$$x_2 \Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \underbrace{-1}_{\text{Eigen value}}\ \underbrace{\overset{x_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}}_{\text{Eigen Vector.}} = (-1)\,x_2$$

$\therefore \quad A x_2 = \lambda x_2$

$A x_2 = (-1)\, x_2$.

Both x_1 & x_2 are eigen Vectors $\Big\{ \therefore$ For Vector x_2, $\lambda_2 = -1$
For Vector x_1, $\lambda_1 = 2$.

Thm 1: If A is $n \times n$ matrix with eigen value λ, then the set of all linearly independent eigen vectors of λ is called eigenspace of λ.

1) If x_1 and x_2 are eigenvectors corresponding to λ then $\underbrace{E\lambda = \{x_1, x_2\}}_{\text{Eigen Space}}$ (linearly independe)

i.e. $(Ax_1 = \lambda x_1, \ \ Ax_2 = \lambda x_2)$

2) Dimension of the E.space is no. of elements in $E\lambda$

* eigenpair $\Rightarrow$ eigenvalues (eigenvector, eigenvalue)
$\Downarrow$
$\underline{(\lambda, v)}$

Algebraic Multiplicity : No. of eigen Values. (Roots of ch eqn)
Geometric Multiplicity : dimension of eigen Space

$$GM \leq AM.$$

If $GM < AM$: defective matrix (prev example)

$A_{n \times n}$ is defective iff it does not have 'n' linearly independent eigenvectors.

* It does not have a complete basis of eigenvectors.

If $GM = 1$ then $AM = 1$.

eg.6)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

M1 ⇒ $R_3 \leftarrow R_3 - R_1$
$R_4 \leftarrow R_4 - R_1$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Eigen values ⇒ 1, 1, 2, 3.

M2 ⇒ $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 5 & -10 \\ 1 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{vmatrix}$$

$$(1-\lambda)^2 (2-\lambda)(3-\lambda) = 0$$

Q.1) $A_{2\times2}$ with $Tr(A) = 3$, $det(A) = 2$.

→ $\lambda_1 + \lambda_2 = 3$

$\lambda_1 \lambda_2 = 2$.

$\lambda_1 = 2, \lambda_2 = 1$.

Diagonalization:-

A sq^r matrix A is diagonalizable if there exists invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(P diagonalizes A)

If $B = P^{-1}AP$ then A & B are similar matrices.

* P has to be obt from Eigen Vectors

Thm 4: Similar matrices have Same Eigen Values.

eg 1) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ $\lambda = -2, 4$

$\lambda_1 = 4, \lambda_2 = -2, \lambda_3 = -2$

Eigen vector wrt $\lambda_1 = 4$ $= P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector wrt $\lambda_2 = -2$ $=$ P_2 $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, P_3 $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thm5: $A_{n \times n}$ is diagonalizable if it has 'n' linearly independent eigenvectors.

Thm6: (Aliter) If $A_{n \times n}$ has 'n' distinct eigen values, then the corresponding eigen vectors are linearly independent & A is diagonalizable.

(Sufficient cond^n, not necessary)

for a $A_{n \times n}$, If eigen values non-distinct - check for all eigen vectors

& if you get 3 linearly independent eigen vectors then $A_{n \times n}$ is diagonalizable.

& if matrix doesn't have 'n' linearly independent vectors then $A_{n \times n}$ is not diagonalizable.

eg 4)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$ check whether diagonalizable or not

$\rightarrow$ $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

$\lambda_1 = 1$

Eigen vector =

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$

variable free

eg. 7) $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$(-3-\lambda)\left[(-\lambda)(1-\lambda)\right] = 0.$

$(-3-\lambda)(-\lambda+\lambda^2) = 0.$

-3×1

$3\lambda - 3\lambda^2 + \lambda^2 - \lambda^3 = 0$

$3\lambda - 2\lambda^2 - \lambda^3 = 0$

$\lambda^3 + 2\lambda^2 - 3\lambda = 0$

$\lambda(\lambda^2 + 2\lambda - 3) = 0$

$\lambda(\lambda+3)(\lambda-1) = 0$

Upper Δer $\therefore$ Eigen Values $\Longrightarrow$ $\lambda = 0, -3, 1$

As Eigen Values are distinct, it is diagonalizable.

(1) Symm matrix $\Rightarrow$ $\boxed{A = A^T}$

Thm 7: Eigen values of Symm matrices

$A_{n\times n}$ is Symm matrix then
1. A is diagonalizable
2. All eigen values of A are real.
3. If λ is eigenvalue of A with multiplicity K (AM = K) then λ has K linearly independent Eigen Vectors.

i.e. Eigenspace of λ has dimension K.

√Imp

Thm 8 ⇒ $P_{n\times n}$ is orthogonal matrix if and only if its column vectors forms an orthogonal set.

$$P_{n\times n} = [P_1 \mid P_2 \mid P_3 \mid \cdots \mid P_n]$$

$$\boxed{P_i \cdot P_j = 0 \quad \forall\, i,j \text{ except } (i=j)}$$

eg 1) $P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix}$ check whether P is orthogonal or not?

