

# A Physicist's Perspective on Quantitative Finance

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**Adrià Garcés<sup>1</sup>**

<sup>1</sup> *Departament de Física de la Matèria Condensada, Universitat de Barcelona, Barcelona, Spain*

*E-mail:* [adria.garces@ub.edu](mailto:adria.garces@ub.edu)

**ABSTRACT:** This project aims to build a comprehensive collection of quantitative finance models from the perspective of a physicist. Each topic is explored through detailed notes, combining analytical tools from stochastic calculus, statistical mechanics, and numerical simulation. The goal is to bridge physical intuition and financial modeling by documenting a range of classical models while progressively incorporating real data analysis. All developments, explanations, and code are shared as a series of standalone "pills" in a structured GitHub repository. Access the repository [here](#).

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## 1 Introduction

This repository gathers a growing collection of projects exploring key concepts in quantitative finance from a physicist's perspective. Each project includes theoretical notes, numerical simulations, and applications to real data, blending physical intuition with financial modeling. Topics range from stochastic processes and option pricing to interest rate models, portfolio theory, and statistical arbitrage. The notes are designed to be rigorous yet intuitive. Some background in differential equations, probability, and basic statistics is recommended, along with familiarity with Python and scientific computing tools.

Este repositorio reúne una colección creciente de proyectos que exploran conceptos clave de finanzas cuantitativas desde la perspectiva de un físico. Cada proyecto incluye apuntes teóricos, simulaciones numéricas y aplicaciones con datos reales, combinando intuición física con modelado financiero. Se abordan temas como procesos estocásticos, valoración de opciones, tipos de interés, teoría de carteras y arbitraje estadístico. Los apuntes buscan ser rigurosos pero accesibles; se recomienda tener conocimientos previos de ecuaciones diferenciales, probabilidad, estadística básica y cierta familiaridad con Python y herramientas de computación científica.

Aquest repositori recull una col·lecció creixent de projectes que exploren conceptes clau de finances quantitatives des del punt de vista d'un físic. Cada projecte combina apunts teòrics, simulacions numèriques i aplicacions a dades reals, unint la intuïció física amb el modelatge financer. Es tracten temes com processos estocàstics, preus d'opcions, tipus d'interès, teoria de carteres i arbitratge estadístic. Els apunts són rigorosos però accessibles; es recomana tenir nocions prèvies d'equacions diferencials, probabilitat i estadística bàsica, així com experiència amb Python i eines de càlcul científic.

Bu depo, bir fizikçinin bakış açısından nicel finansın temel kavramlarını keşfeden projelerden oluşan bir koleksiyondur. Her proje, teorik notlar, sayısal simülasyonlar ve gerçek verilere uygulamalar içerir; fiziksel sezgi ile finansal modellemeyi birleştirir. Ele alınan konular arasında stokastik süreçler, opsiyon fiyatlama, faiz oranı modelleri, portföy teorisi ve istatistiksel arbitraj yer alır. Notlar titiz ama sezgisel bir şekilde hazırlanmıştır. Diferansiyel denklemler, olasılık teorisi ve temel istatistik bilgisi ile Python ve bilimsel hesaplama araçlarına aşinalık önerilir.

## 2 Geometric Brownian Motion (GM)

### 2.1 Introduction

In finance, models capturing the dynamics of the price of an asset in market are essential – they both theoretical understanding and background and have practical applications.

Just like in physics stochastic models are used to understand the dynamics of physical observables of a system, i.e. magnetization of a material in a bath at temperature  $T$  which provides the system with fluctuations, modeling the price of an asset can also be done in a very similar way. The fluctuations (stochastic nature), in this case, is provided by a more complex environment that a single bath at temperature  $T$  – the interactions and interests of individuals who intervene in the market.

Brownian motion (BM) is the core of most stochastic models, in physics and everywhere else. However, it cannot directly be used to model asset prices, since nothing prevents a Brownian variable  $X_t$  to be negative, and asset prices are *strictly positive*. A modified version of Brownian motion which guarantees positiveness of asset prices is instead used.

In the following, let  $S_t$  be the price of an asset at time  $t \in \mathbb{R}^+$ . Positiveness of  $S_t$  is guaranteed if, for example,  $S_t = S_0 e^{Z_t}$ , where  $Z_t$  is a *stochastic* (random) function that depends on  $t$  and might generally depend on the previous history of  $S_t$ , verifying  $Z_t|_{t=0} = 0$ , so that  $S_t|_{t=0} = S_0 > 0$ . A way of achieving this is *geometric Brownian motion* (GBM), which guarantees positiveness of assets at any time  $t$ . Furthermore, GBM incorporates empirically observed phenomena such as the proportionality of an observable called returns,  $R_t = dS_t/S_t$  – fluctuation of an assets price are usually relative to its current price, not absolute.

### 2.2 Theoretical Background

Modeling the stochastic evolution of an asset's price  $S_t$  guaranteeing positiveness and such that fluctuations are relative to the price locally in time can be done using, as mentioned in the introduction, GBM. We say the asset price  $S_t$  performs GBM if its statistical behavior can be drawn by trajectories satisfying the following stochastic differential equation (SDE),

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (2.1)$$

where  $W_t$  is the Wiener process<sup>1</sup>,  $\mu$  is a constant usually named "*percentage drift*" and  $\sigma$  is another constant usually referred to as "*percentage volatility*". Before we keep going with the definition of interesting and practical observables in economics and finance we shall take a look at tools that come from physics that can help us understand the implications of this model.

#### 2.2.1 The Fokker–Planck Equation

Directly from the SDE Eq. (2.1) the Fokker–Planck equation for the probability density function of finding the price to be  $S_t = s$  at some time  $t$  is [Ref]

$$\frac{\partial}{\partial t} p_S(s, t) = -\mu \frac{\partial}{\partial s} [s p_S(s, t)] + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial s^2} [s^2 p_S(s, t)]. \quad (2.2)$$

Unfortunately, Eq. (2.2) is not linear and not readily solved. Using the change of variables  $Z_t = \ln S_t$  ( $z = \ln s$ ) we can find the Fokker–Planck equation not for  $Z_t$ , using that  $p_Z(z, t) =$

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<sup>1</sup>See [?] a brief introduction of what the Wiener process is.

$p_S(s = e^z, t) |\partial s / \partial z| = p_S(s = e^z, t) e^z$  and  $\partial_s = e^{-z} \partial_z$  and  $\partial_s^2 = e^{-2z} (\partial_z^2 - \partial_z)$ , the FP equation becomes,

$$\frac{\partial p_Z(z, t)}{\partial t} = - \left( \mu - \frac{\sigma^2}{2} \right) \frac{\partial p_Z(z, t)}{\partial z} + \frac{1}{2} \sigma^2 \frac{\partial^2 p_Z(z, t)}{\partial z^2}. \quad (2.3)$$

Eq. (2.3) is the FP equation of a *Brownian* particle moving with velocity  $v = -(\mu - \sigma^2/2)$  and diffusion coefficient  $D = \sigma^2/2$ . Note how in the case of a Brownian particle, the diffusion term comes from thermal fluctuations in the bath and its velocity does not depend on it, while here the drift term does depend on the volatility  $\sigma$  – we will see what the implications of this are in the following. Without loss of generality and setting the initial condition to be  $p_S(s, 0) = \delta(s - 1)$  so that  $p_Z(z, 0) \sim \delta(z)$ , the solution of Eq. (2.3) becomes

$$p_Z(z, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left( -\frac{(z - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t} \right). \quad (2.4)$$

This – just as in the case of a Brownian particle moving with velocity  $v$  – means that  $Z_t \sim \mathcal{N}(\mu_t, \sigma_t)$ , with  $\mu_t = (\mu - \sigma^2/2)t$  and  $\sigma_t = \sigma\sqrt{t}$ . Changing variable back again, the probability density function for an asset's price at finite time  $t$  becomes

$$\boxed{p_S(s, t) = \frac{1}{s\sqrt{2\pi\sigma^2 t}} \exp \left( -\frac{(\ln s - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t} \right)}, \quad (2.5)$$

which then makes  $S_t$  to be "lognormally" distributed, that is  $S_t \sim \text{Lognormal}(\mu_t, \sigma_t)$ , also denoted as  $\ln S_t \sim \mathcal{N}(\mu_t, \sigma_t)$ . This follows the general way of defining GBM, since  $S_t \sim e^{Z_t}$  where  $Z_T \sim \mathcal{N}(\mu_t, \sigma_t)$ . Note how the expected price of the asset exponentially increases with time  $\mathbb{E}[S_t] = \exp(\mu_t + \sigma_t^2/2) = e^{\mu t}$ . This can also be directly seen using the Itô's prescription, for which the Brownian the price of the assets at time  $t$  is *not conditioned* to the Weiner process at the same time  $t$  (it does, however, in the case of the Stratonovich discretization scheme). In the Itô prescription,

$$\mathbb{E}[dS_t] = \mathbb{E}[\mu S_t dt] + \mathbb{E}[\sigma S_t dW_t] \Leftrightarrow d\mathbb{E}[S_t] \stackrel{\text{Itô}}{=} \mu \mathbb{E}[S_t] dt + \sigma \mathbb{E}[S_t] \mathbb{E}[dW_t] = \mu \mathbb{E}[S_t] dt. \quad (2.6)$$

so that again  $\mathbb{E}[S_t] = e^{\mu t}$  when  $S_0 = 1$ . It is also important to note that if  $S_t$  follows GBM, then

$$X_t = \frac{Z_t - \mu_t}{\sigma_t} = \frac{\ln S_t - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}} \sim \mathcal{N}(0, 1). \quad (2.7)$$

This basically means histograms of either simulated or real data GBM collapses into a standard gaussian distribution with the proper change of variables. Note how when the initial price of the asset is  $S_0 \neq 1$ , then the only difference is that one has to add  $-\ln S_0$  to the expected value, meaning

$$X_t = \frac{r_t - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}} \sim \mathcal{N}(0, 1), \quad (2.8)$$

where  $r_t = \ln(S_t/S_0)$ . The quantity  $r_t$  is called the logarithmic return and is widely used in econometrics.

### 2.2.2 The Binomial Approximation to GBM

The SDE in Eq. (2.1) can be used as a reference to study time-discrete models which accurately describe GBM. Taking time to be divided in intervals of length  $\Delta t$ , Eq. (2.1) can be written the following way

$$S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \Delta W_t \Leftrightarrow \boxed{S_{t+\Delta t} = S_t (1 + \mu \Delta t + \sigma \Delta W_t)}, \quad (2.9)$$

where  $\Delta W_t := W_{t+\Delta t} - W_t = \int_t^{t+\Delta t} dW_t \sim \mathcal{N}(0, \Delta t)$ . In order to ease notation we now name times starting from  $t_0$  with index  $k \in \mathbb{Z}^+$ , i.e.  $t_k = t_0 + k\Delta t$  and we simply denote  $S_{t_k} := S_k$ . Then,

$$\begin{aligned} S_k &= S_{k-1}(1 + \mu \Delta t + \sigma \Delta W_k) \\ &= S_{k-2}(1 + \mu \Delta t + \sigma \Delta W_{k-1})(1 + \mu \Delta t + \sigma \Delta W_k) \\ &\vdots \\ &= S_0 \prod_{i=1}^k (1 + \mu \Delta t + \sigma \Delta W_i) =: S_0 L_1 L_2 \dots L_k, \end{aligned} \quad (2.10)$$

where we have defined  $L_i = S_i/S_{i-1} = (1 + \mu \Delta t + \sigma \Delta W_i)$ . Note how since  $\Delta W_i \sim \mathcal{N}(0, \Delta t)$ , then  $L_i \sim \mathcal{N}(1 + \mu \Delta t, \sigma^2 \Delta t)$ . This is fundamentally interesting – GBM can be sampled discrete time (with intervals of length  $\Delta$ ) by simply generating a sequence of jumps  $L_i \sim \mathcal{N}(1 + \mu \Delta t, \sigma^2 \Delta t)$  and then multiply them all together. Note how there is no correlation between two different jumps  $L_i, L_j$  with  $j \neq i$ . A sequence of this independent identically distributed jumps  $\{L_i\}_{i=1}^n$  generates a unique “path” for the evolution of the price of the asset,  $S_n = S_0 \prod_{i=1}^n L_i$ . This sets a clear pathway in order to simulate discrete time GBM without requiring the SDE, only using random numbers properly distributed, when  $\Delta t$  is small enough.

However, since  $L_i \sim \mathcal{N}(1 + \mu \Delta t, \sigma^2 \Delta t)$  the jump take any value at a given time step, and this is not very comfortable when looking for analytical results. A very common and natural thing to do in order to simplify the problem, is restricting the possible values of the jumps to take two values (going up or down),  $S_i \rightarrow S_{i+1} = uS_i$  ( $u$  from up) with some probability  $p$  and  $S_i \rightarrow S_{i+1} = dS_i$  ( $d$  from down), with  $u > 1 > d$ , and then taking  $ud = 1$  such that the composition of up and down movement return to the price two steps before  $S_{i+2} = udS_i = duS_i = S_i$ . This is done this way such that it possesses “recombining” property, i.e. the evolution of the asset’s price can be represented as depicted in Fig. (1).

