Decompositions

1 # QR decomposition :-

A matrix $A \in \mathbb{R}^{m\times n}$ with linearly independent columns, ~~can be~~ written as:

$A = QR$ → Upper Δ^{er} matrix

Q → orthogonal matrix

$A = [a_1 | a_2 | \cdots | a_n]$ all a's are linealy independent

$Q \div Q \cdot Q^T = I$

→ i.e. columns of Q has to be orthonormal in nature ie $e_i \cdot e_j = 0 \quad \forall i \neq j$ and $\|e_1\|_2 = 1$.

$A = Q \cdot R$

$$\therefore A = [a_1 | a_2 | \cdots | a_n]$$

Upper Δ^{er} matrix R

$$= \underbrace{[e_1 | e_2 | \cdots | e_n]}_{Q} \underbrace{\begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_2 & \cdots & a_n \cdot e_1 \\ 0 & a_2 \cdot e_2 & \cdots & a_n \cdot e_2 \\ 0 & \vdots & & \vdots \\ 0 & \vdots & & a_n \cdot e_n \end{bmatrix}}_{R}$$

$A = QR$.

(we can say $e_1 = w_1$, $e_2 = w_2$, — $e_n = w_n$ & $x_1 = a_1$, $x_2 = a_2$, ..., $x_n = a_n$) while constructing orthonormal vectors.

orthonormal = orthogonal $\langle x_1 . x_2 \rangle = 0$
+ Unit vector $\|x\| = \cancel{0}\, 1$

eg. $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\langle x_1 . x_2 \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1+1 = 0$

$\|x_1\|_2 = \sqrt{1^2+1^2} = \cancel{1}\ \sqrt{2}$

$\|x_2\|_2 = \sqrt{(-1)^2+(1)^2} = \cancel{1}\ \sqrt{2}$

Then $\{x_1, x_2\}$ is orthonormal set.

① dot product $= 0$
② Norm of each vector $= 1$. (2-norm)

QR decomposition (modified GSOP)

① $v_1 = x_1$

$\therefore w_1 = \dfrac{v_1}{\|v_1\|}$

② $v_2 = x_2 - \langle x_2, w_1 \rangle w_1$

$\therefore w_2 = \dfrac{v_2}{\|v_2\|}$

⋮

$v_n = x_n - \langle x_n, w_1 \rangle w_1 - \cdots \langle x_n, w_{n-1} \rangle w_{n-1}$

$\therefore w_n = \dfrac{v_n}{\|v_n\|}$

Then $w_1, w_2 \ldots w_n$ vectors are orthonormal. Vectors. i.e. Vectors $w_1, w_2 \ldots w_n$ are orthogonal as well as are of unit length. (i.e. norm of each vector is 1)

(norm ≡ 2-norm ≡ Euclidean distance)

(1) QR decomp (MGSOP)

Let $a_1, a_2 \ldots$ can be n linearly independent ~~column~~ vectors then

$$v_1 = a_1$$

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = a_2 - \langle a_2, e_1 \rangle e_1 \quad , \quad e_2 = \frac{v_2}{\|v_2\|}$$

$$\vdots$$

$$v_n = a_n - \langle a_n, e_1 \rangle e_1 - \cdots - \langle a_n, e_{n-1} \rangle e_{n-1}$$

$$e_n = \frac{v_n}{\|v_n\|}$$

then $e_1, e_2, \ldots e_n$ are orthonormal vectors.

Ex. 1) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ find QR decomp of matrix.

1) All columns of A linearly independent
2) Apply MGSOP.

$$u_1 = a_1 = (1, 1, 0)$$

$$e_1 = \frac{u_1}{\|u_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

classmate

$[1,0,1]\begin{bmatrix}1/\sqrt2\\1/\sqrt2\\0\end{bmatrix} = \left(\frac{1}{\sqrt2}\right)$

$u_2 = a_2 - \langle a_2, e_1\rangle e_1$

$= (1,0,1) - \frac{1}{\sqrt2}\begin{bmatrix}1/\sqrt2\\1/\sqrt2\\0\end{bmatrix}$

$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} = \begin{bmatrix}+1/2\\-1/2\\1\end{bmatrix}$

$e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3/2}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \left(\frac{1}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{2}{\sqrt6}\right)$

$(0,1,1)\begin{bmatrix}1/\sqrt2\\1/\sqrt2\\0\end{bmatrix} \Rightarrow 0+\frac{1}{\sqrt2}$

$(0,1,1)\quad -\frac{1}{\sqrt6}+\frac{2}{\sqrt6}\quad \frac{1}{\sqrt6}$

$\sqrt{\frac14+\frac14+1}\quad \sqrt{\frac{1+1+4}{4}}\quad \frac64\quad \sqrt{\frac{2\times3}{2\times2}}$

$u_3 = a_3 - \langle a_3, e_1\rangle e_1 - \langle a_3, e_2\rangle e_2$

$\sqrt{3/2}$

$= (0,1,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, 0\right) - \frac{1}{\sqrt6}\left(\frac{1}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{2}{\sqrt6}\right)$

$\frac{1}{\sqrt{3/2}}\times\frac12\quad \frac{1}{\frac{2\times3}{2}}$

$= (0,1,1) - \left(\frac12, \frac12, 0\right) + \left(-\left\{\frac16, +\frac16, -\frac26\right)\right.$

$= \cancel{(0,1,1) - \left(\frac13, \frac23, \frac{-1}{3}\right)}$

$\frac12 - \frac16\quad \frac{3-1}{6}\quad \frac{4}{6}$

$= \cancel{\frac{-1}{3}, \frac{-1}{3}},\ \left(-\frac12, \frac12, 1\right) - \left(\frac16, -\frac16, \frac13\right)$

$\frac12 \cdot \frac16\quad \frac{3+1}{6}\quad 1+\frac13\quad 1-\frac23\quad \frac{3-2}{3}$

$\Rightarrow \frac{-4}{3}, \frac13, \frac13$

$1+\frac13\quad \frac{3+1}{3}$

$$u_3 = (0,1,1) - \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) - \left(\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{2}{6}\right)$$

$$= \left(-\tfrac{1}{2}, \tfrac{1}{2}, 1\right) - \left(\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{2}{6}\right)$$

$$= \left(\tfrac{-4}{6}, \tfrac{4}{6}, \tfrac{4}{6}\right)$$

$$= (-1, 1, 1)$$

$$e_3 = \frac{u_3}{\|u_3\|} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$Q = [e_1 | e_2 | \cdots | e_n] = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \end{array}$$

$$R = \begin{bmatrix} a_1 e_1 & a_2 e_1 & a_3 e_1 \\ 0 & a_2 e_2 & a_3 e_2 \\ 0 & 0 & a_3 e_3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$a_1 = (1,1,0)$
$a_2 = (1,0,1)$
$a_3 = (0,1,1)$

$$
\begin{array}{ccc} & Q & R \end{array}
$$

$$
\text{Ans } Q \cdot R = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = A.
$$

$\frac{1}{2}+\frac{3}{6}$

$\frac{1}{2}+\frac{1}{6}-\frac{2}{3}$

$\frac{3+1-4}{6}$

$\frac{1}{2}-\frac{1}{6}+\frac{2}{3}$

$\frac{3-1+4}{6}$ $\frac{6}{6}$

$\frac{1}{6}+\frac{2}{3}$

$\frac{2+4}{6}$

A & B similar matrices ← $B = PAP^{-1}$

$A = Q\Lambda Q^{-1}$ → nothing But diagonalizat^n of mat A

classmate
Date
Page
1/10/Eve

(2) Spectral decomposition

$A_{n\times n}$ sqr matrix with 'n' linearly independent eigen vectors q_i ($i = 1, 2, \ldots n$) then A can be factorized as

$$\boxed{A = Q\Lambda Q^{-1}}$$

$Q_{n\times n}$ = contains columns as lin. indep. eigen-vectors of $A_{n\times n}$ (q_i)

Λ = diagonal matrix, whose diagonal elems are corresponding eigen-values $\Lambda_{ii} = \lambda_i$

* Spectrum of matrix ⇒ Collection of all eigen values

* Spectral radius ⇒ Largest absolute value of its eigenvalues.

If matrix is not diagonalizable ~~then~~ (defective matrix) then we can say Spectral decomp doesnt exist.

Ex.1) $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$;

$(2-\lambda)^2 - 1 = 0$

$\cancel{4 + 4\lambda + \lambda^2 - 1 = 0}$

$\cancel{\lambda^2 + 4\lambda + 3 = 0}$

$\cancel{(\lambda+3)(\lambda+1)}$

$4 + 4\lambda + \lambda^2 - 1$

$\lambda^2 + 4\lambda + 3 = 0$

$\cancel{(\lambda+3)(\lambda+1) = 0}$

$(\lambda - 2)^2 - 1 = 0$

$\lambda^2 - 4\lambda + 4 - 1$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\boxed{\lambda = 3, 1}$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$ $Ax = \lambda x$

$\lambda_1 = 1, \quad v_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u_1 = \begin{bmatrix} t \\ -t \end{bmatrix}$

$x_1 = -x_2$ or $x_2 = -x_1$

$u_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ you may have multiple eigen vectors

$\lambda_2 = 3 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$x_1 = x_2$

$u_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$U = \begin{array}{cc} u_1 & u_2 \end{array}$
$U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $U^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$S = U \Lambda U^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = S$ hence proved.

