Integral Calculus

Contents

Introduction

In this section we will review the defintions and rules of integration, and look at some simple examples

Definite Integrals

The definite integral of a function $f(x)$ over the interval $[a, b]$ is (usually) defined as the limit of a sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

where x_i^* is any point in the interval $[a+(i-1)\Delta x, a+i\Delta x]$ and $\Delta x = \frac{b-a}{n}$.

The definite integral is thus the area under the curve $f(x)$ between $x = a$ and $x = b$.

Indefinite Integrals

The indefinite integral of a function $f(x)$ is a function (the anti-derivative) $F(x)$ such that $F'(x) = f(x)$ (i.e. the inverse of differentiation).

Since the derivative of a constant is zero, the indefinite integral of a function is only defined up to a constant. For example, the indefinite integral of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3 + c$. The constant c is called the *constant of integration*.

Fundamental Theorem of Calculus

The fundamental theorem of calculus states that if $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is any anti-derivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This theorem provides a method for evaluating definite integrals by finding an anti-derivative of the function and evaluating it at the limits of integration.

Rules of Integration

Linearity

Integration is a linear operation, i.e. for functions $f(x)$ and $g(x)$ and constants a and b,

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

This is the equivalent of the sum rule for differentiation.

Integration by parts

The integration by parts formula is

$$\int u dv = uv - \int v du$$

where u and v are functions of x. This is a reaarrangement of the product rule for differentiation.

Substitution/Change of variable

The substitution rule is

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$.

This is the equivalent of the chain rule for differentiation.

Examples

Polynomial functions

The indefinite integral of a polynomial function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

is given by

$$\int f(x)dx = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \cdots + \frac{a_n}{n+1}x^{n+1} + c$$

where c is an arbitrary constant.

More generally, the function

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

applies for any real number $n \neq -1$. In that special case, we have

$$\int \frac{1}{x} dx = \ln|x| + c$$

Proving this is easiest by starting (in reverse with the chain rule, applied to the function $g(x) = \exp(\ln x)$:

$\frac{dg}{dx} = \frac{d}{dx}\exp(\ln x) = \exp(\ln x)\frac{d}{dx}\ln x = x\frac{d}{dx}\ln x$.

But $\exp(\ln x) = x$, so $\frac{dg}{dx} = 1 = x\frac{d}{dx}\ln x$, and hence $\frac{d}{dx}\ln x = \frac{1}{x}$.

Trigonometric functions

For the primary functions, these can be derived from the derivatives of rules for differentiation in reverse (not forgetting the constant of integration).

Unlike differentiation, inverse trignometric functions sometimes appear on formula sheets for integration. For example

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c,$$

$$\int \frac{1}{1+x^2} dx = \arctan x + c.$$

These results can generally be proved by appropriate sustitutions, and use of trignometric identities, e.g. setting $x = \sin t$ and using the identity $\sin^2 t + \cos^2 t = 1$.

Integration in multiple dimensions

Double integrals, triple integrals, etc.

The definite integral of a function of two variables $f(x, y)$ over a rectangular region R in the xy-plane is defined as the limit of a sum:

$$\iint_R f(x,y)dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A$$

where x_i^* and y_i^* are any points in a sub rectangle of R which “tiles” the domain and ΔA is the area of the sub rectangle. This idea can be extended to general regions in the plane, and also to higher dimensions.

Green’s Theorem

Green’s theorem is a result in vector calculus that relates a double integral over a region in the plane to a line integral around the boundary of the region. It is a special case of the more general Stokes’ theorem.

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where C is the boundary of the region R, and P and Q are functions of x and y. The left hand side is a line integral around the entire boundary of the region, and the right hand side is a double integral over the region.

Divergence Theorem

The divergence theorem is a result in vector calculus that relates a triple integral over a region in space to a surface integral over the boundary of the region. It is a generalization of Green’s theorem in the plane.

$$\iiint_V (\nabla\cdot\mathbf{F})dV = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS$$

where V is the region in space, S is the boundary of the region, $\hat{\mathbf{n}}$ is an (outward pointing) normal to the boundary, $\mathbf{F}$ is a vector field, and $\nabla\cdot\mathbf{F}$ is the divergence of the vector field. The left hand side is an integral over the entire region, and the right hand side is a surface integral over the entire boundary.

Stokes' Theorem

Stokes' theorem is a result in vector calculus that relates a surface integral of a vector field to a line integral of the vector field's curl. It is a generalization of Green's theorem in the plane.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where C is the boundary of the surface S, $\mathbf{F}$ is a vector field, and $\nabla \times \mathbf{F}$ is the curl of the vector field. The left hand side is a line integral around the entire boundary of the surface, and the right hand side is a surface integral over the entire surface.

Further Reading

- Notes for revision for A-level mathematics can be found at Integral Calculus and Further Integration.
- Notes on integration for the Scottish Higher exam can be found here