DISSERTATION

EMERGENT TOPOLOGICAL PHENOMENA IN LOW-D SYSTEMS INDUCED BY GAUGE POTENTIALS

Submitted by

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ABSTRACT

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Abstract goes here

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DEDICATION

I would like to dedicate this dissertation to my dog Zeta.

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EM gauge potential appears in electronic Hamiltonian in CM

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3. TB Hamiltonian and Peierls phase

Topological phenomena in CM considered in thesis

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   - i Kitaev chain (M— topological invariant). BdG?
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   - ii square lattice — hofstadter butterfly (on other lattices, honeycomb)

STUFF

Chapter 2

Superconducting Triangular Islands as a Platform for Manipulating Majorana Zero Modes

1. Introduction
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2.1 Kitaev Triangle and Peierls substitution

We start with a spinless or spin-polarized *p*-wave superconductor

$$\mathcal{H} = \sum_{\langle j,l \rangle} (-t c_j^\dagger c_l + \Delta e^{i\theta_{jl}} c_j c_l + h.c.) - \sum_j \mu c_j^\dagger c_j, \tag{2.1}$$

where t is the hopping amplitude, Δ is the amplitude of (2D) *p*-wave pairing, μ is the chemical potential, θ_{jl} is the polar angle of $\mathbf{r}_{jl} = \mathbf{r}_l - \mathbf{r}_j$, consistent with $\{c_l^\dagger, c_j^\dagger\} = 0$.

We will now include a gauge potential via a Peierls substitution as

$$
\begin{aligned}
c_j^\dagger &\to c_j^\dagger \exp\left(-\frac{ie}{\hbar}\int_0^{\mathbf{r}_j} \mathbf{A}\cdot d\mathbf{l}\right), \\
c_j^\dagger c_l &\to c_j^\dagger c_l \exp\left(\frac{ie}{\hbar}\int_{\mathbf{r}_j}^{\mathbf{r}_l} \mathbf{A}\cdot d\mathbf{l}\right) \\
&\to c_l^\dagger c_j e^{i\phi_{j,l}}. \\
\phi_{jl} &= \frac{e}{\hbar}\int_{\mathbf{r}_j}^{\mathbf{r}_l} \mathbf{A}\cdot d\mathbf{l} = -\phi_{lj} \qquad (2.2)
\end{aligned}
$$

The modified Hamiltonian is then

$$
\mathcal{H} = \sum_{\langle j,l\rangle}(-te^{i\phi_{jl}}c_j^\dagger c_l + \Delta e^{i\theta_{jl}}c_j c_l + h.c.) - \sum_j \mu c_j^\dagger c_j, \qquad (2.3)
$$

The complex fermion operator can be written in the Majorana Fermion basis, a superposition of two Majorana fermions $c_j = \frac{1}{2}(a_j + ib_j)$. Due to the nature of Majorana fermions, $a_j^\dagger = a_j$, the creation operator is $c_j^\dagger = \frac{1}{2}(a_j - ib_j)$. It is quickly seen after substitution we arrive at

$$
c_j^\dagger c_j = \frac{1}{2}(1 + ia_j b_j), \qquad (2.4)
$$

$$
c_j^\dagger c_l = \frac{1}{4}(a_j a_l + b_j b_l + ia_j b_l - ib_j a_l), \qquad (2.5)
$$

$$
c_j c_l = \frac{1}{4}(a_j a_l - b_j b_l + ia_j b_l + ib_j a_l). \qquad (2.6)
$$

The hopping term in MF basis are

$$
-t(e^{i\phi_{jl}}c_j^\dagger c_l + e^{-i\phi_{jl}}c_l^\dagger c_j) = -\frac{it}{2}(\sin\phi_{jl}(a_j a_l + b_j b_l) + \cos\phi_{jl}(a_j b_l - b_j a_l)), \qquad (2.7)
$$

the order parameter terms are

$$\Delta(e^{i\theta_{jl}}c_j^\dagger c_l^\dagger + e^{-i\theta_{jl}}c_jc_l) = \frac{i\Delta}{2}(\sin\theta_{jl}(a_la_j - b_lb_j) + \cos\theta_{jl}(a_lb_j + b_la_j)). \tag{2.8}$$

