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MFML HMWK 8

2. Let G be the set of zero-mean Gaussian random variables. Since if X and Y are zero-mean Gaussian $aX+bY$ is also zero-mean Gaussian for all $a,b \in \mathbb{R}$, it should be clear that G is a linear vector space. It is easy to check that the correlation $E[XY] = \langle X,Y \rangle$ is a valid inner product on G that induces the norm $\|X\|^2 = \text{var}(X)$.

a) Translate the Pythagorean Theorem and the Cauchy-Schwartz inequality into statements about correlation and variance of pairs of Gaussian random variables. Discuss.

Pythagorean: $\langle X,Y \rangle = 0 \Rightarrow \|X+Y\|^2 = \|X\|^2 + \|Y\|^2$

$$E[XY] = 0 \Rightarrow \text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$$

In general the $\text{var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$

$$= \text{var}(X) + \text{Var}(Y) + 2(E[XY] - E[X]E[Y])$$

$E[XY] = 0 \Rightarrow$ $\quad = \text{var}(X) + \text{Var}(Y) + 2(-E[X]E[Y])$

$E[XY]$ also implies

$E[X]E[Y] = 0$ for a Gaussian

$$= \text{var}(X) + \text{var}(Y) \checkmark$$

For the Gaussian distribution the Pythagorean theorem implies independence if X and Y are uncorrelated.

Cauchy-Schwartz: $|\langle X,Y \rangle| \le \|X\| \, \|Y\|$

$$\begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}$$

$$|E[XY]| \le \sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}$$

$$|E[XY]| \le \sqrt{E[XX^T]E[YY^T]}$$

$$E[XY]^2 \le E[XX^T]E[YY^T]$$

$$\sigma_{XY}^2 \le \sigma_X^2 \cdot \sigma_Y^2$$

The C-S version of Gaussian distributions states that the covariance between X and Y squared can atmost be the variance of X squared times the Variance of Y squared. This makes sense as the covariance between two random variables cannot be higher than the variances of each.

b) Let $X_1, \ldots, X_N \in \mathcal{G}$ with correlations captured in entries of the matrix $R_{ij} = E[X_i X_j]$. Let Y be another arbitrary point in $\mathcal{G}$. Using what you know about finding the closest point (in induced norm) to a subspace, describe how to find the best linear predictor of Y from the $\{X_n\}$. That is, describe how to solve the optimization problem

$$\underset{\omega_1, \ldots, \omega_N}{\text{minimize}} \ \text{var}\left(Y - \sum_{n=1}^{N} w_n X_n\right)$$

using the covariance matrix R and the correlations $E[YX_n]$. Note that we mean "best" above in the mean-square sense, as

$$\text{var}\left(Y - \sum_{n=1}^{N} w_n X_n\right) = E\left[\left(Y - \sum_{n=1}^{N} w_n X_n\right)^2\right]$$

$$= E\left[\left(Y - \sum_{n=1}^{N} w_n X_n\right)\left(Y - \sum_{n=1}^{N} w_n X_n\right)\right] = E\left[YY^T - 2Y\sum_{n=1}^{N} w_n X_n + \left(\sum_{n=1}^{N} w_n X_n\right)^2\right]$$

$$= E[YY^T] - E[2Y w^T X] + E[w^T X X^T w]$$

$$= E[YY^T] - 2w^T E[YX] + w^T E[XX^T] w$$

$$= E[YY^T] - 2\sum_{n=1}^{N} w_n E[YX_n] + w^T R w$$

$$0 = \nabla_w \text{var}\left(Y - \sum_{n=1}^{N} w_n X_n\right) = \nabla_w\left(E[YY^T] - 2\sum_{n=1}^{N} w_n E[YX_n] + w^T R w\right)$$

$$0 = -E[YX] + Rw$$

$$Rw = E[YX]$$

$$\boxed{\hat{w} = R^{-1} E[YX]}$$

c) Let $\hat{\omega}_1, \ldots, \hat{\omega}_N$ be the solution to the optimization program above. Give the simplest expression possible for $E[Y(\hat{\omega}_1 X_1 + \ldots + \hat{\omega}_N X_N)]$.

