermions $c_j = \frac{1}{2}(a_j + ib_j)$. Due to the nature of Majorana fermions, $a_j^\dagger = a_j$, the creation operator is $c_j^\dagger = \frac{1}{2}(a_j - ib_j)$.

$$H = -\frac{i\mu}{4}\sum_j (a_j b_j - b_j a_j) - \frac{i}{4}\sum_{<j,l>}[(t\sin\phi - \Delta\sin\theta)a_l a_j + (t\sin\phi + \Delta\sin\theta)b_l b_j + (t\cos\phi + \Delta\cos\theta)a_l b_j - (t\cos\phi - \Delta\cos\theta)b_l a_j].$$

$$(t\sin\phi_{j,l} - \Delta\sin\theta_{j,l})a_l a_j, \tag{2}$$

$$(t\sin\phi_{j,l} + \Delta\sin\theta_{j,l})b_l b_j, \tag{3}$$

$$(t\cos\phi_{j,l} + \Delta\cos\theta_{j,l})a_l b_j, \tag{4}$$

$$(t\cos\phi_{j,l} - \Delta\cos\theta_{j,l})b_l a_j \tag{5}$$

Gauge potentials in Hamiltonians

Minimal Coupling

$$\mathbf{p}_{\mathsf{can}} = \mathbf{p}_{\mathsf{kin}} + q\mathbf{A} \tag{6}$$

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{p}_{\mathsf{can}} - q\mathbf{A}\right)^2 + qV \tag{7}$$

Peierls Phase

$$c_j^\dagger c_l \to c_j^\dagger c_l \exp\left[\frac{iq}{\hbar}\int_{\mathbf{r}_j}^{\mathbf{r}_j} \mathbf{A}\cdot d\mathbf{l}\right] = e^{i\phi_{jl}} c_j^\dagger c_l \tag{8}$$

$$\mathcal{H} = \sum_{\langle j,l\rangle} t e^{i\phi_{jl}} c_j^\dagger c_l + h.c. \tag{9}$$

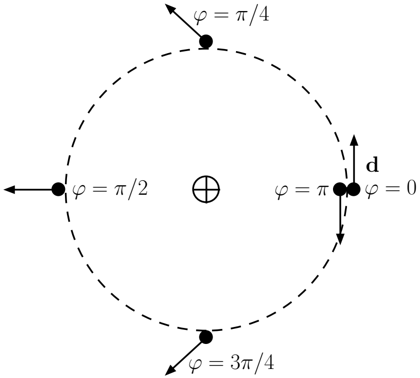
Braiding in a 2D *p*-wave superconductor

- *p*-wave superconductors can exhibit half-quantum vortices.
- Triplet pairing

$$\mathbf{d}(\mathbf{k}) = \Delta e^{i\phi}\langle \cos\alpha, \sin\alpha, 0\rangle (k_x + ik_y)$$

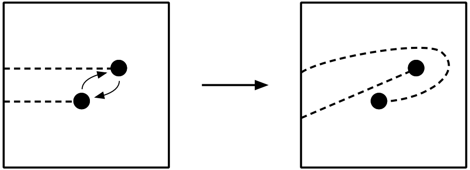
- The order phase ϕ and angle α of $\mathbf{d}$ rotate by $\pi : (\phi, \mathbf{d}) \mapsto (\phi + \pi, -\mathbf{d})$.
- The order parameter θ maps to itself, $(0, 2\pi)$, under the simultaneous change of both d and $\phi : \theta = \phi + \alpha$.

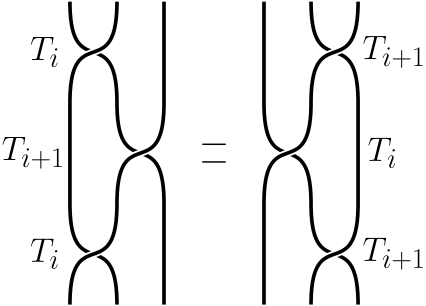
$$\mathcal{H}_\Delta = \int d^2\mathbf{r}\Delta \left[\Psi^\dagger \left[e^{i\theta} * (\partial_x + i\partial_y)\right] \Psi + h.c.\right]$$



- if overall phase shifts by $\theta : \Psi_\alpha \mapsto e^{i\theta/2}\Psi_\alpha$.
- $(u, v) \mapsto (ue^{i\theta/2}, ve^{-i\theta/2})$

Braiding in a 2D p-wave superconductor





- Interchanging two MFs:

$$\gamma_1 \to \gamma_2$$

$$\gamma_2 \to -\gamma_1$$

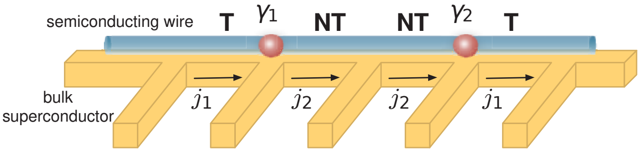
- Exhibit Non-Abelian Statistics
- $a * b \neq b * a$

