Stochastic Processes

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* Stochastic process :- It is a sequence of random variables indexed by a parameter set (eg. time, length, etc)

$\{X_t : t \in T\}$, where T is called the index set / parameter set / Time Space.

Collection of RV : finite, infinite → Countably / Uncountably.

* State Space (S) : Set of all possible values of X_t for each t.

* Time Space (T) : The parameter space is called time space.

* Classification of SP :-

1) Discrete Space, discrete time

syllabus

S and T are finite or countably infinite.

eg. X_t = The no. of people waiting at counter in a line at train station.

$T = \{0, 1, 2, \text{-----} 12\}$

$S = \{0, 1, 2, \text{----} \infty\}$

$\{X_0, X_1 \text{ ---- } X_t\}$

After 1 hr. how many people are waiting & so on when counter closes after 12 hrs no. of people waiting

2) Discrete State, continuous time SP :-

$S \rightarrow$ finite / countably infinite
$T \rightarrow$ continuous (uncountable set)

eg. X_t = no. of people waiting at counter

$\{X_t : t \in T\}$

$T = \{t : 0 \leq t < \infty\}$

$S = \{0, 1, 2, \ldots \ldots\}$

3) Continuous State, discrete time SP :-

$S \rightarrow$ Uncountable set
$T \rightarrow$ finite / countably infinite

eg X_t = selling / buying price of Indian Rupees against dollar in a time interval $[0, t]$

$S = \{x : 0 \leq x < \infty\}$

$T = \{1, 2, 3, \ldots \ldots\}$ days. ie At the end of 1, 2, ... days what was the SP of ₹

$\{X_t : t \in T\}$

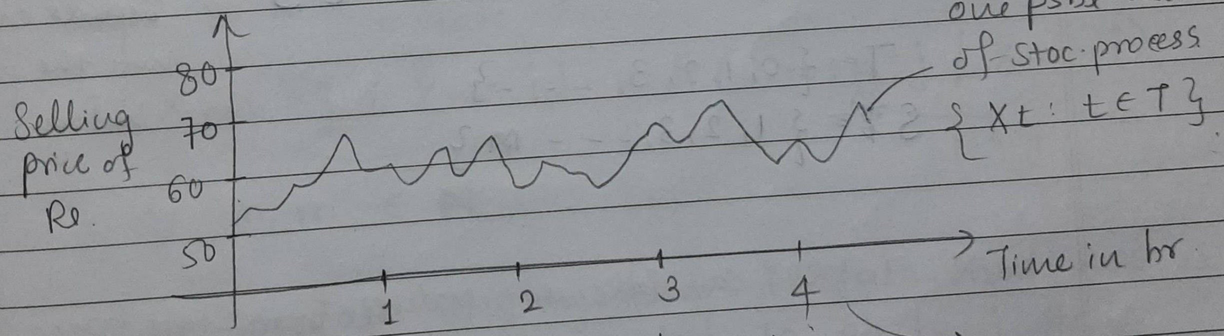
4) Continuous state, continuous time SP :-

S → continuous.
T →

$T = \{ t : 0 \leq t < \infty \}$ where we are interested in each time ~~uniq~~ unit & what is SP selling price & buying price of shares at that unit.

* Sample path :

Mapping $X_t : T \to S$



T → continuous
S → discrete

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* DTDS ~~S~~tochastic Processes ⇒

$\{X_t : t \in T\}$ $\quad \frac{S}{T} >$ finite / countably infinite

Discrete Time Markov chain (DTMC)

A sequence of RV's $\{X_0, X_1, X_2, \ldots\}$ with the state space $S = \{1, 2, \ldots M\}$ is called DTMC, or simply Markov chain if for all $n \geq 0$

independent

$$P(X_{n+1} = j \mid \underbrace{X_n = i, X_{n-1} = i_{n-1}, \cancel{X_n} \ldots X_0 = i_0}$$

state

1 step dependency $\quad = P(X_{n+1} = j \mid X_n = i) = P_{ij}^{(1)}$

step

In future where you will be only depends on present not the past history

where $\quad i, j, i_{n-1}, \ldots i_0 \in S$

$T = \{0, 1, 2, 3, \ldots\}$
$S = \{1, 2, 3, \ldots M\}$.

ie Each $\{X_0, X_1, \ldots\}$ RV's can take any values from state space S

eg. X_n: The no. of customers in Barber shop at the end of n^{th} hour.

→ $\{X_n : \underbrace{n = 0, 1, 2, \ldots}_{\text{hours}}\}$ $\qquad S = \{0, 1, 2, 3, \ldots\}$

$P(X_6 = 10 / X_5 = 9, X_4 = 4, X_3 = 5, X_2 = 8, X_1 = 9, X_0 = 1)$

$= P(X_6 = 10 / X_5 = 9)$

10 customers at 6^{th} hr $\qquad$ 9 customers at 5^{th} hr

$p_{ij}^{(1)}$ = One step transition probability from state i to state j.

$S = \{1, 2, \ldots M\}$.

$$p^{(1)} = \left(p_{ij}^{(1)}\right)_{i,j}$$

One step transn probability matrix =

$$\begin{array}{c|c} & 1\ 2\ 3 \cdots M \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ M \end{matrix} & \left[\begin{matrix} \quad & \quad \\ & \\ & \\ & \end{matrix} \right]_{M \times M} \end{array}$$

Time Homogeneous Markov chain :-

$\{X_n : n \in \mathbb{N}\}$ is said to be a time homogeneous MC if

state

$$P(X_{n+k} = j \mid X_n = i) = P(X_{m+k} = j \mid X_m = i)$$

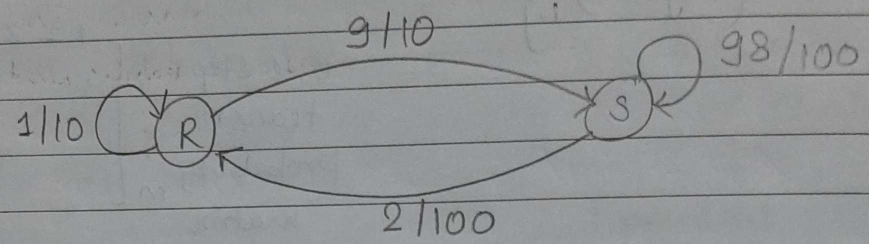
k steps   k steps

$\forall \ n, m \in \mathbb{N}$

When you are starting from i state then how many steps (K) are needed to go to j state. Time Space n, m doesn't matter, only no. of steps & ~~current~~. ~~state~~ matters.