$$E[Y \hat{\omega}^T X] = \hat{\omega}^T E[YX] = (R^{-1}E[YX])^T E[YX] = E[YX]^T R^{-T} E[YX]$$

$$= \boxed{E[YX]^T R^{-1} E[YX]}$$

symmetric

3. Let X_1 and X_2 be Gaussian random variables with

$E[X_1]=E[X_2]=0$ $\quad E[X_1^2]=E[X_2^2]=2$ $\quad E[X_1X_2]=1$.

Let $Y=X_1+X_2$. Suppose we observe $Y=1.5$. Find the conditional densities $f_{X_1}(x|Y=1.5)$ and $f(x_2|Y=1.5)$.

$$\begin{bmatrix} Y \\ X_1 \end{bmatrix} \sim \text{Normal}\left(0, \begin{bmatrix} E[YY] & E[YX_1] \\ E[X_1Y] & E[X_1X_1] \end{bmatrix}\right) \longrightarrow \begin{bmatrix} R_Y & R_{YX} \\ R_{XY} & R_X \end{bmatrix}$$

$E[X_1X_1]=2$

$E[YX_1]=E[(X_1+X_2)X_1]=E[X_1^2+X_2X_1]=E[X_1^2]+E[X_1X_2]=2+1=3$

similarly $E[X_1Y]=3$

$E[YY]=E[(X_1+X_2)(X_1+X_2)]=E[X_1^2]+E[X_2^2]+2E[X_1X_2]=2+2+2=6$

$$\begin{bmatrix} X_o \\ X_h \end{bmatrix} \sim \text{Normal}\left(0, \begin{bmatrix} R_o & R_{oh} \\ R_{oh}^T & R_h \end{bmatrix}\right)$$

$X_h|X_o=x_o \sim \text{Normal}(R_{oh}^T R_o^{-1} x_o, R_h - R_{oh}^T R_o^{-1} R_{oh})$ From pg 23 of notes.

$f_{X_1}(x|Y=1.5) = X_1|Y=\overset{y_o}{1.5} \sim \text{Normal}(R_{YX}^T R_Y^{-1} y_o, R_X - R_{YX}^T R_Y^{-1} R_{YX})$

$R_{YX}^T R_Y^{-1} y_o = 3 \cdot \frac{1}{6} \cdot \frac{3}{2} = \frac{3}{4}$

$R_X - R_{YX}^T R_Y^{-1} R_{YX} = 2 - 3 \cdot \frac{1}{6} \cdot 3 = 2 - \frac{3}{2} = \frac{1}{2}$

$\boxed{f_{X_1}(x|Y=1.5) \sim \text{Normal}\left(\frac{3}{4}, \frac{1}{2}\right)}$

$\boxed{\text{Due to the symmetry of the problem } f_{X_1}(x|Y=1.5) = f_{X_2}(x|Y=1.5) = \text{Normal}\left(\frac{3}{4}, \frac{1}{2}\right)}$

4. Let X be a Gaussian random vector taking values in $\mathbb{R}^N$, let E be a Gaussian random vector taking values in $\mathbb{R}^M$, and let A be a $M \times N$ matrix. We have

$$X \sim \text{Normal}(0, R_x), \quad E \sim \text{Normal}(0, R_e), \quad X, E \text{ independent},$$

We will make observation of the random vector.

$$Y = AX + E$$

a) From your work above, it is clear that Y is a Gaussian random vector in $\mathbb{R}^M$ and that $E[Y]=0$. Find the covariance matrix for the Gaussian random vector $\begin{bmatrix} X \\ Y \end{bmatrix}$ that takes values in R^{N+M}.

