Supplemental Material for "Unveiling quantum Hall effect in spatially inhomogeneous Floquet systems without external magnetic field"

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I. HIGH FREQUENCY (VAN VLECK) EXPANSION FROM DEGENERATE PERTURBATION THEORY

In order to understand the effects of coherent time-periodic modulation of quantum systems, we need an efficient method to obtain the Floquet Hamiltonian $\hat{H}^F$ for a given time-dependent Hamiltonian $\hat{H}(\tau)$. Generally, for the Floquet systems, one would like to obtain a suitable Hamiltonian $\hat{H}(\tau)$ given a desired static Hamiltonian $\hat{H}_{\text{eff}}$. Usually the formal approach in making use of the full eigenstates of a time-dependent model Hamiltonian is not feasible in practice. Therefore, one requires an approximate scheme that still provides a valid description at least on the time-scales and energy-scales. Such an approach is provided by high-frequency approximations [1–6]. Using the Van Vleck expansion within the degenerate perturbation theory as shown in Ref. [5], we can write the explicit expressions for the first few terms with $n = 0, 1, 2$ as required;

$$\begin{aligned}
\hat{H}^{F(0)} &= \hat{H}_0, \qquad (1)\\
\hat{H}^{F(1)} &= \sum_{m\neq 0} \frac{[\hat{H}_m, \hat{H}_{-m}]}{m\hbar\omega},\\
\hat{H}^{F(2)} &= \sum_{m\neq 0} \left(\frac{[\hat{H}_{-m}, [\hat{H}_0, \hat{H}_m]]}{2(m\hbar\omega)^2} + \sum_{m'\neq 0,m} \frac{[\hat{H}_{-m'}, [\hat{H}_{m'-m}, \hat{H}_m]]}{3mm'(\hbar\omega)^2} \right).
\end{aligned}$$

Expressions for higher orders can be found in the above equations and refs. [1–6]. From a practical point of view, and in the cases which we will be considering, one often engineers the time-dependent Hamiltonian in such a way that the approximate Floquet Hamiltonian $\hat{H}_{\text{eff}} = \sum_{n=0}^{m} H^{F(n)}$ corresponds to the desired model Hamiltonian of the effective systems.

A. Model formulation of Floquet Landau Levels in Dirac electrons

As a generic model Hamiltonian to describe 2D systems, we consider Hamiltonian of monolayer graphene,

$$H^D = v_F(\sigma.\mathbf{\Pi}), \qquad (2)$$

where $\mathbf{\Pi} = \mathbf{p} + e\mathbf{A^D}$, here $\mathbf{A^D}$ is the vector potential, $\mathbf{p}$ is the momentum operator, v_F is the Fermi velocity of Dirac fermions, e the absolute value of electron charge, and σ the

Pauli matrices vector in 2D. We have two linearly polarized laser lights with the electric field components

$$\mathbf{E}_1 = E\cos(\omega t)\hat{x}, \mathbf{E}_2 = E\sin(Kx)\sin(2\omega t)\hat{y}, \tag{3}$$

where second light is spatially inhomogeneous. The ω is frequency of light with time t, $K = 2\pi/a$ with a being the spatial period of the electric field with amplitude E. This form of the field leads ($\mathbf{E} = -\frac{\partial \mathbf{A^D}}{\partial t}$) to the following vector potential $\mathbf{A^D}$

$$\mathbf{A^D} = (-V_y \sin(\omega t), V_x \cos(2\omega t), 0), \tag{4}$$

where we have $V_y = \frac{ev_F E}{\omega}, V_x = \frac{ev_F E}{2\omega}\sin(Kx)$. Substituting Eq. (4) into Eq. (2), we arrive at

$$H^D(t) = H_0^D - \sigma_x V_y \sin(\omega t) + \sigma_y V_x \cos 2(\omega t), \tag{5}$$

where $H_0^D = v_F(\sigma_x p_x + \sigma_y p_y)$. Because of the time-translation symmetry through $A(t+T) = A(t)$ with $T = 2\pi/\omega$, one can apply the Floquet theory [4, 5] and obtain an effective Hamiltonian from Eq. (5). According to Floquet approach [1–6], time-dependent components for real physical Hamiltonian written in Eq. (5) are

$$H_n^D = \frac{1}{T}\int_0^T \{-\sigma_x V_y \sin(\omega t) + \sigma_y V_x \cos(2\omega t)\} e^{in\omega t} dt, \tag{6}$$

where $n = \pm 1, \pm 2$, and $H_n^D = (1/T)\int_0^T e^{-in\omega t} H^D(t)dt]$ is the n-th Fourier harmonic of the time-periodic part of the Hamiltonian in Eq. (5).

