This past week we covered norms. In particular we covered properties of norms such as homogeneity and the triangle inequality as well as different kinds of norms like the l-infinity norm and the l2 norm. Norms are relevant because they provide us with a notion of distance, in Euclidean space that we encouter in everyday life the l2 norm is commonly used. The l1 norm is also useful since it is used to measure distances along streets, which can be seen as a grid.

We also covered inner products and their properties. The inner product let's us think about angles and directions of elements in vector spaces. Induced norms are defined with respect to the inner products and have many useful properties such as the Pythagorean Theorem and the Cauchy-Schwarz inequality. Direction is a powerful concept that inner products give to a space and allows discussion on orthogonality. Using these fundamental linear algebra ideas such as norms and inner products, along with spans of linear spaces and basis vectors, gives us the machinery to formulate systems of linear equations on higher dimensions. This in turn allows us to solve a variety of linear problems, such as linear approximation as well as minimization problems, which can be useful for fitting a particular model to a dataset.

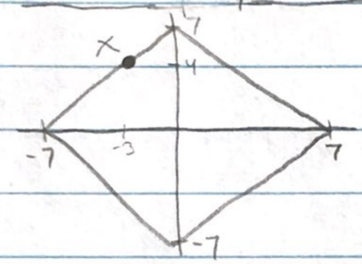
Akash Patel
MFML
HMwk 2

2a). Let $x = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. For $p = 1, 2, \infty$, find $r = \|x\|_p$, and sketch x and rB_p (use different axes for each of the three values of p).

$p=1$

$r = \|x\|_1 = \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\|_1 = \sum_{i=1}^{N} |x_i| = |-3| + |4| = \boxed{7}$

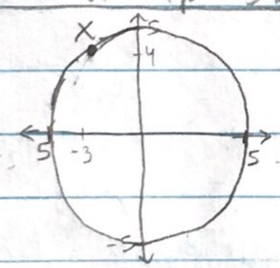
Sketch: $rB_p = 7B_1$



$p=2$

$r = \|x\|_2 = \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\|_2 = \left(\sum_{i=1}^{N} x_i^2 \right)^{\frac{1}{2}} = \left((-3)^2 + (4)^2 \right)^{\frac{1}{2}} = (9+16)^{\frac{1}{2}} = 25^{\frac{1}{2}} = \boxed{5}$

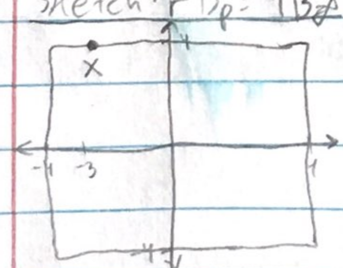
Sketch: $rB_p = 5B_2$



$p = \infty$

$r = \|x\|_\infty = \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\|_\infty = \max_{1 \le i \le N} |x_i| = \boxed{4}$

Sketch: $rB_p = 4B_\infty$



2b) Explain why $\|\cdot\|_{B_b}$ and $\|\cdot\|_{B_c}$ are not valid norms.

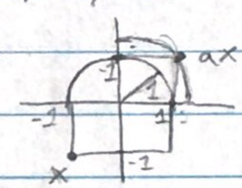
$\|\cdot\|_{B_b}$:

Let us choose the vector, $x = [-1,-1]$.

The $\|x\|_{B_b}$ is 1 by definition (1),

1 is the minimum value $r \geq 0$ such that $x \in rB_b$.

Due to the homogeneity property of a norm, it must be true that $\|ax\|_{B_b}$ equals $|a|\|x\|_{B_b}$. For simplicity let us check this property with $a = -1$.



$1^2+1^2=c^2$
$c=\sqrt{2}$

$|a|\|x\|_{B_b} = |-1| \left\| \begin{bmatrix}-1\\-1\end{bmatrix} \right\|_{B_b} = 1(1) = 1$

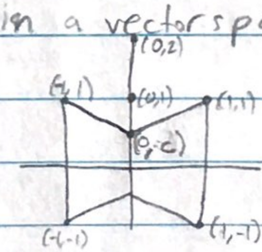
$\|ax\|_{B_b} = \left\| -1 \begin{bmatrix}-1\\-1\end{bmatrix} \right\|_{B_b} = \left\| \begin{bmatrix}1\\1\end{bmatrix} \right\|_{B_b} = \sqrt{2}$

Since $|a|\|x\|_{B_b} \neq \|ax\|_{B_b}$, the definition of $\|\cdot\|_{B_b}$ is $\boxed{\text{not a valid norm}}$

$\|\cdot\|_{B_c}$:

Due to the triangle property of a norm, it must be true that $\|x+y\| \leq \|x\| + \|y\|$ for x,y in a vectorspace.

