Estimation Theory

22nd Sept_M

Non-parametric statistical inference

① Estimating the distributⁿ → directly predicting distributⁿ of random phenomenon.

parametric statistic inference ② Estimating the parameters you know distⁿ form but don't know the parameters. (eg for normal distⁿ μ, σ^2)
eg If distⁿ is Binomial for tossing coin 'n' times then we can estimate the parameter

Predict the parameters based on sample distribution

→ Point Estimation. : You get a single pt
eg. what is mean & variance, It will not give any confidence. Based on sample

→ Interval Estimation. : you get a interval with certain confidence.
eg. there is 99% chance that your True para, will lie in this given interval.
mean(μ) eg $\underbrace{60 \leq \mu \leq 65}_{\text{interval.}}$

① ~~Point Estimation~~ :-

* ~~Estimator~~ :

We know populatⁿ having distⁿ $f(x/\theta)$. & we have to calculate parameters

eg for Normal distⁿ $\theta = (\mu, \sigma^2)$
Binomial $\theta = (n, p)$
Geometric $\theta = (p)$.

& we have a random sample $X_1, X_2, \ldots X_n$ from populatⁿ $f(x/\theta)$. (they are identical & independently distributed Each of them have same distributⁿ)

random sample (If join distⁿ can be written as product of marginal)

(fn of Random Sample.)

* Estimator :- An estimator is a statistics $T(X_1, X_2, ---X_n)$ which is used to estimate the parameter θ.

eg $\theta = (\mu, \sigma^2)$

$T : (\mathbb{R}^n \longrightarrow \mathbb{R}^2)$ (μ, σ^2) fn. i.e. two valued fn

Sample Mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Sample variance $S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

& $S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

for large 'n' both are same.

Max^m Order Statistics $X_{(n)} = \max\{X_1, X_2, -- -X_n\}$ // Max^m Value of Random Sample

$X_{(n)} - X_{(1)}$ // diff betn Max^m & Min^m value

* Estimate :- It is a realized value of estimator

If $X_1 = x_1$, $X_2 = x_2$, ----- $X_n = x_n$

Small $x_1, x_2 --- x_n$ are realized values of estimator

day 1 Temp^r	30°	35°	40
day 2 Temp	40°	39°	39°

$X_1 = y_1$

One of the realized value of estimator is called Estimate

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

Sample variance $\quad S_1^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = 5$ this is our estimate from sample

* For normal distn $N(\mu, \sigma^2)$ → finding Estimator
→ Evaluating Estimator

① Finding Estimator (which estimator to choose)

$X_1 = y_1 \quad X_2 = y_2 \ldots X_n = y_n$

classmate
Date
Page

22/09/2

Point Estimation : ① finding Estimator ② ~~Evaluating Estimator~~

② # Method of Evaluating Estimators :- $T(x_1, x_2 \dots x_n)$

populat? $\rightarrow f(x \mid \theta) \leftarrow$ distribution

Sample $(X_1, X_2, \dots X_n) = \underset{\sim}{X}$ ← vector

There can be any numbes of estimators.

eg $T_1(\underset{\sim}{X}) = \sum_{i=1}^{n} x_i$

$T_2(\underset{\sim}{X}) = \sum_{i=1}^{n} \ln x_i$

$T_3(\underset{\sim}{X}) = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$

then out of many estimators, which estamator is Good? ←

→ If $T(\underset{\sim}{X})$ is very dose to True parameters of θ then thats Best Estimator. Imp

$|T(\underset{\sim}{X}) - \theta|$ if this quantity is Vsmall then we can say $T(\underset{\sim}{X})$ estimator is Good

→ But closeness needs to be established

Measures :- (to decide which estimator is Good)
① Unbiasness
② MSE

You cant compare these two, just use as per you requirement

* Unbiasness :-

An estimator $T(\underset{\sim}{X})$ is said to be unbiased to the parameter θ if :

$$\boxed{E(T(\underset{\sim}{X})) = \theta}$$

Expectⁿ

eg $(X_1, X_2, \ldots X_n) \sim N(\mu, \sigma^2)$

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ & ~~you are estimating~~ you are saying this is your estimate.

$E(\bar{X}) = \mu$ then $\bar{x}$ is unbiased estimator of θ

from $N(\mu, \sigma)$

we know $\bar{X} \sim N(\mu, \sigma^2/\sqrt{n})$

here $\boxed{E(\bar{X}) = \mu}$, on an avg. the $E(\bar{X})$ will take value μ

$\therefore \bar{X}$ is an unbiased estimator of μ.

eg. 1.2, 3.5, 4.5, ------ 0.5

$\bar{x} = \frac{1.2 + 3.5 + 4.5 \ldots + 0.5}{10} = 1.7$

1.7 is mean of the population then

if 1.7 = ~~ECTRW~~ θ then its unbiased estimator.