After performing the Fourier transform of the time-periodicity in Eq. (6), first and second order expansion in Eq. (1) terms leads to the effective Hamiltonian as

$$H_{\text{eff}}^D = H_0^D - \frac{V_y^2 v_F \sigma_y p_y}{\hbar^2\omega^2} - \frac{V_y^2 V_x \sigma_y}{2\hbar^2\omega^2} - \frac{v_F \sigma_x (V_x^2 p_x + p_x V_x^2)}{8\hbar^2\omega^2}. \tag{7}$$

In Eq. (7), first order term in $\hbar\omega$ that leads to gap at the Dirac point in usual circularly polarized light [4, 5] is zero here due to inhomogeneous nature of laser lights. This effective Hamiltonian can be simplified in the long wavelength limit ($\sin(Kx) \to Kx$) to

$$H_{\text{eff}}^D = v'_F \sigma_x p_x + v_F \sigma_y (p_y - eB^D x) \tag{8}$$

In obtaining Eq. (8), last term in Eq. (7) is second order in space and thus zero in the long wavelength limit for the spatially inhomogeneous modulation. Further, we have $v'_F = v_F/C, B^D = \frac{V_y^2 E}{4\hbar^2\omega^3 C}K$, with $C = 1 - V_y^2/(\hbar^2\omega^2)$. In accordance with Eqs. (7) and (8), there is least anisotropy in the Dirac spectrum in addition to zero gap. Diagonalizing the Hamiltonian in Eq. (8), we obtained the eigenvalues for Dirac system as

$$E_n^D = \sqrt{n(v'_F v_F B^D)2e\hbar}, \tag{9}$$

which is shown in the main manuscript.

B. Model formulation of Floquet Landau Levels in Schrödinger electrons

Here, we consider the case of Schrödinger electrons under the application of two linearly polarized laser lights. The unperturbed Hamiltonian for 2DEG is

$$H = \frac{\pi^{\mathbf{2}}}{2m^*}, \tag{10}$$

where m^* is the effective mass of electron. By changing the Hamiltonian into a time-dependent form by applying two linearly polarized lights such that $\pi \to \mathbf{p} - \mathbf{eA}(\mathbf{t})$. Therefore, Eq. (10) is written as

$$H(t) = \frac{1}{2m^*}[p_x + eA_x(t)]^2 + \frac{1}{2m^*}[p_y + eA_y(t)]^2, \tag{11}$$

where the electric field components for two spatially inhomogeneous linearly polarized laser lights are

$$\mathbf{E}_1 = E\cos(\omega t)\hat{x}, \mathbf{E}'_2 = E\cos(Kx)\sin(\omega t)\hat{y}. \tag{12}$$

It is important to note that second electric field in Eq. (12) is different from similar field used for Dirac spectrum given in Eq. (3). This is due to the fact that Schrödinger Hamiltonian is quadratic rather than linear as in case of Dirac electrons. This is basic requirement for the Schrödinger electron spectrum to exhibit LLs. The field give in Eq. (12) lead to the following vector potential $\mathbf{A}(\mathbf{t})$

$$\mathbf{A}(\mathbf{t}) = (-V_1\sin(\omega t), V_2\cos(\omega t), 0), \tag{13}$$

where we have $V_1 = \frac{eE}{\omega}, V_2 = V_1\cos(Kx)$. Employing the Floquet theory similar to Dirac electrons, we obtain the effective Hamiltonian as

$$H_{\text{eff}} = H_0 - \frac{U}{m^*}\sin(Kx)p_y - \frac{U^2}{4m^*}V_1^2\cos(2Kx). \tag{14}$$

In the long wavelength limit ($\sin(Kx) \to Kx$, $\cos(2Kx) \to 1$), Eq. (14) can be simplified to

$$H_{\text{eff}} = \frac{p_x^2}{2m^*} + \frac{1}{2m^*}[p_y + eBx]^2 - \frac{U^2}{4m^*}, \tag{15}$$

where $U = \frac{KV_1^2}{2m^*\omega}$, and the effective magnetic field $B = \frac{K^2V_1^2}{em^*\omega}$. Eq. (15) is a standard LL problem in the presence of an external perpendicular magnetic field. Therefore, by diagonalizing the effective Hamiltonian, the corresponding energy eigenvalues are obtained as

$$E_n = (n + \frac{1}{2})\hbar\omega_c - \frac{U^2}{4m^*}, \tag{16}$$

where $\omega_c = \frac{eB}{m^*}$. This is shown in the main text.

II. INTRODUCTION

In this note we present details of how to set up the tight-binding models for Floquet quantum Hall effect.

A. General framework of Floquet theory

In this section we review the basic results of the Floquet theory and how to recast it into a matrix diagonalization problem. The discussion in this section is mostly following [5].

For a time-periodic Hamiltonian $H(t) = H(t+T)$ with period T, the time evolution of a wavefunction governed by it is described by the Schrödinger equation

$$i\hbar\partial_t\psi(t) = H(t)\psi(t). \tag{17}$$

Floquet theorem states that $\psi(t)$ must satisfy

$$\psi(t+T) = \psi(t)e^{-i\frac{\epsilon T}{\hbar}}, \tag{18}$$

where ϵ is a real number of energy units, or equivalently

$$\psi(t) = e^{-i\frac{\epsilon t}{\hbar}}u_\epsilon(t), \tag{19}$$

where $u_\epsilon(t) = u_\epsilon(t+T)$.