Let us choose $x = [1,1]$ and $y = [-1,1]$.



$c < 1$

$1 = c \cdot \frac{1}{c}$

Then $\|x\|_{B_c} = \|[1,1]\|_{B_c} = 1$,

$\|y\|_{B_c} = \|[-1,1]\|_{B_c} = 1$, and

$\|x+y\|_{B_c} = \|[1,1]+[-1,1]\|_{B_c} = \|[0,2]\|_{B_c} = \frac{2}{c} > 2$

$\|[0,c]\|_{B_c} = c \cdot \frac{1}{c} = 1$

$\|[0,1]\|_{B_c} = 1 \cdot \frac{1}{c} = \frac{1}{c}$

$\|[0,2]\|_{B_c} = 2 \cdot \frac{1}{c} = \frac{2}{c}$

Since $c<1$, $\frac{2}{c} > 2$

Thus $\|x+y\|_{B_c} \not\leq \|x\| + \|y\|$

$(>2) \not\leq 2$

Since the triangle inequality is not satisfied, $\|\cdot\|_{B_c}$ is $\boxed{\text{not a valid norm.}}$

2c) Give a concrete method for computing $\|x\|_{B_a}$, $\|x\|_{B_d}$, and $\|x\|_{B_e}$ for any given vector x. Using your expressions, show that these are indeed valid norms.

$\|x\|_{B_a}$:

$\|x\|_{B_a} = \boxed{\max(\frac{1}{3}|x_1|, |x_2|)}$ Similar to ℓ_∞ norm but scaled on x_1

Prove there are valid norms: (Check for zero condition, homogeneity, and triangle inequality)

Zero: $\|x\|_{B_a} = \max(\frac{1}{3}|x_1|, |x_2|) \geq 0$ ✓

$\|x\|_{B_a} = \max(\frac{1}{3}|x_1|, |x_2|) = 0$ only if $x = 0$ ✓

Let $x=0$ then: $\max(\frac{1}{3}|0|, |0|) = 0$

Homogeneity: $\|ax\|_{B_a} = \max(\frac{1}{3}|ax_1|, |ax_2|) = \max(|a|\frac{1}{3}|x_1|, |a||x_2|)$

$= |a| \max(\frac{1}{3}|x_1|, |x_2|) = |a| \|x\|_{B_a}$

Triangle Inequality: $\|x+y\|_{B_a} = \max(\frac{1}{3}|x_1+y_1|, |x_2+y_2|)$

$\leq \max(\frac{1}{3}(|x_1|+|y_1|), |x_2|+|y_2|)$

$\leq \max(\frac{1}{3}|x_1|, |x_2|) + \max(\frac{1}{3}|y_1|, |y_2|)$

$= \|x\|_{B_a} + \|y\|_{B_a}$

$\|x+y\|_{B_a} \leq \|x\|_{B_a} + \|y\|_{B_a}$

Thus $\|x\|_{B_a}$ is a valid norm.

$\|x\|_{Bd}$:

$\|x\|_{Bd} = \boxed{|x_1| + \frac{1}{2}|x_2|}$ Similar to l_1 norm but with x_2 scaled by $\frac{1}{2}$

Prove this is a valid norm: (Check for zero condition, homogeneity, and triangle inequality)

Zero: $\|x\|_{Bd} = |x_1| + \frac{1}{2}|x_2| \geq 0$ ✓

Let $x = [0,0]$, $\|x\|_{Bd} = |0| + \frac{1}{2}|0| = 0$ ✓

$\|x\|_{Bd} = |x_1| + \frac{1}{2}|x_2| = 0$ only if $x = 0$

Homogeneity: $\|ax\|_{Bd} = |ax_1| + \frac{1}{2}|ax_2| = |a||x_1| + |a|\frac{1}{2}|x_2| = |a|(|x_1| + \frac{1}{2}|x_2|)$

$= |a|\|x\|_{Bd}$ ✓

Triangle Inequality: $\|x+y\|_{Bd} = |x_1 + y_1| + \frac{1}{2}|x_2 + y_2|$

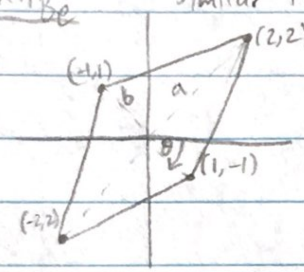
$\leq |x_1| + |y_1| + \frac{1}{2}(|x_2| + |y_2|)$

$= \|x\|_{Bd} + \|y\|_{Bd}$

$\|x+y\|_{Bd} \leq \|x\|_{Bd} + \|y\|_{Bd}$ ✓

Thus $\|x\|_{Bd}$ is a valid norm.

