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MFM2 HMwork5

2. A functional on a Hilbert space $\mathcal{F}: S \to \mathbb{R}$ is bounded if there exists a constant C such that

$$\mathcal{F}(x) \leq C\|x\|_S, \text{ for all } x \in S$$

Argue directly that the sampling (or point evaluation) operator on $L_2([0,1])$

$$\mathcal{F}_\tau(f) = f(\tau)$$

is not bounded.

Let us choose the following class of functions from $L_2([0,1])$

$$f_n(t) = \begin{cases} 0 & \text{if } t < -\frac{1}{2n} \\ \sqrt{n} & \text{if } -\frac{1}{2n} \leq t \leq \frac{1}{2n} \\ 0 & \text{if } t > \frac{1}{2n} \end{cases}$$

$$\|f_n\| = \left(\int_{-\infty}^{\infty} |f(t)|^2 dt\right)^{1/2}$$
$$= \left(\int_{-\frac{1}{2n}}^{\frac{1}{2n}} |\sqrt{n}|^2 dt\right)^{1/2} = \left(\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n \, dt\right)^{1/2} = \left(nt\Big|_{-\frac{1}{2n}}^{\frac{1}{2n}}\right)^{1/2} = \left(n\left(\frac{1}{2n}+\frac{1}{2n}\right)\right)^{1/2} = \left(\cancel{n}\left(\frac{1}{\cancel{n}}\right)\right)^{1/2} = 1.$$

Thus the norm of this class of functions remains constant at 1. However, the sampling operation on $-\frac{1}{2n} \leq t \leq \frac{1}{2n}$ increases as a function of n $(\sqrt{n})$ and is thus not bounded.

$$\mathcal{F}_\tau(f) = f(\tau) \leq C\|f_s\|$$
$$f(\tau) \leq C(1)$$ is not true for all τ thus the sampling functional is not bounded.

3. Let $C'([0,1])$ be the space of functions on $[0,1]$ that is differentiable on $(0,1)$.

(a) Let D_τ be the functional that takes a $f \in C'([0,1])$ and returns the derivative at location τ: $D_\tau(f) = f'(\tau)$. Is D_τ linear? Continuous?

$$D_\tau(af+bg) = (af+bg)'(\tau) = (af)'(\tau) + (bg)'(\tau) = af'(\tau) + bg'(\tau)$$
$$= aD_\tau(f) + bD_\tau(g) \checkmark$$

$$\|f-g\|_s \leq \delta \Rightarrow |D_\tau(f) - D_\tau(g)| \leq \varepsilon$$

Let us show that D_τ is not continuous by showing the above is not true with a counter example.

Let $f_n(t) = \frac{1}{n}\sin(2\pi nt)$ and $f_{n+1}(t) = \frac{1}{n+1}\sin(2\pi(n+1)t)$.

Then
$$\|f_{n+1} - f_n\| = \left(\int_0^1 |f_{n+1}(t) - f_n(t)|^2 dt\right)^{1/2}$$
$$= \left(\int_0^1 (f_{n+1}(t) - f_n(t))^2 dt\right)^{1/2}$$
$$= \left(\int_0^1 (f_{n+1}(t)^2 + f_n(t)^2 - 2f_{n+1}(t)f_n(t))dt\right)^{1/2}$$
$$= \left(\int_0^1 \frac{1}{(n+1)^2}\sin^2(2\pi(n+1)t)dt + \int_0^1 \frac{1}{n^2}\sin^2(2\pi nt)dt - \int_0^1 2\left(\frac{1}{n+1}\right)\left(\frac{1}{n}\right)\sin(2\pi(n+1)t)\sin(2\pi t)dt\right)^{1/2}$$
$$= \left(\frac{1}{(n+1)^2}\int_0^1 \frac{1}{2} - \frac{\cos(2\cdot 2\pi(n+1)t)}{2}dt + \frac{1}{n^2}\int_0^1 \frac{1}{2} - \frac{\cos(2\cdot 2\pi nt)}{2}dt - \left(\frac{1}{n+1}\right)\left(\frac{1}{n}\right)\int_0^1 \cos(2\pi(n+1)t - 2\pi t) - \cos(2\pi(n+1)t + 2\pi t)dt\right)^{1/2}$$
$$= \left[\frac{1}{(n+1)^2}\left[\frac{1}{2}t + \frac{1}{2}(4\pi(n+1))\sin(4\pi(n+1)t)\right]_0^1 + \frac{1}{n^2}\left[\frac{1}{2}t + \frac{1}{2}(4\pi n)\sin(4\pi nt)\right]_0^1 - \frac{1}{(n+1)n}\left[-2\pi nt\sin(2\pi nt) + (2\pi(n+2)t\sin(2\pi(n+2)t)\right]_0^1\right]^{1/2}$$
$$= \left(\frac{1}{2(n+1)^2} + \frac{1}{2n^2}\right)^{1/2}$$
$$\leq \delta \text{ for some arbitrary } n$$

