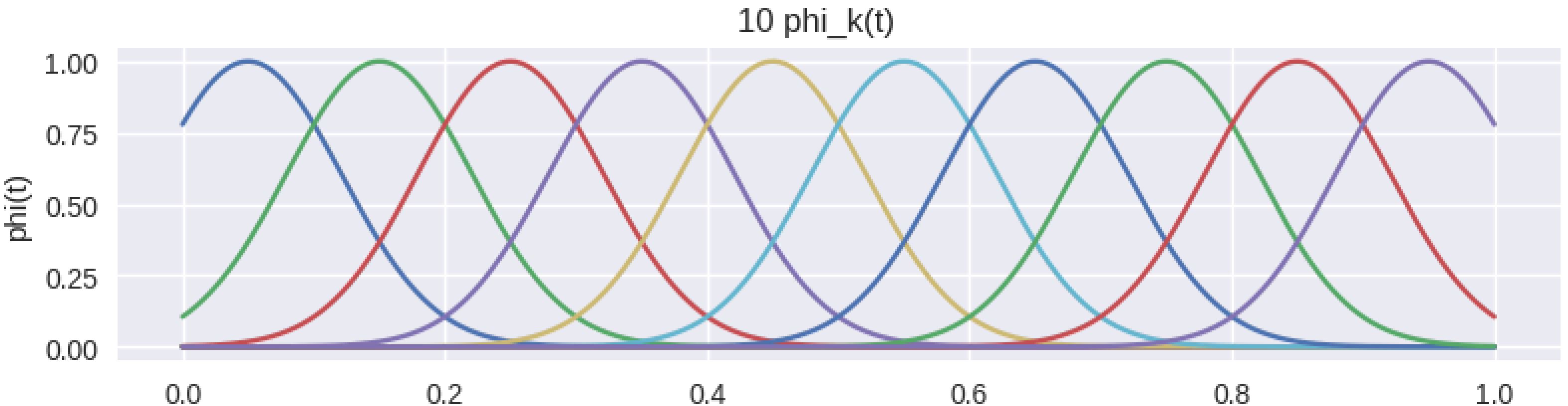
Problem 1

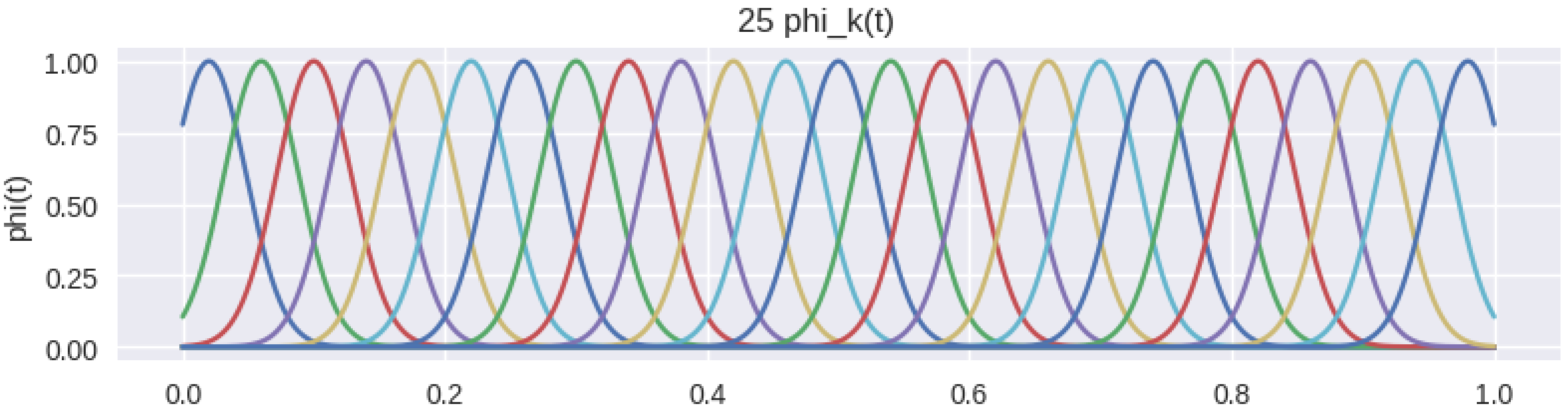
This past week we have discussed orthonormal basis, or in other words, orthobasis. An orthobasis is a basis for a space where <vi, vj> = 1 if i = j and = 0 if i =/= j. An important application of representing a span basis as an orthobasis is the
ability to treat inner products on Hilbert spaces as dot products on Euclidean space (Parseval's Theorem). This gives us a coefficient representation of elements in Hilbert spaces
and lets us manipulate infinite dimensional objects like functions in an intuitive way. This property is examplified when performing linear approximations,
i.e. approximating functions that can be represented with a known set of basis functions. This gives us a very powerful technique of data fitting, one of the biggest concepts we are learning in this class.