$\frac{1}{2} + \frac{3}{2}$

$\frac{4}{2}$

$-\frac{1}{2} + \frac{3}{2} \quad \frac{-1+3}{2}$

$-\frac{1}{2} + \frac{3}{2} \quad \frac{4}{2} \quad \frac{1}{2} + \frac{3}{2}$

$Sv = \lambda v$

If (λ, v) is eigen-pair of S then

$$Sv = \lambda v$$

$$\text{Lhs} = S(\alpha v) = \alpha(Sv)$$
$$= \alpha(\lambda v)$$
$$= \lambda(\alpha \cdot v)$$

i.e. if (λ, v) is eigen-pair then $(\lambda, \alpha v)$ is also eigenpair of S.

disadv of Spectral decomp ⇒ Above concept will create non-uniqueness in decomp as we can take any eigen-vector multiple of given eigen-vector

⇒ Λ is Unique But U and U^{-1} may vary

But Spectral decomp should be unique in nature

Sol^n ⇒ Instead of taking linearly indep vectors as columns of Q, take orthonormal set of vectors

* orthonormal will Reduce the problem of non-Uniq^ness

To remove uniqueness problem ⇒

In spectral decomp if we take $A_{n \times n}$ as symmetric matrix ~~then also~~ ~~Q as linearly indep eigen vectors collection~~ and orthonormal vectors as columns of Q then it resolves uniqueness problem.

$$S = U\Lambda U^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$\Downarrow$ normallizing "U"

$S = Q\Lambda Q^{-1}$ where Q is orthogonal matrix so $\boxed{Q^{-1} = Q^T}$

$Q^{-1} \equiv Q^T$

when columns of Q are orthonormal and orthogonal.

for symmetric matrix S,

$S = Q\Lambda Q^{-1} \Rightarrow S = Q\Lambda Q^T$

$$S = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{Q^T}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{2} \\ -1/\sqrt{2} & 3/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = S. \checkmark$$

* If matrix is symm → eigen vectors are linearly indep and are orthogonal in nature so just normallize them & construct Q.

* If matrix is not symm → eigen vectors are linearly independent Then convert them to orthonormal eigen Vector using MGSOP. & construct Q.

∴ To ensure uniqueness in spectral decomposit^n

$$A = Q \Lambda Q^{-1}$$

where "Q is orthonormal matrix" obtained by (LI eigenvectors + MGSOP.) where uniqueness in Q can be ensured.

Summary :-

If $S \in R^{m \times m}$ is symmetric matrix,

Thm :- There exists a (unique) spectral decompositn where Q is orthogonal.

$$S = Q \Lambda Q^T$$

1) $Q^{-1} = Q^T$

ortho-normal:
2) Columns of Q are normalized eigenvectors.
3) Columns are orthogonal.

4) everything is real

Appl^n + Matrix inverse via spectral decomposition:

$$A^{n \times n} = Q \Lambda Q^{-1}$$

Q having LI eigenvectors as columns

$$|Q| \neq 0$$

$$\cancel{Q^{-1} A = Q^{-1} Q A}$$

$A^{n \times n} = Q \Lambda Q^{-1}$

post multiply by A^{-1}

~~$A^{-1} A = Q$~~

$A \cdot A^{-1} = Q \Lambda Q^{-1} A^{-1}$

$I = Q \Lambda Q^{-1} A^{-1}$

$Q^{-1} = \Lambda Q^{-1} A^{-1}$

$\Lambda^{-1} Q^{-1} = Q^{-1} A^{-1}$

$$\boxed{Q \Lambda^{-1} Q^{-1} = A^{-1}}$$

Appl^n of Spectral decomp.

Spectral decomp

① It is not Unique ⟶ Sol^n Use symmetric matrix

② If eigen values are repeatating then NO LI

Evector's ⇒ So SD may not exist

$U^*AU = T$ (upper Δ^{er} matrix) & diagonals are eigenvalues of A. complex

classmate Date Page

$U^* = U^{-1}$

3-10 Monday

$U^*U = U \cdot U^* = I$

③ Schur Decomposition :-

If $A \in \mathbb{C}^{n \times n}$ then there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

Transpose conjugate of U ← $(\overline{U^T})$ or $(\overline{U})^T$

$$\boxed{\underset{n\times n}{U^*}\,\underset{n\times n}{A}\,\underset{n\times n}{U} = \underset{n\times n}{T}} = \Lambda + T' \text{ (strictly upper } \Delta^{er})$$

$$\begin{bmatrix} & \neq 0 \\ 0 & \end{bmatrix}$$

is upper Δ^{er}.

The diagonal elems of T are the eigenvalues of A.

For real matrices.

$A \cdot A^T = I$.

If $A \in \mathbb{R}^{n \times n}$ then there exists orthogonal matrix $U \in \mathbb{R}^{n \times n}$ s.t.

$$\boxed{\underset{n\times n}{U^T}\,\underset{n\times n}{A}\,\underset{n\times n}{U} = \underset{n\times n}{T}} \text{ is upper } \Delta^{er}.$$

Algorithm

1. find Evalues of A (λ_i)
2. Select one of the Evalue randomly & find corresp. Evector. eg. for (λ_1, v_1) find
3. Normalize obtained Evector $\hat{v}_1 = \frac{v_1}{\|v_1\|}$
   rather than choosing another λ_i to get v_i
4. Find another Evector that belongs to the perp (ω)
5. Repeat ④ till you get desired vectors. (Repeat $(n-1)$ times)
6. Construct U by stacking all eigenvectors
7. find $T = U^T A U$

perp $\Rightarrow$ orthogonal or $\perp$er to remaining vectors.

step 4 $\rightarrow$ 1 $\quad W = \{\hat{v}_1\} \Rightarrow \hat{v}_2$ $\quad$ $\hat{v}_2$ must be $\perp$er or orthogonal to $\hat{v}_1$

4 $\rightarrow$ 2 $\quad W = \{\hat{v}_1, \hat{v}_2\} \Rightarrow \hat{v}_3$

$\hat{v}_3$ must be $\perp$er or orthogonal to both $\hat{v}_1$ and $\hat{v}_2$.

i.e. $\langle \hat{v}_1, \hat{v}_2 \rangle = 0$

$\langle \hat{v}_1, \hat{v}_3 \rangle = 0$

$\langle \hat{v}_2, \hat{v}_3 \rangle = 0$.

$\vdots$

Repeat until you get 'n' vectors.

Perp ($\perp$er complement) :-

Let $W \subseteq \mathbb{R}^k$ where W is non-empty. Then perp is collection of all vectors in $\mathbb{R}^k$, which are orthogonal to all vectors of W.

$$W^\perp = \{ x \in \mathbb{R}^k / \langle x, y \rangle = 0 \ , \forall y \in W \}.$$

$W = \{ v_1, v_2, v_3, \ldots v_n \}$

$v \in W^\perp$ iff $\langle v, v_i \rangle = 0 \ , \forall i = 1, 2, \ldots n.$

Ex.1) find perp corresp. to: $W = \{ (1,1,1)\ (1,2,3) \}$

1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

2) Convert A into Echelon form

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & +1 & 2 \end{bmatrix}$

~~3)~~ * If all vectors of W are LD then
$W^{\perp}$ = perp of single vector

check

3) Solve for $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0$$
$$x_2 = -2x_3$$

$$x_1 - 2x_3 + x_3 = 0$$
$$x_1 = x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ t \neq 0$$

OR

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ x_3 = t$$

Date ______
Page ______

basis of $W^{\perp} = \{(1, -2, 1)\}$

now check whether given vector is $\perp^{er}$ to vectors $(1,1,1)$ and $(1,2,3)$

$v_1 = (1,1,1)$
$v_2 = (1,2,3)$
$v_3 = (1,-2,1)$

$\langle v_1, v_3 \rangle = (1,1,1)(1,-2,1) = 1-2+1 = 0$ ✓

$\langle v_2, v_3 \rangle = (1,2,3)(1,-2,1) = 1-4+3 = 0$ ✓

$\therefore$ v_3 is $\perp^{er}$ to both v_1 and v_2.

3/10 - Eve

Ex.17 find Schur decomp $A = \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}$

→ ① $|A - \lambda I| = 0$

$(7-\lambda)(-3-\lambda) + 24 = 0$
$-21 - 7\lambda + 3\lambda + \lambda^2 + 24 = 0$
$\lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 3)(\lambda - 1) = 0$ $\quad \therefore \lambda = 1, 3$.

Lets take maximum eigenvalue $\lambda = 3$ & find Evector

② $\lambda = 3$. $\quad (A - \lambda I)x = 0$

$$\begin{bmatrix} 4 & -2 \\ 12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Date
Page

$R_2 \leftarrow R_2 + 3R_1$

$$\begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}, t \neq 0.$$

③

do this only if ~~$\|v\|_1 \neq 1$~~ here $\|v\|_1 = 5$ it is not unit vector.

$$\hat{v} = \frac{v}{\|v\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$ Unit Eigen Vector ✓

$$\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = 1 ✓$$

$$\therefore W_1 = \left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

④ find another vector $V_2 \in W_1^\perp$. (which belongs to perp of $W_1^\perp$)

Construct a matrix B_1 & const homo sys of eqⁿ

$B_1 V_2 = 0$

$B1 \Rightarrow$ construct using all elems of W_1

$$\cancel{\tfrac{1}{\sqrt{5}}}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\cancel{\sqrt{5}}} = -\frac{2}{\cancel{\sqrt{5}}}x_2$$

$$x_1 = -2x_2$$

$$x_1 = -2x_2$$

$$V_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}, t \neq 0$$

classmate

Date ____
Page ____

Basis of $\hat{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Not Unit Vector so normalize

Basis of $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\quad \hat{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

matrix

for 3×3

V3

$B_2 = [V_1 \; V_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\;]$

$\omega_1^{\perp} = \{\hat{v}_1, \hat{v}_2\}$

and $\langle \hat{v}_1, \hat{v}_2 \rangle = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = 0$ ✓

(8) construct U by stacking $\hat{v}_1$ and $\hat{v}_2$ vectors

$B3 = [V_1 \; V_2 \; V_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\cancel{U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}$

$\quad\quad v_1 \quad v_2$

$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(7) $T = \cancel{U^{-1} A U} = U^T A U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix} [U]$

$= \frac{1}{\sqrt{5}} \begin{bmatrix} 31 & -8 \\ -2 & +1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 15 & -70 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -14 \\ 0 & 1 \end{bmatrix} = T$ (upper Δer mat) ✓

$\lambda_1 = 3, \; \lambda_2 = 1$

31
28
-2 -12
-14+12
4-6

Schur decomposition continued..

Schur decomp is not unique, as we took $\lambda = 3$, now if we take $\lambda = 1$ then will get diff eigen vector & hence diff matrix U.

→ Schur decomp is unique for a given choice of λ.

① Uniqueness - But for all values of λ, it is not unique.

(In case of choosing v_2, of course there we can take any vector but as we always take vector having min length so this problem is solved)

i.e. length of vector which we construct as a perp should be minimum)

② Uniqueness wrt perp basis → Take basis of perp $w_1^{\perp}$ s.t. length of vector is minimum.

So with pts ① & ② matrix U is unique.

① Choose Absolute Max^m eigen Value

② Choose basis having minimum length.

To ensure unique Schur decomposition.