$\|x\|_{Be}$: Similar to ℓ_1 but scaled and rotated.



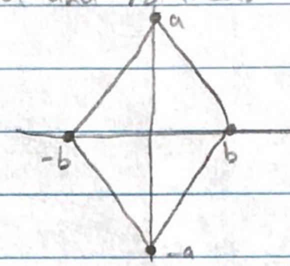
$2^2+2^2=a^2$

$a=\sqrt{8}=2\sqrt{2}$

$\theta=-45°$

$1^2+(-1)^2=b^2$

$b=\sqrt{2}$



Reverse Rotation

scaled norm $= \frac{1}{b}|x_1| + \frac{1}{a}|x_2| = \frac{1}{\sqrt{2}}|x_1| + \frac{1}{2\sqrt{2}}|x_2|$

Rotate vector x by 45° (opposite of -45°)

$$x_r = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

Use rotated values in scaled norm:

$$\frac{1}{b}|x_1\cos\theta - x_2\sin\theta| + \frac{1}{a}|x_1\sin\theta + x_2\cos\theta|$$

Simplify with $a=2\sqrt{2}$, $b=\sqrt{2}$, $\cos\theta=\cos(45°)=\frac{\sqrt{2}}{2}$,

$\sin\theta=\sin(45°)=\frac{\sqrt{2}}{2}$

$$\Rightarrow \frac{1}{\sqrt{2}}\left|\frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}x_2\right| + \frac{1}{2\sqrt{2}}\left|\frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2\right|$$

$$\Rightarrow \boxed{\frac{1}{2}|x_1-x_2| + \frac{1}{4}|x_1+x_2|}$$

Prove this is a valid norm: (Check for zero condition, homogeneity, and triangle inequality)

Zero: $\|x\|_{Be} = \frac{1}{2}|x_1-x_2| + \frac{1}{4}|x_1+x_2| \geq 0$ ✓

Let $x=\mathbf{0}$, $\|x\|_{Be}=0$ ✓

$\frac{1}{2}|x_1-x_2| + \frac{1}{4}|x_1+x_2| = 0$ only if $x=\mathbf{0}$ ✓

Homogeneity: $\|ax\|_{Be} = \frac{1}{2}|ax_1 - ax_2| + \frac{1}{4}|ax_1+ax_2|$

$= \frac{1}{2}|a||x_1-x_2| + \frac{1}{4}|a||x_1+x_2|$

$= |a|\left(\frac{1}{2}|x_1-x_2| + \frac{1}{4}|x_1+x_2|\right)$

$= |a|\|x\|_{Be}$ ✓

Triangle Inequality: $||x+y||_{Be} = \frac{1}{2}|x_1 + y_1 - x_2 - y_2| + \frac{1}{4}|x_1 + y_1 + x_2 + y_2|$

$$= \frac{1}{2}|x_1 - x_2 + y_1 - y_2| + \frac{1}{4}|x_1 + x_2 + y_1 + y_2|$$

$$\leq \frac{1}{2}(|x_1 - x_2| + |y_1 - y_2|) + \frac{1}{4}(|x_1 + x_2| + |y_1 + y_2|)$$

$$= \frac{1}{2}|x_1 - x_2| + \frac{1}{4}|x_1 + x_2| + \frac{1}{2}|y_1 - y_2| + \frac{1}{4}|y_1 + y_2|$$

$$= ||x||_{Be} + ||y||_{Be}$$

$||x+y||_{Be} \leq ||x||_{Be} + ||y||_{Be}$ ✓

Thus $||\cdot||_{Be}$ is a valid norm.

3a) Prove that $|\langle x,y\rangle| \le \|x\|_\infty \cdot \|y\|_1$

$$|\langle x,y\rangle| = \left|\sum_{i=1}^{N} x_i y_i\right| \le \sum_{i=1}^{N} |x_i||y_i|$$

$$\le \sum_{i=1}^{N}\left(|y_i| \max_{1\le j\le N} |x_j|\right)$$

$$= \max_{1\le j\le N}(|x_j|) \cdot \sum_{i=1}^{N} |y_i|$$

$$= \|x\|_\infty \cdot \|y\|_1$$

$|\langle x,y\rangle| \le \|x\|_\infty \cdot \|y\|_1$ ✓

3b) Prove that $\|x\|_1 \le \sqrt{N} \cdot \|x\|_2$ Let $n \in \mathbb{R}^n$ s.t. $n_i = \text{sign}(x_i)$

