Canonical form of second-order linear PDEs

Contents

Here we consider a general second-order PDE of the function $u(x, y)$:

$$au_{xx} + bu_{xy} + cu_{yy} = f(x, y, u, u_x, u_y) \tag{26}$$

Recall from a previous notebook that the above problem is:

- **elliptic** if $b^2 - 4ac > 0$
- **parabolic** if $b^2 - 4ac = 0$
- **hyperbolic** if $b^2 - 4ac < 0$

Any elliptic, parabolic or hyperbolic PDE can be reduced to the following **canonical forms** with a suitable coordinate transformation $\xi = \xi(x, y), \qquad \eta = \eta(x, y)$

1. Canonical form for hyperbolic PDEs: $u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\xi\xi} - u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$
2. Canonical form for parabolic PDEs: $u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$
3. Canonical form for elliptic PDEs: $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

We find the coordinate transformation

$$u_x = u_\xi \xi_x + u_\eta \eta_x, \qquad u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$\text{and similarly for } u_{xx}, u_{xy}, u_{yy}$$

Plugging this back into (26) we get

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta) \tag{27}$$

where

$$\begin{aligned} A &= a(\xi_x)^2 + b\xi_x\xi_y + c(\xi_y)^2 \\ B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y \\ C &= a(\eta_x)^2 + b\eta_x\eta_y + c(\eta_y)^2 \end{aligned}$$

The reader can derive this as partial differentiation practice.

Hyperbolic case

PDE (26) is hyperbolic if $b^2 - 4ac > 0$ so the obvious choice is to set $A = C = 0$ in eq. (27) (note that we could have also chosen for example $A = 1, C = -1$). We get a system of ODEs

$$\begin{aligned} A = a(\xi_x)^2 + b\xi_x\xi_y + c(\xi_y)^2 = 0 \\ C = a(\eta_x)^2 + b\eta_x\eta_y + c(\eta_y)^2 = 0 \end{aligned}$$

Dividing the first equation by $(\xi_y)^2$ and the second by $(\eta_y)^2$ we get

$$\begin{aligned} a\left(\frac{\xi_x}{\xi_y}\right)^2 + b\left(\frac{\xi_x}{\xi_y}\right) + c = 0 \\ a\left(\frac{\eta_x}{\eta_y}\right)^2 + b\left(\frac{\eta_x}{\eta_y}\right) + c = 0 \end{aligned}$$

These are two identical quadratic equations with roots

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $\lambda_1 = \xi_x/\xi_y$ and $\lambda_2 = \eta_x/\eta_y$ and they need to be different for the transformation to make sense. Because $b^2 - 4ac > 0$ we know that they will be two distinct real numbers.

But what do the quantities ξ_x/ξ_y and η_x/η_y actually represent? They are the slopes of the **characteristics** $\xi(x,y) = \text{const.}$ and $\eta(x,y) = \text{const.}$ Notice that if we hadn't divided the equations by ξ_y and η_y we would have

$$\xi_x = \lambda_1 \xi_y, \qquad \eta_x = \lambda_2 \eta_y$$

whose characteristic curves satisfy the ODEs

$$\frac{dy}{dx} = -\lambda_1, \qquad \frac{dy}{dx} = -\lambda_2$$

The solutions of these ODEs are

$$y + \lambda_1 x = c_1, \qquad y + \lambda_2 x = c_2$$

where c_1, c_2 are integration constants, so we choose ξ and η to equal them

$$\xi = y + \lambda_1 x, \qquad \eta = y + \lambda_2 x$$

Finally, going back to the canonical form

$$u_{\xi\eta} = F$$

we integrate w.r.t. η and ξ to get the solution

$$u(\xi, \eta) = \int\int F d\eta d\xi + \phi(\xi) + \psi(\eta)$$

where ϕ and ψ are arbitrary functions.

Example: d'Alembert's solution

The d'Alembert's solution encountered in lectures is an example of the method of characteristics. Here we will show this. Let us transform the 1-D wave equation

$$u_{tt} - v^2 u_{xx} = 0$$

to canonical form. Comparing with (26) we see that $a = 1, b = 0, c = -v^2$. This leads to

$$\lambda_{1,2} = \frac{0 \pm \sqrt{0 + 4v^2}}{2} = \pm v$$

and the characteristics are given by

$$\xi = x + vt, \qquad \eta = x - vt$$

The solution u is given by

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta)$$

or in terms of x and y:

$$u(x, y) = \phi(x + vt) + \psi(x - vt)$$

which is the d'Alembert's solution of the wave equation.

Parabolic case

PDE (26) will be parabolic if $b^2 - 4ac = 0$. We therefore require $B = 0$ and either $A = 0$ or $C = 0$. Let us choose $A = 0$ and $C \neq 0$, so dividing (27) by C we get the canonical form

$$u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

Note: If we chose $C = 0$ and $A \neq 0$ we would get $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$.

Since $A = 0$:

$$A = a\left(\frac{\xi_x}{\xi_y}\right)^2 + b\left(\frac{\xi_x}{\xi_y}\right) + c = 0$$

Therefore the equation

$$a\lambda^2 + b\lambda + c = 0$$

has two equal roots

$$\lambda = \lambda_1 = \xi_x/\xi_y = \eta_x/\eta_y = \lambda_2$$

but we still need ξ and η to be independent for the transformation to make sense. So we let ξ be a solution of

$$\frac{dy}{dx} = -\lambda$$

i.e.

$$\xi = y + \lambda x$$

and we can choose

$$\eta = x$$

so that ξ and η are independent. Then going back to the canonical form and integrating it twice, we get the solution

$$u(\xi, \eta) = \int\int F d\eta d\eta + \eta\phi(\xi) + \psi(\xi)$$

We could have chosen ξ and η the other way around, of course.

Example: $u_{xx} + 2u_{xy} + u_{yy} = 0$

Kreyszig problem set 12.4, question 11.

This is a parabolic PDE because $2^2 - 4 = 0$. Therefore we have a single root

$$\lambda = \frac{-b}{2a} = -1$$

Then $\xi = y - x$ and we can choose $\eta = x$. So the solution is

$$u(\xi, \eta) = \eta\phi(\xi) + \psi(\xi)$$

or in original coordinates

$$u(x, y) = x\phi(y - x) + \psi(y - x)$$

where ϕ and ψ are arbitrary functions.

Example: $u_{xx} - 4u_{xy} + 4u_{yy} = \cos(2x + y)$

The PDE is parabolic and we have a single root

$$\lambda = \frac{-b}{2a} = 2$$

And we choose $\eta = y + 2x$ and $\xi = x$. The canonical form is

$$u_{\xi\xi} = \cos(2x + y) = \cos\eta$$

Integrating twice w.r.t. ξ

$$u(\xi, \eta) = \int\int \cos\eta d\xi d\xi + \xi\phi(\eta) + \psi(\eta)$$

Which is

$$u(\xi, \eta) = \frac{\xi^2}{2}\cos\eta + \xi\phi(\eta) + \psi(\eta)$$

Or in original coordinates

$$u(x, y) = \frac{x^2}{2}\cos(2x + y) + x\phi(x + 2y) + \psi(x + 2y)$$

where ϕ and ψ are arbitrary functions.

Elliptic case

We will not use method of characteristics to solve elliptic equations because the PDE gets only marginally reduced, i.e. the canonical form is the Poisson's equation.