If AM ≥ 1 then try to choose λ having

to get unique eigenvector → AM = 1 & If its not psbl then choose either of eigen values.

We want Unique decomp for
Single value decomⁿ → Rectangular (Non-sqʳ mat)
→ for singular matrix $|A| = 0$ — 5th Oct_Mon
or non-inv matrix

Singular Value Decomposition: (SVD)

Let A be $n \times d$ matrix then

Matrix representⁿ wrt SVD → $A = U \Sigma V^T \Rightarrow A = \sum_{i=1}^{r} \sigma_i . u_i v_i^T$

Vector notation of SVD.

U: Left SV's matrix
Σ: singular values matrix.
VT: Right singular vectors matrix.

Σ is diagonal matrix $\begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix} = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \ \sigma_d \end{bmatrix}$.

$\left.\begin{array}{l} V \cdot V^T = I \\ U \cdot U^T = I \end{array}\right\}$ Both U & V are orthogonal matrix.

A: ~~ma~~

$$[u_1 \ u_2 \ldots u_n] \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \ \sigma_d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}$$

① $\underset{n\times d}{A} = U \Sigma V^T$

$A_{n\times d} = U_{n\times n}\, \Sigma_{n\times d}\, V^T_{d\times d}$

✓ Most common dimension Used i.e. Make U and VT as perfect Square in nature (of size n×n) & make Σ as rectangular diagonal matrix.

② $\underset{n\times d}{A} = U \Sigma V^T$

$A_{n\times d} = U_{n\times d}\, \Sigma_{d\times d}\, V^T_{d\times d}$

$A_{n\times d}$ $\begin{cases} n = \text{no of rows in A} & \Rightarrow U_{n\times n} \\ d = \text{no of columns in A} & \Rightarrow V^T_{d\times d} \end{cases}$ } make these square as matrices.

$\Sigma_{n\times d}$ = Rectangular diagonal matrix

U and V are orthogonal square matrices.

① **Full SVD** : (Pink + Blue)

$$A = \underset{(n \times n)}{U} \; \underset{(n \times d)}{\Sigma} \; \underset{(d \times d)}{V^T}$$

② **Reduced SVD**:- (Pink region only)

$$A = \underset{(n \times r)}{U} \; \underset{(r \times r)}{\Sigma} \; \underset{(r \times d)}{V^T}$$

|| $r = \min(n, d)$
rank

$A_{n \times d} \iff A^T_{d \times n}$

$X = (A \cdot A^T)_{n \times n}$
$(n \times d)\,(d \times n)$

As Size of U is $n \times n$ so we can construct $U_{n \times n}$ using $(A \cdot A^T)_{n \times n}$

$Y = (A^T \cdot A)_{d \times d}$
$(d \times n)\,(n \times d)$

So given a matrix $(A^T \cdot A)_{d \times d}$ we can construct matrix to $V_{d \times d}$.

Date
Page

(1) $A \cdot A^T$ is always Symmetric matrix

(2) $A^T \cdot A$ is always symmetric matrix

eg $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times1} \begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times2}$$

Symmetric

for Symm matrix

(1) λ_i's are real

(2) v_i's are orthogonal (eigen vectors)

We can create set of orthonormal set from orthogonal vectors.

for U ⇒

(1) find spectral decom of $A \cdot A^T$
(λ_i, u_i)

Σ ⇒ $\begin{bmatrix} \ddots & \lambda_i & 0 \\ 0 & & \ddots \end{bmatrix}$

$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$

for $V \Rightarrow$ find spectral decom of $A^T . A$

(μ_i, v_i)

$$\Sigma = \begin{bmatrix} \ddots & & 0 \\ & \mu_i & \\ 0 & & \ddots \end{bmatrix}$$

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_d \\ | & | & & | \end{bmatrix}$$

* $A . A^T$ and $A^T . A$ have same eigen Values and hence $\lambda_i = \mu_i$

i.e. Σ will be same.

classmate

Date
Page

$$SVD \equiv \left(\overset{V^T}{\text{Rotation}} + \overset{\Sigma}{\text{Scaling}} + \overset{U}{\text{Rotation}}\right)$$

U and V are orthogonal matrices, & columns are orthonormal then

This interpretation can be extended to higher dimension as well.

1) A^TA and $A.A^T$ have same eigenvalues:

$B = A^T . A$ $\quad$ $(\cancel{(A^TA)^T = AA^T})$

$B^T = A . A^T$

* B and B^T have same eigenvalues.

<u>proof</u> ① λ is B's E.V., μ is B^Ts eigen values
② B and B^T have same characteristic eq^n.

$|B - \lambda I| = |B - \lambda I|^T$

$\because \det(A) = \det(A^T)$

$(A-B)^T = A^T - B^T$
$(AB)^T = B^T A^T$

$$\det(B - \lambda I) = \det\{(B - \lambda I)^T\}$$
$$= \det\{B^T - \lambda^T I^T\}$$
$$= \det\{B^T - \lambda I\}$$
$$= \det$$

① B and B^T both have same CE (char eq^n)
②
③ A^TA and AA^T — same EV's
" same EVs

$\det(\lambda I - A^T A)$

$= \det(\lambda I - (A^T A)^T$

7th Oct: M

(λ, v) Eigen pair of $A^T A$ then.

$(A^T A) v = \lambda v.$

$A(A^T A)v = A(\lambda v)$

$\underline{(AA^T)}\underbrace{(Av)}_{w} = \lambda(\underbrace{Av}_{w})$

$(A \cdot A^T) w = \lambda \cdot w \qquad w = Av.$

$\therefore (\lambda, w)$ is eigen pair of $A \cdot A^T$

hence proved * $\therefore \lambda$ is the eigenvalue of both AA^T and $A^T A$

Date ______
Page ______

Algo to find SVD :-

1. Construct either AA^T or A^TA
2. Find eigenvalues and eigenvectors of AA^T
3. Construct Σ from Evalues. (Σ)
4. Construct U from Evectors (V)
5. $A = U\Sigma V^T$ to construct (V)

Note:- A^TA or AA^T,
U, Σ, V are going to be unique.

eg. $A = [\quad]_{2\times3}$ $\quad A^T = [\quad]_{3\times2}$

$A \cdot A^T$ and A^TA
2×2 $\quad 3\times3$

($\lambda_1, \lambda_2, 0$ 3rd eigen value)

* For Rectangular matrix, choose A^TA or AA^T depend on whose dim is less.

Ex.1) Find SVD of $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

→ $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad A \cdot A^T$ or A^TA same dim so choose any of them

1. $A \cdot A^T = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}_{2\times2}$

$\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$|A^TA| = 0$ so one of the eigen value $= 0$.

$\lambda_1 = 0 \quad , \quad \lambda_2 = 10$

(2) $\begin{bmatrix} 2-\lambda & 4 \\ 4 & (8-\lambda) \end{bmatrix}$

$(2-\lambda)(8-\lambda) - 16 = 0.$

$\cancel{16} - 2\lambda - 8\lambda + \lambda^2 - \cancel{16} = 0$

$\lambda^2 - 10\lambda = 0$

$\lambda(\lambda - 10) = 0$

$\lambda = 0, \ \lambda = 10$

$\lambda_1 = 0$

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = -2x$

$$v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \cancel{\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$\lambda_2 = 10$

$$\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = 2x_1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

max ε.value

$\lambda_2 = 10_2$ $\lambda = 0_2$

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

v_1 v_2

$V \cdot V^T = I$ orthogonal verify ·cz both U & V are orthogonal matrices

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ✓$$

σ_1

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

σ_2

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}_{2\times 2}$$

$A = U\Sigma V^T$

$U^T A = U^T U \Sigma V^T$

$U^T A = I \Sigma V^T$

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$A = U\Sigma V^T$

$A^T A = (U\Sigma V^T)^T (U\Sigma V^T)$

$= (V\Sigma^T \underbrace{U^T)(U}_{I} \Sigma V^T)$

$= V\Sigma^T I \Sigma V^T$

$= V\Sigma^2 V^T$

$\lambda = \Sigma^2$

$\therefore \Sigma = \sqrt{\lambda}$

$$\boxed{\begin{aligned} \lambda_i &= \sigma_i^2 \\ \sigma_i &= \sqrt{\lambda_i} \end{aligned}}$$

Σ is singular value of A

$A = U\Sigma V^T$

$u_i = \frac{1}{\sigma_i} A v_i$

for non-zero σ_i

inverse of diagonal matrix?

$A = U\Sigma V^T$

$AV = U\Sigma \underbrace{V^T V}_{I}$

$AV = U\Sigma$

$(AV)\Sigma^{-1} = U \Rightarrow u_i = \frac{1}{\sigma_i}(Av_i)$

when $\sigma_i = 0$ we can't use this method

what to do when $\sigma_i = 0$

$u_1 = \frac{1}{\sigma_1}(Av_1) = \frac{1}{\sqrt{10}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}$

$= \frac{1}{\sqrt{10}} \begin{bmatrix} 5/\sqrt{5} \\ 5/\sqrt{5} \end{bmatrix}$

$\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} \quad \frac{5}{}$

$\Rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\sqrt{5}\times\sqrt{5}\times\sqrt{2}$

$\cancel{\sqrt{5}}\times\cancel{\sqrt{5}}\cancel{\sqrt{2}}$

$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$\quad u_1 \quad u_2$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$-\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \quad 0$

$A^T u_3 = 0$

$A_{5\times 2} = \underset{5\times 5}{U} \underset{5\times 2}{\Sigma} \underset{2\times 5}{V^T}$

Ex. 2) find SVD of $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$

1. $A^TA = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\sigma_1 = \sqrt{3} \qquad \sigma_2 = 1$

2. $|A^TA - \lambda I| = 0 \Rightarrow \lambda_1 = 3, \; \lambda_2 = 1$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)^2 - 1 = 0$

$\lambda^2 - 4\lambda + 4 - 1 = 0$ 1

$\cancel{(\lambda-3)}(\lambda-3)(\lambda-1) = 0$

3. $\lambda_1 = 3 \quad (A^TA - \lambda I)v_1 = 0 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \hat{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Date
Page

(2) $\lambda_2 = 1$

$(A^TA - \lambda I)v_2 = 0$ $\quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

(3) $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & \cancel{0} \\ 0 & 1 & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$ No need to add. $\Rightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3×2 ✓

$$\begin{array}{l} V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2\times 2} \\ \qquad\ \ v_1 \qquad\quad v_2 \end{array}$$

(1) full SVD.