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \text{Normal}\left(0, \begin{bmatrix} R_x & R_{xy} \\ R_{xy}^T & R_y \end{bmatrix}\right) \qquad R_x = E[XX^T] = R_x$$

$$R_{XY} = E[XY^T] = E[X(AX+E)^T] = E[X(E^T + X^T A^T)]$$
$$= E[XE^T] + E[XX^T A^T] = E[X]E[E^T] + E[XX^T]A^T = 0 + R_x A^T = R_x A^T$$
$$R_{XY}^T = (R_x A^T)^T = A R_x^T = A R_x$$
$$R_Y = E[YY^T] = E[(AX+E)(AX+E)^T] = E[AXX^TA^T + 2AXE^T + EE^T]$$
$$= AE[XX^T]A^T + 2AE[XE^T] + E[EE^T]$$
$$= AR_xA^T + R_e$$

$$\boxed{\begin{bmatrix} R_x & R_x A^T \\ AR_x & AR_xA^T + R_e \end{bmatrix}}$$

b) Suppose we observe $Y=y$. What is the minimum mean-square error estimate of X given $Y=y$?

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim \text{Normal}\left(0, \begin{bmatrix} AR_xA^T + R_e & AR_x \\ R_xA^T & R_x \end{bmatrix}\right)$$

$$\hat{x} = R_{YX}^T R_Y^{-1} y = \boxed{R_x A^T (AR_xA^T + R_e)^{-1} y}$$ ← pg. 23 of notes

c) Suppose $R_x = \sigma_x^2 I$ and $R_e = \sigma_e^2 I$. In this case, your MMSE estimator should look familiar, and you should see immediately that $\hat{x}_{MMSE}$ is in the row space of A. What are the $\hat{\alpha}_n$ in the expression below?

$\hat{x}_{MMSE} = \sum_{n=1}^{N} \alpha_n v_n$ where the v_n are the right singular vectors of A

$$\hat{x}_{MMSE} = R_x A^T (A R_x A^T + R_e)^{-1} y$$
$$= \sigma_x^2 I A^T (A \sigma_x^2 I A^T + \sigma_e^2 I)^{-1} y$$
$$= \sigma_x^2 A^T (\sigma_x^2 A A^T + \sigma_e^2 I)^{-1} y$$
$$= A^T (A A^T + \frac{\sigma_e^2}{\sigma_x^2} I)^{-1} y$$
$$= A^T (A A^T + \delta I)^{-1} y$$
$$= V \Sigma U^T (U \Sigma V^T V \Sigma U^T + \delta I)^{-1} y$$
$$= V \Sigma U^T (U \Sigma^2 U^T + U \delta U^T)^{-1} y$$
$$= V \Sigma U^T (U (\Sigma^2 + \delta I)^{-1} U^T) y$$
$$= V \Sigma (\Sigma^2 + \delta I)^{-1} U^T y$$
$$= \sum_{n=1}^{N} v_n \underbrace{\left(\frac{\sigma_n}{\sigma_n^2 + \delta}\right) u_n^T y}_{\alpha_n} = \sum_{n=1}^{N} \alpha_n v_n$$

Let $\frac{\sigma_e^2}{\sigma_x^2} = \delta$ for convenience

SVD: $A = U \Sigma V^T$

$A^T = V \Sigma U^T$

Let σ_n be the singular values in $\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \ddots \end{bmatrix}$

$$\boxed{\alpha_n = \left(\frac{\sigma_n}{\sigma_n^2 + \frac{\sigma_e^2}{\sigma_x^2}}\right) u_n^T y}$$

d) Take R_x and R_e as in part c), and assume that A has full column rank. What is MSE $E[\|\hat{x}_{MMSE} - X\|_2^2]$ of the MMSE estimate $\hat{x}_{MMSE}$?