Here we give a proof that is closely analogous to that of the Bloch theorem, based on plane wave expansion. An arbitrary wavefunction can be expanded into plane waves

$$\psi(t) = \sum_\epsilon c_\epsilon e^{-i\frac{\epsilon t}{\hbar}}, \tag{20}$$

where $\epsilon \in \mathbb{R}$, while a time-periodic function $H(t)$ can only be written as a discrete Fourier series

$$H(t) = \sum_n H_n e^{in\omega t}, \tag{21}$$

where $\omega = 2\pi/T$, and $H_n = \frac{1}{T}\int_0^T H(t)e^{-in\omega t}dt$. Substituting the two expansions above into Eq. 17 gives

$$\begin{aligned} 0 &= \sum_\epsilon \left[\sum_n H_n e^{-i\frac{(\epsilon - n\hbar\omega)t}{\hbar}} c_\epsilon - \epsilon c_\epsilon e^{-i\frac{\epsilon t}{\hbar}} \right] \\ &= \sum_\epsilon \left[\sum_n H_n c_{\epsilon+n\hbar\omega} - \epsilon c_\epsilon \right] e^{-i\frac{\epsilon t}{\hbar}}, \end{aligned} \tag{22}$$

which leads to

$$\sum_n H_n c_{\epsilon+n\hbar\omega} - \epsilon c_\epsilon = 0. \tag{23}$$

For an arbitrary $\epsilon \in \mathbb{R}$ we can define $\tilde{\epsilon} \in [-\hbar\omega/2, \hbar\omega/2)$ so that $\epsilon = \tilde{\epsilon} + m\hbar\omega$. It is apparent that Eq. 23 only couples $c_{\tilde{\epsilon}+m\hbar\omega}$ belonging to the same $\tilde{\epsilon}$. We thus define

$$c_{\tilde{\epsilon}+m\hbar\omega} \equiv c_{m\tilde{\epsilon}}, \tag{24}$$

so that Eq. 23 becomes a set of coupled equations for $c_{m\tilde{\epsilon}}$, $m \in \mathbb{Z}$:

$$\sum_n (H_n - m\hbar\omega\delta_{n0}) c_{m+n,\tilde{\epsilon}} = \tilde{\epsilon} c_{m\tilde{\epsilon}}. \tag{25}$$

Eq. 23 is the eigenvalue problem of the infinite-dimensional matrix $\bar{Q}$ with the matrix elements

$$\bar{Q}_{m,m+n} = H_n - m\hbar\omega\delta_{n0}, \tag{26}$$

which is also the quasienergy operator in [5]. In practice the number of eigenvalues $\tilde{\epsilon}$ is determined by the dimension of $H(t)$. The solutions of Eq. 17 are therefore

$$\psi_{\tilde{\epsilon}}(t) = \sum_m c_{m\tilde{\epsilon}} e^{-i\frac{(\tilde{\epsilon}+m\hbar\omega)t}{\hbar}} = e^{-i\frac{\tilde{\epsilon}t}{\hbar}} \sum_m c_{m\tilde{\epsilon}} e^{-im\omega t} \equiv e^{-i\frac{\tilde{\epsilon}t}{\hbar}} u_{\tilde{\epsilon}}(t). \tag{27}$$

The proof above also gives a useful device for calculating the Floquet states $\psi_{\tilde{\epsilon}}(t)$ based on plane wave expansion. In general H_n can be a complicated operator depending on, e.g. position, spin, etc., and $c_{m\tilde{\epsilon}}$ is a function depending on these quantum numbers. One can choose a representation that makes H_0 diagonal, such as the Bloch representation, leading to the eigenvalues ϵ_n of the time-averaged Hamiltonian (H_0). When H_n is 0 for all $n \neq 0$, we have $\tilde{\epsilon} = \epsilon_{nk} - m\hbar\omega$, $m \in \mathbb{Z}$. When H_n is nonzero for any $n \neq 0$ there is in general no simple relationship between $\tilde{\epsilon}$ and ϵ_{nk}. Nonetheless, when H_n, $n \neq 0$ can be viewed as perturbation the spectrum of $\tilde{\epsilon}$ is similar to that of $\epsilon_{nk} - m\hbar\omega$, i.e., the eigenenergies ϵ_{nk} together with infinite number of its copies shifted vertically by $m\hbar\omega$.

The importance of $\tilde{\epsilon}$ is that it completely determines the stroboscopic motion of an arbitrary Floquet wavefunction, i.e.,

$$\psi_{\tilde{\epsilon}}(t + mT) = e^{-i\frac{\tilde{\epsilon}mT}{\hbar}}\psi_{\tilde{\epsilon}}(t), \ \forall m \in \mathbb{Z}. \tag{28}$$

Since $\{\psi_{\tilde{\epsilon}}(t)\}$ is a complete set at time t, the stroboscopic evolution of an arbitrary wavefunction governed by $H(t)$ is

$$\Psi(t + mT) = \sum_{\tilde{\epsilon}} C_{\tilde{\epsilon}} e^{-i\frac{\tilde{\epsilon}mT}{\hbar}}\psi_{\tilde{\epsilon}}(t), \tag{29}$$

where $\Psi(t) = \sum_{\tilde{\epsilon}} C_{\tilde{\epsilon}}\psi_{\tilde{\epsilon}}(t)$. The full time-evolution operator $\hat{U}(t_1, t_0)$ is therefore

$$\hat{U}(t_1, t_0) = \sum_{\tilde{\epsilon}} |\psi_{\tilde{\epsilon}}(t_1)\rangle\langle\psi_{\tilde{\epsilon}}(t_0)| = \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_1)\rangle\langle u_{\tilde{\epsilon}}(t_0)| e^{-i\frac{\tilde{\epsilon}(t_1 - t_0)}{\hbar}}. \tag{30}$$