→ If $P \cdot P^T = I$ then P is orthogonal.

$$P \cdot P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/3 & -2/\sqrt{5} & -2/3\sqrt{5} \\ 2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}$$

$-\frac{2}{15}+\frac{2}{3\sqrt{5}}$

$\frac{1}{3}\ \frac{2}{}$ $\frac{1}{3}+\frac{2}{2}+\frac{2}{3}$ $\frac{4}{9}+\frac{4}{9}$ $\frac{5}{3}$ $\frac{1}{9}+$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\frac{1}{3}\times\frac{-2}{5} + \frac{2}{3}\times\frac{1}{\sqrt{5}}$

$\frac{2}{3}\left[-\frac{1}{5}+\frac{1}{\sqrt{5}}\right] \quad \frac{2}{3}$

Let $P_1 = \begin{bmatrix} 1/3 \\ -2/\sqrt{5} \\ -2/3\sqrt{5} \end{bmatrix}$, $P_2 = \begin{bmatrix} 2/3 \\ 1/\sqrt{5} \\ -4/3\sqrt{5} \end{bmatrix}$, $P_3 = \begin{bmatrix} 2/3 \\ 0 \\ 5/3\sqrt{5} \end{bmatrix}$

$$P_1 . P_2 = \cancel{\begin{bmatrix} 1/3 \\ -2/\sqrt{5} \\ -2/3\sqrt{5} \end{bmatrix}_{3\times 1} \begin{bmatrix} 2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \end{bmatrix}_{1\times 3}}$$

$$= \begin{bmatrix} 1/3 & -2/\sqrt{5} & -2/3\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/\sqrt{5} \\ -4/3\sqrt{5} \end{bmatrix}$$

$$= \frac{2}{9} - \frac{2}{5} + \frac{8}{9\times 5} \Rightarrow \frac{10-18+8}{45} \Rightarrow 0.$$

$\therefore P_1 . P_2 = P_2 . P_3 = P_1 . P_3 = 0$ dot product betn P_1, P_2, P_3 is 0.

length of P_1, P_2, P_3 is 1 $\|P_1\| = \|P_2\| = \|P_3\| = 1$.

$\{P_1, P_2, P_3\}$ is an ~~orthogonal~~ orthonormal set. length of each Vector = 1

$A = A^T$

Thm 9: $A_{n\times n}$ symm matrix and if λ_1 and λ_2 are distinct eigenvalues of A, then the corresponding eigenVectors x_1 and x_2 are orthogonal.

i.e $x_1 . x_2 = 0$

$\|x_1\| = \|x_2\| = 1$

Thm 10 :- Fundamental Th^m of symmetric matrices.

Let $A_{n \times n}$ then A is orthogonally diagonalizable and has real eigen values iff A is symmetric.

* Orthogonal diagonalizat^n of symm matrix (A) :-

Let $A_{n \times n}$ matrix,

① find all eigen Values & determine algebraic multicity of each.

② If AM = 1 i.e. all eigenvalues are distinct then find "unit eigenvector" $u_1 = \frac{v_1}{\|v_1\|}$

③ If AM ≥ 2, find set of K-linearly independ eigen Vectors.
If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

④ Composite of steps ② & ③ produces an orthonormal set of n eigenvectors.

Use eigen Vectors to form columns of P.

$$P^{-1}AP = P^T A P = D$$

If $A_{n \times n}$ Symm $\longrightarrow$ then it is orthogonally diagonalizable.

30th morning

9

Vector Norm :-

A norm is a f^n $\|\cdot\| : \mathbb{R}^m \rightarrow \mathbb{R}^+$ that assigns a real-valued length to each vector.

For all vectors x and y and for all scalars $a \in \mathbb{R}$

A norm must satisfy $foll^n$ 3 conditions

① $\|x\| > 0$ and $\|x\| = 0$ only if $x = 0$ → Positivity $cond^n$.

② $\|x+y\| \leq \|x\| + \|y\|$, Δ^{er} inequality $cond^n$

③ $\|ax\| = |a|\,\|x\|$, Scaling $cond^n$

$z \leq x+y$

* Popular Vector norms :-

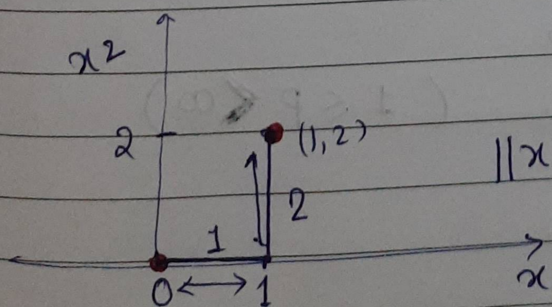
① $\|x\|_1 = \sum_{i=1}^{m} |x_i|$ 1-norm / Manhattan distance / or norm.

one-norm

② eg $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \|x\| = |x_1| + |x_2|$

$x \in \mathbb{R}^2, \; x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

One norm will travel horizontal & then Vertical $direct^n$.



