Tight-binding models of Floquet quantum Hall effect

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1 Introduction

In this note we present details of how to set up the tight-binding models for Floquet quantum Hall effect.

2 General framework of Floquet theory

In this section we review the basic results of the Floquet theory and how to recast it into a matrix diagonalization problem. The discussion in this section is mostly following [1].

For a time-periodic Hamiltonian $H(t) = H(t+T)$ with period T, the time evolution of a wavefunction governed by it is described by the Schrödinger equation

$$i\hbar\partial_t\psi(t) = H(t)\psi(t). \tag{1}$$

Floquet theorem states that $\psi(t)$ must satisfy

$$\psi(t+T) = \psi(t)e^{-i\frac{\epsilon T}{\hbar}}, \tag{2}$$

where ϵ is a real number of energy units, or equivalently

$$\psi(t) = e^{-i\frac{\epsilon t}{\hbar}}u_\epsilon(t), \tag{3}$$

where $u_\epsilon(t) = u_\epsilon(t+T)$.

Here we give a proof that is closely analogous to that of the Bloch theorem, based on plane wave expansion. An arbitrary wavefunction can be expanded into plane waves

$$\psi(t) = \sum_\epsilon c_\epsilon e^{-i\frac{\epsilon t}{\hbar}}, \tag{4}$$

where $\epsilon \in \mathbb{R}$, while a time-periodic function $H(t)$ can only be written as a discrete Fourier series

$$H(t) = \sum_n H_n e^{in\omega t}, \tag{5}$$

where $\omega = 2\pi/T$, and $H_n = \frac{1}{T}\int_0^T H(t)e^{-in\omega t}dt$. Substituting the two expansions above into Eq. 1 gives

$$\begin{aligned} 0 &= \sum_\epsilon \left[\sum_n H_n e^{-i\frac{(\epsilon - n\hbar\omega)t}{\hbar}} c_\epsilon - \epsilon c_\epsilon e^{-i\frac{\epsilon t}{\hbar}} \right] \\ &= \sum_\epsilon \left[\sum_n H_n c_{\epsilon+n\hbar\omega} - \epsilon c_\epsilon \right] e^{-i\frac{\epsilon t}{\hbar}}, \end{aligned} \tag{6}$$

which leads to

$$\sum_n H_n c_{\epsilon+n\hbar\omega} - \epsilon c_\epsilon = 0. \tag{7}$$

For an arbitrary $\epsilon \in \mathbb{R}$ we can define $\tilde{\epsilon} \in [-\hbar\omega/2, \hbar\omega/2)$ so that $\epsilon = \tilde{\epsilon} + m\hbar\omega$. It is apparent that Eq. 7 only couples $c_{\tilde{\epsilon}+m\hbar\omega}$ belonging to the same $\tilde{\epsilon}$. We thus define

$$c_{\tilde{\epsilon}+m\hbar\omega} \equiv c_{m\tilde{\epsilon}}, \tag{8}$$

so that Eq. 7 becomes a set of coupled equations for $c_{m\tilde{\epsilon}}$, $m \in \mathbb{Z}$:

$$\sum_n (H_n - m\hbar\omega\delta_{n0}) c_{m+n,\tilde{\epsilon}} = \tilde{\epsilon} c_{m\tilde{\epsilon}}. \tag{9}$$

Eq. 7 is the eigenvalue problem of the infinite-dimensional matrix $\bar{Q}$ with the matrix elements

$$\bar{Q}_{m,m+n} = H_n - m\hbar\omega\delta_{n0}, \tag{10}$$

which is also the quasienergy operator in [1]. In practice the number of eigenvalues $\tilde{\epsilon}$ is determined by the dimension of $H(t)$. The solutions of Eq. 1 are therefore

$$\psi_{\tilde{\epsilon}}(t) = \sum_m c_{m\tilde{\epsilon}} e^{-i\frac{(\tilde{\epsilon}+m\hbar\omega)t}{\hbar}} = e^{-i\frac{\tilde{\epsilon}t}{\hbar}} \sum_m c_{m\tilde{\epsilon}} e^{-im\omega t} \equiv e^{-i\frac{\tilde{\epsilon}t}{\hbar}} u_{\tilde{\epsilon}}(t). \tag{11}$$

