I. SECOND ORDER IN $\hbar\omega$ FOR DIRAC ($\mathbf{H}_0, H_1, H_2$): LLS FOR SECOND ORDER IN $\hbar\omega$

Graphene is the ever first two-dimensiona stable material, where the lattice structure is highly isotropic along the x and y directions, and thus called the honeycomb structure. The effective Hamiltonian of monolayer graphene [$H_0 = v(\sigma_x p_x + \sigma_y p_y)$] is

$$H = v(\sigma_x \Pi_x + \sigma_y \Pi_y), \tag{1}$$

where we have $\mathbf{\Pi} = \mathbf{p} - e\mathbf{A}/c$, here A is the verctor potential. For circularly polarized light, the potential vector is written as $\mathbf{A} = \mathbf{A}_0(-\sin(\omega t)\hat{x}, \cos(2\omega t)\sin(Kx)\hat{y})$, where ϕ is the phase of light for circular (0,π) or linear ($\pm\pi/2$), $A_0 = E_0/\omega$ with E_0 being the electric field strength and ω the frequency of light. The corresponding electric field is written as $\mathbf{E}$ =$E_0(\cos(\omega t)\hat{x}, \sin(Kx)\sin(2\omega t + \phi)\hat{y})$, which is linked by relation $\mathbf{E} = -\partial_t A$. OR, here we have two linearly polarized light such that [however, there is no differnece between two linearly or a circularly lights]

$$\mathbf{E}_1 = E\cos(\omega t), \mathbf{E}_2 = E\sin(Kx)\sin(2\omega t + \phi), \tag{2}$$

where one is propagating along y-axis and polarization is along x-axis. second is propagating along x-axis and polarization along y-axis. The electric field in terms of vector potential is

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} = \{\mathbf{E_1}\cos(\omega\mathbf{t}), \mathbf{E_2}\sin(\mathbf{Kx})\sin(\mathbf{2}\omega\mathbf{t} + \phi), \mathbf{0}\} \tag{3}$$

and the vector potential A can be obtained as

$$\mathbf{A} = (-\frac{E}{\omega}\sin(\omega t), \frac{E}{2\omega}\cos(2\omega t)\sin(Kx), 0) \tag{4}$$

After using eq. 4 into eq.1, the time periodic part of the Hamiltonian from in eq. 1 is

$$\begin{aligned} H_{n=\pm} &= (1/T)\int_0^T \{-\sigma_x V_y \sin(\omega t) + \sigma_y V_x \cos(2\omega t)\} e^{in\omega t} dt \\ H_\pm &= (1/T)\left[-ni\frac{\pi V_y}{\omega}\sigma_x + \frac{\pi V_x}{\omega}\sigma_y\right], \omega = 2\pi/T \end{aligned} \tag{5}$$

Form above equation 5, we have

$$\begin{aligned} H_{+1} &= \frac{1}{2}\left[-iV_y\sigma_x\right], H_{-1} = \frac{1}{2}\left[iV_y\sigma_x\right], \\ H_{+2} &= \frac{1}{2}\left[V_x\sigma_y\right], H_{-2} = \frac{1}{2}\left[V_x\sigma_y\right] \end{aligned} \tag{6}$$

where

$$V_y = \frac{evE}{\omega}, V_x = \frac{evE}{2\omega}\sin(Kx). \tag{7}$$

Detail derivation for eq. 6 are obtained by using eq.7 as

$$\begin{aligned}
[H_{-1}, H_{+1}] &= H_{-1}H_{+1} - H_{+1}H_{-1} = \frac{1}{4}\{[iV_y\sigma_x]\,[-iV_y\sigma_x] - [-iV_y\sigma_x]\,[iV_y\sigma_x]\} \\
[H_{-2}, H_{+2}] &= H_{-2}H_{+2} - H_{+2}H_{-2} = \frac{1}{4}\{[V_x\sigma_y]\,[V_x\sigma_y] - \frac{1}{2}\,[V_x\sigma_y]\,\frac{1}{2}\,[V_x\sigma_y]\} \\
&= 0
\end{aligned} \tag{8}$$

and Hamiltonian due to ± 2 is nonzero $[H_{\pm 2}]$.