$$\|x\|_1 = \sum_{i=1}^{N} |x_i| = \left|\sum_{i=1}^{N} n_i x_i\right| = |\langle n, x\rangle|$$

$$\le \|n\|_2 \|x\|_2$$ (Cauchy-Schwarz Inequality)

$$= \sqrt{\sum_{i=1}^{N} n_i^2}\, \|x\|_2$$

$$= \sqrt{N}\, \|x\|_2$$ ✓

3c) Let B_2 be the unit ball for the ℓ_2 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on y:

$$\max_{x \in B_2} \langle x, y \rangle = ???$$

$$\max_{x \in B_2} \langle x, y \rangle = \max_{x \in B_2} |\langle x, y \rangle| \leftarrow \exists\, x \text{ s.t. } \operatorname{sign}(x_i) = \operatorname{sign}(y_i) \ \& \ \langle x, y \rangle > 0$$

Therefore $\max \langle x, y \rangle$ will be (+) and $= \max |\langle x, y \rangle|$

$$= \max_{x \in B_2} (\|x\|_2 \|y\|_2) \leftarrow \text{Cauchy-Schwarz Inequality}$$

when $y = ax$ (colinear) turns into equality

$$= \|y\|_2 \max_{x \in B_2} \|x\|_2 = \|y\|_2 \cdot 1 = \boxed{\|y\|_2}$$

$\exists\, x \in B_2$ s.t. $\|x\|_2 = 1$

Describe the vector x which achieves the maximum.

1) $\operatorname{sign}(x_i) = \operatorname{sign}(y_i) \ \forall i \in [1, N]$

2) $\|x\|_2 = 1$

3) $x = ay$, where $a \in \mathbb{R}_+$ (colinear)

$$\sqrt{\Sigma x_i^2} = 1$$

$$\sqrt{\Sigma (ay_i)^2} = 1$$

$$\Sigma (ay_i)^2 = 1$$

$$a^2 \Sigma y_i^2 = 1$$

$$a = \left(\frac{1}{\Sigma y_i^2}\right)^{1/2}$$

$$\Rightarrow \boxed{x = ay = \left(\frac{1}{\Sigma (y_i^2)}\right)^{1/2} y}$$

3d) Let B_∞ be the unit ball for the ℓ_∞ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on y:

$$\max_{x \in B_\infty} \langle x, y \rangle = ???$$

$$\max_{x \in B_\infty} \langle x, y \rangle = \max_{x \in B_\infty} |\langle x, y \rangle| \leftarrow \exists\, x \text{ s.t. } \text{sign}(x_i) = \text{sign}(y_i) \ \& \ \langle x, y \rangle > 0$$

Therefore $\max \langle x, y \rangle$ will be (+) and $= \max |\langle x, y \rangle|$

$$= \max_{x \in B_\infty} \|x\|_\infty \cdot \|y\|_1 \leftarrow \text{Part a) equality holds when } |x_i| = |x_j| \ \forall\, i,j \in [1,N], \text{ and } \text{sign}(x_i) = \text{sign}(y_i)$$

$$= 1 \cdot \|y\|_1 = \boxed{\|y\|_1} \quad \exists\, x \in B_\infty \text{ s.t. } \|x\|_\infty = 1$$

$|\langle x, y \rangle| = |\sum x_i y_i|$

$= \sum |x_i||y_i|$ $\quad \text{sign}(x_i) = \text{sign}(y_i)$

$= \sum |\max_{1 \le i \le N}(x_i)| |y_i|$

if $|x_i| = |\max_{1 \le i \le N}(x_i)| \ \forall\, i \in [1,N]$

which implies

$|x_i| = |x_j| \ \forall\, i,j \in [1,N]$

Describe the vector x which achieves the maximum.

1) $\text{sign}(x_i) = \text{sign}(y_i) \ \forall\, i \in [1,N]$

2) $\|x_\infty\| = 1$

3) $|x_i| = |x_j| \ \forall\, i,j \in [1,N]$

For $\|x_\infty\| = 1$ then there must be atleast 1 element in x equal to 1 or -1. Then by 3) all elements in x must be equal to 1 or -1. Lastly with 1) the sign of $x_i = y_i$.