The price of the asset at time  $n$  thus can be written  $S_n = S_0 \prod_{i=1}^n X_i$  where  $X_i \in \{u, d\}$  is a Bernoulli trial with probabilities  $p(X_i = u) = p$ ,  $p(X_i = d) = 1 - p$ . This makes the asset’s price to be Binomial distributed. Choosing  $ud = 1$  convenient, but in order to recover GBM the expected value and variance of a jump have to be mapped. In the case of GBM, it can be shown that, for fixed  $S_t$

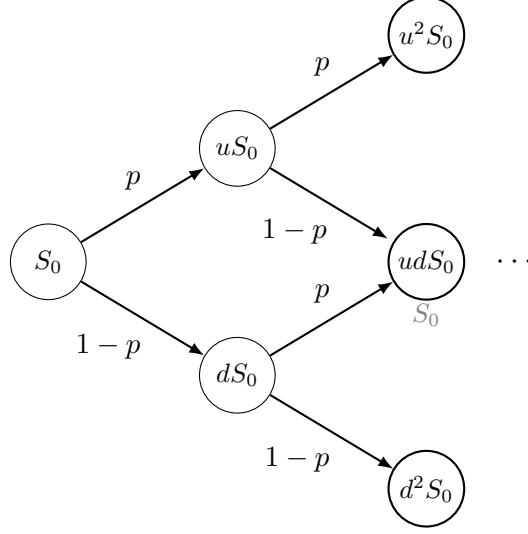
$$\mathbb{E}[S_{t+\Delta t}|S_t] = S_t e^{\mu \Delta t} \quad (2.11)$$

$$\mathbb{E}[S_{t+\Delta t}^2|S_t] = S_t^2 e^{(2\mu + \sigma^2)\Delta t}, \quad (2.12)$$

while for the case of the Binomial approximation, since  $S_{n+1} = uS_n$  with probability  $p$ , and  $S_{n+1} = dS_n$  with probability  $1 - p$ ,

$$\mathbb{E}[S_{n+1}|S_n] = S_n[up + d(1 - p)] \quad (2.13)$$

$$\mathbb{E}[S_{n+1}^2|S_n] = S_n^2[u^2p + d^2(1 - p)], \quad (2.14)$$



**Figure 1:** Recombining binomial evolution of an asset's price when  $ud = 1$ .

then the following system,

$$ud = 1 \quad (2.15)$$

$$up + d(1 - p) = e^{\mu\Delta t} \quad (2.16)$$

$$u^2p + d^2(1 - p) = e^{(2\mu + \sigma^2)t}, \quad (2.17)$$

guarantees that the jumps are going to comply with the "*recombining property*" (essential for  $S_n$  to follow a Binomial distribution) and will have the same expected value and variance as the jumps provided by GBM[?]. The solution to this is a little bit involved but writes,

$$p = \frac{e^{\mu\Delta t} - d}{u - d} \quad (2.18)$$

$$u = \frac{e^{-\mu\Delta t}}{2} \left[ 1 + e^{\sigma^2\Delta t} + \sqrt{(1 + e^{\sigma^2\Delta t})^2 - 4e^{2\mu\Delta t}} \right] \quad (2.19)$$

$$d = 1/u. \quad (2.20)$$

The expression for  $u$  is not very comfortable to work with. Since, however, we expect the discrete time model to resemble to GBM as  $\Delta t \rightarrow 0$ , a rather practical and approximated expressions for  $u$  and  $d$  can be shown to be,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad (2.21)$$

along with  $p = (e^{\mu\Delta t} - d)/(u - d)$ . These expressions for  $u, d$  and  $p$  were introduced by Cox, Ross, and Rubinstein [Ref] in a binomial model for option pricing. A simpler expression for  $u, d$  and  $p$  can be found by simply mapping the expected value and variance of  $X_i$  to the ones of  $L_i \sim \mathcal{N}(1 + \mu\Delta t, \sigma^2\Delta t)$ .

### 2.3 Simulation

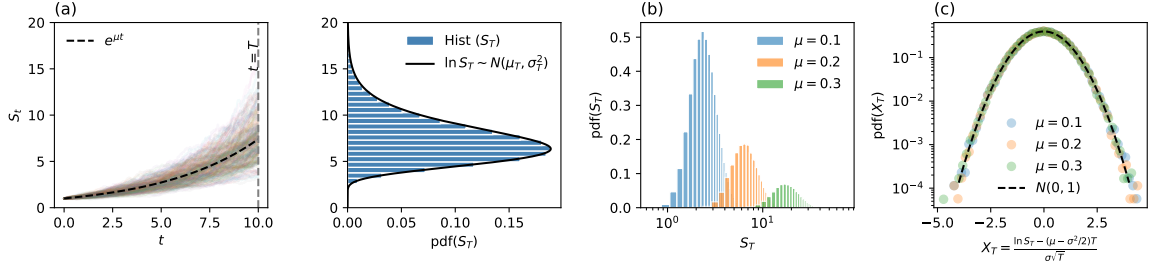
The pill "aa" found in the [GitHub repository](#) includes the numerical integration of the SDE as well as Monte Carlo (MC) simulations using the Binomial approximation and using real data of the price of an asset to test the hypothesis of the model.

### 2.3.1 Numerical integration of the SDE

The SDE of GBM for the price of the price of an asset as time evolves can be integrated numerical using the Euler–Maruyama method. For this case, given the initial condition  $S_0$ ,

$$S_{t+\Delta t} = S_t \left( 1 + \mu\Delta t + \sigma\sqrt{\Delta t}\chi_t \right), \quad (2.22)$$

where  $\chi_t$  is sampled from a standard normal distribution,  $\chi_t \sim \mathcal{N}(0, 1)$ . Without loss of generality and in order to be consistent with the initial condition we used in the theoretical background, we set  $S_0 = 1$ .



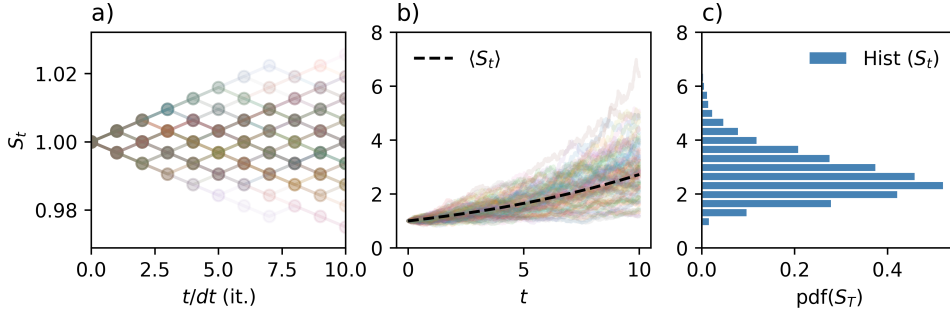
**Figure 2: Geometrical Brownian Motion.** (a) Stochastic evolution of  $G = 250$  independent realizations of the evolution of an asset with initial price  $S_0 = 1$  and  $\mu = 0.1, \sigma = 0.1$ . The black dashed line represents the expected average trajectory  $S_t = S_0 e^{\mu t}$  and the vertical gray dashed line the final time of the simulation,  $T = 10$  ( $\Delta t = 10^{-2}$ ). On the right, the histogram of the values of  $G = 10^5$  differentes trajectories at time  $t = T$  (again  $\Delta t = 10^{-2}$ ) is shown. The black line is the corresponding probability density function of the lognormal distribution with  $\mu_t = (\mu - \sigma^2/2)T$  and  $\sigma_T^2 = \sigma^2 T$ ; (b) Histograms of the price at time  $T = 10$  of  $G = 10^5$  for different values of  $\mu$  for  $\sigma = 0.1$  and  $S_0 = 1$ . The distributions look Gaussian as soon as the  $S_T$  axis is represented in logartihmic scale, as expected since  $\ln S_T \sim \mathcal{N}(\mu_T, \sigma_T^2)$ ; (c) Histograms of the standarized variable  $X_t = (\ln S_T - \mu_T)/\sigma_T$  for the same values of  $\mu$  and  $\sigma$  in (b). As shown, all the distributions collapse into a standard normal distribution of 0 mean and unit variance.

### 2.3.2 Monte Carlo simulation with Binomial approximation

The Binomial approximation of GBM can be implemented using Monte Carlo (MC) dynamics. We start from an initial asset price  $S_0 = 1$ , without loss of generality, and make the price evolve following

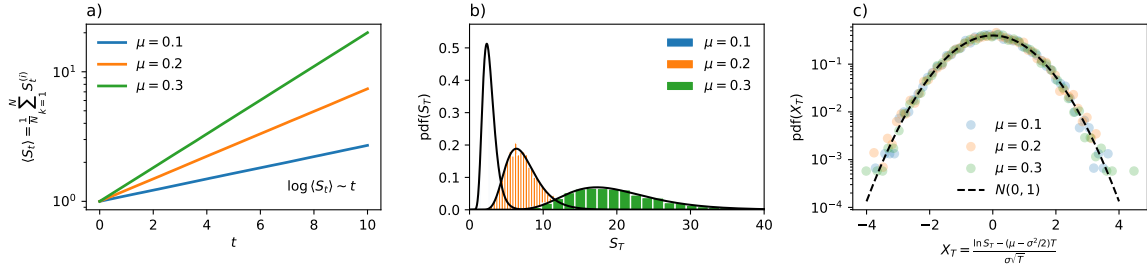
$$S_{k+1} = \begin{cases} uS_k & \text{with probability } p \\ dS_k & \text{with probability } 1 - p, \end{cases} \quad (2.23)$$

where again  $u = e^{\sigma\sqrt{\Delta t}}$ ,  $d = 1/u$  and  $p = (e^{\mu\Delta t} - d)/(u - d)$  so that when  $\Delta t$  is small enough, then at time  $t = n\Delta t$ ,  $S_t \sim \text{Lognormal}(\mu_t, \sigma_t)$ . Each MC step consists on generating a random number uniformly distributed between 0 and 1,  $z \sim U(0, 1)$ , and given the price of the asset at time  $k$ , setting  $S_{k+1} = uS_k$  if  $z < p$  and  $S_k = dS_k$  otherwise. As shown in Fig. (3,a) the dynamics followed by different realizations of the stochastic dynamics portrays the shape of the recombining tree depicted in Fig. (1). At the same time, a bigger time scales the qualitative behaviour of the trajectories simulated through the MC procedure of the Binomial approximation, Fig. (3,b),



**Figure 3: Recombinin tree and MC dynamics.** (a) Evolution of the price of  $G = 10^2$  different realizations over 10 MC steps with  $S_0 = 1$  for  $\mu = 0.1$ ,  $\sigma = 0.1$ ,  $\Delta t = 10^{-2}$ ,  $u = e^{\sigma\sqrt{\Delta t}}$ ,  $d = 1/u$  and  $S_0 = 1$ . (b) Evolution of the asset price of  $G = 10^4$  different realization for the same values of  $\mu, \sigma, \Delta t, p, u$  and  $d$  now at bigger time scales. The dynamics of the asset price now resembles more the one obtained through the direct numerical integration of the SDE. The dashed line is the noise average trajectory  $\langle S_t \rangle = (1/N) \sum_{i=1}^N S_t^i$ . (c) Histogram of the asset's price after time  $T = 10$ . The shape still resembles the one of a log-normally distributed asset price.

as well as the histogram at time  $T = 10$ , Fig. (3,c) qualitatively resemble the ones obtained through the direct numerical integration of the SDE. The dashed line in Fig. (3) represents the trajectory averaged over the noise  $\langle S_t \rangle = (1/N) \sum_{i=1}^N S_t^i$  and looks, exponential. In order to quantify how good the Binomial approximation is we perform simulations for different values of  $\mu$ , keeping  $\sigma$  and  $\Delta t$  the same. Indeed, as seen in Fig. (4,a)  $\log \langle S_t \rangle \sim \mu t$ , recovering what we know from GBM. As also shown in Fig. (4,b-c), the lognormal distribution with parameters  $\mu, \sigma$  and  $\Delta t$  perfectly fit the distributions at time  $T = N\Delta t$  ( $N$  is the number of MC steps). The data can be made to collapse, again, into a standard normal distribution with the proper change of variables.

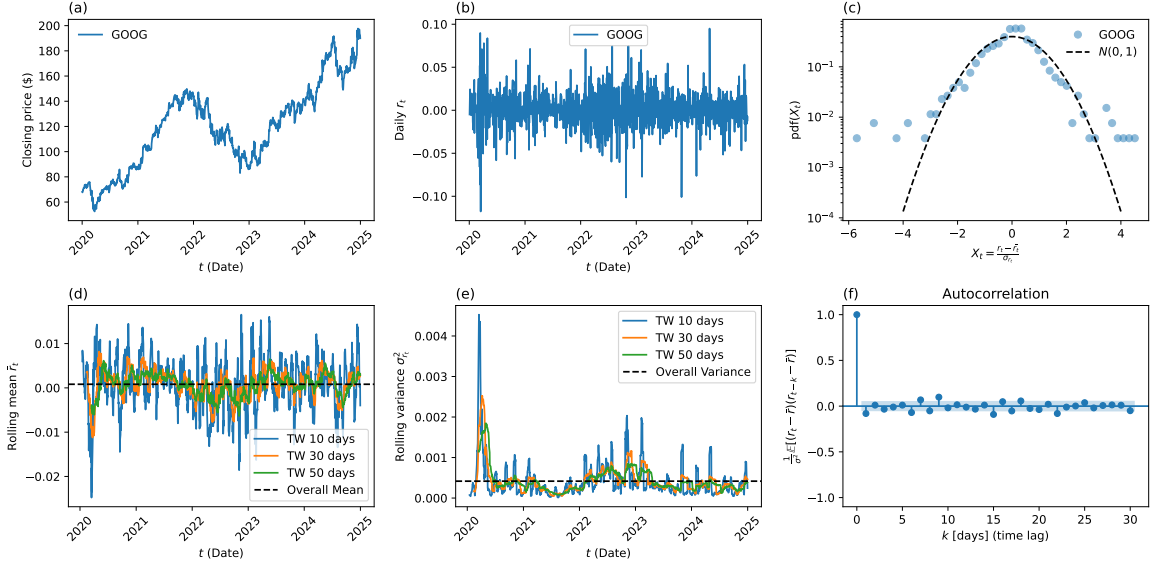


**Figure 4: Binomial approximation and GBM.** (a) Average trajectory of  $G = 10^4$  different stochastic realizations for different values of  $\mu$  and fixed  $\sigma = 0.1$ ,  $\Delta t = 10^{-2}$  and initial price  $S_0 = 1$ . The logarithmic scale of the price shows how  $\log \langle S_t \rangle \sim \mu t$ , as expected from GBM. (b) Same parameters, now showing the histogram obtained after a running time  $T = 10$ . The black lines are the probability density function obtained from a log-normal distribution with mean  $\mu T = (\mu - \sigma^2/2)T$  and variance  $\sigma_T^2 = \sigma^2 T$  (no fitting parameters). (c) Collapse of the distributions in (b) into a single distribution. The black dashes line is a normal gaussian distribution with mean 0 and unit variance.