II. SECOND ORDER DERIVATION FOR DIRAC CASE

Here we have

$$\begin{aligned}
\hat{H}^3_{eff} = &\frac{[\hat{H}_{-1},[\hat{H}_0,\hat{H}_1]]}{2(\hbar\omega)^2} + \frac{[\hat{H}_1,[\hat{H}_0,\hat{H}_{-1}]]}{2(\hbar\omega)^2} - \frac{[\hat{H}_1,[\hat{H}_{-2},\hat{H}_1]]}{3(\hbar\omega)^2} \\
&- \frac{[\hat{H}_{-1},[\hat{H}_{+2},\hat{H}_{-1}]]}{3(\hbar\omega)^2} + \frac{[\hat{H}_{-1},[\hat{H}_{-1},\hat{H}_2]]}{6(\hbar\omega)^2} + \frac{[\hat{H}_1,[\hat{H}_{+1},\hat{H}_{-2}]]}{6(\hbar\omega)^2} \\
&+ \frac{[\hat{H}_{-2},[\hat{H}_0,\hat{H}_2]]}{8(\hbar\omega)^2} + \frac{[\hat{H}_2,[\hat{H}_0,\hat{H}_{-2}]]}{8(\hbar\omega)^2}
\end{aligned} \tag{9}$$

Now the first commutation in eq.7 is solved as $[V_y = \frac{evE_1}{\omega}\cos(Ky), V_x = \frac{evE_2}{2\omega}\cos(Kx), V_0 = evE/\omega = emvs/sm = ev]$.

III. FIRST COMMUTATION IS

Here we have $[H_{+1} = \frac{1}{2}[-iV_y\sigma_x], H_{-1} = \frac{1}{2}[iV_y\sigma_x], H_{+2} = \frac{1}{2}[V_x\sigma_y], H_{-2} = \frac{1}{2}[V_x\sigma_y]]$ is $[[\hat{H}_{-1},[\hat{H}_0,\hat{H}_1]$, and $[\hat{H}_{+1},[\hat{H}_0,\hat{H}_{-1}]]$

$$\begin{aligned}
[\hat{H}_{-1},[\hat{H}_0,\hat{H}_1] &= v\{\sigma_x p_x + \sigma_y p_y\} \times \frac{1}{2}[-iV_y\sigma_x] - \frac{1}{2}[-iV_y\sigma_x]\,v\{\sigma_x p_x + \sigma_y p_y\} \\
&= -\frac{1}{2}ivV_y\sigma_0 p_x - \frac{1}{2}V_y v\sigma_z p_y + \frac{1}{2}iV_y v\sigma_0 p_x - \frac{1}{2}vV_y\sigma_z p_y \\
&= -\frac{1}{2}[iV_y\sigma_x]\,V_y v\sigma_z p_y + V_y v\sigma_z p_y\frac{1}{2}[iV_y\sigma_x] \\
&= -\frac{1}{2}V_y^2 v\sigma_y p_y - \frac{1}{2}V_y^2 v\sigma_y p_y = -V_y^2 v\sigma_y p_y
\end{aligned} \tag{10}$$

and

$$
\begin{aligned}
[\hat{H}_{+1},[\hat{H}_0,\hat{H}_{-1}] &= v\{\sigma_x p_x+\sigma_y p_y\}\times\frac{1}{2}\left[iV_y\sigma_x\right]-\frac{1}{2}\left[iV_y\sigma_x\right]v\{\sigma_x p_x+\sigma_y p_y\} \\
&= \frac{1}{2}iV_y v\sigma_0 p_x+\frac{1}{2}vV_y\sigma_z p_y-\frac{1}{2}ivV_y\sigma_0 p_x+\frac{1}{2}vV_y\sigma_z p_y \\
&= \frac{1}{2}\left[-iV_y v\sigma_x\right]V_y\sigma_z p_y - V_y v\sigma_z p_y\frac{1}{2}\left[-iV_y\sigma_x\right] \\
&= -\frac{1}{2}V_y^2 v\sigma_y p_y-\frac{1}{2}V_y^2 v\sigma_y p_y = -V_y^2 v\sigma_y p_y
\end{aligned}
\tag{11}
$$

