Aarti Pol

Optimization

15th - evening

n-vector → n-tuple → $(a_1, a_2, \ldots, a_n)$ $a_i \in \mathbb{R}^n$

$\uparrow$ $\mathbb{R}^n$

Equality of vectors → ff each entry is same

$$a = b$$

$$(a_1\ a_2 \ldots a_n) = (b_1, b_2 \ldots b_n)$$

iff $a_i = b \quad \forall\ 1 \leq i \leq n$.

Addition of vectors :-

$$(a_1, a_2 \ldots a_n) \pm (b_1, b_2 \ldots b_n) = (a_1 \pm b_1, a_2 \pm b_2 \ldots)$$

$(\mathbb{R}^n, +)$ forms Abelian group.

① Associativity $(a+b)+c = a+(b+c)$; $a, b, c \in \mathbb{R}^n$

② Commutativity $(a+b) = (b+a)$

Zero vector ③ $\exists\ (0, 0 \ldots 0) \in \mathbb{R}^n$, $0 + a = a = a + 0$, $a \in \mathbb{R}^n$.

Additive inverse ④ $\forall\ a \in \mathbb{R}^n \longrightarrow -a$

$(a_1\ a_2 \ldots a_n)$ $(-a_1, -a_2 \ldots -a_n)$

$$a + (-a) = 0 = (-a) + a \quad ; \quad a \in \mathbb{R}^n.$$

* Scalar multiplicat^n of $\mathbb{R}^n$; a, ~~x~~ $a \in \mathbb{R}^n, \lambda \in \mathbb{R}$

$$\lambda a = \lambda (a_1, a_2, \dots a_n) = (\lambda a_1, \lambda a_2 \dots \lambda a_n)$$

i) $\lambda(a+b) = \lambda a + \lambda b$, $a, b \in \mathbb{R}^n, \lambda \in \mathbb{R}$

ii) $(\lambda_1 + \lambda_2) a = \lambda_1 a + \lambda_2 a$

iii) $(\lambda_1 \lambda_2) a = \lambda_1 (\lambda_2 a)$ $\quad \lambda_1, \lambda_2 \in \mathbb{R}^n$

(($\mathbb{R}^n$, +) form abelian group & Scalar multiplicat^n)

then $\mathbb{R}^n$ is vector space over $\mathbb{R} \Rightarrow \mathbb{R}^n_{\mathbb{R}}$

* Inner product on $\mathbb{R}^n$ (dot product)

$a = a_1, a_2 \dots a_n$
$b = b_1, b_2 \dots b_n$

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Symm, +ve definite, & Bilinear map.

Bilinear map $\Rightarrow$

$$\left.\begin{aligned} \langle a+b, c\rangle &= \langle a,c\rangle + \langle b,c\rangle \\ \langle a, b+c\rangle &= \langle a,b\rangle + \langle a,c\rangle \end{aligned}\right\} \; a,b,c \in \mathbb{R}^n.$$

Symmetric $\Rightarrow$

$$\langle a,b\rangle = \langle b,a\rangle$$

+ve definite $\Rightarrow$

$$\langle a,a\rangle \geq 0$$

$$\langle a,a\rangle = a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0.$$

$$\langle a,a\rangle = 0 \text{ iff } a = 0.$$

$$\underbrace{\Big(\cancel{\text{Over}}\underbrace{\big(\underbrace{(\mathbb{R}^n,+)}_{\text{abelian group}},\ \text{Scalar multipl}^n\big)}_{\text{Vector space}},\ \underbrace{\langle\,,\rangle}_{\text{dot product}}\Big)}_{\text{Inner product space.}}$$

* length of Vector $\Rightarrow$

$$a = (a_1, a_2 \ldots a_n) \in \mathbb{R}^n$$

$$\|a\| = \sqrt{a * a} = \sqrt{\langle a, a \rangle}$$

Unit Vector:- if $\|a\| = 1$.

* linear comb^n of vectors $\Rightarrow$

$$a, b \in \mathbb{R}^n$$

$$L.C. \text{ of } a, b \rightarrow \{ \lambda a + \beta b \;/\; \lambda, \beta \in \mathbb{R} \}$$

set of vectors.

$$a_1, a_2 \ldots a_m \in \mathbb{R}^n$$

L.C of m vectors $\{a_1, \ldots a_m\}$

$$= \{ \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m \;/\; \lambda_i \in \mathbb{R} \}$$

3 eq 2 variables

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 + 2c_2 + c_3 = 0$$
$$c_1 + 4c_2 + 0 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 + 2c_3 = 0 \\ c_2 - \frac{1}{2} c_3 = 0 \end{bmatrix}$$

If you are in $\mathbb{R}^2$ are you have
3 vectors then they must be dependent
cz in 2D at max 2 independent vectors are psbl.

* linear dependency $\Rightarrow$

$\{a_1, \ldots a_m\}$ be set of vectors in $\mathbb{R}^n$ then
it is called linear dep. set s.t.

$\exists \ \lambda_i \in \mathbb{R}$ (not all zero)

$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m = 0$

* Linear independency $\Rightarrow$

$\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m = 0$

$\cdot \ \lambda_i = 0 \ \forall i$

then $\{a_1, a_2 \ldots a_m\}$ set is LI.

eg. $\{(1,1), (2,4), (1,0)\}$ in $\mathbb{R}^2$ LD or LI ?

$\alpha(1,1) + \beta(2,4) + \nu(1,0) = 0$. $\alpha =$
$\beta =$
$\nu =$

(you can get 3rd vector by using
other two).

eg. $\{(1,0,0), (0,1,0), (0,0,1)\}$ in $\mathbb{R}^3$ is LI

$\alpha(1,0,0) + \beta(0,1,0) + \nu(0,0,1) = 0$

$= (\alpha, \beta, \nu) = (0,0,0)$

$x_1, x_2, x_3 \in \mathbb{R}$

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

Basis $\Rightarrow$ LI set 'A' of $\mathbb{R}^n$ is called basis if each vector of $\mathbb{R}^n$ can be expressed as LC of vectors in set A.

e.g. $\{(1,0,\ldots 0), (0,1,\ldots 0), \ldots (0,\ldots 0,1)\}$
$\quad e_1 \qquad e_2 \qquad e_n$

std basis of $\mathbb{R}^n$

then $e_i = (0,\ldots 0, 1, 0, \ldots 0)$ — ith pos.

* Every vector space is inner product space $\mathbb{R}^n$

Optimization ⇒

16th morning

Electric ckt — Capacitors, Registers, transisters.

	Requirement for ckt 1 (x_1)	Reqᵐ of ckt 2 (x_2)	Stock.
① T	2	4	20
② R	2	2	12
③ C	4	0	16.
profit per unit (or ckt)	2	3	

How many units of ckts of ckt 1 & 2 produced to maximize profit?

⇒ $x_1 \longrightarrow$ no. of units company should produce for ckt 1

$x_2 \longrightarrow$ —— " —— ckt 2.

$p \longrightarrow$ profit $= 2x_1 + 3x_2$.

~~Then no. of transisters reqd.~~

$P = 2x_1 + 3x_2$

$2x_1 + 4x_2 \leq 20$ // Transistor

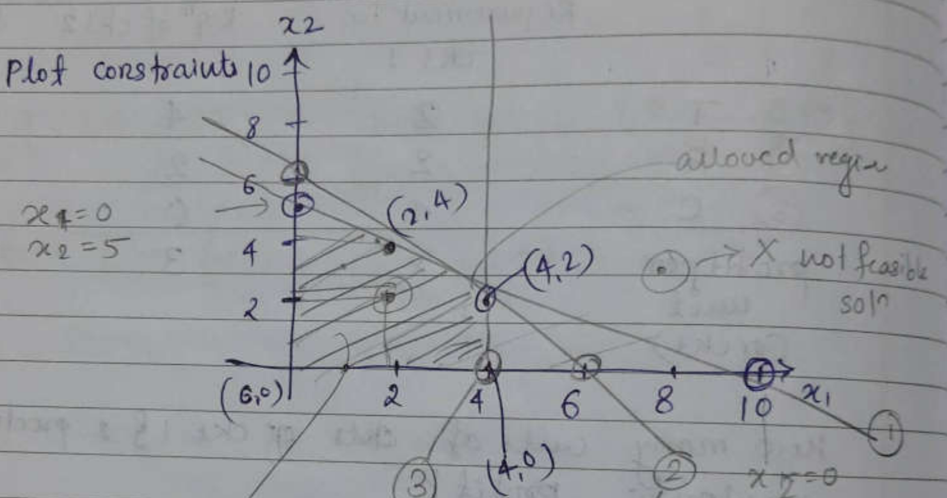
$2x_1 + 2x_2 \leq 12$ // Register

$4x + 0x_2 \leq 16$ // Capacitor

Maximize, $P = 2x_1 + 3x_2$

constraints $\begin{cases} x_1 + 2x_2 \leq 10 & \text{— (1)} \\ x_1 + x_2 \leq 6 & \text{— (2)} \\ x_1 \leq 4 & \text{— (3)} \end{cases}$ $x_1 \geq 0$ and $x_2 \geq 0$

Plot constraints



feasible solution ⇐ Allowed region where all constraints satisfied ✓

These solns are called feasible soln.

maximize profit

So find x_1 & x_2 lying in region s.t.

$P = 2x_1 + 3x_2$ is max^m

↓

will come come boundary pts

$P = 2x_1 + 3x_2$

$P = 0$ at $(0,0)$

$P = 8$ at $(4,0)$

$P = 14$ at $(4,2)$

$P = 16$ at $(2,4)$

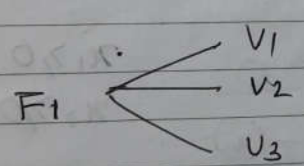
$P = 15$ at $(0,5)$

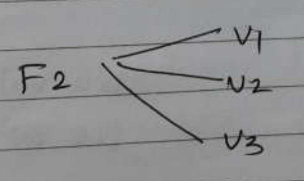
$\frac{12}{4}$

∴ ckt comp should produce 2 units of ckt 1 & 4 units of ckt 2 for maximizing profit.

For large no. of variables, graphical method is not useful.

2) Diet problem $\quad V_1 \quad V_2 \quad V_3$





	(mg) Vit 1	Vit 2	Vit 3	cost
x_1 F1	1 mg	100 mg	10 mg	1 Re.
x_2 F2	1 mg	10 mg	100 mg	1.5 Rs.
daily requirement	1 mg	50 mg	10 mg	

x_1 = # units of F1

x_2 # units of F2

minimize cost, $C = x_1 + 1.5x_2$

$x_1 + x_2 \geq 1$ vitamin 1

$100x_1 + 10x_2 \geq 50$

$10x_1 + 100x_2 \geq 10$

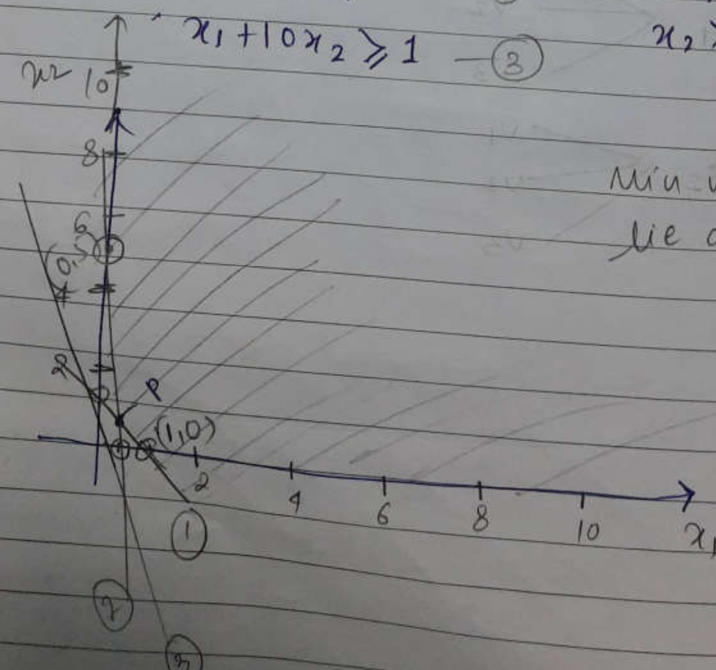
$x_1 + x_2 \geq 1$ — ①

$10x_1 + x_2 \geq 5$ — ②

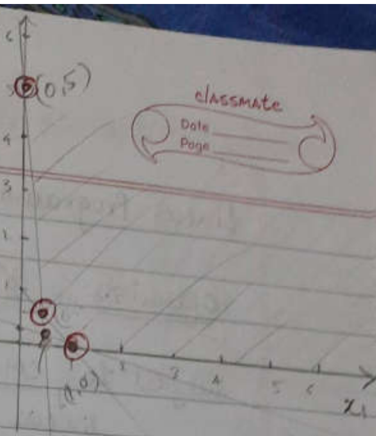
$x_1 + 10x_2 \geq 1$ — ③

$x_1 \geq 0$

$x_2 \geq 0$



Min value will lie on boundary



$10x_1 + x_2 = 5$

$x_1 + 10x_2 = 1$

at $(0,5)$ $C = 0 + 7.5$

$C = 7.5$

we want minimum cost $\Rightarrow$

$C = x_1 + 1.5x_2$

$C = 1.4$

$\begin{array}{cc} x_1 & x_2 \end{array}$

$(4/9, 5/9) \rightarrow$

(0.56)

Boundary pts $\Rightarrow$ $(1,0)$ $(0.44, 0.556)$

$C_1 = 1 * x_1 + 0 = 1$ — (1)

$C_2 = 0.44 \times 1 + 0.556 \times 1.5 = 1.278$ — (2).

$\therefore$

Linear Programming problem ⇒ LPP

Optimize $z = \sum_{i=1}^{n} c_i x_i$ linear f^n

subject to constraints having linear equality or inequality
(linear)

LPP
$$\left.\begin{aligned} \sum a_{ij} x_j &\le b_i \\ &\text{or } \ge \\ &\text{or } = \end{aligned}\right\} \text{ linear constraint } (*)$$

* feasible solutions ⇒

$(x_1, x_2, \ldots x_n) \in \mathbb{R}^n$
which satisfies all constraints of (*)
(linear)

* Optimal solutions ⇒

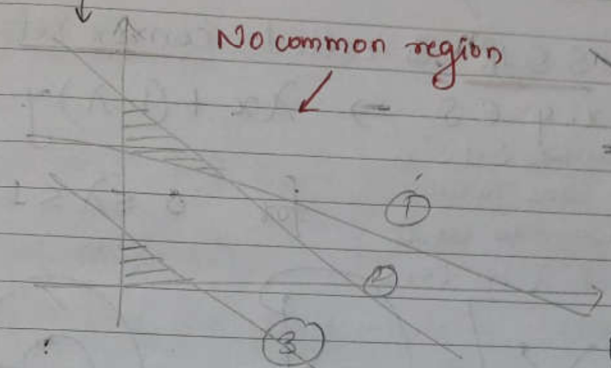
vector
$(x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ which satisfies all constraints of (*)
& 'z' takes max^m/min value at $(x_1, x_2 \ldots x_n)$
as per requirement

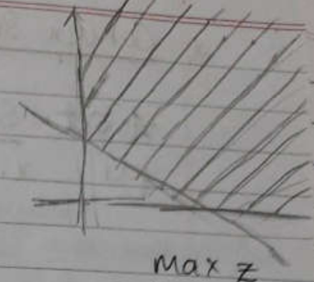
Every feasible sol^n is optimal sol^n.

There might be no optimal sol^n eventhough feasible sol^n exists.

If no feasible sol^n then there is No optimal solution.

LPP ⟹ feasible sol^n doesn't exist





Max z
feasible sol^n
exists but
No optimal sol^n

Constraint set / feasible set ⟹ collection of all vectors which are feasible solutions

$$T = \{ (x_1 x_2 \dots x_n) \in \mathbb{R}^n \mid (x_1, x_2 \dots x_n) \text{ is a feasible solution} \}.$$

To study nature of constraint set ⟹

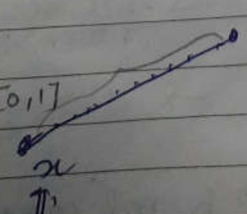
* line seg joining two pts $x, y \in \mathbb{R}^n$

$x = (x_1, x_2 \dots x_n)$, $y = (y_1, y_2 \dots y_n)$

is the set of pts given by

$$\{ z \mid z = \lambda x + (1-\lambda) y \;; 0 \le \lambda \le 1 \}$$

all values of λ lying betn $[0,1]$ will give these points on the line.



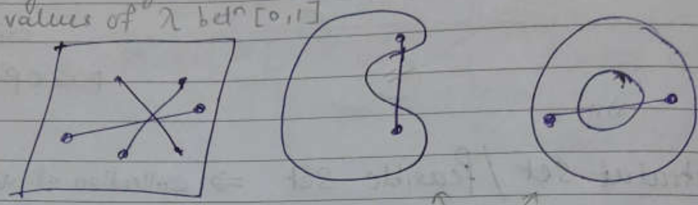
R^n · Euclidean Space

* **Convex Set** ⇒

A set $S \subseteq \mathbb{R}^n$ is called convex set if two pts $x, y \in S \Rightarrow \lambda x + (1-\lambda) y \in S$ for $0 \leq \lambda \leq 1$

(x, y belong to Set S')

implies that the line joining pts x and y belongs to set S for all values of λ betⁿ $[0,1]$



A — Convex set

B, C — Not convex set

* **Convex combination** ⇒ Used for more than two pts

A vector $x \in \mathbb{R}^n$ is called convex combⁿ of vectors $\{x_1, x_2, \ldots x_n\} \in \mathbb{R}^n$ if there exists (scalars in $\mathbb{R}$)

$\exists \lambda_i \in \mathbb{R}$ satisfying 3 properties

① $x = \sum_{i=} \lambda_i x_i$ → x is linear combⁿ of vectors.

② $\lambda_i \geq 0$ → All scalars are +ve

③ $\sum \lambda_i = 1$ → Sum of all scalars is 1

So every linear combⁿ is not convex combⁿ, to become convex combⁿ it has to satisfy properties ② & ③ extra of scalars.

* A line segment is convex combination of end pts
* All pts on a line is a convex combⁿ of end pts $(x \,\&\, y)$ of the line.

* Convex Set ⇒ :

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2$$

① Intersectⁿ of <u>any collection</u> of convex set (finite, infinite, uncountable) is a convex set.

