Linear Differential Equations

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Linear ODEs

Definition. An ODE is **linear** if it is of the form $\mathcal{L}_x[u] = f$, where $\mathcal{L}_x$ is the linear differential operator

$$\mathcal{L}_x = A_n(x)\frac{d^n}{dx^n} + A_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + A_1(x)\frac{d}{dx} + A_0(x)$$

Linear ODEs of the form $\mathcal{L}[u] = 0$ are termed **homogeneous**, otherwise **inhomogeneous**.

We will often want to work with the linear operator in *standard form*, meaning that the factor multiplying the highest derivative is 1:

$$\mathcal{L}_x = \frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

Solutions

Solutions of a homogeneous ODE satisfy the **superposition principle** (or *linearity principle*): if u_1 and u_2 are solutions of a linear ODE $\mathcal{L}[u] = 0$ then any linear combination $\alpha u_1 + \beta u_2$ is also a solution:

$$\mathcal{L}[\alpha u_1 + \beta u_2] = \alpha\mathcal{L}[u_1] + \beta\mathcal{L}[u_2] = 0 + 0 = 0$$

A **general solution** of an ODE is a solution $u = \alpha u_1 + \beta u_2$ where u_1 and u_2 are linearly independent and they are called a **basis** of the set of solutions. A **particular solution** is obtained by assigning specific values to constants.

Every element of the general solution of an ODE is a function of the form $u = u_p + u_h$, where u_h is an element of the general solution of the homogeneous ODE and u_p is a **particular integral**, the solution of the inhomogeneous problem:

$$\mathcal{L}[u_p + u_h] = \mathcal{L}[u_p] + \mathcal{L}[u_h] = f + 0 = f$$

Solution methods

- **Separation of variables** when we can algebraically transform the equation to the form $f(u)du = g(x)dx$ which we can integrate directly
- **Variation of parameters** where we find the solution to the homogeneous problem and use it as solution *ansatz* for a particular solution by allowing constants to become new unknown functions which we then seek
- **Undetermined coefficients**
- **Green's functions**
- **Numerical methods**

Linear PDEs