Now we introduce two operators

$$\hat{U}^F(t_1, t_0) \equiv \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_1)\rangle\langle u_{\tilde{\epsilon}}(t_0)|, \tag{31}$$

and

$$\hat{H}^F_{t_0} \equiv \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_0)\rangle\tilde{\epsilon}\langle u_{\tilde{\epsilon}}(t_0)|, \tag{32}$$

which allows us to rewrite Eq. 30 as

$$\hat{U}(t_1, t_0) = \hat{U}_F(t_1, t_0) \exp\left[-i\frac{(t_1 - t_0)\hat{H}^F_{t_0}}{\hbar}\right] = \exp\left[-i\frac{(t_1 - t_0)\hat{H}^F_{t_1}}{\hbar}\right]\hat{U}_F(t_1, t_0). \tag{33}$$

Namely, the full time evolution is separated into two parts: $\hat{H}^F_{t_0}$ governs the stroboscopic evolution *with the starting time* t_0, since

$$\exp\left[-i\frac{mT\hat{H}^F_{t_0}}{\hbar}\right]\psi_{\tilde{\epsilon}}(t_0) = e^{-i\frac{mT\tilde{\epsilon}}{\hbar}}\psi_{\tilde{\epsilon}}(t_0) = \psi_{\tilde{\epsilon}}(t_0 + mT), \tag{34}$$

while $\hat{U}_F(t_1, t_0)$ evolves the periodic part of the Floquet wavefunctions. $\hat{H}^F_{t_0}$ and $\hat{U}_F(t_1, t_0)$ are respectively called the Floquet Hamiltonian and the micromotion operator.

The most unsettling property of $\hat{H}^F_{t_0}$ is its dependence on t_0. To get rid of it we note that Eq. 27 implies

$$|u_{\tilde{\epsilon}}(t)\rangle = \sum_{\alpha} \left(\sum_{m} c^{\alpha}_{m\tilde{\epsilon}} e^{-im\omega t} \right) |\alpha\rangle \equiv \sum_{\alpha} |\alpha\rangle U_{\alpha,\tilde{\epsilon}}(t), \tag{35}$$

where the time-independent basis $|\alpha\rangle$ spans the Hilbert space of $H(t)$, and $U(t)$ is a time-dependent unitary matrix satisfying $U(t+T) = U(t)$. Substituting this $|u_{\tilde{\epsilon}}(t)\rangle$ into Eq. 17 gives

$$\mathrm{Diag}[\{\tilde{\epsilon}\}] = U^\dagger H(t) U - i\hbar U^\dagger \partial_t U = U^\dagger \bar{Q} U, \tag{36}$$

where $\mathrm{Diag}[\{\tilde{\epsilon}\}]$ is a diagonal matrix with its eigenvalues being $\tilde{\epsilon}$. Comparing this with the effect of a time-dependent unitary transformation of the wavefunction $\psi' = U^\dagger \psi$ in the Schrödinger equation:

$$i\hbar \partial_t \psi' = (U^\dagger H U - i\hbar U^\dagger \partial_t U)\psi' \equiv H'\psi', \tag{37}$$

we can see that U essentially transforms $H(t)$ to an effective Hamiltonian $H' = U^\dagger \bar{Q} U$ which is time independent. The time evolution of ψ can thus obtained as

$$\begin{aligned} \psi(t_1) &= U(t_1)\psi'(t_1) = U(t_1) \exp\left[-i \frac{H'(t_1 - t_0)}{\hbar} \right] \psi'(t_0) \\ &= U(t_1) \exp\left[-i \frac{H'(t_1 - t_0)}{\hbar} \right] U^\dagger(t_0)\psi(t_0) \\ &= \hat{U}(t_1, t_0)\psi(t_0). \end{aligned} \tag{38}$$

We therefore define

$$\hat{H}_F \equiv U^\dagger \bar{Q} U = H' \tag{39}$$

as the Floquet effective Hamiltonian, which gives the time-evolution operator

$$\hat{U}(t_1, t_0) = U(t_1) \exp\left[-i \frac{\hat{H}_F(t_1 - t_0)}{\hbar} \right] U^\dagger(t_0). \tag{40}$$

Intuitively, this means that the time evolution is obtained by first doing a gauge transformation to the time-independent gauge, evolving the system, and finally gauge-transforming back to the original gauge.

