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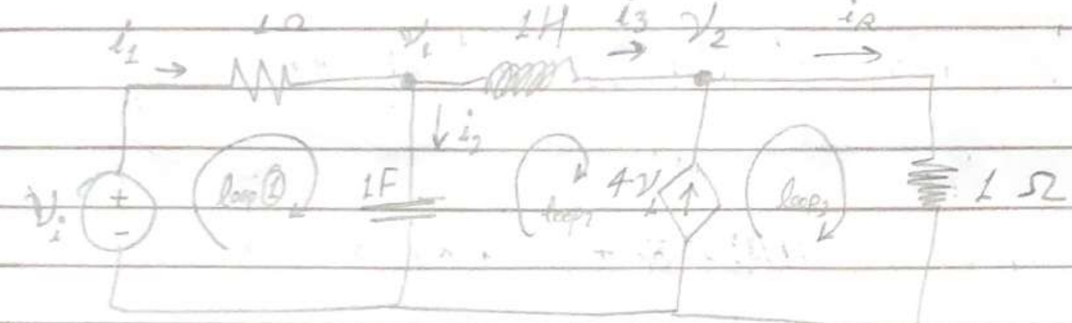
Sheet 3

Question 1:



$x_1 = V_1$, $x_2 = i_3$, output $= i_R(t)$

at node (1):

$$i_1 = i_2 + i_3$$
$$i_1 = 1 \times \frac{dv_1}{dt} + i_3 \rightarrow (1)$$

at loop (1)

$$v_i(t) = 1 \times i_1 + v_1 \rightarrow (2)$$

at loop (2)

$$v_1(t) = 1 \times \frac{di_3}{dt} + v_2 \rightarrow (3)$$

at loop (3)

$v_2 = 1 \times i_R$ substituting in (3)

$$v_1(t) = \frac{di_3}{dt} + i_R(t) \rightarrow (4)$$

at node (2)

$$i_3 + 4v_1 = i_R \rightarrow (5)$$

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in the Form $\dot{X}(t) = A\,x(t) + B\,u(t)$

$y(t) = C\,X(t) + D\,u(t)$

from equation (1): $i_1 = \dot{X}_1 + X_2 \rightarrow (6)$

from equation (2): $u(t) = i_1 + X_1$ subsitute by (6)

$$u(t) = \dot{X}_1 + X_2 + X_1$$

$$\dot{X}_1 = u(t) - X_2(t) - X_1(t) \rightarrow (7)$$

from equation (4):

$$X_1 = \dot{X}_2 + y \longrightarrow (8)$$

from equation (5):

$$y = X_2 + 4X_1 \quad \text{subsitute in } (8)$$

$$X_1 = \dot{X}_2 + X_2 + 4X_1$$

$$\dot{X}_2(t) = -3X_1(t) - x_2(t) \longrightarrow (9)$$

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix}_{2\times1} = \begin{bmatrix} -1 & -1 \\ -3 & -1 \end{bmatrix}_{2\times2} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}_{2\times1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times1} \begin{bmatrix} u(t) \end{bmatrix}_{1\times1}$$

$$\begin{bmatrix} y(t) \end{bmatrix}_{1\times1} = \begin{bmatrix} 4 & 1 \end{bmatrix}_{1\times2} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}_{1\times1} \begin{bmatrix} u(t) \end{bmatrix}_{1\times1}$$

$$A = \begin{bmatrix} -1 & -1 \\ -3 & -1 \end{bmatrix}_{2\times2} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times1}$$

$$C = \begin{bmatrix} 4 & 1 \end{bmatrix}_{1\times2} \qquad D = \begin{bmatrix} 0 \end{bmatrix}_{1\times1}$$