The proof above also gives a useful device for calculating the Floquet states $\psi_{\tilde{\epsilon}}(t)$ based on plane wave expansion. In general H_n can be a complicated operator depending on, e.g. position, spin, etc., and $c_{m\tilde{\epsilon}}$ is a function depending on these quantum numbers. One can choose a representation that makes H_0 diagonal, such as the Bloch representation, leading to the eigenvalues $\epsilon_{n\boldsymbol{k}}$ of the time-averaged Hamiltonian (H_0). When H_n is 0 for all $n \neq 0$, we have $\tilde{\epsilon} = \epsilon_{n\boldsymbol{k}} - m\hbar\omega$, $m \in \mathbb{Z}$. When H_n is nonzero for any $n \neq 0$ there is in general no simple relationship between $\tilde{\epsilon}$ and $\epsilon_{n\boldsymbol{k}}$. Nonetheless, when H_n, $n \neq 0$ can be viewed as perturbation the spectrum of $\tilde{\epsilon}$ is similar to that of $\epsilon_{n\boldsymbol{k}} - m\hbar\omega$, i.e., the eigenenergies $\epsilon_{n\boldsymbol{k}}$ together with infinite number of its copies shifted vertically by $m\hbar\omega$.

The importance of $\tilde{\epsilon}$ is that it completely determines the stroboscopic motion of an arbitrary Floquet wavefunction, i.e.,

$$\psi_{\tilde{\epsilon}}(t + mT) = e^{-i\frac{\tilde{\epsilon}mT}{\hbar}} \psi_{\tilde{\epsilon}}(t), \ \forall m \in \mathbb{Z}. \tag{12}$$

Since $\{\psi_{\tilde{\epsilon}}(t)\}$ is a complete set at time t, the stroboscopic evolution of an arbitrary wavefunction governed by $H(t)$ is

$$\Psi(t + mT) = \sum_{\tilde{\epsilon}} C_{\tilde{\epsilon}} e^{-i\frac{\tilde{\epsilon}mT}{\hbar}} \psi_{\tilde{\epsilon}}(t), \tag{13}$$

where $\Psi(t) = \sum_{\tilde{\epsilon}} C_{\tilde{\epsilon}} \psi_{\tilde{\epsilon}}(t)$. The full time-evolution operator $\hat{U}(t_1, t_0)$ is therefore

$$\hat{U}(t_1, t_0) = \sum_{\tilde{\epsilon}} |\psi_{\tilde{\epsilon}}(t_1)\rangle\langle\psi_{\tilde{\epsilon}}(t_0)| = \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_1)\rangle\langle u_{\tilde{\epsilon}}(t_0)| e^{-i\frac{\tilde{\epsilon}(t_1 - t_0)}{\hbar}}. \tag{14}$$

Now we introduce two operators

$$\hat{U}^F(t_1, t_0) \equiv \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_1)\rangle\langle u_{\tilde{\epsilon}}(t_0)|, \tag{15}$$

and

$$\hat{H}^F_{t_0} \equiv \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_0)\rangle \tilde{\epsilon} \langle u_{\tilde{\epsilon}}(t_0)|, \tag{16}$$

which allows us to rewrite Eq. 14 as

$$\hat{U}(t_1, t_0) = \hat{U}_F(t_1, t_0) \exp\left[-i\frac{(t_1 - t_0)\hat{H}^F_{t_0}}{\hbar}\right] = \exp\left[-i\frac{(t_1 - t_0)\hat{H}^F_{t_1}}{\hbar}\right] \hat{U}_F(t_1, t_0). \tag{17}$$

Namely, the full time evolution is separated into two parts: $\hat{H}^F_{t_0}$ governs the stroboscopic evolution *with the starting time* t_0, since

$$\exp\left[-i\frac{mT\hat{H}^F_{t_0}}{\hbar}\right] \psi_{\tilde{\epsilon}}(t_0) = e^{-i\frac{mT\tilde{\epsilon}}{\hbar}} \psi_{\tilde{\epsilon}}(t_0) = \psi_{\tilde{\epsilon}}(t_0 + mT), \tag{18}$$

while $\hat{U}_F(t_1, t_0)$ evolves the periodic part of the Floquet wavefunctions. $\hat{H}^F_{t_0}$ and $\hat{U}_F(t_1, t_0)$ are respectively called the Floquet Hamiltonian and the micromotion operator.