### 2.3.3 Real data and GBM

In order to test the real applicability of the model studied we can directly take data from the real price of an asset of the market. In this case, we will take a look at the evolution of the price of an asset of ALPHABET INC. (GOOG).

Accessing financial data can be done using Python, by installing the library [YahooFinance](#). A very basic tutorial on accessing data is found inside the pill of this topic, check out the [GitHub repository](#). We particularly access data of the price of an asset between 2020 and 2025. We keep record only of the closing asset price of each day, and thus have data for every single (trading) day between 01/01/2020 and 01/01/2025. The daily return is computed as follows  $r_{d,k} = \ln(S_k/S_{k-1})$  where  $S_k$  is the closing price of day  $k$  and  $S_{k-1}$  is the closing price of the previous day. As shown in Fig. (5,c) the distribution of logarithmic returns seems to be normally distributed, with numerical means and variance  $\bar{r}$  and  $\sigma_r$  (how these are computed is specified in the pill). Interestingly enough, the distribution of daily log returns have "fat" tails, that diverge from the lognormal behaviour.



**Figure 5: Real data & GBM.** (a) evolution of the daily closing price of an asset of Alphabet Inc. between 01/01/2020 and 01/01/2025, (b) evolution of the logarithmic daily return  $r_{d,k} = \log(S_k/S_{k-1})$  in the same time gap as the one in (a). (c) Distribution of the logarithmic daily returns. The dashed black line presents a normal distribution with mean 0 and unit variance. (d-e) Evolution of the mean and variance of the logarithmic daily returns over time windows of 10, 30 and 50 years along the 5 years of data. The dashed lines represent the overable average and variance. (f) Autocorrelation function of the logarithmic daily returns for different time lags  $k$ .

Key hypothesis of GBM include the fact that returns are independently and identically sampled, with constant mean and variance, e.g. recall that for GBM  $L_k = S_k/S_{k-1} \sim \mathcal{N}(1 + \mu\Delta t + \sigma\Delta W_k)$ . We put this to test by computing the average value and variance of the logarithmic daily return over different time windows of 10, 30 and 50 days over the 5 years of data. In Fig. (5,d-e) the average value over these time window gaps are shown. The mean logarithmic daily return seems to fluctuate homogeneously around the global mean value, wheres the variance

presents peaks of activity in certain periods of time. While assuming the mean value of daily log-returns is more or less acceptable, the hypothesis does not seem to directly hold for the variance. In order to check the no-correlation between logarithmic daily returns we compute the autocorrelation function  $C_k = (1/\sigma^2)\langle(r_t - \bar{r})(r_{t-k} - \bar{r})\rangle$ , where  $k$  is called time lag. As shown in Fig. (5,f), logarithmic daily returns do not seem to be too correlated. Another way to compute autocorrelations in a time series is by using the Ljung-Box test<sup>2</sup>. A direct implementation in Python (see `pill`) shows that indeed returns are not completely independent, meaning this hypothesis also breaks.

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<sup>2</sup>Find more information on what the Ljung-Box test is and what criteria is applied here.

### 3 Option Pricing – Binomial Model

#### 3.1 Introduction

In this section we will take a look at a fundamental topic in finance – *pricing of derivative*. Since there are many different kinds of derivatives, we will focus on options, which are a very common financial instrument traded in the market.

For this section, I will assume the reader has some previous knowledge on what an option is, the different kinds (European, American, ...) and the mechanisms behind them. The reader is suggested to take a look at chapters 10 and 11 of [Ref]. I will also assume the reader has previous knowledge on other financial instruments such as bonds and (non-) arbitrage.

We will mainly focus on European options simply because they are easy to study analytically – and the purpose of this manuscript is to be able to compare theory with simulations. We will use the following notation,

- $S_0$  is the current stock price of the *underlying asset* of an option.
- $K$  is the strike price of the option.
- $T$  is the "*maturity*" (time) – the time at which the rights of the owner of the option will be executed.
- $S_T$  is the price of the *underlying asset* at maturity.
- $r$  is the risk-free interest rate.
- $\sigma$  is the (percentage) volatility.

In order to simplify things, we will first take a look at what option pricing looks like with simple models such as the binomial model, which we have already studied in the previous section. Again, we discretize time  $t_k = k\Delta t$  for  $k \in \mathbb{Z}^+$  and we define  $T = n\Delta t$  ( $k \leq n$ ). As discussed in the following theoretical context section, in order to price options extensive use of the "*risk neutral*" valuation and measure is used.

#### 3.2 Theoretical Background

The *risk neutral valuation* claims that when valuing derivatives, e.g. options, it can be assumed that the price of the underlying asset grows at the risk-free rate. This means the following; take an asset with current stock price  $S_0$ . We will imagine that the price of the stock evolves in two different worlds, i) the real world – which involves very complex dynamics, i.e. a Langevin equation with a drift term and noise,  $dS_t = f_\mu(S_t, t)dt + \sigma(S_t, t)dW_t$  where  $f_\mu$  is the drift term controlled by a set of parameters  $\mu$  and  $\sigma$  represents a generalized volatility, and ii) the *risk-neutral world* where we simply forget about the existence of drift and some generalized volatility, and assume that statistically speaking the stock price grows with the risk-free rate  $r$ . Then, the price of the derivative in both worlds, the real one and the risk-free one is the same.

The risk-neutral valuation can be equivalently stated by saying investors do not mind volatility when pricing – meaning an option with an underlying asset with low volatility and one with a high volatility one have the same price under risk-neutral valuation, provided they have the same risk-free rate  $r$ . In general (not specifically for derivative pricing), risk-neutral applies to an investor who does not take into account the potential risk (volatility) associated with an investment when making a decision.

### 3.2.1 Risk-neutral measure for the Binomial model

Consider a the Binomial model but instead of probabilities  $p$  and  $1 - p$  (real-world probabilities) now probabilities  $q$  and  $1 - q$ . Just as in the binomial model, the evolution of the asset's price with measure  $q$  is given  $S_k$  then  $S_{k+1} = uS_k$  with probability  $q$  and  $S_{k+1} = dS_k$  with probability  $1 - q$ , with  $d < 1 < u$ . The measures (probabilities)  $q$  and  $1 - q$  are said to be risk-neutral if,

$$\mathbb{E}_q[S_{k+1}|S_k] := S_k[ug + d(1 - q)] = S_k e^{r\Delta t}, \quad (3.1)$$

where the subindex in  $\mathbb{E}_q$  simply means the average is taken under the risk-neutral probability  $q$ . Eq. (3.1) implies then that,

$$q = \frac{e^{r\Delta t} - d}{u - d}. \quad (3.2)$$

Note how  $q$  is not a real probability. Indeed in order for  $q$  to be between 0 and 1 it is required that  $d < e^{r\Delta t} < u$ .

Now consider a multi-period Binomial tree stopping at time  $T = n\Delta t$ . Through each Binomial repetition of a trajectory, the final price of the stock  $S_n$  can take values  $S_n = S_0 u^k d^{n-k}$ , where  $k = 0, 1, \dots, n$  is a Binomial variable,  $p(k) = \binom{n}{k} q^k (1 - q)^{n-k}$ . Thus,

$$\begin{aligned} \mathbb{E}_q[S_n|S_0] &= \sum_{k=0}^n S_0 u^k d^{n-k} p(k) = S_0 \sum_{k=0}^n \binom{n}{k} [uq]^k [d(1 - q)]^{n-k} \\ &= S_0 [uq + d(1 - q)]^n = S_0 (e^{r\Delta t})^n = \boxed{S_0 e^{rT}}, \end{aligned} \quad (3.3)$$

since  $q$  is the risk-neutral measure so  $ud + d(1 - q) = e^{r\Delta t}$  and where we have used the binomial theorem. In other words, the price of the asset statistically grows with the risk-free rate  $r$ . This also implies that the initial price of the stock can be obtained by simply discounting the risk-free rate of the statistical average price of the stock at time  $T$ ,  $S_0 = e^{-rT} \mathbb{E}_q[S_n|S_0]$  when using the risk-free measure.

### 3.2.2 Pricing of European options with no dividends and the Put-Call parity

In this section we analytically price both call and put European options with no dividends under the assumption of risk-neutral valuations, that is, assuming that the price of these objects is the same in the real worlds and the risk neutral one.

Consider a European call option, which in the absence of transaction fees as payoff  $\max(S_T - K, 0)$ , where  $S_T$  is the price of the underlying asset at maturity and  $K$  is the strike price. We denote by  $c_t$  the price of the call option at some time  $t$ . Then, in the risk-free world we know that, just as we wrote for the price of an asset, the current price of the option – that is, the ammount of money you should pay for it in a exchange market – is related to the value of the same option at maturity through

$$c_0 = e^{-rT} \mathbb{E}_q[c_T|c_0]. \quad (3.4)$$

Now, since at maturity the option is simply worth its payoff – that is, it is worthless (you make no money out of it) if  $S_T < K$  and can instantly sell the asset to ensure profit  $S_T - K$  if  $S_T > K$  – meaning  $c_T = \max(S_T - K, 0)$ . Then,

$$\mathbb{E}_q[c_T|c_0] = \mathbb{E}_q[\max(S_T - K, 0)|S_0] = \sum_{k=0}^n \binom{n}{k} q^k (1-q)^{n-k} \times \max(S_0 u^k d^{n-k} - K, 0), \quad (3.5)$$

since the price at maturity is Binomially distributed taking values  $S_T = S_0 u^X d^{n-X}$  for  $X = 0, \dots, n$  – remember that  $T = n\Delta t$ . The previous sum can be computed noting that only the terms for which

$$S_0 u^k d^{n-k} - K > 0 \Leftrightarrow k > \frac{-n \ln d + \ln(K/S_0)}{\ln(u/d)} =: \alpha \quad (3.6)$$

are different than zero. This simply means that

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} q^k (1-q)^k \times \max(S_0 u^k d^{n-k} - K, 0) &= \sum_{k>\alpha} \binom{n}{k} q^k (1-q)^{n-k} (S_0 u^k d^{n-k} - K) \\ &= S_0 U_1 - K U_2, \end{aligned} \quad (3.7)$$

where we have defined

$$U_1 = \sum_{k>\alpha} \binom{n}{k} q^k (1-q)^{n-k} u^k d^{n-k} \quad (3.8)$$

$$U_2 = \sum_{k>\alpha} \binom{n}{k} q^k (1-q)^{n-k} =: \mathbb{P}_q(X > \alpha), \quad (3.9)$$

where  $X \sim \text{Binom}(n, q)$  and  $\mathbb{P}_q(\circ)$  denotes the probability measure of some event under the risk-free measure. While the term  $U_2$  directly represents the probability of  $X$  being bigger than some value  $\alpha$  (which might be integer or not), while  $U_1$  can be shown to be related to  $U_2$ . Indeed, defining  $\tilde{q} = uq/(uq + d(1-q))$  – remember that  $\mathbb{E}_q[S_{k+1}|S_k = 1] = uq + d(1-q) \equiv e^{r\Delta t}$  – it follows that

$$U_1 = e^{rT} \sum_{k>\alpha} \binom{n}{k} (q^*)^k (1-q^*)^{n-k} \equiv e^{rT} p_{\tilde{q}}(X > \alpha), \quad (3.10)$$

where now  $X \sim \text{Binom}(n, \tilde{q})$ . As a result, the price of call option at present is simply,

$$c \equiv c_0 = e^{-rT} \mathbb{E}_q[c_T|c_0] = S_0 \mathbb{P}_{\tilde{q}}(X > \alpha) - K e^{-rT} \mathbb{P}_q(X > \alpha), \quad (3.11)$$

Note how the probability multiplying  $S_0$  is computed using the measure  $\tilde{q}$  and the one multiplying the  $K$  using the normal risk-free measure.

The same thing can be done for a European put option, for which the the payoff is instead  $p_T = \max(K - S_T, 0)$ . Following the same procedure, it follows that

$$p \equiv p_0 = e^{-rT} \mathbb{E}_q[p_T|p_0] = K e^{-rT} \mathbb{P}_q(X < \alpha) - S_0 \mathbb{P}_{\tilde{q}}(X < \alpha). \quad (3.12)$$

This presents a very interesting result. Indeed, we have not said anything about  $\alpha$ , which is for sure a real number. If it is a decimal number, then for  $c$  we would need to compute the

probabilities  $\mathbb{P}_{q,\tilde{q}}(X > \alpha) = \mathbb{P}_{q,\tilde{q}}(X \geq \lfloor \alpha \rfloor + 1)$ , while we would need to compute  $\mathbb{P}_{q,\tilde{q}}(X < \alpha) = \mathbb{P}_{q,\tilde{q}}(X \leq \lfloor \alpha \rfloor)$  for the put option  $p$ . It is clear, then, that

$$\begin{aligned} p &= Ke^{-rT} \mathbb{P}_q(X \leq \lfloor \alpha \rfloor) - S_0 \mathbb{P}_{\tilde{q}}(X \leq \lfloor \alpha \rfloor) \\ &= Ke^{-rT} [1 - \mathbb{P}_q(X \geq \lfloor \alpha \rfloor + 1)] - S_0 [1 - \mathbb{P}_{\tilde{q}}(X \geq \lfloor \alpha \rfloor + 1)] \\ &= Ke^{-rT} - S_0 + \underbrace{S_0 \mathbb{P}_{\tilde{q}}(X \geq \lfloor \alpha \rfloor + 1) - Ke^{-rT} \mathbb{P}_q(X \geq \lfloor \alpha \rfloor + 1)}_{\text{Call option, } c}, \end{aligned} \quad (3.13)$$

or, samewise,

$$\boxed{c + Ke^{-rT} = p + S_0.} \quad (3.14)$$

This last result is known as the put–call parity, which relates the price of European put and call options when they share strike price and risk-free interest rate, as well as present (underlying) stock price  $S_0$ .