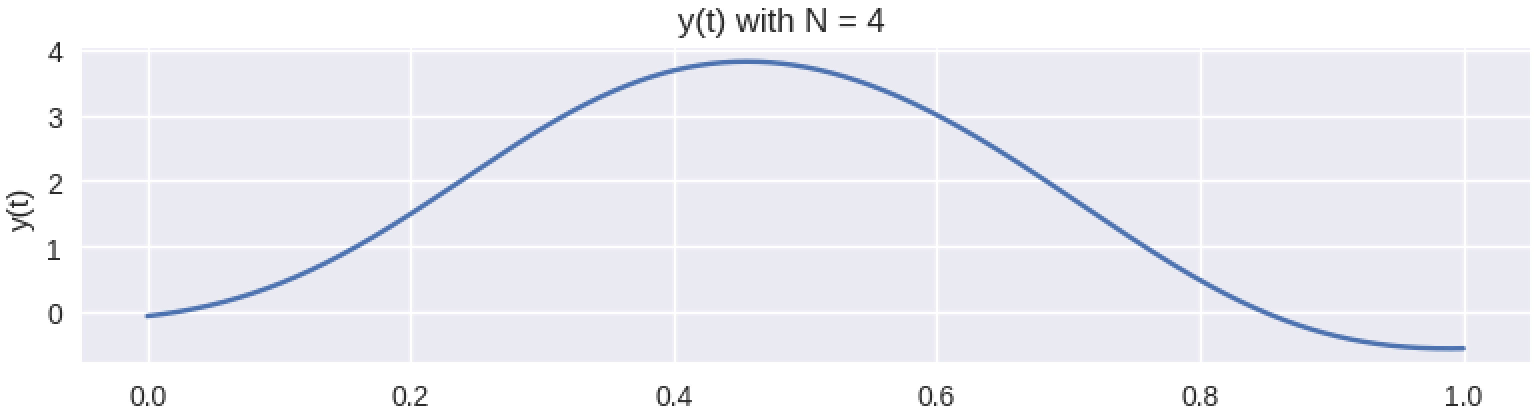
```
import sys
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
import scipy.integrate as integrate

mpl.style.use('seaborn')

phi = lambda z: np.exp(-z**2)

def plot_all_phi(N):
    t = np.linspace(0,1,1000)

    fig = plt.figure()
    fig.suptitle(str(N) + " phi_k(t)")
    ax = fig.add_subplot(111)
    for kk in range(N):
        ax.plot(t, phi(N*t - (kk + 1) + 0.5))

    ax.set_xlabel("t")
    ax.set_ylabel("phi(t)")

    plt.show()

def part_a():
    N_list = [10, 25]
    for N in N_list:
        plot_all_phi(N)

part_a()

def plot_lin_comb_of_phi(N, a):
    t = np.linspace(0,1,1000)
    y = np.zeros(1000)
    for i in range(N):
        y = y + a[i]*phi(N*t - (i + 1) + 0.5)

    fig = plt.figure()
    fig.suptitle("y(t) with N = " + str(N))
    ax = fig.add_subplot(111)

    ax.plot(t, y)

    ax.set_xlabel("t")
    ax.set_ylabel("y(t)")

    plt.show()

def part_b():
    a = [-1/2, 3, 2, -1]
    N = len(a)
    plot_lin_comb_of_phi(N, a)

part_b()

def estimate_f(N):
    t = np.linspace(0,1,1000)

    f = lambda z: (z < 0.25) * (4 * z) + (z >= 0.25) * (z < 0.5) * \
            (-4 * z + 2) - (z >= 0.5) * np.sin(14 * np.pi * z)

    f_phik = lambda z: f(z) * phi(N*z - (i + 1) + 0.5)
    phij_phik = lambda z: phi(N*z - (j + 1) + 0.5) * phi(N*z - (i + 1) + 0.5)
```

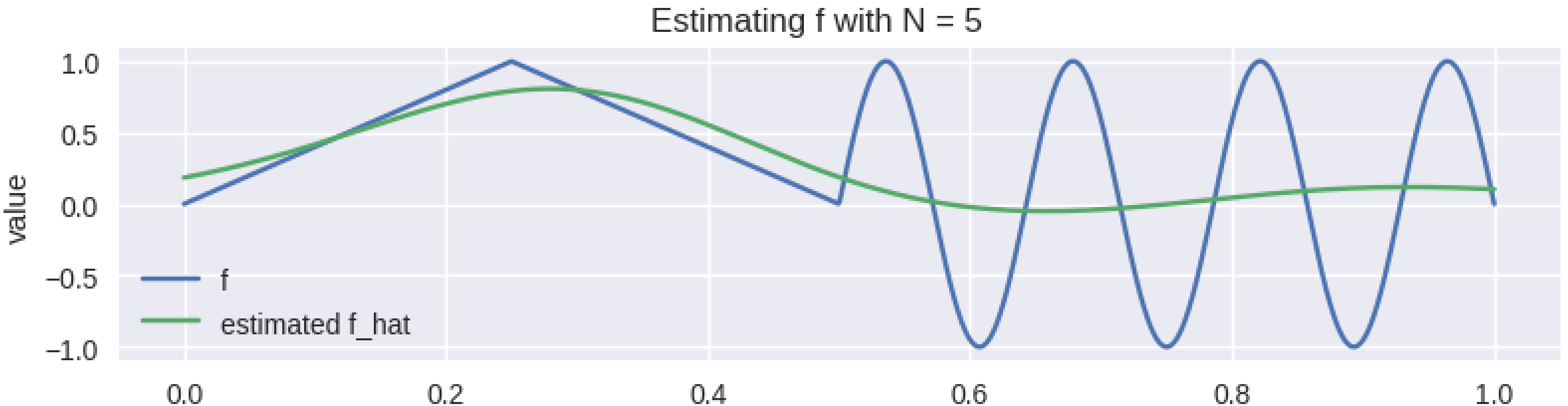
```
    G = np.ones(shape=(N,N))
    b = np.ones(shape=(N,1))

    for i in range(N):
        for j in range(N):
            G[i, j] = integrate.quad(phij_phik, 0, 1)[0]

        b[i, :] = [integrate.quad(f_phik, 0, 1)[0]]

    a = np.linalg.inv(G) @ b

    f_hat = np.zeros(1000)
    for i in range(N):
        f_hat = f_hat + a[i]*phi(N*t - (i + 1) + 0.5)

    fig = plt.figure()
    fig.suptitle("Estimating f with N = " + str(N))
    ax = fig.add_subplot(111)

    ax.plot(t, f(t), label="f")
    ax.plot(t, f_hat, label="estimated f_hat")

    ax.set_xlabel("t")
    ax.set_ylabel("value")
    ax.legend()

    plt.show()

def part_c():
    N_list = [5, 10, 20, 50]

    for N in N_list:
        estimate_f(N)

part_c()
```