$\boxed{\text{Therefore } x_i = 1 \cdot \text{sign}(y_i) \ \forall\, i \in [1,N]}$

3e) Let B_1 be the unit ball for the ℓ_1 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on y:

$$\max_{x \in B_1} \langle x, y \rangle = ????$$

$$\max_{x \in B_1} \langle x, y \rangle = \max_{x \in B_1} |\langle x, y \rangle| \leftarrow \exists x \text{ s.t. } \text{sign}(x_i) = \text{sign}(y_i) \ \& \ \langle x, y \rangle > 0$$

Therefore $\max \langle x, y \rangle$ will be (+) and $= \max |\langle x, y \rangle|$

$$= \max_{x \in B_1} |\langle y, x \rangle| \leftarrow \text{conjugate property of norm: } \langle x, y \rangle = \langle y, x \rangle$$

$$= \max_{x \in B_1} \|y\|_\infty \|x\|_1 \leftarrow \text{See point 3) for this reasoning}$$

if $\text{sign}(x_i) = \text{sign}(y_i) \ \forall i \in [1, N]$ & $\sum |y_i x_i| = \|y\|_\infty$

$$= \|y\|_\infty \max_{x \in B_1} \|x\|_1 = \|y\|_\infty \cdot 1 = \boxed{\|y\|_\infty} \quad \exists x \in B_1 \text{ s.t. } \|x\|_\infty = 1$$

Describe the vector x which achieves the maximum

1) $\text{sign}(x_i) = \text{sign}(y_i) \ \forall i \in [1, N]$
2) $\|x\|_1 = 1$
3) For $\max_{x \in B_1} |\langle y, x \rangle| = \max_{x \in B_1} \|y\|_\infty \|x\|_1$

$$\max_{x \in B_1} \left| \sum y_i x_i \right| = \max_{x \in B_1} \sum |y_i x_i| \quad \text{if } \text{sign}(x_i) = \text{sign}(y_i) \ \forall i \in [1, N]$$

$$= \max_{x \in B_1} \|y\|_\infty \cdot 1 \quad \text{if } \underline{\sum |y_i x_i| = \|y\|_\infty \cdot 1}$$

$$= \max_{x \in B_1} \|y\|_\infty \cdot \|x\|_1 \quad \text{if } x \in B_1 \quad 1 = \max \|x\|_1$$

The vector x has an ℓ_1 norm of 1 and the elements x_i are distributed in any way such that $\text{sum}(|x_j|) = 1 \ \forall j$ where $y_j = \max_{1 \le i \le N} |y_i|$ and $x_{i \ne j} \ \forall i \in [1, N] = 0$ and the $\text{sign}(x_i) = \text{sign}(y_i)$.

4) For a given ~~matrix~~ $N \times N$ matrix Q, set

$\langle x, y \rangle_Q = y^T Q x$ for vectors $x, y \in \mathbb{R}^N$.

a) Prove that if Q has an entry along its diagonal that is ≤ 0, then $\langle \cdot, \cdot \rangle_Q$ cannot be a valid inner product on $\mathbb{R}^N$.

Let Q_{mm} be the diagonal entry of Q where $Q_{mm} \leq 0$.

Let a be a vector in $\mathbb{R}^N$ where all elements are 0 except for the m^{th} element equaling 1.

$$\langle a, a \rangle_Q = a^T Q a = \sum_i^N \left(a_i \sum_j^N Q_{ij} a_j\right) = (1) \sum_j^N Q_{mj} a_j = (1)\, Q_{mm} (1)$$

$$= \underline{Q_{mm} \leq 0} \quad X$$

By the definition of an inner product $\langle a, a \rangle > 0$, since $a \neq 0$, which $\langle a, a \rangle_Q$ does not obey when Q has an entry along its diagonal that is ≤ 0.

4b) Prove that if Q is not symmetric, $Q^T \neq Q$, then $\langle \cdot,\cdot \rangle_Q$ cannot be a valid inner product on $\mathbb{R}^N$.

To be a valid norm, $\langle \cdot,\cdot \rangle_Q$ must satisfy the following:

$\langle x,y \rangle = \overline{\langle y,x \rangle}$

Let us deal with just the conjugate since we only want to prove the inner product on $\mathbb{R}^N$. $\langle x,y \rangle_Q = \langle y,x \rangle_Q$

$$\langle x,y \rangle_Q = \sum_{i=1}^{N} \left(y_i \sum_{j=1}^{N} Q_{ij} x_j \right)$$

$$= \sum_{i}^{N} \sum_{j}^{N} y_i Q_{ij} x_j$$

$$= \sum_{j}^{N} \sum_{i}^{N} y_i Q_{ij} x_j$$

$$= \sum_{j}^{N} \sum_{i}^{N} x_j Q_{ij} y_i$$

$$= \sum_{j}^{N} \left(x_j \sum_{i}^{N} Q_{ij} y_i \right)$$

$$= x^T Q^T y \qquad (4b.1)$$

$$= x^T Q^T y \neq x^T Q y \quad \text{if } Q^T \neq Q. \quad \checkmark$$

$$= \langle y,x \rangle_{Q^T} \neq \langle y,x \rangle_Q$$

4c) A symmetric positive definite matrix matrix (sym+def) is an $N \times N$ matrix Q that is symmetric ($Q = Q^T$) and obeys

$$x^T Q x > 0, \text{ for all } x \in \mathbb{R}^N, x \neq 0$$

Prove that $\langle \cdot, \cdot \rangle_Q$ is a valid inner product on $\mathbb{R}^N$ if and only if Q is symmetric positive definite.