Eqs. 10 and 11 can be simplified by

$$
\begin{aligned}
\sigma_x\sigma_z &= \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=-i\sigma_y \\
\sigma_z\sigma_x &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=i\sigma_y
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\sigma_x\sigma_x=\sigma_y\sigma_y=\sigma_z\sigma_z \\
&\sigma_x\sigma_y = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}i&0\\0&-i\end{pmatrix}=i\sigma_z \\
&\sigma_y\sigma_x = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}-i&0\\0&i\end{pmatrix}=-i\sigma_z
\end{aligned}
\tag{13}
$$

Final result is

$$
\frac{[\hat{H}_{-1},[\hat{H}_0,\hat{H}_1]]}{2(\hbar\omega)^2}+\frac{[\hat{H}_1,[\hat{H}_0,\hat{H}_{-1}]]}{2(\hbar\omega)^2}=\frac{1}{2(\hbar\omega)^2}\{-2V_y^2 v\sigma_y p_y\}
\tag{14}
$$

IV. SECOND COMMUTATION IS

we have $[H_{+1}=\frac{1}{2}\left[-iV_y\sigma_x\right], H_{-1}=\frac{1}{2}\left[iV_y\sigma_x\right], H_{+2}=\frac{1}{2}\left[V_x\sigma_y\right], H_{-2}=\frac{1}{2}\left[V_x\sigma_y\right]]$ is $[[\hat{H}_1,[\hat{H}_{-2},\hat{H}_1]]]$

$$
[H_{-2},H_{+1}]=H_{-2}H_{+1}-H_{+1}H_{-2}=\frac{1}{2}\left[V_x\sigma_y\right]\frac{1}{2}\left[-iV_y\sigma_x\right]-\frac{1}{2}\left[-iV_y\sigma_x\right]\frac{1}{2}\left[V_x\sigma_y\right]
\tag{15}
$$

$$
=-\frac{1}{4}V_xV_y\sigma_z-\frac{1}{4}V_xV_y\sigma_z=-\frac{1}{2}V_xV_y\sigma_z,
$$

Therefore, using eqs. 13 and 14, we arrive at

$$H' = -\frac{1}{2} V_y V_x \sigma_z \tag{16}$$

Now we have $[H_- = \frac{1}{2}\left[iV_y\sigma_x + V_x\sigma_y\right], H_+ = \frac{1}{2}\left[-iV_y\sigma_x + V_x\sigma_y\right]]$

$$[[\hat{H}_1, [\hat{H}_{-2}, \hat{H}_1]]] = [\hat{H}_1, H'] = \hat{H}_1 H' - H'\hat{H}_1, \tag{17}$$

for eq.16, we have

$$\begin{aligned} &\frac{1}{2}\left[-iV_y\sigma_x\right] \times -\frac{1}{2} V_y V_x \sigma_z + \frac{1}{2} V_y V_x \sigma_z \frac{1}{2}\left[-iV_y\sigma_x\right] \\ &= \frac{1}{2} V_y^2 V_x \sigma_y \end{aligned} \tag{18}$$

Above eq.17 can be simplified to

$$\begin{aligned} \sigma_x\sigma_z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y \\ \sigma_y\sigma_z &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x \\ \sigma_z\sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y \\ \sigma_z\sigma_y &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_x \end{aligned} \tag{19}$$

Here we have $[H_{+1} = \frac{1}{2}\left[-iV_y\sigma_x\right], H_{-1} = \frac{1}{2}\left[iV_y\sigma_x\right], H_{+2} = \frac{1}{2}\left[V_x\sigma_y\right], H_{-2} = \frac{1}{2}\left[V_x\sigma_y\right]]$ is $[[\hat{H}_{-1}, [\hat{H}_{+2}, \hat{H}_{-1}]]]$

$$\begin{aligned} [H_{+2}, H_{-1}] &= H_{+2}H_{-1} - H_{-1}H_{+2} = \frac{1}{2}\left[V_x\sigma_y\right]\frac{1}{2}\left[iV_y\sigma_x\right] - \frac{1}{2}\left[iV_y\sigma_x\right]\frac{1}{2}\left[V_x\sigma_y\right] \\ &= \frac{1}{4}V_xV_y\sigma_z + \frac{1}{4}V_xV_y\sigma_z \\ &= \frac{1}{2}V_xV_y\sigma_z, \end{aligned} \tag{20}$$