The most unsettling property of $\hat{H}^F_{t_0}$ is its dependence on t_0. To get rid of it we note that Eq. 11 implies

$$|u_{\tilde{\epsilon}}(t)\rangle = \sum_{\alpha} \left(\sum_m c^{\alpha}_{m\tilde{\epsilon}} e^{-im\omega t}\right) |\alpha\rangle \equiv \sum_{\alpha} |\alpha\rangle U_{\alpha,\tilde{\epsilon}}(t), \tag{19}$$

where the time-independent basis $|\alpha\rangle$ spans the Hilbert space of $H(t)$, and $U(t)$ is a time-dependent unitary matrix satisfying $U(t + T) = U(t)$. Substituting this $|u_{\tilde{\epsilon}}(t)\rangle$ into Eq. 1 gives

$$\mathrm{Diag}[\{\tilde{\epsilon}\}] = U^\dagger H(t) U - i\hbar U^\dagger \partial_t U = U^\dagger \bar{Q} U, \tag{20}$$

where $\mathrm{Diag}[\{\tilde{\epsilon}\}]$ is a diagonal matrix with its eigenvalues being $\tilde{\epsilon}$. Comparing this with the effect of a time-dependent unitary transformation of the wavefunction $\psi' = U^\dagger \psi$ in the Schrödinger equation:

$$i\hbar\partial_t \psi' = (U^\dagger H U - i\hbar U^\dagger \partial_t U)\psi' \equiv H'\psi', \tag{21}$$

we can see that U essentially transforms $H(t)$ to an effective Hamiltonian $H' = U^\dagger \bar{Q} U$ which is time independent. The time evolution of ψ can thus obtained as

$$\begin{aligned} \psi(t_1) &= U(t_1)\psi'(t_1) = U(t_1) \exp\left[-i\frac{H'(t_1 - t_0)}{\hbar}\right] \psi'(t_0) \\ &= U(t_1) \exp\left[-i\frac{H'(t_1 - t_0)}{\hbar}\right] U^\dagger(t_0)\psi(t_0) \\ &= \hat{U}(t_1, t_0)\psi(t_0). \end{aligned} \tag{22}$$

We therefore define

$$\hat{H}_F \equiv U^\dagger \bar{Q} U = H' \tag{23}$$

as the Floquet effective Hamiltonian, which gives the time-evolution operator

$$\hat{U}(t_1, t_0) = U(t_1) \exp\left[-i\frac{\hat{H}_F(t_1 - t_0)}{\hbar}\right] U^\dagger(t_0). \tag{24}$$

Intuitively, this means that the time evolution is obtained by first doing a gauge transformation to the time-independent gauge, evolving the system, and finally gauge-transforming back to the original gauge.

Although we have been assuming that $U(t)$ diagonalizes $\bar{Q}$, this is not necessary. Any time-independent unitary transformation multiplied to $U(t)$ can still make $\hat{H}_F$ time independent. To make connection between the t_0 dependent Floquet Hamiltonian $\hat{H}^F_{t_0}$ in Eq. 16 and the effective Hamiltonian $\hat{H}_F$, we use a minimal $U(t)$ that is independent of the basis of $\hat{H}(t)$:

$$U_F(t) = \sum_m c_m e^{-im\omega t}, \tag{25}$$

which is a time-dependent scalar function. In the matrix form of $\bar{Q}$, this $U_F(t)$ block-diagonalizes $\bar{Q}$. All the diagonal blocks have the form $H_F - m\hbar\omega\mathbb{1}$. Here we removed the hat of H_F to indicate that it is a matrix written in certain representation instead of an operator. In this particular representation or gauge, $|\alpha(t)\rangle = |\alpha\rangle U_F(t)$. We thus have

$$\hat{H}^F_{t_0} = \sum_{\tilde{\epsilon}} |u_{\tilde{\epsilon}}(t_0)\rangle \tilde{\epsilon} \langle u_{\tilde{\epsilon}}(t_0)| = \sum_{\alpha\beta} U_F(t_0)|\alpha\rangle (H_F)_{\alpha\beta} \langle\beta| U^\dagger_F(t_0). \tag{26}$$

Or loosely speaking $\hat{H}^F_{t_0} = U_F(t_0)\hat{H}_F U^\dagger_F(t_0)$. Therefore the t_0 dependence in $\hat{H}^F_{t_0}$ is only due to a gauge transformation and is not physical. The complete information of time evolution can be obtained from H_F *and* U_F according to Eq. 24.

