Numerical Calculation of the Base Inertial Parameters of Robots

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This article presents a new approach to the problem of determining the minimum set of inertial parameters of robots. The calculation is based on numerical QR and SVD factorizations and on the scaling procedure of matrices. It proceeds in three steps:

- eliminate standard inertial parameters which have no effect on the dynamic model,
- determine the number of base parameters,

and

- determine a set of base parameters by regrouping some standard parameters to some others in linear relations.

Different models, linear in the inertial parameters are used: a complete dynamic model, a simplified dynamic model, and an energy model. The method is general, it can be applied to open loop, or graph-structured robots. The algorithms are easy to implement. An application for the PUMA 560 robot is given.

この論文では、ロボットの慣性パラメータの最小セットを定義するときの問題に対する新しい試みについて説明している。 この計算は、計算によるQR及びSVD factorizationとマトリックスの評価手順を基本としている。 この計算は、次の３つのステップを実行する。

- 運動モデルに影響を与えない標準慣性パラメータの消去。
- ベース・パラメータの定義。
- 正比例関係にある他のいくつかのパラメータに対するある標準パラメータの再グループ化による、ベース・パラメータのセットの定義。

完全な運動モデル、簡素化された運動モデル、エネルギー・モデルなど、慣性パラメータについて直線性を持つ異なったモデルが使われている。 一般にこの方法は、オープン・ループもしくはグラフ構造のロボットに応用されている。 このアルゴリズムの導入は容易である。 PUMA 560ロボットでの応用例が示されている。

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INTRODUCTION

An exact dynamic model of the robot is required to control or simulate its motion. The model is characterized by 11 inertial parameters per link, which are called the standard inertial parameters.

Because of the redundancy of this representation, there are infinite sets of standard parameters which satisfy the dynamic model. In order to reduce the computational cost of the dynamic model and to facilitate the identification process, a minimum set of inertial parameters, which are called also base parameters, must be used to determine the dynamic model.[1-16] Dynamic model, using Newton-Euler or Lagrange, or the energy formulation model, linear in the inertial parameters, can be used to study this problem.

From a linear algebra point of view, this problem appears to be a rank deficiency problem which we call structural because it depends only on the symbolic expressions of the dynamic model and not on the trajectories used in the analysis. The study is carried out in three steps:

- eliminate the standard parameters which have no effect on the model;
- find the rank of a linear system, this gives the number of base parameters; and
- choose the base parameters from the standard ones by eliminating some of the ones which are regrouped to the others in linear relations and, in this step, the regrouping relations will be determined.

Previous Work

Several authors have studied the problem using two principal approaches: Symbolic and Numerical.

Symbolic Approach

- In refs. 1–5, a case-by-case method using the symbolic expression of the dynamic model have been presented.
- In refs. 6–10, using the energy formulation, we have developed a general and direct method to determine most of the set of minimum inertial parameters of serial- or tree-structured robots.
- At the same time, similar results concerning the special case of robots whose successive axes are perpendicular or parallel have been given by Mayeda et al.[12,13]

Numerical Approach

- Atkenson et al.[14] use ridge regression and singular value decomposition, SVD, to solve the rank deficiency problem. The dynamic model using Newton-Euler formulation is used.
- Sheu and Walker[15,16] use SVD and the energy model. A minimum norm solution is obtained and a singular value analysis is proposed to obtain the

identifiable parameter space.[16] We think this approach is limited because results depend on the trajectory which is used in the analysis.

These two methods did not explicitly give the minimum set of inertial parameters, nor the linear relations which define them. They cannot be applied to simplify the dynamic model.

In this article we propose to use the QR decomposition, which serves equally well as the SVD, and at less cost,[18] and the SVD itself to solve the rank-deficiency problem.[11] We find the number of base parameters and we give a method to define them. Then, we give an algorithm to compute their numerical values from those of the standard parameters. The method has been applied to three linear models, the dynamic model, a simplified dynamic model, and an energy model.

An example treating the PUMA 560 robot type and an example of closed-loop robot are given. The results are the same as those obtained by our symbolic approach.

DYNAMIC MODEL

Standard Inertial Parameters

The system considered is an open-loop structure, simple or tree structure, mechanism. The description of the system will be carried out by using the modified Denavit and Hartenberg notation.[19,20] The system is composed of n joints and $n + 1$ links, link 0 is the base while link n is the end-effector. A coordinate frame j is assigned fixed with respect to link j. The standard inertial parameters are composed of the following elements:

- $\mathbf{J}^j$ the inertia matrix of link j about the origin of frame j, referred to frame j;
- $\mathbf{MS}^j$ the first moments of link j about the origin of frame j, referred to frame j;
- M_j the mass of link j; and
- Ia_j the rotor actuator inertia referred to joint side.

Let:

$$\mathbf{J}^j = \begin{bmatrix} XX_j & XY_j & XZ_j \\ XY_j & YY_j & YZ_j \\ XZ_j & YZ_j & ZZ_j \end{bmatrix} \tag{1}$$

$$\mathbf{MS}^j = [MX_j \quad MY_j \quad MZ_j]^T \tag{2}$$

The 11 standard inertial parameters of link j are given by the vector $\mathbf{X}^j$, which is denoted as:

$$\mathbf{X}^j = [XX_j \quad XY_j \quad XZ_j \quad YY_j \quad YZ_j \quad ZZ_j \quad MX_j \quad MY_j \quad MZ_j \quad M_j \quad Ia_j]^T \quad (3)$$

The inertial parameters of the robot will be represented by the vector $\mathbf{X}$ containing the inertial parameters of all the links. Thus, the dimension of $\mathbf{X}$ is equal to $11n$.

$$\mathbf{X} = [\mathbf{X}^{1^T} \quad \mathbf{X}^{2^T} \ldots \mathbf{X}^{n^T}]^T \quad (4)$$

Dynamic Model Formulation

From the Lagrangian or Newton-Euler equations, we obtain a dynamic model linear in the inertial parameters[1–3]:

$$\Gamma = \mathbf{d}\,\mathbf{X} = \sum_{i=1}^{11n} \mathbf{d}_{:,i} X_i \quad (5)$$

where:

$\mathbf{d}$ is a $n \times 11n$ matrix function of $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ and the constant geometric parameters
$\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ are the $n \times 1$ vectors of joint positions, velocities, and accelerations respectively
$\mathbf{d}_{:,i}$ is the ith column of $\mathbf{d}$
Γ is the $n \times 1$ vector of joint torques.

Parameters Not Affecting the Dynamic Model

If a column $\mathbf{d}_{:,i}$ is equal to zero for any value of $(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$, then the corresponding parameter X_i does not affect the model.

These columns can be easily detected and eliminated by calculating $\mathbf{d}$ using random values for $(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$. In the following, we consider a matrix $\mathbf{d}$ which is reduced to its c no-zero columns, with $c \leq 11n$.