Eq. 12 can be simplified to [for upper diagonal elements]

$$\begin{aligned}
\sigma_x\sigma_x &= \sigma_y\sigma_y = \sigma_z\sigma_z \\
\sigma_x\sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z \\
\sigma_y\sigma_x &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z
\end{aligned} \tag{21}$$

and lower diagonal elements of eq. 20 are

$$H' = \frac{1}{2}V_yV_x\sigma_z \tag{22}$$

Now we have $[H_{-1} = \frac{1}{2}[iV_y\sigma_x]]$

$$[[\hat{H}_{-1}, [\hat{H}_{+2}, \hat{H}_{-1}]]] = [\hat{H}_{-1}, H'] = \hat{H}_{-1}H' - H'\hat{H}_{-1}, \tag{23}$$

for eq.16, we have

$$\begin{aligned}
&= \frac{1}{2}[iV_y\sigma_x] \times \frac{1}{2}V_yV_x\sigma_z - \frac{1}{2}V_yV_x\sigma_z\frac{1}{2}[iV_y\sigma_x] \\
&= \frac{1}{2}V_y^2V_x\sigma_y
\end{aligned} \tag{24}$$

Above eq.17 can be simplified to

$$\begin{aligned}
\sigma_x\sigma_z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y \\
\sigma_z\sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y
\end{aligned} \tag{25}$$

Final result is

$$-\frac{[\hat{H}_1, [\hat{H}_{-2}, \hat{H}_1]]}{3(\hbar\omega)^2} - \frac{[\hat{H}_{-1}, [\hat{H}_{+2}, \hat{H}_{-1}]]}{3(\hbar\omega)^2} = \frac{1}{3(\hbar\omega)^2}\{-V_y^2V_x\sigma_y\} \tag{26}$$

V. THIRD COMMUTATION IS

the second two terms in eq. 10 are

$$\frac{[\hat{H}_{-1}, [\hat{H}_{-1}, \hat{H}_2]]}{6(\hbar\omega)^2} + \frac{[\hat{H}_1, [\hat{H}_{+1}, \hat{H}_{-2}]]}{6(\hbar\omega)^2} \tag{27}$$

Here we have first term $[H_{+1} = \frac{1}{2}\left[-iV_y\sigma_x\right], H_{-1} = \frac{1}{2}\left[iV_y\sigma_x\right], H_{+2} = \frac{1}{2}\left[V_x\sigma_y\right], H_{-2} = \frac{1}{2}\left[V_x\sigma_y\right]]$ as

$$\begin{aligned}
[\hat{H}_{-1},[\hat{H}_{-1},\hat{H}_2]] &= \frac{1}{2}\left[iV_y\sigma_x\right]\frac{1}{2}\left[V_x\sigma_y\right] - \frac{1}{2}\left[V_x\sigma_y\right]\frac{1}{2}\left[iV_y\sigma_x\right] \\
&= -\frac{1}{2}\{V_yV_x\}\sigma_z \\
[\hat{H}_{-1},[\hat{H}_{-1},\hat{H}_2]] &= -\frac{1}{2}\left[iV_y\sigma_x\right]\frac{1}{2}\{V_yV_x\}\sigma_z + \frac{1}{2}\{V_yV_x\}\sigma_z\frac{1}{2}\left[iV_y\sigma_x\right] \\
&= -\frac{1}{4}V_y^2V_x\sigma_y - \frac{1}{4}V_y^2V_x\sigma_y \\
&= -\frac{1}{2}V_y^2V_x\sigma_y
\end{aligned} \tag{28}$$

and the second term is

$$\begin{aligned}
[\hat{H}_1,[\hat{H}_{+1},\hat{H}_{-2}]] &= \frac{1}{2}\left[-iV_y\sigma_x\right]\frac{1}{2}\left[V_x\sigma_y\right] - \frac{1}{2}\left[V_x\sigma_y\right]\frac{1}{2}\left[-iV_y\sigma_x\right] \\
&= \frac{1}{2}\{V_yV_x\}\sigma_z \\
[\hat{H}_1,[\hat{H}_{+1},\hat{H}_{-2}]] &= \frac{1}{2}\left[-iV_y\sigma_x\right]\frac{1}{2}\{V_yV_x\}\sigma_z - \frac{1}{2}\{V_yV_x\}\sigma_z\frac{1}{2}\left[-iV_y\sigma_x\right] \\
&= -\frac{1}{4}V_y^2V_x\sigma_y - \frac{1}{4}V_y^2V_x\sigma_y \\
&- \frac{1}{2}V_y^2V_x\sigma_y
\end{aligned} \tag{29}$$

