Higher order Linear ODEs

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Wronskian

We define a Wronskian of two functions as the determinant

$$W[f,g](x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

Now consider the Wronskian of two solutions of a homogeneous ODE $y'' + p(x)y' + q(x)y = 0$:

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$$

Differentiating it

$$\begin{aligned} \frac{dW}{dx} &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\ &= y_1 y_2'' - y_2 y_1'' \\ &= y_1(-p y_2' - q u_2) - u_2(-p u_1' - q u_1) \\ &= -pW \end{aligned}$$

In other words, the Wronskian satisfies the first-order linear ODE

$$\frac{dW}{dx} + pW = 0$$

whose solution is the **Abel's identity**

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(s)ds}$$

In general, the Wronskian of a set of functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is defined by

$$W[f_1, \ldots, f_n](x) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Second-order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

Reduction of Order - homogeneous

Often we can find one solution of an ODE from inspection, which we can then use to find the other linearly independent solution.

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W[y_1, y_2]}{y_1^2}$$

And integrating

$$y_2(x) = y_1(x)\left(C + \int_{x_0}^{x} \frac{W(s)}{y_1^2(s)} ds\right)$$

Can set C = 0 since we write the solution as a linear combination of u1 u2 anyway

Variation of parameters - inhomogeneous

Let the general solution of the corresponding homogeneous equation be known

$$y_h(x) = Ay_1(x) + By_2(x) \tag{10}$$

As we have done before, we use variation of parameters to set the ansatz for the particular solution:

$$y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x) \tag{11}$$

where we have "promoted" the constants C_1, C_2 to functions $A(x), B(x)$. Differentiating it once:

$$y' = C_1'y_1 + C_1y_1' + C_2'y_2 + C_2y_2'$$

We also require that

$$C_1'y_1 + C_2'y_2 = 0 \tag{12}$$

so eq. (10) becomes

$$y' = C_1y_1' + C_2y_2'. \tag{13}$$

Differentiating again

$$y'' = C_1'y_1' + C_1y_1'' + C_2'y_2' + C_2y_2'' \tag{14}$$

Now we substitute (13) and (14) into eq. (9):

$$\begin{aligned} C_1'y_1' + C_1y_1'' + C_2'y_2' + C_2y_2'' + p(C_1y_1' + C_2y_2') + q(C_1y_1 + C_2y_2) &= r \\ (C_1'y_1' + C_2'y_2') + \underbrace{C_1(y_1'' + py_1' + qy_1) + C_2(y_2'' + py_2' + qy_2)}_{=0 \quad eq.(20)} &= r \end{aligned}$$

We get

$$C_1'y_1' + C_2'y_2' = r \tag{15}$$

The solution of the system of equations (11) and (15) is given by

$$C_1' = -\frac{y_2}{W[y_1, y_2]}r, \quad C_2' = \frac{y_1}{W[y_1, y_2]}r \tag{16}$$

and after integrating we get

$$C_1(x) = -\int^x \frac{y_2(s)}{W(s)}r(s)\,ds, \quad C_2(x) = \int^x \frac{y_1(s)}{W(s)}r(s)\,ds \tag{17}$$

where the choice of the lower bounds of integration is irrelevant since it only changes the coefficients A and B (see below). Therefore, the solution of the inhomogeneous second-order ODE (9) is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= Ay_1(x) + By_2(x) + C_1(x)y_1(x) + C_2(x)y_2(x) \\ &= [A + C_1(x)]y_1(x) + [B + C_2(x)]y_2(x) \end{aligned}$$

General n-th order linear ODE

We can directly extend the method of variation of parameters we used to solve second-order linear ODE to n-th order ODEs, which are of the form

$$\mathcal{L}_x[y] = y^{(n)} + p_n(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{18}$$

Let $\{y_1, y_2, \ldots, y_n\}$ be solutions of the homogeneous ODE, i.e. $\mathcal{L}[y_i] = 0$. We seek solutions of the inhomogeneous ODE in the form of ansatz

$$y_p(x) = \sum_{i=1}^{n} C_i(x)y_i(x) \tag{19}$$

where $C_i(x)$ are functions that we need to find. As before, we differentiate and define conditions:

$$
\begin{aligned}
y' &= \sum_{i=1}^{n} C_i' y_i + C_i y_i', \quad \text{assume: } \sum_{i=1}^{n} C_i' y_i = 0 \\
y'' &= \sum_{i=1}^{n} C_i' y_i' + C_i y_i'', \quad \text{assume: } \sum_{i=1}^{n} C_i' y_i' = 0 \\
&\vdots \\
y^{(n-1)} &= \sum_{i=1}^{n} C_i' y_i^{(n-2)} + C_i y_i^{(n-1)}, \quad \text{assume: } \sum_{i=1}^{n} C_i' y_i^{(n-2)} = 0
\end{aligned}
$$

$$
y^{(n)} = \sum_{i=1}^{n} C_i' y_i^{(n-1)} + C_i y_i^{(n)}
$$

These n equations define a system of linear equations with unknowns $C_1, \ldots, C_n$:

$$
\begin{bmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \vdots & & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{bmatrix}
\begin{bmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ r \end{bmatrix}
\tag{20}
$$