Regrouping the Inertial Parameters

The study of the minimum inertial parameters, also called base parameters, can be carried out by studying the dependence of the symbolic expressions of the elements of the columns $\mathbf{d}_{:,i}$.[1–3]

Numerically, this is equivalent to studying the space spanned by the columns of a $r \times c$ matrix $\mathbf{W}$, with r independent rows and $r \geq c$,[10,11] which is calculated using a sequence of e random matrices $(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})_{(i)}$, $i = 1, \ldots, e$, as follows:

$$\mathbf{W} = \begin{bmatrix} \mathbf{d}(1) \\ \cdots \\ \mathbf{d}(e) \end{bmatrix}, \; r = n \times e \geq c \tag{6}$$

$$\mathbf{d}(i) = \mathbf{d}((\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})_{(i)}) \tag{7}$$

It is known[1-3] that it is equivalent to study the base parameters from the matrix $\mathbf{d}$ or from a simpler matrix $\mathbf{d}^0$ obtained from $\mathbf{d}$ by putting $\dot{\mathbf{q}} = \mathbf{0}_{n\times 1}$, as follows:

$$\mathbf{d}^0 = \mathbf{d}(\mathbf{q}, \mathbf{0}_{n\times 1}, \ddot{\mathbf{q}}). \tag{8}$$

$\mathbf{0}_{n\times m}$ is a $n \times m$ matrix of zeros.

Numerically, an alternative to eq. (6) is to use the following matrix, $\mathbf{W}$:

$$\mathbf{W} = \begin{bmatrix} \mathbf{d}^0(1) \\ \cdots \\ \mathbf{d}^0(e) \end{bmatrix} \tag{9}$$

Details about the derivation of $\mathbf{d}$ can be found in refs. 1, 2, 3, and 10.

Energy Formulation of the Dynamic Model

It is known that the total energy, H, is linear in the inertial parameters[6-10]:

$$H(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{hX} = \sum_{i=1}^{11n} h_i X_i \tag{10}$$

$H = E + U$
$E(\mathbf{q},\dot{\mathbf{q}})$ is the kinetic energy, $U(\mathbf{q})$ is the potential energy.

$\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}})$ is a row vector composed of the $11n$ energy functions h_i which define the contribution of each standard inertial parameters X_i to the total energy H. Details about the derivation of $\mathbf{h}$ can be found in refs. 6–10. The energy theorem applied between times t_a and t_b gives the energy formulation of the dynamic model, called the energy model:

$$\int_{t_a}^{t_b} \Gamma^T \dot{\mathbf{q}} \, dt = [\mathbf{h}(\mathbf{q}(t_b), \dot{\mathbf{q}}(t_b)) - \mathbf{h}(\mathbf{q}(t_a), \dot{\mathbf{q}}(t_a))]\mathbf{X} \tag{11}$$

Parameters Not Affecting the Dynamic Model

If a function h_i is equal to a constant value for any value $(\mathbf{q}, \dot{\mathbf{q}})$, then the corresponding parameter X_i does not affect the energy model (11). These functions can be easily detected and eliminated by calculating $\mathbf{h}$ using random values for $(\mathbf{q}, \dot{\mathbf{q}})$. In the following we consider a matrix $\mathbf{h}$ which is reduced to its c no-constant functions, with $c \leq 11n$.

Regrouping the Inertial Parameters

The search of the minimum inertial parameters can be carried out by studying the dependence of the functions h_i from their symbolic expressions.[6–10]

Numerically, this is equivalent to studying the space spanned by the columns of a $r \times c$ matrix, $\mathbf{W}$, with r independent rows and $r \geq c$, which is calculated using a sequence of r random matrices $(\mathbf{q}, \dot{\mathbf{q}})_{(i)}$, $i = 1, \ldots, r$, as follows:

$$\mathbf{W} = \begin{bmatrix} h(1) \\ \cdots \\ \mathbf{h}(r) \end{bmatrix} \tag{12}$$

$$\mathbf{h}(i) = \mathbf{h}((\mathbf{q}, \dot{\mathbf{q}})_{(i)})$$

Theoretically, it is equivalent to study the space span by the columns of a matrix, $\mathbf{W}$, obtained from $\mathbf{d}$ or $\mathbf{d}^0$, or $\mathbf{h}$, but this does not imply the numerical equivalence because of different scalings of the three formulations of $\mathbf{W}$. The proprieties to study are:

- the rank b of $\mathbf{W}$ which is the dimension of the space and the number of base parameters;
- the choice of $(c\text{-}b)$ columns to be deleted and of b independent columns constituting a base of the space, and which define the base parameters;
- the determination of $(c\text{-}b)$ linear relations between the base columns and the deleted ones; and
- the determination of the values of the b base parameters from those of the standard ones.

We propose to use a QR and a SVD decomposition of $\mathbf{W}$ to solve these problems.

BASE PARAMETERS BY *QR* DECOMPOSITION

QR Decomposition

The $r \times c$ matrix, $\mathbf{W}$, can be decomposed into the following form using QR decomposition[17–19]:

$$\mathbf{Q}^T\mathbf{W} = \begin{bmatrix} \mathbf{R} \\ \\ \mathbf{0}_{(r-c)\times c} \end{bmatrix} \tag{13}$$

where:

$\mathbf{Q}$ is a $r \times r$ orthogonal matrix, that is:
$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_r$
$\mathbf{I}_r$ is the $r \times r$ identity matrix
$\mathbf{R}$ is a $c \times c$ upper triangular matrix.

Rank Deficiency: QR with Column Pivoting (QRΠ)

If $\mathbf{W}$ is rank deficient ($b < c$), then $\mathbf{W}$ does not have a unique QR decomposition. However, there is a permutation matrix Π (i.e., an identity matrix with its columns permuted) such that $\mathbf{W}\Pi$ has a unique decomposition:

$$\mathbf{Q}^T(\mathbf{W}\Pi) = \begin{bmatrix} \mathbf{T} \\ \\ \mathbf{0}_{(r-c)\times c} \end{bmatrix} = \begin{bmatrix} \mathbf{T1} & \mathbf{T2} \\ \\ \mathbf{0}_{(r-b)\times b} & \mathbf{0}_{(r-b)\times(c-b)} \end{bmatrix} \tag{14}$$

where:

$\mathbf{Q}$ is orthogonal
$\mathbf{T2}$ is a $b \times (c\text{-}b)$ matrix
$\mathbf{T1}$ is a $b \times b$ upper triangular and regular matrix so that the diagonal elements T_{ii} of $\mathbf{T1}$ are in nonincreasing order.

$$|T_{11}| > |T_{22}| > \ldots > |T_{bb}| > 0$$

The numerical rank is defined with a tolerance $\tau \neq 0$, because of round-off errors.[22]

$$\text{Rank}(\mathbf{W}) = \text{b} \Leftrightarrow |T_{ii}| \leq \tau, \text{ for } i = b + 1, \ldots, c$$

τ can be chosen as $\tau = \text{r}.|T_{11}|.\varepsilon$, where ε is the machine precision.[18]
When the values $|T_{ii}|$ are clustered in two groups, one near $|T_{11}|$ and the other less than τ, there is no problem in determining the rank of $\mathbf{W}$. This finding has always been the case in our experiments. QR, with pivoting, gives the rank of $\mathbf{W}$ which is equal to the number of base parameters.