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Question 2

c) $y''' + 5y'' + 7y' + 3y = u'' + 3u' + 2u$

$$S^3 Y(s) + 5S^2 Y(s) + 7S Y(s) + 3Y(s) = S^{?} u(s) + 3S u(s) + 2u(s)$$

$$Y(s)(S^3 + 5S^2 + 7S + 3) = u(s)(S^3 + 3S + 2)$$

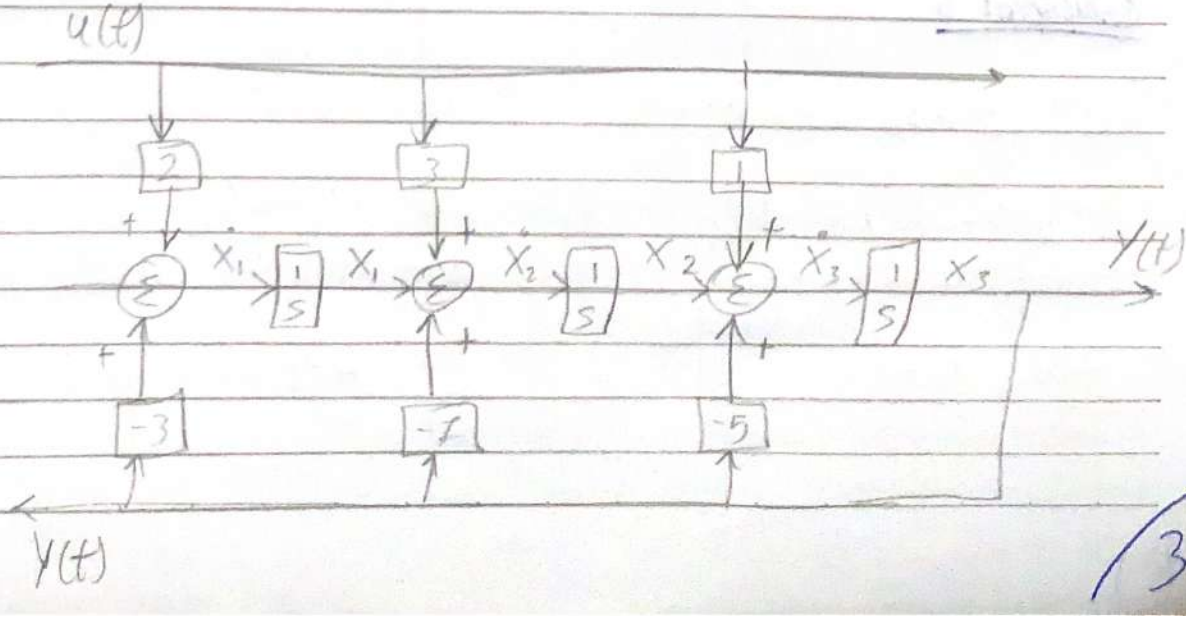
$$\frac{Y(s)}{u(s)} = \frac{S^3 + 3S + 2}{S^3 + 5S^2 + 7S + 3}$$

The Observer Form :

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -7 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 0 \quad 1] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\dot{X}_1 = -3X_3 + 2u(t)$$
$$\dot{X}_2 = X_1 - 7X_3 + 3u(t)$$
$$\dot{X}_3 = X_2 - 5X_3 + u(t)$$



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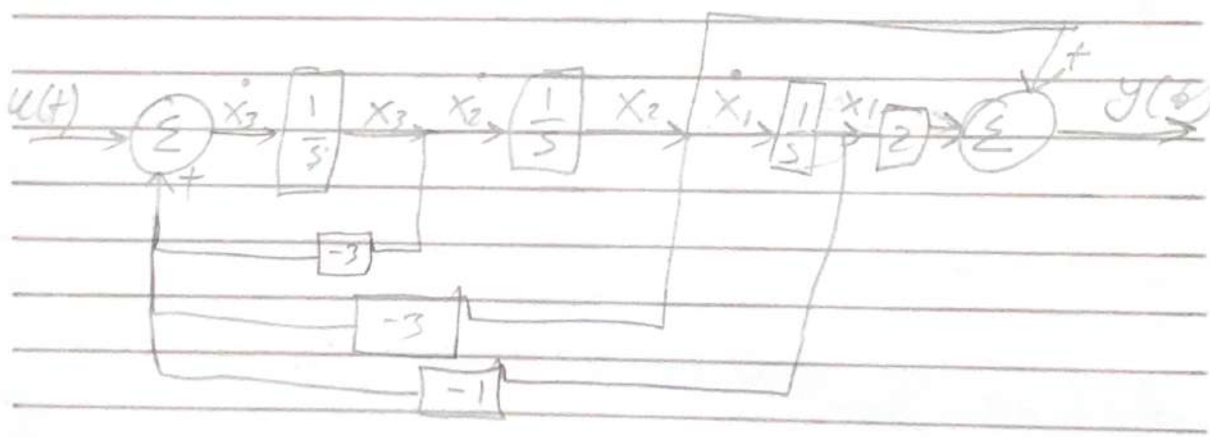
d) $$\frac{Y(s)}{U(s)} = \frac{s+2}{(s^2+2s+1)(s+1)} = \frac{s+2}{s^3+3s^2+3s+1}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
$\dot{x}_3 = -x_1 - 3x_2 - 3x_3$
$y = 2x_1 + x_2$



Question 3

$y''' + 3y'' + 3y' + y = u' + u$

$s^3 y(s) + 3s^2 y(s) + 3s\, y(s) + y(s) = s\,u(s) + u(s)$
$y(s)(s^3 + 3s^2 + 3s + 1) = u(s)(s+1)$

$$\frac{y(s)}{u(s)} = \frac{s+1}{s^3+3s^2+3s+1} = \frac{s+1}{(s+1)^3}$$

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$$\frac{Y(s)}{U(s)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+1)^2} + \frac{K_3}{(s+1)^3}$$

$$s+1 = K_1(s+1)^2 + K_2(s+1) + K_3 \qquad K_3 = 0 \quad K_2 = 1 \quad K_1 = 0$$

$$\frac{y(s)}{u(s)} = \frac{0}{s+1} + \frac{1}{(s+1)^2} + \frac{0}{(s+1)^3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u(t)$$