$\|x\| = |1| + |2| = 3.$

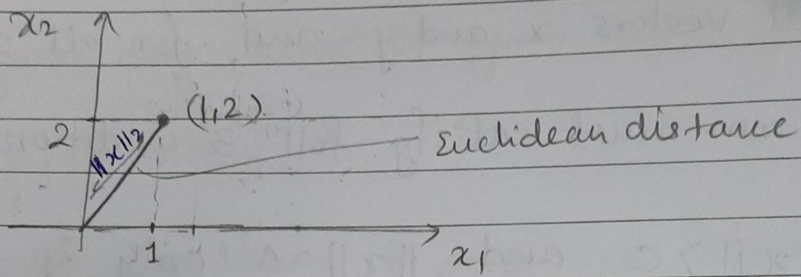
$\therefore$ one-norm is distance bet^n (0,0) and (1,2)

Two-Norm:

(2) $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \sqrt{x^* x}$

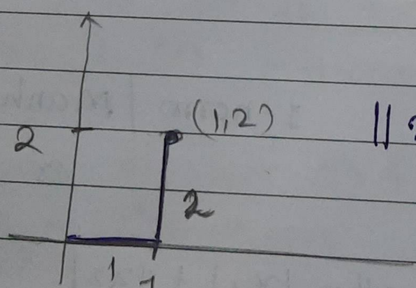
norm of a vector

2-norm or Euclidean norm/distance.



(3) Infinity norm/ Max-norm

$$\|x\|_\infty = \max_{1 \le i \le m} |x_i|$$



$\|x\|_\infty = 2$.

Maximum of all the distances covered in any direction.

(4) p-norm / lp-norm : Generalization of all norms.

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} \qquad (1 \le p < \infty)$$

find x_1 and x_2 such that

$\|x\|_1 = 1$, $\|x\|_2 = 1$, $\|x\|_\infty = 1$

classmate

Date: 30th - evening

$x_1 = [-1, 1]$

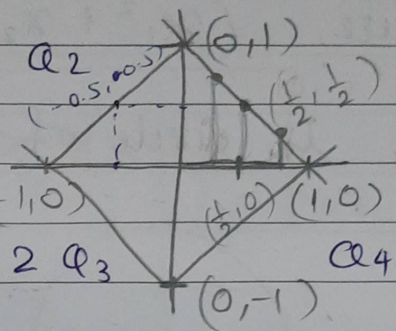
$x_2 = [-1, 1]$

Geometric interpretation :-

Consider closed unit ball $x \in \mathbb{R}^2 : \|x\| \leq 1$.

Imp cond^n for below discussions

① $\|x\|_1 = \sum_{i=1}^{m} |x_i| \Rightarrow$



$\|x\| \leq 1$

$\|(0.8, x_2)\|_1 = 1 \Leftrightarrow x_2 = 0.2$ Q3

Q4 Here every Vector is going to have length / distance 1 wrt norm-1, it is Rhombus ✓