② eg. $\{ S_\alpha \,/\, \alpha \in \lambda \}$ is a convex set. then
(λ: $\mathbb{N}$ or $\mathbb{R}$)

we have to prove their intersectⁿ is also convex set.

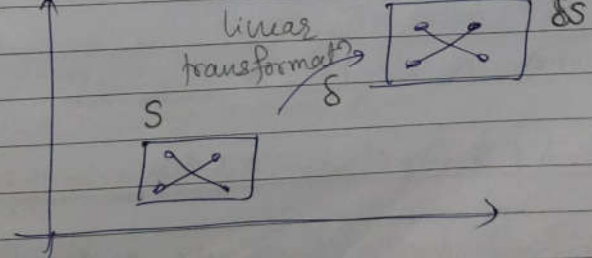
$$x, y \in \bigcap_{\alpha \in \lambda} S_\alpha .$$

$\Rightarrow \; x, y \in S_\alpha , \; \forall \; \alpha \in \lambda$

$\Rightarrow \; \lambda x + (1-\lambda) y \in S_\alpha , \; \forall \; \alpha, \lambda$

$\Rightarrow \; \lambda x + (1-\lambda) y \in \bigcap_{\alpha \in \lambda} S_\alpha$

② If S is a convex set then its linear transformⁿ $\delta \in \mathbb{R}$, $\delta S = \{ \delta x \,/\, x \in \mathbb{R}^n \}$ is also convex set (shifting S by δ)



③ If S & T are two convex sets then their sum, $S+T = \{ s+t \mid s \in S \text{ \& } t \in T \}$ is also a convex set.

proofe ⇒

Theorem ⇒

(46)

The constraint set

$T = \{ (x_1, x_2 \dots x_n) \in \mathbb{R}^n \mid (x_1, x_2 \dots x_n)$ is feasible $sol^n \}$

set of all vectors that satisfy all constraints

is a convex set

proof ⇒ Consider the linear constraints are foll^n eq^s

linear constraint $\left\{ \sum_{j=1}^{n} a_{ij} x_j \sim b_i \right.$ $\quad \sim \in \{=, \leq, \geq\}$

$1 \leq i \leq m$

$T_i = \{ x \in \mathbb{R}^n / (x_1 \dots x_n) = x$ s.t. x satisfies only ith constraint $\}$.

$\{x_1, x_2 \dots x_n\}$

set of vectors which satisfies ith constraint

$1 \leq i \leq m$

$T_i \longrightarrow$ satisfying only ith constraint

$T = \bigcap_{i=1}^{m} T_i$ (which satisfy all constraints)

set of vectors which satisfy all constraints

To prove T is convex set it is enough to prove

To Prove ⇒ T_i is a convex set $\forall i$

$x = (x_1, x_2 \dots x_n) \in T_i$, $\sum a_{ij} x_j \sim b_i$

$y = (y_1, y_2 \dots y_n) \in T_i$, $\sum a_{ij} y_j \sim b_i$

To prove ⇒ any pt lying on the line joined by x & y also belong to set T_i

* Any pt lying on the line joined by x & y belongs to T_i

$\lambda x + (1-\lambda) y \in T_i$ — (A)

$\sum a_{ij} x_j \sim b_i$ — ①

$\sum a_{ij} y_i \sim b_i$ — ②

eqⁿ ① $\times \lambda$, eqⁿ ② $\times (1-\lambda)$

$\sum a_{ij} \lambda x_j \sim \lambda b_i$

$\sum a_{ij} (1-\lambda) y_j \sim (1-\lambda) b_i$

$\sum a_{ij} (\lambda x_j + (1-\lambda) y_j) \sim \lambda \cancel{b_i} + b_i - \cancel{\lambda b_i}$

$\cancel{\sum a_{ij} \lambda x_j + y_j - \lambda y_j}$

$\therefore \sum a_{ij} (\lambda x_j + (1-\lambda) y_j) \sim b_i$

$(\lambda x_1 + (1-\lambda) y_1, \lambda x_2 + (1-\lambda) y_2 - - - \lambda x_n + (1-\lambda) y_n)$

$\lambda X + (1-\lambda) Y \in T_i$

T_i is convex set which proves each & their intersectⁿ is also convex set

$\therefore T = \bigcap_{i=1}^{m} T_i$ is convex set

* **Convex Hull ⇒**

Let $A \subset \mathbb{R}^n$ then the intersection of all convex sets of $\mathbb{R}^n$ containing A is called convex hull.

* Convex hull is smallest convex set containing A.

$\langle A \rangle = \bigcap W_i$; where W_i is ~~vector subs~~ subset of $\mathbb{R}^n$ $W_i \subseteq \mathbb{R}^n$ and has two properties ⇒

Convex hull of A

① W_i is convex set

② $A \subset W_i$ contains A

① for two pts $\therefore \curvearrowright$ convex hull is a line

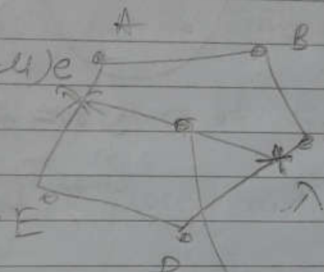
② for three pts $\therefore \curvearrowright$ $\langle A \rangle$ Δ^{el} is convex hull.

④ $\curvearrowright$ $\langle A \rangle$

$A = \{x_1, x_2 \dots x_n\}$ then a pt $x \in \langle A \rangle$ iff x is a convex combination of points in A

$$x = \sum_{i=1}^{n} \lambda_i x_i \quad , \lambda_i \geq 0 , \quad \Sigma \lambda_i = 1 .$$

any pt in plane can be written in this for



$\mu a + (1-\mu)e$

$(\mu \geq 0)$

$\lambda d + (1-\lambda) c \quad , (\lambda \geq 0)$

$\delta(\mu a + (1-\mu)e) + 1-\delta$

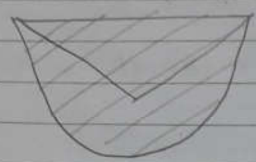
$+ (1-\delta)(\lambda d + (1-\lambda)c)$

$= 1$

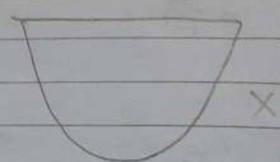
* convex polytope $\Rightarrow$ Convex hull of finitely many points

Let A be convex set then you find a minimum (smallest set) $S \subseteq \mathbb{R}^n$ such that convex hull of S is A

$$\boxed{\langle S \rangle = A}$$



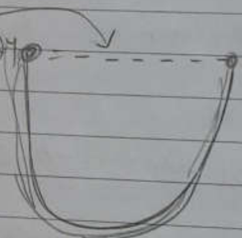
A



$\langle S \rangle = A$

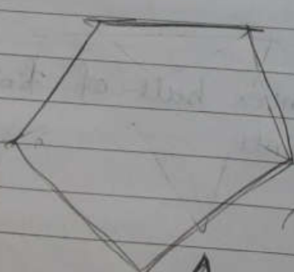
smallest subset

will come automatically

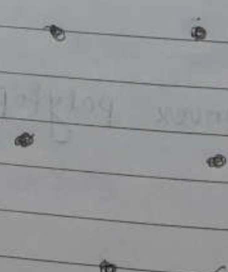


✓ Boundary is enough

S



A



convex hull

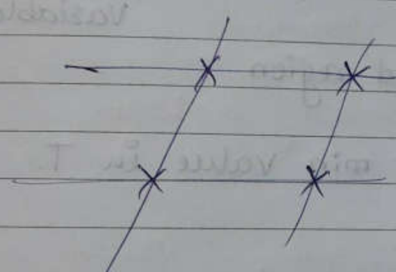
$\langle S \rangle = A$

✓

* LPP →

constraint set → linear Boundary

no. of vertices of finite constraint set is finite (If it is closed and bounded).



* A constraint set is always a convex set.
* Constraint set is always a convex hall of finitely many points.

in LPP

Fundamental $Th^m \Rightarrow$

Let constraint set T be non-empty, closed & bounded, then the optimal sol^n exist and it is attained at a vertex of T.

Proof $\Rightarrow$ max or min $z = c^t x$

$$c = (c_1, c_2 - - - - c_n) \in \mathbb{R}^n$$

$$x = (x_1, x_2 - - - x_n) \text{ — vertex Vector of Variables}$$

T is closed & bounded region

$\therefore$ Z will have max or min value in T.

$\hookrightarrow$ optimal sol^n exists.

* optimal sol^n is attained at vertex point of T.

① T has finitely many vertex points

$$\{y_1, y_2 - - - y_K\}$$

$(y_{11}, y_{12} - y_{1n})$

$(y_{K1}, y_{K2} - - - y_{Kn})$

(2) T is convex hull of vertices $\{y_1, y_2 - - y_K\}$.

$y \in T$ $(y_1^*, y_2^*, - - - y_n^*)$

y is vector

$$y = \sum_{i=1}^{K} \lambda_i y_i, \ \lambda_i \geq 0, \ \sum_{i=1}^{K} \lambda_i = 1$$

$$\min, \ z = c^t x$$

$$\max, \ z = c^t x.$$

converted to

$$\min -z = -c^t x$$

* Standard linear programming problem

A

Minimize $z = c^t x$

Subject to $Ax = b$ — Equality constraints

$x \geq 0$ — non-negativity constraints

where $b \geq 0$ is said to a linear pgm in standard form

SLPP

$A \rightarrow m \times n$ is coefficient matrix

$b = (b_1, b_2 \dots b_m)^T$ is vector of constants

$c = (c_1, c_2 \dots c_n)$ are cost factors of variables

$x = (x_1, x_2 \dots x_n)^T \in \mathbb{R}^n$ is the vector of variables.

NS Kambo

60-62
3rd chapter

Column vectors of matrix A are referred to as activity vectors.

* How to convert a LPP to SLPP ⇒

i) If you are asked to maximize z then

$$\text{Max}\,(z = c^T x) = -\text{Min}\,(-z = -c^T x)$$

ii) Conversion of to equality constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 \qquad + a_{mn}x_n = b_m$$

-1 $\quad b_i < 0$

if $b_i < 0$ then multiply i^{th} eqn with -1

iii) $\sum a_{ij} x_j \leq b_i$

$$\sum a_{ij} x_j + y_i = b_i$$

$$y_i \geq 0 \qquad y_i = \text{slack variable.}$$

$$\sum a_{ij} x_j \geq b_i$$

$$\sum a_{ij} x_j - y_i = b_i, \quad y_i = \text{Surplus variable}$$

③ Let $x_i \le 0 \longrightarrow x_i' = -x_i$

If some x_i is free variable then it can take both +ve & -ve values

then $x_i = u_i - v_i$, $u_i, v_i \geq 0$

Ex.1) Min $z = x_1 + 2x_2 - 2x_3$ Reduce to SLPP

Subject to $2x_1 + x_2 - x_3 \geq 2$

$-x_1 + x_2 + 3x_3 \le 3$

$x_2 \geq 0$, $x_3 \geq 0$

→ all const +ve → change inequality to equality

$2x_1 + x_2 - x_3 - x_4 = 2$ surplus var. $(x_4 \geq 0)$

$-x_1 + x_2 + 3x_3 + x_5 = 3$ slack var. $(x_5 \geq 0)$

$x_1 = u_1 - v_1$, $u_1, v_1 \geq 0$ (x_1 is free variable)

min $z = u_1 - v_1 + 2x_2 - 2x_3$ — (1)

$2(u_1 - v_1) + x_2 - x_3 - x_4 = 2$ — (2)

$-u_1 + v_1 + x_2 + 3x_3 + x_5 = 3$. — (3)

$u_1, v_1, x_2, x_3, x_4, x_5 \geq 0$.

from ② & ③

Coeff matr. $A = \begin{array}{cccccc} u_1 & v_1 & x_2 & x_3 & x_4 & x_5 \end{array}$
$$A = \begin{pmatrix} 2 & -2 & 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 3 & 0 & 1 \end{pmatrix}$$

$x = (u_1, v_1, x_2, x_3, x_4, x_5)$

$C = (1, -1, 2, -2, 0, 0)^T$ from (*) _ from ①

$b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{vmatrix} +2 & 1 \\ -1 & 1 \end{vmatrix} = -2+1$
$= -1 \neq 0$

Rank(A) = 2. (check)

(*) $Ax = b$ — m constr., n variables.

$A \to m \times n$ → m rows, n columns.

Rank(A) = m

then $\exists$ some $m \times m$ submatrix B of A s.t. $|B| \neq 0$.

Let x_B be vector of variables associated with columns in B then x_B is Basic vector.

x_{NB} — Remaining vectors.

$$B = \begin{array}{cc} ① & ③ \end{array}$$
$$B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \qquad x_B = (u_1, x_2)^t$$

$(|B| \neq 0)$ $\qquad x_{NB} = (v_1, x_3, x_4, x_5)^t$

$$Ax = b$$

$$[B, R] \begin{bmatrix} x_B \\ x_{NB} \end{bmatrix} = b$$

$a_1x_1 + a_2x_2 + a_3x_3$

$a_1x_1 + a_3x_3 + a_4x_4$

$$Bx_B + Rx_{NB} = b$$

$$Bx_B = b - Rx_{NB}$$

$$x_B = B^{-1}b - B^{-1}Rx_{NB}$$

Basic solⁿ to the system

$$\boxed{\begin{array}{l} x_B = B^{-1}b \\ x_{NB} = 0 \end{array}}$$

— Basic solution

min value for x_{NB}

* Basic solⁿ may not follow non--ve constraints.

* If Basic solⁿ follows non-negativity constraint then it is called Basic feasible solⁿ.

$$x_B \geq 0.$$

* If $x_B > 0$ Non-degenerate Basic solⁿ. if all the basic variables are non-zero.

* no of basic feasible solutions $\leq$ number of basic solutions $\leq$ ${}^nC_m$

$$Bx_B = b$$

$$\begin{pmatrix}2 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}u_1\\ x_2\end{pmatrix} = \begin{pmatrix}2\\ 3\end{pmatrix}$$

Basic feasible solⁿ

$$D = \begin{array}{c} \begin{matrix} u_1 & x_5 \end{matrix} \\ \begin{pmatrix}2 & 0\\ -1 & 1\end{pmatrix} \end{array}, \quad |D| \neq 0$$

Basic feasible solⁿ
non-deg solⁿ

21st

Basic solⁿ $X_B, X_{NB} = 0$, maynot be optimal solⁿ

$$\min(z) = x_1 - x_2$$

Subject to constraints $\left.\begin{array}{l} x_1 + x_2 = 1 \\ x_1 \geq 0, \; x_2 \geq 0 \end{array}\right\}$ SLPP.

$$A = \begin{array}{cc} x_1 & x_2 \\ [1 & 1] \end{array}$$

$$B = \begin{array}{c} x_1 \\ [1] \end{array}$$

$$x_B = (x_1 = B^{-1} b = 1)$$

Basic solⁿ :- $x_1 = 1, \quad x_2 = 0.$ // Basic feasible solⁿ

but it is not giving optimal solⁿ.

Simplex Algorithm ⇒

step 1: Check whether the objective of LPP is to be maximized or minimized. If it is to be maximized, then convert it to minimization problem, using

$$\text{Max } Z = -\text{Min}(-Z)$$

step 2:- check whether all b_i (coefficients of constraint vector) are non-negative. If some b_i is negative then multiply that constraint by -1, so that we get ~~nor~~ all b_i non-negative.

step 3:- Convert all inequalities into equalities by adding slack & surplus variables. Put cost of these variables zero. Convert it to SLPP.

step 4:- ~~convert it to standard~~ Obtain initial basic feasible solution to the problem

$$x_B = B^{-1} b$$

Replacement ratio

$\theta = \dfrac{b}{\text{entering variable}}$

Simplex table

	$C_j \rightarrow$						
C_B ↓	x_B ↓ variables	x_1	x_2	S_1	S_2	b Initial basic sol^n	θ
	$Z_j = C_B x_j$						
	$C_j - Z_j$ Net Evaluat^n						

CB^{x_i}

Soln) ~~Maximize $z = 40 x_1$~~

Step5:- Compute net evaluations $c_j - z_j$

① If all $(c_j - z_j) \geq 0$, then this basic soln is optimal soln

optimal

② If $\exists$ atleast one j, such that $c_j - z_j < 0$ and all entries in the column (except cost value c_j) are negative then the soln is Unbounded

Unbounded

③ If none of ① and ② holds, then choose the most negative value of $(c_j - z_j)$. Let it be $c_r - z_r$ for some $j = r$

Step6:- compute replacement ratio $\theta = b / x_r$, choose min non-negative value of θ and be it y_s basis vector. (row)

$\theta = b/x_r$ min +ve

$y_s \rightarrow$ leaves the basis

$y_r \rightarrow$ enters the basis

(Update c_B after updating basis). Intersectn of y_s and y_r is pivot elem.

Step7:- Convert pivot elem to unity and all other elems in column to zero by elem row opns. (including column b)

Step8:- Compute values of z_j and $c_j - z_j$ and repeat step 5 onwards until optimal soln is obtained or there is indication of unbounded soln.

Ex.17 maximize $Z = 107x_1 + x_2 + 2x_3$

Subject to constraints

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5 \qquad + x_5 = 5$$

$$3x_1 - x_2 - x_3 \leq 0 \qquad + x_6 = 0$$

$$x_1, x_2, x_3, x_4 \geq 0 \qquad x_5, x_6 \geq 0$$

→ Step 1: minimize $-Z = -107x_1 - x_2 - 2x_3$

Step 3: $16x_1 + \frac{1}{2}x_2 - 6x_3 + x_5 = 5$ $(x_5 \geq 0)$

$3x_1 - x_2 - x_3 + x_6 = 0$ $(x_6 \geq 0)$

Check for non-negativity constraint (are variables are +ve non-negative)

Step 4:

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 14 & 1 & -6 & 3 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rank(A) = 3.

$$B = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$|B| \neq 0$.

$\boxed{x_B = B^{-1} b}$

$x_4 = 7/3$, $x_5 = 5$, $x_6 = 0$

Basic soln $\Rightarrow$ $x_B \Rightarrow$ $x_4 = 7/3$, x_5, x_6

$x_{NB} \Rightarrow x_1 = 0$, $x_2 = 0$, $x_3 = 0$

} Initial basic feasible solution.

Start forming simplex table.