$$E[\|\hat{x}_{MMSE} - X\|_2^2 \mid Y = y] = \mathrm{trace}(R_x - R_{yx}^T R_y^{-1} R_{yx})$$
$$= \mathrm{trace}(R_x - R_x A^T (A R_x A^T + R_e)^{-1} A R_x)$$
$$= \mathrm{trace}(\sigma_x^2 I - \sigma_x^2 I A^T (A \sigma_x^2 I A^T + \sigma_e^2 I)^{-1} A \sigma_x^2 I)$$
$$= \mathrm{trace}(\sigma_x^2 I - A^T (A A^T + \delta I)^{-1} A \sigma_x^2)$$
$$= \sigma_x^2 (\mathrm{trace}(I - A^T (A A^T + \delta I)^{-1} A))$$

Let $\delta = \frac{\sigma_e^2}{\sigma_x^2}$

$$A = U\Sigma V^T \quad A^T = V\Sigma U^T$$

$$= \sigma_x^2 \left(\mathrm{trace}\left(I - V\Sigma U^T \left(U\Sigma V^T V \Sigma U^T + U\delta U^T \right)^{-1} U \Sigma V^T \right)\right)$$

$$= \sigma_x^2 \left(\mathrm{trace}\left(I - V\Sigma U^T \left(U\Sigma^2 U + U\delta U^T \right)^{-1} U \Sigma V^T \right)\right)$$

$$= \sigma_x^2 \left(\mathrm{trace}\left(I - V\Sigma U^T \left(U (\Sigma^2 + \delta I)^{-1} U^T \right) U \Sigma V^T \right)\right)$$

$$= \sigma_x^2 \left(\mathrm{trace}\left(I - V\Sigma (\Sigma^2 + \delta I)^{-1} \Sigma V^T \right)\right)$$

$$= \sigma_x^2 \left(\mathrm{trace}(I) - \mathrm{trace}\left(V\Sigma (\Sigma^2 + \delta I)^{-1} \Sigma V^T \right)\right)$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_N \end{bmatrix}$$

$$= \sigma_x^2 \left(\mathrm{trace}(I) - \mathrm{trace}\left(\Sigma V^T V \Sigma (\Sigma^2 + \delta I)^{-1} \right)\right)$$

$$= \sigma_x^2 \left(\mathrm{trace}(I) - \mathrm{trace}\left(\Sigma^2 (\Sigma^2 + \delta I)^{-1} \right)\right)$$

$$= \sigma_x^2 N - \sigma_x^2 \sum_{n=1}^{N} \frac{\sigma_n^2}{\sigma_n^2 + \frac{\sigma_\varepsilon^2}{\sigma_x^2}}$$

$$= \boxed{\sum_{n=1}^{N} \sigma_x^2 \left(1 - \frac{\sigma_n^2}{\sigma_n^2 + \frac{\sigma_\varepsilon^2}{\sigma_x^2}} \right)}$$

5. Let A be an $M \times N$ matrix. Suppose we have computed $P = (AA^T + \delta I)^{-1}$ and used it to form the ridge estimate $\hat{x} = A^T P y$ for some observation vector $y \in \mathbb{R}^M$. We now add a row to A, and an entry to y, forming

$$A' = \begin{bmatrix} A \\ a_{M+1}^T \end{bmatrix}, \quad y' = \begin{bmatrix} y \\ y_{M+1} \end{bmatrix}.$$

Using the Schur complement, describe how to form the updated ridge estimate $\hat{x}' = A'^T (A' A'^T + \delta I)^{-1} y'$ using only a few matrix-vector multiplications, dot products, and vector additions (and no additional matrix inversions).

$$P' = (A' A'^T + \delta I_{M+1})^{-1} = \left(\begin{bmatrix} A \\ a_{M+1}^T \end{bmatrix} \begin{bmatrix} A^T & a_{M+1} \end{bmatrix} + \delta I_{M+1} \right)^{-1}$$

$$= \left(\begin{bmatrix} AA^T + \delta I & A a_{M+1} \\ a_{M+1}^T A^T & a_{M+1}^T a_{M+1} + \delta \end{bmatrix} \right)^{-1}$$

$$P'^{-1} = \begin{bmatrix} AA^T + \delta I & A a_{M+1} + \delta \\ a_{M+1}^T A^T + \delta & a_{M+1}^T a_{M+1} + \delta \end{bmatrix}$$