Choice of the Columns to Be Deleted

The choice of $(c\text{-}b)$ columns to be deleted is not unique. As we did in our previous symbolic approach,[1-3,6-9] we choose to eliminate columns $\mathbf{W}_{:,i}$ with

the larger subscript such that the parameters of link j will be regrouped on those of links j-1, . . . , 1. The solution is very easy to find from a QR decomposition of **W** without pivoting, [eq. (13)]. The (c-b) diagonal elements $|R_{ii}| \leq \tau$ give the subscripts i of the columns $\mathbf{W}_{:,i}$ to be deleted, which are the subscripts of the standard parameters to be regrouped. We deduce the subscripts of the b base parameters.

It is notable that this method at the same time gives the number and the choice of base parameters, taking the value $\tau = r.|R_{ii}|_{\max}.\varepsilon$, where $|R_{ii}|_{\max}$ is the largest value of $|R_{ii}|$, $i = 1, \ldots c$. However, the QRΠ method is generally numerically more efficient to calculate the rank of **W**.

Linear Relations Between the Columns of W

From the previous step we deduce a permutation matrix **P** such that:

$$\mathbf{P}^T\mathbf{X} = \begin{bmatrix} \mathbf{X1} \\ \\ \mathbf{X2} \end{bmatrix} \tag{15}$$

X2 is a (c-b) vector of standard parameters to be regrouped on the b standard parameters of the vector **X1**.

Permuting the columns of **W** we obtain:

$$(\mathbf{W}\ \mathbf{P}) = [\mathbf{W1}\ \mathbf{W2}] \tag{16}$$

W1 is a set of b independent columns,
W2 is a set of (c-b) columns to be deleted.
A QR decomposition of (**WP**) gives:

$$[\mathbf{W1}\ \mathbf{W2}] = [\mathbf{Q1}\ \mathbf{Q2}] \begin{bmatrix} \mathbf{R1} & \mathbf{R2} \\ \\ \mathbf{0}_{(r-b)\times b} & \mathbf{0}_{(r-b)\times(c-b)} \end{bmatrix} = [\mathbf{Q1.R1}\ \mathbf{Q1.R2}]$$

R1 is a b × b regular matrix. Then it comes:

$$\mathbf{W1} = \mathbf{Q1.R1} \text{ and } \mathbf{W2} = \mathbf{Q1.R2}$$

We deduce the relation:

$$\mathbf{W2} = \mathbf{W1}\beta \tag{17}$$

with:

$$\beta = \mathbf{R1}^{-1}\ \mathbf{R2}$$

Relation (17) expresses the (c-b) columns of **W2** as linear combinations of the b independent columns of **W1**.

Explicit Relations for the Base Parameters

Let **XR** be any solution which assumes the invariancy of the model:

$$\mathbf{W\,X} = \mathbf{W\,P\,P}^T\mathbf{X} = \mathbf{W\,P\,P}^T\mathbf{XR}$$

with:

$$\mathbf{P}^T\mathbf{XR} = \begin{bmatrix}\mathbf{XR1}\\ \\ \mathbf{XR2}\end{bmatrix},\ \mathbf{P}^T\mathbf{X} = \begin{bmatrix}\mathbf{X1}\\ \\ \mathbf{X2}\end{bmatrix}$$

Rewriting the invariancy:

$$[\mathbf{W1\ W2}]\begin{bmatrix}\mathbf{X1}\\ \\ \mathbf{X2}\end{bmatrix} = [\mathbf{W1\ W2}]\begin{bmatrix}\mathbf{XR1}\\ \\ \mathbf{XR2}\end{bmatrix} \tag{18}$$

we obtain:

$$\mathbf{XR1} = \mathbf{X1} + \beta(\mathbf{X2} - \mathbf{XR2}) \tag{19}$$

There are an infinite number of values of **XR1,** depending on arbitrary values in **XR2,** which satisfy eq. (18). A basis solution, named **XB,** corresponds to $\mathbf{XB2} = \mathbf{0}_{(c\text{-}b)\times 1}$. The relation which gives the numerical values of the b base parameters **XB1** from those of the c standard parameters is given by:

$$\mathbf{P}^T\mathbf{XB} = \begin{bmatrix}\mathbf{XB1}\\ \\ \mathbf{XB2}\end{bmatrix} = \begin{bmatrix}\mathbf{X1} + \beta\,\mathbf{X2}\\ \\ \mathbf{0}_{(c\text{-}b)\times 1}\end{bmatrix} \tag{20}$$

The $(c\text{-}b)$ columns of **W2** are eliminated from the model, which takes a simplified expression:

$$\mathbf{W\,X} = \mathbf{W1\,XB1}$$

The dynamic model [eq. (5)] and the energy model [eq. (10)], take the simplified expressions:

$$\Gamma = \mathbf{d1\,XB1}$$

$$H = \mathbf{h1\,XB1}$$

where:

$$(\mathbf{d\ P}) = [\mathbf{d1\ d2}],\ (\mathbf{h\ P}) = [\mathbf{h1\ h2}]$$

This choice allows us to reduce the computational cost of the dynamic model using a customized method.[1-3]

BASE PARAMETERS USING *SVD* DECOMPOSITION

Singular Value Decomposition SVD

Another efficient method to solve rank deficiency is to use a SVD decomposition of **W**. It is known that a $r \times c$ matrix, **W**, can be factorized by the following expression.[21-23]

$$\mathbf{U}^T\mathbf{W} = \mathbf{W}\mathbf{V}^T \tag{21}$$

$$\mathbf{S} = \begin{bmatrix} \Sigma \\ \mathbf{0}_{(r-c)\times c} \end{bmatrix}$$

where:

U is a $r \times r$ orthogonal matrix
V is a $c \times c$ orthogonal matrix
Σ is a $c \times c$ diagonal matrix whose elements σ_i are in nonincreasing order: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_c \geq 0$. The elements σ_i are the singular values of **W**.

Explicit Relations for the Base Parameters

The numerical rank of **W** can be deduced from the singular values of **W**. If σ_{b+1} is less than a tolerance τ, then $\sigma_{b+1}, \ldots, \sigma_c$ are considered to be zero, and rank(**W**) = b, with respect to τ which can be chosen as $\tau = r.\varepsilon.\sigma_1$.[18]

When **W** is rank deficient, the SVD takes the form:

$$\mathbf{W}[\mathbf{V1}\ \mathbf{V2}] = \mathbf{U}\begin{bmatrix} \Sigma & \mathbf{0}_{b\times(c-b)} \\ \mathbf{0}_{(r-b)\times b} & \mathbf{0}_{(r-b)\times(c-b)} \end{bmatrix} \tag{22}$$

where:

V1 is a $c \times b$ matrix,
V2 is a $c \times (c\text{-}b)$ matrix,
Σ is a $b \times b$ diagonal matrix,
b = rank(**W**), $b < c$.
From eq. (22), it comes:

$$\mathbf{W}\ \mathbf{V2} = \mathbf{0}_{r\times(c-b)} \tag{23}$$

The columns of **V2** define the $(c\text{-}b)$ linear relations between the columns of **W**. A column $\mathbf{W}_{:,i}$ corresponding to a zero row $\mathbf{V2}_{i,:}$ is independent of all the other

columns. The problem is to choose $(c\text{-}b)$ dependent columns $\mathbf{W}_{:,k}$ of largest subscript to be deleted. From eq. (23) and the orthogonality of $\mathbf{V}$, we have:

$$\mathbf{W\,X} = \mathbf{W\,V\,V}^T\mathbf{X} + \mathbf{W\,V2\,X_a} \tag{24}$$

$$\mathbf{W\,V\,V}^T = \mathbf{W\,V1\,V1}^T$$

$$\mathbf{W\,X} = \mathbf{W}\,[\mathbf{V1\,V1}^T\mathbf{X} + \mathbf{V2\,X_a}]$$

We define:

$$\mathbf{XR} = \mathbf{V1\,V1}^T\mathbf{X} + \mathbf{V2\,X_a} = [\mathbf{V1\ V2}]\begin{bmatrix}\mathbf{V1}^T\mathbf{X}\\ \mathbf{X_a}\end{bmatrix}$$

$$\mathbf{XR} = \mathbf{V}\begin{bmatrix}\mathbf{V1}^T\mathbf{X}\\ \mathbf{X_a}\end{bmatrix} \tag{25}$$

where $\mathbf{X_a}$ is a $(c\text{-}b) \times 1$ arbitrary vector.
There are an infinite number of values $\mathbf{XR}$ that keep invariant the linear system:

$$\mathbf{W\,X} = \mathbf{W\,XR}$$

A common idea is to use the minimal two-norm solution $\mathbf{XM}$[14–16]. The two-norm is invariant with respect to $\mathbf{V}$ multiplication:

$$\|\mathbf{XR}\| = \|\mathbf{V1}^T\mathbf{X}\| + \|\mathbf{X_a}\|$$

The minimal two-norm vector [eq. (25)] is obtained with $\mathbf{X_a} = \mathbf{0}_{(c\text{-}b)\times 1}$:

$$\mathbf{XM} = \mathbf{V1\,V1}^T\mathbf{X}$$

This solution has c components and does not contribute to simplification of the dynamic model.
We are looking for a solution of the form eq. (20), given by the permutation matrix, $\mathbf{P}$ [eqs. (15) and (16)]. We define the permutation of the rows of $\mathbf{V2}$ as:

$$\mathbf{P}^T\mathbf{V2} = \begin{bmatrix}\mathbf{V21}\\ \mathbf{V22}\end{bmatrix} \tag{26}$$

where:

$\mathbf{V21}$ is a $b \times (c\text{-}b)$ matrix
$\mathbf{V22}$ is a $(c\text{-}b) \times (c\text{-}b)$ matrix.

From eqs. (16) and (23), we obtain:

$$\mathbf{W}\ \mathbf{P}\ \mathbf{P}^T\mathbf{V2} = \mathbf{0}_{r\times(c-b)} = \mathbf{W1}\ \mathbf{V21} + \mathbf{W2}\ \mathbf{V22}$$

The linear relation [eq. (23)] takes the explicit form:

$$\mathbf{W2} = \mathbf{W1}\ \beta$$

$$\beta = -\mathbf{V21}\ \mathbf{V22}^{-1} \tag{27}$$

P is not unique but must be chosen for **V22** to be regular. Starting from the last row of **V2** we extract the first regular $(c\text{-}b) \times (c\text{-}b)$ matrix, which gives the subscript of the columns $\mathbf{W}_{:,k}$ to be deleted, and defines the matrix **P.**
The numerical values of the base parameters are given by eq. (20):

$$\mathbf{XB1} = \mathbf{X1} + \beta\ \mathbf{X2} \tag{28}$$

APPLICATION

A Serial-Link Manipulator

A six-joint serial link manipulator, similar to the PUMA 560, is chosen (Fig. 1). The geometric parameters are given in Table I. There are 66 standard inertial parameters. The matrices $\mathbf{d}(q,\dot{\mathbf{q}},\ddot{\mathbf{q}})$, $d^0(\mathbf{q},\ddot{\mathbf{q}})$, and $\mathbf{h}(\mathbf{q},\dot{\mathbf{q}})$ can be calculated directly, numerically. We have preferred to make use of the symbolic expressions, automatically computed using the software package SYMORO.[24]

There are 11 functions h_i which are constant, and which correspond to 11 zero columns of **d** or $\mathbf{d}_0$. These columns and the corresponding parameters are eliminated because they do not affect the dynamic model. So, **W** has 55 columns ($c = 55$).

Scaling of **W**

The samples $(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})_{(i)}$, $i = 1, \ldots, e$ or $(\mathbf{q},\dot{\mathbf{q}})_{(i)}$, $i = 1, \ldots, r$, can be generated by random sequences. To study the rank deficiency we can choose $r = c = 55$. To avoid numerical difficulties during SVD or QR decomposition, we propose to look for a sequence of samples which scales **W**. This procedure

Table I. The geometric parameters of the serial link robot.

j	σ_j	α_j	d_j	θ_j	r_j
1	0	0	0	θ_1	0
2	0	−90	0	θ_2	0
3	0	0	$D3 = 0.5$ m	θ_3	$R3 = 0.2$ m
4	0	−90	$D4 = 0.02$ m	θ_4	$R4 = 0.6$ m
5	0	90	0	θ_5	0
6	0	−90	0	θ_6	0

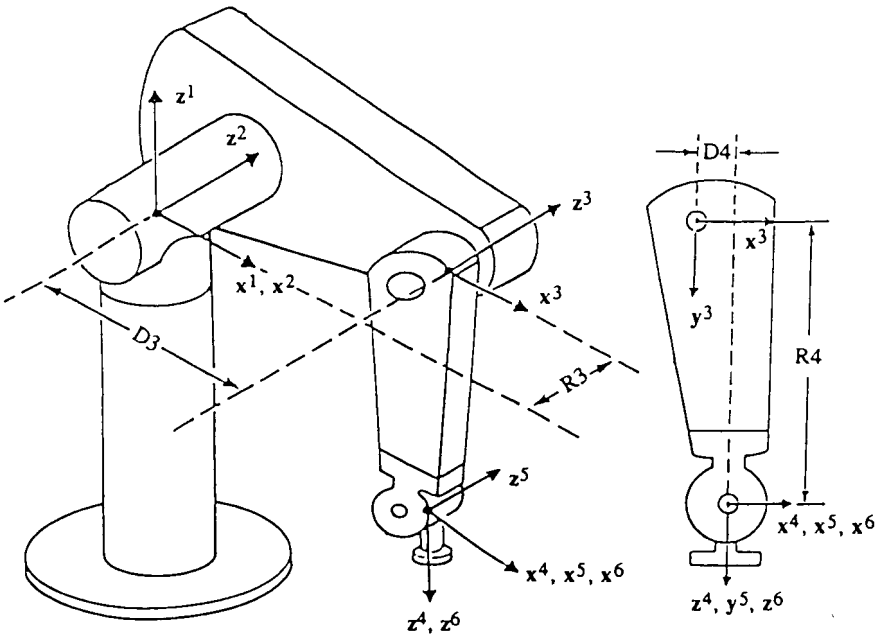

Figure 1. PUMA 560 Robot.

is generally not necessary using double precision, but can be useful in treating particular robots having a large number of parameters, such as graph-structured robots.