$\|(0.5, x_2)\|_1 = 1 \Leftrightarrow x_2 = 0.5$

$\|(x_1, x_2)\| = 1 \Leftrightarrow |x_1| + |x_2| = 1$

② $\cancel{\|x\|_2 = \sqrt{x * x}}$

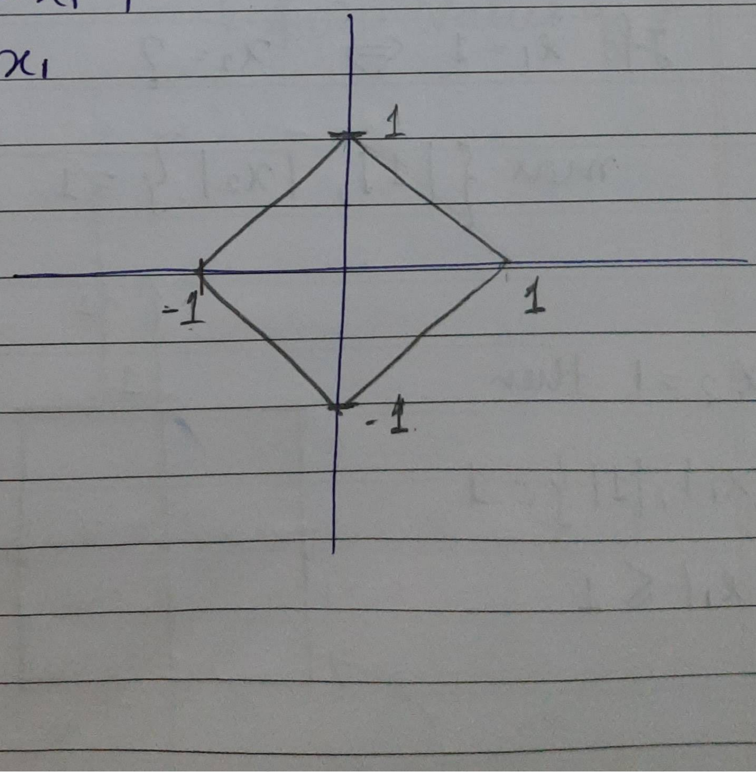
In Q1, $x_2 = 1 - x_1$

In Q2, $x_2 = 1 + x_1$

In Q3, $x_2 = -x_1 - 1$

In Q4, $x_2 = x_1$

$1 - 0.5 = 0.5$ ✓



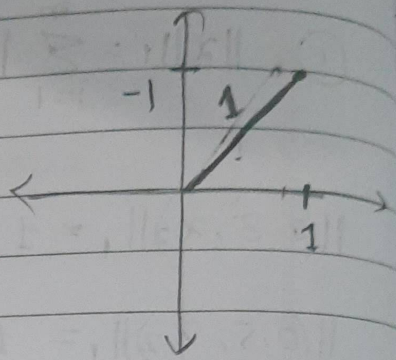
2-norm

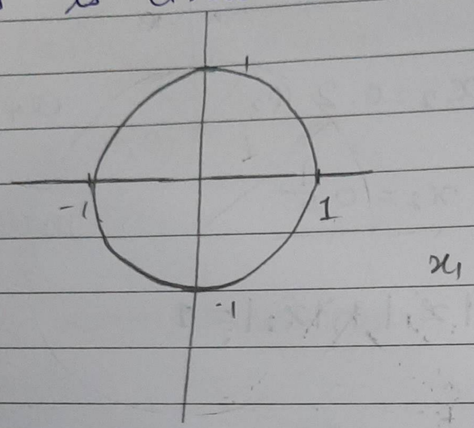
② $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2}$ we want $\|x\|_2 = 1$

(Sqrt of sum of the squares of individual elements)

So here, $(x_1^2 + x_2^2)^{1/2} = 1$

this is circle of radius 1.





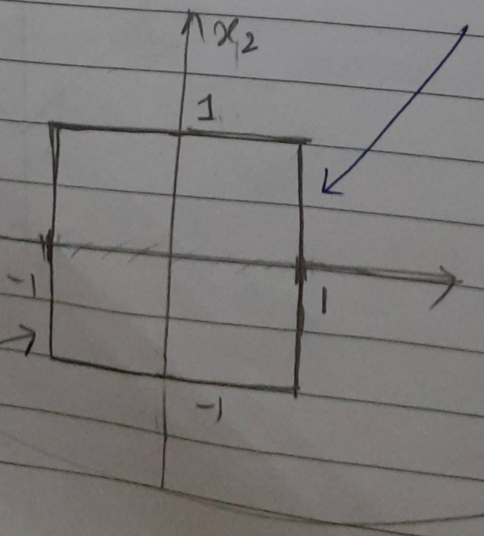
Unit circle for 2-norm $\geqslant$ 1-norm

③ $\|x\|_\infty = \max\limits_{1 \leq i \leq m} |x_i|$ max^m value of individual elem

If $x_1 = 1 \iff x_2 = ?$

$\max\{|1|, |x_2|\} = 1 \quad \forall\, |x_2| \leq 1$

If $x_2 = 1$ then $\max\{|x_1|, |1|\} = 1$ $\forall\, |x_1| \leq 1$

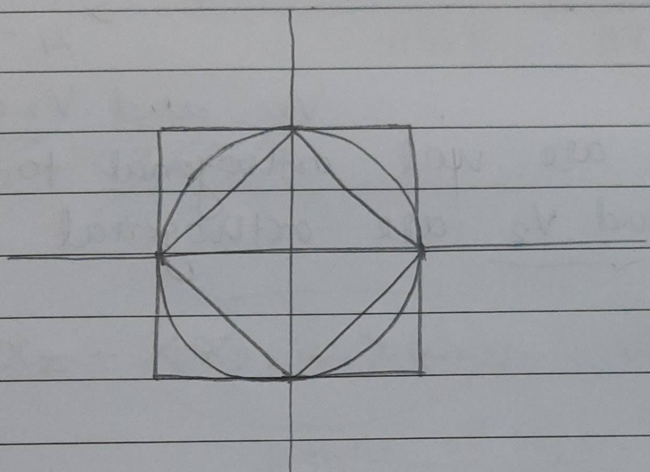


If $x_1 = -1$ then $\max\{|-1|, |x_2|\} = 1$ $\forall\, |x_2| \leq 1$

The unit circle for $\|x\|_1$ is smaller than $\|x\|_2$ and $\|x\|_\infty$

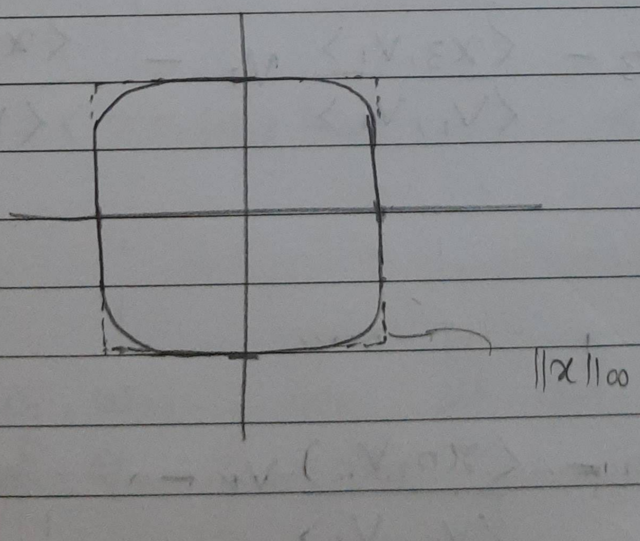
But the norm values for (x_1, x_2) are bigger.

$$\boxed{\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty}$$



(4) $\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}$ $(1 \leq p < \infty)$

finite values of p.