$A = U\Sigma V^T$

3×2 $\quad 3\times 2 \quad 2\times 2 \quad 2\times 2$

$\cancel{3\times 2}$ 3×2

full SVD

$A = \underset{3\times 3}{U} \ \underset{3\times 2}{\Sigma} \ \underset{2\times 2}{V^T}$

Rectangular diagonal matrix

(4) $\boxed{u_i = \frac{1}{\sigma_i} A v_i}$

$$u_1 = \frac{1}{\sqrt{3}} \underset{3\times 2}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}} \overset{v_1}{\underset{2\times 1}{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$u_2 = \frac{1}{1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

from v_1 v_2 ?

$$U = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$$

$A = U \Sigma V^T$

$A^T = V \Sigma^T U^T$

$\boxed{A^T = V \Sigma U^T}$ to get final vector u_3

Utilize A^T to find U^T & then get U → Next lect Friday →

Solve for reduced SVD ⇒ HoWo

$A^T u_j = 0$ $\langle u_i, u_j \rangle = 0$ for full SVD.

$A^T u_3 = 0$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = 0$$

$-u_{31}$ $u_{31} = u_{33}$

$u_{32} = u_{33}$.

$$\begin{bmatrix} u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{u_3}, \quad t \neq 0$$

Date
Page

$$u_3 = \frac{u_3}{||u_3||} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$U = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$$

$$U = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

check $\boxed{U \cdot U^T = I}$

(rows < colums)

$A \Rightarrow m < n$

$$A = U \Sigma V^T$$

$$= \underbrace{\begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}}_{3\times3} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{3\times2} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{2\times2}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{2} & 0 \end{bmatrix} \Leftarrow \underbrace{\begin{bmatrix} -\sqrt{3}/\sqrt{6} & 1/\sqrt{2} \\ -\sqrt{3}/\sqrt{6} & -1/\sqrt{2} \\ 2\sqrt{3}/\sqrt{6} & 0 \end{bmatrix}}_{3\times2} \times \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{2\times2}$$

$\sqrt{6} = \sqrt{2} \times \sqrt{3}$

$$\cancel{\times 1/\sqrt{2}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} = A$$

$$\frac{-\cancel{\sqrt{3}}}{\cancel{\sqrt{3}} \times \sqrt{2} \times \sqrt{2}} \qquad \frac{-1}{\sqrt{4}} \quad -\frac{1}{2}$$

$$A = UP$$

$$= (U_A V_A^T)(V_A \Sigma_A V_A^T)$$

for 4X2 ⇒ same eqⁿ will give more than one solutions ✓

~~more unknowns, less eqns~~ ✓

Note

write

Reduced SVD

Full SVD

a) $A_{m\times n}$, $m>n$

a) $A_{m\times n}$, $m>n$

$$\underset{m\times n}{A} = \underset{m\times n}{U}\ \underset{n\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

2) $A_{m\times n}$, $m<n$?

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

Date
Page

* Polar Decomposition :-

A matrix $A \in \underset{\mathbb{R}^{m\times n}}{\underset{or}{\mathbb{C}^{m\times n}}}$ $\exists$ $U \in \underbrace{\mathbb{C}^{m\times n}}_{\mathbb{R}^{m\times n}}$ with orthonormal columns and P such that

orthogonal matrix

any matrix

$$\underset{m\times n}{A} = \underset{m\times n}{U}\,\underset{n\times n}{P}.$$

To uniquely find PD, we use SVD.

Proof :- Let us consider SVD of matrix A as

$$A = U_A \Sigma_A V_A^T$$

$$= U_A I \Sigma_A V_A^T$$

$$= (U_A V_A^T)(V_A \Sigma_A V_A^T)$$

U_A and V_A are orthogonal in nature

$U_A \cdot U_A^T = I$

$V_A \cdot V_A^T = I$

orthogonal matrix

Spectral decomposition

$\boxed{A = UP}$ unique ∵ SVD is unique.

$U_A V_A^T$ = multiplicatn of 2 orthogonal matrices is orthogonal matrix

$U_A \cdot V_A^T = U$ is orthogonal.

↓

$U \cdot U^T = I$

$U_A \cdot \underbrace{V_A^T \cdot V_A}_{I} \cdot U_A^T$

$U_A \cdot U_A^T = I$

$I = I$ ✓

Moore - Penrose Pseudoinverse :-

Given $\underline{m \times n}$ matrix A, the pseudoinverse is defined as $(A^+)_{n \times m}$

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^T = AA^+$
4. $(A^+A)^T = A^+A$

eg

$$A_{m \times n} \begin{cases} m = n \Rightarrow A^{-1} \text{ (Inv)} \\ m \neq n \Rightarrow A^+ \text{ (P Inv)} \end{cases}$$

check → for ~~non~~ singular matrix $|A| = 0$ calculate A^+

1. $A \quad A^+ \quad A = A$

$$\underset{m \times n}{A} \; \underset{n \times m}{A^+} \; \underset{m \times n}{A} = \underset{m \times n}{A}$$

∴ If $A_{\underline{m \times n}}$ then $A^+_{\underline{n \times m}}$

2. $A^+ \quad A \quad A^+ = A^+$

$$\underset{n \times m}{A^+} \; \underset{m \times n}{A} \; \underset{n \times m}{A^+} = \underset{n \times m}{A^+}$$

* Pseudo-Inverse in terms of SVD :-

Let $A_{m \times n}$ then $\exists$ U, Σ, V such that

$$A = \underbrace{U \Sigma V^T}_{\text{replacing all by respective inverses}} \quad \text{——— (1)}$$

SVD defn.

Taking Inv on both sides

$$A^+ = (U \Sigma V^T)^{-1}$$

$(AB)^{-1} = B^{-1} A^{-1}$

$$= (V^T)^{-1} \Sigma^{-1} U^{-1}$$

$U \cdot U^T = I$

$U^T = U^{-1}$

also

$V \cdot V^T = I$

$V^T = V^{-1}$

$\underbrace{(V^T)^{-1}}_{V} = (V^T)^T = V$

$$\boxed{A^+ = V \Sigma^{-1} U^T} \quad \text{——— (2)}$$

$$A^+ = V \left(\begin{array}{c|c} S^{-1} & 0 \\ \hline 0 & 0 \end{array} \right) U^T \quad \text{——— (3)}$$

S : diagonal matrix.

S^{-1} : Take reciprocal of non-zero diagonal elements.

$= \dots$

$= n \times m$

Notes :-

① $\dim(A^+)_{n\times m} = \dim(A^T)_{n\times m}$.

② If A is sqr invertible matrix then
$A^+ = A^{-1}$

③ If A is a full column rank matrix, then A^+ is called left inverse of A. $\boxed{A^+A = I_{n\times n}}$

④ If A is a full row rank matrix, then A^+ is called Right inverse of A.

$$\boxed{AA^+ = I_{m\times m}}$$

⑤ Types of Inverses ⇒

1) Inverse :- For a matrix A, ∃ A^{-1} such that

$$A \cdot A^{-1} = I = A^{-1}A$$
(A is square & non-singular) i.e. A is invertible

2) Generalized inverse :- A matrix 'A^+' is called generalized inverse if it satisfies

$$AA^+A = A$$

3) Reflexive Generalized inverse :- Matrix A^+ is called RGI if

1) $AA^+A = A$

2) $A^+AA^+ = A^+$

4) Pseudo inverse :- A^+ is PI if it satisfies all four conditions.

Ex. 1) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1/6 & -1/6 \\ -1/6 & 1/6 \\ 1/6 & -1/6 \end{bmatrix}$, $B_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $(B3)_{2\times2}$ is neither GI, RGI nor PI as it does'nt satisfy dim cond^n.

① $B_1 \Rightarrow$

① If $A B_1 A = A$ then B_1 is GI.

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}_{2\times3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}_{3\times2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}_{2\times3}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}_{2\times2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}_{2\times3}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} = A$$

$\therefore$ B_1 is GI

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \neq B_1$$

② $B_1 A B_1 = B_1$ then B_1 is RGI

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

→ B_1 is only GI and not RGI and PI

② $B_2 \Rightarrow$

1) $A B_2 A = A$

2) $B_2 A B_2 = B_2$.

B_2 is GI as well as RGI. Now check for PI.

3) $(A B_2)^T = A B_2$ $\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

GI, RGI also PI

4) $(B_2 A)^T = B_2 A \quad \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

Ex. 2) H.W.

* Generalized Inv (GI) and RGI} are not unique

* GI, RGI and PI are not unique

But if we obtain these inverses via SVD $\Big\downarrow$ SVD

(6) To get unified SVD.

then using SVD, we can get RGI / GI / PI are going to be unique.

Sol^n to $AX = Y$ Gaussian eliminatn Method

Ex.1) $3x_1 - 2x_2 = 4$

$6x_2 + 4x_2 = -8$ 2eq^n, 2 unknows

$$\Rightarrow \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

Augmented matrix.

$$\left(\begin{array}{cc:c} 3 & -2 & 4 \\ 6 & 4 & -8 \end{array}\right) \Rightarrow \left(\begin{array}{cc:c} 3 & -2 & 4 \\ 0 & 8 & -16 \end{array}\right)$$

4-

8-8

$$\left(\begin{array}{cc:c} 3 & -2 & 4 \\ 0 & 0 & 0 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$8 + 4(-2)$ $-16 + 16$

$$3x_1 - 2x_2 = 4$$

$Ax = y$ can be categorized into

I) Consistent System ⇒ System has a solution

$\text{rank}(A) = \text{rank}(A \vdots y)$

II) Inconsistent System ⇒ sys has No Sol^n..

$\text{rank}(A) < \text{rank}(A|y)$

Note 1: All real life problems will lead to an overdetermined system

eq^{n}s > unknows (para)

When ~~eq~~ sys has more eq^{n}s & less unknowns.

Note2: Every overdetermined system is inconsistent

Almost

$$x_2 \Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \underbrace{-1}_{\text{Eigen Value}} \overset{x_2}{\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{Eigen Vector}}} = (-1)\, x_2$$

$$\therefore \quad A x_2 = \lambda x_2 .$$

$$A x_2 = (-1)\, x_2 .$$

Both x_1 & x_2 are eigen Vectors $\Big\{ \therefore$ For Vector x_2, $\lambda_2 = -1$
For Vector x_1, $\lambda_1 = 2$.