C_j = coefficient of (x_1) in z.
cost vector

6 variables → so 10 columns.

from Z

cost of Basis vector

		C_j	-107	-1	-2	0	0	0	
C_B		x_B	x_1	x_2	x_3	x_4	x_5	x_6	b
0		x_4	14	1	-6	3	0	0	7 → 0
0		x_5	16	1/2	-6	0	1	0	5
0	x_j	x_6	3	-1	-1	0	0	1	0
		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	-107	-1	-2	0	0	0	

step 5 (ii)

c_j is increase in z

C_j measure is increase in cost

* Z_j measure is decrease in cost

$$Z_j = C_B \cdot x_j . \quad \text{// compute for all columns.}$$

$$= (0,0,0)\begin{pmatrix}14\\16\\3\end{pmatrix}$$

$$Z_j = 0$$

* $c_j - z_j$ = Net effect of variables on cost.
* Examine sign of net evaluation $c_j - z_j$.
* from steps 5, (ii), sol^n for LPP is unbounded.

Ex. 2) maximize $z = 4x_1 + 10x_2$

subject to constraints

$$2x_1 + x_2 \le 50$$
$$2x_1 + 5x_2 \le 100$$
$$2x_1 + 3x_2 \le 90$$
$$x_1, x_2 \geq 0$$

→ S1 ⇒ maximize $z = 4x_1 + 10x_2$

convert to minimizatⁿ problem.

$$\text{Max } z = -\text{Min}(-z)$$

$$\therefore \; -z = -4x_1 - 10x_2$$

S2 ⇒ all b_i's are non-negative

S3 ⇒ convert inequalities to equalities

$$2x_1 + x_2 + x_3 = 50$$
$$2x_1 + 5x_2 + x_4 = 100$$
$$2x_1 + 3x_2 + x_5 = 90$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0. \qquad x_i \geq 0 \;\forall\, 1 \le i \le 5$$

Step 4 ⇒

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rank (A) = 3

$$B = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

S4 ⇒ $x_B = B^{-1} b$ initial basic feasible sol^n

Basic feasible sol^n ⇒ $x_B \Rightarrow x_3 = 50, \quad x_4 = 100, \quad x_5 = 90$

$x_{NB} \Rightarrow x_1 = 0, \quad x_2 = 0$

Simplex table

	C_j	-4	-10	0	0	0	0	b/x2
C_B ↓	x_B ↓	x_1	x_2	x_3	x_4	x_5	b	θ =
0	x_3	2	1	1	0	0	50	50 (45)
0	x_4	2	pivot 5	0	1	0	100	20 → leaves Basis
0	x_5	2	3	0	0	1	90	30
	Z_j	0	0	0	0	0		
	$C_j - Z_j$	-4	-10 ↑ enter basis (y₂)	0	0	0		

$Z_j = C_B \cdot x_j$

$= 0$

Step 5 (ii) $C_j - Z_j < 0$ but not unbounded. sol^n

(iii) choose most non-negative value of $C_j - Z_j$ i.e. −10 and will replace basis.

Step 6 ⇒ Replacement ratio θ ⇒

$\theta = b/x_2$ → (आधार)

* update C_B after updating basis

pivot elem ⇒ common elem in 2th & 5th column

Convert pivot elem to unity & all other elems to zero

		C_j	−4	−10	0	0	0	= b/x_2
	C_B	x_B	x_1	x_2	x_3	x_4	x_5	θ
R_1	0	x_3	8/5	0	−1/5	0	0	30
R_2	−10	x_2	2/5	1	[illegible]	1/5	0	20
R_3	0	x_5	2 4/5	[illegible]	0	[illegible] −3/5	1	30
		Z_j	−4	−10	0	−2	0	
		$C_j - Z_j$	0	0	0	2	0	

$\frac{2}{5}$ $\frac{10-2}{5}$

$R_2 \leftarrow R_2/5$

$R_1 \leftarrow R_1 - R_2$

$R_3 \leftarrow R_3 - 3R_2$

$4 - \frac{6}{5}$ $\frac{20-6}{5}$ $\frac{14}{5}$

$\frac{6}{5}$ $\frac{10-6}{5}$

$z_j = C_B \cdot x_j$ $\quad \frac{-10 \times 2}{8}$

$= (0, -10, 0) \begin{pmatrix} \\ \\ \end{pmatrix}$

* $c_j - z_j \geq 0$ then basic sol^n is optimal sol^n.

$x_2 = 20$ and ~~$x_3 = 30$~~ $x_3 = 30$

$x_1 = 0$ ~~$x_4 = 30$~~ $x_5 = 30$

$\min(-z) = -4x_1 - 10x_2$

$= 0 - 10 \times 20$

$= -30$

HW. Q:- Solve by simplex method.

Ex. 1) Maximize $z = 6x_1 - 2x_2$

subject to $2x_1 - x_2 \le 2$

$x_1 \le 4$

$x_1, x_2 \geq 0$

→ ① Minimize $-z = -6x_1 + 2x_2$

② all b_i's are non-negative

③ $2x_1 - x_2 + x_3 = 2$

$x_1 + x_4 = 4$

$x_i \geq 0, \quad \forall \; 1 \le i \le 4$

x_3, x_4 are slack variables

④
$$A = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$A = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Rank(A) = 2

$$\text{Basic matrix } B = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \quad |B| \neq 0$$

$x_B = \{x_3, x_4\}$

$x_B \Rightarrow x_3 = 2, \quad x_4 = 4.$

$x_{NB} \Rightarrow x_1 = 0, x_2 = 0.$

Simplex table .

	c_j	-6	2	0	0		leaving basic
C_B	x_B	x_1	x_2	x_3	x_4	b	$\theta = b/x_1$
0	x_3	2	-1	1	0	2	1 → (2/2)
0	x_4	1	0	0	1	4	4 (4/1)
	z_j	0	0	0	0		
	$c_j - z_j$	-6	2	0	0		min non-negative value
		↑ entering basic					

$z_j = C_B . x_j$
$= 0$ as $C_B = (0,0)^T$.

steps (iii) $c_j - z_j = -6$ most negative value.
$\boxed{c_1 - z_1 = -6}$

$\therefore$ $\boxed{\theta = b/x_1}$ and then update C_B vector.*

	c_j	-6	2	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	b	θ
* -6	x_1	2 pivot	-1	1	0	2	1
0	x_4	1	0	0	1	4	4
	z_j	0	0	0	0		
	$c_j - z_j$	-6	2	0	0		

convert pivot elem to unity & all other elems in the column as zero. by elem row operation perform operations only on constraint eq^{ns}.

$R_1 \leftarrow R_1/2$, $R_2 \leftarrow R_2 - R_1$

		c_1	c_2	c_3	c_4		
	C_j	-6	2	0	0	b	θ
C_B	x_B	x_1	x_2	x_3	x_4		
R_1 -6	x_1	1	$-1/2$	$1/2$	0	1	$1 \rightarrow$
R_2 0	x_4	0	$1/2$	$-1/2$	1	3	4
	Z_j	-6	3	-3	0		
	$C_j - Z_j$	0	-1	3	0		

$Z_j = C_B^T \cdot x_j$

$$Z_1 = (-6\ 0)\begin{pmatrix}1\\0\end{pmatrix} = -6$$

$$Z_2 = (-6\ 0)\begin{pmatrix}-1/2\\1/2\end{pmatrix} = 6/2 = 3$$

$$Z_3 = -6 \times \frac{1}{2} = -3$$

$$Z_4 = 0$$

This is not optimal sol^n. so choose most -ve value of $C_j - Z_j = -1$ $(C_2 - Z_2 = -1)$

$\theta = b/x_2$

$= b/x_2$

C_B	C_j / x_B	-6 / x_1	2 / x_2	0 / x_3	0 / x_4	b	θ
-1	x_2	1	$-1/2$	$1/2$	0	1	-2
0	x_4	0	$1/2$	$-1/2$	1	3	6
	Z_j						
	$C_j - Z_j$	0	-1	3	0		

$R_2 \leftarrow R_2 + R_1$

$R_1 \leftarrow R_1 \times (-2)$, $R_2 \leftarrow R_2 + \frac{1}{2} R_1$

	C_j	-6	2	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	b	$\theta = b/x_2$
-1	x_2	-2	1	-1	0	-2	$1 \rightarrow$
0	x_4	$1/2$	0	0	1	4	2 min +ve
	Z_j	2	-1	1	0		
	$C_j - Z_j$	-8	3	-1	0		
		$\uparrow$ most -ve					

$\theta = b/x_4$ $R_1 \leftarrow R_1 \times (-2)$, $R_2 \leftarrow R_2 - \frac{1}{2} R_1$

	C_j	-6	2	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	b	$\theta = b/x_1$
-1	x_2	-2	1	-1	0	-2	$1 \rightarrow$
0	x_4	1	0	0	1	4	4
	Z_j	2	-1	1	0	2	
	$C_j - Z_j$	-8	3	-1	0		
		$\uparrow$					

$R_1 \leftarrow R_1 / (-2)$, $R_2 \leftarrow R_2 - 2R_1$

	C_j	-6	2	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	b	$\theta = b/x_2$
-8	x_1	1	$-1/2$	$1/2$	0	1	-2
0	x_4	0	-2	0	1	2	-1
	Z_j	-8	4	-4	0	-8	
	$C_j - Z_j$	$+2$	-2	4	0		soln is unbounded.
			$\uparrow$				

$3 - \frac{1}{2}$ $3 - \frac{1}{2} + \frac{1}{2}(-1)$ $-\frac{1}{2} - \frac{1}{2}$

$-\frac{1}{2} - \frac{1}{2}(-1)$

$1 - \frac{1}{2}(0)$

$3 - \frac{1}{2}(-2)$

$0 - 2(1)$ $1 - 2 \times \frac{1}{2}$

$4 - 2$

Ex 2) minimize $z = x_1 - 3x_2 + 3x_3$

subject to

$$3x_1 - x_2 + 2x_3 \le 7$$
$$2x_1 + 4x_2 \ge -12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

→ (1) min $z = x_1 - 3x_2 + 3x_3$

(2) all b_i's should be non-negative.

$-2x_1 - 4x_2 \le 12$

(3)

$$3x_1 - x_2 + 2x_3 + x_4 = 7$$
$$-2x_1 - 4x_2 + x_5 = 12.$$
$$-4x_1 + 3x_2 + 8x_3 + x_6 = 10.$$

$$x_i \ge 0 \quad \forall \; 1 \le i \le 6.$$

(4)

$$A = \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$A = \begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & -4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{array}{ccc} x_4 & x_5 & x_6 \end{array}$$
$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Rank}(A) = 3.$$

$x_B \Rightarrow x_4 = 7, \quad x_5 = 12, \quad x_6 = 10.$

$x_{NB} \Rightarrow x_1 = 0, \; x_2 = 0, \; x_3 = 0.$

	C_j	1	-3	3	0	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	b	θ
0	x_4	3	-1	2	1	0	0	7	-7
0	x_5	-2	-4	0	0	1	0	12	-3
0	x_6	-4	3 pivot	8	0	0	1	10	10/3 →
	Z_j	0	0	0	0	0	0		
	$C_j - Z_j$	1	-3 ↑	3	0	0	0		

$\theta = b/x_2$

		C_j	1	-3	3	0	0	0		
	C_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	b	$\theta = b/x_1$
R_1	0	x_4	5/3	0	14/3	1	0	1/3	31/3	31/5 →
R_2	0	x_5	-22/3	0	32/3	0	1	4/3	76/3	-38/11
R_3	-3	x_2	-4/3	1	8/3	0	0	1/3	10/3	-5/2
		Z_j	4	-3	-8	0	0	-1		
		$C_j - Z_j$	-3 ↑	0	11	0	0	1		

$R_3 \leftarrow R_3/3$, $R_2 \leftarrow R_2 + 4R_3$, $R_1 \leftarrow R_1 + R_3$

$\theta = b/x_1$

$R_1 \leftarrow R \times \frac{3}{5}$ — ①

$R_2 \leftarrow R_2 + \frac{22}{3} R_1$ — ② $R_3 \leftarrow R_3 + \frac{4}{3} R_1$

	C_j	1	-3	3	0	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	b	θ
1	x_1	1	0	14/5	3/5	0	1/5	31/5	
0	x_5	0	0	70/3	27/5	1	14/5	354/5	
-3	x_2	0	1	82/5	4/5	0	3/5	58/5	
	Z_j	1	-3	-82/5	-9/5	0	-8/5		
	$C_j - Z_j$	0	0	97/5	9/5	0	8/5		

Optimal Sol^n

$x_1 = 31/5, \quad x_2 = 58/5, \quad x_3 = 0$

$Z = x_1 - 3x_2 + 3x_3$

$= 31/5 -$

$= -143/5$

Charnes m-method

22nd morning

$$\min z = 12x_1 + 20x_2$$

$$6x_1 + 8x_2 \geq 100$$

$$7x_1 + 12x_2 \geq 120$$

$$x_1, x_2 \geq 0$$

SLPP

$$z = 12x_1 + 20x_2$$

$$6x_1 + 8x_2 - x_3 = 100$$

$$7x_1 + 12x_2 - x_4 = 120$$

$$x_i \geq 0, \quad \forall x \quad 1 \leq i \leq 4$$

x_3, x_4 are surplus variables

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 6 & 8 & -1 & 0 \\ 7 & 12 & 0 & -1 \end{pmatrix} \end{array}$$

$$B = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

Rank $(A) = 2$.

$x_B \Rightarrow x_3 = -100, \quad x_4 = -120$

$x_{NB} \Rightarrow x_1 = 0, \quad x_2 = 0.$

Basic sol^n is not feasible.

So we can't start simplex method.

Basic feasible solⁿ → If it satisfies non-negativity constraints ($x_B \geq 0$)

Charne's m - method ⇒

Add artificial variables

$$\min \quad z = 12x_1 + 20x_2 + Mw_1 + Mw_2$$

$$6x_1 + 8x_2 - x_3 + w_1 = 100$$
$$7x_1 + 12x_2 - x_4 + w_2 = 120$$

M is very big number.

$x_i \geq 0, \quad \forall \ 1 \leq i \leq 4$, $w_1, w_2 \geq 0$

Artificial var

$$A = \begin{array}{cccccc} & & & & w_1 & w_2 \end{array}$$
$$\begin{pmatrix} 6 & 8 & -1 & 0 & 1 & 0 \\ 7 & 12 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$x_B \Rightarrow w_1 = 100, \quad w_2 = 120$

$x_{NB} \Rightarrow x_i = 0 \quad \forall \ 1 \leq i \leq 4$

Basic feasible Solution

	C_j	12	20	0	0	M	M		
C_B	x_B	x_1	x_2	x_3	x_4	w_1	w_2	b	$\theta = b/x_2$
M	w_1	6	8	-1	0	1	0	100	12.5
M	w_2	7	12	0	-1	0	1	120	10 →
	Z_j	13M	20M	-M	-M	M	M		
	$C_j - Z_j$	12-13M	20-20M	M	M	0	0		
			↑						

Choose most -ve entry.

$R_2 \leftarrow R_2/12$ $R_1 \leftarrow R_1 - 8R_2$

<table>
<tr><td></td><td>12</td><td>20</td><td>0</td><td>0</td><td>0</td><td>M</td><td>M</td><td></td><td></td></tr>
<tr><td>C_B</td><td>x_B</td><td>x_1</td><td>x_2</td><td>x_3</td><td>x_4</td><td>w_1</td><td>$\cancel{w_2}$</td><td>b</td><td>θ</td></tr>
<tr><td>M</td><td>w_1</td><td>4/3</td><td>0</td><td>-1</td><td>0</td><td>1</td><td>$\cancel{2/3}$</td><td>20</td><td></td></tr>
<tr><td>20-20M</td><td>x_2</td><td>7/12</td><td>1</td><td>0</td><td>-1/12</td><td>0</td><td>$\cancel{1/12}$</td><td>10</td><td></td></tr>
<tr><td></td><td>Z_j</td><td>1.3M +</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>$C_j - Z_j$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

$x_1 = 15$

$x_2 = 3/4$

min z =

primal – original problem

Dual – Associated problem

Rule of thumb →
Additional constraints require more computational effort than additional variables.

23rd M.

Max $Z = 6x_1 + 5x_2$

subject to

$y_1 \leftarrow x_1 + x_2 \leq 5$ — ⓐ

$y_2 \leftarrow 3x_1 + 2x_2 \leq 12$ — ⓑ

$x_1, x_2 \geq 0$.

* without solving LPP, find an upper bound on Z.

* If primal should have large no. of constraints & relatively few variables, then its dual would require less computational effort due to interchange in the var & constraints.

Maximization problem

① $Z = \infty$ (If no constraint is there)

we want to improve this upper bound in terms of constraints so that it could give useful information.

Aim →

for eq ⓑ $3x_1 + 2x_2 \leq 12$ what is UB?

$\times 3$ $9x_1 + 6x_2 \leq 36$

$Z = 6x_1 + 5x_2 \leq 9x_1 + 6x_2 \leq 36$. improved upper bound.

for eqn ⓐ $x_1 + x_2 \leq 5$ $\times 6$

$6x_1 + 6x_2 \leq 30$

$Z = 6x_1 + 5x_2 \leq 6x_1 + 6x_2 \leq 30$ improved UP for both eq^n's ① and ② what is improved UB?

multiplying eq^ns ⓐ & ⓑ by y_1 and y_2

$y_1 \leftarrow x_1 + x_2 \le 5$ — ⓐ $\qquad z = 6x_1 + 5x_2$

$y_2 \leftarrow 3x_1 + 2x_2 \le 12$ — ⓑ

constraint number

$2\times ⓑ + ⓐ \Rightarrow 6x_1 + 4x_2 \le 24$

$+ \quad x_1 + x_2 \le 5$

$z = 6x_1 + 5x_2 \le 7x_1 + 5x_2 \le 29$ improved UB

$3\times ⓐ + ⓑ$

$3x_1 + 3x_2 \le 15$

$+ \; 3x_1 + 2x_2 \le 12$

$6x_1 + 5x_2 \le 27$

optimal sol^n Improved UB

ⓐ y_1 + ⓑ y_2

$y_1 x_1 + y_1 x_2 \le 5y_1$

$3y_2 x_1 + 2y_2 x_2 \le 12y_2$

$(y_1 + 3y_2)x_1 + (y_1 + 2y_2)x_2 \le 5y_1 + 12y_2 = \omega$

while finding the upper bound → we want to find min ω i.e. lower the UB, better.

min $\omega = 5y_1 + 12y_2$ such that

Subject to

$y_1 + 3y_2 \ge 6$

$z = 6x_1 + 5x_2$

$y_1 + 2y_2 \ge 5$ $\qquad y_1, y_2 \ge 0$

Primal LPP → original problem *

Max $z = 6x_1 + 5x_2$

subject to

$x_1 + x_2 \le 5$

$3x_1 + 2x_2 \le 12$

$A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$

$x_1, x_2 \ge 0$

Dual of LPP *

Min $\omega = 5y_1 + 12y_2$

sub to

$y_1 + 3y_2 \ge 6$

$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ $\quad y_1 + 2y_2 \ge 5$

$y_1, y_2 \ge 0$

Q17) Min $Z = 8x_1 + 5x_2 + 4x_3$.

subject to

$$4x_1 + 2x_2 + 8x_3 = 12$$
$$7x_1 + 5x_2 + 6x_3 \geq 9$$
$$8x_1 + 5x_2 + 4x_3 \leq 10$$
$$3x_1 + 7x_2 + 9x_3 \geq 7.$$

$x_1 \geq 0$, $x_2 \rightarrow$ unrestricted, $x_3 \leq 0$.