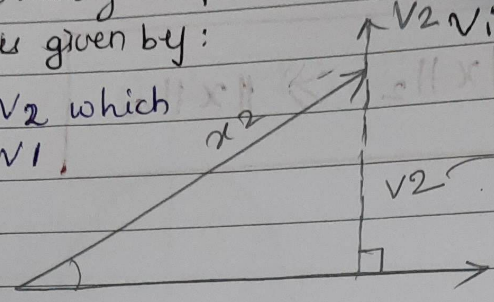
Gram-Schmidt orthogonalization process :-

$$\{x_i \,/\, i = 1, 2, \ldots n\} \longrightarrow \underbrace{\{V_i \,/\, i = 1, 2, \ldots n\}}_{\text{orthogonal}}$$

$V_i \cdot V_j = 0 \;\forall\, i \neq j$

Let $x_1, x_2, \ldots x_n$ are linearly independent vectors then, GSOP is given by:

Look for V_2 which is $\perp^{er}$ to V_1.



projection of vector x_2 on x_1 (V_1)

V_1 and V_2 are $\perp^{er}$ to each other

$\because$ x_1 & x_2 are not orthogonal to each other but $\underbrace{V_1 \text{ and } V_2}$ are orthogonal

$$V_1 = x_1$$

$$V_2 = x_2 - \frac{\langle x_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1 \qquad // \text{ projection of vector } x_2 \text{ on } V_1$$

$\perp^{er}$ to both V_1 & V_2

$$V_3 = x_3 - \frac{\langle x_3, V_1\rangle}{\langle V_1, V_1\rangle} V_1 - \frac{\langle x_3, V_2\rangle}{\langle V_2, V_2\rangle} V_2$$

$\vdots$

$$V_n = x_n - \frac{\langle x_n, V_1\rangle}{\langle V_1, V_1\rangle} V_1 - \cdots - \frac{\langle x_n, V_{n-1}\rangle}{\langle V_{n-1}, V_{n-1}\rangle} V_{n-1}$$

Then $V_1, V_2, \ldots V_n$ is an orthogonal Basis for V.

dot product zero

orthonormal = orthogonal + Unit length

Orthogonalizat^n + Normalization

Let $x_1, x_2, \ldots x_n$ are 'n' linearly independent Vectors then modified GSOP is given as :

$$V_1 = x_1 \quad , \quad w_1 = \frac{V_1}{\|V_1\|} = \frac{x_1}{\|x_1\|}$$

$$\cancel{V_2 = x_2 - \frac{\langle x_2, V_1 \rangle}{\langle V_1, V_1 \rangle}}$$

$$V_2 = x_2 - \langle x_2, w_1 \rangle w_1 \, , \quad w_2 = \frac{V_2}{\|V_2\|}$$

project^n of x_2 on w_1

$$V_3 = x_3 - \langle x_3, w_1 \rangle w_1 - \langle x_3, w_2 \rangle w_2 , \quad w_3 = \frac{V_3}{\|V_3\|}$$

$$\vdots$$

$$V_n = x_n - \langle x_n, w_1 \rangle w_1 - \cdots - \langle x_n, w_{n-1} \rangle w_{n-1}$$

$$w_n = \frac{V_n}{\|V_n\|}$$

then $w_1, w_2, \ldots w_n$ is an orthonormal Basis ~~of~~ ~~q~~. - for V.

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orthogonal diagonalization.

eg Ex 9) find ~~orthol matrix~~ orthogonal matrix P that diagonalizes A

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

① find eigen values & eigen vectors & construct P.

→ $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix} = 0$$

$$2-\lambda\left[(-1-\lambda)^2 - 4\right] - 2\left[\underbrace{(-2-\lambda)+8}_{-\lambda+6}\right] + (-2)\left[8+2(-1-\lambda)\right]$$

$6-2\lambda$

$8-2+2\lambda$

$\Rightarrow (2-\lambda)\left[-(\lambda^2+2\lambda+1)-4\right] + 2\lambda - 12 + 12 + 4\lambda$

$(2-\lambda)(-\lambda^2-2\lambda-5)$

$\Rightarrow \cancel{-2\lambda^2} - \cancel{4\lambda} - 10 + \lambda^3 + \cancel{2\lambda^2} + 5\lambda + 2\lambda - 12 - \cancel{12} + \cancel{4\lambda}$

$\lambda^3 + 7\lambda - 34 = 0$

8+14
10
22

$$\begin{vmatrix} (2-\lambda) & 2 & -2 \\ 2 & -(1+\lambda) & 4 \\ -2 & 4 & -(1+\lambda) \end{vmatrix} = 0$$

$(2-\lambda)\left[(1+\lambda)^2 - 16\right] - 2\left[-2(1+\lambda)+8\right] - 2\left[8-2(1+\lambda)\right]$

$(2-\lambda)(\lambda^2+2\lambda+1-16) - 2[-2-2\lambda+8] - 2[8-2\lambda-2]$

$(2-\lambda)(\lambda^2+2\lambda-15) + 8\lambda - 24 \quad + 4 + 4\lambda - 16 - 16 + 4\lambda + 4$

$\cancel{2\lambda^2} + \cancel{4\lambda} - 30 - \cancel{\lambda^3} - \cancel{3\lambda^2} + 15\lambda + \cancel{2\lambda} - 24 \quad 8\lambda + 8\lambda - 24)$