Thm 1: If A is $n \times n$ matrix with eigen value λ, then the set of all linearly independent eigen vectors of λ is called eigen space of λ.

1) If x_1 and x_2 are eigenvectors corresponding (linearly independe) to λ then $\underbrace{E\lambda = \{x_1, x_2\}}_{\text{Eigen Space}}$

i.e. $(Ax_1 = \lambda x_1, \quad Ax_2 = \lambda x_2)$

2) Dimension of the E·space is no. of elements in $E\lambda$

* eigenpair $\Rightarrow$ eigenvalues (eigenvector, eigenvalue)

$\Downarrow$

(λ, v)

Algebraic Multiplicity : No. of eigen Values. (Roots of ch eqn)

Geometric Multiplicity : dimension of eigen Space

$$GM \leq AM.$$

If $GM < AM$: defective matrix (prev example)

If $GM = 1$ then $AM = 1$.

Anxn is defective iff it does not have 'n' linearly independent eigenvectors.

* It does not have a complete basis of eigenvectors.

eg.6)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

M1 ⇒ $R_3 \leftarrow R_3 - R_1$

$R_4 \leftarrow R_4 - R_1$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Eigen values ⇒ 1, 1, 2, 3.

M2 ⇒ $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 5 & -10 \\ 1 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{vmatrix}$$

$$(1-\lambda)^2 (2-\lambda)(3-\lambda) = 0$$

Q.1) $A_{2\times 2}$ with $Tr(A) = 3$, $\det(A) = 2$.

→ $\lambda_1 + \lambda_2 = 3$

$\lambda_1 \lambda_2 = 2$.

$\lambda_1 = 2$, $\lambda_2 = 1$.

Diagonalization :-

A sq^r matrix A is diagonalizable if there exists invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(P diagonalizes A)

If $B = P^{-1}AP$ then A & B are similar matrices.

* P has to be obt from eigen vectors

Thm 4: Similar matrices have same Eigen values.

eg 1) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ $\lambda = -2, 4$

$\lambda_1 = 4$, $\lambda_2 = -2$, $\lambda_3 = -2$

Eigen vector wrt $\lambda_1 = 4$ = $P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector wrt $\lambda_2 = -2$ = $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (P_2, P_3)

Thm 5: $A_{n\times n}$ is diagonalizable if it has 'n' linearly independent eigenvectors.

Thm 6: (Aliter) If $A_{n\times n}$ has 'n' distinct eigen values, then the corresponding eigen vectors are linearly independent & A is diagonalizable.

(Sufficient condn, not necessary)

for a $A_{n\times n}$, If eigen values non-distinct - check for all eigen vectors

& if you get 3 linearly independent eigen vectors then $A_{n\times n}$ is diagonalizable.

& if matrix doesn't have 'n' linearly independent vectors then $A_{n\times n}$ is not diagonalizable.

eg 4)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$ check whether diagonalizable or not

$\rightarrow$ $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

$\Leftrightarrow \lambda_1 = 1$

Eigen vector =

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$

variable free

eg. 7) $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$(-3-\lambda)[(-\lambda)(1-\lambda)] = 0.$

$(-3-\lambda)(-\lambda+\lambda^2) = 0.$

$-3\times'$

$3\lambda - 3\lambda^2 + \lambda^2 - \lambda^3 = 0$

$3\lambda - 2\lambda^2 - \lambda^3 = 0$

$\lambda^3 + 2\lambda^2 - 3\lambda = 0$

$\lambda(\lambda^2 + 2\lambda - 3) = 0$

$\lambda(\lambda+3)(\lambda-1) = 0$

Upper Δer ∴ Eigen Values $\Longrightarrow \lambda = 0, -3, +1$

As Eigen Values are distinct, it is diagonalizable.

① Symm matrix $\Rightarrow$ $\boxed{A = A^T}$

Thm 7: Eigen values of Symm matrices

$A_{n\times n}$ is symm matrix then

① A is diagonalizable

② All eigen values of A are real.

③ If λ is eigenvalue of A with multiplicity K (AM = K) then λ has K linearly independent Eigen Vectors.

i.e. Eigenspace of λ has dimension K.

Date
Page

√Imp

Thm 8 ⇒ $P_{n\times n}$ is orthogonal matrix if and only if its column vectors forms an orthogonal set.

$$\underset{n\times n}{P} = [P_1 \mid P_2 \mid P_3 \mid \cdots \mid P_n]$$

$$\boxed{P_i \cdot P_j = 0 \quad \forall\, i,j \text{ except } (i=j)}$$

eg 1) $P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix}$ check whether P is orthogonal or not?

→ If $P \cdot P^T = I$ then P is orthogonal.

$$P \cdot P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/3 & -2/\sqrt{5} & -2/3\sqrt{5} \\ 2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}$$

$-\frac{2}{15} + \frac{2}{3\sqrt{5}}$

$\frac{1}{3}$ $\frac{1}{3}+\frac{2}{2}+\frac{2}{3}$ $\frac{4}{9}+\frac{4}{9}$ $\frac{5}{3}$ $\frac{1}{9}+$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\frac{1}{3} \times -\frac{2}{5} + \frac{2}{3} \times \frac{1}{\sqrt{5}}$

$\frac{2}{3}\left[-\frac{1}{5} + \frac{1}{\sqrt{5}}\right]$ $\frac{2}{3}$

Let $P_1 = \begin{bmatrix} 1/3 \\ -2/\sqrt{5} \\ -2/3\sqrt{5} \end{bmatrix}$, $P_2 = \begin{bmatrix} 2/3 \\ 1/\sqrt{5} \\ -4/3\sqrt{5} \end{bmatrix}$, $P_3 = \begin{bmatrix} 2/3 \\ 0 \\ 5/3\sqrt{5} \end{bmatrix}$

$$P_1 . P_2 = \cancel{\begin{bmatrix} 1/3 \\ -2/\sqrt{5} \\ -2/3\sqrt{5} \end{bmatrix}_{3\times 1}} \cancel{\begin{bmatrix} 2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \end{bmatrix}_{1\times 3}}$$

$$= \begin{bmatrix} 1/3 & -2/\sqrt{5} & -2/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/\sqrt{5} \\ -4/3\sqrt{5} \end{bmatrix}$$

$$= \frac{2}{9} - \frac{2}{5} + \frac{8}{9\times 5} \Rightarrow \frac{10-18+8}{45} \Rightarrow 0.$$

$\therefore P_1 . P_2 = P_2 . P_3 = P_1 . P_3 = 0$ dot product betn P_1, P_2, P_3 is 0.

length of P_1, P_2, P_3 is 1 $\quad ||P_1|| = ||P_2|| = ||P_3|| = 1.$

$\{P_1, P_2, P_3\}$ is an ~~orthogonal~~ orthonormal set. length of each vector = 1

$A = A^T$

Thm 9: $A_{n\times n}$ symm matrix and if λ_1 and λ_2 are distinct eigenvalues of A, then the corresponding eigenvectors x_1 and x_2 are orthogonal.

i.e. $x_1 . x_2 = 0$

$$||x_1|| = ||x_2|| = 1$$

Thm 10 :- Fundamental Thm of symmetric matrices.

Let $A_{n \times n}$ then A is orthogonally diagonalizable and has real eigenvalues iff A is Symmetric.

* Orthogonal diagonalizatn of symm matrix (A) :-

Let $A_{n \times n}$ matrix,

① Find all eigen values & determine algebraic multiplicity of each.

② If $AM = 1$ i.e. all eigenvalues are distinct then find "unit eigenvector" $u_1 = \frac{v_1}{\|v_1\|}$

③ If $AM \geq 2$, find set of K-linearly independ eigen vectors.
If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

④ Composite of steps ② & ③ produces an orthonormal set of n eigenvectors.

Use eigen vectors to form columns of P.

$$P^{-1}AP = P^{T}AP = D$$

If $A_{n \times n}$ Symm $\longrightarrow$ then it is orthogonally diagonalizable.

30th morning 7

Vector Norm :-

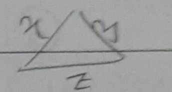
A norm is a f^n $\|\cdot\| : \mathbb{R}^m \rightarrow \mathbb{R}^+$ that assigns a real-valued length to each vector.

For all vectors x and y and for all scalars $a \in \mathbb{R}$

A norm must satisfy foll^n 3 conditions

① $\|x\| > 0$ and $\|x\| = 0$ only if $x = 0$ → Positivity cond^n.

② $\|x+y\| \leq \|x\| + \|y\|$, Δ^{er} inequality cond^n



$Z \leq x+y$

③ $\|ax\| = |a|\,\|x\|$, Scaling cond^n

* Popular Vector norms :-

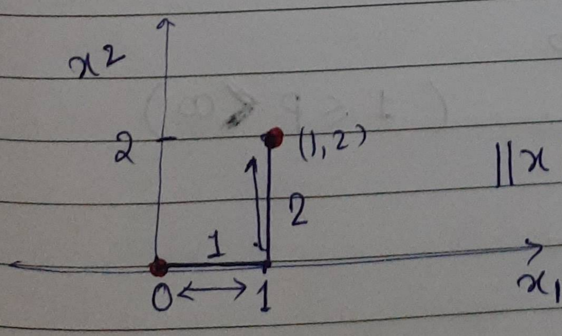
① $\|x\|_1 = \sum_{i=1}^{m} |x_i|$ 1-norm / Manhattan distance / or norm.

one-norm

② eg $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \|x\| = |x_1| + |x_2|$

$x \in \mathbb{R}^2, \; x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

One norm will travel horizontal & then Vertical direct^n.