Although we have been assuming that $U(t)$ diagonalizes $\bar{Q}$, this is not necessary. Any time-independent unitary transformation multiplied to $U(t)$ can still make $\hat{H}_F$ time independent. To make connection between the t_0 dependent Floquet Hamiltonian $\hat{H}^F_{t_0}$ in Eq. 32

and the effective Hamiltonian $\hat{H}_F$, we use a minimal $U(t)$ that is independent of the basis of $\hat{H}(t)$:

$$U_F(t) = \sum_m c_m e^{-im\omega t}, \tag{41}$$

which is a time-dependent scalar function. In the matrix form of $\bar{Q}$, this $U_F(t)$ block-diagonalizes $\bar{Q}$. All the diagonal blocks have the form $H_F - m\hbar\omega\mathbf{1}$. Here we removed the hat of H_F to indicate that it is a matrix written in certain representation instead of an operator. In this particular representation or gauge, $|\alpha(t)\rangle = |\alpha\rangle U_F(t)$. We thus have

$$\hat{H}^F_{t_0} = \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_0)\rangle\tilde{\epsilon}\langle u_{\tilde{\epsilon}}(t_0)| = \sum_{\alpha\beta} U_F(t_0)|\alpha\rangle(H_F)_{\alpha\beta}\langle\beta|U_F^\dagger(t_0). \tag{42}$$

Or loosely speaking $\hat{H}^F_{t_0} = U_F(t_0)\hat{H}_F U_F^\dagger(t_0)$. Therefore the t_0 dependence in $\hat{H}^F_{t_0}$ is only due to a gauge transformation and is not physical. The complete information of time evolution can be obtained from H_F *and* U_F according to Eq. 40.

In practice, to obtain the quasienergy spectrum or H_F we simply start from the eigenvalue problem Eq. 23 for $\bar{Q} \equiv \bar{H} + \bar{Q}_0$, where $\bar{H}_{m,m+n} = H_n$ and $(\bar{Q}_0)_{m,m+n} = -m\hbar\omega\delta_{n0}$. We can either use perturbation theory and treat $\bar{H}$ as perturbation, which is accurate in the high-frequency limit, or directly diagonalize $\bar{Q}$ with a large enough cutoff. The first several terms in the perturbation series of H_F are given in Eqs. 86-89 in [5] (m there should be $-m$ in our notation).

B. Including a spatially and temporally varying vector potential in a tight-binding model

In this section we discuss how to include a spatially and temporally varying vector potential in a tight-binding model and to set up the matrix of $\bar{Q}$ for numerical diagonalization.

A tight-binding Hamiltonian is in general written as a polynomial of creation and annihilation operators of Wannier states, denoted by $a^\dagger_{i,s}$ and $a_{i,s}$, where i labels sites, and s labels internal degrees of freedom. Assume that the external electromagnetic fields represented by a vector potential $A(r,t)$ vary smoothly in space and time, the fields can be included in the tight-binding model through a Peierls phase

$$a^\dagger_{i,s} \to a^\dagger_{i,s} \exp\left[-i\frac{e}{\hbar}\int_{r_0}^{r_i} A(r,t)\cdot dl\right], \tag{43}$$

which leads to a change of the hopping term

$$t_{ij,ss'}a^\dagger_{i,s}a_{j,s'} \to t_{ij,ss'} \exp\left[-i\frac{e}{\hbar}\int_{r_j}^{r_i} A(r,t)\cdot dl\right] a^\dagger_{i,s}a_{j,s'} \equiv \tilde{t}_{ij,ss'}a^\dagger_{i,s}a_{j,s'}. \tag{44}$$

The phase factor is path dependent. This is not a problem if the spatial variation of A is smooth on the lattice scale. In the limit of smooth variation we can approximate $\tilde{t}_{ij,ss'}$ by

$$\tilde{t}_{ij,ss'} \approx t_{ij,ss'} \exp\left[-i\frac{e}{\hbar}A(r_{ij},t)\cdot d_{ij}\right] \equiv t_{ij,ss'}e^{-i\phi_{ij}(t)}, \tag{45}$$

where $r_{ij} \equiv (r_i + r_j)/2$, and $d_{ij} \equiv r_i - r_j$. Eq. 45 is the main result to be used in the next section.

C. Tight-binding models

In this section we give two tight-binding models of the Floquet quantum Hall effect, respectively for Schrödinger and Dirac electrons.

1. *Schrödinger electron*

We consider a nearest-neighbor single-orbital tight-binding model on a square lattice

$$H_S = -t\sum_{\langle ij\rangle} c^\dagger_i c_j + \text{h.c.} \tag{46}$$

where $\langle ij\rangle$ means sites i, j are nearest neighbors. $t > 0$. We assume the lattice constant is a and the lattice sites have coordinates

$$r_i = x_i a\hat{x} + y_i a\hat{y},\ \ x_i, y_i \in \mathbb{Z}. \tag{47}$$

In the case that the system is infinite in both directions one can Fourier transform the Hamiltonian and obtain the eigenenergy $\epsilon_k = -4t\cos(k_x a + k_y a)$, with $k_x, k_y \in [-\pi/a, \pi/a]$. For long wavelength $|k| \ll 1/a$ we have $\epsilon_k \approx 2ta^2k^2 - 4t$, same as that of a Schrödinger electron with mass $m = \hbar^2/(4ta^2)$.