$\frac{32}{4}$ $4 - \frac{1}{2}$ 8 $\frac{-32}{+8}$

3
6

classmate

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$\cancel{-\lambda^3 + 11\lambda + 54 = 0}$

$-\lambda^3 + 27\lambda - 54 = 0$

$(3)^3 = 27$

$-27 + 71 - 54 \neq 0$

$-(\lambda^3 - 27\lambda + 54) = 0$

$27 - 81 + 54$

$-54 + 54 = 0$

$-(\lambda - 3)(\lambda^2 + 3\lambda - 18) = 0$

$+(\lambda - 3)(\lambda + 6)(\lambda - 3) = 0$

$$\begin{array}{r} \lambda^2 + 3\lambda - 18 \\ \lambda - 3 \overline{)\lambda^3 - 27\lambda + 54} \\ -\lambda^3 - 3\lambda^2 \\ \hline +3\lambda^2 - 27\lambda + 54 \\ -3\lambda^2 + 9\lambda \\ \hline -18\lambda + 54 \\ -18\lambda + 54 \\ \hline 0 \end{array}$$

$\therefore \lambda_1 = 3$ (has multiplicity 2)

$\lambda_2 = -6$ — (AM=1) so find eigenVector

AM ≥ 2 then find set of k-linearly independent eigen-vectors.

1) $\lambda_2 = -6$,

$(A - \lambda I)x = 0$

$$\left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 2 & 5 & 4 & 0 \\ -2 & 4 & 5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ 0 & 9 & 9 \end{bmatrix}$$

$R_1 \leftarrow \frac{1}{8} R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1/4 & -1/4 & 0 \\ 2 & 5 & 4 & 0 \\ -2 & 4 & 5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 9/2 & 9/2 \\ -2 & 4 & 5 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$5 - \frac{2}{4} \cdot \frac{1}{2}$ $\frac{10-1}{2}$

$4 - \frac{1}{2}$ $\frac{8-1}{2}$

$5 - \frac{1}{2}$

$R_3 \leftarrow R_3 + 2R_1$

$4 + \frac{2}{4} \cdot \frac{1}{2}$

$\frac{2}{9} \times \frac{9}{2} \times \frac{1}{4}$

$4 - \frac{1}{2}$ $\frac{8-1}{2}$ $5 - \frac{1}{2}$ $\frac{10-1}{2}$

$\frac{2}{9} \times \frac{9}{2}$

$R_2 \leftarrow R_2 \cdot 2/9\, R_2$

$$\begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 1 \\ -2 & 4 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 1 \\ 0 & 9/2 & 9/2 \end{bmatrix}$$

$\frac{7}{2} \times$ $\frac{81}{9} \times \frac{1}{4}$

$\frac{-32}{+8}$

$$\left[\begin{array}{ccc|c} \cancel{1} & 1/4 & -1/4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$R_1 \leftarrow R_1 - \frac{1}{4} R_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = \frac{1}{2} x_3$.

$x_2 = -x_3$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ -t \\ t \end{bmatrix}$$

$$t\begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0 \Rightarrow t\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$\lambda_1 = -6, \; V_1 = (1, -2, 2)$

$$u_1 = \frac{V_1}{\|V_1\|} \Rightarrow \frac{1}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}$$

$\sqrt{1+4+4}$
$9 = 3$

$$\therefore u_1 = \frac{V_1}{\|V_1\|} = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right) \text{ unit Eigen Vector}$$

2) $\lambda_2 = 3$

$$(A - \lambda I)x = 0 \quad \left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \\ -2 & 4 & -4 & 0 \end{array}\right]$$

Gaussian Elimination $\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 2 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 0.$$

$$x_1 = 2x_2 - 2x_3$$
$$x_2 = x_2$$
$$x_3 = x_3.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s - 2t \\ s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 2s - 2t \\ s \\ t \end{bmatrix} \Rightarrow \left\{ s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis for eigenspace with $\lambda_2 = 3$ $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ $v_2 = (2, 1, 0)$ $\quad$ $V_3 = (-2, 0, 1)$ and they are linearly independent.

If $P_1 \cdot P_2 = 0$ and $\|P_1\| = \|P_2\| = 1$ } orthonormal Vectors.

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$V_2 = (2,1,0)$, $V_3 = (-2,0,1)$

$V_2 \cdot V_3 = (2,1,0)\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = -4+0+0 = -4 \neq 0$

$\therefore V_2$ & V_3 are not orthogonal vectors.

Applying GSOP.

~~$V_2 = \alpha t$~~

If $W_2 = V_2$ then

~~$u_2 t$~~ $u_2 = \dfrac{W_2}{\|W_2\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} \leftarrow$ Unit Vector ✓

~~If $W_3 = V_3$ then~~

~~$u_2 =$~~

Basic GSOP + divide by unit vector

$\{V_2, V_3\} \xrightarrow{GSOP} \{u_2, u_3\}$

$W_3 = V_3 \Rightarrow$ ~~$W_3 = V_3 - \dfrac{(W_2, V_3)}{(V_3, V_3)} V_3$~~

$W_3 = V_3 - \dfrac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2$ $\qquad W_2 = V_2 = (2,1,0)$

$=$ ~~$(2,1,0)$~~ ~~$\{(-2,0,1)[2$~~

$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \dfrac{4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2-8/5 \\ -8/5 \\ 1 \end{bmatrix} =$