Time homogeneous

Ex.1) R → Rainy day. → 0
S → Sunny day → 1



X_n : The weather on nth day. $\cancel{T}=\{1,2,\ldots\}$

$S = \{0, 1\}$.
$T = \{1, 2, 3, \ldots\}$.

1step prob. Trans^n matrix.

$$P^{(1)} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1/10 & 9/10 \\ 1 & 2/100 & 98/100 \end{array}$$

$P_{00}^{(1)}$, $P_{01}^{(1)}$, $P_{10}^{(1)}$, $P_{11}^{(1)}$

*** Observations :-**

1) Sum of each row is 1

$$\sum_{j=1}^{M} P_{ij} = 1, \quad \forall\ i = 1, 2, \ldots M$$

2) $P_{ij}^{(1)} \geq 0, \ \forall\ i,j$

m-step transition probability matrix

1) $P^{(m)} = P_{ij}^{(m)}$ $S = \{1, 2, \dots, M\}$.

$$= \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ M \end{array} \begin{array}{c} 1 \quad 2 \quad \dots \quad M \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & \end{array} \right]_{M \times M} \end{array}$$

m-step trans^n probability from state 'i' to state 'j'

2) $P_{ij}^{(0)} = \begin{cases} 1, & \text{if } \boxed{i = j} \\ 0, & \text{if } i \neq j \end{cases}$

3) $0 \leq P_{ij}^{(m)} \leq 1$, $\forall i, j$

4) $\sum_{j=1}^{M} P_{ij}^{(m)} = 1$, $\forall i = 1, 2, \dots, M$ $\forall i \in S$.

Chapman - Kolmogorov Eq^{n}s :-

$P^{(n)} = P^{(n-1)} P^{(1)}$ $P^{(3)} = P^{(2)} \cdot P^{(1)}$

$P^{(m+n)} = P^{(m)} \cdot P^{(n)}$ $P^{(2)} = P^{(1)} \cdot P^{(1)}$

$P^{(1)} = P$

14th_M.

$S = \{0, 1, 2, \dots M\}$

$p(m) = \left(p_{ij}^{(m)}\right)_{(M+1)\times(M+1)}$ M-step trans^n prob. matrix.

As 0 to M $\Rightarrow$ (M+1) states.

* Initial distribution:-

If you have a MC $\{X_n ; n \in N\}$ with state space $S = \{0, 1, 2, \dots M\}$. The probability distribution $\Pi = (\Pi_0, \Pi_1, \Pi_2, \dots \Pi_M)$ with

$\boxed{\Pi_i = P(X_0 = i)}$ what is the Probability that in 0 step, you will be in state i

is called the initial distribution.

Start $S = \{0, 1\}$ when MC is starting then what is the probability of being in state 0

eg. $P(X_0 = 0) = \frac{1}{2}$, $P(X_0 = 1) = \frac{1}{2}$ $S = \{0, 1\}$

then $(\frac{1}{2}, \frac{1}{2})$ is called initial distribution.

At start of MC.
P(being in state 1)

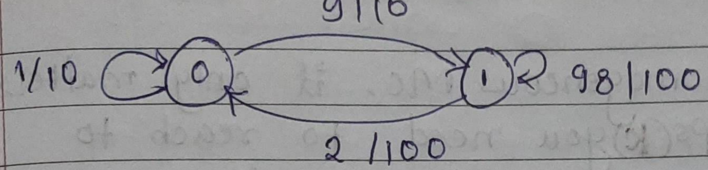
* $p^{(2)} = p^{(1)} \cdot p^{(1)}$

$p^{(3)} = p^{(2)} \cdot p^{(1)}$

$p^{(6)} = p^{(3)} \cdot p^{(3)}$

$= p^{(4)} \cdot p^{(1)} \cdot p^{(1)}$

eg.1) Rainy Sunny Example :-



$$p^{(1)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1/10 & 9/10 \\ 2/100 & 98/100 \end{bmatrix} \end{array}$$

$$p^{(2)} = p^{(1)} \cdot p^{(1)} = \begin{bmatrix} 1/10 & 9/10 \\ 2/100 & 98/100 \end{bmatrix} \begin{bmatrix} 1/10 & 9/10 \\ 2/100 & 98/100 \end{bmatrix}$$

$$p^{(2)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{280}{10000} & \frac{9820}{10000} \\ \frac{216}{10000} & \frac{9784}{10000} \end{bmatrix} \end{array}$$

$P_{00}^{(2)}$ $P_{01}^{(2)}$ $P_{10}^{(2)}$ $P_{11}^{(2)}$

$P_{00}^{(2)}$ = If today is rainy day then probability that day after tomorrow will be rainy day

→ Conditional probability.

* $P(X_{n+2}=0 / X_n = 0) = P_{00}^{(2)}$

* $P(X_2=0)$ = Probability that day after tomorrow will be rainy (0) day. and we dont know whats the state today

Unconditional probability.

$P(X_2=0/X_0=0) \equiv P(X_{n+2}=0/X_n=0) = P_{00}^{(2)}$

Can be calculated using Total probability

1) $P(X_2=0) = P(X_2=0/X_0=0)\,P(X_0=0)$ Initial day Rainy
$+ P(X_2=0/X_0=1)\,P(X_0=1)$ Initial day Sunny

As it is time homogeneous MC, it only matters how many steps (K) you need to reach to desired state, your current state does not matter

i.e. $P(X_{n+k}=j / X_n=i) \equiv P(X_{m+k}=j / X_m=i)$

current state

i.e. after K steps you will reach to step j, current state doesnt matter.