### 3.2.3 Large $n$ and the BSM pricing of European options

Deep inside the “*geometric Brownian motion*” (GBM) regime, for small  $\Delta$ , that is, large enough  $n$ , one can approximate the probabilities involved in the pricing of both European call and put options using the normal approximation of the Binomial distribution. Indeed, for large enough  $n$ ,  $X \sim \text{Binom}(n, p)$  can be approximated by  $Y \sim \mathcal{N}(np, np(1-p))$ .

In Sec. (2) it was shown that Binomial trees can be made converge in the  $n$  large limit – or  $\Delta t$  vanishing limit – to GBM using the Cox–Ross–Rubenstein measure  $q = (e^{r\Delta t} - d)/(u - d)$  and setting  $u = d^{-1} = e^{\sigma\sqrt{\Delta t}}$ , where again  $r$  is the risk-free rate and  $\sigma$  is the percentage volatility. It is straight forward to see that in this case,

$$\alpha = \frac{n}{2} - \frac{1}{2\sigma\sqrt{\Delta t}} \ln \left( \frac{S_0}{K} \right). \quad (3.15)$$

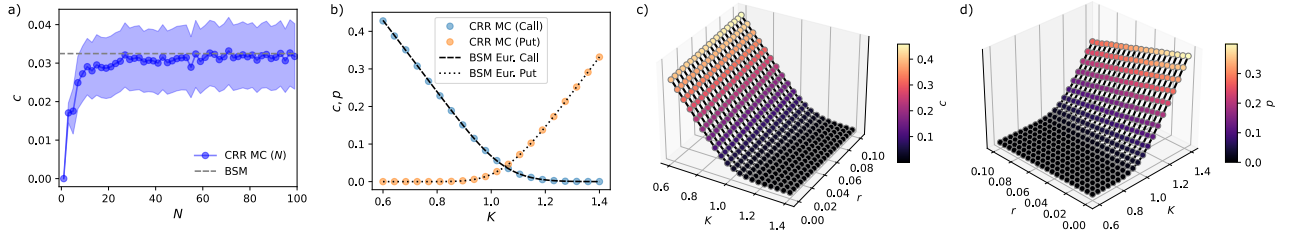
Furthermore, since for  $n$  large enough,  $X \sim \text{Binom}(n, p)$  can be approximated normally with  $Y \sim \mathcal{N}(np, \sqrt{np(1-p)})$ , meaning for instance – in the case of the call option –  $\mathbb{P}_q(X > \alpha) \approx \mathbb{P}(Y > \alpha)$  where  $Y \sim \mathcal{N}(nq, \sqrt{nq(1-q)})$ , and, at the same time,  $\mathbb{P}(Z > \frac{\alpha - nq}{\sqrt{nq(1-q)}})$  where  $Z \sim \mathcal{N}(0, 1)$  is the standardization of the variable  $Y$ ,  $Z = (Y - \mu_Y)/\sigma_Y$ . Using now that  $\Delta t = T/n$ , with  $n \rightarrow \infty$ , it can be shown that  $q \approx 1/2 + (1/2\sigma)(r - \sigma^2/2)\sqrt{T/n} + \mathcal{O}(T/n)$ , so that  $q(1-q) \approx \frac{1}{4} + \mathcal{O}(T/n)$  while  $\sqrt{n}(1/2 - q) \approx -\frac{1}{2\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{T} + \mathcal{O}(1/\sqrt{n})$  so that

$$\lim_{n \rightarrow \infty} \frac{\alpha - nq}{\sqrt{nq(1-q)}} = -\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \equiv -d_2. \quad (3.16)$$

Hence, we have  $\mathbb{P}_q(X > \alpha) \approx \mathbb{P}(Z > (\alpha - nq)/\sqrt{nq(1-q)}) = 1 - \mathbb{P}(Z < (\alpha - nq)/\sqrt{nq(1-q)}) \approx 1 - \Phi(-d_2) = \Phi(d_2)$ , where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z dz' e^{-z'^2/2}, \quad (3.17)$$

and where we have used that for large enough  $n$ ,  $(\alpha - nq)/\sqrt{nq(1-q)} \approx -d_2$ . Furthermore,  $\mathbb{P}_{\tilde{q}}(X > \alpha) \approx \mathbb{P}(Z > (\alpha - n\tilde{q})/\sqrt{n\tilde{q}(1-\tilde{q})})$ . Using now  $\tilde{q} = uqe^{-r\Delta t} \approx 1/2 + (1/2\sigma)(r +$



**Figure 6: Monte Carlo simulations of European options' prices.** a) Plot showing the convergence of the price estimated using the BSM formula and the simulated price of a call option as the size of the tree  $N$  increases. The simulation was performed for  $S_0 = 1$ ,  $K = 1.2$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $T = 1$  and  $N_r = 10^4$  independent realizations. The shaded blue area corresponds to  $\pm\sigma$  with respect to the average value. b) Comparison of simulated prices of both a European call ( $c$ ) and put ( $p$ ) options for  $S=1.0$ ,  $r = 0.05$ ,  $\sigma = 0.1$ ,  $T = 1$ ,  $N_r = 10^4$  and  $N = 10^3$  for different values of the strike price  $K$ . The dashed and dotted black lines represent the approximation of the price using the BSM formula. c) Comparison of the simulated price of a call option and the approximated one using the BSM formula for a range of parameters ( $K, r$ ) for  $S_0 = 1$ ,  $\sigma = 0.1$ ,  $T = 1$ ,  $N_r = 10^4$  and  $N = 10^3$ . d) Same as in c), but now for a European put option.

$\sigma^2/2)\sqrt{T/n} + \mathcal{O}(T/n)$ , so that again  $\tilde{q}(1 - \tilde{q}) \approx 1/4 + \mathcal{O}(T/n)$  and  $\sqrt{n}(1/2 - \tilde{q}) \approx -(1/2\sigma)(r + \sigma^2/2)\sqrt{T} + \mathcal{O}(1/\sqrt{n})$ ,

$$\lim_{n \rightarrow \infty} \frac{\alpha - n\tilde{q}}{\sqrt{n\tilde{q}(1 - \tilde{q})}} = -\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \equiv -d_1, \quad (3.18)$$

meaning  $\mathbb{P}_{\tilde{q}}(X > \alpha) \approx \mathbb{P}(Z > (\alpha - n\tilde{q})/\sqrt{n\tilde{q}(1 - \tilde{q})}) \approx 1 - \Phi(-d_1) = \Phi(d_1)$ , where again we have used that for large enough  $n$ ,  $(\alpha - n\tilde{q})/\sqrt{n\tilde{q}(1 - \tilde{q})} \approx -d_1$ . The price of a call option, then, can be approximated as

$$c = S_0\mathbb{P}_{\tilde{q}}(X > \alpha) - Ke^{-rT}\mathbb{P}_q(X > \alpha) \stackrel{n \rightarrow \infty}{\approx} \boxed{S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)}. \quad (3.19)$$

It is important to note that  $d_2 = d_1 - \sigma\sqrt{T}$ . The pricing of a European call option  $c$  as boxed in Eq. (3.19) is known as the Black–Scholes–Merton (BSM) formula. The price of a European put option can be approximated the same way. A smarter way, however, is to use the put–call parity – which applies also in the  $n$  large limit since it was shown to apply for all  $n$  – which gives,

$$\boxed{p = Ke^{-rT}(1 - \Phi(d_2)) - S_0(1 - \Phi(d_1))}. \quad (3.20)$$

As we will see in the next section regarding Monte Carlo simulations and option pricing, these closed expression for  $c$  and  $p$  fit simulated prices when  $\Delta t$  becomes small.

### 3.3 Simulation

Simulating the price of both European call and put options using Monte Carlo dynamics is trivially done once one know how to perform Monte Carlo simulations of the evolution of the price of an asset using the Binomial model. This was already discussed on the previous chapter. Once the Binomial model is used to sample stochastic evolutions of the price until maturity

$T$ ,  $S_{T,\alpha}$ , the payoffs are computed at maturity, i.e.  $c_{T,\alpha} = \max(S_{T,\alpha} - K, 0)$ , to be later on discounted until present and then the average is performed,

$$c = \langle e^{-rT} c_{T,\alpha} \rangle = \frac{e^{-rT}}{N_r} \sum_{\alpha=1}^{N_r} \max(S_{T,\alpha} - K, 0) \quad (3.21)$$

$$p = \langle e^{-rT} p_{T,\alpha} \rangle = \frac{e^{-rT}}{N_r} \sum_{\alpha=1}^{N_r} \max(K - S_{T,\alpha}, 0), \quad (3.22)$$

where  $N_r$  is the number of independent realizations. The pill corresponding to numerical simulation of European call and put option prices and its comparison and convergence with the Black–Scholes–Merton formula has been added to the [GitHub repository](#). See some of the results shown in the pill in Fig. (6).

## 4 The Black–Scholes Equation

### 4.1 Introduction

In this short – yet relevant – chapter we will take a look at option pricing now through the well-known Black–Scholes equation. The concepts introduced in the previous chapter will also be used here, and the reader will be assumed to know the bare basics, i.e. the payoffs of European call and put options.

Two different derivations of the Black–Scholes equation will be presented, as well as the formal resolution of the equation. A first introduction to the risk-free measure using stochastic differential equations will be showcased. The first derivation will be using a technique frequently used in finance – the replicating portfolio. The second derivation consists on a direct application of the Feynman – Kac theorem, which links the formal solution of parabolic differential equations and stochastic processes. A practical Python Notebook pill that numerically integrates the Black–Scholes equation will be provided. The same pill will include finding the solution of the Black–Scholes equation generating stochastic trajectories, following the Feynman–Kac theorem.

### 4.2 Theoretical background

In this section, we will go through the bare basics required in order to understand a full derivation of the Black–Scholes equation. This implies knowing how to define the risk-neutral measure for GBM using stochastic differential equations, which involves some technical details that will be quickly revised. The proof of these already existing results for stochastic processes and probability measures are out of the scope of the manuscript and thus we will only go over the results. Previous knowledge on stochastic processes and differential equations is recommended.

#### 4.2.1 Risk-neutral measure & GBM

Transforming to the "*risk-neutral*" world is not as simple as shown in the previous chapter. Indeed, traveling to the risk-neutral world has the additional cost of changing the probability measure, which the case of Binomial trees is not very involved. This is not the case, however, when using stochastic differential equations to describe GBM. Indeed, Brownian motion possesses its own physical measure  $\mathbb{P}$  – the one we use to describe generalized GBM with the probability density function  $p_S(s, t)$  solution of Eq. (2.2). We thus start from the usual from usual GBM,

$$dS_t = \mu S_t + \sigma S_t dW_t^p, \quad (4.1)$$

where the superscript  $p$  denotes that the Wiener process (or diffusion) is under probability measure  $\mathbb{P}$  – with some filtration  $(\mathcal{F}_t)_{0 \leq t < \infty}$  –, and want to build the risk-free measure  $\mathbb{P}_q$  so that, as pointed out in the previous chapter, under  $\mathbb{P}_q$  the process evolves at the risk-free rate  $r$ .

One condition that be imposed when transforming into the risk-free world is that the probability measures are equivalent. Recall that two probability measures  $\mathbb{P}$  and  $\mathbb{P}'$  are said to be equivalent if for any event  $A$ ,  $\mathbb{P}(A) = 0$  if and only if  $\mathbb{P}'(A) = 0$ . The Radon–Nikodym theorem shows that two measures  $\mathbb{P}$  and  $\mathbb{P}'$  are equivalent if and only if there exists a random variable  $Z > 0$  such that  $\mathbb{E}_p[Z] = 1$  and then  $d\mathbb{P}' = Z d\mathbb{P}$ . This is however, not always strictly necessary when it comes to risk-neutral measures. This can be generalized for parameter dependent transformations, such as time. In this case, the new measure  $\mathbb{P}'_t$  is related to the old one  $\mathbb{P}$  through

$d\mathbb{P}'_t = Z_t d\mathbb{P}$ , where  $Z_t$  is now a martingale [Ref].

The most relevant result when transforming to the risk-free world is the (Cameron–Martin) Garsinov theorem, which is indeed a generalization of the Cameron–Martin theorem proposed for stochastic processes generated via Brownian motion with constant drift. Indeed, the Girsanov theorem applies to nearly all probability measures  $\mathbb{P}'$  such that  $\mathbb{P}$  and  $\mathbb{P}'$  are *mutually absolutely continuous*, but this is out of the scope of this manuscript. Indeed, consider first a standard Brownian motion  $W_t$  under probability measure  $\mathbb{P}$  with filtration  $(\mathcal{F}_t)_{0 \leq t < \infty}$ . Under  $\mathbb{P}$ , for all  $\lambda \in \mathbb{R}$  (a scalar, real number) the process  $Z_\lambda(t) = \exp(\lambda W_t - \lambda^2 t/2)$  is a martingale with respect to  $\mathcal{F}_t$ . These martingales are used to build the probability measures  $\mathbb{P}_{\theta,t}$  through  $d\mathbb{P}_{\theta,t} = Z_\theta(t) d\mathbb{P}$  in the Cameron–Martin theorem. These martingales belong a broader class of martingales called exponential martingales. Here's where Garsinov's theorem emerges from. Let  $\{\lambda_s\}$  be an adapted process so that the Itô integrals  $\int_0^t \lambda_s dW_s$  are well defined. We define

$$Z_t = \exp \left( \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right). \quad (4.2)$$

It can be shown that  $Z_t$  is a martingale if either

$$\mathbb{E}_p \left[ \exp \left( \frac{1}{2} \int_0^t \lambda_s^2 ds \right) \right] < \infty \text{ or } \mathbb{E}_p \left[ \exp \left( \frac{1}{2} \int_0^t \lambda_s dW_s \right) \right] < \infty. \quad (4.3)$$

These conditions are called the Novikov and Kazamaki conditions, which, when verified, then for each  $t \geq 0$ ,  $\mathbb{E}_p[Z_t] = 1$  and  $Z_t$  is a positive martingale. The *Girsanov theorem* then states that given a standard Wiener process  $W_t^p$  under probability measure  $\mathbb{P}$ , the stochastic process

$$W_t^q = W_t^p - \int_0^t \lambda_s ds \quad (4.4)$$

is a standard Wiener process under the probability measure  $\mathbb{P}_q$  when  $d\mathbb{P}_q = Z_t d\mathbb{P}$  and  $Z_t$  is a martingale [Ref]. It is then enough that the adapted process  $\lambda_s$  verifies either the Novikov or Kazamaki conditions, for  $W_t^q$  to be standard Wiener process under the new probability measure  $\mathbb{P}_q$ . A proof of Novikov's condition and the Garsinov theorem are shown in [galton uchicago lalley courses 390].