We can split this statement into two:

Prove $\langle \cdot, \cdot \rangle_Q$ is a valid inner product on $\mathbb{R}^N$ if Q is sym+def

and Q is sym+def if $\langle \cdot, \cdot \rangle_Q$ is a valid inner product.

The contrapositive of these statements can be proven instead and is as follows:

$\langle \cdot, \cdot \rangle$ is not a valid inner product if Q is not sym+def (1)

Q is not sym+def if $\langle \cdot, \cdot \rangle_Q$ is not a valid inner product (2)

(1) If Q is not sym+def then $\langle \cdot, \cdot \rangle$ is not a valid inner product.

Using results from 4b) up till (4b.1):

$$\langle x, y \rangle_Q = x^T Q^T y$$

$$\langle y, x \rangle_Q = x^T Q y$$

$x^T Q^T y$ does not equal $x^T Q y$ if Q is not (sym+def)

(1) ✓

(2) If $\langle \cdot, \cdot \rangle_Q$ is not a valid inner product then Q is not sym+def

If $\langle \cdot, \cdot \rangle_Q$ is not a valid inner product then $\langle x, y \rangle_Q \neq \langle y, x \rangle_Q$

In other words $\underbrace{x^T Q^T y}_{(4b.1)} \neq x^T Q y. \Rightarrow Q^T \neq Q$

Thus Q is not sym. (a). ✓

Let a be a vector in $\mathbb{R}^N$ where all elements are 0 except for the m^{th} element equaling 1.
If $\langle \cdot,\cdot \rangle_Q$ is not a valid inner product then $\langle a,a \rangle \not> 0$ $(a \neq 0)$, by definition.

$$\langle a,a \rangle_Q = \sum_i^N a_i \sum_j^N Q_{ij} a_j = (1) \sum_j^N Q_{mj} a_j = (1) Q_{mm} (1) = Q_{mm}$$

If $\langle \cdot,\cdot \rangle_Q$ is not a valid inner product then $Q_{mm} \leq 0$.
If $Q_{mm} \leq 0$ then $\langle a,a \rangle_Q = a^T Q a \leq 0$. Thus Q is not a positive definite matrix. (b) ✓

(a) & (b) show that if $\langle \cdot,\cdot \rangle_Q$ is not a valid inner product then Q is not sym+def. (2) ✓

(1) & (2) of the contrapositive prove the original statement that $\langle \cdot,\cdot \rangle_Q$ is a valid inner product on $\mathbb{R}^N$ iff Q is (sym+def). ✓

4d) Define the norm on $\mathbb{R}^2$

$$\|x\| = \|Ax\|_2 , \quad A = \begin{bmatrix} 3 & 3 \\ -1/2 & 1/2 \end{bmatrix}$$

Find Q so that $\langle \cdot , \cdot \rangle_Q$ induces this norm.

Induced norm: $\langle x, x \rangle_Q = \|x\|^2$

$$x^T Q x = \|x\|^2$$

$$x^T Q x = \|Ax\|_2^2$$

$$x^T Q x = \left\| \begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2^2$$

$$x^T Q x = \left\| \begin{bmatrix} 3x_1 + 3x_2 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 \end{bmatrix} \right\|_2^2$$

$$x^T Q x = \left(\left((3x_1 + 3x_2)^2 + (-\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2)^2 \right)^{\cancel{1/2}} \right)^{\cancel{2}}$$

$$x^T Q x = 9x_1^2 + 9x_2^2 + 18x_1x_2 + \tfrac{1}{4}x_1^2 + \tfrac{1}{4}x_2^2 - \tfrac{2}{4}x_1x_2$$

$$\begin{array}{l|l|l|l} x_1 Q_{11} x_1 & x_1 Q_{11} x_1 & \frac{36}{4}x_1^2 + \frac{1}{4}x_1^2 & \frac{37}{4}x_1^2 \\ + x_1 Q_{12} x_2 = & + 2x_1 Q_{12} x_2 = & + \frac{72}{4}x_1x_2 - \frac{2}{4}x_1x_2 = & + 2\left(\frac{35}{4}x_1x_2\right) \\ + x_2 Q_{21} x_1 & + x_2 Q_{22} x_2 & + \frac{36}{4}x_2^2 + \frac{1}{4}x_2^2 & + \frac{37}{4}x_2^2 \\ + x_2 Q_{22} x_2 & & & \end{array}$$