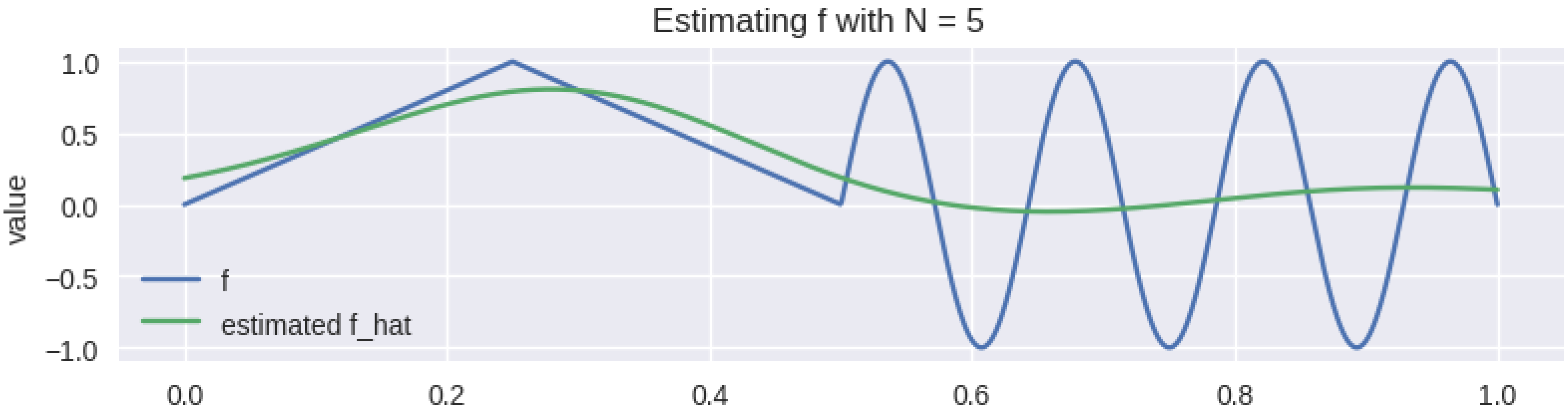
10 phi_k(t)
phi(t)
1.00
0.75
0.50
0.25
0.00
0.0
0.2
0.4
0.6
0.8
1.0