$P(X_0=0)=\frac{1}{2}$ $P(X_0=1)=\frac{1}{2}$

$P(X_2=0) = P_{00}^{(2)}\cdot\frac{1}{2} + P_{10}^{(2)}\cdot\frac{1}{2}$

(UNCD. probability)

$= \frac{\cancel{280}\ 140}{10000}\times\frac{1}{\cancel{2}} + \frac{\cancel{216}\ 108}{10000}\times\frac{1}{\cancel{2}}$

$= \frac{140+108}{10000} = \frac{248}{10000}$

2) $P(X_2=1) = P(X_2=1/X_0=0)\,P(X_0=0)$
$+ P(X_2=1/X_0=1)\,P(X_0=1)$

$= P_{01}^{(2)}\,\overset{\pi_0}{P(X_0=0)} + P_{11}^{(2)}\,\overset{\pi_1}{P(X_0=1)}$

$= \frac{\cancel{9820}\ 4910}{10000}\times\frac{1}{\cancel{2}} + \frac{\cancel{9784}\ 4892}{10000}\times\frac{1}{\cancel{2}}$

$= \frac{9802}{10000}$

$S = \{0, 1, 2, \ldots M\}$ states and initial distⁿ is

$\pi = (\pi_0, \pi_1, \ldots \pi_M)$

$$P(X_n = K) = \sum_{j=0}^{M} P_{jK}^{(n)} \pi_j$$

probability that after n steps it will be in state 'K'

$$P(X_2 = 1) = \sum_{j=0}^{1} P_{jK}^{(2)} \pi_j$$

$$P(X_2 = 1) = P_{01}^{(2)} \pi_0 + P_{11}^{(2)} \pi_1$$

$\pi_0 = P(X_0 = 0)$, $\pi_1 = P(X_0 = 1)$ ← Initial distribution

Gamblers Ruin Problem

eg.2> Two players A and B.

9.15 A's capital $= x \in \mathbb{N}$ dollars

B's capital $= y \in \mathbb{N}$ dollars.

Let $a = x + y = 3$ (any number)

$P(A \text{ wins } \$1 \text{ from } B)$ ~~with~~ $= p$

$P(B \text{ wins } \$1 \text{ from } A) = \cancel{1-p}\ q\ (1-p)$

~~p+q=1~~

The game goes on until one of the players loses all of his capital, that is until $X_n = 0$ or $X_n = a$.

$X_n =$ capital of the player A in the nth round.

$T = \mathbb{N}$, $S = \{0, 1, 2, \ldots a\}$.

$\{X_1, X_2, X_3 \ldots\}$ forms a MC.

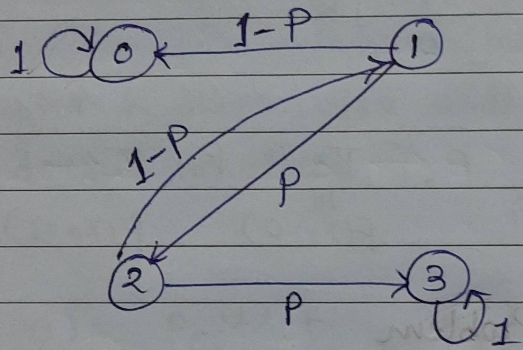
X_1: capital of A at end of 1st game

X_n = capital of player A at end of nth round.

Game stops

$S = \{0, 1, 2, 3\}$ as $x + y = 3$.

$$p^{(1)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



n-step transition probability matrix $p(n) = ?$

$$\lim_{n \to \infty} p(n) = (\qquad)$$ OR whether this limit exists or not.

→ After long run

long run behaviour of Markov chain.

limiting distribution

i.e. after maybe 1M games whats the probability that player A will be in state 0, 1, 2 or 3 ?

To know whether limiting distribution exists OR not & even if exists how to calculate that limiting distribution → Classification of State.

14th Eve.

* Classification of States :

We have a MC , $\{X_n : n \geq 0\}$ with state space S.

(1) Accessible State:-

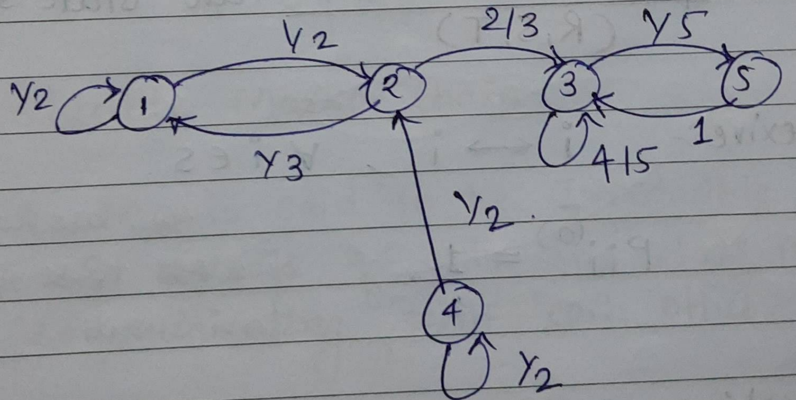
In a MC, a state $j \in S$ is said to be accessible from a state $i \in S$ if.

$$P_{ij}^{(n)} > 0 \quad \text{for some } \underline{n \geq 0}.$$

(then state j is accessible from state i)

$$j \leftarrow i$$

eg. $S = \{1, 2, 3, 4, 5\}$.



$P_{42}^{(1)} = 1/2$, state 2 can be accessed from state 4 with probability 1/2

$2 \leftarrow 4$ but $4 \not\leftarrow 2$.

② Communicating States :-

In a MC, two states $i, j \in S$ are said to be communicating states if

$$i \leftarrow j \text{ and } j \leftarrow i \equiv i \leftrightarrow j$$

eg. In prev state diagrams below state are are communicating with each others.

$(1 \leftarrow 2,\ 2 \leftarrow 1),\ (5 \leftarrow 3,\ 3 \leftarrow 5)$

$(1 \leftrightarrow 2),\ (5 \leftrightarrow 3)$

③ Result :- The relation $i \leftrightarrow j$ forms an equivalence relation in state space S.
(R, S, T)

i) Reflexive :- $i \leftrightarrow i,\ \forall i \in S$.

$P_{ii}^{(0)} = 1$ You are not moving & you are there forever

certain event.

ii) Symmetric :- $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

$\exists\ n \geq 0$ such that $P_{ij}^{(n)} > 0$.

$P_{ji}^{(n)} > 0$

iii) Transitivity :- $i \leftrightarrow j$ & $j \leftrightarrow k \Rightarrow i \leftrightarrow k$.

(3) Communicating Classes :-

Every equivalence relation forms an Equivalence Class.
(Makes a partition in State space S)

* C(i) :- Communicating class of i.
is set of all all those state spaces such that i is communicating to j.

$$C(i) = \{ j \in S \,/\, i \leftrightarrow j \}.$$

$$= C(j)$$

eg. From prev state tr. diag. $S = \{1, 2, 3, 4, 5\}$

$C(1) = \{1, 2\} = C(2)$

$C(3) = \{3, 5\} = C(5)$.