These sets the road to understanding how we can transform to the risk-free world using stochastic differential equations. Formally speaking the risk-neutral measure  $\mathbb{P}_q$  is an equivalent measure to  $\mathbb{P}$  (of GBM) under which the discounted stock price  $e^{-\int_0^t r_s ds} S_t$  is a martingale [gautam, match cmu]. Indeed,

$$\begin{aligned} d \left( e^{-\int_0^t r_s ds} S_t \right) &= -r_t e^{-\int_0^t r_s ds} S_t dt + e^{-\int_0^t r_s ds} dS_t \\ &= (\mu - r_t) e^{-\int_0^t r_s ds} S_t dt + e^{-\int_0^t r_s ds} \sigma S_t dW_t \\ &= \sigma \left( e^{-\int_0^t r_s ds} S_t \right) \left( \frac{\mu - r_t}{\sigma} dt + dW_t^p \right) \equiv \sigma \left( e^{-\int_0^t r_s ds} S_t \right) dW_t^q, \end{aligned} \quad (4.5)$$

where  $dW_t^q \equiv \frac{\mu - r_t}{\sigma} dt + dW_t^p \equiv dW_t^p - \lambda_t dt$ , with  $\lambda_t = -\frac{\mu - r_t}{\sigma}$ . The quantity  $-\lambda_t = \frac{\mu - r_t}{\sigma}$  is known as market price of risk, a commonly used quantity in finance. Since there is no drift term in Eq. (4.5) is a local martingale. It is easy to see that for  $r_t = r$  constant, Novikov's condition holds

meaning  $Z_t$  will be a martingale with  $\lambda = -(\mu - r)/\sigma$  and thus the risk-free measure will, indeed, exist, and be provided by  $d\mathbb{P}_q = Z_t d\mathbb{P}$ . This is by definition what one expects in the risk-free world: the discounted price is a martingale,  $S_0 = \mathbb{E}_q[e^{-\int_0^t r_s ds} S_t]$ .

Indeed, when  $r_t = r$  is constant, then as expected by the definition of the risk-free measure, the prices evolve under  $\mathbb{P}_q$  at the risk-free interest rate  $r$ ,

$$dS_t = \mu S_t dt + \sigma S_t dW_t^p = \mu S_t dt + \sigma S_t \left( dW_t^q - \frac{\mu - r}{\sigma} \right) \equiv \boxed{r S_t dt + \sigma S_t dW_t^q}. \quad (4.6)$$

It is straightforward to show this is a martingale. The formal solution (using Itô's lemma) can be shown to be  $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t^q)$ . Thus, the discounted price is  $S_t^d = e^{-rt} S_t = S_0 \exp(-\frac{1}{2}\sigma^2 t + \sigma W_t^q)$  (the super-index  $d$  is used to denote the discounted price), and its expected value is  $\mathbb{E}_q[S_t^d] = S_0 e^{-\sigma^2 t/2} \mathbb{E}_q[e^{\sigma W_t^q}] = S_0$ , since  $W_t^q \sim \mathcal{N}(0, t)$  (in  $\mathbb{P}_q$ ) and thus  $\mathbb{E}_q[e^{\sigma W_t^q}] = e^{\sigma^2 t/2}$ . Example options such as European call and put options can be directly priced to find the Black-Scholes—Merton formula. This is not the goal of this chapter, since the pricing was already discussed in the previous section. The goal now, having build the risk-free measure for GBM using stochastic differential equations, will be finding our way into the famous Black-Scholes equation.

#### 4.2.2 Black-Scholes through a replicating portfolio

Consider the market consists of 3 different items; assets of price  $S_t$ , bonds  $B_t$  and the option to buy assets the assets,  $V(t, S_t)$ . We will consider this option to be a forward contract for simplicity. We want to know what the price of the option in the present,  $t = 0$ , knowing that at maturity the option is worth its payoff,  $V(T, S_T) \equiv \varphi_K(S_T) = \max(S_T - K, 0)$  in the case of a European call option (remember that  $K$  is the strike price). We work at the risk free world since in order to price the option at present we know we can discount the interest rate,  $V_0 = \mathbb{E}_q[e^{-rT} \varphi_K(S_T)]$ .

In order to price  $V$  we will used a commonly used technique in finance, the "*replicating portfolio*" along with no arbitrage opportunity. The first step into the pricing of  $V$  at present is to build one portfolio by buying  $n_S$  units of the stock at price  $S_0$ , and some units  $n_B$  of bond at price  $B_0$ . Note how  $n_S, n_B$  are not required to be positive numbers, and they might evolve in time. Negative  $n_B$  means units of bond are build sold, that it, money is being borrowed. We let the stock evolve in the risk-free neutral measure so that, Let then  $S_t$  evolve under the risk-free measure, while  $B_t$  is a riskless bond, that is,

$$dS_t = r S_t dt + \sigma S_t dW_t^q \quad (4.7)$$

$$dB_t = r B_t. \quad (4.8)$$

since bonds evolve deterministically at the risk-free interest rate. The total value of the portfolio at time  $t$  will be  $\Pi_t^{(1)} = n_S S_t + n_B B_t$ . We will consider now a second portfolio that consists only on the European call option. Its value at time  $t$  will be  $\Pi_t^{(2)} = V(t, S_t)$ . Since the option only allows for payment only at present time,  $t = 0$ , and at maturity,  $t = T$ , the replicating portfolio of the option will also not allow in or out flow of money for  $t \in (0, T)$  (ends excluded). This means that, if at a given time  $t$  we decide to sell  $dn_S(t)$  units of stock, the ammount we received,  $S_t dn_S(t)$  is fully invested in buying  $dn_B(t)$  units of bond for cost  $-dn_B(t) B_t$ , meaning  $S_t dn_S(t) = -dn_B(t) B_t$  [uc3m, halweb]. This is called the "*self-financing*" condition, and it ensures that a porfolio's performance arises from market movements, and not for external factos

(an isolated system, this can be considered a conservation law). Using Itô's lemma, the evolutions for  $S_t$  and  $B_t$  in Eqs. (4.7, 4.8) and the self-financing condition imposed over the replicating portfolio  $\Pi_t^{(1)}$ , it follows that

$$d\Pi_t^{(1)} = n_S dS_t + n_B dB_t = (n_S r S_t + n_B r B_t) dt + n_S \sigma S_t dW_t^q \quad (4.9)$$

$$d\Pi_t^{(2)} = \left( \frac{\partial V}{\partial t} + r S_t \frac{\partial V}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial V}{\partial S_t} dW_t^q. \quad (4.10)$$

Since the replicating portfolio  $\Pi_t^{(1)}$  "replicates" the portfolio that only consists on the option,  $\Pi_t^{(2)}$  any change in value on one has to correspond to a change on the other one, meaning  $d\Pi_t^{(1)} = d\Pi_t^{(2)}$ . This takes to

$$n_S = \frac{\partial V}{\partial S_t}(\delta) \quad (4.11)$$

$$n_B = \frac{1}{r B_t} \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right). \quad (4.12)$$

Now, again, since the whole market consists of these objects and the allow no arbitrage, then the actual values of both portfolios has to coincide, since if that was not the case there would be arbitrage opportunity. Hence, since  $\Pi_t^{(1)} = \Pi_t^{(2)} = V(t, S_t)$ , plugging the above expressions for  $n_S, n_B$ , it follows that

$$V(t, S_t) = n_S S_t + n_B B_t = \frac{\partial V}{\partial S_t} S_t + \frac{1}{r} \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right), \quad (4.13)$$

or,

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + r s \frac{\partial V}{\partial s} - r V = 0}, \quad (4.14)$$

where we have simply chaged  $S_t$  by  $s$  in order to ease notation. Equation Eq. (4.14) is known as the Black-Scholes equation. The boundary conditions in this case apply at maturity, since we know that  $V(T, S) = \varphi_K(s) = \max(s - K, 0)$  for European call options. Since the boundary condition is applied at maturity, it feel natural to make the change of variables  $\tau = T - t$  so that now the boundary condition is applied at  $\tau = 0$ , i.e.  $V(\tau = 0, s) = \varphi_K(s)$ .

### 4.2.3 Solution to the Black-Scholes Equation

Solving the Black-Scholes equation Eq. (4.14) without a clear starting point is hard. It is possible to use a wise change of variables and a transformation to  $V$ , that turns the Black-Scholes equation into the diffusion equation. Since it is easier to use initial conditions rather than termianl, we define  $\tau = T - t$  so that  $V(T, s) \equiv V(\tau = 0, s)$ . Furthermore, since we see each derivative with respect to  $s$  comes together with an  $s$  multiplying, i.e. there's only terms of the shape  $s^n \frac{\partial^n}{\partial s^n}$ , the change of variables  $x = \ln s$  will simplify the PDE. Indeed, after applying these two change of vairables, one obtains

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 \left( \frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right) + r \left( \frac{\partial V}{\partial x} - V \right) = 0, \quad (4.15)$$

with now  $V(x, \tau = 0) = \max(e^x - K, 0)$ . The next step into getting closer to a diffusion equation is by making Gauge transformation on  $V(x, t)$ , defining  $v(x, t) = e^{z_1 x + z_2 t} V(x, t)$ , with  $z_1, z_2$  real numbers. Using the transformation, the equation for  $v(x, t)$  writes

$$-\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left(-\sigma^2 z_1 + r - \frac{1}{2}\sigma^2\right) \frac{\partial v}{\partial x} + \left(\frac{1}{2}\sigma^2 z_1^2 - (r - \sigma^2/2)z_1 - r + z_2\right) v = 0. \quad (4.16)$$

Since the transformation is, in principle, arbitrary and valid for all  $z_1, z_2$  real numbers, we can fix  $z_1$  and  $z_2$  to be the values for which the prefactor multiplying the terms  $\partial_x v$  and  $v$  to vanish, meaning,

$$z_1 = \frac{r}{\sigma^2} - \frac{1}{2} \quad (4.17)$$

$$z_2 = r + \left(r - \frac{1}{2}\sigma^2\right) z_1 - \frac{1}{2}\sigma^2 z_1^2. \quad (4.18)$$

When this is the case, the equation for  $v(x, \tau)$  becomes,

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2}, \quad (4.19)$$

with initial condition  $v(x, 0) = e^{z_1 x} V(x, 0) = e^{z_1 x} \max(e^x - K, 0)$ . Note how since  $s \geq 0$ ,  $x = \log s \in (-\infty, \infty)$ . Transforming to Fourier space by taking the convention  $\hat{h}(k) = \mathfrak{F}[h(x)] = \int e^{-ikx} h(x)$  and the antitransform  $h(x) = \mathfrak{F}^{-1}[\hat{h}(k)] = \int \frac{dk}{2\pi} e^{ikx} \hat{h}(k)$  the equation for  $v(x, \tau)$  in Fourier space becomes,

$$\frac{\partial \hat{v}(k, \tau)}{\partial \tau} + \frac{1}{2}\sigma^2 k^2 \hat{v}(k, \tau) = 0. \quad (4.20)$$

The solution to Eq. (4.20) is formally written as  $\hat{v}(k, \tau) = \hat{v}(k, 0) e^{-\frac{1}{2}\sigma^2 k^2 \tau}$ , where  $\hat{v}(k, 0)$  is the initial condition in Fourier space. Using that in real space  $v(x, \tau = 0) = e^{z_1 x} \max(e^x - K, 0)$  it is easily shown that the full solution in Fourier space writes,

$$\hat{v}(k, \tau) = \int_{-\infty}^{\infty} dx e^{-ikx} e^{z_1 x} \max(e^x - K, 0) e^{-\frac{1}{2}\sigma^2 k^2 \tau}. \quad (4.21)$$

Transforming back to real space, the full solution can be shown to be

$$v(x, \tau) = \int_{-\infty}^{\infty} dx' e^{z_1 x'} \max(e^{x'} - K, 0) G(x' - x), \quad G(z) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{z^2}{2\sigma^2\tau}\right). \quad (4.22)$$

In order to obtain this result, a Gaussian integral is performed in momentum space. Note how the presence of  $\max(e^{x'} - K, 0)$  makes the integral identically zero for any  $x'$  real such that  $e^{x'} < K$ . Thus, the integral can be separated in to two,

$$v(x, \tau) = \int_{\ln K}^{\infty} dx' e^{z_1 x'} (e^{x'} - K) G(x' - x) \equiv I_1 - K I_2, \quad (4.23)$$

where we have defined

$$I_1 = \int_{\ln K}^{\infty} dx' e^{(1+z_1)x'} G(x' - x) \quad (4.24)$$

$$I_2 = \int_{\ln K}^{\infty} dx' e^{z_1 x'} G(x' - x'). \quad (4.25)$$

These two integrals both have the shape of the following generalized integral,

$$\mathfrak{I}(m) = \int_a^\infty e^{mx'} G(x' - x) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_a^\infty dx' \exp\left(mx' - \frac{(x' - x)^2}{2\sigma^2\tau}\right), \quad (4.26)$$