25 phi_k(t)
phi(t)
1.00
0.75
0.50
0.25
0.00
0.0
0.2
0.4
0.6
0.8
1.0

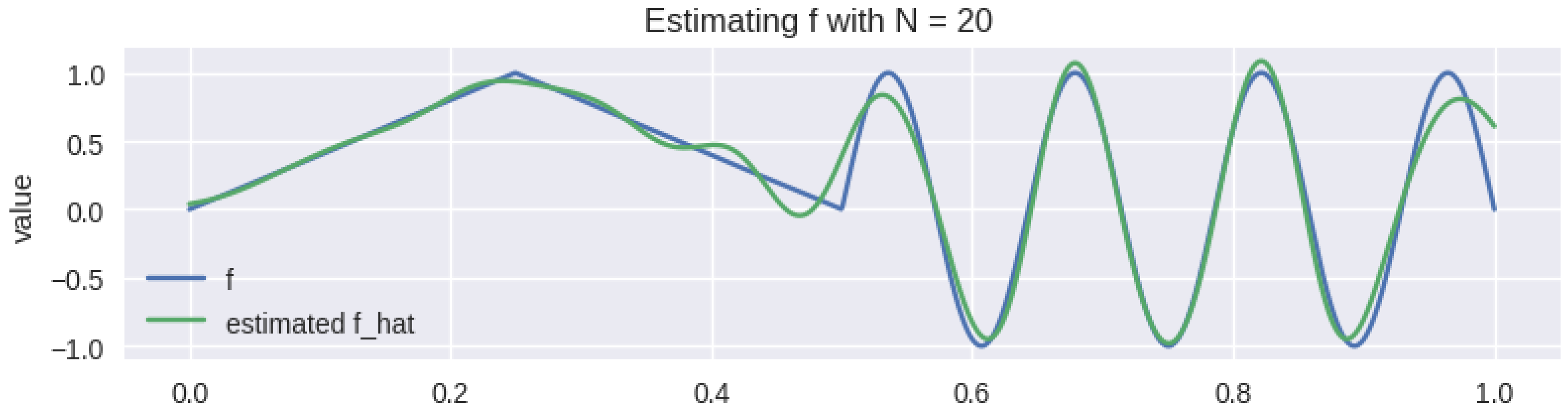

y(t) with N = 4
y(t)
4
3
2
1
0
0.0
0.2
0.4
0.6
0.8
1.0

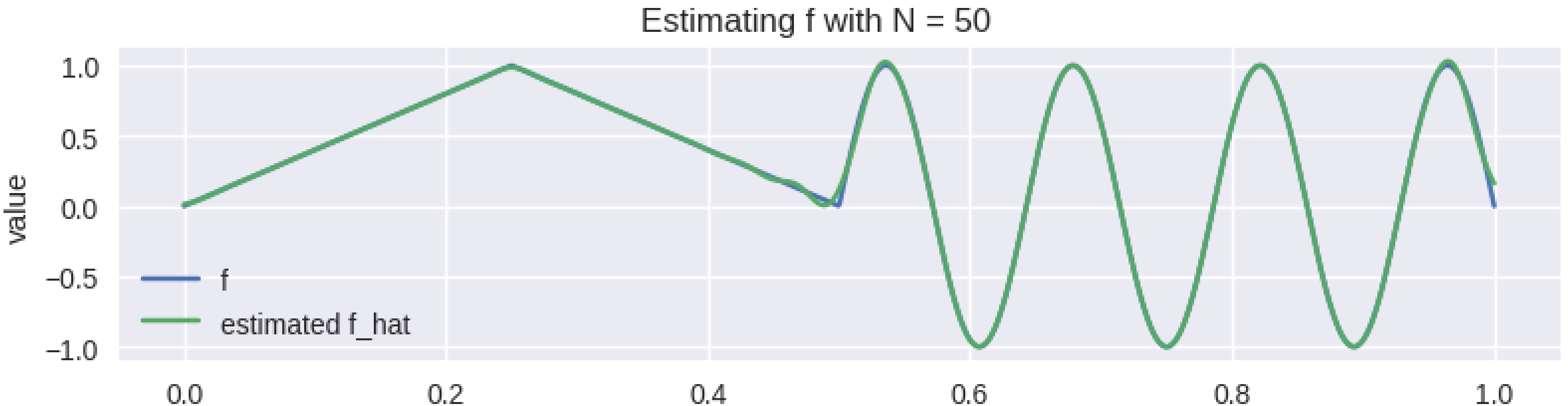


Estimating f with N = 5
value
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
f
estimated f_hat



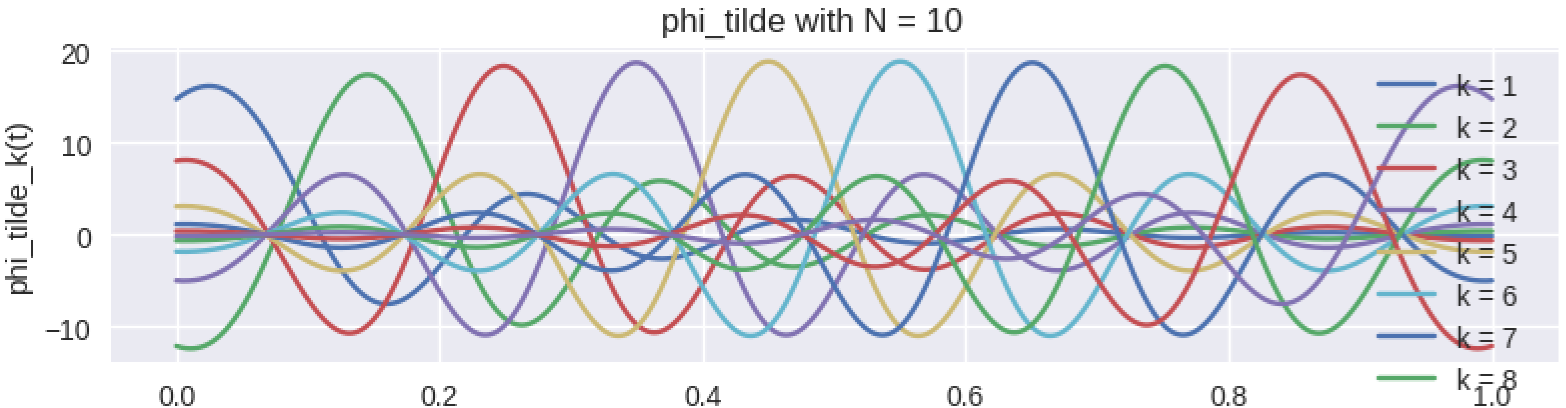
Estimating f with N = 5
value
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
f
estimated f_hat