Let $M = P'^{-1}$ then $M^{-1} = P'$

$M_{11} = AA^T + \delta I = P^{-1}$

$M_{12} = A a_{M+1}$

$M_{21} = a_{M+1}^T A^T$

$M_{22} = a_{M+1}^T a_{M+1} + \delta$

$M_{11}^{-1} = P$

Schur Complement:

$$P' = M^{-1} = \begin{bmatrix} M_{11}^{-1} + M_{11}^{-1} M_{12} S^{-1} M_{21} M_{11}^{-1} & -M_{11}^{-1} M_{12} S^{-1} \\ -S^{-1} M_{21} M_{11}^{-1} & S^{-1} \end{bmatrix}$$

$$S = M_{22} - M_{21} M_{11}^{-1} M_{12}$$

S is scalar in our case:

$$S = a_{M+1}^T a_{M+1} + \delta - a_{M+1}^T A^T P A a_{M+1}$$

$$\underline{S = \delta + a_{M+1}^T (I - A^T P A) a_{M+1}}$$

$$\frac{1}{a+b} = \frac{1}{b} + \frac{1}{a}$$

$$P'_{11} = P + P a_{M+1}^T A^T S^{-1} A a_{M+1} P$$

$$P'_{12} = -P A a_{M+1} S^{-1}$$

$$P'_{21} = -S^{-1} a_{M+1}^T A^T P$$

$$P'_{22} = S^{-1}$$

$$\hat{x}' = A'^T P' y'$$
$$= \begin{bmatrix} A^T & a_{m+1} \end{bmatrix} \begin{bmatrix} P'_{11} & P'_{12} \\ P'_{21} & P'_{22} \end{bmatrix} \begin{bmatrix} y \\ y_{m+1} \end{bmatrix}$$
$$= \begin{bmatrix} A^T & a_{m+1} \end{bmatrix} \begin{bmatrix} P'_{11} y + P'_{12} y_{m+1} \\ P'_{21} y + P'_{22} y_{m+1} \end{bmatrix}$$
$$= A^T(P'_{11} y + P'_{12} y_{m+1}) + a_{m+1}(P'_{21} y + P'_{22} y_{m+1})$$

$$\hat{x} \quad \boxed{\begin{aligned} &= A^T\left[(P + P a^T_{m+1} A^T S^{-1} A a_{m+1} P) y - P A a_{m+1} S^{-1} y_{m+1}\right] \\ &\quad + a_{m+1}\left[-S^{-1} a^T_{m+1} A^T P y + S^{-1} y_{m+1}\right] \end{aligned}}$$

In addition to abusing the Schur complement to remove additional matrix inversions many of the above martrix-matrix, matrix-vector computations can be precomputed.

Let $A^T P = C_1$

$A^T P y = C_1 y = C_2$

$a^T_{m+1} A^T = C_3$

$A a_{m+1} = C_3^T = C_4$

$A^T P A = C_1 A = C_5$

$S = \delta + a^T_{m+1}(I - A^T P A) a_{m+1} = \delta + a^T_{m+1}(I - C_5) a_{m+1} = C_6 \quad S^{-1} = \frac{1}{C_6} = C_7$

Then

$$\boxed{\hat{x} = C_2 + C_7\left(C_1 C_3 C_4 P y - P C_4 y_{m+1} - a_{m+1}\left(a^T_{m+1} C_2 + y_{m+1}\right)\right)}$$