Our Hamiltonian in MF basis is then

$$\begin{aligned}\mathcal{H} = -\frac{i}{2}\sum_{\langle j,l\rangle}[&(t\sin\phi_{jl} - \Delta\sin\theta_{jl})a_ja_l + (t\sin\phi_{jl} + \Delta\sin\theta_{jl})b_jb_l \\ &+(t\cos\phi_{jl} - \Delta\cos\theta_{jl})a_jb_l - (t\cos\phi_{jl} + \Delta\cos\theta_{jl})b_ja_l] \\ -\frac{i\mu}{2}\sum_j a_jb_j&\end{aligned} \tag{2.9}$$

For concreteness we consider a 1-D chain in the Kitaev limit $t = \Delta$, $\mu = 0$, and choose $phi_{jl} = 0$ (either zero or a perpendicular gauge potential). The Kitaev chain is resultant with $\mathcal{H} = \sum_{j,j+1} -itb_ja_{j+1}$ and hosting MZM a_1 and b_N.

2.2 Conditions for MZMs on equilateral triangular islands

We want to now use a gauge potential to tune our system into having zero modes located at the base corners of a triangular lattice. Consider first forming a minimal Kitaev triangle in the positive *y*-axis, with only 3-sites such that its base, with sites 1 and 2, are along the *x*-axis. While still considering the Kitaev limit in this minimal model, as previously stated, sites 1 and 2 form a Kitaev chain. In order for the MZM to persist in the presence of site 3, one can choose ϕ_{23} and ϕ_{31} so that all terms involving these Majorana operators cancel out. For example, consider the 2–3 bond, for which $\theta_{23} = 2\pi/3$, we require

$$\sin\phi_{jl} + \sin\frac{2\pi}{3} = \cos\phi_{jl} + \cos\frac{2\pi}{3} = 0 \tag{2.10}$$

which means $\phi_{23} = -\pi/3$. Similarly one can find $\phi_{31} = -\phi_{13} = -\pi/3$. The three Peierls phases can be realized by the following staggered vector potential

$$\mathbf{A} = [1 - 2\Theta(x)]\frac{2\pi}{3\sqrt{3}}\hat{\mathbf{y}}. \tag{2.11}$$

Which is derived in the following subsection

2.2.1 Staggered vector potential

First, naively consider a constant vector potential field. For sites 1–2 we want the field to be perpendicular to their axis this tells usto start with $\mathbf{A} = A\hat{\mathbf{y}}$. From Eq. 2.2, set $e = \hbar = 1$ and the path integral for ϕ_{13} becomes

$$\begin{aligned}
\phi_{13} &= \int_{\mathbf{r_1}}^{\mathbf{r_3}} \mathbf{A} \cdot d\mathbf{l} \\
&= A \int_{y_1}^{y_3} \hat{\mathbf{y}} \cdot d\mathbf{l} \\
&= A \int_{0}^{\sqrt{3}a/2} dy \\
&= \frac{\sqrt{3}Aa}{2} \\
&= \pi/3.
\end{aligned}$$

We find that we need

$$A = \frac{2\pi}{3\sqrt{3}a}. \tag{2.12}$$

Now let us check if this allows for $\phi_{23} = -\pi/3$.

$$\begin{aligned}
\phi_{23} &= \int_{\mathbf{r_2}}^{\mathbf{r_3}} \mathbf{A} \cdot d\mathbf{l} \\
&= A \int_{y_2}^{y_3} \hat{\mathbf{y}} \cdot d\mathbf{l} \\
&= A \int_{0}^{\sqrt{3}a/2} dy \\
&= \frac{\sqrt{3}Aa}{2} \\
&= \frac{\sqrt{3}a}{2} \frac{2\pi}{3\sqrt{3}a} \\
&= \pi/3 \neq -\pi/3.
\end{aligned}$$