In practice, to obtain the quasienergy spectrum or H_F we simply start from the eigenvalue problem Eq. 7 for $\bar{Q} \equiv \bar{H} + \bar{Q}_0$, where $\bar{H}_{m,m+n} = H_n$ and $(\bar{Q}_0)_{m,m+n} = -m\hbar\omega\delta_{n0}$. We can either use perturbation theory and treat $\bar{H}$ as perturbation, which is accurate in the high-frequency limit, or directly diagonalize $\bar{Q}$ with a large enough cutoff. The first several terms in the perturbation series of H_F are given in Eqs. 86-89 in [1] (m there should be $-m$ in our notation).

3 Including a spatially and temporally varying vector potential in a tight-binding model

In this section we discuss how to include a spatially and temporally varying vector potential in a tight-binding model and to set up the matrix of $\bar{Q}$ for numerical diagonalization.

A tight-binding Hamiltonian is in general written as a polynomial of creation and annihilation operators of Wannier states, denoted by $a^\dagger_{i,s}$ and $a_{i,s}$, where i labels sites, and s labels internal degrees of freedom. Assume that the external electromagnetic fields represented by a vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$

vary smoothly in space and time, the fields can be included in the tight-binding model through a Peierls phase

$$a_{i,s}^{\dagger} \to a_{i,s}^{\dagger} \exp\left[-i\frac{e}{\hbar}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_i} \boldsymbol{A}(\boldsymbol{r},t)\cdot d\boldsymbol{l}\right], \tag{27}$$

which leads to a change of the hopping term

$$t_{ij,ss'}a_{i,s}^{\dagger}a_{j,s'} \to t_{ij,ss'}\exp\left[-i\frac{e}{\hbar}\int_{\boldsymbol{r}_j}^{\boldsymbol{r}_i} \boldsymbol{A}(\boldsymbol{r},t)\cdot d\boldsymbol{l}\right] a_{i,s}^{\dagger}a_{j,s'} \equiv \tilde{t}_{ij,ss'}a_{i,s}^{\dagger}a_{j,s'}. \tag{28}$$

The phase factor is path dependent. This is not a problem if the spatial variation of $\boldsymbol{A}$ is smooth on the lattice scale. In the limit of smooth variation we can approximate $\tilde{t}_{ij,ss'}$ by

$$\tilde{t}_{ij,ss'} \approx t_{ij,ss'}\exp\left[-i\frac{e}{\hbar}\boldsymbol{A}(\boldsymbol{r}_{ij},t)\cdot\boldsymbol{d}_{ij}\right] \equiv t_{ij,ss'}e^{-i\phi_{ij}(t)}, \tag{29}$$

where $\boldsymbol{r}_{ij} \equiv (\boldsymbol{r}_i + \boldsymbol{r}_j)/2$, and $\boldsymbol{d}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$. Eq. 29 is the main result to be used in the next section.

4 Tight-binding models

In this section we give two tight-binding models of the Floquet quantum Hall effect, respectively for Schrödinger and Dirac electrons.

4.1 Schrödinger electron

We consider a nearest-neighbor single-orbital tight-binding model on a square lattice

$$H_S = -t\sum_{\langle ij\rangle} c_i^{\dagger}c_j + \text{h.c.} \tag{30}$$

where $\langle ij\rangle$ means sites i,j are nearest neighbors. $t>0$. We assume the lattice constant is a and the lattice sites have coordinates

$$\boldsymbol{r}_i = x_i a\hat{x} + y_i a\hat{y},\ \ x_i, y_i \in \mathbb{Z}. \tag{31}$$

In the case that the system is infinite in both directions one can Fourier transform the Hamiltonian and obtain the eigenenergy $\epsilon_{\boldsymbol{k}} = -4t\cos(k_x a + k_y a)$, with $k_x, k_y \in [-\pi/a, \pi/a]$. For long wavelength $|\boldsymbol{k}| \ll 1/a$ we have $\epsilon_{\boldsymbol{k}} \approx 2ta^2k^2 - 4t$, same as that of a Schrödinger electron with mass $m = \hbar^2/(4ta^2)$.