To get the Floquet QHE effect we consider a vector potential due to two linearly polarize light

$$E_1 = E\cos(\omega t)\hat{x},\ \ E_2 = E\cos(Kx)\sin(\omega t)\hat{y}, \tag{48}$$

which is

$$A(t) = -\frac{E}{\omega}\sin(\omega t)\hat{x} + \frac{E}{\omega}\cos(Kx)\cos(\omega t)\hat{y}. \tag{49}$$

The hopping term ($t_{i,j}$) in this case is

$$t_{i,j} = -\exp\left[-\frac{ie}{\hbar}A(\frac{r_i + r_j}{2}, t)\cdot d_{ij}\right]. \tag{50}$$

With the help of vector potential, above equation can be written as

$$t_{i,j} = \begin{cases} -\exp\left[\mp\frac{ie}{\hbar}\left(-\frac{Ea}{\omega}\sin(\omega t)\right)\right] \equiv \exp\left[\pm i\theta\right], & \text{if } r_j - r_i = \pm a\hat{x} \\ -\exp\left[\mp\frac{ie}{\hbar}\left(\frac{Ea}{\omega}\cos(Kx_i)\cos(\omega t)\right)\right] \equiv \exp\left[\pm i\phi_x\right], & \text{if } r_j - r_i = \pm a\hat{y} \end{cases} \tag{51}$$

where we have

$$\phi_x = -\frac{e}{\hbar}\left(\frac{Ea}{\omega}\cos(Kx_i)\cos(\omega t)\right), \theta = \frac{e}{\hbar}\left(\frac{Ea}{\omega}\sin(\omega t)\right), \phi_0 = \frac{eEa}{\hbar\omega} \tag{52}$$

The Hamiltonian is written ($r_i = x_i a\hat{x} + y_i a\hat{y}$) as

$$\begin{aligned} H_S^F = &\sum_x\sum_y \left[C_{x,y}^\dagger C_{x,y+a}\exp\left[i\phi_x\right] + C_{x,y}^\dagger C_{x,y-a}\exp\left[-i\phi_x\right]\right] \\ &+ \sum_x\sum_y \left[C_{x,y}^\dagger C_{x+a,y}\exp\left[+i\theta\right] + C_{x,y}^\dagger C_{x-a,y}\exp\left[-i\theta\right]\right] \end{aligned} \tag{53}$$

Using eigenstates of the form

$$C_{x,y}^\dagger = \sum_k e^{iky} C_{x,k}^\dagger, \tag{54}$$

For fixed k, we arrive at

$$H_S^F(k) = \sum_x \left[2\cos[\phi_x - ka]C_{x,k}^\dagger C_{x,k} + C_{x,k}^\dagger C_{x+a,k}\exp\left[+i\theta\right] + C_{x,k}^\dagger C_{x-a,k}\exp\left[-i\theta\right]\right] \tag{55}$$

Now in terms of $x = ja$, above equation can be written as

$$\begin{aligned} H_{j,j}(k) &= -2\cos\left[\frac{e}{\hbar}\frac{Ea}{\omega}\cos(Kaj)\cos(\omega t) + ka\right] \\ H_{j,j+1}(k) &= -\exp\left[i\frac{e}{\hbar}\left(\frac{Ea}{\omega}\sin(\omega t)\right)\right] \\ H_{j,j-1}(k) &= -\exp\left[-i\frac{e}{\hbar}\left(\frac{Ea}{\omega}\sin(\omega t)\right)\right] \end{aligned} \tag{56}$$

Now we need to perform time Fourier transform of above equation as

$$
\begin{aligned}
H_{j,j,n} &= \frac{1}{T}\int_0^T H_{j,j}(k)e^{-in\omega t}dt & (57)\\
&= \frac{-1}{2\pi}\int_0^{2\pi} 2\cos\left[\phi_0\cos(Kaj)\cos(\tau)+ka\right]e^{-in\tau}d\tau
\end{aligned}
$$

(58)

we have used the property of Bessel function

$$
J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\tau - in\tau}d\tau \Longrightarrow \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\tau - in\tau}d\tau = J_n(x)e^{\frac{in\pi}{2}} \tag{59}
$$

and the fact that $\tau \to \tau + \pi/2; \sin\tau = \sin\tau + \pi/2 = \cos\tau$. Therefore, we arrive at

$$
H_{j,j,n} = -\left[J_n\left(\phi_0\cos(Kaj)\right)e^{ika} + J_n\left(-\phi_0\cos(Kaj)\right)e^{-ika}\right]e^{\frac{in\pi}{2}} \tag{60}
$$

similarly, we have

$$
\begin{aligned}
H_{j,j+1,n} &= -\frac{1}{2\pi}\int_0^{2\pi} e^{i\phi_0\sin\tau - in\tau}d\tau = -J_n(\phi_0) & (61)\\
H_{j,j-1,n} &= -J_n(-\phi_0)
\end{aligned}
$$

We can now construct the matrix of $\bar{Q}$. To this end we choose a cutoff for m ($m\hbar\omega$ in the diagonal blocks):

$$
|m| \leq m_c, \tag{62}
$$

where m_c is a positive integer. This means that there are $N_m = 2m_c + 1$ diagonal blocks, and each block is a $N_S \times N_S$ matrix. Therefore $\bar{Q}$ is a $N_m N_S \times N_m N_S$ matrix. Each $N_S \times N_S$ block, labeled by $\bar{Q}_{m,m+n}$, is

$$
\bar{Q}_{m,m+n} = H_S^F(k,n) - m\hbar\omega\delta_{n0}\mathbf{1}_{\mathbf{N_S}\times\mathbf{N_S}}, \tag{63}
$$

where the $N_S \times N_S$ matrix H_n has matrix elements shown as

$$
H_S^F(k,n) = \frac{1}{T}\int_0^T H_S^F(k,t)e^{-in\omega t}dt \tag{64}
$$

To make convergence with respect to m_c faster one can choose $\hbar\omega \gg 8t$, where $8t$ is the band width of the tight-binding model. For $m_c = 4$, $r_c = 7$, the dimension of $\bar{Q}$ is $N_m N_S = 2025$.