→ Min $Z = 8x_1 + 5x_2 + 4x_3$.

$$4x_1 + 2x_2 + 8x_3 = 12$$
$$7x_1 + 5x_2 + 6x_3 - x_4 = 9$$
$$8x_1 + 5x_2 + 4x_3 + x_5 = 10$$
$$3x_1 + 7x_2 + 9x_3 - x_6 = 7.$$

$x_1, x_4, x_5, x_6 \geq 0.$, $x_2 \rightarrow$ free, $x_3 \leq 0$.

$x_2 = u_2 - v_2$, $u_2, v_2 \geq 0$

$x_3 = -x_3'$, $x_3' \geq 0$

* SLPP ⇒ Min Z $= 8x_1 + 5u_2 - 5v_2 - 4x_3' + 0x_4 + 0x_5 + 0x_6$

$y_1 \leftarrow$ $4x_1 + 2u_2 - 2v_2 - 8x_3' = 12$

$y_2 \leftarrow$ $7x_1 + 5u_2 - 5v_2 - 6x_3' - x_4 = 9$

$y_3 \leftarrow$ $8x_1 + 5u_2 - 5v_2 - 4x_3' + x_5 = 10$

$y_4 \leftarrow$ $3x_1 + 7u_2 - 7v_2 - 9x_3' - x_6 = 7$

$x_1 \geq 0$, $u_2, v_2 \geq 0$, $x_3' \geq 0$.

dual problem:- Max $z = 12y_1 + 9y_2 + 10y_3 + 7y_4$

from cost

$4y_1 + 7y_2 + 8y_3 + 3y_4 \le 8$

$\boxed{\begin{array}{l} 2y_1 + 5y_2 + 5y_3 + 7y_4 \le 5 \\ -2y_1 - 5y_2 - 5y_3 - 7y_4 \le -5 \end{array}}$ * $2y_1 + 5y_2 + 5y_3 + 7y_4 = 5$

$-8y_1 - 6y_2 - 4y_3 - 9y_4 \le -4 \Rightarrow 8y_1 + 6y_2 + 4y_3 + 9y_4 \ge 4$ *

all constraints will be $\le$ since we started with minimization problem

$-y_2 \le 0$ x_4

$y_3 \le 0$ x_5

$-y_4 \le 0$ x_6

from cost

from simplex

$\left.\begin{array}{l} y_2 \ge 0 \\ y_3 \le 0 \\ y_4 \ge 0 \end{array}\right\}$ y_1 is free variable as no condition on y_1

* As primal was minimization problem, in dual all constraints will be $\le$ type.

* If primal is maximization problem then in dual all constraints will be $\ge$ type.

* simplify constraints → by combining constraints

Simplify constraints such that no. of constraints in dual problem is equal to number of variables in given primal problem.

y_4

$2+$

$_4 = 5$

y_2+

$y_4 \geq 4$

y_1 is free variable as no ondition n y.

No. of new variables in dual problem = No. of constraints in the primal problem.

primal coeff matrix

$$A = \begin{array}{ccccccc} x_1 & u_2 & v_2 & x_3' & x_4 & x_5 & x_6 \end{array}$$
$$\begin{pmatrix} 4 & 2 & -2 & -8 & 0 & 0 & 0 \\ 7 & 5 & -5 & -6 & 1 & 0 & 0 \\ 8 & 5 & -5 & -4 & 0 & 1 & 0 \\ 3 & 7 & -7 & -9 & 0 & 0 & 1 \end{pmatrix}$$

Primal

$$\text{Max } Z \xrightarrow{SLPP} \text{Min } z \xrightarrow{\text{dual}} \underset{\text{primal}}{\text{Max } W} \longrightarrow \underset{\text{dual}}{\text{min } w}$$

$\max w = -\min(-w)$

* Coefficient matrix of dual problem = Transpose of coefficient matrix of primal problem.

* Primal - Dual Relationship ⇒ $\max(Z) \leq \min(w)$, $\min(z) \geq \max(w)$

① Weak Duality theorem :- For a maximization primal problem, every feasible solution to the dual has an objective value greater than or equal to the objective value of primal problem at any of its (primal) feasible solution.

② Optimal Criterion Theorem :- If a primal problem (max Z) has a feasible solution X and dual problem (min w) has a feasible sol^n Y. such that value of objective function Z at X is same as value of objective f^n W at Y.

primal (x_1, x_2, x_3) max $z \to z^*$

Dual (y_1, y_2) min $w \to w^*$

If $\boxed{z^* = w^*}$ then (x_1, x_2, x_3) is optimal sol^n of primal & (y_1, y_2) is optimal sol^n of dual problem

(2)

Then X and Y are optimal solutions of primal problem and dual problem

① weak duality → (max) primal problem objective f^n val $\leq$ dual problem objective f^n value

Ex. Primal

Max $z = 6x_1 + 5x_2$

y_1 — $x_1 + x_2 \leq 5$

y_2 — $3x_1 + 2x_2 \leq 12$

$x_1, x_2 \geq 0$

Dual

Min $w = 5y_1 + 12y_2$

$y_1 + 3y_2 \geq 6$

$y_1 + 2y_2 \geq 5$

$y_1, y_2 \geq 0$

⇒ some feasible solutions to

Primal problem

$\leq 5, \leq 12$

x_1, x_2		z
(0,0)	→	0
(1,1)	→	11
(2,2)	→	22
(2,3)	→	27

$x_1 + x_2 = 4 \leq 5$

$3x_1 + 2x_2$

$6 + 4 = 10 \leq 12$

optimal

Dual problem

$\geq 6, \geq 5$ $y_1\ y_2$		z
(0,3)	→	36
(1,2)	→	29
~~(2,1)~~	→	~~22~~
(3,1)	→	27

$4 \geq 5$

optimal

w is giving UB of z so all these values should be $\geq$ primal(z) optimal criteria thm is telling that

(1) $(2,3)$ is optimal solⁿ for primal problem

$(3,1)$ —————— " —— dual problem.

primal is giving lower Bound, dual is giving UB of z.

Objective was to find UB when primal → dual conv.

Dual of ω will give LB of ω.

once LB of some fⁿ is same as UB means we have attained the optimal value.

26th Morning

* Mathematical Explaination to Dual ⇒

Primal Problem	Dual Problem.
Max $Z = 6x_1 + 5x_2$	Min $\omega = 5y_1 + 12y_2$
$y_1 \leftarrow x_1 + x_2 \le 5 + \delta$	$y_1 + 3y_2 \ge 6$
$y_2 \leftarrow 3x_1 + 2x_2 \le 12 + \nu$	$y_1 + 2y_2 \ge 5$
$x_1, x_2 \ge 0$	$y_1, y_2 \ge 0$
optimal sol^n ⇒ $x_1 = 2,\ x_2 = 3$	optimal sol^n ⇒ $y_1 = 3$, $y_2 = 1$
optimal value ⇒ $Z = 27$	optimal value ⇒ $\omega = 27$

* A small change δ in ~~constaint~~ constaint of first constraint of primal problem gives the foll^n optimal sol^n of primal problem

$x_1 = 2 - 2\delta, \quad x_2 = 3 + 3\delta$

$Z = 27 + 3\delta$

$12 - 12\delta + 15 + 15\delta$

$27 + 3\delta$

* A small change ν in ~~constraint~~ of constant of 2nd constraint of primal prob^m gives foll^n optimal sol^n

$x_1 = 2 + \nu, \quad x_2 = 3 - \nu$

$Z = 27 + 1\nu$

$12 + 6\nu + 15 - 5\nu$

$27 + \nu$

classmate
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6th Morning
em.
$12y_2$
$y_2 = 1$
7
nstraint
sol^n
$-15+15\delta$
δ
of
ptimal
$+15-5\gamma$
$+\gamma$

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* The value of the variable y_i in the optimal sol^n of dual problem gives us rate of change of objective f^n of primal problem for a small change in the value of constant in the constraint of primal, corresponding to variable y_i.

① $(x_1 + x_2 = 5 + \delta) * 3 \rightarrow 3x_1 + 3x_2 = 15 + 3\delta$

$3x_1 + 2x_2 = 12$ $\qquad - \; 3x_1 + 2x_2 = 12$

$$\boxed{x_2 = 3 + 3\delta}$$

~~$x_1 + x_2 =$~~

$x_1 + 3 + 3\delta = 5 + \delta$

$$\boxed{x_1 = 2 - 2\delta}$$

$12 - 12\delta + 15 + 15\delta$

$\Rightarrow z = 6(2 - 2\delta) + 5(3 + 3\delta)$

$$\boxed{z = 27 + 3\delta}$$

② $(x_1 + x_2 = 5) * 3 \Rightarrow 3x_1 + 3x_2 = 15$

$3x_1 + 2x_2 = 12 + \gamma$ $\qquad 3x_1 + 2x_2 = 12 + \gamma$

$$\boxed{x_2 = 3 - \gamma}$$

~~$3x_1 + 2 - \gamma = 5$~~

$2(3-\gamma)$

$3x_1 + 6 - 2\gamma = 12 + \gamma$

$3x_1 = 6 + 3\gamma$

$$\boxed{x_1 = 2 + \gamma}$$

$\Rightarrow$ ~~$z = 5y_1 + 12y_2$~~

$12 + 6\gamma + 15 - 5\gamma$

$z = 6(2 + \gamma) + 5(3 - \gamma)$

$x_1 = 2 + \gamma, \; x_2 = 3 - \gamma \Rightarrow \boxed{z = 27 + \gamma}$

26th _ m

* ~~Primal Dual Relationship~~ ⇒

* Physical meaning of dual ⇒

Let the company manufacture 2 types of ckt

	Type I	Type II	Available Stock
C	1	1	5
T	3	2	12
profit	6	5	

compute no. of type I & type II ckts to get maximum profit

$\max z = 6x_1 + 5x_2$

$x_1 + x_2 \leq 5$

$3x_1 + 2x_2 \leq 12$

⇒ $x_1 \rightarrow$ Type I ckt

$x_2 \rightarrow$ Type II ckt

$\text{Max } z = 6x_1 + 5x_2$

Subject to

$x_1 + x_2 \leq 5$ — C

$3x_1 + 2x_2 \leq 12$ — T

$x_1, x_2 \geq 0$

we can't make -ve no. of ckts so both variables to +ve

* Let manufacturer has some extra funds to buy more capacitors & transistors & he wants to find optimum value at which he should by C and T so that he should make +ve profit.

→ Now if you buy 1 capacitor i.e $\delta = 1$ then profit will change by 3δ

i.e. $Z = 27 + 3\delta$.

So now if you by 1 capacitor for Rs.4 $(\delta = 1)$ then profit will change by $3 * \delta$

i.e. 1 capacitor $\delta = 1$ for Rs. 4.

$$\text{profit} = 27 + \underbrace{3 * 1}$$
$$= \text{Rs. } 30$$

← So if you buy 1 capacitor at Rs. 4 profit will increase by Rs. 3. i.e we are in loss.

So we should not buy capacitor for more than Rs.3 to gain profit

→ Now if you buy 1 transister than profit will change by Rs. 1. So we should not pay more than Rs.1 to buy transister.

i.e. 1 transister $\gamma = 1$ for Rs. 1.

$$\text{profit} = 27 + \gamma$$
$$= \text{Rs. } 28$$

So we should not buy for more than Rs. 1 to get +ve profit

∴ LPP's and duals decides the rate in the market.

∴ Seller should sell capacitors close to Rs. 3

if SP = Rs. 2 ~~then~~ & company getting 3S profit so company might buy capacitors in Bulk to get profit

Dual Simplex Method :- used to solve any LPP

→ Simplex method

① SLPP

② Basic feasible solⁿ

$x_B \geq 0 \Rightarrow x_1 = b_1, x_2 = b_2$

~~If~~ $b_1, b_2 \geq 0$

If $b_1 < 0$, $b_2 < 0$ then its not Basic feasible solⁿ so use Charnes M method. & if you can't find solⁿs easily then use Dual Simplex method

Solving to obtain optimality → In simplex method we start with basic feasible solution which is not optimal & we do iterations till we get optimal feasible solutions

→ In dual simplex method we start with basic ~~non feasible~~ infeasible solution, which is actually better than optimal solution.

⇓

Solving to obtain feasibility

Start with Basic feasible sol^n

Start with basic infeasibility

simplex

dual simplex

$x_i \geq 0$ Basic variables

$x_i \leq 0$ for some basic vector.

$C_j - Z_j \geq 0$ optimality $cond^n$

$C_j - Z_j \rightarrow$ optimal $cond^n$

To retain optimal $cond^n$ make all basic variables +ve

$x_i \geq 0$ feasible. sol^n.

Ex.1) Dual simplex method

$$\text{Min } Z = 4x_1 + 7x_2$$

$$2x_1 + 3x_2 \geq 5$$

$$x_1 + 7x_2 \geq 9$$

$$x_1, x_2 \geq 0$$

→ $$\text{Min}(Z) = 4x_1 + 7x_2.$$

$$2x_1 + 3x_2 - x_3 = 5$$

$$x_1 + 7x_2 - x_4 = 9$$

$$A = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{pmatrix} 2 & 3 & -1 & 0 \\ 1 & 7 & 0 & -1 \end{pmatrix}$$

Rank$(A) = 2$

$$B = \begin{array}{cc} x_3 & x_4 \end{array}$$
$$\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

; x_3 & x_4 are Basic variables

In basic solⁿ → non-basic variables take zero value

Basic variables ∴ $x_B \Rightarrow x_3 = -5$, $x_4 = -9$ Not feasible i.e. Infeasible solⁿ.

Non-basic variables $x_{NB} \Rightarrow x_1 = 0$, $x_2 = 0$

($\because x_3, x_4 \geq 0$ in SLPP.

* Dual-simplex method ⇒

26th evening

	c_j	4	7	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	b	
0	x_3	2	3	-1	0	5	
0	x_4	1	7	0	-1	9	
	z_j	0	0	0	0		pivot entry ↓ values of Basic variables
	c_j-z_j	4	7	0	0		

In dual simplex I Starting with optimal but infeasible solⁿ

				pivot				
		✓	✓			b		θ
0	x_3	-2	-3	1	0	-5	$R_1 \leftarrow R_1 \times (-1)$	-4
0	x_4	-1	-7	0	1	-9	$R_2 \leftarrow R_2 \times (-1)$ → x_4 leaves basis	-1 →
	z_j	0	0	0	0	0	Step (f)	
	c_j-z_j	+4	7	0	0			
	θ	-4	↑ -1					

		4	7	0	0		
C_B	x_B	x_1 ✓	x_2	x_3	x_4 ✓	b	-ve so not optimum feasible solⁿ
0	x_3	-11/7	0	1	-3/7	-8/7	→
7	x_2	1/7	1	0	-1/7	9/7	
	z_j	1	7	0	-1		
	c_j-z_j	3	0	0	1		
	θ	-21/11 ↑	-	-	-7/3		

replacement ratio ⇒ Only for non-basic variables

$$\theta = \left\{ \frac{c_j - z_j}{x_{rj}} \Big/ x_{rj} < 0 \right\}$$ choose max^m value.

$\theta_1 = \frac{4}{-1} = -4$

$\theta_2 = \frac{7}{-7} = -1 \leftarrow$

choose max^m value & the corresponding vector x_r will enter the basis.

$\therefore x_2$ will enter the basis.

pivot = -7 → make it 1 & all other entries in its column as zero's.

$R_2 \leftarrow R_2 / (-7)$

$R_1 \leftarrow R_1 + 3R_2$

* Now x_3 leaves the basis. as $-8/7$ is most negatively.

All entries in x_3 are not non-negative so no infeasibility i.e. we still can get feasible sol^n

Replacement Ratio only for Non-basic Variables (x_1, x_4)

x_B: x_3, x_2 Basic variables

x_{NB}: x_1, x_4 Non-basic variables

$\theta_1 = 3/-\cancel{11}/7 = -21/\cancel{11} = -\cancel{1}.99$ Max^n Value

$\theta_4 = 1/-3/7 = -7/3$. ~~least ve~~. $= -2.33$

So x_1 will enter basis and x_3 will leave basis. Make pivot as 1 & other entries 0.

		4	7	0	0		
C_B	x_B	x_1	x_2	x_3	x_4	b	
4	x_1	1	0	$-7/11$	$3/11$	$8/11$	$R_1 \leftarrow R_1 \times -\frac{7}{11}$
7	x_2	0	1	$1/11$	$-2/\cancel{7}11$	$13/11$	
	Z_j	4	7	$-21/11$	$-2/11$		$R_2 \leftarrow R_2 - \frac{1}{7}R_1$
	$C_j - Z_j$	0	0	$21/11$	$2/11$		

As $C_j - Z_j \geq 0$ and $b_i \geq 0$ then this is optimum basic feasible sol^n

$x_1 = 8/11, \; x_2 = 13/11, \; x_3 = 0, \; x_4 = 0$

Min $Z = 4 \times 8/11 + 7 \times 13/11$

$Z = 123/11$ optimal Solution.

Practice problem $\Rightarrow$

Q. Use dual simplex method to solve LPP.

Min $Z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

subject to

$$5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12.$$

$$x_2 + 5x_3 - 6x_4 \geq 10$$

$$2x_1 + 5x_2 + x_3 + x_4 \geq 8.$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

28th morning.

Integer Programming Problem ⇒

* Special class of linear prog. problem (LPP) + all or some variables are constrainted to be integers.