$\|x\| = |1| + |2| = 3.$

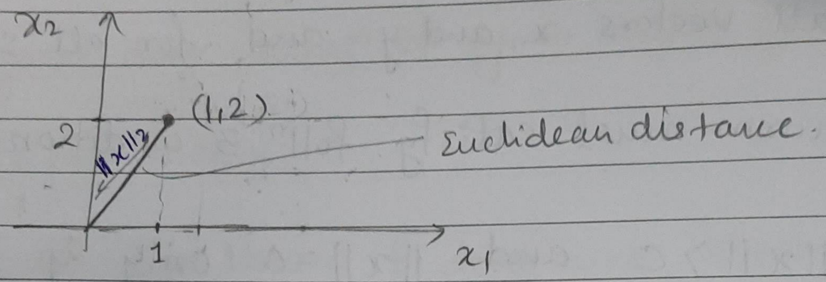
∴ one-norm is distance bet^n (0,0) and (1,2)

Two-Norm:

(2) $$\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \sqrt{x^* x}$$

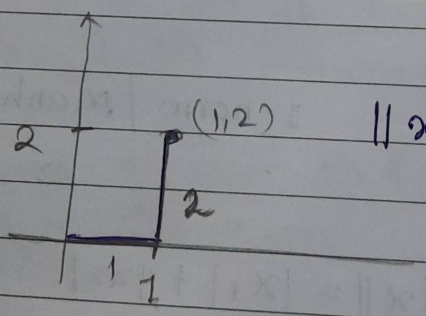
norm of a vector

2-norm or Euclidean norm/ distance.



(3) Infinity norm/ Max-norm

$$\|x\|_\infty = \max_{1 \le i \le m} |x_i|$$



$\|x\|_\infty = 2$.

Maximum of all the distances covered in any direction.

(4) p-norm / lp-norm : Generalization of all norms.

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} \qquad (1 \le p < \infty)$$

find x_1 and x_2 such that

$\|x\|_1 = 1$, $\|x\|_2 = 1$, $\|x\|_\infty = 1$

Geometric interpretation :-

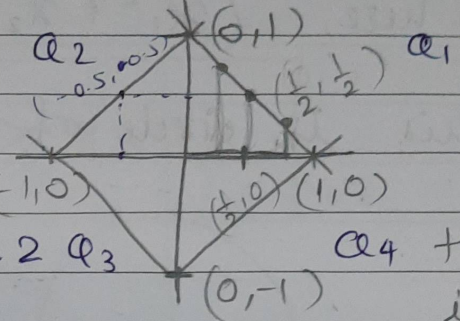
$x_1 = [-1, 1]$

$x_2 = [-1, 1]$

Consider closed unit ball $x \in \mathbb{R}^2 : \|x\| \leq 1$.

Imp cond^n for below discussions

① $\|x\|_1 = \sum_{i=1}^{m} |x_i| \Rightarrow$



$\|x\| \leq 1$

$\|(0.8, x_2)\|_1 = 1 \Leftrightarrow x_2 = 0.2$ Q3

Q4 Here every vector is going to have length / distance 1 wrt norm-1, it is Rhombus ✓

$\|(0.5, x_2)\|_1 = 1 \Leftrightarrow x_2 = 0.5$

$\|(x_1, x_2)\| = 1 \Leftrightarrow |x_1| + |x_2| = 1$

② $\cancel{\|x\|_2 = \sqrt{x * x}}$

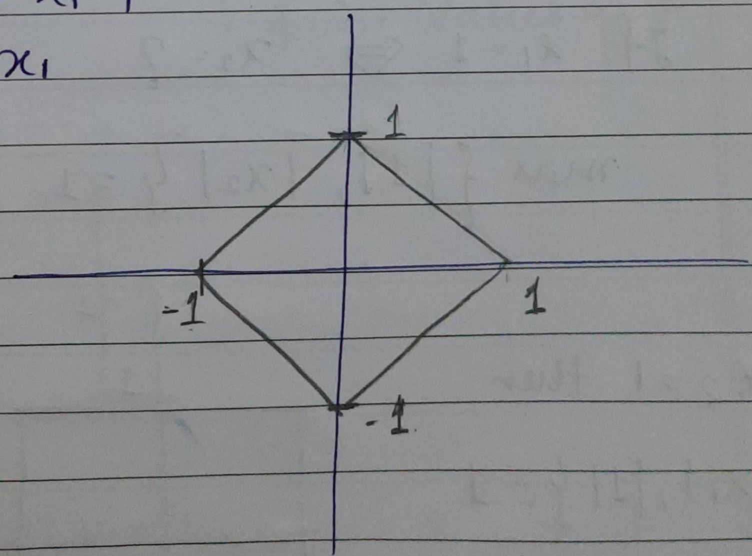
In Q1, $x_2 = 1 - x_1$

In Q2, $x_2 = 1 + x_1$

In Q3, $x_2 = -x_1 - 1$

In Q4, $x_2 = x_1$

$1 - 0.5 = 0.5$ ✓



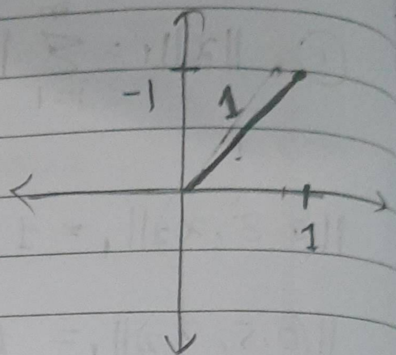
2-norm

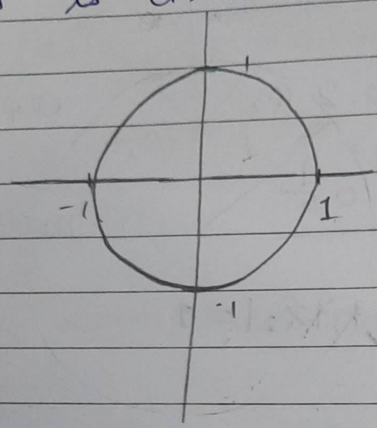
② $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2}$ we want $\|x\|_2 = 1$

(Sqrt of sum of the squares of individual elements)

So here, $(x_1^2 + x_2^2)^{1/2} = 1$

this is circle of radius 1.





Unit circle for

2-norm $\geq$ 1-norm

③ $\|x\|_\infty = \max_{1 \leq i \leq m} |x_i|$ max^m value of individual elems

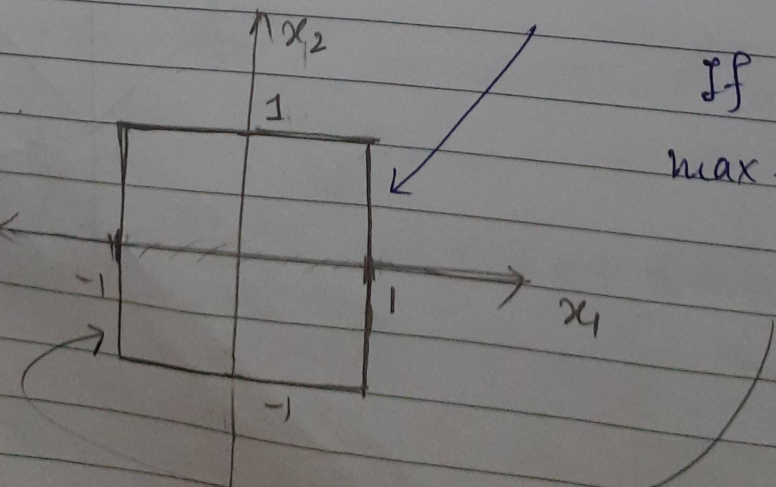
If $x_1 = 1 \iff x_2 = ?$

$\max\{|1|, |x_2|\} = 1 \quad \forall\ |x_2| \leq 1$

If $x_2 = 1$ then

$\max\{|x_1|, |1|\} = 1$

$\forall\ |x_1| \leq 1$



If $x_1 = -1$ then

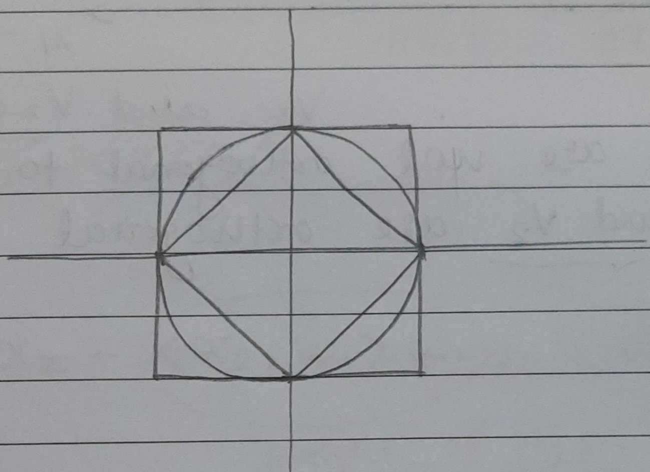
$\max\{|-1|, |x_2|\} = 1$

$\forall\ |x_2| \leq 1$

The unit circle for $\|x\|_1$ is smaller than $\|x\|_2$ and $\|x\|_\infty$

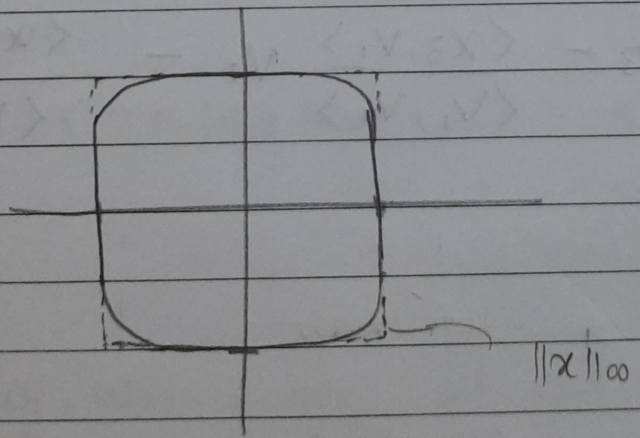
But the norm values for (x_1, x_2) are bigger.

$$\boxed{\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty}$$



④ $\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} \quad (1 \leq p < \infty)$

finite values of p.