2. Dirac electron

We consider a nearest-neighbor single-orbital tight-binding model

$$H_D = -t_{i\alpha,j\beta} \sum_{\langle i\alpha,j\beta\rangle} c^{\dagger}_{i\alpha} c_{j\beta} + \text{h.c.} \tag{65}$$

where $\langle i\alpha, j\beta\rangle$ means sites i, j are nearest neighbors with sublattices α, β and $t > 0$ being the hopping parameter. We assume the lattice constant is a and the lattice sites have coordinates

$$r_{i\alpha} = m_i a_1 \hat{x} + n_i a_2 \hat{y} + \tau_\alpha,\ m_i, n_i \in \mathbb{Z}. \tag{66}$$

To get the Floquet QHE effect we consider a vector potential due to two linearly polarize light

$$E_1 = E\cos(\omega t)\hat{x},\ E_2' = E\sin(Kx)\sin(2\omega t)\hat{y}, \tag{67}$$

which is

$$A'(t) = -\frac{eE}{\omega}\sin(\omega t)\hat{x} + \frac{eE}{2\omega}\sin(Kx)\cos(2\omega t)\hat{y}. \tag{68}$$

Note that $\nabla \cdot A = 0$. For simplicity we consider the long wavelength limit

$$A'(t) \approx -\frac{eE}{\omega}\sin(\omega t)\hat{x} + \frac{eE}{2\omega}(Kx)\cos(2\omega t)\hat{y}. \tag{69}$$

To include A' in the tight-binding model, we consider a finite system defined by

$$\max(|x_{i\alpha}|, |y_{i\beta}|) \le r_c, \tag{70}$$

where r_c is a positive integer. The Hamiltonian H_D in the tight-binding basis is a $N_S \times N_S$ square matrix with $N_S = (2r_c + 1)^2$ and its matrix elements

$$H_{i\alpha,j\beta} = -t_{i\alpha,j\beta},\ \text{if } |r_{i\alpha} - r_{j\beta}| = a \tag{71}$$

and 0 otherwise.

Including the vector potential using Eq. 45 corresponding to replacing $H_{i\alpha,j\beta}$ by

$$H_{i\alpha,j\beta} = -t\exp\left\{-i\frac{eEa}{\hbar\omega}\left[-(x_{i\alpha} - x_{j\beta})\sin(\omega t) + \frac{1}{2}\left(\sin(\frac{Ka(x_{i\alpha} + x_{j\beta})}{2})\right)(y_{i\alpha} - y_{j\beta})\cos(2\omega t)\right]\right\}, \tag{72}$$

if $|r_{i\alpha} - r_{j\beta}| = a$, and $H_{i\alpha,j\beta} = 0$ otherwise. For simplicity we use a as the length unit and t as the energy unit. K is thus in units of $1/a$. Eq. 72 is then simplified as

$$H_{i\alpha,j\beta} = -\exp\left\{-i\phi_0\left[-(x_{i\alpha} - x_{j\beta})\sin(\omega t) + \frac{1}{2}\left(\sin(\frac{K(x_i + x_j)}{2})\right)(y_{i\alpha} - y_{j\beta})\cos(2\omega t)\right]\right\} \tag{73}$$

where $\phi_0 \equiv eEa/\hbar\omega = (eEa/t)/(\hbar\omega/t)$ is dimensionless. Here we essentially use t/ea as the units of E and $t/\hbar$ as the units of ω.

We next construct the quasienergy operator $\bar{Q}$. For this we first need to calculate $H_{i\alpha,j\beta,n}$:

$$H_{i\alpha,j\beta,n} = \frac{1}{T}\int_0^T H_{i\alpha,j\beta} e^{-in\omega t} dt \tag{74}$$