Here we see that a constant vector potential does not meet the condition for MZMs, it's off by a sign factor. This is remedied by using the Heaviside function instead from equation 2.11

$$\mathbf{A} = [1 - 2\Theta(x)] \frac{2\pi}{3\sqrt{3}} \hat{\mathbf{y}}.$$

2.2.2 Linear vector potential

While the simplest vector potential one can use in the minimal Kitaev triangle is a staggered potential it remains to be seen if other odd functions also work. Again, we want the Peierls phase for sites 1–2 to have no contribution, let $\mathbf{A} = Ax\hat{\mathbf{y}}$. Simlarly, for sites 1–3 we have

$$\begin{aligned}\phi_{13} &= \int_{\mathbf{r_1}}^{\mathbf{r_3}} \mathbf{A} \cdot d\mathbf{l} \\ &= \int_{y_1}^{y_3} Axdy \\ &= \int_{x_1}^{x_3} Ax\frac{dy}{dx}dx \\ &= \sqrt{3}A\int_{-a/2}^{0} xdx \\ &= -\frac{\sqrt{3}Aa^2}{8} \\ &= \pi/3.\end{aligned}$$

The magnitude is then

$$A = -\frac{8\pi}{3\sqrt{3}a^2}. \tag{2.13}$$

Check if $\phi_{23} = -\pi/3$:

$$\begin{aligned}\phi_{23} &= \int_{x_2}^{x_3} Ax\frac{dy}{dx}dx \\ &= -\sqrt{3}A\int_{a/2}^{0} xdx \\ &= A\left(\frac{\sqrt{3}a^2}{8}\right) \\ &= -\frac{8\pi}{3\sqrt{3}a^2}\left(\frac{\sqrt{3}a^2}{8}\right) \\ &= -\pi/3.\end{aligned}$$

We have shown a linear vector potential (symmetric/centered about the y-axis) can host MZMs on a minimal Kitaev triangle's base corners. In general, this should be true for any odd function used

Larger Kitaev triangles

When considering larger Kitaev triangles we need to adjust the vector potential strength for linear and higher order vector potentials. Start with the botton left corner point, x_j, and look at its nearest neighbor along $\theta = \pi/3$, we denote this point with position x_l. If we look back at our integral formulation for the phase we have the general form of

$$\begin{aligned}\phi_{jl} &= A\int_{x_j}^{x_l} \frac{dy}{dx} x dx \\ &= \frac{\sqrt{3}A}{2}(x_l^2 - x_j^2) = \pi/3.\end{aligned}$$

We can rearrange to get

$$A = \frac{2\pi}{3\sqrt{3}} \frac{1}{x_l^2 - x_j^2}. \tag{2.14}$$

A more simplified solution follows. We use `nr` to denote the number of rows the triangle has, it is one of the first few defined variables in a given script. The positions x_j and x_l have simple linear relations in regards to `nr`. Due to the equilateral nature of our triangle and centering about the y-axis

$$x_l = \frac{-a}{2}(\mathtt{nr} - 1). \tag{2.15}$$

It's easy to see that $x_l = x_j + a/2$ which gives

$$x_l = \frac{-a}{2}(\mathtt{nr} - 2). \tag{2.16}$$

Now, the difference of the squares is

$$x_l^2 - x_j^2 = \frac{-a^2}{4}(2\texttt{nr} - 3). \tag{2.17}$$

Plugging back into our expression we find

$$\frac{8\pi}{3\sqrt{3}a^2(2\texttt{nr} - 3)}. \tag{2.18}$$

This is expression is easy to implement in code.

While I'm here might as well say how we calculate the phase term. Since the integral is easy to solve by hand with a simple quadratic expression we directly calculate it. There is a dy/dx term, both of those terms are calculated already for nearest neighbor and angle calculations, thus the $\pm$ is already accounted for. Therefore, our phase is simply

$$\phi = -\frac{B}{2}\frac{dy}{dx}[x_l^2 - x_j^2] \tag{2.19}$$

Chapter 3

Floquet Landau Levels

1. Introduction (Tahir's intro is fine, maybe in my own words, Floquet engineering)

   i Time dependent, motivation—QAHE gap but not QHE gap

   ii Floquet Theorem— quasi-energy spectrum

2. Results

   i square + A(t) (Tahir's perturbative calc)

   ii honeycomb + A(t) (Tahir's perturbative calc)

3. Discussion and future

Chapter 4

Conclusion and Discussion

What makes gauge potential unique in creating/tuning/manipulating new topoglical systems

Applications

Appendix A

Suitable Name

1. Majorana Number derivation

2. Other derivations not included in introduction