Importance ⇒

(Physical problem)

when Fractional values of x_i doesnt make any sense. $x_i \xrightarrow{LPP}$ No. of persons, No of vehicles

Solve LPP & round it to nearest integer

3^{x+3}

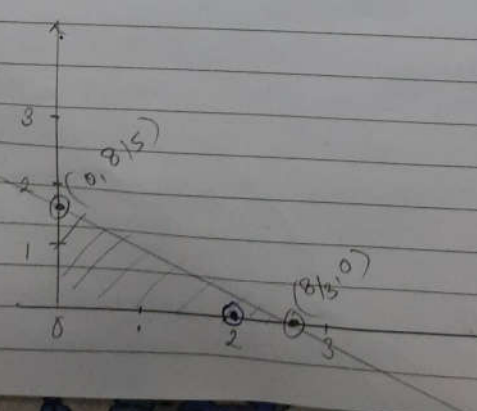
Ex.1) Max $Z = 3x + 4y$

subject to

$5y + 3x \leq 8$

$x, y \geq 0$, x & y are integers

→



$x = 8/3 = 2.66$

$y = 8/5 = 1.6$

vertex $\quad z = 3x + 4y$

$(0,0) \quad 0$

$(8/3, 0) \quad 8$ optimal

$(0, 8/5) \quad \frac{32}{5} = 6.4$

$x = 8/3, \; y = 0$

$x = 2.66 \begin{cases} 2 & (2,0) \\ 3 & (3,0) \end{cases}$ Nearest integer solutions

Vertex $\quad Z = 3x + 4y$.

$(2,0) \quad 6 \leq 8$ Is it optimal sol^n to IPP?

$(3,0) \quad 9 \nleq 8$ Infeasible. $\quad 5x + 3y \leq 8$

Better than $(2,0)$

$(1,1) \quad 7$ — can't be guessed by rounding off.

$8 \leq 8$ ✓ constraint satisfied.

By rounding off sol^n of LPP, you can't solve IPP effectively (no optimal sol^n).

Types of IPP $\begin{cases} \to ① \\ \to ② \end{cases}$

IPP $\Rightarrow$

The LPP,

Max or Min $z = c^t x$

$c = \{c_i\}$ cost factor

$x \longrightarrow$ vector of variables.

subject to

$A x = \underbrace{b}$ constant

$x \geq 0$ non-negativity

$X = \{x_1, x_2, \ldots x_n\}$

Some variables are constrainted to take integer values.

some $x_{j_1}, x_{j_2} \ldots x_{j_K}$ $K \geq 1$

$\forall j \in \{1, 2 \ldots n\}$

All IPP $\Rightarrow$ x_i has to take Integer values $\forall\ 1 \leq i \leq n$

Mixed IPP $\Rightarrow$ which is IPP but not all IPP.

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* Gomory's cut constraint method $\Rightarrow$

(1) For given IPP $\rightarrow$ write corresponding LPP and use
1) Simplex
2) Charnes M
3) Dual simplex

method to solve given IPP.

Ex $\Rightarrow$ Max $z = 3x + 4y$.

$$3x + 5y \leq 8$$

$$x, y \geq 0.$$

$\rightarrow$ $-\min(-z) = -3x - 4y$.

$$3x + 5y + s = 8.$$

$$x, y, s \geq 0$$

$$A = \begin{array}{ccc} x & y & s \\ \end{array}$$
$$\begin{pmatrix} 3 & 5 & 1 \end{pmatrix}$$

Rank(A) = 1.

$$B = \begin{array}{c} s \\ (1) \end{array}$$

Basic sol^n $\rightarrow$ $x_B \Rightarrow s = 8$

$x_{NB} \Rightarrow x = 0, y = 0$

$\left.\begin{array}{l} \\ \end{array}\right\}$ Basic sol^n, ≥ 0 — Basic feasible solution So use simplex method

Non-basic / solution

C_B	c_j / x_B	-3 / x	-4 / y	0 / s	b	$\theta =$
0	s	3	5	1	8	$8/5$
	$c_j - z_j$	-3	-4 ↑	0	0	

C_B	x_B	-3 / x	-4 / y	0 / s	b	$\theta = \frac{b}{(x)}$
-4	y	$3/5$	1	$1/5$	$8/5$	$8/3$
	z_j	$-12/5$	-4	$-4/5$		
	$c_j - z_j$	$-3/5$ ↑	0	$4/5$		

$\frac{3}{5}x$

C_B	x_B	-3 / x	-4 / y	0 / s	b	
-3	x	1	$5/3$	$1/3$	$8/3$	$\Rightarrow x + \frac{5}{3}y + \frac{1}{3}s = \frac{8}{3}$
	z_j	-3	-5	-1		
	$c_j - z_j$	0	1	1		optimal solution

Giving optimum Basic feasible soln which is Non-integral

should be integer

$x = 8/3,\ y = 0,\ z = 0$

$x + 5/3y + 1/3$

$x + \frac{5}{3}y + \frac{1}{3}s = \frac{8}{3}$.

* if $b_{11} > 0$ (optimal feasible sol^n)

let $b_{11} = I_{11} + f_{11}$, $I_{11} \geq 0$,

$\underbrace{I_{11}}_{\text{integer part}} + \underbrace{f_{11}}_{\text{fractional part}}$ $0 < f_{11} < 1$

here $b_{11} = 8/3$.

$\therefore$ $b_{11} = \underbrace{2}_{\text{Int.}} + \underbrace{2/3}_{\text{fract}^n}$

here $x + 5/3 y + \frac{1}{3}s = \underbrace{8/3}_{b_{11}}$.

$x + (1 + 2/3)y + (0 + \frac{1}{3})s = 2 + 2/3$ — (1)

Separate integer values on LHS & fractional on RHS.

$x + y + 2/3 y + \frac{1}{3}s = 2 + 2/3$

$\underbrace{x + y - 2}_{+0.s} = \frac{2}{3} - \frac{2}{3}y - \frac{1}{3}s$. — (2)

add constraint such that fractional value ≤ 0

$\frac{2}{3} - \frac{2}{3}y - \frac{1}{3}s \leq 0$

$$\left.\begin{aligned} \tfrac{2}{3} - \tfrac{2}{3}y - \tfrac{1}{3}s &\le 0 \\ -\tfrac{2}{3} + \tfrac{2}{3}y + \tfrac{1}{3}s &\ge 0 \end{aligned}\right) (-1)$$

$$\boxed{\tfrac{2}{3}y + \tfrac{1}{3}s \ge \tfrac{2}{3}}$$

* Using this Gromony's constraint of last simplex table form new LPP.

New LPP with Gromony's constraint

$$\min z = -3x - 4y + 0s$$

subto

$$x + \tfrac{5}{3}y + \tfrac{1}{3}s = \tfrac{8}{3}$$

$$\tfrac{2}{3}y + \tfrac{1}{3}s \ge \tfrac{2}{3}$$

$$x, y, s \ge 0.$$

i) conv to SLPP.

$$\min Z = -3x - 4y + 0s + 0g$$

Sub to $x + \tfrac{5}{3}y + \tfrac{1}{3}s = \tfrac{8}{3}$

$$\tfrac{2}{3}y + \tfrac{1}{3}s - \underline{\underline{g}} = \tfrac{2}{3}$$ gomory's variable

$$x, y, s, g \ge 0$$

$$A = \begin{array}{cccc} x & y & s & g \\ \end{array}$$
$$A = \begin{pmatrix} 1 & 5/3 & 1/3 & 0 \\ 0 & 2/3 & -1/3 & -1 \end{pmatrix} \qquad \text{Rank}(A) = 2$$

$$\begin{array}{cc} x & g \end{array}$$
$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$x_B \Rightarrow x = 8/3$, $\qquad x_{NB} \Rightarrow$ Non-Basic variable $\quad y = 0$, $S = 0$

Basic variables (x, g) x_B $\qquad g = -2/3$

↓

Sol^n is optimal but not feasible. so use dual simplex method to solve given LPP.

Dual Simplex table ⇒

	C_j	-3	-4	0	0		
C_B	x_B	x	y	s	g	b	
-3	x	1	$5/3$	$1/3$	0	$8/3$	1) pivot = 1.
0	g	0	$-2/3$	$-1/3$	-1	$-2/3$	→ ×(-1) most -ve leaves.
	z_j	-3	-5	-1	0	$(+1)$	
	$C_j - z_j$	0	1	1	0		← optimal sol^n but infeasible as one of b is -ve.
	θ	—	$-3/2$	-3	—		

$\theta = \dfrac{C_j - z_j}{g_j}$ for non basic y, s

↑ max val enters.

y enters g leaves the basis

$R_2 \leftarrow -\frac{3}{2} R_2$, $R_1 \leftarrow R_1 - \frac{5}{3} R_2$

$R_1 \leftarrow \frac{3}{5} R_1$

		-3	-4	0	0		
C_B	x_B	x	y	s	g	b	
-3	x	1	0	-3/2	5/2	1	} ≥ 0 feasible soln
-4	y	0	1	1/2	-3/2	1	
	Z_j	-3	-4	-1/2	-3/2		
	$C_j - Z_j$	0	0	1/2	3/2	] ≥ 0	optimum soln

$x = 1,\ y = 1,\ s = 0,\ g = 0$

LPP ⟶ SLPP ⟶ simplex Method —

New SLPP ← New LPP ← Add Gromory's constraint

non-feasible better than optimal soln

dual simplex Method ⟶ Integral soln is obtained.

$\} \geq 0$ feasible soln

ptimum soln

constraint

Q17 Max $Z = x_1 + x_2$

Sub to

$$3x_1 + 2x_2 \leq 5$$

$$x_2 \leq 2$$

$x_1 \geq 0$ and $x_2 \geq 0$ & are integers.

→ 1) SLPP

$-\text{Min}(-Z) = -x_1 - x_2$

Sub to

$$3x_1 + 2x_2 + x_3 = 5$$

$$x_2 + x_4 = 2$$

$$x_1, x_2, x_3, x_4 \geq 0$$

2)

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$ Rank(A) = 2.

$$B = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Basic soln $x_B \Rightarrow x_3 = 5, \; x_4 = 2$

$x_{NB} \Rightarrow x_1 = 0, \; x_2 = 0.$

Simplex Table.

c_B	c_j / x_B	-1 / x_1	-1 / x_2	0 / x_3	0 / x_4	b	$\theta = b/x_1$
0	x_3	3	2	1	0	5	5/3 →
0	x_4	0	1	0	1	2	–
	z_j	0	0	0	0		
	$c_j - z_j$	-1 ↑	-1	0	0		not optimal.

choose most -ve value → $c_j - z_j$

$f\text{cale } \theta = b/x_r$

c_B	c_j / x_B	-1 / x_1	-1 / x_2	0 / x_3	0 / x_4	b	$\theta = b/x_2$
-1	x_1	1	2/3	1/3	0	5/3	~~t/2~~ 5/2 →
0	x_4	0	1	0	1	2	2 →
	z_j	-1	$-2/3$	$-1/3$	0		
	$c_j - z_j$	0	$-1/3$ ↑	1/3	0		

Eq^n Max^m fractio part

	c_B	c_j / x_B	-1	-1	0	0	b	
$R_1 \leftarrow R_1 \times (3/2)$	-1	x_2	3/2	1	1/2	0	5/2	not feasible
$R_2 \leftarrow R_2 - R_1$	0	x_4	$-3/2$	0	$-1/2$	1	$-1/2$	
		z_j	$-3/2$	-1	$-1/2$	0		
		$c_j - z_j$	1/2	0	1/2	0		→ optimal.

Apply dual-simplex method

$= b/x_1$

$73 \rightarrow$

ot

ptimal.

b/x_2

$5/2 \rightarrow$

$2 \rightarrow$

ible

al.

$R_1 \leftarrow R_1 - \frac{2}{3} R_2$

		-1	-1	0	0	
C_B	x_B	x_1	x_2	x_3	x_4	b
-1	x_1	1	0	$+1/3$	$-2/3$	$1/3$
-1	x_2	0	1	0	1	2
	Z_j	-1	-1	$-1/3$	$-4/3$	
	$C_j - Z_j$	0	0	$+1/3$	$4/3$	

$1/3$ → not integer. → R_1^* → R_2^*

optimal feasible sol^n for LPP.

Unbounded

$x_1 = 1/3, \quad x_2 = 2.$

eq^n with Max^m fractional part → $R_1^* \rightarrow x_1 + 1/3\, x_3 - 2/3\, x_4 = 1/3 \rightarrow (0 + 1/3)$

choose eq^n with Max^m fractional part

$R_2^* \rightarrow x_2 + x_4 = 2 \rightarrow (2+0)$

Integer part can be +ve/-ve but fractional part should be +ve.

$-2/3 \Rightarrow (-1 + \frac{1}{3})$

$x + \frac{1}{3} x_3$

$(1+0)x_1 + (0 + 1/3)x_3 + (-1 + 1/3)x_4 = 0 + 1/3$

$x_1 + 1/3\, x_3 - x_4 + 1/3\, x_4 = 1/3$

$x_1 - x_4 = 1/3 - 1/3\, x_3 - 1/3\, x_4$

Integral values — fractional values

Gomory's Constraint $\Rightarrow$ Make fractional values -ve

$$-\frac{1}{3}x_3 - \frac{1}{3}x_4 + \frac{1}{3} \leq 0$$

$$-\frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{1}{3} \qquad * (-3)$$

$$\frac{1}{3}x_3 + \frac{1}{4}x_4$$

$$x_3 + x_4 \geq 1$$

$$x_3 + x_4 - g = 1$$

gomory's variable is always a surplus variable

New SLPP $\Rightarrow$ Add above constraint to current constraints.

$$\min(-Z) = -x_1 - x_2$$

sub to $\quad x_1 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{1}{3}$

$$x_2 + x_4 = 2$$

$$x_3 + x_4 - g = 1 \quad \leftarrow \text{added this}$$

$$x_i \geq 0 \quad 1 \leq i \leq 4, \quad g \geq 0$$

Now use dual simplex method

simplex

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & g \end{array} \\ \begin{pmatrix} 1 & 0 & 1/3 & -2/3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Rank(A) = 3

Basis Vector =

$$B = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & g \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

$-g = 1$

$g = -1$

$x_B \Rightarrow x_1 = 1/3,\ x_2 = 2,\ g = -1$

$x_{NB} \Rightarrow x_3 = 0,\ x_4 = 0.$

Not feasible but optimal better than as more constraints

* Gomory's cut constraint ($\leq$) $(-g)$ leads to infeasibility & that why Gomory's cut constraint method calls for Dual simplex method.

$\boxed{\begin{array}{c} x_1 = 1/3 \\ x_2 = 2 \end{array}}$

prev.

Optimal sol^n of less constraint

$\boxed{\begin{array}{c} x_1 = 1/3 \\ x_2 = 2 \\ g = -1 \end{array}}$

now

Better than optimal.

As we are putting more constraints

$-z = -1/3 - 2 = -7/3$

$-z = -x_1 - x_2$

$-z = -x_1 - x_2 + g$

$-z = -1/3 - 2 - 1$

$\Rightarrow -10/3$

Dual simplex method ⇒

Make pivot = 1

C_B	c_j / x_B	x_1 (−1)	x_2 (−1)	x_3 (0)	x_4 (0)	g (0)	b
−1	x_1	1	0	1/3	−2/3	0	1/3
−1	x_2	0	1	0	1	0	2
0	g	0	0	−1 (pivot)	−1	+1	−1 →
	Z_j	−1	−1	−1/3	−1/3	0	
	$c_j - Z_j$	0	0	1/3	1/3	0	
	$\theta = \frac{c_j - Z_j}{g}$	–	–	−1/3 ↑	−1/3	—	

not feasible

Better the optimal soln ←

$R_3 \leftarrow R_3 \div (-1)$

$R_1 \leftarrow R_1 - \frac{1}{3} R_3$

C_B	c_j / x_B	x_1 (−1)	x_2 (−1)	x_3 (0)	x_4 (0)	g (0)	b
−1	x_1	1	0	0	−1	1/3	0
−1	x_2	0	1	0	1	0	2
0	x_3	0	0	1	1	−1	1
	Z_j	−1	−1	0	0	−1/3	
	$c_j - Z_j$	0	0	0	0	1/3	

R

feasible

optimal

$x_1 = 0,\ x_2 = 2,\ \cancel{x_3 = 1}$

Slack

$\text{Min}(-Z) = -x_1 - x_2$

$\text{Min}(-Z) = -2$

$\text{Max}(Z) = x_1 + x_2$

$= 2$

(OR $-\text{Min}(-Z) = -(-2) = 2$

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31st M

* Non-linear Programming Problems

linvar x.

In LPP, we have

Objective $f^n \rightarrow Z = C^T x$
constraints $\rightarrow \begin{cases} Ax = B \\ x \geq 0 \end{cases}$

when all these are linear fns, it is LPP.

All real life problems may not be LPP.

* General Non-linear Programming Problem $\Rightarrow$

Let Z be a real valued f^n of n variables defined by

$$Z = f(x_1, x_2, x_3 \dots x_n) \quad \text{— (A)}$$

and let $\{b_1, b_2, \dots b_n\}$ be set of constants s.t we have set of constraints

'm' constraints

$$g^1(x_1, x_2, \dots x_n) \{\leq \text{ or } \geq \text{ or } =\} b_1$$
$$\vdots$$
$$g^m(x_1, x_2, \dots x_n) \{\leq \text{ or } \geq \text{ or } =\} b_m$$

— (B)

where all these g^i's are real valued functions of 'n' variables $x_1, x_2, \dots x_n$ and

set $x_i \geq 0 \ \forall \ i \in (0 \text{ to } n)$ — (C)

* If either objective f^n $f(x_1, x_2 \dots x_n)$ OR some constraint f^n $g^i(x_1, x_2 \dots x_n)$ are non-linear f^{ns} ~~are~~ or Both then the problem of determining . n-tuple $(x_1, x_2 \dots x_n)$ where z takes max^m / min value and satisfying constraints (B) and (C) is called General NLPP.

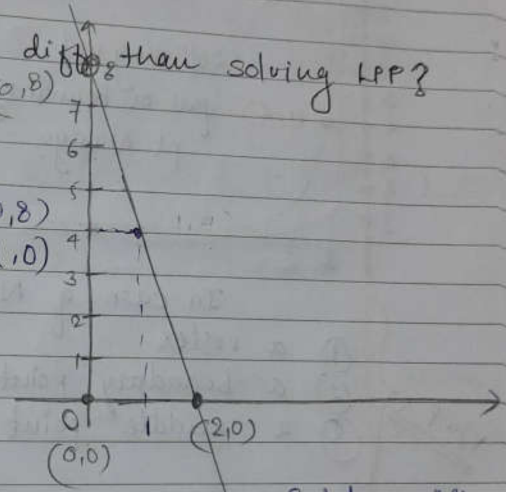
How is solving NLPP differ than solving LPP?