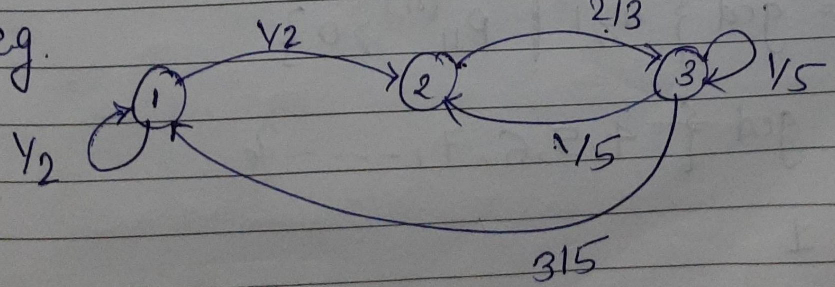
$C(4) = \{4\}$.

partition of state space S. with 3 diff classes.

(4) Irreducible Markov Chain :-

A MC is said to be Irreducible if the state space S consists of only one class ie. all states are communicating with each other.

eg.



$C(1) = \{1, 2, 3\} = C(2) = C(3)$

Access in 1 step 2 steps.

⑤ Absorbing State :-

In a given MC, a state $i \in S$ is said to be absorbing state if

once you go there then no coming back

$$p_{ii}^{(n)} = 1, \ \forall n \geq 0$$

$P_{00}^{(1)} = 1$
$P_{00}^{(2)} = 1$

i.e. if you are in state i then you are in that state forever.

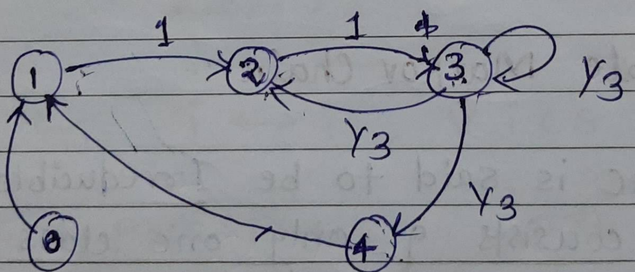
eg. In Gamblers ruin problem, states 0 and 3 are absorbing states.

⑥ Periodicity :-

The periodicity of state $i \in S$ in MC, is defined as

$$\lambda(i) = GCD \{ n \geq 1 \mid P_{ii}^{(n)} > 0 \}$$

eg.



$$\lambda(1) = gcd \{ n \geq 1 \mid P_{11}^{(n)} > 0 \}$$

$$= gcd \{ 4, 5, 6, 7, \text{----} \}$$

$$\lambda(1) = 1$$

$$\lambda(3) = gcd \{ 1, 2, 4, 5, \text{---} \}$$
$$= 1$$

Remarks: If $P_{ii}^{(n)} = 0$, $\forall\, n \geq 1$ then the periodicity of the state i is defined as
$\lambda(i) = 0$.
i.e. if you start from state i and cannot come back then periodicity of that state is 0.

here, $\lambda(0) = 0$ (state)

⑦ Aperiodic State :-
A state $i \in S$ is said to be aperiodic if its periodicity is 1 i.e. $\lambda(i) = 1$.

* Results ⇒ one communicating class

1) In an irreducible MC, all the states have same periodicity.

2) Defination: A chain is said to be aperiodic if all of its states are aperiodic.

⇓

Aperiodic MC.

16th morning

* First Entrance / Visit :-

$MC = \{X_n : n \geq 0\}$, $S = \{1, 2, \dots m\}$.

The first entrance is defined as

$f_{jK}^{(n)}$ = P(Entering to the state 'K' for the first time in 'n' steps given that it was in the state j)

probability of first time return

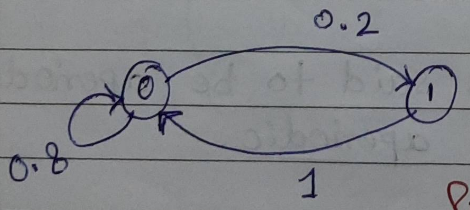
$= P(X_{n+m} = K, X_{n+m-1} \neq K \dots X_{m+1} \neq K \,/\, X_m = j)$

Result :-

$$\boxed{P_{jK}^{(n)} = \sum_{r=0}^{n} f_{jK}^{(r)} \underline{P_{KK}^{(n-r)}}}$$

Imp

eg. 1)



$$P^{(1)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 1 & 0 \end{bmatrix} \end{array}$$

first time i.e. you have to go & come back

$P_{00}^{(2)} = f_{00}^{(0)} P_{00}^{(2)} + f_{00}^{(1)} P_{00}^{(1)} + f_{00}^{(2)} P_{00}^{(0)}$

$j = K = 0$
$n = 2$

$= 0 + 0.8 * 0.8 + 0.2 \times 1 * 1$

$\Rightarrow 0.64 + 0.2 \Rightarrow 0.84$

here $P_{00}^{(2)}$ & $f_{00}^{(2)}$ are $= 0.2 \times 1$

Start from 0, land up in 0, in 0 step for first time that prob. is zero

Not going anywhere i.e. always staying there

$f_{00}^{(0)} = 0, \; P_{00}^{(0)} = 1$

* Probability of ever return :-

For fixed j and k,

f_{jk} = P(the process will ever return to the state k when it started from state j)

$$= \sum_{n=0}^{\infty} f_{jk}^{(n)}$$

If $j=k$ then

$$\boxed{f_{jj} = \sum_{n=0}^{\infty} f_{jj}^{(n)}}$$

starting from state j, probability that you will ever return to state j

* Recurrent / Persistent state :-

$\sum_{n=0}^{\infty} f_{jj}^{(n)} = 1$

A state $j \in S$ is said to be recurrent state if

$f_{jj} = 1$

if you start from j 100% chance that you will return to state j.

i.e. P(ever return) = 1

* Transient State :-

A state $j \in S$ is said to be transient if

$0 \leq f_{jj} < 1$

i.e. There is some % chance that you will never come back to same state.

* Mean Recurrent time :- On an avg how much time will be required to start from State j and reach to state k

$$\mu_{jK} = \sum_{n=0}^{\infty} n \cdot f_{jK}^{(n)}$$

if $j = k$, $\boxed{\mu_{jj} = \sum_{n=0}^{\infty} n \cdot f_{jj}^{(n)}}$ come back to same state j

* Null Recurrent :-

If $\mu_{jj} = \infty$ then state j is called Null Rec.