To get the Floquet QHE effect we consider a vector potential due to two linearly polarize light

$$\boldsymbol{E}_1 = E\cos(\omega t)\hat{x},\ \ \boldsymbol{E}_2 = E\cos(Kx)\sin(\omega t)\hat{y}, \tag{32}$$

which is

$$\boldsymbol{A}(t) = -\frac{E}{\omega}\sin(\omega t)\hat{x} + \frac{E}{\omega}\cos(Kx)\cos(\omega t)\hat{y}. \tag{33}$$

The hopping term ($t_{i,j}$) in this case is

$$t_{i,j} = -\exp\left[-\frac{ie}{\hbar}\boldsymbol{A}(\frac{\boldsymbol{r}_i+\boldsymbol{r}_j}{2},t)\cdot\boldsymbol{d}_{ij}\right]. \tag{34}$$

With the help of vector potential, above equation can be written as

$$t_{i,j} = \begin{cases} -\exp\left[\mp\frac{ie}{\hbar}\left(-\frac{Ea}{\omega}\sin(\omega t)\right)\right] \equiv \exp\left[\pm i\theta\right], & \text{if } \boldsymbol{r}_j-\boldsymbol{r}_i = \pm a\hat{x} \\ -\exp\left[\mp\frac{ie}{\hbar}\left(\frac{Ea}{\omega}\cos(Kx_i)\cos(\omega t)\right)\right] \equiv \exp\left[\pm i\phi_x\right], & \text{if } \boldsymbol{r}_j-\boldsymbol{r}_i = \pm a\hat{y} \end{cases} \tag{35}$$

where we have

$$\phi_x = -\frac{e}{\hbar}\left(\frac{Ea}{\omega}\cos(Kx_i)\cos(\omega t)\right), \theta = \frac{e}{\hbar}\left(\frac{Ea}{\omega}\sin(\omega t)\right), \phi_0 = \frac{eEa}{\hbar\omega} \tag{36}$$

The Hamiltonian is written ($\boldsymbol{r}_i = x_ia\hat{x}+y_ia\hat{y}$) as

$$\begin{aligned} H_S^F &= \sum_x\sum_y\left[C^\dagger_{x,y}C_{x,y+a}\exp\left[i\phi_x\right]+C^\dagger_{x,y}C_{x,y-a}\exp\left[-i\phi_x\right]\right] \\ &+ \sum_x\sum_y\left[C^\dagger_{x,y}C_{x+a,y}\exp\left[+i\theta\right]+C^\dagger_{x,y}C_{x-a,y}\exp\left[-i\theta\right]\right] \end{aligned} \tag{37}$$

Using eigenstates of the form

$$C^\dagger_{x,y} = \sum_k e^{iky}C^\dagger_{x,k}, \tag{38}$$

For fixed k, we arrive at

$$H_S^F(k) = \sum_x\left[2\cos[\phi_x-ka]C^\dagger_{x,k}C_{x,k}+C^\dagger_{x,k}C_{x+a,k}\exp\left[+i\theta\right]+C^\dagger_{x,k}C_{x-a,k}\exp\left[-i\theta\right]\right] \tag{39}$$

Now in terms of $x = ja$, above equation can be written as

$$\begin{aligned} H_{j,j}(k) &= -2\cos\left[\frac{e}{\hbar}\frac{Ea}{\omega}\cos(Kaj)\cos(\omega t)+ka\right] \\ H_{j,j+1}(k) &= -\exp\left[i\frac{e}{\hbar}\left(\frac{Ea}{\omega}\sin(\omega t)\right)\right] \\ H_{j,j-1}(k) &= -\exp\left[-i\frac{e}{\hbar}\left(\frac{Ea}{\omega}\sin(\omega t)\right)\right] \end{aligned} \tag{40}$$