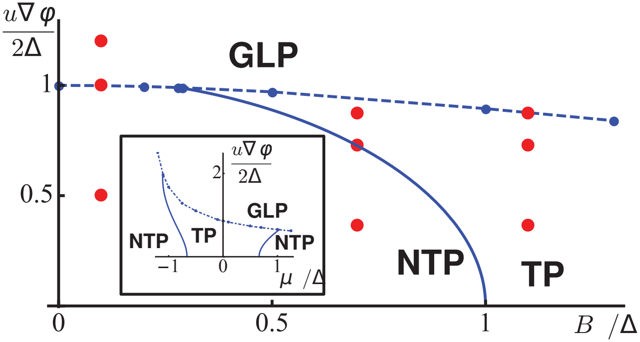
$$T_i T_j = T_j T_i$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

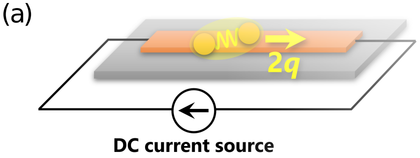
Ivanov, *PRL* **86**, 268 (2001).

Topological phase transition induced by a supercurrent

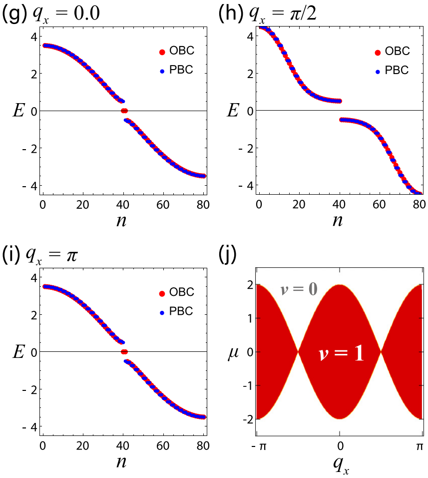




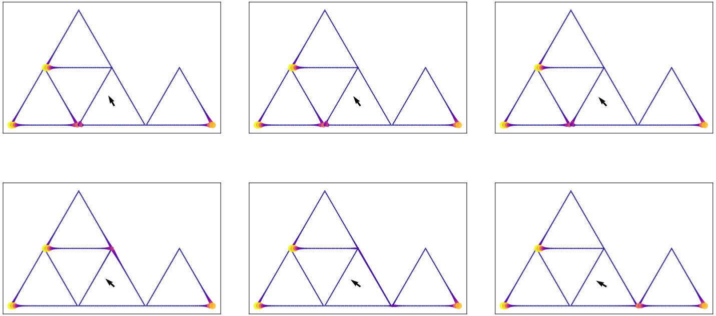
Romito et al., *PRB* **85**, 020502(R) (2012).

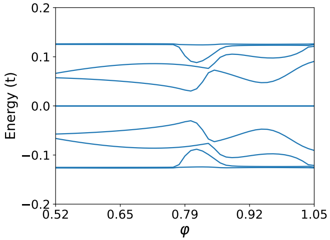


Takasan et al., *PRB* **106**, 014508 (2022).



Braiding 2 of 4 MZMs eigenstate and spectral flow





Dirac systems

$$\mathcal{H}(t) = v_F \boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}(t)) \tag{10}$$

$$\begin{aligned} \mathbf{E}_1 &= E\cos(\omega t)\hat{\mathbf{x}}, \\ \mathbf{E}_2 &= E\sin(Kx)\sin(2\omega t)\hat{\mathbf{y}} \end{aligned} \tag{11}$$

Perform Fourier time-transform, HF approximation, and the limit $Kx \ll 1$

$$\mathcal{H}^D_{\text{eff}} = v_F \sigma_x p_x + v_F \sigma_y \left(C p_y + eB^D x\right), \tag{12}$$

where $C = 1 - \left(\frac{v_F e E}{\hbar\omega^2}\right)^2$ and

$$B^D = \frac{K v_F^2 e^2 E^3}{4\hbar^2\omega^5}, \tag{13}$$

$$\epsilon_n^D = \pm v_F^2 \sqrt{\frac{nKe^3E^3}{2\hbar\omega^5}} \tag{14}$$

2DEG systems

$$\mathcal{H}(t) = \frac{1}{2m^*}(\mathbf{p} + e\mathbf{A}(t))^2 \tag{15}$$

$$\begin{aligned} \mathbf{E}_1 &= E\cos(\omega t)\hat{\mathbf{x}}, \\ \mathbf{E}_2 &= -E\cos(Kx)\sin(\omega t)\hat{\mathbf{y}} \end{aligned} \tag{16}$$

Perform Fourier time-transform, HF approximation, apply periodic potential $V(x)$, and the limit $Kx \ll 1$

$$\mathcal{H}_{\text{eff}}^{\text{2DEG}} = \frac{1}{2m^*}\left[p_x^2 + \left(p_y - eB^{\text{2DEG}}x\right)^2\right], \tag{17}$$

where

$$B^{\text{2DEG}} = \frac{K^2 e E^2}{m^* \omega^3}, \tag{18}$$

$$\epsilon_n^{\text{2DEG}} = \frac{\hbar K^2 e^2 E^2}{m^{*2}\omega^3}\left(n + \frac{1}{2}\right) \tag{19}$$