$$\begin{aligned}
\sigma_x\sigma_x &= \sigma_y\sigma_y = \sigma_z\sigma_z \\
\sigma_x\sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z \\
\sigma_y\sigma_x &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z
\end{aligned} \tag{30}$$

The Final result is

$$\frac{[\hat{H}_{-1},[\hat{H}_{-1},\hat{H}_2]]}{6(\hbar\omega)^2} + \frac{[\hat{H}_1,[\hat{H}_{+1},\hat{H}_{-2}]]}{6(\hbar\omega)^2} = \frac{1}{6(\hbar\omega)^2}\{-V_y^2V_x\sigma_y\} \tag{31}$$

VI. FOURTH COMMUTATION IS $H_{+2} = \frac{1}{2}\left[V_x\sigma_y\right], H_{-2} = \frac{1}{2}\left[V_x\sigma_y\right]$

$$+\frac{[\hat{H}_{-2},[\hat{H}_0,\hat{H}_2]]}{8(\hbar\omega)^2} + \frac{[\hat{H}_2,[\hat{H}_0,\hat{H}_{-2}]]}{8(\hbar\omega)^2} \tag{32}$$

second missing

$$
\begin{aligned}
[\hat{H}_{-2},[\hat{H}_0,\hat{H}_2] &= v\{\sigma_x p_x+\sigma_y p_y\}\times\frac{1}{2}\left[V_x\sigma_y\right]-\frac{1}{2}\left[V_x\sigma_y\right]v\{\sigma_x p_x+\sigma_y p_y\} \\
&= i\sigma_z\frac{1}{2}vV_xp_x+\frac{1}{2}\sigma_z vK\cos Kx(V_0\hbar)+v\sigma_0p_y\frac{1}{2}V_x+i\sigma_z\frac{1}{2}vV_xp_x-v\sigma_0p_y\frac{1}{2}V_x \\
&= +vV_xi\sigma_zp_x+\frac{1}{2}\sigma_z vK\cos Kx(V_0\hbar) \\
[\hat{H}_{-2},[\hat{H}_0,\hat{H}_2] &= \frac{1}{2}\left[V_x\sigma_y\right]\{vV_xi\sigma_zp_x+\frac{1}{2}\sigma_z vK\cos Kx(V_0\hbar)\}-\{vV_xi\sigma_zp_x+\frac{1}{2}\sigma_z vK\cos Kx(V_0\hbar)\}\frac{1}{2}\left[V_x\sigma_y\right] \\
&= -vV_x^2\sigma_xp_x+i\frac{1}{2}\left[V_x\right]\sigma_x vK\cos Kx(V_0\hbar)\}-\frac{1}{2}vV_x\sigma_x(-i\hbar KV_0\cos Kx) \\
&= -vV_x^2\sigma_xp_x+iV_xvK\cos Kx(V_0\hbar)\sigma_x
\end{aligned} \tag{33}
$$

where

$$
\begin{aligned}
\sigma_x\sigma_z &= \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=-i\sigma_y \\
\sigma_z\sigma_x &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=i\sigma_y
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
\sigma_x\sigma_x &= \sigma_y\sigma_y=\sigma_z\sigma_z \\
\sigma_x\sigma_y &= \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}i&0\\0&-i\end{pmatrix}=i\sigma_z \\
\sigma_y\sigma_x &= \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}-i&0\\0&i\end{pmatrix}=-i\sigma_z
\end{aligned} \tag{35}
$$

with

$$
\begin{aligned}
\sigma_y\sigma_z &= \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&i\\i&0\end{pmatrix}=i\sigma_x \\
\sigma_z\sigma_y &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}=-i\sigma_x
\end{aligned} \tag{36}
$$