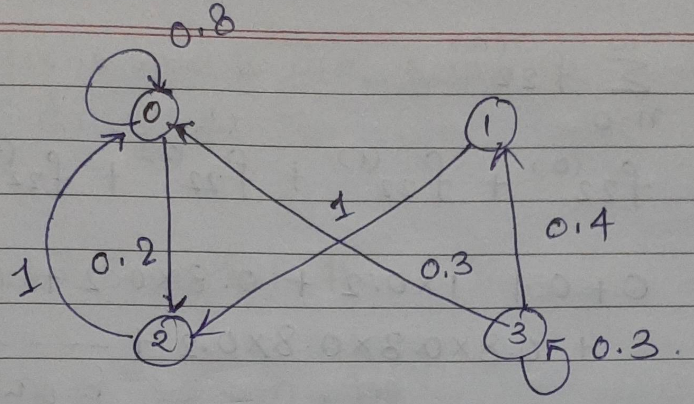
* Positive Recurrent :-

If $\mu_{jj} < \infty$ (finite) then state j is said to be positive recurrent.

eg.

$$p(1) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ 0 & 0.8 & 0 & 0.2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0.3 & 0.4 & 0 & 0.3 \end{array}$$

$S = \{0, 1, 2, 3\}$



Irreducible MC = one class

Absorbing = no moving.

periodicity = gcd (how many steps you can come back)

Aperiodic ⇒ gcd = 1

Every irreducible MC is aperiodic.

Recurrent = $f_{jj} = 1$. forever returns ✓.

Transient = $f_{jj}^{\infty} < 1$

1> check for Irr-Reducible MC

$C(0) = \{0, 2\} = C(2)$ Reducible MC.

$C(1) = \{1\}$

$C(3) = \{3\}$

2> check for Recurrence. on all states.

$f_{00} =$ ~~0.8~~ $f_{jj} = \sum_{n=0}^{\infty} f_{jj}^{(n)}$

$$f_{00} = \sum_{n=0}^{3} f_{00}^{(n)} = f_{00}^{(0)} + f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} + \dots$$

$= 0 + 0.8 + 0.2 \times 1 + 0$

$= 0.8 + 0.2$

$= 1$

∴ State 0 is recurrent state.

$$f_{22} = \sum_{n=0}^{\infty} f_{22}^{(n)}$$

$$= f_{22}^{(0)} + f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + \dots$$

$$= 0 + 0 + 1 \times 0.2 + 0.8 \times 0.2 + 0.8 \times 0.8 \times 0.2 + 0.8 \times 0.8 \times 0.8 \times 0.2 + \dots$$

$$= 0.2 + 0.8 \times 0.2 + 0.8^2 \times 0.2 + 0.8^3 \times 0.2$$

$$= \cancel{0.2 + 0.8 \times 0.2 (1 + 0.8}$$

$$= 0.2 (1 + 0.8 + 0.8^2 + 0.8^3 + \dots)$$

$$= \cancel{0.2 + 0.2 (0.8}$$

$$=$$

$$\left(\frac{a}{1-r}\right) \quad = 0.2 \times \left(\frac{1}{1-0.8}\right) = \cancel{0.2} \times \frac{1}{\cancel{0.2}} = 1$$

$\therefore f_{22} = 1$ $\therefore$ State 2 is also recurrent state.

$$f_{33} = \sum_{n=0}^{\infty} f_{33}^{(n)}$$

$$= f_{33}^{(0)} + f_{33}^{(1)} + f_{33}^{(2)} + \dots$$

$$= 0 + 0.3 + 0 + 0 \dots$$

$$= 0.3 < 1$$ $\therefore$ State 3 is not recurrent

if state 3 is Transient state

$$f_{44} = \sum_{n=0}^{\infty} f_{44}^{(n)} = f_{44}^{(0)} + f_{44}^{(1)} + f_{44}^{(2)} + \dots$$

$$= 0 + 0 + \dots$$

$$=$$

$$= 0$$ Transient state.

$\therefore$ States 0 and 2 are Recurrent and states 1 and 3 are transient.

(3) Mean Recurrent time

$$\mu_{jj} = \sum_{n=0}^{\infty} n \cdot f_{jj}^{(n)}$$

$$\mu_{00} = 0 + 1 \times f_{00}^{(1)} + 2 \times f_{00}^{(2)} + 3 \times f_{00}^{(3)} + \cdots$$

$$= 0 + 0.8 + 2 \times 0.2 + 3 \times 0 + 0 + \cdots$$

$$= 0.8 + 0.4$$

$$\boxed{\mu_{00} = 1.2}$$

i.e. On an avg you need 1.2 states to come back to the same state '0' if you start from it.

As $\mu_{jj} < \infty$ i.e. its finite So state 0 is positive Recurrent.

States 0 & 2 are Recurrent & its finite MC so these states are positive recurrent.

Result :- In a finite MC, every recurrent state is a positive recurrent state.

* Ergodic State :-

A state 'j' is said to be Ergodic if it is positive recurrent & aperiodic.

eg. Above proble.

periodicity of state 0

$\lambda(0) = \gcd(1, 2, 3, 4 \ldots\ldots)$

$= 1$ i.e. state 0 is periodic state

& it is +ve Recurrent. So state 0 is Ergodic state.

* A MC is said to be ~~E~~ Ergodic if all of its states are Ergodic (+ve rec & aperiodic)

Imp

State j

- Recurrent state
  $f_{jj} = 1$
  - positive recurrent
    $\mu_{jj} < \infty$
    - Aperiodic
      $\lambda(j) = 1$
      - Ergodic state
        (+ve rec & aperiodic)
    - Periodic
      $\lambda(j) \neq 1$
  - Null recurrent
    $\mu_{jj} = \infty$
- Transient state.
  $0 \leq f_{jj} < 1$.

Results ⇒

A) Suppose that two states 'i' and 'j' are such that $i \leftrightarrow j$, Then 'i' is recurrent iff 'j' is recurrent. *

Corollary: $i \leftrightarrow j$, then state 'i' is transient iff 'j' is transient.