$-2+\frac{8}{5}$

$-10+8$

$\frac{4}{25}+$

$\frac{-10-8}{5}$

$w_2 = v_2 = (2, 1, 0)$

$v_3 = (-2, 0, 1)$

$\langle v_3, w_2 \rangle = (-2, 0, 1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = -4$

$\langle w_2, w_2 \rangle = (2, 1, 0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 5$

$$w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= (-2, 0, 1) - \left(-\frac{4}{5}\right) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

This is not unit vector so normalize it

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{w_3}{3\sqrt{5}/5} = \left(\frac{-2}{\cancel{5}} \times \frac{\cancel{5}}{3\sqrt{5}}\right) = \frac{-2}{3\sqrt{5}}$$

$$\left(\frac{4}{\cancel{5}} \times \frac{\cancel{5}}{3\sqrt{5}}\right) = \frac{4}{3\sqrt{5}}$$

$$\left(1 \times \frac{5}{3\sqrt{5}}\right) = \frac{5}{3\sqrt{5}}$$

$$\therefore u_3 = \begin{bmatrix} -2/3\sqrt{5} \\ 4/3\sqrt{5} \\ 5/3\sqrt{5} \end{bmatrix}$$

$\frac{-2+8}{5}$

$\frac{-10+8}{5}$

$\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2$

$\frac{4}{25} + \frac{16}{25} + 1$

$\sqrt{\frac{4+16+25}{25}}$

$\sqrt{\frac{45}{25}}$

$\frac{3\sqrt{5}}{5} \times \frac{5}{5}$

$\frac{-10-8}{5}$

So we converted all vectors to unit ~~unique~~ eigen vector

$$P = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}$$

Orthogonal diagonalization of matrix A

Now $P^{-1}AP \equiv P^TAP$.

$$\overset{P^T}{\begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & 4/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & 1 \end{bmatrix}}$$

$$\frac{2}{3} - \frac{4}{3} - \frac{4}{3} \Rightarrow \overset{P^T.A}{\begin{bmatrix} -2 & 4 & -8/3 \\ 5/\sqrt{5} & 3/\sqrt{5} & 0 \\ -2/\sqrt{5} & 4/\sqrt{5} & 4/\sqrt{5} \end{bmatrix}} \overset{P}{\begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{bmatrix}}$$

$\frac{2-8}{3}$

2

$\frac{2}{3} + \frac{2}{3} + \frac{8}{3}$ $\quad 4+4+5$

12

$\frac{12}{3}$ 4

$$\Rightarrow \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$-\frac{2}{3} - \frac{8}{3} + \frac{2}{3}$

$\frac{-8}{3}$

$\frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}}$ $\quad \frac{-4}{\sqrt{5}} + \frac{4}{\sqrt{5}}$

$\frac{4}{\sqrt{5}} - \frac{1}{\sqrt{5}}$

$\frac{4}{\sqrt{5}} + \frac{8}{3\sqrt{5}}$ -10

$-4+8-10$

$4 - 4 + 20$

$-8+20$

16

$4-10$

$\frac{6}{3\sqrt{5}}$

$4 - 4 + 20$ $\quad 12$

$-7-8-16$

$2-4-4$

$2-8$ -

$-6-8-16$

$\frac{-14-20}{3}$

$\frac{34}{3}$

$\frac{-6}{9} - \frac{8}{3} - \frac{16}{9}$

$\frac{-6-24-16}{9}$

$-\frac{2}{3} + \frac{2}{3} - \frac{2}{3}$

$-4+4$

$-4+16-40$

$12-40$

$\frac{-4}{3\sqrt{5}} + \frac{16}{3\sqrt{5}} - \frac{40}{3\sqrt{5}}$

$\frac{12-40}{3\sqrt{5}} \Rightarrow \frac{-28}{3\sqrt{5}}$

If matrix is symmetric $\longrightarrow$ then it is Orthogonally diagonalizable

If not symm $\longrightarrow$ We can't say anything about diagonalization.

$\downarrow$

find eigen values $\rightarrow$ check whether distinct or not & so on the entire procedure.

Matrix Norm

A norm is a fn $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ that assigns a real valued length to each matrix. For all matrices A and B and for all scalars $a \in \mathbb{R}$

1. $\|A\| \geq 0$ and $\|A\| = 0$ only if $A = 0$ positivity condn
2. $\|A+B\| \leq \|A\| + \|B\|$ △er inequality.
3. $\|aA\| = |a| \, \|A\|$ Scaling cond?
4. $\|AB\| \leq \|A\| \cdot \|B\|$ Compatibility cond?. applicable to sqr matrix

eg 2)

over column

* popular matrix Norms :-

1. $\|A\|_1 = \max_{1\le j\le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$ 1-norm / l_1-norm

max over columns

rows

over rows

eg $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$

(1) (5)

$\max(1, 5) = 5 \leftarrow$ is 1-norm of A $\Rightarrow \|A\|_1 = 5$

2. $\|A\|_E = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}$ 2-norm / l_2-norm / E-norm

Imp

$\sqrt{(1)^2 + (0)^2 + (2)^2 + (3)^2} \Rightarrow \sqrt{14} \leftarrow \|A\|_E$

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same defⁿ as one norm
traverse through columns.