Estimating f with N = 20
value
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
f
estimated f_hat



Estimating f with N = 50
1.0
0.5
0.0
−0.5
−1.0
value
0.0
0.2
0.4
0.6
0.8
1.0
f
estimated f_hat

```
import sys
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
import scipy.integrate as integrate

mpl.style.use('seaborn')

phi = lambda z: np.exp(-z**2)

def plot_all_phi_tilde(N):
    t = np.linspace(0,1,1000)

    phij_phik = lambda z: phi(N*z - (j + 1) + 0.5) * phi(N*z - (i + 1) + 0.5)

    G = np.ones(shape=(N,N))

    for i in range(N):
        for j in range(N):
            G[i, j] = integrate.quad(phij_phik, 0, 1)[0]

    H = np.linalg.inv(G)

    fig = plt.figure()
    fig.suptitle("phi_tilde with N = " + str(N))
    ax = fig.add_subplot(111)

    for i in range(N):
        phi_tilde_k = np.zeros(1000)
        phi_tilde_k = sum([H[i, l] * phi(N*t - (l + 1) + 0.5) for l in
range(N)])
        ax.plot(t, phi_tilde_k, label="k = " + str(i +1))

    ax.set_xlabel("t")
    ax.set_ylabel("phi_tilde_k(t)")
    ax.legend()

    plt.show()

def prob7():
    N = 10
    plot_all_phi_tilde(N)

prob7()
```


phi_tilde with N = 10
phi_tilde_k(t)
20
10
0
−10
0.0
0.2
0.4
0.6
0.8
1.0
k = 1
k = 2
k = 3
k = 4
k = 5
k = 6
k = 7
k = 8

2.) As you know, a square $N \times N$ matrix G is invertible if

$$x_1 \neq x_2 \iff Gx_1 \neq Gx_2$$

That is, Gx is different for every different x. So if you can show that $Gx = 0$ only if $x = 0$, then you have shown that G is invertible.

a) Let $v_1, \dots, v_N$ be N linearly independent vectors in Hilbert Space, and let $T = \text{span}\{v_1, \dots, v_N\}$. Show that if $z \in T$ and $\langle v_n, z \rangle = 0$ for all $n = 1, \dots, N$, then it must be true that $z = 0$.

$$z = \sum_{n=1}^{N} \alpha_n v_n$$

$$\langle z, z \rangle = \langle \sum_{n=1}^{N} \alpha_n v_n, z \rangle$$

$$= \sum_{n=1}^{N} \alpha_n \langle v_n, z \rangle$$

$$= \sum_{n=1}^{N} \alpha_K (0)$$

$$= 0$$

$$\boxed{\langle z, z \rangle = 0 \text{ iff } z = 0 \quad \text{Thus } z = 0}$$

b) Show that if $v_1, \dots, v_N$ are N linearly independent vectors in a Hilbert Space, then the Gram Matrix

$$G = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_N, v_1 \rangle \\ \vdots & \ddots & \\ \langle v_1, v_N \rangle & & \langle v_N, v_N \rangle \end{bmatrix}$$

is invertible.

Let x be a vector in the Hilbert Space and let the elements of x be the set $\{\alpha_1, \dots, \alpha_N\}$ where $\{\alpha_1, \dots, \alpha_N\}$ satisfies the following linear combination:

$$x = \sum_{n=1}^{N} \alpha_n v_n$$

$$Gx = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_N, v_1 \rangle \\ \vdots & & \\ \langle v_1, v_N \rangle & & \langle v_N, v_N \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = 0$$

$$\sum_{n=1}^{N} \alpha_n \langle v_n, v_k \rangle = 0 \quad \forall\, k \in [1, N]$$

$$\langle \sum_{n=1}^{N} \alpha_n v_n, v_k \rangle = 0 \quad \forall\, k \in [1, N] \rightarrow \langle x, v_k \rangle = 0 \quad \forall\, k \in [1, N]$$

Using our result from part a)

$$\langle x, v_k \rangle = 0 \Rightarrow x = 0$$

Thus $Gx = 0$ only if $x = 0$

4.) a) Argue that for the u_2 produced above that $\|u_2\| > 0$, and so Ψ_2 is well defined.

By definition of the norm $\|u_2\|$ must be ≥ 0. For Ψ_2 to be welldefined, $\|u_2\| > 0$. $\|u_2\| > 0$ when u_2 is not the zero vector.

$$u_2 = v_2 - \sum_{l=1}^{2-1} \langle v_2, \Psi_l \rangle \Psi_l \qquad \text{Let } \alpha = \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2}$$

$$= v_2 - \langle v_2, \Psi_1 \rangle \Psi_1 = v_2 - \langle v_2, \frac{v_1}{\|v_1\|} \rangle \frac{v_1}{\|v_1\|} = v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 = v_2 - \alpha v_1$$

Since v_2 and v_1 are linearly independent, there is no such scalar, α, that makes the statement $v_2 = \alpha v_1$ true. Thus $u_2 = v_2 - \alpha v_1$ will never be equal to the zero vector. Thus $\|u_2\| > 0$.

b) Argue that $\text{Span}\{\Psi_1, \Psi_2\} = \text{Span}\{v_1, v_2\}$, and show that Ψ_1 and Ψ_2 are orthonormal. Hence $\{\Psi_1, \Psi_2\}$ is an orthonormal basis for $\text{Span}\{v_1, v_2\}$.