$Q_{12} = Q_{21}$ $\Rightarrow$ $\boxed{Q = \begin{bmatrix} \frac{37}{4} & \frac{35}{4} \\ \frac{35}{4} & \frac{37}{4} \end{bmatrix}}$

5) Show that there is no inner product on $\mathbb{R}^N$ that induces the l_1 norm or the l_∞ norm.
(Hint: Parallelogram Law)

If the l_1 norm has an induced norm then it must obey the Parallelogram Law: $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$

Let us choose a vector $e_i \in \mathbb{R}^N$ where the ith element in e_i equals 1 and all other elements are zero.

Let us also chose a vector $e_j \in \mathbb{R}^N$ where the j^{th} element in e_j equals 1 and all other elements are zero. Here $j \neq i$.

Evaluating both sides of the Parallelogram Law (using the l_1 norm) we obtain the following:

$$\|e_i+e_j\|_1^2 + \|e_i-e_j\|_1^2 = 2\|e_i\|_1^2 + 2\|e_j\|_1^2$$

$$(2)^2 + (2)^2 = 2(1)^2 + 2(1)^2$$

$$8 = 4 \quad \text{not true. } \times$$

The l_1 norm does not satisfy the Parallelogram Law and hence is not a valid norm.

Let us evaluate the Parallelogram Law using the l_∞ norm with e_i & e_j.

$$\|e_i+e_j\|_\infty^2 + \|e_i-e_j\|_\infty^2 = 2\|e_i\|_\infty^2 + 2\|e_j\|_\infty^2$$

$$(1)^2 + (1)^2 = 2(1)^2 + 2(1)^2$$

$$2 = 4 \quad \text{not true } \times$$

The l_∞ norm does not satisfy the Parallelogram Law and hence is not a valid norm.

6a) From class: $z = \tau x + (1-\tau)y \Rightarrow \|x-y\| = \|x-z\| + \|y-z\|$

Show that the converse does not hold for the l_1 norm.

$\|x-y\|_1 = \|x-z\|_1 + \|y-z\|_1 \not\Rightarrow z = \tau x + (1-\tau)y$

Let us choose x to be a vector in $\mathbb{R}^N$ whose i^{th} element $=1$ and all other elements equal 0. Let us choose y to be a vector in $\mathbb{R}^N$ whose j^{th} element $= 1$ and all other elements equal 0, where $j \neq i$. Let z be the $\mathbf{0}$ vector.

The triangle equality holds:

$\|x-y\|_1 = \|x-z\|_1 + \|y-z\|_1$

$2 = 1 + 1$ ✓

Evaluating $z = \tau x + (1-\tau)y$ we notice that no $\tau \in [0,1]$ will evaluate this expression as true. $\tau x + (1-\tau)y > 0$ whereas $z = 0$. ✗

Therefore for the l_1 norm, $\|x-y\|_1 = \|x-z\|_1 + \|y-z\|_1 \not\Rightarrow z = \tau x + (1-\tau)y$

6b) Show that the converse does indeed hold true if $\|\cdot\|$ is an induced norm. That is, if $\|x\| = \sqrt{\langle x,x \rangle}$ for some valid inner product $\langle \cdot,\cdot \rangle$, then

$$\|x-y\| = \|x-z\| + \|y-z\| \Rightarrow z = \tau x + (1-\tau)y \text{ for some } \tau \in [0,1]$$

The contrapositive of the above statement is logically equivalent, so let us prove the contrapositive for ease of proof.

$$z \neq \tau x + (1-\tau)y \Rightarrow \|x-y\| < \|x-z\| + \|y-z\|$$

To simplify the proof let us take $x=0$ without loss of generality. We can do this since the expressions on the inequality are all norms, i.e. distances instead of the actual value of x. Since we only care about the distance between the vectors and not the vectors themselves we can "shift" our problem and let $x=0$, without losing the structure of the problem and maintaining the norms.

The contrapositive is then simplified to:

$$z \neq (1-\tau)y \Rightarrow \|-y\| < \|-z\| + \|y-z\|$$

$$z \neq (1-\tau)y \Rightarrow \|y\| < \|z\| + \|y-z\| \Rightarrow \|y\| - \|z\| < \|y-z\|$$

Let us split this problem into two cases. One case where $\|z\| > \|y\|$ and case 2 where $\|z\| \leq \|y\|$.