The Final rsult is

$$
+\frac{[\hat{H}_{-2},[\hat{H}_0,\hat{H}_2]]}{8(\hbar\omega)^2}+\frac{[\hat{H}_2,[\hat{H}_0,\hat{H}_{-2}]]}{8(\hbar\omega)^2}=\frac{2}{8(\hbar\omega)^2}\{-vV_x^2\sigma_xp_x+iV_xvK\cos Kx(V_0\hbar)\sigma_x\} \tag{37}
$$

VII. FINAL RESULT OF EQS. 14, 26, 31 AND 37 INTO EQ. 9

where we arrive at

$$H_{eff} = H_0 + \frac{1}{2(\hbar\omega)^2}\{-2V_y^2 v\sigma_y p_y\} + \frac{1}{3(\hbar\omega)^2}\{-V_y^2 V_x\sigma_y\} + \frac{1}{6(\hbar\omega)^2}\{-V_y^2 V_x\sigma_y\} \\ + \frac{2}{8(\hbar\omega)^2}\{-vV_x^2\sigma_x p_x + iV_x vK\cos Kx(V_0\hbar)\sigma_x\} \tag{38}$$

Here, first 4 terms in eq. 38 are simplified as $[V_y = \frac{evE}{\omega}, V_x = \frac{evE}{2\omega}\sin Kx]$, $[\frac{evE}{2\omega} = \frac{eVm}{ms}s = eV]$, $[\{1 - \frac{V_y^2}{\hbar^2\omega^2}\} = C = 1 - 0.02 = 1$, for graphene experiments]

$$H_{eff} = v\sigma_x p_x + v\sigma_y p_y - \frac{V_y^2 v}{\hbar^2\omega^2}\sigma_y p_y - \frac{V_y^2 V_x}{2\hbar^2\omega^2}\sigma_y, \frac{e^2v^2E^2}{\omega^2}\frac{vp_y}{\hbar^2\omega^2} = \frac{e^2V^2m^2}{m^2s^2}s^2\frac{eV}{e^2V^2} \\ = \frac{v}{C}\sigma_x p_x + v\sigma_y p_y - \frac{V_y^2\frac{evE}{2\omega}Kx}{2\hbar^2\omega^2 C}\sigma_y, \frac{V_y^2 V_x}{2\hbar^2\omega^2} = \frac{e^2v^2E^2}{\omega^2}\frac{\frac{evE}{2\omega}Kx}{\hbar^2\omega^2} = \frac{evE}{2\omega}Kx \\ = \frac{v}{C}\sigma_x p_x + v\sigma_y\{p_y - \frac{V_y^2\frac{eE}{2\omega}}{2\hbar^2\omega^2 C}Kx\}, \tag{39}$$

where magnetic field B is

$$B = \frac{V_y^2\frac{eE}{2\omega}}{2\hbar^2\omega^2 C}K, \frac{eEK}{\omega} = e\frac{Vs}{m^2}, eB = e\frac{V_y^2\frac{E}{2\omega}}{2\hbar^2\omega^2 C}K \tag{40}$$

Therefore leat anisotropy and zero gap Dirac spectrum is still good to go. The magnetic field strength is [100nm and 3 times electric field]

$$B = \frac{V_y^2\frac{E}{2\omega}}{2\hbar^2\omega^2}K = 1Tesla \tag{41}$$

The last term in eq. 38 is $[V_x = \frac{evE}{2\omega}\sin Kx]$ simplified as;

$$\frac{1}{4(\hbar\omega)^2}\{-vV_x^2\sigma_x p_x + iV_x vK\cos Kx(V_0\hbar)\sigma_x\} \\ = \frac{v\sigma_x}{4(\hbar\omega)^2}\frac{1}{2}\{-V_x^2 p_x - p_x V_x^2\} = 0 \tag{42}$$

The proof of eq. 41 is

$$A^+ - B = A + B, \frac{1}{2}\{A - A^+\} = -B \\ A^+ = A + 2B, \frac{1}{2}\{A + A^+\} + B \\ A = \frac{1}{2}\{A + A^+\} + \frac{1}{2}\{A - A^+\} \\ = \frac{1}{2}\{A + A^+\} - B \\ A + B = \frac{1}{2}\{A + A^+\} \tag{43}$$