In our problem, the elements of **W** are computed to high accuracy with only the rounding errors. In this case, the error in an element of **W** is proportional to its size,[18] and the strategy reduces to scale **W** so that all of its elements are roughly equal. The criterion to minimize is:

$$J = \lambda_1 \|\mathbf{W}\|_F + \lambda_2 S$$

$$S = |W_{ij}|_{\max} / |W_{ij}|_{\min}$$

$|W_{ij}|_{\max}$ and $|W_{ij}|_{\min}$ are the maximum and minimum absolute values of the coefficients W_{ij} of **W**, and $|W_{ij}|_{\min} \neq 0$. $\|.\|_F$ is the Frobenius norm:

$$\|\mathbf{W}\|_F = \left(\sum_{i=1}^{r} \sum_{j=1}^{c} W_{ij}^2 \right)^{1/2}$$

λ_1 and λ_2 are two weighting scalars.

The nonlinear optimization problem is characterized by a large number of degrees of freedom n_d. Using the matrix **h** to calculate **W**, with $r = c = 55$, we have $n_d = (2 * 6\text{-}1) * 55 = 605$. Also, symbolic expressions of the derivative of J cannot be obtained.

Table II (Part 1). Determination of the number and choice of the b base parameters.

Number of base parameters (b)		Choice of b base parameters			
QRΠ T_{ii}	SVD σ_i	QR R_{ii}	SVD **V2 N**	i	X_i
6.8553D + 01	1.2082D + 02	−3.8203D + 00	1.4128D − 01	1	ZZ_1
5.9540D + 01	9.6715D + 01	**7.9726D − 17**	1.2633D + 00	**2**	$\mathbf{Ia_1}$
−5.6210D + 01	7.8129D + 01	−1.1072D + 00	−6.4517D − 01	3	XX_2
−5.3820D + 01	7.0393D + 01	−2.7443D + 00	−3.0548D − 16	4	$\mathbb{XY}_2$
−5.2934D + 01	5.9110D + 00	−1.5815D + 00	−1.7779D − 01	5	XZ_2
4.5777D + 01	5.5097D + 01	**−8.3251D − 17**	−5.2031D − 01	**6**	$\mathbf{YY_2}$
4.1811D + 01	4.9868D + 01	−2.3261D + 00	4.7093D − 16	7	$\mathbb{YZ}_2$
−4.0388D + 01	4.6651D + 01	1.2721D + 00	2.5507D − 02	8	ZZ_2
−3.8433D + 01	4.4702D + 01	−2.8195D + 01	2.4972D − 01	9	MX_2
3.6752D + 01	4.0420D + 01	−3.6526D + 01	−9.2725D − 18	10	$\mathbb{MY}_2$
3.5341D + 01	3.9754D + 01	**−8.1084D − 18**	9.9354D − 02	**11**	$\mathbf{Ia_2}$
−3.2487D + 01	3.7589D + 01	−1.2027D + 00	−6.8264D − 01	12	XX_3
3.1803D + 01	3.5828D + 01	1.7946D + 00	8.6334D − 03	13	XY_3
3.0019D + 01	3.2022D + 01	−4.9349D + 00	1.6515D − 16	14	$\mathbb{XZ}_3$
2.6775D + 01	2.8993D + 01	**−8.8418D − 17**	−3.5387D − 01	**15**	$\mathbf{YY_3}$
2.5016D + 01	2.6636D + 01	5.3286D + 00	−3.6194D − 16	16	$\mathbb{YZ}_3$
−2.2815D + 01	2.3792D + 01	−5.7767D + 00	−3.2732D − 01	17	ZZ_3
−1.9409D + 01	2.0918D + 01	2.2033D + 01	3.6423D − 02	18	MX_3
1.7306D + 01	1.9197D + 01	2.8448D + 01	−4.3167D − 01	19	MY_3
−1.5057D + 01	1.7571D + 01	**3.1513D − 17**	−2.5570D − 01	**20**	$\mathbf{MZ_3}$
−1.4006D + 01	1.4026D + 01	**6.7436D − 16**	1.3217D + 00	**21**	$\mathbf{M_3}$
−9.1768D + 00	9.9962D + 00	−6.3825D − 01	2.6388D − 15	22	$\mathbb{a}_3$
7.9791D + 00	9.5516D + 00	3.6147D + 00	−3.5180D − 01	23	XX_4
7.7318D + 00	8.1208D + 00	7.3713D + 00	2.9775D − 17	24	$\mathbb{XY}_4$
6.1533D + 00	5.9656D + 00	−1.4435D + 01	−9.7873D − 17	25	$\mathbb{XZ}_4$
5.1126D + 00	5.0137D + 00	**−2.4113D − 16**	−8.4556D − 01	**26**	$\mathbf{YY_4}$
4.4065D + 00	4.6084D + 00	9.9817D + 00	−1.1408D − 16	27	$\mathbb{YZ}_4$

Owing to these constraints, the numerical method developed by Powell,[25] and implemented in the subroutine VA04 of the Harwell[26] software package, is adequate for this problem. It is a gradient conjugate-type method which approaches the conjugate directions by an iterative procedure. Because of the large number of variables the algorithm fails for the 55 × 55 matrix **W** of the PUMA.

An alternative is to find r sequences of optimum points for the scaling of the $1 \times c$ row matrix **h.** In fact, we just want a solution which gives an acceptable scaling value of **h** (around 10). Starting the optimization from r different random sequences for the n_d variables, we obtain r acceptable scaled matrices **h,** which give the 55 rows of the 55 × 55 matrix **W,** with a value of S about 100. Using a random sequence, without scaling, we obtain a matrix **W** with a value of S about 10^8.

Results

Results are given using **h** to calculate a 55 × 55 no-scaled **W** matrix and the QR decomposition. Similar results were obtained with scaling, that is, for this

Table II (Part 2). Determination of the number and choice of the b base parameters.