Completing the square inside the exponential,  $mx' - (x' - x)^2/2\sigma^2\tau = -(x' - \mu)^2/2\sigma^2\tau + \frac{1}{2}m^2\sigma^2\tau + mx$  with  $\mu = x + m\sigma^2\tau$ , the integral becomes,

$$\mathfrak{I}(m) = \exp\left(mx + \frac{1}{2}m^2\sigma^2\tau\right) \Phi\left(\frac{\mu - a}{\sigma\sqrt{\tau}}\right), \quad (4.27)$$

where  $\Phi(\circ)$  is the CDF of the standardized gaussian, as in Eq. (3.17). Here  $I_1 = \mathfrak{I}(1 + z_1)$ ,  $I_2 = \mathfrak{I}(z_1)$  with  $a = \ln K$ , and thus, remembering that  $x = \log s$  and that  $\mu = x + m\sigma^2\tau$ ,

$$v(x, \tau) = e^{(1+z_1)x} e^{\frac{1}{2}(1+z_1)^2\sigma^2\tau} \Phi\left(\frac{\ln(s/K) + (1+z_1)\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - K e^{z_1x} e^{\frac{1}{2}z_1^2\sigma^2\tau} \Phi\left(\frac{\ln(s/K) + z_1\sigma^2\tau}{\sigma\sqrt{\tau}}\right). \quad (4.28)$$

Transforming back now to  $V(x, \tau) = e^{-(z_1x+z_2\tau)}v(x, \tau)$  and applying again that  $e^x = s$ ,

$$V(s, \tau) = s e^{\frac{1}{2}(1+z_1)^2\sigma^2\tau} e^{-z_2\tau} \Phi\left(\frac{\ln(s/K) + (1+z_1)\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - K e^{\frac{1}{2}z_1^2\sigma^2\tau} e^{-z_2\tau} \Phi\left(\frac{\ln(s/K) + z_1\sigma^2\tau}{\sigma\sqrt{\tau}}\right). \quad (4.29)$$

Plugging now the definitions of  $z_1, z_2$  in Eqs. (4.17, 4.18), it follows that  $(1+z_1)\sigma^2\tau = (r+\sigma^2/2)\tau$  and  $z_1\sigma^2\tau = (r-\sigma^2/2)\tau$ , and also that  $\frac{1}{2}(1+z_1)^2\sigma^2\tau - z_2\tau = 0$ , while  $\frac{1}{2}z_1^2\sigma^2\tau - z_2\tau = -r\tau$ . Hence,

$$\boxed{V(s, \tau) = s\Phi\left(\frac{\ln(s/K) + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - K e^{-r\tau} \Phi\left(\frac{\ln(s/K) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)}. \quad (4.30)$$

This last expresion evaluated at present,  $t = 0$ , so  $\tau = T$  returns the one discussed in the previous chapter,  $V_0(s) = s\Phi(d_1) - K e^{-rT} \Phi(d_2)$ .

#### 4.2.4 Feynman–Kac Theorem

The Feynman–Kac theorems and formula are a set of provable statements which set a link between parabolic partial differential equations and stochastic processes. The results of the theorem present a very useful method to approximate solutions to parabolic partial differential equations by generating random path of a specific stochastic process. The Feynman–Kac formula can be also used to, samewise, compute the expection and momenta of a given observable of a stochastic process using partial differential equations. This will work particularly great in our advantage because the Black–Scholes (BS) equation is indeed, parabolic.

The theorem states the following – the solution to the partial differential equation

$$\frac{\partial u(x, t)}{\partial t} + \mu(x, t) \frac{\partial}{\partial x} u(x, t) + \frac{1}{2}\sigma^2(x, t) u(x, t) \frac{\partial^2}{\partial x^2} - h(x, t) u(x, t) = z(x, t), \quad (4.31)$$

defined for  $x$  real and  $t \in [0, T]$  and subject to the terminal condition  $u(x, T) = \Phi(x)$  can be written as a conditional expectation under the probability measure  $\mathbb{P}_q$ ,

$$u(x, t) = \mathbb{E}_q \left[ e_h(t, T) \Phi(X_T) - \int_t^T e_z(t, t') z(X_{t'}, t') dt' \middle| X_t = x \right], \quad (4.32)$$

where  $X_t$  is the Itô process

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t^q, \quad (4.33)$$

with  $W_t^q$  a Wiener process under  $\mathbb{P}_q$  and where,

$$e_h(t, T) = \exp\left(-\int_t^T h(X_{t'}, t')dt'\right), \quad e_z(t, t') = \exp\left(-\int_t^{t'} h(X_{t''}, t'')dt''\right). \quad (4.34)$$

The particular case of the BS equation is simply obtained by directly applying the theorem for  $\mu(x, t) = rx$ ,  $\sigma(x, t) = \sigma x$ ,  $h(x, t) = r$  and  $z(x, t) = 0$ . Under this conditions, the formal solution to the BS equation simply writes

$$u(x, t) = e^{-r(T-t)}\mathbb{E}_q[\Phi(X_T)|X_t = x], \quad (4.35)$$

where now the terminal condition is simply the payoff,  $\Phi(x_T) \equiv \Phi_K(X_T) = \max(X_T - K, 0)$  in the case of a European call option (note how the price now is represented by the variable  $x$ ). Since we are mostly interested in the price of the option at present time, we simply have to evaluate the  $u(x, t)$  at  $t = 0$ . Changing now the name of the variables back to  $s$  and  $S_t$  for the stochastic process, we have

$$u(s, 0) \equiv c(s) = e^{-rT}\mathbb{E}_q[\max(S_T - K, 0)|S_0 = s]. \quad (4.36)$$

Eq. (4.36) then simply says the the solution of the BS equation at time  $t = 0$  is simply the discounted value of the payoff of the option at maturity in the risk free probability measure  $\mathbb{P}_q$ .

### 4.3 Numerical integration

#### 4.3.1 Crank–Nicolson & the Black–Scholes Equation

A very commonly used method in order to solve partial differential equations is the Crank–Nicolson (CN) method [Ref?]. We will not discuss the technicities here, the reader is suggested to visit [Refs] for a brief introduction to the CN method. That being said, the CN can be applied to numerically integrate the BS equation. In order to do so, we first discretize space (price of the underlying) and time by setting  $s_i = s_0 + i\Delta s$  for  $i = 0, \dots, N_s$  where  $N_s$  is the number of discretized intervals for the price of the underlying  $s$ , and  $t_j = j\Delta t$  with  $j = 0, \dots, N_t$  where  $N_t$  is the number of discretization intervals for  $t$ .

The choice for the discretization of  $s$  to start at  $s_0$  is taken so that both  $s_0$  and  $s_{N_s}$  are close to the strike price. We will particularly take  $s_0 = K/3$  and  $s_{N_s} = 2K$ . The time evolves, of course, between 0 and maturity  $T$ , i.e  $t_0 = 0$  and  $t_{N_t} = T$ . The implementation of the solver of the BS equation in the pill of this chapter is an adaptation of [this one](#) [Ref], which directly applies CN to the BS equation and uses already implemented tools in Python for European call options. The CN scheme consists on solving the set

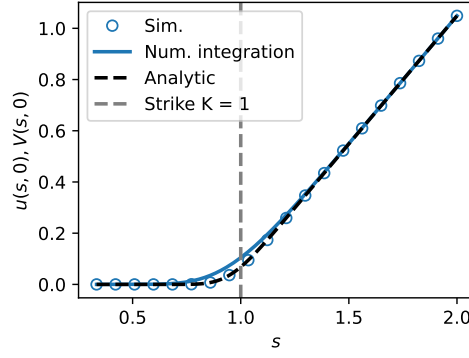
$$-\alpha_i V_{i+1}^{j+1} + (1 - \beta_i) V_i^{j+1} - \gamma_i V_{i-1}^{j+1} = \alpha_i V_{i+1}^j + (1 + \beta_i) V_i^j + \gamma_i V_{i-1}^j, \quad (4.37)$$

where  $V_i^j \equiv V(x_i, t_j)$  and

$$\alpha_i = \frac{1}{4}\Delta t \left[ \sigma^2 \frac{(s_0 + i\Delta s)^2}{\Delta s^2} + r \frac{s_0 + i\Delta s}{\Delta s} \right] \quad (4.38)$$

$$\beta_i = -\frac{1}{2}\Delta t \left[ \sigma^2 \frac{(s_0 + i\Delta s)^2}{\Delta s^2} + r \right] \quad (4.39)$$

$$\gamma_i = \frac{1}{4}\Delta t \left[ \sigma^2 \frac{(s_0 + i\Delta s)^2}{\Delta s^2} - r \frac{s_0 + i\Delta s}{\Delta s} \right], \quad (4.40)$$



**Figure 7: Solution to the Black–Scholes equation.** European call option prices at present generated through simulation of stochastic paths following geometric brownian motion growing at the risk–free interest rate  $r = 0.05$ , with percentage volatility  $\sigma = 0.2$  for initial prices between  $s \in [K/3, 2K]$ , where  $K = 1$  is the srtrike price. The maturity time is  $T = 1$  (blue circle), as well as the solution of the BS equation integrated using the CN method for the same parameters (solid blue line) and the analytical expression for the price of a European call option (black dashed line). The code used to generate the plot can be found in this chapter’s pill.

while applying the terminal condition  $V_i^{N_t} = \max(s_i - K, 0)$  and the following boundary conditions; i)  $V_0^j = 0$  for all  $j = 0, \dots, N_t$  (for all time) since the option is worthless when  $s \ll K$  (here we assume that  $s_0 \ll K$ ), and ii)  $V_{N_s}^j \approx s_{N_s} - Ke^{-r(T-j\Delta t)}$ , since for  $s \gg K$  it the price of the option can be shown to approximately be  $V(s, t) \approx s - Ke^{-r(T-t)}$ .

#### 4.3.2 Approximating solutions of the BS equation using Feynman–Kac’s theorem

As stated in previous section, the Feynman–Kac theorem and formula can be used to find solutions to the BS equation. As discussed, for the BS equation,  $u(x, 0)$  – the price of the option at present – is simply the discounted value fo the payoff of the function in the risk free probability measure. It is enough, then, to generate different paths following geometric Brownian motion (GBM) as done in the first chapter, computing the value of the payoff at maturity of each path, computing the average and then discounting the risk free interest rate.

## 5 Modern Portfolio Theory, Efficient Frontier & CAPM

### 5.1 Introduction

On this chapter we will go over the pillars of modern portfolio theory (MPT), attributed to Markowitz [AG: Ref.] and the capital asset pricing model (CAPM), which was developed by Sharpe, Lintner and Mossin in the 60s and is sustained through the theory developed by Markowitz. While modern portfolio theory aims to explain a portfolio's performance in terms of average return and volatility, the CAPM is a model which has been proven to be useful when pricing assets in the real market. With simple words, it states that the price of an assets is strictly related to its contribution to the total market's volatility.

The simulation section aims to explain how the theory is implemented. In the case of MPT, we mainly focus on the modelling and understanding the nature of a rebalanced portfolio as we all the the modelling of buy-and-hold ones. As of the CAPM, we look at monthly asset returns in the american market, i.e. GOOGL and AAPL, taking as market reference S&P 500. Bonds are tracked through american T-bills.

### 5.2 Theoretical background

#### 5.2.1 Modern portfolio theory & efficient frontier

Consider a portfolio  $\Pi_t$  that consists that consists of units of  $p$  different assets, e.g.  $n_{1,t}$  units of asset 1, with price  $S_{1,t}$ ,  $n_{2,t}$  units of asset 2, with price  $S_{2,t}$  and so on. The value of the portfolio at time  $t$  is given by,

$$\Pi_t = n_{1,t}S_{1,t} + \dots + n_{p,t}S_{p,t} = \sum_{\alpha=1}^p n_{\alpha,t}S_{\alpha,t}. \quad (5.1)$$

The simple return over the period of time  $t \in [t_1, t_2]$  with  $t_1 < t_2$  defined as  $R_{1,2} \equiv \frac{S_{t_2} - S_{t_1}}{S_{t_1}}$  for the portfolio reads,

$$R_{1,2} = \frac{\Pi_{t_2} - \Pi_{t_1}}{\Pi_{t_1}} = \sum_{\alpha=1}^p \left( \frac{n_{\alpha,t_2}}{\sum_{\beta=1}^p n_{\beta,t_1} S_{\beta,t_1}} \right) S_{\alpha,t_2} - 1 \equiv \sum_{\alpha=1}^p x_{\alpha,t_2} R_{\alpha}^{1,2}, \quad (5.2)$$

where  $R_{\alpha}^{1,2} = \frac{S_{\alpha,t_2}}{S_{\alpha,t_1}} - 1$  is the return over  $[t_1, t_2]$  of asset  $\alpha$ ,

$$x_{\alpha,t_2} = \frac{n_{\alpha,t_2} S_{\alpha,t_1}}{\sum_{\beta=1}^p n_{\beta,t_1} S_{\beta,t_1}} \equiv \frac{n_{\alpha,t_2} S_{\alpha,t_1}}{\Pi_{t_1}} \quad (5.3)$$

is the proportion of the investment at time  $t_1$  on asset  $\alpha$ , and where we have used that  $\sum_{\alpha} x_{\alpha,t} = 1$  for all  $t$ . Note how all the individual returns  $R_{\alpha}^{1,2}$  are random variables that may be correlated to one another – making the return of the portfolio a statitstically rich object. Once the value of the prices of the assets are fixed – after an observation – at time  $t_1$ , the expected value (under the measure attributed to the stochastic sampling of the asset prices) and variance of the return of the portfolio are

$$\mathbb{E}[R_{1,2}|\mathbf{S}_{t_1}] = \sum_{\alpha} x_{\alpha,t_1} \mathbb{E}[R_{\alpha}^{1,2}|\mathbf{S}_{t_1}], \quad \mathbb{V}[R_{1,2}|\mathbf{S}_{t_1}] = \sum_{\alpha,\beta} x_{\alpha,t_1} x_{\beta,t_1} \text{Cov}(R_{\alpha}^{1,2}, R_{\beta}^{1,2}|\mathbf{S}_{t_1}), \quad (5.4)$$

where the average  $\mathbb{E}[\circ|\mathbf{S}_{t_1}]$  and the variance  $\mathbb{V}[\circ|\mathbf{S}_{t_1}]$  are evaluated considering the prices at time  $t_1$  to be frozen and where  $\mathbf{S}_{t_1} = (S_{1,t_1}, S_{2,t_1}, \dots, S_{p,t_1})$ .