B) In a finite MC, all states cannot be transient i.e. There exist atleast one recurrent state. *

C) All states of a finite irreducible MC (only one class) are recurrent (i.e. no transient state)

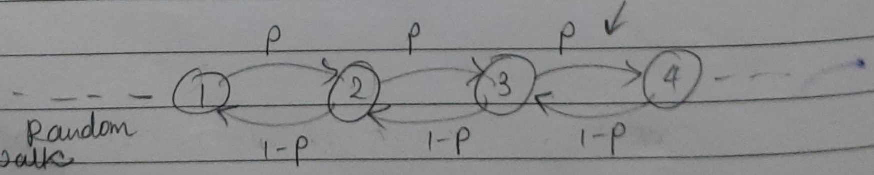
D) Every recurrent state in a finite MC is positive recurrent

E) All states of finite irreducible MC are positive recurrent.

(finite + infinite)

F) In an irreducible MC, one of the following holds

a) All states are positive recurrent
b) All states are Null recurrent
c) All states are transient (Infinite MC)



Simple Random walk

G) Suppose $i \leftrightarrow j$ and state 'i' has periodicity 'd' then state 'j' also has the same periodicity 'd'

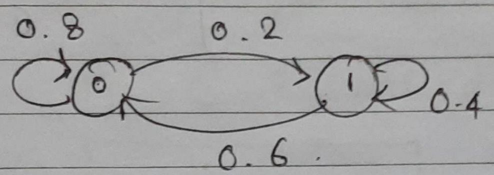
(one class)
Corollary: In an irreducible MC, if one state is aperiodic then all other states are aperiodic

Limiting distribution

Rainy Sunny ⇒

$$* \; P^{(1)} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.8 & 0.2 \\ 1 & 0.6 & 0.4 \end{array}$$

0 : Rainy
1 : Sunny.



$$P^{(2)} = P^{(1)} P^{(1)} = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{pmatrix}$$

$$P^{(3)} = P^{(2)} . P^{(1)} = \begin{pmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{pmatrix}$$

$$P^{(4)} = P^{(3)} . P^{(1)} = \begin{pmatrix} 0.75 & 0.25 \\ 0.74 & 0.251 \end{pmatrix}$$

$$P^{(5)} = P^{(4)} . P^{(1)} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.75 & 0.25 \\ 1 & 0.75 & 0.25 \end{array}$$

only matters.

If today is rainy then 75% chance that after 6 days it will be rainy day.

→ 1 — Sunny.

if whether today is rainy or Sunny, it doesn't matter as after 6 days 75% chance it will be rainy (0)

* i.e. For a long run, initial state does not matter.

$\pi_0 = 0.75$, $\pi_1 = 0.25$ } limiting distribution

$\pi = (\pi_0, \pi_1)$

* After long run whats the probability that I will land up in state 0 or 1.

⇓

limiting distribut^n

* Stationary distribution :-

$P_{00}^{(n)} \Rightarrow 0.75$, $P_{11}^{(n)} \rightarrow 0.25$

$P_{10}^{(n)} \rightarrow 0.75$, $P_{01}^{(n)} \rightarrow 0.25$

$\pi \cdot P^{(5)} = \pi$

$(\pi_0, \pi_1) P^{(5)} = \pi$

LHS $\Rightarrow$ $\underset{1\times2}{(0.75 \quad 0.25)} \underset{2\times2}{\begin{pmatrix} 0.75 & 0.25 \\ 0.75 & 0.25 \end{pmatrix}} = \underset{1\times2}{\begin{bmatrix} 0.75 & 0.25 \end{bmatrix}}$

= RHS.

$$\boxed{\pi \cdot P^{(n)} = \pi}$$

Stationary distribution $\Rightarrow$

A MC $\{X_n : n \geq 0\}$ and $S = \{0, 1, 2, \ldots\}$ then a probability distribution $\pi = \{\pi_0, \pi_1, \ldots\}$

where $\pi_i > 0$ is called the stationary dist^n

if $\boxed{\pi \cdot P^{(1)} = \pi}$ $\leftarrow$ Stationary dist^n

when $\boxed{\sum_{i=0}^{\infty} \pi_i = 1 \quad \text{and} \quad 0 < \pi_i < 1}$

$$\pi \cdot P^{(2)} = \pi \cdot P^{(1)} \cdot P^{(1)}$$
$$= \pi \cdot P^{(1)}$$
$$= \pi$$

$$\pi \cdot P^{(3)} = \pi \cdot P^{(2)} \cdot P^{(1)} = \pi \cdot P^{(1)} = \pi$$

$$\vdots$$

$$\boxed{\pi \cdot P^{(n)} = \pi}$$

$\therefore$ If you start with initial distribution π then over the time it doesn't change.

Th: Chr 88

Th: Existence & Uniqueness theorem :-

A) If a MC is Irreducible (one class) then there is a sol^n to the system (*) iff the MC is +ve Recurrent

$\mu_{jj} < \infty$

$$\left.\begin{aligned} &\pi \cdot P^{(1)} = \pi \\ &\text{and } \sum_{i=0}^{\infty} \pi_i = 1 \end{aligned}\right\} \text{Unique stationary dist}^n$$

B) If a sol^n exists, then it is unique.

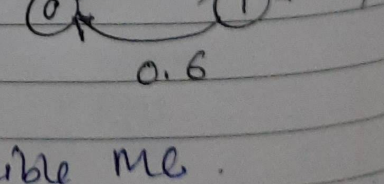
$$\boxed{\text{Corollary:- An irreducible +ve recurrent MC has unique stationary distribution}}$$

~~Irreducible finite MC, every Recurrent state is +ve Rec,~~

\# Irreducible finite recurrent (+ve rec) MC has unique dist^n. and

eg Rainy Sunny

$$P^{(1)} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.8 & 0.2 \\ 1 & 0.6 & 0.4 \end{array}$$



$C(0) = \{0,1\} = C(1)$ Irreducible MC.

Result G

$$\left.\begin{aligned} \lambda(0) &= \gcd\{1,2,3,4\ldots\} = 1 \\ \lambda(1) &= \gcd\{1,2,3\ldots\} = 1 \end{aligned}\right\} \text{Aperiodic.}$$

finite MC — atleast one recurrent state.

~~$d_{jj} = \sum_{n=0} n \cdot f_{jj}$~~

~~$\mu_{00} = 0 \times f_{00}^{(0)} + 1 \times f_{00}^{(1)} + 2 \times f$~~

$$\sum_{n=0}^{\infty} f_{jj}^{(n)} = 1$$

$$f_{00} = f_{00}^{(0)} + f_{00}^{(1)} + f_{00}^{(2)} - -$$
$$= 0 + 0.8 + 0.2 \times 0.6 + 0.2 \times 0.4 \times 0.6 + 0.2 \times 0.4 \times 0.4 \times 0.6 +$$

$0 \leftrightarrow 1$

then if 0 is recurrent then 1 is recurrent.