Number of base parameters (b)		Choice of b base parameters			
QRΠ Diag. de **T**	SVD Diag. de **S**	QR Diag. de **R**	SVD **V2 N**	i	X_i
−3.9558D + 00	4.1533D + 00	1.4250D + 01	4.9376D − 01	28	ZZ_4
−3.0364D + 00	3.4905D + 00	1.8716D + 01	1.8375𝔻 − 17	29	$\mathbb{MX}_4$
2.9987D + 00	2.9391D + 00	−2.0361D + 01	8.1228D − 01	30	MY_4
−2.7431D + 00	2.8430D + 00	**2.5541D − 15**	1.5243D − 00	**31**	$\mathbf{MZ_4}$
−1.7903D + 00	2.5664D + 00	**1.0295D − 15**	−1.6854D + 00	**32**	$\mathbf{M_4}$
1.5204D + 00	2.0108D + 00	3.4103D + 00	1.1091𝔻 − 16	33	$\mathbb{a}_4$
1.4065D + 00	1.6119D + 00	−7.1551D + 00	−1.0276 + 00	34	XX_5
3187D + 00	1.2859D + 00	−1.0192D + 01	1.5193𝔻 − 18	35	$\mathbb{XY}_5$
−8.7418D − 01	1.1429D + 00	−1.5635D + 01	1.4820𝔻 − 17	36	$\mathbb{XZ}_5$
−7.7411D − 01	8.3973D − 01	**7.4052D − 16**	−4.9376D − 01	**37**	$\mathbf{YY_5}$
5.3603D − 01	6.8671D + 01	1.3316D + 17	9.0862𝔻 − 17	38	$\mathbb{YZ}_5$
−4.7190D − 01	4.1455D − 01	−1.3966D + 01	−5.3387D − 01	39	ZZ_5
−2.5204D − 01	2.8153D − 01	2.0472D + 01	9.1535𝔻 − 17	b = 40	$\mathbb{MX}_5$
−1.6199D − 15	1.3828D − 15	−1.6395D + 01	2.4349D − 01	41	MY_5
−9.0819D − 16	9.5098D − 16	**6.8609D − 16**	8.1228D − 01	**42**	$\mathbf{MZ_5}$
−7.6112D − 16	8.0914D − 16	**5.7533D − 16**	**8.2822D** − 02	**43**	$\mathbf{M_5}$
−5.7027D − 16	7.0702D − 16	3.9575D + 00	1.8610𝔻 − 16	44	$\mathbb{a}_5$
−4.5733D − 16	6.5005D − 16	−1.0858D + 01	5.3387D − 01	45	XX_6
−2.9198D − 16	5.6727D − 16	−2.0111D + 01	1.2192𝔻 − 17	46	$\mathbb{XY}_6$
−2.6984D − 16	5.0297D − 16	1.9148D + 01	1.6842𝔻 − 17	47	$\mathbb{XZ}_6$
−2.4353D − 16	4.2402D − 16	**−1.6120D − 15**	5.3387D − 01	**48**	$\mathbf{YY_6}$
2.2046D − 16	3.8045D − 16	−1.7955D + 01	−3.5353𝔻 − 17	49	$\mathbb{XZ}_6$
1.3862D − 16	3.0940D − 16	1.0806D + 01	2.6972𝔻 − 17	50	$\mathbb{ZZ}_6$
9.5979D − 17	2.5514D − 16	−1.2469D + 01	2.0261𝔻 − 18	51	$\mathbb{MX}_6$
5.1004D − 17	2.0519D − 16	1.2599D + 01	3.7307𝔻 − 17	52	$\mathbb{MY}_6$
2.4312D − 17	1.4772D − 16	**7.8398D − 16**	−2.4349D − 01	**53**	$\mathbf{MZ_6}$
1.0142D − 18	3.0844D − 17	**9.9760D − 17**	−2.1858D − 01	**54**	$\mathbf{M_6}$
0.0000D + 00	2.1179D − 18	5.4424D − 01	3.5993𝔻 − 17	55	$\mathbb{a}_6$

example, scaling is not necessary. However, scaling improves the ratio $|T_{40,40}|/|T_{41,41}|$ by a factor of 10, that makes easier the determination of the rank of **W**. Results were obtained with large facilities using the software Ctrl-C [21]. Table II gives the diagonal elements of the matrix **T** resulting from the QRΠ decomposition. It can be seen that the elements T_{ii} are clustered in two separate groups, with $\tau = 5.23\ 10^{-14}$. So, there is no doubt about the value of rank (**W**) which equals 40. We give the diagonal elements of the matrix **R** of the QR decomposition without pivoting. The subscript of the 15 elements R_{kk} which equal zero, ($|R_{kk}| \leq \tau$, bold case), give the subscript of the columns $\mathbf{W}_{:,k}$ to be deleted, and define the permutation matrix **P.**

Then, we proceed to a QR factorization of (**WP**) to calculate the matrix $\beta = \mathbf{R1}^{-1}\mathbf{R2}$. For each zero row $\beta_{i,:}$ corresponds an independent function h_i. A fast and necessary, but not sufficient, test can be done by looking for the zero elements of the (40 × 1) vector **K** with $\mathbf{K} = \beta\mathbf{N}$, where **N** is 15 × 1 vector whose elements are random values (Table III). If $K_i = 0$, we check that the corresponding row $\beta_{i,:} = \mathbf{0}_{1\times 15}$. We obtain 23 functions, h_i, which are independent of

Table III. Values of the 40 base parameters.

X_i	i	βN	P^TX X1	P^TXB XB1
ZZR_1	1	2.1458	0.5000	5.0186
XXR_2	3	−0.7925	1.0000	−2.0500
XY_2	*4*	*0.0000*	*0.7000*	*0.7000*
XZR_2	5	−0.3449	−0.3000	−1.0700
YZ_2	*7*	*0.0000*	*0.6500*	*0.6500*
ZZR_2	8	0.6250	0.5000	6.5500
MXR_2	9	0.7113	1.2000	4.3000
MY_2	*10*	*0.0000*	*0.6000*	*0.6000*
XXR_3	12	0.1707	0.1000	0.7634
XYR_3	13	−0.0157	0.7000	0.6872
XZ_3	*14*	*0.0000*	*0.5500*	*0.5500*
YZ_3	*16*	*0.0000*	*−0.6000*	*−0.6000*
ZZR_3	17	0.8036	0.2000	0.9646
MXR_3	18	0.0101	0.5000	0.5280
MYR_3	19	0.7843	0.5000	1.1400
Ia_3	*22*	*0.0000*	*1.0000*	*1.0000*
XXR_4	23	0.3711	0.0600	−0.4200
XY_4	*24*	*0.0000*	*0.0200*	*0.0200*
XZ_4	*25*	*0.0000*	*0.0200*	*0.0200*
YZ_4	*27*	*0.0000*	*0.0150*	*0.0150*
ZZR_4	28	0.4148	0.0500	0.0700
MX_4	*29*	*0.0000*	*0.0200*	*0.0200*
MYR_4	30	−0.2806	0.0100	−0.0700
Ia_4	*33*	*0.0000*	*0.3000*	*0.3000*
XXR_5	34	0.3635	0.0200	0.0200
XY_5	*35*	*0.0000*	*0.0100*	*0.0100*
XZ_5	*36*	*0.0000*	*0.0100*	*0.0100*
YZ_5	*38*	*0.0000*	*0.0100*	*0.0100*
ZZR_5	39	0.7783	0.0400	0.0600
MX_5	*40*	*0.0000*	*0.0200*	*0.0200*
MYR_5	41	0.2119	0.0100	0.0300
Ia_5	*44*	*0.0000*	*0.3000*	*0.3000*
XXR_6	45	−0.7783	0.0200	0.0000
XY_6	*46*	*0.0000*	*0.0100*	*0.0100*
XZ_6	*47*	*0.0000*	*0.0100*	*0.0100*
YZ_6	*49*	*0.0000*	*0.0100*	*0.0100*
ZZ_6	*50*	*0.0000*	*0.0200*	*0.0200*
MX_6	*51*	*0.0000*	*0.0100*	*0.0100*
MY_6	*52*	*0.0000*	*0.0100*	*0.0100*
Ia_6	*55*	*0.0000*	*0.3000*	*0.30000*
			X2	**XB2**
Ia_1	2		1.0000	0.0000
YY_2	6		1.5000	0.0000
Ia_2	11		4.5000	0.0000
YY_3	15		0.1000	0.0000
MZ_3	20		0.3000	0.0000
M_3	21		4.8000	0.0000
YY_4	26		0.5000	0.0000
MZ_4	31		−0.2000	0.0000
M_4	32		0.8000	0.0000
YY_5	37		0.0200	0.0000
MZ_5	42		0.0800	0.0000
M_5	43		0.5000	0.0000
YY_6	48		0.0200	0.0000
MZ_6	53		0.0200	0.0000
M_6	54		0.1000	0.0000