⇒ Max $z = x + y$

$4x + y \le 8$ → (0,8)

$x \ge 0,\ y \ge 0$. → (2,0)

$x = 1,\ y = 4$



~~Max(Z) = 1+4~~
~~= 5~~

vertex	Z
(0,0)	= 0
(2,0)	= 2
(0,8)	= 8 ← Max(Z).

Solution will be at one of the vertex for LPP.

NLPP

→ Max(Z) = xy.

$4x + y \le 8$

$x \ge 0,\ y \ge 0$

Vertex	Z
(0,0) →	0
(0,0)	0
(2,0)	0
(0,8)	
(1,4) →	4

This f^n is not taking sol^n on vertex.

Plot for various values of objective f^n

$xy = k$

eg $xy = 1 \longrightarrow$ Rectangular hyperbola

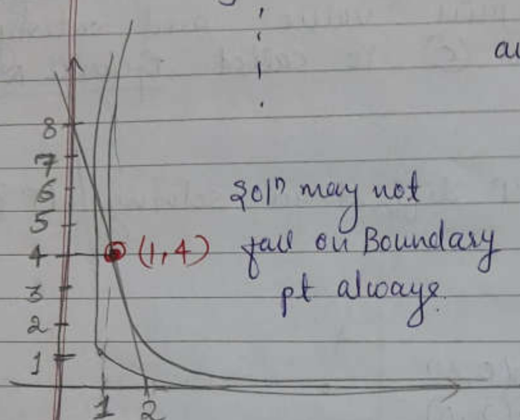
$xy = 2 \longrightarrow$

$xy = 3 \longrightarrow$

$xy = 4 \longrightarrow x = 1, y = 4$

$\vdots$

and for $xy = k$ where $k > 4$ the hyperbola will lie outside the region.



sol^n may not fall on Boundary pt always.

Max(Z) = 4 at (1,4)

In case of NLPP, solution may be

1. a vertex
2. a boundary point
3. a middle point

V Imp

* There are infinitely many possibilities for solution

* Type of functions ⇒

1) ~~Concave~~ Convex function ⇒ Let S be a non-empty convex subset of $\mathbb{R}^n$. A f^n $f(x)$ on S is said to be convex, if for any two pts.

$x_1 = (x_{11}, x_{12} - - - x_{1n}) \in S$

$x_2 = (x_{21}\ x_{22} - - - x_{2n}) \in S$, we have.

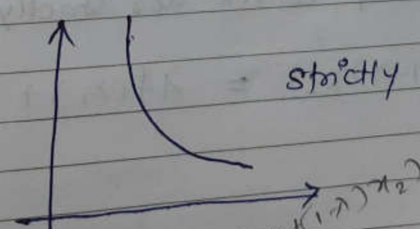
f^n operated on convex comb^n is

$$\underbrace{f(\lambda x_1 + (1-\lambda)x_2)}_{f(x)} \leq \lambda f(x_1) + (1-\lambda) f(x_2) \quad \text{—①}$$

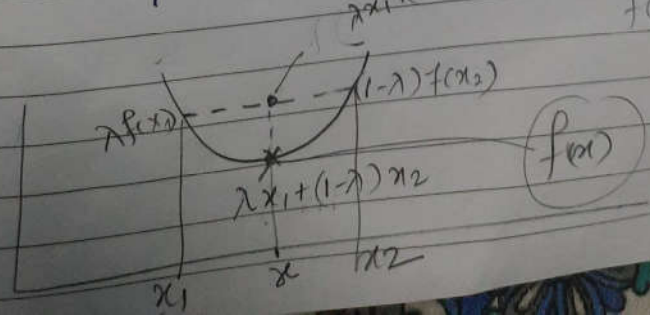
for all $\boxed{0 \leq \lambda \leq 1}$

A convex f^n is called strictly convex f^n if above inequality in eq^n ① is strict

$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$ strictly convex f^n.



$f(x) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$



Concave $f^n \Rightarrow$

convex set

f^n $f(x)$ defined on $S \subseteq \mathbb{R}^n$ is called concave

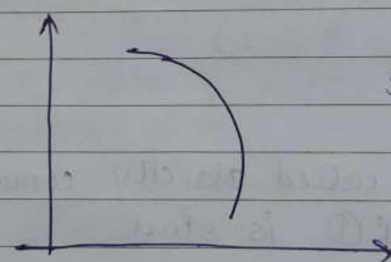
if $-f(x)$ is convex function.

for concave f^n.

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$

$f(x)$

$>$ strictly concave f^n.



strictly concave.

linear function $\Rightarrow$ Both convex and concave but neither strictly convex nor strictly concave.

$$f(\lambda x_1 + (1-\lambda) x_2) = \lambda f(x_1) + (1-\lambda) f(x_2)$$

linear f^n.

Examples ⇒

① $f(x,y) = xy.$
$x, y \geqslant 0.$

convex.

② $f(x,y) = x^2+y^2$
$x \geqslant 0,\ y \geqslant 0$

concave

$\boxed{y = w_0 + w_1 x_1}$

$x_0 = 1$

$y = b + wx$ $\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix}$

$y \langle \underline{w}, \underline{x} \rangle < 0$

$y \langle \underline{w}, \underline{x} \rangle \geqslant 0$

Global Minimum ⇒

A Global min of f^n $f(x)$ is said to be attained at x_0 if

$$f(x_0) \le f(x) \quad \forall\, x \text{ in the feasible region.}$$

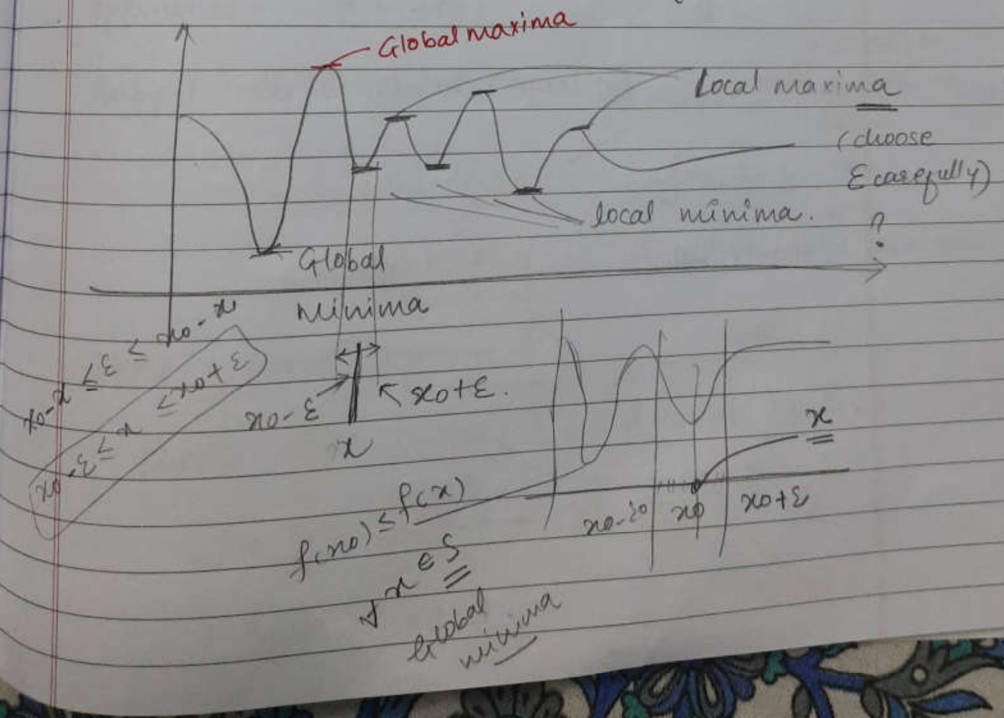
Local minima ⇒

A local minimum of f^n $f(x)$ is said to be attained at x_0 if

$\exists\ \varepsilon > 0$ such that $f(x_0) \le f(x)$ for all x in the feasible region which also satisfy the condition

$$|x_0 - x| \le \varepsilon.$$

Thing are in neighbourhood of epsilon



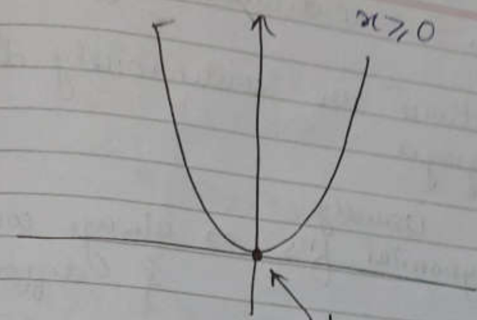
There can be more than one local minima.

$f(x_0) \leq f(x)$

Local Maxima $\Rightarrow$

Ex 1) $f(x) = x^2$

$x \geq 0$



Maxima does not exist.

Local minima
Global minima

* Langrange Multiplier Method. (to solve NLPP)

→ constraint ⇒ all constraints are equalities. Imp.

optimize $z = f(x)$; $x \in \mathbb{R}^n$.

Subject to constraints (equality) — constraints. make it equal to zero

$$h^j(x) = 0 \qquad j = 1, 2, \ldots m$$

always equality constraints & all variables are +ve.

Step 1: Write langrange f^n.

choose some scalar (λ) & equate to $f(x)$.

$$L(x, \lambda) = f(x) - \sum_{j=1}^{m} \lambda_j - h^j(x)$$

$(\lambda_1, \lambda_2 \ldots \lambda_n)$ — Objective f^n — constraint f^n

f, h^j can be continuous differentiable fns.

Condn ⇒ all are functions are continuously differentiable fn. in langrages.

usually.
$h^j \Rightarrow$ polynomial fns. ⇒ always continuous & differentiable.

Step 2 ⇒

Necessary condn for local extrema

$x^0 = (x_1^0, x_2^0, \ldots x_n^0) \in \mathbb{R}^n$ is a local extrema of $f(x)$ subject to constraints

Local Extrema

$h^j(x) = 0$, $x \geq 0$

eq ~~constraints~~.

if $\exists$ constants $\lambda^0 = (\lambda_1, \lambda_2, \ldots \lambda_m) \in \mathbb{R}^m$ such that (x^0, λ^0) satisfies.

i) $\dfrac{\partial L}{\partial x_i} = 0$, $1 \leq i \leq n$

ii) $h^j(x) = 0$, $1 \leq j \leq m$

$$\left.\begin{array}{l} \text{i) } \dfrac{\partial L}{\partial x_i} = 0, \quad 1 \leq i \leq n \\ \text{ii) } h^j(x) = 0, \quad 1 \leq j \leq m \end{array}\right\} - (1)$$

Ex.1)

→

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...ferentiable fns.

...ly differentiable

continuous
...ifferentiable.

local
...traints

$\mathbb{R}^m$

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1) Min $z = x_1^2 + x_2^2 + x_3^2$.

s.t. $x_1 + x_2 + 3x_3 = 2$ $\rightarrow x_1 + x_2 + 3x_3 - 2 = 0$

$5x_1 + 2x_2 + x_3 = 5$ $\rightarrow 5x_1 + 2x_2 + x_3 - 5 = 0$

$x_1, x_2, x_3 \geq 0$

$\rightarrow L = f(x) + \lambda_1 h'(x) + \lambda_2 h^2(x)$

$= (x_1^2 + x_2^2 + x_3^2) + \lambda_1 \underbrace{(x_1 + x_2 + 3x_3 - 2)}_{h'(x)} +$

$\lambda_2 \underbrace{(5x_1 + 2x_2 + x_3 - 5)}_{h^2(x)}$

$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0.$

$\frac{\partial L}{\partial \lambda_1} = h'(x) = 0 \quad \frac{\partial L}{\partial \lambda_2} = h^2(x) = 0.$

$\frac{\partial L}{\partial x_1} = 2x_1 + \lambda_1 + 5\lambda_2 = 0$

$\frac{\partial L}{\partial x_2} = 2x_2 + \lambda_1 + 2\lambda_2 = 0$

$\frac{\partial L}{\partial x_3} = 2x_3 + 3\lambda_1 + \lambda_2 = 0.$

? $\left(\frac{\partial L}{\partial \lambda_1} = h'(x) = 0\right)$ $\cancel{x_1 + x_2 + 3x_3}$ $\frac{\partial L}{\partial \lambda_2} = h^2(x) = 0$

$x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

x_1, x_2, x_3 ?
λ_1, λ_2 ?

$$\left.\begin{aligned} x_1 + x_2 + 3x_2 &= 2 \\ 5x_1 + 2x_2 + x_3 &= 5 \end{aligned}\right\} \quad \partial h^i(x) = 0$$

$$\left.\begin{aligned} 2x_1 + \lambda_1 + 5\lambda_2 &= 0 \\ 2x_2 + \lambda_1 + 2\lambda_2 &= 0 \\ 2x_3 + 3\lambda_1 + \lambda_2 &= 0 \end{aligned}\right\} \quad \frac{\partial L}{\partial x_i} = 0$$

5 eq^{n}s &
5 unknowns
$(x_1, x_2, x_3, \lambda_1, \lambda_2)$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & \lambda_1 & \lambda_2 \end{array}$$
$$\begin{pmatrix} 1 & -1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

x^0: $x_1 = 37/46$, $x_2 = 8/23$, $x_3 = 13/46$

λ_0: $\lambda_1 = -2/23$, $\lambda_2 = -7/23$

$$\Downarrow$$

$x_0 = \left(\frac{37}{46}, \frac{8}{23}, \frac{13}{46}\right)$ local Extrema

$'(x) = 0$

$= 0$

eq^n's &
unknowns.
$(x_1, x_2, x_3, \lambda_1, \lambda_2)$

0

step 2 → How to check whether given local extrema (x_0, λ_0) by conditions ① is maxima or minima.

$$G = \Big(\quad \Big)_{n \times n}$$

① Compute G matrix $_{n \times n}$

② Compute H matrix $_{n \times m}$

$$x = (x_1, x_2 \dots x_n) \in \mathbb{R}^n$$

$$G = \left(\frac{\partial^2 L}{\partial x_i \partial x_j} (x_0, \lambda_0) \right)_{n \times n}.$$

$$H = \left(\frac{\partial h^j}{\partial x_i} (x_0) \right)_{n \times m}.$$

m = no. of equality constraints.

$(i \times j)$th entry.

$$P(\lambda) = \begin{vmatrix} G - \lambda I & H \\ H^t & 0 \end{vmatrix}$$

If all roots of $P(\lambda)$
+ve → local minima
−ve → local maxima
+ve & −ve → neither max nor minima

If all roots of eq^n $P(\lambda) = 0$ are +ve then

pt x_0 is Local minima

If ——"—— −ve then

pt x_0 is local maxima.

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$$G = \left(\frac{\partial^2 L}{\partial x_i \partial x_j} (x_0, \lambda_0) \right)$$

$$= \begin{pmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{pmatrix}_{3\times 3}$$

$\frac{\partial^2 f(x)}{\partial x^2} < 0$ Maxima

$$G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\frac{\partial^2 f(x)}{\partial x^2} > 0$ Minima

$$H = \begin{pmatrix} \frac{\partial h^1}{\partial x_1} & \frac{\partial h^2}{\partial x_1} \\ \frac{\partial h^1}{\partial x_2} & \frac{\partial h^2}{\partial x_2} \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h^2}{\partial x_3^2} \end{pmatrix}_{3\times 2}$$

$h^1 \rightarrow x_1 + x_2 + 3x_3 - 2 = 0$

$h^2 \rightarrow 5x_1 + 2x_2 + x_3 - 5 = 0$

↳ 2 equality constraints

$$H = \begin{pmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$P(\lambda) = \begin{pmatrix} G - \lambda I & H \\ H^T & 0 \end{pmatrix}$$ ← if some values are +ve or -ve.

$$= \begin{vmatrix} 2-\lambda & 0 & 0 & 1 & 5 \\ 0 & 2-\lambda & 0 & 1 & 2 \\ 0 & 0 & 2-\lambda & 3 & 1 \\ 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{vmatrix}$$ – 3 roots.

$$= -230\lambda + 460$$

$P(\lambda) =$

$$\boxed{\lambda = 2}$$

* If all roots are -ve then x_0 is local maxima

* If all roots are +ve then x_0 is local minima

compute value of z at x_0 & that will be minimum value.

$$x_0 \downarrow \left(\frac{37}{46}, \frac{8}{23}, \frac{13}{46}\right)$$

min $f(x) = 3x_1^2 + 4x_2^2 + 5x_3^2$

s.t $\quad x_1 + x_2 + x_3 - 10 = 0 \quad h'(x)$

$x_i \geq 0$

$\rightarrow \quad L = (3x_1^2 + 4x_2^2 + 5x_3^2) + \lambda(x_1 + x_2 + x_3 - 10)$

$\frac{\partial L}{\partial x_1} = 6x_1 + \lambda$

$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 = 10$

$\frac{\partial L}{\partial x_2} = 8x_2 + \lambda$

$\frac{\partial L}{\partial x_3} = 10x_3 + \lambda$

$6x_1 + \lambda = 0$

$x_1 = -\lambda/6$

$-\frac{\lambda}{6} - \frac{\lambda}{8} - \frac{\lambda}{10} = 10$

$8x_2 + \lambda = 0$

$-\frac{47}{120}\lambda = 10$

$x_2 = -\lambda/8$

$\lambda = -12 \times 47$

$10x_3 + \lambda = 0$

$x_3 = -\lambda/10$

$$\boxed{\lambda = -\frac{1200}{47}}$$

$x_1 = \frac{200}{47}$

$x_2 = \frac{150}{47}$

$x_3 = \frac{120}{47}$

$$G = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

$h'(x) = x_1 + x_2 + x_3 - 10$

$$H = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \left(\frac{\partial h^j}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} 6-\lambda & 0 & 0 & 1 \\ 0 & 8-\lambda & 0 & 1 \\ 0 & 0 & 10-\lambda & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$P(\lambda) = 6-\lambda\big((8-\lambda)(-1) + 1(-(10-\lambda))\big)$$

$$= 6-\lambda(-8+\lambda-10+\lambda) - (8-\lambda)(10-\lambda)$$

$$= 6-\lambda(2\lambda-18)$$

$$= 12\lambda - 108 - 2\lambda^2$$

$$= -2\lambda^2 + 12\lambda - 108 - (80 - 8\lambda - 10\lambda + \lambda^2)$$

$$=$$

$= 12\lambda - 2\lambda^2 - 108 + 18\lambda - 80 + 10\lambda + 8\lambda - \lambda^2$

$= -3\lambda^2 + 48\lambda - 188$

$\lambda = \cancel{(8)(\lambda)}\ 8 \pm \frac{4\sqrt{3}}{6}$

$\lambda_1 = 9.15$

$\lambda_2 = 6.845$

2

* Karaush Kuhn Tucker Conditions $\Rightarrow$

Consider NLPP

$$\left.\begin{aligned} &\text{Minimize } f(x) \quad ; \quad x \in \mathbb{R}^n \\ &\text{subject to } \quad g_i(x) \leq C \\ &\qquad\qquad x \geq 0 \quad ; \quad i = 1,2,\ldots m \\ &\text{where } f, g_i(x) \; ; \; 1 \leq i \leq m \text{ are all continuously} \\ &\text{differentiable functions.} \end{aligned}\right] \quad (1)$$

Convex programming problem $\Rightarrow$ The above problem (1) is called convex pgmg problem if $f(x); g_i(x) \; ; \; 1 \leq i \leq m$ are all convex functions.

matrix Th $\Rightarrow$

$A_{m \times n}$ over field F then char. polynomial of A is.