A sufficient condition for the existence of a solution of the system of equations (20) is that the determinant of coefficients is nonzero for each value of x. This is equal to the Wronskian $W[y_1, \ldots, y_n]$ and we know it is nonzero for all x because the set $\{y_1, \ldots, y_n\}$ is linearly independent. We determine $C_1', \ldots, C_n'$ using Cramer's rule:

$$
C_1'(x) = \frac{W_i(x)}{W(x)} r(x)
$$

where W_i is the determinant obtained from W by replacing i-th column with the RHS column $(0, 0, \ldots, r)^T$. We integrate this and substitute the result in eq. (19) to get the particular solution of (18)

$$
y_p = \sum_{i=1}^{n} y_i(x) \int_{x_0}^{x} \frac{W_i(s)}{W(s)} r(s)\, ds
$$

where x_0 is arbitrary.

Linear ODEs with Constant Coefficients

In this section we focus on second-order ODEs where functions $p(x), q(x)$ and $r(x)$ are now constants. We therefore write Eq. `inhmg2ndode` as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{21}$$

Homogeneous case - method of characteristic polynomial

We use the method of **characteristic polynomial** which can be used for any such ODE of any order. A homogeneous linear ODE of first order with constant coefficient a_0

$$y' + a_0 y = 0$$

is separable and its integral is $y(x) = Ce^{\lambda x}$, where λ is an unknown constant. We find it by plugging y back into the ODE:

$$\begin{aligned} \lambda Ce^{\lambda x} + a_0 Ce^{\lambda x} &= 0 \\ \lambda + a_0 &= 0 \end{aligned}$$

This is the **characteristic equation** of the ODE. We find $\lambda = -a_0$.

Consider now a second-order ODE with constant coefficients a_1, a_0

$$y'' + a_1 y' + a_0 y = 0$$

Inspired by the solution of the first-order equation, we assume the solution of the form $y(x) = Ce^{\lambda x}$. We plug it back into the ODE and get the characteristic equation

$$\lambda^2 + a_1\lambda + a_0 = 0$$

The roots of this quadratic equation are

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{D}}{2}, \qquad D = a_1^2 - 4a_0$$

so we have three cases depending whether $D > 0, D = 0$ or $D < 0$, keeping in mind that we need linearly independent solutions to form the basis:

1. $\lambda_{1,2} \in \mathbb{R}, \lambda_1 \neq \lambda_2 : \quad y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
2. $\lambda_{1,2} \in \mathbb{R}, \lambda_1 = \lambda_2 : \quad y(x) = (C_1 + C_2 x) e^{\lambda_1 x}$
3. $\lambda_{1,2} \in \mathbb{C}, \lambda_2 = \lambda_1^* : \quad y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_1^* x}$

Hint: Diagonalisation

The method of characteristic polynomial might remind us of the process of finding eigenvalues. Indeed, we can think of finding solutions in the form of an exponential function as a type of diagonalisation.

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}$$

The exponential function can be thought of as an **eigenvector** of the derivative operator and λ as the **eigenvalue**. If λ_1, λ_2 are roots of the characteristic equation, we can write

$$\left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right) y(x) = 0$$

Let us now generalise this to an n-th order ODE. The characteristic equation of eq. (21) is again obtained by substituting $y = e^{\lambda x}$:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

1. For any distinct real root λ, one solution is $y = e^{\lambda x}$
2. For any complex root $\lambda = \gamma + i\omega$, its conjugate $\lambda^* = \gamma - i\omega$ is also a root:

$$y_1 = e^{\gamma x} \cos \omega x, \quad y_2 = e^{\gamma x} \sin \omega x$$

3. Multiple real roots: if a real root is repeated m times, the m corresponding linearly independent solutions are

$$e^{\lambda x},\ xe^{\lambda x},\ \ldots\ , x^{m-1} e^{\lambda x}$$

4. Multiple complex roots: if a complex root $\lambda = \gamma + i\omega$ is repeated, the corresponding linearly independent solutions are

$$e^{\gamma x} \cos \omega x, \quad e^{\gamma x} \sin \omega x, \quad xe^{\gamma x} \cos \omega x, \quad xe^{\gamma x} \sin \omega x,\ \ldots$$

Inhomogeneous case