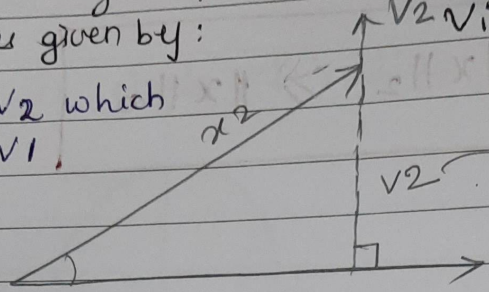
Gram-Schmidt orthogonalization process :-

$$\{x_i \mid i = 1, 2, \ldots n\} \longrightarrow \underbrace{\{V_i \mid i = 1, 2, \ldots n\}}_{\text{orthogonal}}$$

$V_i \cdot V_j = 0 \ \forall \ i \neq j$

Let $x_1, x_2, \ldots x_n$ are linearly independent vectors then, GSOP is given by:

Look for V_2 which is $\perp^{er}$ to V_1.



projection of vector x_2 on x_1 (V_1)

$x_1 = V_1$ (Assume)

V_1 and V_2 are $\perp^{er}$ to each other

$\therefore x_1$ & x_2 are not orthogonal to each other but $\underbrace{V_1 \text{ and } V_2}$ are orthogonal

$$V_1 = x_1$$

$$V_2 = x_2 - \frac{\langle x_2, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1$$ // projection of vector x_2 on V_1

$\perp^{er}$ to both V_1 & V_2

$$V_3 = x_3 - \frac{\langle x_3, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 - \frac{\langle x_3, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2$$

$$\vdots$$

$$V_n = x_n - \frac{\langle x_n, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 - \cdots - \frac{\langle x_n, V_{n-1} \rangle}{\langle V_{n-1}, V_{n-1} \rangle} V_{n-1}$$

Then $V_1, V_2, \ldots V_n$ is an orthogonal basis for V.

dot product zero

orthonormal = orthogonal + unit length

Orthogonalizatⁿ + Normalization

Let $x_1, x_2, \ldots x_n$ are 'n' linearly independent vectors then modified GSOP is given as :

$$V_1 = x_1 \quad , \quad w_1 = \frac{V_1}{\|V_1\|} = \frac{x_1}{\|x_1\|}$$

$$\cancel{V_2 = x_2 - \frac{\langle x_2, V_1 \rangle}{\langle V_1, V_1 \rangle}}$$

$$V_2 = x_2 - \langle x_2, w_1 \rangle w_1 \; , \quad w_2 = \frac{V_2}{\|V_2\|}$$

projectⁿ of x_2 on w_1

$$V_3 = x_3 - \langle x_3, w_1 \rangle w_1 - \langle x_3, w_2 \rangle w_2 \; , \quad w_3 = \frac{V_3}{\|V_3\|}$$

$$\vdots$$

$$V_n = x_n - \langle x_n, w_1 \rangle w_1 - \cdots - \langle x_n, w_{n-1} \rangle w_{n-1}$$

$$w_n = \frac{V_n}{\|V_n\|}$$

then $w_1, w_2, \ldots w_n$ is an orthonormal Basis ~~of~~ ~~4.~~ - for V.

Date
Page

orthogonal diagonalization.

eg Ex 9) find ~~orthol matrix~~ orthogonal matrix P that diagonalizes A

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

① find eigen values & eigen vectors & construct P.

→ $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix} = 0$$

$$2-\lambda\left[(-1-\lambda)^2-4\right] - 2\left[(-2-\lambda)+8\right] + (-2)\left[8+2(-1-\lambda)\right]$$

$-\lambda+6$ $\quad$ $6-2\lambda$ $\quad$ $8-2+2\lambda$

$\Rightarrow (2-\lambda)\left[-(\lambda^2+2\lambda+1)-4\right] + 2\lambda - 12 + 12 + 4\lambda$

$(2-\lambda)(-\lambda^2-2\lambda-5)$

$\Rightarrow \cancel{-2\lambda^2} - \cancel{4\lambda} - 10 + \lambda^3 + \cancel{2\lambda^2} + 5\lambda + 2\lambda - 12 - \cancel{12} + \cancel{4\lambda}$

$\lambda^3 + 7\lambda - 34 = 0$

$8+14$, $\frac{18}{22}$

$$\begin{vmatrix} (2-\lambda) & 2 & -2 \\ 2 & -(1+\lambda) & 4 \\ -2 & 4 & -(1+\lambda) \end{vmatrix} = 0$$

$(2-\lambda)\left[(1+\lambda)^2-16\right] - 2\left[-2(1+\lambda)+8\right] - 2\left[8-2(1+\lambda)\right]$

$(2-\lambda)(\lambda^2+2\lambda+1-16) - 2\left[-2-2\lambda+8\right] - 2\left[8-2\lambda-2\right]$

$(2-\lambda)(\lambda^2+2\lambda-15) + 8\lambda + 24$

$\cancel{2\lambda^2} + 4\lambda - 30 - \cancel{\lambda^3} - \cancel{3\lambda^2} + 15\lambda + \cancel{2\lambda} - 24$ $\quad + 4 + 4\lambda + 16 - 16 + 4\lambda + 4$

$8\lambda + 8\lambda - 24)$

$4 - \frac{1}{2}$, 8, $\left(\frac{-32}{+8}\right)$

classmate
Date
Page

$\cancel{-\lambda^3 + 11\lambda + 54 = 0}$

$-\lambda^3 + 27\lambda - 54 = 0$ $\qquad (3)^3 = 27$

$\qquad -27 + 71 - 54 \neq 0.$

$-(\lambda^3 - 27\lambda + 54) = 0$

$27 - 81 + 54$

$-54 + 54 = 0.$

$-(\lambda - 3)(\lambda^2 + 3\lambda - 18) = 0$

$$\begin{array}{r} \lambda^2 + 3\lambda - 18 \\ \lambda - 3 \,\big)\, \lambda^3 - 27\lambda + 54 \\ -\lambda^3 - 3\lambda^2 \\ \hline +3\lambda^2 - 27\lambda + 54 \\ -3\lambda^2 + 9\lambda \\ \hline -18\lambda + 54 \\ -18\lambda + 54 \\ \hline 0 \end{array}$$

$+(\lambda - 3)(\lambda + 6)(\lambda - 3) = 0$

$\therefore \lambda_1 = 3$ (has multiplicity 2)

$\lambda_2 = -6$. — (AM=1) so find eigenVector

AM ≥ 2 then find set of k-linearly independent eigen-vectors.

1) $\lambda_2 = -6$,

$(A - \lambda I)x = 0$. $\left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 2 & 5 & 4 & 0 \\ -2 & 4 & 5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ 0 & 9 & 9 \end{bmatrix}$

$R_1 \leftarrow \frac{1}{8} R_1$ $\qquad R_2 \leftarrow R_2 - 2R_1$ $\qquad 5 - \frac{2}{4}\frac{1}{2} \quad \frac{10-1}{2}$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1/4 & -1/4 & 0 \\ 2 & 5 & 4 & 0 \\ -2 & 4 & 5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 9/2 & 9/2 \\ -2 & 4 & 5 \end{bmatrix}$ $\qquad 4 - \frac{1}{2} \quad \frac{8-1}{2} \quad 5 - \frac{1}{2}$

$R_3 \leftarrow R_3 + 2R_1$ $\qquad 4 + \frac{2}{4}\frac{1}{2}$

$\frac{2}{9} \times \frac{9}{2} \times \frac{1}{4}$ $\qquad 4 - \frac{1}{2} \quad \frac{8-1}{2} \quad 5 - \frac{1}{2} \quad \frac{10-1}{2} \quad \frac{2}{9} \times \frac{9}{2}$

$R_2 \leftarrow R_2 \times 2/9\, R_2$

$\begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 1 \\ -2 & 4 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 1 \\ 0 & 9/2 & 9/2 \end{bmatrix}$ $\qquad \frac{7}{2} \times \frac{8}{9} \times \frac{-1}{4}$

$\left(\frac{-32}{+8}\right)$

Date
Page

$$\left[\begin{array}{ccc|c} \cancel{1} & 1/4 & -1/4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$R_1 \leftarrow R_1 - \frac{1}{4} R_2.$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = \frac{1}{2} x_3.$

$x_2 = -x_3.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ -t \\ t \end{bmatrix}$$

$$t \begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0 \Rightarrow t \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$\lambda_1 = -6, \quad V_1 = (1, -2, 2)$

$$u_1 = \frac{V_1}{\|V_1\|} \Rightarrow \frac{1}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}$$

$\sqrt{1+4+4} = \sqrt{9} = 3$

$$\therefore u_1 = \frac{V_1}{\|V_1\|} = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$$ unit Eigen Vector

2) $\lambda_2 = 3$

$$(A - \lambda I)x = 0 \quad \left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \\ -2 & 4 & -4 & 0 \end{array}\right]$$

Gaussian Elimination →

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 2 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 0.$$

$$x_1 = 2x_2 - 2x_3$$
$$x_2 = x_2$$
$$x_3 = x_3.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s - 2t \\ s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 2s - 2t \\ s \\ t \end{bmatrix} \Rightarrow \left\{ s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis for eigenspace with $\lambda_2 = 3$ $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\therefore v_2 = (2, 1, 0)$ $\quad V_3 = (-2, 0, 1)$ and they are linearly independent.

If $p_1 \cdot p_2 = 0$ and $\|p_1\| = \|p_2\| = 1$ } orthonormal vectors.

$V_2 = (2,1,0)$, $V_3 = (-2,0,1)$

$V_2 \cdot V_3 = (2,1,0)\begin{bmatrix}-2\\0\\1\end{bmatrix} = -4+0+0 = -4 \neq 0$

$\therefore V_2$ & V_3 are not orthogonal vectors.

Applying GSOP.