Question 4

$$G(s) = \frac{4s^2 + 22s + 18}{s^3 + 8s^2 + 19s + 12} = \frac{4s^2 + 22s + 18}{(s+1)(s+3)(s+4)}$$

$$= \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+4} = \frac{0}{s+1} + \frac{6}{s+3} + \frac{-2}{s+4}$$

$$4s^2 + 22s + 18 = A(s+3)(s+4) + B(s+1)(s+4) + C(s+1)(s+3)$$
$$= A(s^2+7s+12) + B(s^2+5s+4) + C(s^2+4s+3)$$

$4 = A + B + C \qquad 22 = 7A + 5B + 4C \qquad 18 = 12A + 4B + 3C$

$A = 0 \qquad B = 6 \qquad C = -2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u(t)$$

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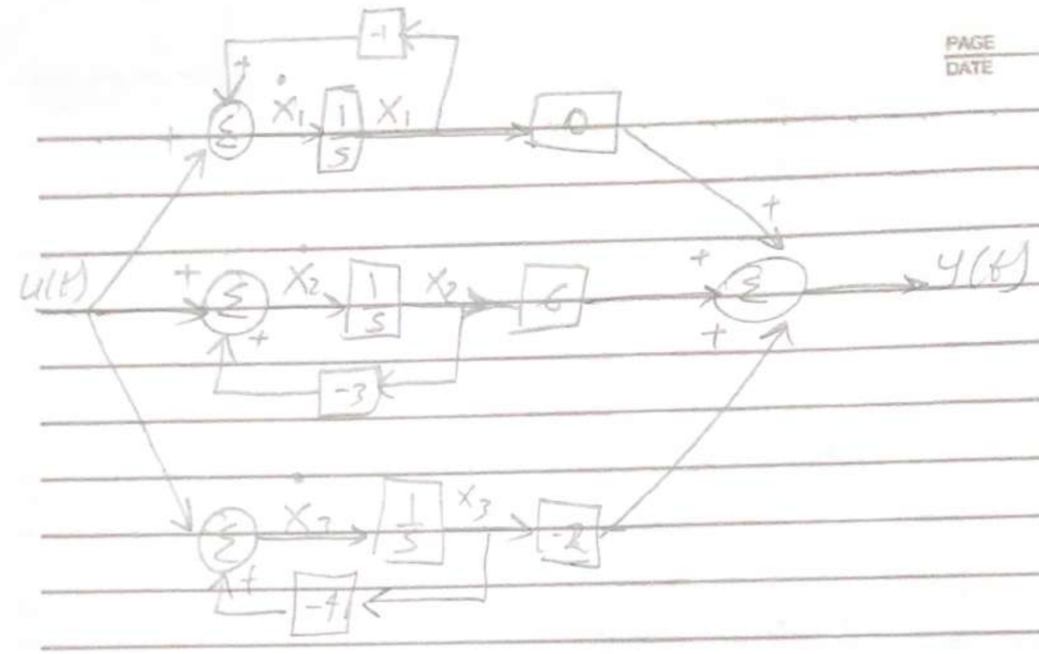

Question 5

$$A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$$

a) Laplace

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$e^{AT} = \mathcal{L}^{-1}\left[(SI - A)^{-1}\right]$$

$$(SI - A) = \begin{bmatrix} S & -1 \\ 9 & S-6 \end{bmatrix}$$

$$(SI - A)^{-1} = \frac{1}{S^2 - \cancel{8}6S + 9} \begin{bmatrix} S-6 & 1 \\ -9 & S \end{bmatrix} = \begin{bmatrix} \frac{S-6}{S^2-6S+9} & \frac{1}{S^2-6S+9} \\ \frac{-9}{S^2+6S+9} & \frac{S}{S^2-6S+9} \end{bmatrix}$$

$$e^{AT} = \begin{bmatrix} e^{3t} - e^{3t} \cdot 3t & e^{3t}\, t \\ -9e^{3t}\, t & e^{3t} + e^{3t} \cdot 3t \end{bmatrix}$$

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b) Cayley - Hamilton

$$e^{At} = b_0 I + b_1 A = \begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix} + \begin{bmatrix} 0 & b_1 \\ -9b_1 & 6b_1 \end{bmatrix}$$

$$e^{AT} = \begin{bmatrix} b_0 & b_1 \\ -9b_1 & b_0 + 6b_1 \end{bmatrix}$$

$\det |\lambda I - A| = 0$

$= -\lambda(6-\lambda) + 9$

$= \lambda^2 - 6\lambda + 9$

$\lambda = 3$ #

repeated values then

$$\frac{\partial}{\partial \lambda}(e^{\lambda t}) = \frac{\partial}{\partial \lambda}(b_0 + b_1 \lambda) \rightarrow t e^{\lambda t} = b_1$$

at $\lambda = 3$ $\therefore$ $\underline{\underline{t e^{3t} = b_1}}$

& $e^{\lambda_i t} = b_0 + b_1 \lambda_i$ $\therefore$ $e^{3t} = b_0 + 3b_1$

$e^{3t} = b_0 + 3te^{3t}$

$\boxed{b_0 = e^{3t}(1-3t)}$

$$\therefore e^{At} \begin{bmatrix} e^{3t}(1-3t) & te^{3t} \\ -9te^{3t} & e^{3t}(1-3t) + 6te^{3t} \end{bmatrix}$$

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