In finite MC, every recurrent state is +ve rec.

∴ for irreducible +ve Rec MC, there exists a unique stationary distribution. — corollary

$\pi = (\pi_0, \pi_1)$ only two states 0 & 1.

$S = \{0, 1\}$

$\pi \cdot P^{(1)} = \pi$ — ①

and $\pi_0 + \pi_1 = 1$ — ②.

$$\underset{\pi}{(\pi_0 \quad \pi_1)} \underset{P^{(1)}}{\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}} = \underset{\pi}{(\pi_0 \quad \pi_1)} \quad \text{— ①}$$

$$\boxed{\pi_0 + \pi_1 = 1} \quad \text{— ②}$$

solving system of eqn.

Stationary distribution ⇒

$$(0.8\pi_0 + 0.6\pi_1 \quad 0.2\pi_0 + 0.4\pi_1) = (\pi_0, \pi_1)$$

$$0.8\pi_0 + 0.6\pi_1 = \pi_0 \quad \text{— (1)}$$
$$0.2\pi_0 + 0.4\pi_1 = \pi_1 \quad \text{— (2)}$$
$$\pi_0 + \pi_1 = 1 \quad \text{— (3) — Imp}$$

3 eqns, 2 unknowns.

(2)/0.2 $\quad 0.2 \times (3)$.

$\cancel{\pi_0 + 0.2\pi_1 = 0.2\pi_1}$

$\cancel{0.2\pi_0 + 0.4\pi_1 = \pi_1}$

$$0.2\cancel{\pi_0} + 0.4\pi_1 = \pi_1$$
$$-\ 0.2\cancel{\pi_0} + 0.2\pi_1 = 0.2$$

$$0.2\pi_1 = \pi_1 - 0.2$$

$\cancel{0.2 = 1.2\pi_1}$ $\quad 0.2 = 0.8\pi_1$

$\cancel{\pi_1 =}$

So we have unique stationary distributⁿ. for ~~for~~ the Rec MC ← Unique solution $\left\{ \begin{array}{l} \boxed{\pi_1 = 0.25} \\ \boxed{\pi_0 = 0.75} \end{array} \right.$

∴ Stationary distⁿ ⇒ $(\pi_0 = 0.75,\ \pi_1 = 0.25)$

⇓

$$\pi = (0.75, 0.25)$$

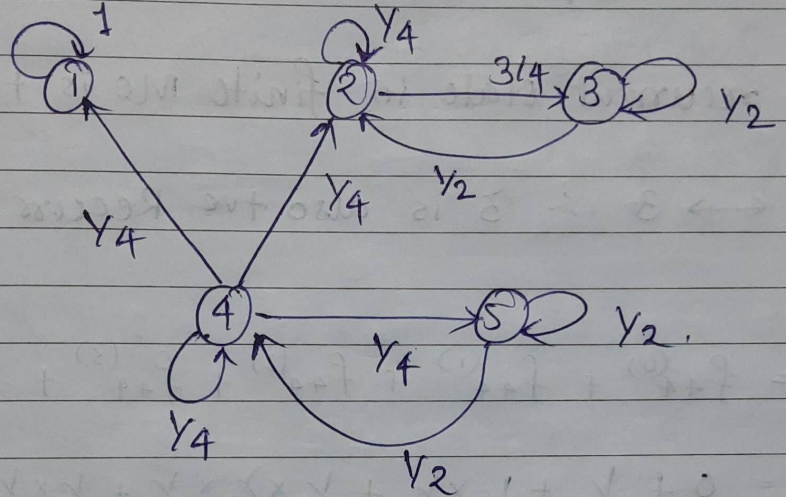
considering a MC which is not irreducible.

Ex.1)

$$P^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{array}.$$

$S = \{1,2,3,4,5\}$

→



(1) $C(1) = \{1\}$
$C(2) = \{2,3\} = C(3)$
$C(4) = \{4,5\} = C(5)$

} Not irreducible MC

(2) Recurrent $\sum_{n=0}^{\infty} f_{jj}^{(n)} = 1$.

$$f_{11} = \sum_{n=0}^{\infty} f_{11}^{(n)} = f_{11}^{(0)} + f_{11}^{(1)} + f_{11}^{(2)} + \cdots$$
$$= 0 + 1 + 0 + 0 + \cdots$$
$$= 1$$

~~Re~~ Recurrent state. as finite mc +ve

$$f_{22} = \sum_{n=0}^{\infty} f_{22}^{(n)} = f_{22}^{(0)} + f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + \cdots$$

$$= 0 + 1/4 + 3/4 \times 1/2 + 3/4 \times 1/2 \times 1/2 +$$
$$3/4 \times 1/2 \times 1/2 \times 1/2 + \cdots$$

$$f_{22} = \cancel{\frac{1}{4}}\frac{1}{4} + \frac{3}{4}\times\frac{1}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\right)$$

$$\Rightarrow \frac{1}{4} + \frac{3}{8} \times \frac{1}{1-\frac{1}{2}}$$

$$\Rightarrow \frac{1}{4} + \frac{3}{\cancel{8}4}\times\cancel{2} = \frac{4}{4} = 1.$$ +ve Recurrent

Every recurrent state in finite MC is +ve Recurr.

$2 \leftrightarrow 3$ ∴ 3 is also +ve Recurr

$$f_{44} = f_{44}^{(0)} + f_{44}^{(1)} + f_{44}^{(2)} + f_{44}^{(3)} + \ldots$$

$$= 0 + \frac{1}{4} + \frac{1}{4}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \ldots$$

$$= \frac{1}{4}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots\right)$$

$$= \frac{1}{4}\times 2 = \frac{1}{2} < 1$$ ∴ State 4 is not recurrent i.e. it is transient

$4 \leftrightarrow 5$ ∴ states 4 & 5 are transient states.