$\Psi_1 = \frac{v_1}{\|v_1\|}$ $\quad \Psi_2 = \frac{u_2}{\|u_2\|}$ $\quad$ By constriction Ψ_k are normal vectors. ✓

Let us check orthogonality.

$$\langle \Psi_1, \Psi_2 \rangle = \langle \frac{v_1}{\|v_1\|}, \frac{v_2}{\|u_2\|} - \frac{\langle v_2, v_1 \rangle}{\|u_2\| \|v_1\|^2} v_1 \rangle = \langle v_1, v_2 \rangle \frac{1}{\|v_1\| \|u_2\|} - \langle \frac{v_1}{\|v_1\|}, \frac{\langle v_2, v_1 \rangle v_1}{\|v_1\|^2 \|u_2\|} \rangle$$

$$= \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|u_2\|} - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^3 \|u_2\|} \langle v_1, v_1 \rangle = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|u_2\|} - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^3 \|u_2\|} \|v_1\|^2 = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|u_2\|} - \frac{\langle v_2, v_1 \rangle}{\|v_1\| \|u_2\|} = 0 ✓$$

Let us show: $\text{Span}\{\Psi_1, \Psi_2\} = \text{Span}\{v_1, v_2\}$

$\Psi_1 = \frac{v_1}{\|v_1\|} = \alpha v_1 \quad$ Let $\alpha = \frac{1}{\|v_1\|}$

$$\Psi_2 = \frac{u_2}{\|u_2\|} = \frac{v_2 - \langle v_2, \frac{v_1}{\|v_1\|} \rangle \frac{v_1}{\|v_1\| \|u_2\|}}{\|u_2\|} = \frac{v_2}{\|u_2\|} - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \frac{v_1}{\|u_2\|} = \beta v_2 - \gamma v_1$$

Let x be any vector in the span of $\{v_1, v_2\}$, and be represented as follows:

$$x = \sum_{i=1}^{2} a_i v_i = a_1 v_1 + a_2 v_2$$

Let us try to represent x as a linear combination of $\{\Psi_1, \Psi_2\}$:

$$x = \sum_{i=1}^{2} b_i \Psi_i = b_1 \Psi_1 + b_2 \Psi_2 = b_1 \alpha v_1 + b_2 \beta v_2 + b_2 \gamma v_1$$

b_1 and b_2 can be choosen such that $b_1 \alpha + b_2 \gamma = a_1$ and $b_2 \beta = a_2$.

Thus the span $\{\Psi_1, \Psi_2\}$ is equal to the span of $\{v_1, v_2\}$.

4.c | Use induction to show that $\{\psi_1, \dots, \psi_N\}$ is an orthobasis for T. Part of this argument will be ensuring that $u_k \neq 0$.

We have proven the base case ($n=k=2$) for this statement from 4a and 4b.

Let us now prove the induction step, i.e.

if the $n=k$ case holds true then the $n=k+1$ case is true.

Let us first show that $\|u_k\| > 0$ so that ψ_k can be well defined.

$$u_k = v_k - \sum_{l=1}^{k-1} \langle v_k, \psi_l \rangle \psi_l$$

$= v_k - \sum_{l=1}^{k-1} \alpha_l \psi_l$ Let us rewrite $\langle v_k, \psi_l \rangle$ as a constant α_l.

$= v_k - \sum_{l=1}^{k-1} \beta_l v_l$ $\sum_{l=1}^{k-1} \alpha_l \psi_l$ can be rewritten as $\sum_{l=1}^{k-1} \beta_l v_l$.

This can be done since $\{\psi_1, \dots, \psi_{k-1}\}$ and $\{v_1, \dots, v_{k-1}\}$ span the same subspace.

Since v_k is linearly independent from $\{v_1, \dots, v_{k-1}\}$, there is no such set $\{\beta_1, \dots, \beta_{k-1}\}$ in which $\sum_{l=1}^{k-1} \beta_l v_l$ will equal v_k. Therefore $v_k - \sum_{l=1}^{k-1} \beta_l v_l$ does not equal zero and $\|v_k - \sum_{l=1}^{k-1} \beta_l v_l\| = \|u_k\|$ must be greater than zero, since $u_2 \neq 0$.

Now let us show that the span $\{\psi_1, \dots, \psi_k\} = \text{span}\{v_1, \dots, v_k\}$

By construction ψ_k is $\frac{u_k}{\|u_k\|}$ and is a normal vector. ✓

Show $\langle \psi_i, \psi_k \rangle = 0 \ \forall i \in [1, k-1]$ and is thus an orthogonal vector.