$|A - \lambda I|$ = ~~0~~

└ n-degree polynomial

* Eigen values of $A_{n \times n}$ $\Longrightarrow$ Roots of chara eq^n. $|A - \lambda I|$

* Let A be $n \times n$ matrix & $\lambda_1, \lambda_2 \ldots \lambda_n$ be Evalues of A (counting up to multiplicity)

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Positive definite matrix $\Rightarrow$ If all $\lambda_i > 0 \quad \forall\ 1 \leq i \leq n$

Positive semi-definite matrix $\Rightarrow$ all $\lambda_i \geq 0 \quad \forall\ i$

Negative definite $\Rightarrow$ $\lambda_i < 0$

Negative semi-definite matrix $\Rightarrow$ $\lambda_i \leq 0$

* Hession matrix of a f^n f $\Rightarrow$

Let f be a twice differentiable f^n of n-variables The hessian matrix of 'f' at 'a' is

$$H(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1 x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_1 x_n}(a) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 f}{\partial^2 x_2^2}(a) & \cdots & \frac{\partial^2 f}{x_2 x_n}(a) \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n x_1}(a) & \frac{\partial^2 f}{\partial x_n x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix}$$

$f(x) = 3x_1^2 + x_2^2 + 4x_3^2 + x_1x_2 + 2x_1x_3 + x_3$

~~$H(x) = \left(6x_1 + x_2 + 2x_3 \right.$~~

$2x_2 + x_1$

$$H = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} x_1 & x_1x_2 & x_1x_3 \end{array}$$

$$H = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$

x_1	x_1x_2	x_1x_3
x_2x_1	x_2	x_2x_3
x_3x_1	x_3x_2	x_3^2

$8x_3 + 2x_1 + 1$

Now how the hessian matrix tells you whether given f^n $f(x)$ is convex or concave.

Result → Let f be twice diff f^n of n variable defined on convex set S and let $H(x)$ denote hessian mat of f at $x \in S$ then

1) f is concave f^n iff $H(x)$ is -ve semidefinite $\forall x \in S$

2) f is strictly concave iff $H(x)$ is -ve definite $\forall x \in S$.

3) — " — convex — " — $H(x)$ is +ve semidefinite $\forall x \in S$

4) — " — strictly convex — " — $H(x)$ is +ve definite $\forall x \in S$.

① compute H matrix (A) for all $x \in S$.

② compute chara poly of A

③ compute eigen values of A

④ check sign of EV & tell whether f^n is convex, concave OR (mixed eigen values) f^n is neither convex nor concave.

⑤ Compute maxima/minima using NLPP → KKT cond^n

Ex.1) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ —— +ve definite

Strictly convex.

↓

Global minima exist

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \lambda = 2, 2, 2.$$

KKT conditions ⇒ Necessary conditions

Sufficient conditions.

Applicable to any NLPP of (≤) type (dgn)

↓

* If given problem is convex programming problem then necessary conditions become sufficient.

Max problem(P)		Min Problem(D)
ith eqⁿ	$\leq$ type	ith variable ≥ 0
ith eqⁿ	$\geq$ type	ith variable ≤ 0
ith eqⁿ	$=$ type	ith variable is unrestricted
ith var	≥ 0	ith eqⁿ $\geq$ type
ith var	≤ 0	ith eqⁿ $\leq$ type
ith var	is unsatisfied	ith eqⁿ is equality.

(A) $\boxed{\text{Dual Simplex}}$ most -ve leaves, Max^m val of θ enters

1) conv pivot = 1

2) Most -ve leaves, Max^m value of variable with θ enters.

3) If all values in leaving row are non-negative (+ve) then no feasible sol^n.

4) If atleast one val in row -ve, compute $\theta = \frac{c_j - z_j}{x_{rj}}$ for non-basic variables

5) Max^m value θ, variable enters basis.

6) Make pivot = 1 & all other columns = 0

7) Repeat.

(B) Simplex : most -ve enters, min non-negative θ → leaves

least +ve leaves

least +ve leaves

most -ve enters

Dual simplex ⇒ Max^m val of θ enters, most -ve leaves.

Simplex ⇒ most -ve enter, least +ve θ leaves.

Ex.1) Compute Dual of following problems

(1) primal | dual.

primal	dual
Max $z = 100x_1 + 15x_2$	min $w = 20y_1 + 35y_2$
Subject to	sub to
$x_1 + 4x_2 \leq 20$	$y_1 + 4y_2 \geq 100$
$4x_1 + x_2 \leq 35$	$4y_1 + y_2 \geq 15$
$x_1, x_2 \geq 0$	$y_1, y_2 \geq 0$

G SLPP.

$y_1 \leftarrow x_1 + 4x_2 + x_3 = 20$

$y_2 \leftarrow 4x_1 + x_2 + x_4 = 35$

$x_1, x_2, x_3, x_4 \geq 0$

$y_1 + 4y_2 = 100$

$4y_1 + y_2 = 15.$

$y_3 = 20$

$y_4 = 35$

min $z = 20y_1 + 35y_2$

max $z = 6x_1 + 14x_2 + 13x_3$

$\frac{1}{2}x_1 + 2x_2 - x_3 \leq 24$

$x_1 + 2x_2 + 4x_3 \leq 60$

$x_1, x_2, x_3 \geq 0$

⇓

$w = 24y_1 + 60y_2.$

$\frac{1}{2}y_1 + y_2 \geq 6$

$2y_1 + 2y_2 \geq 14$

$-y_1 + 4y_2 \geq 13.$

$y_1, y_2 \geq 0$

Ex. 27 Min

$Z = 3x_1 - 2x_2 + 4x_3$

sub to

$y_1 \leftarrow 3x_1 + 5x_2 + 4x_3 \geq 7$

$y_2 \leftarrow 6x_1 + x_2 + 3x_3 \geq 4$

$y_3 \leftarrow 7x_1 - 2x_2 - x_3 \leq 10$ $\times$

$y_4 \leftarrow x_1 - 2x_2 + 5x_3 \geq 3$

$y_5 \leftarrow 4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

$\Rightarrow$ max

$w = 7x_1 + 4x_2 + 10x_3 + 3x_4 + 2x_5$

$3y_1 + 6y_2 + 7y_3 + y_4 + 4y_5 \leq 3$

$5y_1 + y_2 - 2y_3 - 2y_4 + 7y_5 \leq \cancel{4} -2$

$4y_1 + 3y_2 - y_3 + 5y_4 - 2y_5 \leq 4$

$y_1, y_2, y_4, y_5 \geq 0$

$y_3 \leq 0$

(3) Maximize $Z = 5x_1 + 12x_2 + x_3$

subject to

$y_1 \leftarrow \quad x_1 + 2x_2 + x_3 \leq 5$

$y_2 \leftarrow \quad 2x_1 - x_2 + 3x_3 = 3$

$x_1, x_2 \geq 0 \quad , x_3 \geq 0.$

$\rightarrow$ Min $w = 5y_1 + 3y_2$

$y_1 + 2y_2 \geq 5$

$2y_1 - y_2 \geq 12$

$y_1 + 3y_2 \geq 1$

$y_1 \geq 0$

y_2 is free variable

(4) min

$Z = x_1 + x_2 + 2x_3$

Sub to

$y_1 \quad x_1 + 2x_2 \geq 3$

$y_2 \quad x_2 + 7x_3 \leq 6$

$y_3 \quad x_1 - 3x_2 + 5x_3 = 5$

$\rightarrow$ Max $w = 3y_1 + 6y_2 + 5y_3$

$y_1 + y_3 \geq 1$ $\quad\quad y_1 \geq 0$

$2y_1 + y_2 - 3y_3 \geq 1$ $\quad\quad y_2 \leq 0$

$5y_3 \geq 1$ $\quad\quad y_3$ free variable.

Ex. 4) min $z = x_1 + x_2 + 2x_3$

subject to

$x_1 + 2x_2 \geq 3$ $\qquad x_1, x_2 \geq 0$

$x_2 + 7x_3 \leq 6$ $\qquad x_3 = \text{free}$

$x_1 - 3x_2 + 5x_3 = 5$

$\Rightarrow$ Max $w = 3y_1 + 6y_2 + 5y_3$

$y_1 + y_3 \leq 1$

$2y_1 + 8y_2 - 3y_3 \leq 1$

$7y_2 + 5y_3 \cancel{\leq} 2$

$y_1 \geq 0, \; y_2 \leq 0, \; y_3 = \text{free}$

Ex. 5) min $z = x_1 - x_2$

$y_1 \quad 2x_1 + 3x_2 - x_3 + x_4 \leq 0$

$y_2 \quad 3x_1 + x_2 + 4x_3 - 2x_4 \geq 8$

$y_3 \quad -x_1 - x_2 + 2x_3 + x_4 = 6$

$x_1 \leq 0, \; x_2, x_3 \geq 0, \; x_4 \in \mathbb{R}$

Dual

Max $= 3y_2 + 6y_3$

$2y_1 + 3y_2 - y_3 \leq 1$

$3y_1 + 2y_2 - y_3 \leq -1$

$-y_1 + 4y_2 + 2y_3 \leq 0$

$y_1 - 2y_2 + y_3 = 0$

$y_1 \leq 0, \; y_2 \geq 0, \; y_3 \in \mathbb{R}$

* Fundamental Theorem $\Rightarrow$

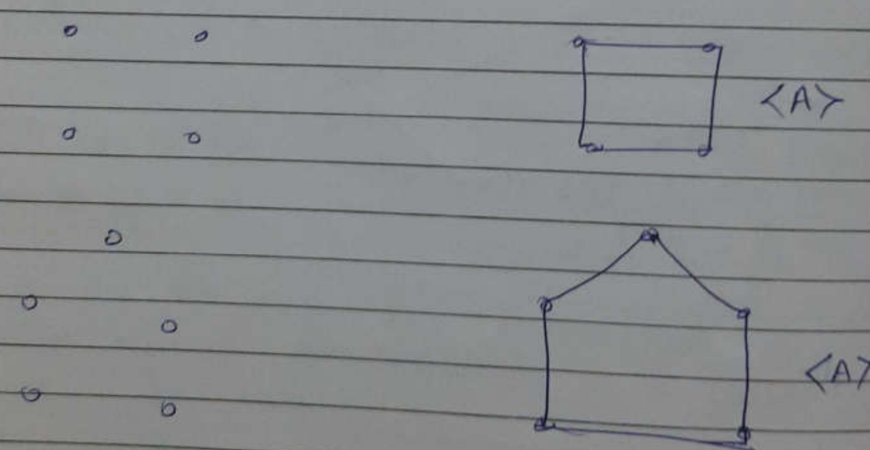
Let T be a constraint set (set of feasible sol^{n}s) satisfying three constrains

→ Non-empty
→ Closed and ~~bounded~~
→ bounded

then the optimal sol^n for LPP exists & it will be attained as some vertex of T.

i.e. It is enough to check values on the vertices to see whether optimal sol^n exists OR not.

points convex hull

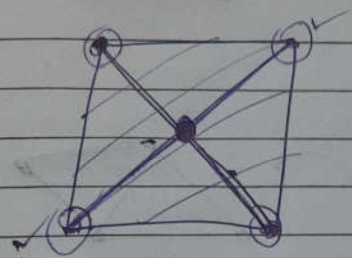


If you have n points $A = \{x_1, x_2 \ldots x_n\}$ then a pt $x \in \langle A \rangle$ (x belongs to convex hull of A)

iff x is convex comb^n of pts in A.

i.e. x is linear comb^n of x_i's where constants satisfies the properties that each const is +ve or 0 & $\Sigma \lambda_i = 1$

$$x = \sum_{i=1}^{n} \lambda_i x_i \ , \quad \lambda_i \geq 0, \quad \Sigma \lambda_i = 1$$

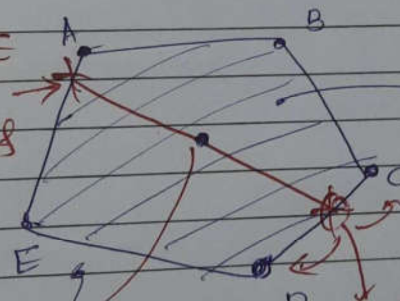


(A)

$\mu A + (1-\mu) E$

convex comb^n of A & E

$\mu \geq 0$



any pt in this plane can be written as $x = \sum_{i=1}^{n} \lambda_i x_i$

convex comb^n of C & D

$\delta(\mu A + (1-\mu) E) + (1-\delta)(\lambda D + (1-\lambda) C)$

$\lambda D + (1-\lambda) C$

$\lambda \geq 0$

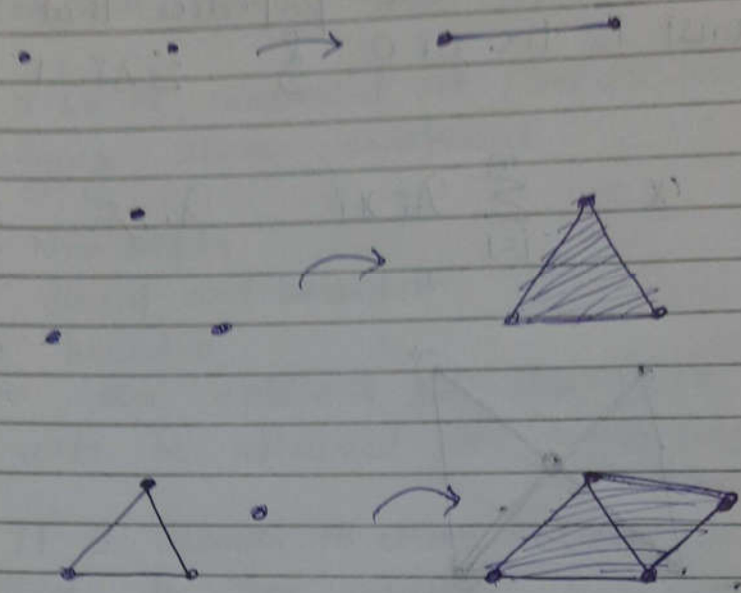
$\delta \geq 0$

then

hull

Convex hull

points



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KKT

applicable to

applicable to all

Neccessary cond^n $\longrightarrow$ any LPP.

Sufficient cond^n $\longrightarrow$ convex PP.

* KKT conditions $\Rightarrow$ derived from langrange multiplier method.

$$\min (z) = f(x)$$

s.t.

$$g_i(x) \le C \qquad 1 \le i \le m.$$

$$x \ge 0$$

f, g_i are conti diff. f^n.

problem ①

①

(i) $g_i(x) = g_i(x) - c$ — $g_i(x) \le 0$ — ③

we add slack var S_i^2 in ith constraint

we get equality constraints

$$g_i(x) + S_i^2 = 0 \qquad (\forall\ \underline{1 \le i \le m})$$

$$\min\ z = f(x) + \underbrace{0S_1 + 0S_2 + \dots + 0S_m}$$

$$\boxed{g_i(x) + S_i^2 = 0}\ ,\ x \ge 0,\ S_i \ge 0.$$

problem with equality constraints Use lagrange multiplier

slack var have 0 weightage in objective f^n

so we can use Lagrange Multiplier Method

$$L = f(x) + \sum_{i=1}^{m} \lambda_i (g_i(x) + S_i^2)$$

$$L = f(x) + \sum_{i=1}^{m} \lambda_i (g^i(x) + s_i^2)$$

$(x_1, x_2, \ldots x_n)$ $+ 0s_1 + 0s_2 + \ldots + 0s_m$

$$\frac{\partial L}{\partial x_j} = \frac{\partial L}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g^i(x)}{\partial x_j}$$

Take partial derivative wrt $(n+m)$ variables

(1) $$\frac{\partial L}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \left(\frac{\partial g^i(x)}{\partial x_j} \right) = 0$$ — 'n' equations

$1 \leq j \leq n$

(2) $$\frac{\partial L}{\partial s_k} = 2\lambda_k s_k = 0 \quad 1 \leq k \leq m.$$ — 'm' equations.

(3) $$\frac{\partial L}{\partial \lambda_l} = g^l(x) + s_l^2 = 0. \quad 1 \leq l \leq m$$ — 'm' eq^ns.

$\boxed{(n+2m) \text{ conditions.}}$ or obtained from Laugraunge Mult method

(2) ⇒ either $\lambda_k = 0$ or $s_k = 0$

If $s_k = 0$ then (if $l = k$ in eq^n (3)) gives us.

$$g^k(x) = 0$$

combining (2) & (3) we get

either $\lambda_k = 0$ OR $g^k(x) = 0$

i.e. product $\boxed{\lambda_k \cdot g^k(x) = 0}$ $\quad 1 \leq k \leq m$

Necessary Part ⇒

This implies that KKT conditions are.

① $$\frac{\partial L}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g^i(x)}{\partial x_j} = 0$$
$, \forall\ 1 \le j \le n$

② $\lambda_K g^K(x) = 0 \quad , \quad \forall \quad 1 \le K \le m$

③ $g^K(x) \le 0 \quad , \quad \forall \quad 1 \le K \le m$ (all g^i's are less than equ to zero).

④ $x \geq 0$ & $\lambda_i \geq 0$. $, \forall\ 1 \le i < m$

* Any point satisfying all these KKT conditions is Local minima.

⑤ obtain values of λ which satisfy all above eqns

⑥ construct Hessian matrix for given fn & check whether it is +ve / -ve definite / semi-definite ie. whether it is convex OR concave fn.

⑦ According to that declare it as strict convex/concave or just convex/concave

⑧ put the values obtained in optimization fn.f.