all the others, and they correspond to the standard parameters affecting the dynamic model separately. Table III gives the values of base parameters given the values of standard parameters. We conclude that the 66 standard parameters are reduced to 40 base parameters. Further: 11 standard parameters are eliminated because they do not affect the model, 15 standard parameters are regrouped to 17 others, and 23 standard parameters are not changed.

Table IV gives the 17 rows of β, which defines the coefficients of the 17 regrouping relations. The ith row, $\beta_{i,:}$, defines the following relation:

$$XB1_i = X1_i + \beta_{i,:}\mathbf{X2}$$

The standard parameters **X2** are reported at the top of the table.
These results agree completely with those obtained by our symbolic method that we recall in Table V.[6-8] As an example, we can verify the first three regrouping relations of Table V, which are calculated using the numerical values of the geometric parameters given in Table I.

$$ZZR_1 = ZZ_1 + YY_2 + YY_3 + 0.4\,MZ_3 + 0.2904\,(M_4 + M_5 + M_6) + 0.29\,M_3 + Ia_1$$

$$XXR_2 = XX_2 - YY_2 - 0.25(M_3 + M_4 + M_5 + M_6)$$

$$XZR_2 = XZ_2 - 0.5\,MZ_3 - 0.1(M_3 + M_4 + M_5 + M_6)$$

These three regrouping relations agree with those given by the first three rows of Table IV (coefficients are in bold).

The same results are obtained using SVD. Table II shows that the singular values σ_i of **W** are clustered in two separate groups. So, there is no doubt about the value of rank (**W**), which equals 40. We detect the 17 zero rows of matrix **V2,** eq. (23), corresponding to unchanged standard parameters, by calculating the matrix **V2 N,** where **N** is a 15×1 random matrix. We then look for a regular matrix **V22,** eq. (26), starting from the last row of **V2,** that defines the matrix, **P,** and we calculate the matrix β, eq. (27), and the base parameters, eq. (28).

Without the numerical problem, we prefer to use the QR method, which is simpler than SVD and less costly. But, the two methods are very easy to implement using the Ctrl-C package.[27]

Also, the same results are obtained using matrices **d** or $\mathbf{d}^0$. The comparison concerns the scaling. Using the same random sequences for $(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$, we find that $\mathbf{d}^0$ ($\mathbf{S} \cong 10^4$) is the best, then **d** ($S \cong 10^6$), and lastly, **h** ($S \cong 10^8$). But, using double precision, the difference is not so dramatic, and we prefer to use **h** because of the simplicity of the functions h_i. However, in the case of numerical problem, it would be better to use $\mathbf{d}^0$, with an eventual scaling procedure.

Application to Graph-Structured Robots

Our method is applicable to the analysis of graph-structured mechanisms with explicit constraint relations, as those given by parallelogram loops. In the

Table IV. No-zero rows of β, coefficients of regrouping relations.

	Ia_1	YY_2	Ia_2	YY_3	MZ_3	M_3	YY_4	MZ_4	M_4	YY_5	MZ_5	M_5	YY_6	MZ_6	M_6
$\mathbb{ZZR}_1$	**1.0000**	**1.0000**	0.0000	**1.0000**	**0.4000**	**0.2900**	0.0000	0.0000	**0.2904**	0.0000	0.0000	**0.2904**	0.0000	0.0000	**0.2904**
$\mathbb{XXR}_2$	0.0000	**−1.000**	0.0000	0.0000	0.0000	**−0.250**	0.0000	0.0000	**−0.250**	0.0000	0.0000	**−0.250**	0.0000	0.0000	**−0.250**
$\mathbb{XZR}_2$	0.0000	0.0000	0.0000	0.0000	**−0.500**	**−0.100**	0.0000	0.0000	**−0.100**	0.0000	0.0000	**−0.100**	0.0000	0.0000	**−0.100**
ZZR_2	0.0000	0.0000	1.0000	0.0000	0.0000	**0.2500**	0.0000	0.0000	**0.2500**	0.0000	0.0000	**0.2500**	0.0000	0.0000	**0.2500**
MXR_2	0.0000	0.0000	0.0000	0.0000	0.0000	**0.5000**	0.0000	0.0000	**0.5000**	0.0000	0.0000	**0.5000**	0.0000	0.0000	**0.5000**
XXR_3	0.0000	0.0000	0.0000	**−1.000**	0.0000	0.0000	**1.0000**	**1.2000**	**0.3596**	0.0000	0.0000	**0.3596**	0.0000	0.0000	**0.3596**
XYR_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**−0.020**	**−0.012**	0.0000	0.0000	**−0.012**	0.0000	0.0000	**−0.012**
ZZR_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.2000**	**0.3604**	0.0000	0.0000	**0.3604**	0.0000	0.0000	**0.3604**
MXR_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.0200**	0.0000	0.0000	**0.0200**	0.0000	0.0000	**0.0200**
MYR_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**1.0000**	**0.6000**	0.0000	0.0000	**0.6000**	0.0000	0.0000	**0.6000**
XXR_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**−1.000**	0.0000	0.0000	**1.0000**	0.0000	0.0000	0.0000	0.0000	0.0000
ZZR_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**1.0000**	0.0000	0.0000	0.0000	0.0000	0.0000
MYR_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**−1.000**	0.0000	0.0000	0.0000	0.0000
XXR_5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**−1.000**	0.0000	0.0000	**1.0000**	0.0000	0.0000
ZZR_5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**1.0000**	0.0000	0.0000
MYR_5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**1.0000**	0.0000
XXR_6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**−1.000**	0.0000	0.0000

equivalent opened-tree structure robot, we define:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_p \end{bmatrix}, \dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_p \end{bmatrix}$$

where:

$\mathbf{q}_a$ and $\dot{\mathbf{q}}_a$ are the position and velocity vectors of active joints, and are independent and can be chosen as random sequences.
$\mathbf{q}_p$ and $\dot{\mathbf{q}}_p$ are the position and velocity vectors of passive joints.