Now we need to perform time Fourier transform of above equation as

$$H_{j,j,n} = \frac{1}{T}\int_0^T H_{j,j}(k)e^{-in\omega t}dt \tag{41}$$

$$= \frac{-1}{2\pi}\int_0^{2\pi} 2\cos\left[\phi_0\cos(Kaj)\cos(\tau)+ka\right]e^{-in\tau}d\tau \tag{42}$$

we have used the property of the Bessel function

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\tau - in\tau} d\tau \Longrightarrow \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\tau - in\tau} d\tau = J_n(x) e^{\frac{in\pi}{2}} \tag{43}$$

and the fact that $\tau \to \tau + \pi/2; \sin\tau = \sin\tau + \pi/2 = \cos\tau$. Therefore, we arrive at

$$H_{j,j,n} = -\left[J_n\left(\phi_0\cos(Kaj)\right)e^{ika} + J_n\left(-\phi_0\cos(Kaj)\right)e^{-ika}\right]e^{\frac{in\pi}{2}} \tag{44}$$

similarly, we have

$$\begin{aligned} H_{j,j+1,n} &= -\frac{1}{2\pi}\int_0^{2\pi} e^{i\phi_0\sin\tau - in\tau} d\tau = -J_n(\phi_0) \\ H_{j,j-1,n} &= -J_n(-\phi_0) \end{aligned} \tag{45}$$

We can now construct the matrix of $\bar{Q}$. To this end we choose a cutoff for m ($m\hbar\omega$ in the diagonal blocks):

$$|m| \le m_c, \tag{46}$$

where m_c is a positive integer. This means that there are $N_m = 2m_c + 1$ diagonal blocks, and each block is a $N_S \times N_S$ matrix. Therefore $\bar{Q}$ is a $N_m N_S \times N_m N_S$ matrix. Each $N_S \times N_S$ block, labeled by $\bar{Q}_{m,m+n}$, is

$$\bar{Q}_{m,m+n} = H_S^F(k,n) - m\hbar\omega\delta_{n0}\mathbb{1}_{N_S\times N_S}, \tag{47}$$

where the $N_S \times N_S$ matrix H_n has matrix elements shown as

$$H_S^F(k,n) = \frac{1}{T}\int_0^T H_S^F(k,t)e^{-in\omega t}dt \tag{48}$$

To make convergence with respect to m_c faster one can choose $\hbar\omega \gg 8t$, where $8t$ is the band width of the tight-binding model. For $m_c = 4$, $r_c = 7$, the dimension of $\bar{Q}$ is $N_m N_S = 2025$.

4.2 Dirac electron

We consider a nearest-neighbor single-orbital tight-binding model

$$H_D = -t_{i\alpha,j\beta}\sum_{\langle i\alpha,j\beta\rangle} c_{i\alpha}^{\dagger}c_{j\beta} + \text{h.c.} \tag{49}$$

where $\langle i\alpha, j\beta\rangle$ means sites i, j are nearest neighbors with sublattices α, β and $t > 0$ being the hopping parameter. We assume the lattice constant is a and the lattice sites have coordinates

$$\boldsymbol{r}_{i\alpha} = m_i\boldsymbol{a}_1\hat{x} + n_i\boldsymbol{a}_2\hat{y} + \boldsymbol{\tau}_\alpha,\ m_i, n_i \in \mathbb{Z}. \tag{50}$$

To get the Floquet QHE effect we consider a vector potential due to two linearly polarize light

$$\boldsymbol{E}_1 = E\cos(\omega t)\hat{x},\ \boldsymbol{E}_2' = E\sin(Kx)\sin(2\omega t)\hat{y}, \tag{51}$$

which is

$$\boldsymbol{A}'(t) = -\frac{eE}{\omega}\sin(\omega t)\hat{x} + \frac{eE}{2\omega}\sin(Kx)\cos(2\omega t)\hat{y}. \tag{52}$$

Note that $\nabla \cdot \boldsymbol{A} = 0$. For simplicity we consider the long wavelength limit

$$\boldsymbol{A}'(t) \approx -\frac{eE}{\omega}\sin(\omega t)\hat{x} + \frac{eE}{2\omega}(Kx)\cos(2\omega t)\hat{y}. \tag{53}$$

To include $\boldsymbol{A}'$ in the tight-binding model, we consider a finite system defined by

$$\max(|x_{i\alpha}|, |y_{i\beta}|) \leq r_c, \tag{54}$$

where r_c is a positive integer. The Hamiltonian H_D in the tight-binding basis is a $N_S \times N_S$ square matrix with $N_S = (2r_c+1)^2$ and its matrix elements

$$H_{i\alpha,j\beta} = -t_{i\alpha,j\beta},\ \text{if } |\boldsymbol{r}_{i\alpha} - \boldsymbol{r}_{j\beta}| = a \tag{55}$$

and 0 otherwise.