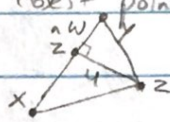
Case 1: If $z \neq (1-\tau)y$ (i.e. not collinear) and if $\|z\| > \|y\|$ then $\|y\| - \|z\| < 0$. Since $\|y-z\|$ is a norm by definition, and $y \neq z$, it is > 0. Thus $\|y\| - \|z\| < 0 < \|y-z\|$ and $\|y\| - \|z\| < \|y-z\|$ if $\|z\| > \|y\|$ and $z \neq (1-\tau)y$

Case 2: $\|z\| \le \|y\|$

Let us first start by proving $(\|y\|-\|z\|)^2 < \|y-z\|^2$ (1)

Expanding out: $\|y\|^2 + \|z\|^2 - 2\|y\|\|z\| < \|y-z\|^2$

let $\hat{z}$ be the closest point to z in the 1D subspace spanned by y.



Let $w = y - \hat{z}$ and $u = z - \hat{z}$

$\hat{z} = \frac{\langle z,y\rangle}{\langle y,y\rangle} y$ (The projection onto y)

Thus $\hat{z}$ is colinear with y and w is colinear with y.

We rewrite z as $z = \hat{z} + u$ where $\hat{z} = (1-\tau)y$ for some $\tau \in [0,1]$

This is true for the $\|z\| \le \|y\|$ case. $\hat{z}$ will be colinear with y and have a smaller magnitude than y. Since $z \ne (1-\tau)y$ and $\hat{z} = (1-\tau)y$

$u = z - \hat{z}$ is a non-zero vector and $\|z - \hat{z}\| > 0$ (2)

$u = z - \frac{\langle z,y\rangle}{\langle y,y\rangle} y = z - \frac{\langle z,y\rangle}{\|y\|^2} y \Rightarrow \langle u,y\rangle = \langle z,y\rangle - \frac{\langle z,y\rangle}{\|y\|^2}\|y\|^2 = 0$

→ From the details of Cauchy-Schwarz proof

$\langle u,y\rangle$ is thus orthogonal $= 0$.

We can also do a similar analysis with $\langle u,w\rangle$ since w is colinear with y.

Since $\langle u,w\rangle = 0$ we can use the Pythagorean Theorem and state

$$\|w\|^2 + \|u\|^2 = \|w-u\|^2$$

$$\|y-\hat{z}\|^2 + \|z-\hat{z}\|^2 = \|y - \hat{z} - z + \hat{z}\|^2 = \|y-z\|^2$$

Plugging $\|y-z\|^2$ back into the inequality (1) we have:

$\|y\|^2 + \|z\|^2 - 2\|y\|\|z\| < \|y-\hat{z}\|^2 + \|z-\hat{z}\|^2$ Plugging in $\hat{z} = (1-\tau)y$ for some $\tau \in [0,1]$

$\|y\|^2 + \|z\|^2 - 2\|y\|\|z\| < \|y - (1-\tau)y\|^2 + \|z-\hat{z}\|^2$

$\|y\|^2 + \|z\|^2 - 2\|y\|\|z\| < \|\tau y\|^2 + \|z-\hat{z}\|^2$

$\|y\|^2 + \|z\|^2 - 2\|y\|\|z\| < \|y\|^2 + \|z-\hat{z}\|^2$

$\|z\|^2 - 2\|y\|\|z\| < \|z-\hat{z}\|^2$

$\|z\|^2 < 2\|y\|\|z\| + \|z-\hat{z}\|^2$

$\|z\| < 2\|y\| + \frac{\|z-\hat{z}\|^2}{\|z\|}$

max τ can be is 1

$\underbrace{2\|y\|}_{>\|y\|} + \underbrace{\frac{\|z-\hat{z}\|^2}{\|z\|}}_{>0 \text{ by definition of norm and (2)}}$

Based on our case: $\|z\| \le \|y\|$

$\|z\| \le \|y\| < 2\|y\| + \frac{\|z-\hat{z}\|^2}{\|z\|}$ ✓

Since $(\|y\|-\|z\|)^2 < \|y-z\|^2$ is true,

$\|y\|-\|z\| < \|y-z\|$ is also true.

$\|y\| < \|z\| + \|y-z\|$

The simplified contrapositive is then proven and earlier it was explained that $\|y\| < \|z\| + \|y-z\| \Rightarrow \|x-y\| < \|x-z\| + \|y-z\|$ without loss of generality, below is a more mathematical reasoning.

Let $y' = y-x$ and $z' = z-x$

$\|y-x\| < \|z-x\| + \|y-x-z+x\|$

$\Rightarrow \|x-y\| < \|x-z\| + \|y-z\|$

Thus no matter x, with a simple change of variables we arrive at the same result.