$$\langle \psi_i, \psi_k \rangle = \langle \frac{\psi_i}{\|\psi_i\|}, \frac{v_k}{\|u_k\|} - \sum_{l=1}^{k-1} \langle v_k, \psi_l \rangle \psi_l \rangle = \frac{\langle \psi_i, v_k \rangle}{\|\psi_i\| \|u_k\|} - \langle \frac{\psi_i}{\|\psi_i\|}, \sum_{l=1}^{k-1} \frac{\langle v_k, \psi_l \rangle \psi_l}{\|\psi_l\|^2 \|u_k\|} \rangle$$

$$= \frac{\langle \psi_i, v_k \rangle}{\cancel{\|\psi_i\|}^{1} \|u_k\|} - \sum_{l}^{k-1} \frac{\langle v_k, \psi_l \rangle}{\cancel{\|\psi_l\|^2}^{1} \|u_k\|} \cancel{\langle \psi_l, \psi_l \rangle}^{1} = \frac{\langle \psi_i, v_k \rangle}{\|u_k\|} - \sum_{l}^{k-1} \frac{\langle v_k, \psi_l \rangle}{\|u_k\|} = \frac{\langle \psi_i, v_k \rangle}{\|u_k\|} - \frac{\langle v_k, \psi_i \rangle}{\|u_k\|} = 0 \checkmark$$

other terms are 0 based on previous $\langle v_k, \psi_{l \neq i} \rangle = 0$

Let us show: Span $\{\psi_1, \dots, \psi_k\}$ = Span $\{v_1, \dots, v_k\}$ Let $\psi_i = \sum_{n=1}^{i} c_{in} v_n \ \forall i \in [1, k]$

Let x be any vector in the span of $\{v_1, \dots, v_k\}$ and be represented as follows:

$x = \sum_{i=1}^{k} a_i v_i$ Let us try to represent x as a linear combination of $\{\psi_1, \dots, \psi_k\}$:

$x = \sum_{i=1}^{k} b_i \psi_i = \sum_{i=1}^{k} b_i \sum_{n=1}^{i} c_{in} v_n$. b_i can be choosen such that $b_i \sum_{n=1}^{i} c_{in} = a_i$. Thus the span $\{\psi_1, \dots, \psi_k\}$ is equal to the span of $\{v_1, \dots, v_k\}$.

5. a) The vector space $L_2([0,1]^2)$ is the space of signals of two variables, $x(s,t)$ with $s,t \in [0,1]$ such that

$$\int_0^1 \int_0^1 |x(s,t)|^2 \, ds\, dt < \infty$$

Let $\{\psi_k(t), k \geq 0\}$ be an orthobasis for $L_2([0,1])$. Define

$$v_{k,l}(s,t) = \psi_k(s)\psi_l(t), \quad k,l \geq 0$$

Show that $\{v_{k,l}(s,t), k,l \geq 0\}$ is an orthobasis for $L_2([0,1]^2)$. You need to argue that the $v_{k,l}$ are orthonormal and that they span $L_2([0,1]^2)$.

Let us first show that the $v_{k,l}$ are orthonormal.

For any $v_{m,n}$ and $v_{o,p}$ in the $v_{k,l}$, $\langle v_{m,n}, v_{o,p}\rangle = \begin{cases} 1 & \text{if } m,n \neq o,p \\ 0 & \text{if } m,n \neq o,p \end{cases}$

$$\langle v_{m,n}, v_{m,n}\rangle = \int_0^1\int_0^1 v_{m,n}(s,t)\cdot v_{m,n}(s,t)\, ds\, dt$$

$$= \int_0^1\int_0^1 \psi_m(s)\psi_n(t)\psi_m(s)\psi_n(t)\, ds\, dt$$

$$= \int_0^1 \psi_m(s)\cdot\psi_m(s)\, ds \cdot \int_0^1 \psi_n(t)\psi_n(t)\, dt \qquad \int_0^1 \psi_m(s)\cdot\psi_m(s)\, ds = 1$$

$$= 1\cdot 1 = 1 \ \checkmark \qquad \text{since } \{\psi_k\} \text{ is an orthonormal basis.}$$

$$\langle v_{m,n}, v_{o,p}\rangle = \int_0^1\int_0^1 v_{m,n}(s,t)\, v_{o,p}(s,t)\, ds\, dt$$

$$= \int_0^1\int_0^1 \psi_m(s)\psi_n(t)\psi_o(s)\psi_p(t)\, ds\, dt$$

$$= \int_0^1 \psi_m(s)\psi_o(s)\, ds \cdot \int_0^1 \psi_n(t)\psi_p(t)\, dt$$

$$= 0\cdot 0 = 0 \ \checkmark$$

Therefore the $v_{k,l}$ are an orthonormal set.

Let $x(s,t)$ be any function that $: [0,1]^2 \to \mathbb{R}$. $x(s,t)$ can be written as the following: $x(s,t) = \sum_l \alpha_l(s)\psi_l(t)$. Noticing that $\alpha_l(s)$ itself is a function it can be written as follows: $\alpha_l(s) = \sum_k \beta_k \psi_k(s)$. Thus $x(s,t)$ can be written as $\sum_l\sum_k \beta_k \psi_k(s)\psi_l(t) = \sum_l\sum_k \beta_k v_{k,l}(s,t)$

$x(s,t) = \sum_l\sum_k \beta_k v_{k,l}(s,t) = \sum_l\sum_k \beta_k \psi_k(s)\psi_l(t)$ i.e. the set $v_{k,l}(s,t)$ spans the space, $[0,1]^2 \to \mathbb{R}$.

Let us show that $x(s,t)$ is also a function in $L_2([0,1]^2)$ to complete the proof.