$2 \leftrightarrow 3$ ∴ states 2 & 3 are +ve recurrent

and 1 is +ve recurrent

* But chain is not +ve Recurrent

so stationary distn may or maynot exist & may or maynot be unique

③ limiting distribution

$\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$

$\pi . P^{(1)} = \pi$.

$$(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$$

$\pi_1 + \frac{1}{4}\pi_4 = \pi_1$ —①

$\frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 + \frac{1}{4}\pi_4 = \pi_2$ —②

$\frac{3}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_3$ —③

$\frac{1}{4}\pi_4 + \frac{1}{2}\pi_5 = \pi_4$. —④

$\frac{1}{4}\pi_4 + \frac{1}{2}\pi_5 = \pi_5$ —⑤.

$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$ —⑥ * Include always

⑥ eqns ⑤ ~~variegns~~ variables.

$$\begin{array}{r} \pi_4 + 2\pi_5 = 4\pi_4 \\ - \; \pi_4 + 2\pi_5 = 4\pi_5 \\ \hline \pi_4 = \pi_5 \end{array}$$

$\pi_5 + 2\pi_5 = 4\pi_4$

~~$3\pi_5 =$~~

$\boxed{\pi_4 = 0 = \pi_5}$

Unique $sol^{n}s$

$m = n$

3 $eq^{n}s$

3 variables

$eq^{n}s$: $m < n$: variables

∞ solutions or

No sol^{n}.

$\frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_2$

$\frac{3}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_3$.

$\cancel{\pi_2 + \frac{1}{2}\pi_3 = \frac{1}{4}\pi_2}$

$\pi_4 = \pi_5 = 0$ —— (1)

$\cancel{\pi_3 = \pi_2}$

$\frac{1}{2}\pi_3 = \frac{3}{4}\pi_2$ —— (2)

Both are same.

$\frac{3}{4}\pi_2 = \frac{1}{2}\pi_3$ —— (4).

$\pi_1 + \pi_2 + \pi_3 + \overset{0}{\cancel{\pi_4}} + \overset{0}{\cancel{\pi_5}} = 1$

$\pi_1 + \pi_2 + \pi_3 = 1$ —— (6)

$\frac{3\pi_2}{4} = \frac{\pi_3}{2}$

$\left.\right\}$ 2 $eq^{n}s$, 3 variables

$eq^{n}s$ < variables

$\hookrightarrow \infty$ solutions.

$\pi_4 = 0, \quad \pi_5 = 0$. i.e. infinite solutions

eg $\pi_1 = 1$

$(1, 0, 0, 0, 0)$ is sol^{n} of this sys of eq^{n}

$(1/6, 1/3, 1/2, 0, 0)$ is ——"——

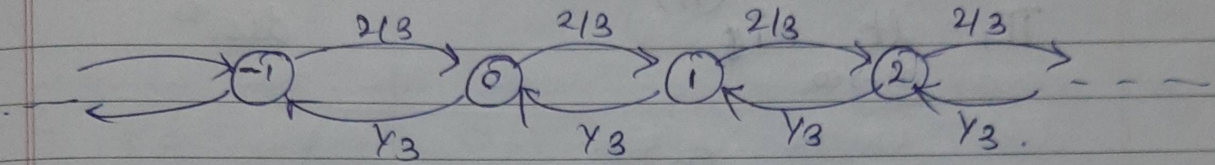
i.e., the stationary distribution ~~are~~ is not unique.

For an infinite MC ⇒

if each state Null Rec — for each state

Result:- If a MC is Null Recurrent ~~and~~ OR transient then there does not exist any Stationary distribution.

eg. $S = \{ ---- -2, -1, 0, 1, 2, ---- \}$.



Transient MC. i.e. more probability that you will go forward & less probability that you will come back

∴ So if you remove any of Irreducibility OR +ve recurrent then

→ Stationary distⁿ may not exist

→ If —"— Exist, its not Unique.

→ Stationary distⁿs exists uniquely.

* Stationary distⁿ for Rainy Sunny (for long run)

$$\lim_{n\to\infty} P_{i0}^{(n)} = 0.75$$

$$\lim_{n\to\infty} P_{i1}^{(n)} = 0.25$$

↑ Limiting distributⁿ $(0.75, 0.25)$

Probability vector $\Pi = (\Pi_0, \Pi_1, \ldots)$ is called limiting dist^n if

limiting

$$\Pi_j = \lim_{n\to\infty} P_{ij}^{(n)} \quad \forall j = 0,1,2,\ldots$$

$\forall i \in S.$

$$\left.\begin{aligned} \Pi_0 &= \lim_{n\to\infty} P_{i0}^{(n)} \\ \Pi_1 &= \operatorname{lt}_{n\to\infty} P_{i1}^{(n)} \\ \Pi_2 &= \operatorname{lt}_{n\to\infty} P_{i2}^{(n)} \end{aligned}\right\} \text{limiting dist}^n$$

LD & SD are in gn diff.

eg

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1"] \end{tikzcd}$$

$$P(1) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

$$(\Pi_0, \Pi_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\Pi_0 \ \Pi_1)$$

$\Pi_1 = \Pi_0$

$\Pi_0 = \Pi_1$

$\Pi_0 + \Pi_1 = 1$

$\Pi_0 = \Pi_1 = 1/2$

Stationary dist^n = $(1/2, 1/2)$

(Π_0, Π_0)

$$P^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^{3} = P^{(2)} P^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P^{4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^{5} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\vdots$$

$$P^{(2m)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$P^{(2m+1)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$$P_{00}^{(n)} = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$P_{11}^{(n)} = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

limiting distⁿ

$\lt_{n\to\infty} P_{i0}^{(n)}$ & $\lt_{n\to} P_{i1}^{(n)}$ doesn't Exist

Even if stationary exist limiting may or maynot exist

* for an Irreducible Ergodic MC, the limiting distⁿ exists uniquely & is equal to stationary distⁿ

Irreducible, +ve Recurrent & aperiodic

$\Downarrow$

limiting distⁿ = Stationary dist

$\pi . P^{(1)} = \pi$

& $\sum_{i=0}^{\infty} \pi_i = 1$, $0 \le \pi_i \le 1$

classmate
Date
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rainy Sunny.

$\pi = (0.75 \quad 0.25)$ Stationary distⁿ.

$$p^{(1)} = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

Irreducible + Aperiodic finite MC (Removed Recurrent condⁿ)

$\Downarrow$

lim distⁿ = Stationary

Sufficient but not necessary.

$\pi \cdot p^{(1)} = \pi$

$\pi_0 + \pi_1 + \pi_2 = 1$

All have

Irreducible → Same periodicity

P_{10}^{2}

$P_{10}^{(2)} \Rightarrow$