6. In this problem, $X \in \mathbb{R}^D$ is a Gaussian random vector with $E[X]=0$ and $E[XX^T]=R$.

(a) Suppose that R is block diagonal in that it can be written

$$R=\begin{bmatrix} R_a & 0 \\ 0 & R_b \end{bmatrix},$$

where is R_a is a $D_a \times D_a$ symmetric positive definite matrix, R_b is a $D_b \times D_b$ sym+def matrix, $D_a+D_b=D$, and the zero matrices above are the appropriate sizes so that the dimensions work out. Argue that R^{-1} is also block diagonal and the joint pdf can be factored as,

$$f_{X_1,\ldots,X_D}(x_1,\ldots,x_D)=f_{X_1,\ldots,X_{D_a}}(x_1,\ldots,x_{D_a}) \cdot f_{X_{D_a+1},\ldots,X_D}(x_{D_a+1},\ldots,x_D)$$

meaning that the random vectors

$$X_A=\begin{bmatrix} X_1 \\ \vdots \\ X_{D_a} \end{bmatrix} \quad X_B=\begin{bmatrix} X_{D_a+1} \\ \vdots \\ X_D \end{bmatrix}$$

are independent.

R is invertible if R_a and R_b are invertible.

$$R^{-1}=\begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix} \qquad RR^{-1}=\begin{bmatrix} R_a & 0 \\ 0 & R_b \end{bmatrix}\begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix}=\begin{bmatrix} R_aR_a^{-1} & 0 \\ 0 & R_bR_b^{-1} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

✓

$$f_X(x) \propto \exp(-\tfrac{1}{2}x^TR^{-1}x)$$

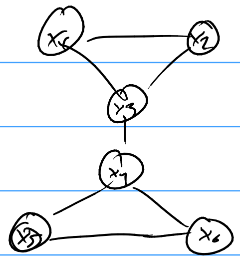
$$\propto \exp\left(-\tfrac{1}{2}\begin{bmatrix} x_A \\ x_b \end{bmatrix}^T\begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix}\right)$$

$$\propto \exp(-\tfrac{1}{2}x_A^TR_a^{-1}x_a - \tfrac{1}{2}x_b^TR_b^{-1}x_b)$$

$$\propto \exp(-\tfrac{1}{2}x_a^TR_a^{-1}x_a)\exp(-\tfrac{1}{2}x_b^TR_b^{-1}x_b)$$

Thus $f_X(x)=f_{X_a}(x_a)\cdot f_{X_b}(x_b) \Rightarrow$ X_a, X_b are independent.

b) Suppose that X has inverse covariance R^{-1} whose non-zero entries are described by the graph.



Argue as rigorously as you can that the random vector $(X_1, X_2)|X_3$ is independent of the random vector $(X_4, X_5, X_6)|X_3$.

Let us construct the corresponding matrix of the graph:

$$R^{-1} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array} \\ \begin{bmatrix} X & X & X & O & O & O \\ X & X & X & O & O & O \\ X & X & X & X & O & O \\ O & O & X & X & X & X \\ O & O & O & X & X & X \\ O & O & O & X & X & X \end{bmatrix} \end{array}$$

'X' refers to non-zero and 'O' refers to a zero entry

Now if we are conditioning on X_3 then the corresponding X_3 row and column entries are not considered, thus our updated R^{-1} matrix would be block diagonal and look like the following:

$$R^{-1} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array} \\ \begin{bmatrix} X & X & \cancel{X} & O & O & O \\ X & X & \cancel{X} & O & O & O \\ \cancel{X} & \cancel{X} & \cancel{X} & \cancel{X} & \cancel{O} & \cancel{O} \\ O & O & \cancel{X} & X & X & X \\ O & O & \cancel{O} & X & X & X \\ O & O & \cancel{O} & X & X & X \end{bmatrix} \end{array}$$

Let R^{-1}_{12} be the first block and R^{-1}_{456} be the second block

$$\therefore R^{-1} = \begin{bmatrix} R^{-1}_{12} & 0 \\ 0 & R^{-1}_{456} \end{bmatrix}$$

With the current matrix form we can use the results from part a) and show that $f_{X_{1,2,4,5,6}}(X_1, X_2, X_4, X_5, X_6 | X_3) = f_{X_1,X_2}(X_1, X_2 | X_3) \cdot f_{X_4,X_5,X_6}(X_4, X_5, X_6 | X_3)$

Thus $(X_1, X_2 | X_3)$ is independent of $(X_4, X_5, X_6 | X_3)$. ✓