These variables are calculated using the explicit constraint equations:

$$\mathbf{q}_p = \mathbf{F}(\mathbf{q}_a)$$

$$\dot{\mathbf{q}}_p = \mathbf{G}(\mathbf{q}_a)\, \dot{\mathbf{q}}_a$$

$\mathbf{G}(\mathbf{q}_a)$ is the Jacobian matrix of $\mathbf{F}$.

Table V. PUMA 560, symbolic results of base parameters.

1. The 11 parameters not affecting the dynamic model are:
XX_1, XY_1, XZ_1, YY_1, YZ_1, MX_1, MY_1, MZ_1, M_1, MZ_2, M_2.

2. The 15 parameters to be eliminated owing to the regrouping are:
$\mathbf{X2} = [Ia_1\ YY_2\ Ia_2\ YY_3\ MZ_3\ M_3\ YY_4\ MZ_4\ M_4\ YY_5\ MZ_5\ M_5\ YY_6\ MZ_6\ M_6]^T$

3. There are 40 base parameters.

23 base parameters are not changed by the regrouping:
XY_2, YZ_2, MY_2, XZ_3, YZ_3, Ia_3, XY_4, XZ_4, YZ_4, MX_4, Ia_4, XY_5, XZ_5, YZ_5, MX_5, Ia_5, XY_6, XZ_6, YZ_6, ZZ_6, MX_6, MY_6, Ia_6.

17 base parameters are given by the regrouping relations:

$$ZZR_1 = ZZ_1 + YY_2 + YY_3 + 2\,R3\,MZ_3 + (R3^2 + D3^2 + D4^2)(M_4 + M_5 + M_6) + (R3^2 + D3^2)\,M_3 + Ia_1$$
$$XXR_2 = XX_2 - YY_2 - D3^2(M_3 + M_4 + M_5 + M_6)$$
$$XZR_2 = XZ_2 - D3\,MZ_3 - D3\,R3(M_3 + M_4 + M_5 + M_6)$$
$$ZZR_2 = ZZ_2 + D3^2(M_3 + M_4 + M_5 + M_6) + Ia_2$$
$$MXR_2 = MX_2 + D3(M_3 + M_4 + M_5 + M_6)$$
$$XXR_3 = XX_3 - YY_3 + YY_4 + 2\,R4\,MZ_4 + (R4^2 - D4^2)(M_4 + M_5 + M_6)$$
$$XYR_3 = XY_3 - D4\,MZ_4 - D4\,R4(M_4 + M_5 + M_6)$$
$$ZZR_3 = ZZ_3 + YY_4 + 2\,R4\,MZ_4 + (D4^2 + R4^2)(M_4 + M_5 + M_6)$$
$$MXR_3 = MX_3 + D4(M_4 + M_5 + M_6)$$
$$MYR_3 = MY_3 + MZ_4 + R4(M_4 + M_5 + M_6)$$
$$XXR_4 = XX_4 - YY_4 + YY_5$$
$$ZZR_4 = ZZ_4 + YY_5$$
$$MYR_4 = MY_4 - MZ_5$$
$$XXR_5 = XX_5 - YY_5 + YY_6$$
$$ZZR_5 = ZZ_5 + YY_6$$
$$MYR_5 = MY_5 + MZ_6$$
$$XXR_6 = XX_6 - YY_6$$

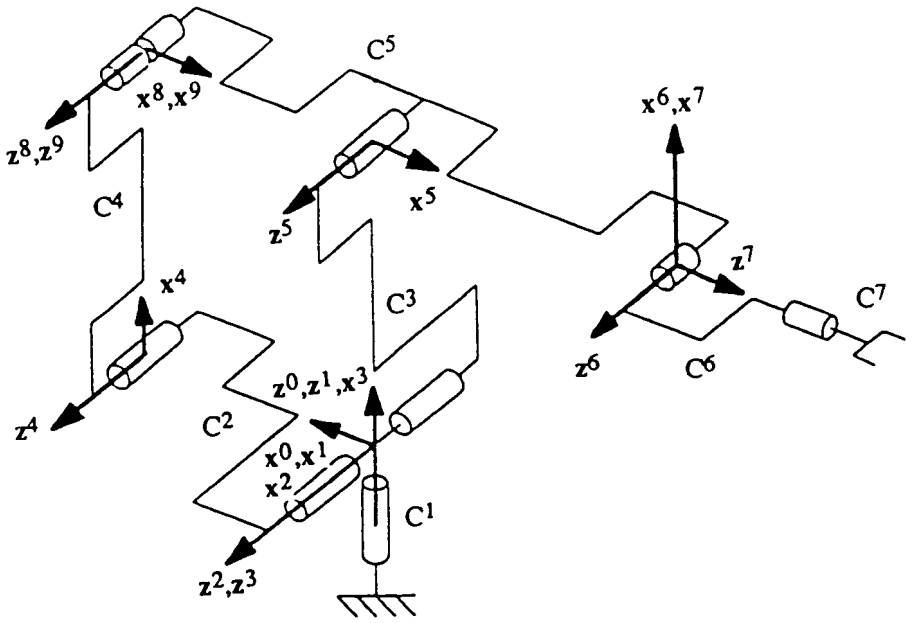

Figure 2. HITACHI-HPR Robot.

This method has been applied to the five degrees of freedom HITACHI-HPR robot (Fig. 2). Details for the description of this robot can be found in ref. 29. We have:

$$\mathbf{q}_a = [\theta_1\ \theta_3\ \theta_4\ \theta_6\ \theta_7]^T$$

$$\mathbf{q}_p = [\theta_2\ \theta_5]^T$$

$$\begin{bmatrix}\theta_2\\ \theta_5\end{bmatrix} = \begin{bmatrix}\theta_3-\theta_4\\ 3\pi-\theta_4\end{bmatrix}$$

$$\begin{bmatrix}\dot{\theta}_2\\ \dot{\theta}_5\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}\begin{bmatrix}\dot{\theta}_3\\ \dot{\theta}_4\end{bmatrix}$$

The results completely agree with those obtained using the symbolic approach given in ref. 29.

CONCLUSIONS

Two numerical methods are given to determine the set of base parameters of the dynamic model of robots. Starting from the standard inertial parameters,

each method gives the number, the choice and the numerical values of the base parameters, and are based on QR or SVD decompositions.

The numerical method is universal, it can be applied to open-loop or graph-structured robots. It is also very easy to implement using commercial software packages such as Ctrl-C, MATLAB,[30] or BASILE[31] for the numerical tools, and software SYMORO[24] to obtain the symbolic expressions of h_i. The only limitations could be from numerical problems and, to avoid these problems, we proposed a scaling strategy.

All the results agree with those of the symbolic method whose great advantages remain its insensitivity to numerical problems and the insight into the physical meaning of regrouping. Our final conclusion is to at first apply the symbolic general relations as given in refs. 6–11 and 29 to obtain most of the regrouped inertial parameters, and then to use the numerical algorithm on this reduced model to confirm the result or to eventually find supplementary regrouping relations.

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