The eigenstate of the Hamiltonian is written as

$$C^\dagger_{i,\alpha} = C^\dagger_{m,n,\alpha} = \sum_{k_y} e^{ik_y(3an)} C^\dagger_{m,k_y,\alpha} \tag{75}$$

and we can write the diagonal and off-diagonal parts of the Hamiltonian $H_{j,j}$,

$$H_{j,j} = -\exp\left[-i\phi_0\left\{-(\tau_x^{A1} - \tau_x^{B1})\sin(\omega t) + \frac{1}{2}\sin\left(K\frac{2ja\sqrt{3} + \tau_x^{A1} + \tau_x^{B1}}{2}\right)(\tau_y^{A1} - \tau_y^{B1})\cos(2\omega t)\right\}\right] \tag{76}$$

$$H_{j,j} = -\exp\left[-i\phi_0\left\{-(\tau_x^{B1} - \tau_x^{A2})\sin(\omega t) + \frac{1}{2}\sin\left(K\frac{2ja\sqrt{3} + \tau_x^{B1} + \tau_x^{A2}}{2}\right)(\tau_y^{B1} - \tau_y^{A2})\cos(2\omega t)\right\}\right] \tag{77}$$

$$H_{j,j} = -\exp\left[-i\phi_0\left\{-(\tau_x^{A2} - \tau_x^{B2})\sin(\omega t) + \frac{1}{2}\sin\left(K\frac{2ja\sqrt{3} + \tau_x^{A2} + \tau_x^{B2}}{2}\right)(\tau_y^{A2} - \tau_y^{B2})\cos(2\omega t)\right\}\right] \tag{78}$$

and $H_{j,j+1}$,

$$H_{j,j+1} = -\exp\left[-i\phi_0\left\{-(\tau_x^{B1} - \tau_x^{A1})\sin(\omega t) + \frac{1}{2}\sin\left(K\frac{(2j+1)a\sqrt{3} + \tau_x^{A1} + \tau_x^{B1}}{2}\right)(\tau_y^{B1} - \tau_y^{A1})\cos(2\omega t)\right\}\right. \tag{79}$$

$$H_{j,j+1} = -\exp\left[-i\phi_0\left\{-(\tau_x^{B2} - \tau_x^{A2})\sin(\omega t) + \frac{1}{2}\sin\left(K\frac{(2j+1)a\sqrt{3} + \tau_x^{B2} + \tau_x^{A2}}{2}\right)(\tau_y^{B2} - \tau_y^{A2})\cos(2\omega t)\right\}\right. \tag{80}$$

$$H_{j,j+1} = -\exp\{-ik_y 3a\}\exp\left[-i\phi_0\left\{-(\tau_x^{B2} - \tau_x^{A1})\sin(\omega t) + \frac{1}{2}\sin\left(K\frac{(2j+1)a\sqrt{3} + \tau_x^{B2} + \tau_x^{A1}}{2}\right)(\tau_y^{B2} - \tau_y^{A\right.\right. \tag{81}$$

and $H_{j,j-1}$,

$$H_{j,j-1} = -\exp\Big[-i\phi_0\Big\{-(\tau_x^{A1}-\tau_x^{B1})\sin(\omega t)+\frac{1}{2}\sin\Big(K\frac{(2j-1)a\sqrt{3}+\tau_x^{A1}+\tau_x^{B1}}{2}\Big)(\tau_y^{A1}-\tau_y^{B1})\cos(2\omega t)\Big\}\Big] \tag{82}$$

$$H_{j,j-1} = -\exp\{ik_y 3a\}\exp\Big[-i\phi_0\Big\{-(\tau_x^{A1}-\tau_x^{B2})\sin(\omega t)+\frac{1}{2}\sin\Big(K\frac{(2j-1)a\sqrt{3}+\tau_x^{A1}+\tau_x^{B2}}{2}\Big)(\tau_y^{A1}-\tau_y^{B2}) \tag{83}$$

$$H_{j,j-1} = -\exp\Big[-i\phi_0\Big\{-(\tau_x^{A2}-\tau_x^{B2})\sin(\omega t)+\frac{1}{2}\sin\Big(K\frac{(2j-1)a\sqrt{3}+\tau_x^{A2}+\tau_x^{B2}}{2}\Big)(\tau_y^{A2}-\tau_y^{B2})\cos(2\omega t)\Big\}\Big] \tag{84}$$

We can now construct the matrix of $\bar{Q}$. To this end we choose a cutoff for m ($m\hbar\omega$ in the diagonal blocks):

$$|m| \le m_c, \tag{85}$$

where m_c is a positive integer. This means that there are $N_m = 2m_c + 1$ diagonal blocks, and each block is a $N_S \times N_S$ matrix. Therefore $\bar{Q}$ is a $N_m N_S \times N_m N_S$ matrix. Each $N_S \times N_S$ block, labeled by $\bar{Q}_{m,m+n}$, is

$$\bar{Q}_{m,m+n} = H_n - m\hbar\omega\delta_{n0}1_{N_S\times N_S}, \tag{86}$$

where the $N_S \times N_S$ matrix H_n has matrix elements.

To make convergence with respect to m_c faster one can choose $\hbar\omega \gg 8t$, where $8t$ is the band width of the tight-binding model. For $m_c = 4$, $r_c = 7$, the dimension of $\bar{Q}$ is $N_m N_S = 2025$.

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[1] J. H. Shirley, Phys. Rev. 138, B979 (1965).

[2] H. Sambe, Phys. Rev. A 7, 2203 (1973).

[3] M. Grifoni and P. Hänggi, Phys. Rep. 304, 229 (1998).

[4] M. Bukov, L. D'Alessio, and A. Polkovnikov, Adv. Phys. 64, 139 (2015).

[5] Eckardt and E. Anisimovas, New J. Phys. 17, 093039 (2015).

[6] N. Goldman and J. Dalibard, Phys. Rev. X 4, 031027 (2014).