**Remak** – *Modern portfolio assumptions.*

In MPT, the proportions  $x_\alpha$  are chosen at the time of investment and mainted fixed throughout time by continuously rebalancing<sup>a</sup> —shortly— selling and buying units of assets. Under this assumption, the return of a given portfolio  $\Pi_t$  is a linear combination of the individual returns of each asset the portfolio contains, while the variance of the portfolio returns is fully characterized by the covariance matrix of the individual returns of the assets.

<sup>a</sup>What rebalancing means is fruther explained on the "Simulation" section.

We will usually set  $t_1 = 0$  and  $t_2 = t$  for convenience, without loss of generality. Considering that the expected returns of each individual asset is known until time  $t$  provided the inital condition  $\mathbf{S}_0 = (S_{1,0}, S_{2,0}, \dots, S_{p,0})$ ,  $r_\alpha = \mathbb{E}[R_\alpha^t|\mathbf{S}_0]$  is known and considering we have full access to covariance matrix of the individual asset returns – we know the invididual variances and the correlations between individual returns  $R_\alpha^t$  and  $R_\beta^t$  for  $\alpha \neq \beta$  – then the average return of the portfolio  $r_p = \mathbb{E}[R_t|\mathbf{S}_0]$  and its variance  $\sigma_p^2 = \mathbb{V}[R_t|\mathbf{S}_0]$  are provided by

$$r_p = \sum_{\alpha=1}^p x_{\alpha,0} r_\alpha \quad (5.5)$$

$$\sigma_p^2 = \sum_{\alpha=1}^p x_{\alpha,0}^2 \sigma_\alpha^2 + 2 \sum_{\alpha < \beta} x_{\alpha,0} x_{\beta,0} \sigma_\alpha \sigma_\beta \rho_{\alpha\beta}, \quad (5.6)$$

where  $\rho_{\alpha\beta} = \text{Cov}(R_\alpha^t, R_\beta^t|\mathbf{S}_0)/\sigma_\alpha \sigma_\beta$  is the correlation matrix. Note how  $\sigma_p^2$  can be written only in terms of the correlation matrix since its diagonal is full of ones,  $\rho_{\alpha\alpha} = 1$  for all  $\alpha$ . In what follows we denote  $x_{\alpha,0} \equiv x_\alpha$  in order to ease notation. A given portfolio's performance is provided by its expected returns and variance (or volatility), i.e.  $(r_p, \sigma_p)$ . Optimizing a portfolio is done by adjusting the weights  $x_\alpha$  such that the portfolio  $\Pi_t$  has desired expected return and volatility. A very usual thing to do is to build portfolios that minize risk, that is, minize volatility, so that investments become as *safe* (non-risky) as possible. We thus want to minimize  $\sigma_p$ , or similarly  $\sigma_p^2$  while imposing  $\sum_\alpha x_\alpha = 1$  and  $r_p = \sum_\alpha x_\alpha r_\alpha$ . In order to do that we use lagrange multipliers  $\lambda_1$  and  $\lambda_2$  and define the Lagrange function

$$\mathcal{L}_p = \sigma_p^2 + \lambda_1 \left( \sum_{\alpha=1}^p x_\alpha - 1 \right) + \lambda_2 \left( \sum_{\alpha=1}^p x_\alpha r_\alpha - r_p \right), \quad (5.7)$$

so that the proportions optimizing  $\mathcal{L}_p$  can be shown to be provided by,

$$\frac{\partial}{\partial x_\mu} \mathcal{L}_p = 2 \sum_{\beta} Z_{\mu\beta} x_\beta + \lambda_1 + \lambda_2 r_\mu = 0 \implies \boxed{x_\alpha = -\frac{1}{2} (Z^{-1})_{\alpha\mu} [\lambda_1 e_\mu + \lambda_2 r_\mu]}, \quad (5.8)$$

where we have defined the matrix  $(Z)_{\alpha\beta} \equiv \sigma_\alpha \sigma_\beta \rho_{\alpha\beta}$  and where  $\mathbf{e} = (1, \dots, 1)^T$  is a vector consisting of  $p$  ones, i.e.  $e_\mu = 1$  for all  $\mu = 1, \dots, p$ <sup>3</sup>. The proportions depend on the Lagrange

<sup>3</sup>This is easier to see if one writes the equation in vectorial shape, i.e.  $\mathbf{x} = -\frac{1}{2} Z^{-1} [\lambda_1 \mathbf{e} + \lambda_2 \mathbf{r}]$

multipliers, which are obtained through the application of the corresponding constraints. In order to ease notation, we write the conditions now using the vector  $\mathbf{e}$  and Einstein's summation convention, so that the first constraint writes  $e^\alpha x_\alpha = 1$  and the second one  $r_p = r^\alpha x_\alpha$ . Thus

$$e^\alpha x_\alpha = 1 \implies \lambda_1 e^\alpha (Z^{-1})_\alpha^\mu e_\mu + \lambda_2 e^\alpha (Z^{-1})_\alpha^\mu r_\mu = -2 \quad (5.9)$$

$$r_p = r^\alpha x_\alpha \implies \lambda_1 r^\alpha (Z^{-1})_\alpha^\mu e_\mu + \lambda_2 r^\alpha (Z^{-1})_\alpha^\mu r_\mu = -2r_p. \quad (5.10)$$

Defining  $a = e^\alpha (Z^{-1})_\alpha^\mu e_\mu$ ,  $b = e^\alpha (Z^{-1})_\alpha^\mu r_\mu = r^\alpha (Z^{-1})_\alpha^\mu e_\mu$  and  $c = r^\alpha (Z^{-1})_\alpha^\mu r_\mu$  the solutions to Eqs. (5.9, 5.10) are

$$\lambda_1 = -\frac{2}{ac - b^2}(c - br_p), \quad \lambda_2 = -\frac{2}{ac - b^2}(ar_p - b). \quad (5.11)$$

Consequently, the proportions become

$$x_\alpha = (Z^{-1})_\alpha^\mu \left[ \left( \frac{c - br_p}{ac - b^2} \right) e_\mu + \left( \frac{ar_p - b}{ac - b^2} \right) r_\mu \right], \quad (5.12)$$

meaning the volatility of the portfolio is provided by

$$\sigma_p^2 = x^\alpha Z_\alpha^\beta x_\beta = (Z^{-1})_\alpha^\mu \left[ \left( \frac{c - br_p}{ac - b^2} \right) e_\mu + \left( \frac{ar_p - b}{ac - b^2} \right) r_\mu \right] Z_\alpha^\beta (Z^{-1})_\beta^\nu \left[ \left( \frac{c - br_p}{ac - b^2} \right) e_\nu + \left( \frac{ar_p - b}{ac - b^2} \right) r_\nu \right]. \quad (5.13)$$

Using that  $Z_\alpha^\beta (Z^{-1})_\beta^\nu = \delta_\alpha^\nu$  and reorganizing terms, the optimal volatility reads

$$\sigma_p^2 = \frac{ar_p^2 - 2br_p + c}{ac - b^2}, \quad r_p = \frac{b}{a} = \frac{r^\alpha (Z^{-1})_\alpha^\beta e_\beta}{e^\alpha (Z^{-1})_\alpha^\beta e_\beta}. \quad (5.14)$$

Note how these solution exists provided that  $ac - b^2 \neq 0$ , and how exactly at the point of minimum volatility, i.e.  $r_p = \frac{b}{a}$ , then  $\lambda_2 = 0$ . This implies that exactly at the point of minimal volatility (MV), the proportions are provided by

$$x_{\alpha(\text{MV})} = x_\alpha \Big|_{r_p = \frac{b}{a}} = (Z^{-1})_\alpha^\mu \left( \frac{c - b(b/a)}{ac - b^2} \right) e_\mu = \frac{1}{a} (Z^{-1})_\alpha^\mu e_\mu. \quad (5.15)$$

Similarly, the variance of the portofflio at the MV is simply provided by the replacemete of  $r_p$  in  $\sigma_p^2$  in Eq. (5.14),

$$\sigma_{p(\text{MV})}^2 = \sigma_p^2 \Big|_{r_p = \frac{b}{a}} = a^{-1} \equiv (\mathbf{e}^T Z^{-1} \mathbf{e})^{-1}, \quad (5.16)$$

where now we have simply expressed  $a \equiv e^\alpha (Z^{-1})_\alpha^\mu e_\mu$  in matricial form (omitting Einstein's summation convention) in order to simplify the result.

**Example – Two asset portfolio.**

Consider a portfolio consists of two assets  $A$  and  $B$ , with initial proportions  $x_A, x_B$ , satisfying  $x_A + x_B = 1$ . The assets posses average returns  $r_A, r_B$  and volatilities  $\sigma_A, \sigma_B$ . In this case  $\mathbf{e} = (1 \ 1)^T$  and

$$Z = \begin{pmatrix} \sigma_A^2 & \sigma_A \sigma_B \rho_{AB} \\ \sigma_A \sigma_B \rho_{A,B} & \sigma_B^2 \end{pmatrix} \Rightarrow Z^{-1} = \frac{1}{\sigma_A^2 \sigma_B^2 (1 - \rho_{A,B}^2)} \begin{pmatrix} \sigma_B^2 & -\sigma_A \sigma_B \rho_{AB} \\ -\sigma_A \sigma_B \rho_{A,B} & \sigma_A^2 \end{pmatrix}$$

As a consequence, exactly at the MV, the propotions can be shown to be

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix}_{(\text{MV})} = \frac{1}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{A,B}} \begin{pmatrix} \sigma_B^2 - \sigma_A \sigma_B \rho_{AB} \\ \sigma_A^2 - \sigma_A \sigma_B \rho_{AB} \end{pmatrix} \quad (5.17)$$

while the variance of the potrfolio takes the minimum value

$$\sigma_{p(\text{VM})}^2 = (\mathbf{e}^T Z^{-1} \mathbf{e})^{-1} = \frac{\sigma_A^2 \sigma_B^2 (1 - \rho_{AB}^2)}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{AB}}. \quad (5.18)$$

This happens when the portfolio's expected return is

$$r_{p(\text{MV})} = \frac{\mathbf{r}^T Z^{-1} \mathbf{e}}{\mathbf{e}^T Z^{-1} \mathbf{e}} = \frac{(\sigma_B^2 - \sigma_A \sigma_B \rho_{AB})r_A + (\sigma_A^2 - \sigma_A \sigma_B \rho_{AB})r_b}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{AB}}, \quad (5.19)$$

where we denote in vector notation  $\mathbf{r} = (r_A \ r_B)^T$ . The efficient frontier and shape for different correlations  $\rho_{AB}$  are shown in Fig. (8, a).

The  $(r_p, \sigma_p^2)$  framework derived here does not properly describe buy-and-hold protfolios, where the investor fixed the number of asset units at the initial time of investment without rebalancing afterwards. In the following subsection we briefly discuss this case, and we will show how generally the variance of the portfolio does not generally take a quadratic form with the portfolio's average return.

### 5.2.2 Comment on buy-and-hold portfolios

Consider an investor who does not rebalance the proportions  $x_\alpha$  in order to keep them constant over time, i.e. the investor buys  $n_\alpha$  pieces of stock  $\alpha$  at some initial price  $S_{\alpha,0}$  for  $\alpha = 1, \dots, p$  maintaining  $n_\alpha$  constant. Under this assumption, the portfolio's value at any time  $t$  is  $\Pi_t = \sum_{\alpha=1}^p n_\alpha S_{\alpha,t}$ , but now the investment proportions  $x_{\alpha,t} = n_\alpha S_{\alpha,t} / \Pi_t$  are also random numbers. The exptec return of the portfolio after a time  $T$  can be written as

$$\mathbb{E}[R_T] \equiv \mathbb{E} \left[ \frac{\Pi_T}{\Pi_0} \right] - 1 = \mathbb{E} \left[ \sum_{\alpha=1}^p \frac{n_\alpha}{\sum_{\beta=1}^p n_\beta S_{\beta,0}} S_{\alpha,T} \right] - 1 \equiv \sum_{\alpha=1}^p x_{\alpha,0} \mathbb{E}[R_{\alpha,T}], \quad (5.20)$$

which, just as was the case under the MPT, remains linear with the expected returns of each individual asset,  $R_{\alpha,T} = \frac{S_{\alpha,T}}{S_{\alpha,0}} - 1$ . The form of the variance stays as stated in Eq. (5.4), however and unlike in the rebalaced case, the variance of a buy-and-hold portfolio cannot be appropriately written solely in terms of the covariance matrix  $\rho_{\alpha\beta}$  of asset returns. This is related to the fact that indeed now the return of the portfolio is now a linear combination of the shape  $R_T = \sum_{\alpha=1}^p x_{\alpha,0} R_{\alpha,T}$  where the investment proportions are fixed and deterministic. When asset

prices follow GBM the individual returns are log-normally distributed, hence the portfolio's return  $R_T$  becomes the sum of  $p$  log-normally distributed random variables, which makes finding explicit analytical expressions for the variance an impossible task. As a consequence, the efficient frontier of a buy-and-hold portfolio is not generally a parabola in  $(r_p, \sigma_p^2)$  space, in contrast with the predictions of MPT, i.e. the Markowitz efficient frontier – which only holds under rebalancing and in the limit of infinitesimal time horizons. Recent research shows how buy-and-hold portfolios are outperformed by rebalanced portfolios on average [1]. While little can be said about the shape of the efficient frontier in buy-and-hold portfolios, the fact that the expected value remains the same has some implications. For instance, for a two-asset portfolio where assets  $A$  and  $B$  undergo through correlated brownian motion,