Including the vector potential using Eq. 29 corresponding to replacing $H_{i\alpha,j\beta}$ by

$$H_{i\alpha,j\beta} = -t\exp\left\{-i\frac{eEa}{\hbar\omega}\left[-(x_{i\alpha} - x_{j\beta})\sin(\omega t) + \left(\frac{Ka(x_{i\alpha}+x_{j\beta})}{2}\right)(y_{i\alpha} - y_{j\beta})\cos(2\omega t)\right]\right\}, \tag{56}$$

if $|\boldsymbol{r}_{i\alpha} - \boldsymbol{r}_{j\beta}| = a$, and $H_{i\alpha,j\beta} = 0$ otherwise. For simplicity we use a as the length unit and t as the energy unit. K is thus in units of $1/a$. Eq. 56 is then simplified as

$$H_{i\alpha,j\beta} = -\exp\left\{-i\phi_0\left[-(x_{i\alpha} - x_{j\beta})\sin(\omega t) + \left(\frac{K(x_i+x_j)}{2}\right)(y_{i\alpha} - y_{j\beta})\cos(2\omega t)\right]\right\}, \tag{57}$$

where $\phi_0 \equiv eEa/\hbar\omega = (eEa/t)/(\hbar\omega/t)$ is dimensionless. Here we essentially use t/ea as the units of E and $t/\hbar$ as the units of ω. If the long-wavelength limit is not taken at this stage we have instead of Eq. 57

$$H_{i\alpha,j\beta} = -\exp\left\{-i\phi_0\left[-(x_{i\alpha} - x_{j\beta})\sin(\omega t) + \frac{1}{2}\sin\left(K(x_{i\alpha}+x_{j\beta})\right)(y_{i\alpha} - y_{j\beta})\cos(2\omega t)\right]\right\}. \tag{58}$$

We will use this expression below, since one can always get the long-wavelength limit from it.

We next construct the quasienergy operator $\bar{Q}$. For this we first need to calculate $H_{i\alpha,j\beta,n}$:

$$\begin{aligned} H_{i\alpha,j\beta,n} &= \frac{1}{T}\int_0^T H_{i\alpha,j\beta}e^{-in\omega t}dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\exp\left[iX_1\sin(\tau) + iX_2\cos(2\tau) - in\tau\right]d\tau \\ &= -J_n(X)e^{in\phi}, \end{aligned} \tag{59}$$

where we have used the property of the Bessel function

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin(\tau)-in\tau}d\tau. \tag{60}$$

In the result of $H_{i\alpha,j\beta,n}$ we have defined

$$X_1 \equiv \phi_0(x_{i\alpha} - x_{j\beta}), \quad X_2 \equiv -\phi_0\left(\frac{K(x_i + x_j)}{2}\right)(y_{i\alpha} - y_{j\beta}), \tag{61}$$

which gives

$$\cos(2x) = 2\cos(x)\cos(x) - 1. \tag{62}$$

Using this result it is easy to calculate $e^{in\phi} = (\cos\phi + i\sin\phi)^n$ numerically.

We can now construct the matrix of $\bar{Q}$. To this end we choose a cutoff for m ($m\hbar\omega$ in the diagonal blocks):

$$|m| \leq m_c, \tag{63}$$

where m_c is a positive integer. This means that there are $N_m = 2m_c + 1$ diagonal blocks, and each block is a $N_S \times N_S$ matrix. Therefore $\bar{Q}$ is a $N_m N_S \times N_m N_S$ matrix. Each $N_S \times N_S$ block, labeled by $\bar{Q}_{m,m+n}$, is

$$\bar{Q}_{m,m+n} = H_n - m\hbar\omega\delta_{n0}\mathbb{1}_{N_S\times N_S}, \tag{64}$$

where the $N_S \times N_S$ matrix H_n has matrix elements shown in Eq. **??**

$$(H_n)_{i\alpha,j\beta} = \begin{cases} -J_n(X)e^{in\phi}, & \text{if } |\boldsymbol{r}_{i\alpha} - \boldsymbol{r}_{j\beta}| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{65}$$

To make convergence with respect to m_c faster one can choose $\hbar\omega \gg 8t$, where $8t$ is the band width of the tight-binding model. For $m_c = 4$, $r_c = 7$, the dimension of $\bar{Q}$ is $N_m N_S = 2025$.

References

[1] A. Eckardt and E. Anisimovas **17**, 093039 (2015).