In fact as $n \to \infty$ then δ can be taken closer to zero. In otherwords $\delta \to 0$ as $n \to \infty$.

$$|D_\tau(f) - D_\tau(g)| = |f_{n+1}'(\tau) - f_n'(\tau)|$$
$$= \left|\tfrac{1}{n+1}(\cdot 2\pi(n+1))\cos(2\pi(n+1)\tau) - \tfrac{1}{n}(2\pi n)\cos(2\pi n\tau)\right|$$
$$= |2\pi\cos(2\pi(n+1)\tau) - 2\pi\cos(2\pi n\tau)|$$
$$= \left|2\pi \cdot 2\sin\left(\frac{2\pi(2n+1)\tau}{2}\right)\cdot \sin\left(\frac{2\pi\tau}{2}\right)\right|$$
$$= |4\pi\sin(\pi(2n+1)\tau)\cdot\sin(\pi\tau)| = \gamma(n) \quad 0 \le \gamma \le 4\pi \quad \forall\, n$$

As n approaches infinity, $\|f_{n+1}(t) - f_n(t)\|$ will be bounded by a smaller and smaller δ, but $|D_\tau(f_{n+1}) - D_\tau(f_n)|$ will oscillate between values of 0 and 4π and not approach 0 as $n \to \infty$. Thus D_τ is not continuon

(b) Let ℓ_τ be the functional that takes a $f \in C^1([0,1])$ and returns the definite integral

$$\ell_\tau(f) = \int_0^\tau f(t)\,dt$$

Is ℓ_τ linear? Continuous?

$$\ell_\tau(af+bg) = \int_0^\tau (af+bg)(t)\,dt = \int_0^\tau af(t)+bg(t)\,dt$$
$$= \int_0^\tau af(t)\,dt + \int_0^\tau bg(t)\,dt = a\,\ell_\tau(f) + b\,\ell_\tau(g) \checkmark$$

$\|f-g\| \le \delta \Rightarrow |\ell_\tau(f) - \ell_\tau(g)| \le \varepsilon$

$$|\ell_\tau(f) - \ell_\tau(g)| = \left|\int_0^\tau f(t)\,dt - \int_0^\tau g(t)\,dt\right|$$
$$= \left|\int_0^\tau f(t)-g(t)\,dt\right|$$
$$= \left|\int_0^\tau (f(t)-g(t))(1)\,dt\right|$$
$$= |\langle f-g, 1\rangle_\tau| \qquad \langle f,g\rangle_\tau = \int_0^\tau f(t)g(t)\,dt$$
$$= (|\langle f-g, 1\rangle_\tau|^2)^{1/2}$$
$$\le (\langle f-g, f-g\rangle_\tau \cdot \langle 1,1\rangle_\tau)^{1/2} \text{ Cauchy-Schwarz}$$
$$= \left(\int_0^\tau |f(t)-g(t)|^2\,dt \int_0^\tau |1|^2\,dt\right)^{1/2}$$
$$\le \left(\int_0^1 |f(t)-g(t)|^2\,dt \int_0^1 |1|^2\,dt\right)^{1/2} \text{ Since } \tau \le 1$$
$$= (\|f-g\|)^{1/2}$$
$$\le \delta^{1/2}$$

Letting $\varepsilon = \delta^{1/2}$ we can say if $\|f-g\| \le$ some δ then there exists an ε s.t. $|\ell_\tau(f) - \ell_\tau(g)| \le \varepsilon$. Thus $\ell_\tau(\cdot)$ is continuous. $\checkmark$