$$dS_{\alpha,t} = \mu_{\alpha} S_{\alpha,t} dt + \sigma_{\alpha} S_{\alpha,t} dW_{\alpha,t}, \quad (5.21)$$

with  $\alpha = A, B$  where now  $\mathbb{E}[dW_{\alpha,t}] = 0$ ,  $\mathbb{E}[dW_{\alpha,t}^2] = dt$  and  $\mathbb{E}[dW_{A,t} dW_{B,t}] = \rho_{AB} dt$ , the portfolio  $\Pi_t = n_A S_{A,t} + n_B S_{B,t}$  goes through specific points in the  $(r_p, \sigma_p^2)$  space in the cases in which the portfolio consists of a unique asset, i.e. when  $x_A = 1$  or when  $x_A = 0$ . Indeed, the specific mean value and variance of each asset's individual return  $R_{\alpha,T} = \frac{S_{\alpha,T}}{S_{\alpha,0}} - 1$  have

$$\mathbb{E}[R_{\alpha,T}] = e^{\mu_{\alpha} T} - 1, \quad \mathbb{V}[R_{\alpha,T}] = (e^{\sigma_{\alpha}^2 T} - 1) e^{2\mu_{\alpha} T}, \quad (5.22)$$

so that when  $x_A = 1$  ( $x_A = 0$ ) the portfolio's return is  $\Pi_t = n_A S_{A,t}$  ( $\Pi_t = n_B S_{B,t}$ ), and thus the return  $R_T = R_{A,T}$  ( $R_T = R_{B,T}$ ), meaning

$$\mathbb{E}[R_T] = e^{\mu_{A(B)} T} - 1, \quad \mathbb{V}[R_T] = (e^{\sigma_{A(B)}^2 T} - 1) e^{2\mu_{A(B)} T}. \quad (5.23)$$

As a consequence, a two-asset portfolio always should in principle go through the points  $(r_A, \tilde{\sigma}_A^2)$  and  $(r_B, \tilde{\sigma}_B^2)$  in  $(r_p, \sigma_p^2)$ , where  $r_{\alpha} = e^{\mu_{\alpha} T} - 1$  and  $\tilde{\sigma}_{\alpha} = (e^{\sigma_{\alpha}^2 T} - 1) e^{2\mu_{\alpha} T}$  in  $(r_p, \sigma_p^2)$  space.

### 5.2.3 Capital asset pricing model (CAPM), capital market line (CML) & security market line (SML)

The capital asset pricing model (CAPM), developed by Sharpe, Lintner and Mossin in the 60s, represents the birth of modern financial economics and asset pricing. Its success comes from the model's ability to measure risk, and provides a good theoretical relation between expected return and risk [AG: Fama]. Both Sharpe and Markowitz received the 1990 Nobel Prize in Economics for their contributions to portfolio theory and asset pricing.

The CAPM is directly built on Markowitz's mean-variance  $(r_p, \sigma_p^2)$  portfolio theory and under somewhat ideal assumptions, such as assuming that investors take only into consideration mean and variance of return when investing, the assumption that every investor optimizes and holds diversified portfolios. The risk under these circumstances is always measured with respect to the general's market behaviour. It is also assumed that all the investors possess homogeneous expectations. As a consequence, assets which behave like the market offer high expected returns, while assets that behave independent to the general market earn the risk-free rate.

The CAPM pricing equation can be directly derived from the Markowitz  $(r_p, \sigma_p^2)$  theory, but this is not the goal here<sup>4</sup>. Consider a market with two kinds of objects, risky assets and a risk-less asset which grows at the risk-free rate  $r_f$ . Any investor, thus, may hold a combination of the risk-free asset and risky assets from the market. Let  $R_M$  be the return of the "market

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<sup>4</sup>Maybe in the near future.

*potfolio*”, that is, the set of all the risky assets in the market, and  $R_\alpha$  the return of a single asset in the market. Then [AG: cite Sharpe, Lintner, Mossin],

$$\mathbb{E}[R_\alpha] = r_f + \beta_\alpha(\mathbb{E}[R_M] - r_f), \quad \beta_\alpha = \frac{\text{Cov}(R_\alpha, R_M)}{\mathbb{V}[R_M]}, \quad (5.24)$$

where again  $\mathbb{V}[R_M]$  is simply the variance of  $R_M$ . A direct consequence of Eq. (5.24) is that, if the risky asset  $\alpha$  moves independent to the market, i.e.  $\text{Cov}(R_\alpha, R_M) = 0$ , its expected return is nothing but the risk free rate. Contrarily, if the risky asset  $\alpha$  behaves like the market, i.e.  $\text{Cov}(R_\alpha, R_M) \approx \mathbb{V}[R_M]$  or  $\beta_{\alpha 1} \approx 1$ , then its expected return is nothing but the expected return of the market. Note how in principle  $\beta_\alpha$  may be greater than 1, and also negative. The subtraction of the risk-free rate at both sides of Eq. (5.24) results into what’s know as the *security market line* (SML), i.e.  $\mathbb{E}[R_\alpha] - r_f = \beta_\alpha(\mathbb{E}[R_M] - r_f)$ . Now, consider a portfolio that consists of both the market potfolio and bonds growing at the risk-free rate, so that the total expected return is, using the mean-variance framework  $\mathbb{E}[R_p] = x_B r_f + x_M \mathbb{E}[R_M]$ , where  $x_B$  and  $x_M$  are again the proportion investments on the bonds and on the market portfolio, and hence  $x_B + x_M = 1$ . The variance, becomes, in this case  $\mathbb{V}[R_p] = x_M^2 \sigma_M^2$ , where  $\sigma_M^2 \equiv \mathbb{V}[R_M]$ . As a consequence one has,  $\sigma_p = x_M \sigma_M$  (the proportions are always positive), meaning  $\mathbb{E}[R_p] = (1 - \frac{\sigma_p}{\sigma_M})r_f + \frac{\sigma_p}{\sigma_M} \mathbb{E}[R_M]$ , or

$$\mathbb{E}[R_p] = r_f + \frac{\sigma_p}{\sigma_M}(\mathbb{E}[R_M] - r_f), \quad (5.25)$$

which is know as the *capital market line* (CLM) and is the efficient frontier of a portfolio consisting on bonds growing at the risk-free rate and the market portfolio. Eq. (5.25) is identical to Eq. (5.24) when  $\text{Cov}(R_p, R_M) = x_M \mathbb{V}[R_M]$ .

The main idea provided by CAPM is that the expected returns of an individual asset are determined and proportional to their contribution to the risk of the market portfolio, and not their standalone risk.

### 5.3 Simulation

#### 5.3.1 Rebalancing a portfolio

Rebalancing a portfolio is something that can be done manually, by selling and buying assets accordingly, in order to leave the proportions  $x_\alpha, t$  constant in time, considering the of a given asset  $\alpha$  at a previos time step. It is straight forward to see that, for a self-financing portfolio for which  $\sum_{\alpha=1}^p dn_{\alpha,t} S_{\alpha,t} = 0$ , for which the proportions  $x_{\alpha,t} \equiv x_\alpha$  are rebalanced in order to remain constant in time, then

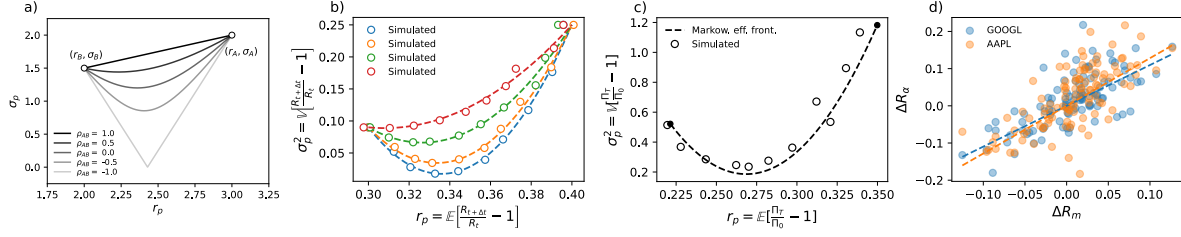
$$\frac{d\Pi_t}{\Pi_t} = \sum_{\alpha=1}^p x_\alpha \frac{dS_{\alpha,t}}{S_{\alpha,t}}. \quad (5.26)$$

Assuming the assets follow GBM with drift  $\mu_\alpha$ , percentage volatility  $\sigma_\alpha^2$  and correlation matrix  $\mathbb{E}[dW_{\alpha,t} dW_{\beta,t}] = \rho_{\alpha\beta} dt$  as in Eq. (5.21), it is striaght forward to show that  $\Pi_t$  also follows GBM,

$$d\Pi_t = \left( \sum_{\alpha=1}^p x_\alpha x_\alpha \mu_\alpha \right) \Pi_t dt + \Pi_t \sum_{\alpha=1}^p x_\alpha \sigma_\alpha dW_{\alpha,t} \quad (5.27)$$

since the sum of correlation Wiener processes can be re-written as

$$\sigma_p W_{p,t} = \sum_{\alpha=1}^p x_\alpha W_{\alpha,t} \xLeftrightarrow{\text{only if}} \sigma_p^2 = \sum_{\alpha,\beta=1}^p x_\alpha x_\beta \sigma_\alpha \sigma_\beta \rho_{\alpha\beta} \quad (5.28)$$



**Figure 8: Markowitz efficient frontier and example CAPM.** a) Sketch of the Markowitz efficient frontier for a two-asset portfolio with individual returns  $r_A = 3, r_B = 2$  and volatilities  $\sigma_A = 2, \sigma_B = 1.5$  for different values of the correlation  $\rho_{AB}$ . b) Simulated expected local returns generating mean and variance of a two-asset portfolio with drifts  $\mu_A = 0.3, \mu_B = 0.4$  and volatilities  $\sigma_A = 0.3, \sigma_B = 0.5$  and different values of correlation  $\rho_{AB} = -0.75$  (blue),  $-0.5$  (orange),  $0$  (green) and  $0.5$  (red). The simulations are done generating a GBM for the portfolio following Eq. (5.29). c) *Idem*, but now integrating the individual assets following GBM and computing the average return and variance of a buy-and-hold portfolio consisting of  $n_A + n_B = 100$  initial units of assets and initial price  $S_{A,0} = S_{B,0} = 1$ , for different  $n_A = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$ . d) Test of the CAPM via a scatter plot of monthly excess returns of two individual assets, GOOGL and AAPL, versus market returns using the S&P 500 index. The risk-free rate is obtained from U.S. Treasury bills.

where  $W_{p,t}$  is a standard Wiener process, i.e.  $\mathbb{E}[dW_{p,t}] = 0$ ,  $\mathbb{E}[dW_{p,t}^2] = dt$ , when the variance is properly mapped. When this is the case, then,

$$d\Pi_t = \mu_p \Pi_t dt + \sigma_p \Pi_t dW_{p,t}, \quad \text{with } \mu_p = \sum_{\alpha=1}^p x_\alpha \mu_\alpha, \quad \sigma_p^2 = \sum_{\alpha,\beta=1}^p x_\alpha x_\beta \sigma_\alpha \sigma_\beta \rho_{\alpha\beta}. \quad (5.29)$$

Since, in this case, the portfolio evolves following GBM, everything discussed in Chapter 1 holds. The simulation then is performed by generating GBM trajectories specifying the initial proportions  $x_\alpha$  and the initial value of the portfolio value  $\Pi_0 = \sum_{\alpha=1}^p n_{\alpha,0} S_{\alpha,0}$ . While the underlying dynamics of the individual assets are absorbed into the dynamics of the portfolio, the individual assets and their corresponding initial proportions have direct impact in the performance of the portfolio, since the portfolio's drift and volatility depend on both the individual drifts  $\mu_\alpha$ , the individual percentage volatilities  $\sigma_\alpha$  and on each asset's individual initial proportion. The returns of the portfolio, in this case, are log-normally distributed. Note how Markowitz's efficient frontier holds for small time horizons, and thus only locally in time.

### 5.3.2 Buy-&-hold portfolios

The dynamics of the portfolio in this case does not follow GBM, and this is because in the case of buy-and-hold portfolio the proportions  $x_{\alpha,t}$  are not constant in time; the investor buys  $n_\alpha$  of assets with price  $S_{\alpha,t}$  at initial time  $t = 0$  and holds the investment. Since now the proportions do depend on time, the trick used in the previous section to write the dynamics of the portfolio cannot be done, and so the dynamics become a little bit more complex. As a result, the log-returns of the portfolio will not be normally distributed.

The log-normality of the returns is only respected exceptionally, for instance, when looking at very short time horizons of the performance of the portfolio. Buy-and-hold portfolios are thus simulated by generating individual correlated GBM trajectories of each asset by fixing the units

of asset bought initially  $n_\alpha$ , so the evolution of the portfolio is provided by  $\Pi_t = \sum_{\alpha=1}^p n_\alpha S_{\alpha,t}$ . The implementation of this case can be found at this chapter's pill. **[AG: Comment figure]**.

### 5.3.3 Examples of CAPM with real data

We download available ticker data from `GOOGL`, `AAPL`, the market ticker for S&P 500 and track bonds through the available data on american T-bills on YahooFinance, in Python. From the closing price at the last day of each month, we compute monthly returns and compare the excess return of the market ticker provided by S&P 500 to the individual ones of `GOOGL` and `AAPL` in order to test Eq. (5.24) by simply computing the covariance between individual asset returns and market returns, as well as the market's total variance, from years 2015 to 2023.

## References

- [1] P. A. Forsyth, “A buy and hold portfolio loses diversification,” 2024.