$\int_0^1\int_0^1 |x(s,t)|^2 ds\,dt = \int_0^1\int_0^1 \left|\sum_l\sum_k \beta_k\psi_k(s)\psi_l(t)\right|^2 ds\,dt = \int_0^1\int_0^1 \left|\sum_l\sum_k \beta_k\psi_k(s)\psi_l(t)\right|\left|\sum_l\sum_k \beta_k\psi_k(s)\psi_l(t)\right| ds\,dt$

$< \infty$. Since $\beta_k < \infty \ \forall k$ and the integral is closed on $[0,1]$ for $\psi_k(s)$ and $\psi_l(t)$.

5 b) Given an orthobasis for $L_2([0,1])$, describe how to construct an orthobasis for $L_2([0,1]^D)$ - the space of functions of D continuous-valued variables $x(t)$ such that

$$\int_0^1 \cdots \int_0^1 |x(t)|^2 \, dt_1 \cdots dt_D < \infty.$$

Based on our results from 5a) we have shown that constructing an orthobasis for $L_2([0,1]^2)$ can be done as a composition of orthobasis functions on the $L_2([0,1])$ space. Specifically $v_{k,l} = \psi_k(s)\psi_l(t)$ is an orthobasis for $L_2([0,1]^2)$. Extending this logic, $v_{k_1 \ldots k_D}$ can be an orthobasis for $L_2([0,1]^D)$ where $\boxed{v_{k_1 \ldots k_D} = \prod_{i=1}^{D} \psi_{k_i}(t_i).}$ Essentially there is an orthobasis function corresponding to each input dimension.

6.) Let $v_1,\dots,v_k$ be a set of vectors in a subspace T of an inner product space. Prove that if the only vector in T that is orthogonal to all of the v_k is 0, then $\{v_1,\dots,v_k\}$ is a basis for T.

For this proof, let us prove the equivalent contrapositive:

If $\{v_1,\dots,v_k\}$ is not a basis for T, then there is a non-zero vector in T that is orthogonal to all of the v_k.

Let T' be the space spanned by $\{v_1,\dots,v_k\}$.

Let x be any vector in T, but not in T'!

Let us define $\hat{x}$ as the projection of x onto T':

$$\hat{x} = \sum_k \frac{\langle x, v_k\rangle v_k}{\|x\|\|v_k\|}$$

We also know that since $\hat{x}$ is a projection it lies in T' and can be written as a linear combination of $\{v_1,\dots,v_k\}$ as such:

$$\hat{x} = \sum_k \beta_k v_k$$

We can also write it as follows, considering the norm of x and v_k.

$$\hat{x} = \sum_k \frac{\alpha_k}{\|x\|\|v_k\|} v_k$$

Let us set the two expressions of $\hat{x}$ equal to each other to gain some insight.

$\hat{x} = \hat{x}$ component wise

$$\frac{\langle x, v_k\rangle v_k}{\|x\|\|v_k\|} = \frac{\alpha_k}{\|x\|\|v_k\|} v_k \quad \forall k$$

$$\langle x, v_k\rangle = \alpha_k \quad \forall k \qquad \alpha_k = \sum_\ell \beta_\ell \langle v_\ell, v_k\rangle$$

$$\langle x, v_k\rangle = \sum_\ell \beta_\ell \langle v_\ell, v_k\rangle = \langle \sum_\ell \beta_\ell v_\ell, v_k\rangle = \langle \hat{x}, v_k\rangle \quad \forall k$$

$$\langle x, v_k\rangle - \langle \hat{x}, v_k\rangle = 0 \quad \forall k$$

$$\langle x - \hat{x}, v_k\rangle = 0 \quad \forall k$$

Thus the vector $(x-\hat{x})$ is orthogonal to all v_k and is non-zero since $x \neq \hat{x}$. (x is not in T', but $\hat{x}$ is)

7.) Fix $N=10$ and compute the dual basis vectors of the bump basis from Problem 3.

That is find $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_{10}$ so that if

$$x(t) = \sum_{k=1}^{10} \alpha_k \Phi_k(t)$$

we can compute $\{\alpha_k\}$ using

$$\alpha_k = \int_0^1 x(t)\, \tilde{\Phi}_k(t)\, dt$$

$$\tilde{\Phi}_k = \sum_{\ell=1}^{10} H_{k,\ell}\, \Phi_\ell(t)$$

$$\boxed{\tilde{\Phi}_k = \sum_{\ell=1}^{10} (G^{-1})_{k,\ell}\, \Phi_\ell(t)}$$ where $G = \begin{bmatrix} \langle \Phi_1, \Phi_1 \rangle & \cdots & \langle \Phi_1, \Phi_N \rangle \\ \langle \Phi_2, \Phi_1 \rangle & \ddots & \vdots \\ \vdots & & \\ \langle \Phi_N, \Phi_1 \rangle & \cdots & \langle \Phi_N, \Phi_N \rangle \end{bmatrix}$

and $\langle \Phi_j, \Phi_k \rangle = \int_0^1 \Phi_j(t)\, \Phi_k(t)\, dt$

Turn in a plot of each of the 10 $\tilde{\Phi}_k(t)$.

→ See next page.