(3) $\|A\|_\infty = \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |a_{ij}| \right)$ max infinity norm.

max over rows

eg A = $\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$ $\cancel{\max(2,0) = (2 = \|A\|_\infty}$

$\max(\underbrace{3,3}) = 3 = \|A\|_\infty$

Take norm.

$(1+2), (0+3) = \max(3,3)$.

eg2) $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

over columns $\|A\|_1 = \max(1, 4) = 4$ (~~rows~~ (column)

$\|A\|_E = \sqrt{\underset{1}{(1)^2} + \underset{4}{(2)^2} + \underset{4}{(2)^2}} \Rightarrow \sqrt{9} = 3.$

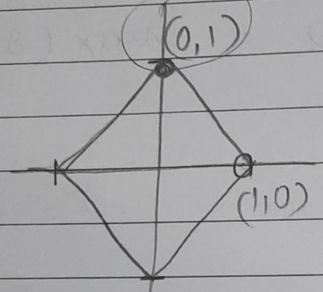
over rows $\|A\|_\infty = \max(3, 2) = 3.$ (row) (~~columns~~)

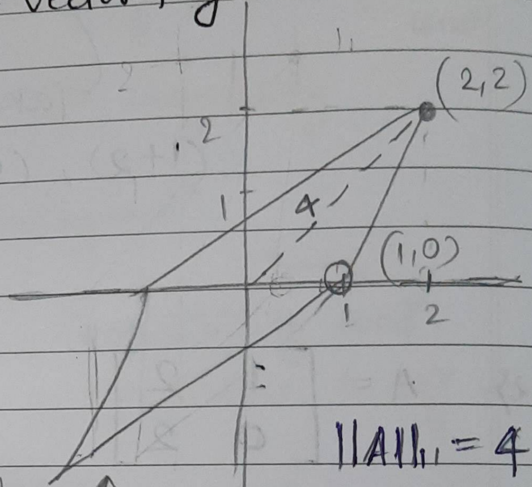
Geometric interpretation :-

The closed unit ball $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\qquad Ax = y.$ norm of $\|A\|$ will define amplificatn of ~~vector of~~ y to x.

① 1-norm





$\|A\|_1 = 4$

$(2,2)$

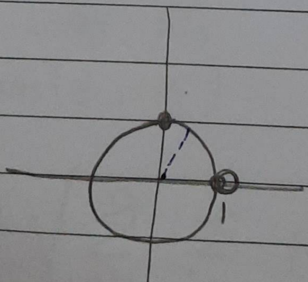
$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{2\times 2} \overset{x}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times 1} = \overset{y}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

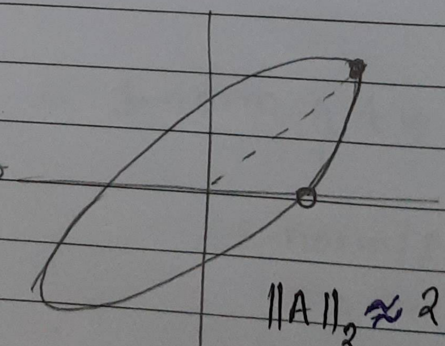
→ Amplified

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ so this vector doesnt change

max^m amplificatn you can get is given by 1-norm.

② 2-norm

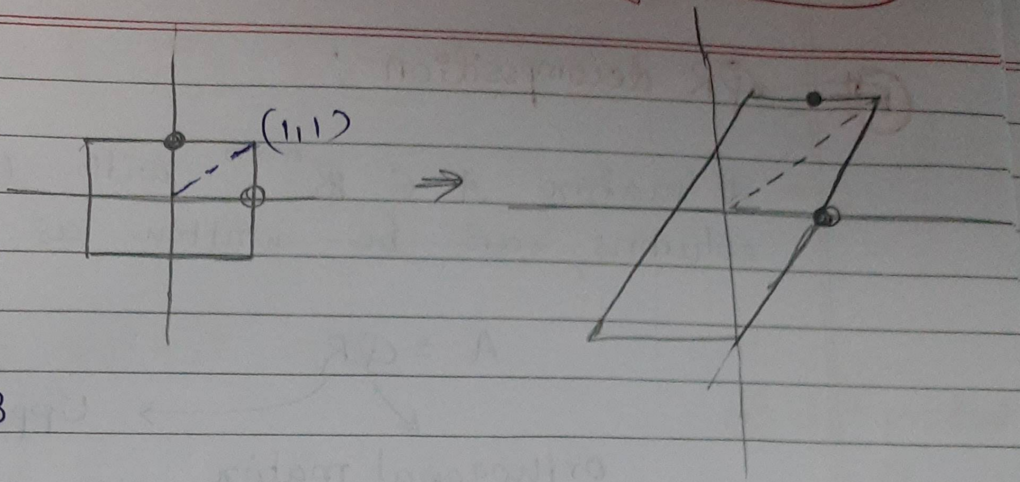




$\|A\|_2 \approx 2.9208$

max^m amplif^n factor under 2-norm

(3) ∞-norm $\Rightarrow$



$\|A\|_\infty = 3$

Vector Norm $\Rightarrow$ distance

Matrix Norm $\Rightarrow$ Amplification of original vector