* when problem is NLPP & all constraints equality $\Rightarrow$ Lagrange multiplier method

* when problem is NLPP & all const are not equality then KKT using Lagrange.

$\min z = x_1^2 - 2x_1 - x_2$ — objective f^n → [illegible]

subject to

$$2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

constraint $f^n \Rightarrow$ linear f^n so it is both convex & concave

first write lagrange $f^n \Rightarrow$

$$L(x, \lambda) = (x_1^2 - 2x_1 - x_2) + \lambda_1 (2x_1 + 3x_2 - 6) + \lambda_2 (2x_1 + x_2 - 4)$$

① $\frac{\partial L}{\partial x_1} = 2x_1 + 2\lambda_1 + 2\lambda_2 - 2 = 0$ — ⓐ

$\frac{\partial L}{\partial x_2} = -1 + 3\lambda_1 + \lambda_2 = 0$ — ⓑ

② $2x_1 + 3x_2 \le 6$ — ⓒ $\quad x \ge 0$

$2x_1 + x_2 \le 4$ — ⓓ

③ $\lambda_1 (2x_1 + 3x_2 - 6) = 0$ — ⓔ

$\lambda_2 (2x_1 + x_2 - 4) = 0$ — ⓕ

④ $\lambda_1, \lambda_2 \ge 0$. Now find pts which satisfy ①, ②, ③, ④ conditions.

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Case I $\Rightarrow$ $\lambda_1 = 0,\ \lambda_2 = 0$

(b) $-1 = 0$ contradicⁿ.

Case II $\Rightarrow$ $\lambda_1 = 0,\ \lambda_2 \neq 0.$

(f) $2x_1 + x_2 = 4$

(a) $2x_1 - 2 + 2\lambda_2 = 0$

(b) $-1 + \lambda_2 = 0$ $\boxed{\lambda_2 = 1}$

$2x_1 - \cancel{2} + \cancel{2} = 0$

$\boxed{x_1 = 0}$

$\boxed{x_2 = 4}$ using (f)

$\therefore$ $x_1 = 0,\ x_2 = 4,\ \lambda_2 = 1,\ \lambda_1 = 0$

(c) $2x_1 + 3x_2 \leq 6.$

$12 \nleq 6$ not satisfying. so it is not extrema

$\lambda_1 \neq 0, \quad \lambda_2 = 0$

(e) $2x_1 + 3x_2 = 6$

(a) $2x_1 - 2 + 2\lambda_1 = 0$

(c) $-1 + 3\lambda_1 = 0 \quad \lambda_1 = 1/3$

$2x_1 - 2 + \frac{2}{3} = 0.$

$x_1 = \frac{\cancel{4}\,2}{3\,\cancel{2}}$

$\boxed{x_1 = 2/3}$

$\cancel{2 \times \frac{2}{3} - 2 + 2\lambda_1 = 0}$ $\qquad$ $2 \times \frac{2}{3} + 3x_2 = 6.$

$\cancel{2\lambda_1 = \frac{2}{3}}$ $\qquad$ $3x_2 = 6 - \frac{4}{3}$

$\cancel{\boxed{\lambda_1 = 1/3}}$ $\qquad$ $x_2 = \frac{14}{3 \times 3}$

$\boxed{x_2 = 14/9}$

local minima $\boxed{\begin{array}{c} x_1 = 2/3, \; x_2 = 14/9 \\ \lambda_1 = 1/3, \; \lambda_2 = 0. \end{array}}$

check whether it satisfies all condns.

(b) $2x_1 + x_2 \leq 4$

$2 \times \frac{2}{3} + \frac{14}{9} \leq 4$

$2.88 \leq 4$ ✓

case IV $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

$$2x_1 + 3x_2 = 6$$
$$-\ 2x_1 + x_2 = 4$$

$$2x_2 = 2$$

$$\boxed{x_2 = 1}$$

$$\boxed{x_1 = 3/2}$$

(a) ~~$2x_1 - 2 + \lambda_1 = 0$~~

~~$2 \times \frac{3}{2} - 2 + \lambda_1 = 0$~~

~~$1 + \lambda_1 = 0$~~

~~$\boxed{\lambda_1 = -1} \neq 0$~~

$2\lambda_1 + 3/2$

$2x_1 - 2 + 2\lambda_1 + 2\lambda_2 = 0$

$3 - 2 + 2\lambda_1 + 2\lambda_2 = 0$

$2\lambda_1 + 2\lambda_2 = -1$

$\lambda_1 + \lambda_2 = -1/2$

$-\ 3\lambda_1 + \lambda_2 = 1$

$-2\lambda_1 = -3/2$

If Both objective (f) constraint (g) f^ns are convex f^ns

To check whether given f^n is convex / concave build hessian matrix.

$\lambda_1 = 3/4$

$\therefore \lambda_2 = -1/2 - 3/4 < 0$ so not satisfying.

Constraint f^n is convex & concave f^n as it is linear function, so now

Vimp

check whether $f(x)$ is convex ~~$f(x_1, x_2) = -2x_1 + x_1^2 - 2x_2$~~

$$f(x) = -2x_1 + x_1^2 - x_2$$

$$H_f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad 2, 0 \geq 0.$$

+ve semidefinite.

↳ convex f^n ✓.

So this is convex PP & the local min which satisfy KKT is global minima also. as here no other cond^n satisfied.

$x_1 = 2/3, \quad x_2 = 14/9,$

$Z = (2/3)^2$ Global min value.

Sufficient Part ⇒

If both objective & constraint f^{ns} are convex f^{ns}.

If $(\bar{x}, \bar{\lambda})$ satisfy all KKT cond^{ns} given (1 to 4) and f, g_i, $1 \leq i \leq m$ be differentiable convex functions (check by building hessian matrix) then $\bar{x}$ is Global minima of problem ①.

Convex PP ⇒ all f and g_i are convex functions.

If the problem is not convex PP, there might be a pt which satisfy all the cond^ns But still that pt is not Global min/maxima

KKT conditions (sufficient cond^ns) (If we are solving convex PP then Necessary cond^ns bcm sufficient)

NLPP

$$\left.\begin{aligned} &\text{Min } f = x_1^2 + x_2^2 - 2x_1 \\ &x_1^2 + x_2 - 1 \le 0 \\ &x_1, x_2 \ge 0. \end{aligned}\right\}$$ So this is convex pgmg problem

convex or concave

$f = x_1^2 + x_2^2 - 2x_1$ — (strictly convex)

$H_f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ +ve def. → strictly convex. $(\lambda = 2, 2)$

$g = x_1^2 + x_2 - 1$ — (convex)

$H_g = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ +ve semi-definite — convex. $(\lambda = 2, 0)$

As f and gi's are convex f^ns given problem is convex PP so

∴ KKT cond^ns are necessary & sufficient.

$L(x, \lambda) = (x_1^2 + x_2^2 - 2x_1) + \lambda_1 (x_1^2 + x_2 - 1)$ — (a)

① $\frac{\partial L}{\partial x_1} = (2x_1 - 2) + \lambda_1 (2x_1) = 0$ — (a)

$\frac{\partial L}{\partial x_2} = (2x_2 + \lambda_1) = 0$ — (b)

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(2) $x_1^2 + x_2 - 1 \le 0$ — (c)

(3) $\lambda_1 (x_1^2 + x_2 - 1) = 0$ — (d)

(4) $\lambda \ge 0$

telling that

$\therefore$ either $\lambda = 0$ or $x_1^2 + x_2 = 1$.

* find pts (x_1, x_2, λ) satisfying all (4) conditions

case I $\Rightarrow$ $\lambda_1 = 0$

$$2x_2 + \cancel{\lambda_1}^{0} = 0 \quad \text{— from (a)}$$

$\boxed{x_2 = 0}$ satisfying (a), (b), (c), (d).

$$2x_1 - 2 = 0 \quad \text{— from}$$

$\boxed{x_1 = 1}$

so we got pot (1,0) satisfy all cond^ns of KKT

$(x_1, x_2) = (1, 0)$

convex

$$\boxed{(\bar{x}, \bar{\lambda}) = ((1,0), 0)}$$

$\therefore$ Global minima at (1,0)

satisfying all KKT conditions & all fns are differential convex f^n so given $(\bar{x}, \bar{\lambda})$ is Global minima of the problem.

* KKT conditions → for minima

If max problem given convert to minimization.

(m+1) f^n

Without convexity ~~conditions~~ assumptions on f, g_i, $1 \le i \le m$, KKT conditions are not sufficient for a point $\bar{x}$ to be local minima / Global minima

so out of (m+1) f^ns if any one f^n is concave then KKT conditions are not sufficient for $\bar{x}$ to be global/local minima

NLPP + Not Convex PP ⟹ $\bar{x}$ satisfying KKT cond^n may/maynot be Local/Global minima.

NLPP + not convex PP ⟹ If $\bar{x}$ is local/Global minima then $\bar{x}$ definitely satisfies KKT conditions.

NLPP + convex PP ⟹ (Strictly convex f^ns – f & g_i are)

* If $\bar{x}$ is local minima of then $\bar{x}$ satisfies KKT conditions corresponding to problem

* If $\bar{x}$ satisfies KKT conditions then $\bar{x}$ is global minima.

Ex.1) Min $f(x) = -x_2$ $\longrightarrow$ linear so convex f^n.

Subject to

$$x_1^2 + x_2^2 \leq 4$$

$$-x_1^2 + x_2 \leq 0$$

$$x_1, x_2 \geq 0.$$

$\Rightarrow$ $g^1(x) = x_1^2 + x_2^2 - 4 = 0$

$g^2(x) = -x_1^2 + x_2 = 0$

$Hg^1(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad 2, 2 > 0$ strictly convex f^n.

$Hg^2(x) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad -2, 0 \Rightarrow$ concave f^n. -ve semi-def.

So this is not convex PP.

$L(x, \lambda) = (-x_2) + \lambda_1 (x_1^2 + x_2^2 - 4)$
$+ \lambda_2 (-x_1^2 + x_2)$

~~$\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x}$~~

(i) $\frac{\partial L}{\partial x_1} = 2x_1\lambda_1 - 2x_1\lambda_2$

$\frac{\partial L}{\partial x_2} = -1 + 2x_2\lambda_1 + \lambda_2$

~~$\frac{\partial L}{\partial \lambda_1}$~~

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$x_1^2 + x_2^2 \le 4$ — (c)

(2) $-x_1^2 + x_2 \le 0$ — (d)

$\frac{\partial L}{\partial x_1} =$

$2x_1\lambda_1 - 2x_1\lambda_2 = 0$ — (a)

$-1 + 2x_2\lambda_1 + \lambda_2 = 0$ — (b)

(3) $\lambda_1(x_1^2 + x_2^2 - 4) = 0$ — (e)

$\lambda_2(-x_1^2 + x_2) = 0.$ — (f)

(4). $\lambda_1, \lambda_2 \geq 0$

case I $\Rightarrow \lambda_1 = 0, \lambda_2 = 0$

from (b) $-1 = 0$ contradict^n

case II $\Rightarrow \lambda_1 = 0, \lambda_2 \neq 0$

from (f) $\cancel{-x_1^2 = -x_2}$

$\cancel{x_2 = x_1^2}$

(b) $\Rightarrow -1 + \lambda_2 = 0$

$\boxed{\lambda_2 = 1}$ $\boxed{x_1 = 0}$

(f) $\Rightarrow -x_1^2 + x_2 = 0$

$\cancel{-2x_1 = 0}$

$\cancel{\boxed{x_1 = 0}}$

$-2x_1\cancel{\lambda_2}^{1} = 0$

$-2x_1 = 0$

$\boxed{x_1 = 0}$

$\therefore \quad x_2 = x_1^2$

$\therefore \boxed{x_2 = 0}$

$(x_1 = 0, x_2 = 0, \lambda_1 = 0, \lambda_2 = 1)$

(a) ✓, (b) ✓, (c) ✓

$-1+1=0$ ✓

(d) ✓, (e) ✓, (f) ✓

$\therefore (0,0)$ satisfies KKT cond^n

This pt is neither local minima / maxim nor Global minima → 0 case III : $\lambda_1 \neq 0$, $\lambda_2 = 0$

(e) $x_1^2 + x_2^2 = 4$.

(a) $2x_1 \lambda_1 = 0$

$\boxed{x_1 = 0.}$

$\therefore \quad x_2^2 = 4 \qquad x_2 = \pm 2$ discard $x_2 = -2$ Not feasible

$\boxed{x_2 = 2}$

(b) $-1 + 2 \times 2 \lambda_1 = 0$ (a) $2x_1 \cdot \lambda_1 = 0$

$4\lambda_1 = 1 \qquad 0 = 0.$

$\boxed{\lambda_1 = 1/4}$ —— (e) satisfied.

$-1 + 2 \times 2 \times \frac{1}{4} = -\lambda_2$

$\boxed{\lambda_2 = 0}$ (f) satisfied

(c) satisfied.

(d) $2 \leq 0$ ✗ not satisfied.

→ Use graphical method ⇒

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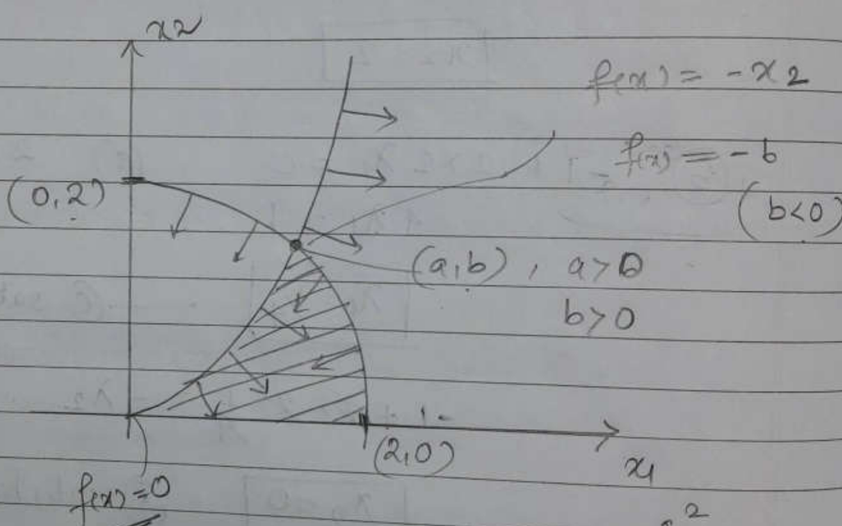
$\min f(x_1, x_2) = -x_2$

s.t.

$x_1^2 + x_2^2 \leq 4$ — circle of radius 4 — $(0, \pm 2)$, $(\pm 2, 0)$

$-x_1^2 + x_2 \leq 0$ — $x_1^2 = x_2$ parabola

$x_1, x_2 \geq 0.$



$-x_1^2 + x_2 \leq 0$

$-4 + x_2 \leq 0$

$x_2 = 4$

case IV $\Rightarrow \lambda_1 \neq 0, \lambda_2 \neq 0$

$\cancel{x_1^2 + x_2^2 = 4}$

$+ \cancel{-x_1^2 + x_2 = 0}$

$\cancel{x_2^2 + x_2 = 4}$

(e) $x_1^2 + x_2^2 - 4 = 0$

ei $x_1^2 + x_2^2 = 0$.

(f) $-x_1^2 + x_2 = 0$

$\cancel{x_1^2 = x_2}$

$$\boxed{x_1^2 = x_2}$$

$x_2 + x_2^2 - 4 = 0$.

$x_2^2 + x_2 - 4 = 0$. $a^2 + a - 4 = 0$

$x_2 =$

Practice Problems $\Rightarrow$

Min $f(x_1, x_2) = 2x_1 + x_2$

subject to $x_1^2 + x_2^2 \leq 4$

$x_1 - x_2 \leq 0$

① $L(x, \lambda) = (2x_1 + x_2) + \lambda_1 (x_1^2 + x_2^2 - 4)$

$+ \lambda_2 (x_1 - x_2)$

$\frac{\partial L}{\partial x_1} = 2 + 2\lambda_1 x_1 + \lambda_2 = 0$ —— ⓐ

$\frac{\partial L}{\partial x_2} = 1 + 2\lambda_1 x_2 - \lambda_2 = 0$ —— ⓑ

$\cancel{\frac{\partial L}{\partial \lambda_1}} =$

② $x_1^2 + x_2^2 \leq 4$ —— ⓒ

$x_1 - x_2 \leq 0$ —— ⓓ

③ $\lambda_1 (x_1^2 + x_2^2 - 4) = 0$ —— ⓔ

$\lambda_2 (x_1 - x_2) = 0$ —— ⓕ

④ $\lambda_1, \lambda_2 \geq 0$

c III $\Rightarrow$ $\lambda_1 = 0, \quad \lambda_2 \neq 0$

$\boxed{\lambda_2 = -2}$ $\boxed{\lambda_1 = 0}$

(d) $\Rightarrow$ ~~Qu~~ $x_1 = x_2$

$2 + 0 + \lambda_2 = 0$

(a) $\Rightarrow$ $\boxed{\lambda_2 = -2}$ — (a)

// contradiction

(b) $\Rightarrow$ $\boxed{\lambda_2 = -1}$

c IV $\Rightarrow$ ~~$x_1 \neq 0, \lambda_2 \neq 0$~~

$\boxed{\lambda_1 \neq 0, \quad \lambda_2 = 0}$

(b) $\Rightarrow$ $1 + 2\lambda_1 x_2 = 0$

$\boxed{x_2 = -1/2\lambda_1}$

(a) $\Rightarrow$ $1 + 2\lambda_1 x_1 = 0$

$x_1 = -\cancel{2}/2\lambda_1 = -1/\lambda_1$

$\boxed{x_1 = -1/\lambda_1}$

$$x_1^2 + x_2^2 = 4$$

$$\frac{1}{4\lambda_1^2} + \frac{1}{\lambda_1^2} = 4$$

$$\frac{1+4}{4\lambda_1^2} = 4.$$

$$\boxed{\lambda_1^2 = 5/16}$$

$$\boxed{\lambda_1 = \sqrt{5}/4}$$

$$\boxed{x_1 = -4/\sqrt{5}}$$

$$\boxed{x_2 = -2/\sqrt{5}}$$

$$2 - 2 \times \frac{\cancel{\sqrt{5}}}{\cancel{4}} \times \frac{4}{\cancel{\sqrt{5}}} = 0 \checkmark$$

all condns satisfied.

$$Hg_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{strictly convex}$$

$$Hg^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{convex.}$$

Optimiz^n (Z) f^n is convex so this convex LPP & KKT condns are sufficient & $\left(\frac{-4}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$ is Global minima