~~$V_2 = u_2$~~

If $w_2 = V_2$ then

~~$u_2 =$~~ $u_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\\0\end{bmatrix} \leftarrow$ Unit Vector

~~If $w_3 = V_3$ then~~

~~$u_2 =$~~

Basic GSOP + divide by unit vector

$\{V_2, V_3\} \xrightarrow{GSOP} \{u_2, u_3\}$

$w_3 = V_3 \Rightarrow$ ~~$w_3 = V_3 - \frac{(w_2, V_3)}{(V_3, V_3)} V_3$~~

$w_3 = V_3 - \frac{\langle V_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$ $\qquad w_2 = V_2 = (2,1,0)$

~~$= (2,1,0)\{(-2,0,1)[2$~~

$= \begin{bmatrix}-2\\0\\1\end{bmatrix} - \frac{4}{5}\begin{bmatrix}2\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}-2-8/5\\-8/5\\1\end{bmatrix} =$

$-2+\frac{8}{5}$

$-10+8$ / 5

$\frac{4}{25}+$

$-10-8$ / 5

$w_2 = v_2 = (2, 1, 0)$

$v_3 = (-2, 0, 1)$

$\langle v_3, w_2 \rangle = (-2, 0, 1)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = -4$

$\langle w_2, w_2 \rangle = (2, 1, 0)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 5$

$$w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= (-2, 0, 1) - \left(-\frac{4}{5}\right)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

This is not unit vector so normalize it

$\frac{-2+8}{5}$

$\frac{-10+8}{5}$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{w_3}{3\sqrt{5}/5} = \left(\frac{-2}{\cancel{5}} \times \frac{\cancel{5}}{3\sqrt{5}}\right) = \frac{-2}{3\sqrt{5}}$$

$$\left(\frac{4}{\cancel{5}} \times \frac{\cancel{5}}{3\sqrt{5}}\right) = \frac{4}{3\sqrt{5}}$$

$$\left(1 \times \frac{5}{3\sqrt{5}}\right) = \frac{5}{3\sqrt{5}}$$

$\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2$

$\frac{4}{5} + \frac{1}{5}$

$\frac{4}{25} + \frac{16}{25} + 1$

$\sqrt{\frac{4+16+25}{25}}$

$\sqrt{\frac{45}{25}}$

$\frac{3\sqrt{5}}{5} \times \frac{5}{5}$

$\therefore$ $u_3 = \begin{bmatrix} -2/3\sqrt{5} \\ 4/3\sqrt{5} \\ 5/3\sqrt{5} \end{bmatrix}$

$\frac{-10-8}{5}$

So we converted all vectors to Unit ~~unique~~ eigen vector

$$P = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}$$

Now $P^{-1}AP \equiv P^TAP$. Orthogonal diagonalization of matrix A

$$\begin{array}{cc} P^T & A \\ \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & 4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix} & \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} P^T.A & P \\ \begin{bmatrix} -2 & 4 & -8/3 \\ 5/\sqrt{5} & 3/\sqrt{5} & 0 \\ -2/\sqrt{5} & 4/\sqrt{5} & 4/\sqrt{5} \end{bmatrix} & \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\frac{2}{3} - \frac{4}{3} - \frac{4}{3} \Rightarrow \frac{2-8}{3}$

$\frac{2}{3} + \frac{2}{3} + \frac{8}{3}$ $\quad 4+4+5$ $\quad 12$ $\quad \frac{12}{3}$ 4

$-\frac{2}{3} - \frac{8}{3} + \frac{2}{3}$ $\quad \frac{-8}{3}$

$\frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}}$ $\quad \frac{-4}{\sqrt{5}} + \frac{4}{\sqrt{5}}$

$\frac{4}{\sqrt{5}} - \frac{1}{\sqrt{5}}$

$\frac{4}{\sqrt{5}} + \frac{8}{3\sqrt{5}} - 10$

$-4+8-10$ $\quad 4-10$ $\quad \frac{6}{3\sqrt{5}}$

$4 - 4 + 20$ $\quad -8+20$ $\quad 16$

$4 - 4 + 20$ $\quad 12$

$-7 - 8 - 16$

$2 - 4 - 4$

$2 - 8$

$-6 - 8 - 16$

$\frac{-14-20}{3}$ $\quad \frac{34}{3}$

$\frac{-6}{9} - \frac{8}{3} - \frac{16}{9}$

$\frac{-6-24-16}{9}$

$-\frac{2}{3} + \frac{2}{3} - \frac{2}{3}$

$-4+4$

$-4+16-40$

$12-40$

$$\frac{-4}{3\sqrt{5}}+\frac{16}{3\sqrt{5}}-\frac{40}{3\sqrt{5}}$$

$$\frac{12-40}{3\sqrt{5}} \Rightarrow \frac{-28}{3\sqrt{5}}$$

If matrix is symmetric $\longrightarrow$ then it is Orthogonally diagonalizable

If not symm $\longrightarrow$ We can't say anything about diagonalization.

find eigen values $\rightarrow$ check whether distinct OR not & so on the entire procedure.

Matrix Norm

A norm is a f^n $\|\cdot\| : \mathbb{R}^{m\times n} \rightarrow \mathbb{R}$ that assigns a real valued length to each matrix. For all matrices A and B and for all scalars $a \in \mathbb{R}$

① $\|A\| \geq 0$ and $\|A\| = 0$ only if $A = 0$ positivity cond^n

② $\|A+B\| \leq \|A\| + \|B\|$ Δ^{er} inequality.

③ $\|aA\| = |a| \, \|A\|$ Scaling cond^n.

④ $\|AB\| \leq \|A\| \cdot \|B\|$ Compatibility cond^n.
applicable to sq^r matrix

eg 2)

* popular matrix Norms :-

over column

① $\|A\|_1 = \underset{1 \leq j \leq n}{\overset{\text{max over / columns}}{\max}} \left(\sum_{i=1}^{n} |a_{ij}| \right)$ 1-norm / l_1-norm

over row

eg $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$

(1) (5)

$\max(1, 5) = 5 \leftarrow$ is 1-norm of $A \Rightarrow \|A\|_1 = 5$

② $\|A\|_E = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}$ 2-norm / l_2-norm / E-norm

Imp

$\sqrt{(1)^2 + (0)^2 + (2)^2 + (3)^2} \Rightarrow \sqrt{14} \leftarrow \|A\|_E$

1 4 9

classmate

Date

Page

same defⁿ as one norm
traverse through columns.

(3) $\|A\|_\infty = \max\limits_{1 \le i \le n} \left(\sum\limits_{j=1}^{n} |a_{ij}| \right)$ ~~max~~ infinity norm.

max over rows

eg A = $\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$ ~~max(2, 0) = (2) = ‖A‖∞~~

max(3, 3) = 3 = ‖A‖∞

take norm.

(1+2), (0+3) = max (3, 3).

eg2) A = $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

over columns $\|A\|_1 = \max(1, 4) = 4$ (~~rows~~ (column)

$\|A\|_E = \sqrt{(1)^2+(2)^2+(2)^2} \Rightarrow \sqrt{9} = 3.$

1 4 4

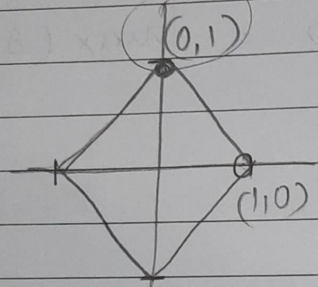
over rows $\|A\|_\infty = \max(3, 2) = 3.$ (row) (~~columns~~)

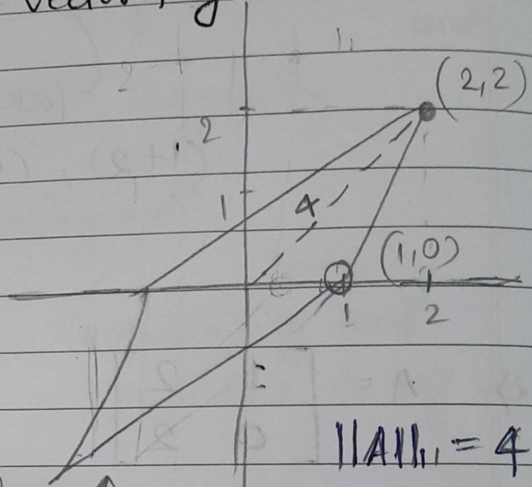
Geometric interpretation :-

The closed unit ball $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\quad Ax = y.$ norm of $\|A\|$ will define amplificatn of vector y to x.

① 1-norm





$\|A\|_1 = 4$

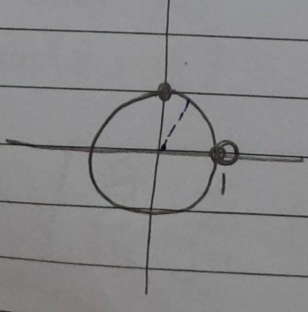
(2,2)

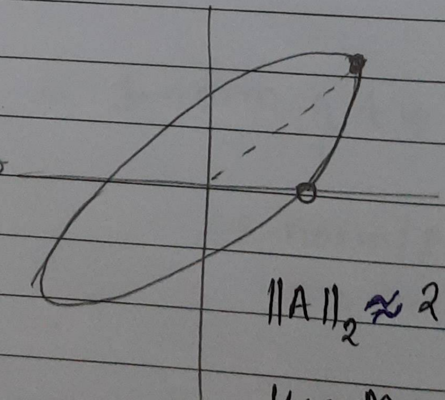
$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{2\times 2} \overset{x}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times 1} = \overset{y}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \rightarrow \text{Amplified}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ so this vector doesn't change

max^m amplificatn you can get is given by 1-norm.

② 2-norm

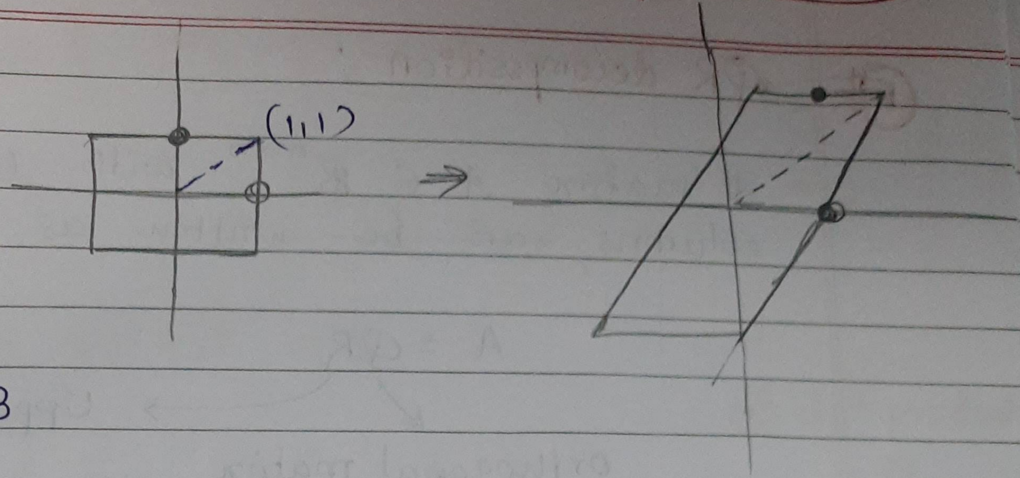




$\|A\|_2 \approx 2.9208$

max^m amplif^n factor under 2-norm

(3) ∞-norm $\Rightarrow$



$\|A\|_\infty = 3$

Vector Norm $\Rightarrow$ distance

Matrix Norm $\Rightarrow$ Amplification of original vector