$(x_1, x_2, \ldots x_n) \sim N(\mu, \sigma^2)$ sample variance

Ex.2) One method is saying S_1^2 can be used as estimator of σ^2
Other ——— " S_2^2 ——— " ——— σ^2

Then S_1^2 is good or S_2^2 ?

$$S_1^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$S_2^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

So calculate whether

$E(S_1^2) = \sigma^2$ or $E(S_2^2) = \sigma^2$

If $Y \sim \chi^2_n$
then $E(Y) = n$
$Var(Y) = 2n$

we know

$$\boxed{\frac{(n-1)S_1^2}{\sigma^2} \sim \chi^2_{n-1}} \quad \text{——— (1)}$$

Now we have a result from Chi-sq. distⁿ theory that
$Y \sim \chi^2_n$ $E(Y) = n$ (If Y is having Chi-sq. distⁿ with n-doF then)
$Var(Y) = 2n$

using this result

$$\therefore E\left(\frac{(n-1)S_1^2}{\sigma^2}\right) = n-1$$

// $E(aT(X)) = aE(X)$
$\Downarrow$
$E(a\chi^2_{(n-1)}) = E(a\chi^2_{(n-1)})$

$$\frac{(\cancel{n-1})}{\sigma^2} E(S_1^2) = (\cancel{n-1})$$

If $E(\chi^2_n) = n$
then $E(\chi^2_{n-1}) = n-1$

$$\therefore E(S_1^2) = \sigma^2$$

$\therefore$ S_1^2 is unbiased estimator of σ^2

classmate

Date

Page

σ^2

σ^2

$$S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 = \frac{(n-1)}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$$

S_1^2

$$\boxed{S_2^2 = \frac{(n-1)}{n}\cdot S_1^2}$$

$$E(S_2^2) = E\ \frac{n-1}{n}E(S_1^2) = \frac{(n-1)}{n}\cdot\sigma^2$$

$\therefore E(S_2^2) \neq \sigma^2$. $\therefore S_2^2$ is not unbiased estimator

$\therefore S_1^2$ satisfies the unbiasness constraint.

(S_2^2 is called asymptotically unbiased)

(2) Mean Squared Error (MSE):- how close our estimator is to the True paramet

MSE of an estimator $T(\underset{\sim}{X})$ for a paramtr θ is defined as

$$MSE(T) = E(\underbrace{T-\theta}_{\text{Error}})^2 \qquad T(\underset{\sim}{X}) \equiv T$$

Take the sqr & then take the mean

$$= E(\underbrace{T-E(T)}_{a} + \underbrace{E(T)-\theta}_{b})^2$$

$$= E(T-E(T))^2 + (E(T)-\theta)^2$$

$a^2+2ab+b^2$

$$\boxed{MSE(T) = var(T) + Bias^2(T)}$$

$$\therefore \boxed{Bias(T) = E(T)-\theta}$$

classmate
Date
Page

Unbias: $E(T) = \theta$

$E(T) - \theta < 0$ Under Estimate or Bias

Bias

$E(T) - \theta > 0$ Over Estimate

write

Let T is unbiased estimator then
$MSE(T) = Var(T)$

If var is close to zero i.e. $MSE \simeq 0$ then that's Good estimator

eg. $(X_1, X_2, \ldots, X_n) \sim N(\mu, \sigma^2)$

$E(T) - \theta = \mu - \mu = 0$

$E(\bar{X}) = \mu$ then Bias $= 0$

$\bar{X} \sim N(\mu, \sigma^2/n)$

$E(\bar{X}) = \mu$

$var(\bar{X}) = \sigma^2/n$

$MSE(\bar{X}) = var(\bar{X}) = \sigma^2/n$

i.e. As you take more samples $(n\uparrow)$ MSE will be minimum.

Compare $MSE(S_1^2)$ vs $MSE(S_2^2)$ to judge the estimator (MSE criteria is used to judge the estimator)

1) $MSE(S_1^2)$

→ S_1^2 is unbiased estimator of σ^2
i.e. $Bias(S_1^2) = 0$

(For unbiased estimator Bias $= 0$ then $MSE(T) = var(T)$)

$\therefore MSE(S_1^2) = Var(S_1^2) + \overbrace{Bias^2(S_1^2)}^{0}$

$S_1^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})$

$= 2(n-1)$

$var\left(\frac{(n-1)S_1^2}{\sigma^2}\right) = 2(n-1)$

$\frac{(n-1)S_1^2}{\sigma^2} \sim \chi^2_{n-1}$

If $Y \sim \chi^2_n$
$E(Y) = n$
$var(Y) = 2n$

$var(ax) = a^2 var(x)$

$$\frac{(n-1)^2}{\sigma^4} var(S_1^2) = 2(n-1)$$

$$\frac{2\cancel{(n-1)} \times \sigma^4}{\cancel{(n-1)}^2 (n-1)}$$

$$\therefore MSE(S_1^2) = Var(S_1^2) = \frac{2\sigma^4}{n-1}$$

2) $MSE(S_2^2)$.

$$\rightarrow \quad var(S_2^2) = var\left(\frac{n-1}{n} S_1^2\right)$$

$$= \left(\frac{n-1}{n}\right)^2 var(S_1^2)$$

$$= \frac{\cancel{(n-1)}^2}{n^2} \times \frac{2\sigma^4}{\cancel{(n-1)}}$$

$$var(S_2^2) = \frac{2\sigma^4(n-1)}{n^2}$$

$$\therefore MSE(S_2^2) = var(S_2^2) + \overbrace{\left[E(S_2^2) - \sigma^2\right]^2}^{Bias^2(T)}$$

$$= \frac{2(n-1)\sigma^4}{n^2} + \left[\frac{(n-1)\sigma^2}{n} - \sigma^2\right]^2$$

$$= \frac{2(n-1)\sigma^4}{n^2} + \frac{((n-1)\sigma^2 - n\sigma^2)^2}{(n)^2}$$

$$= \frac{2(n-1)\sigma^4 + \sigma^4}{n^2}$$

$\cancel{n\sigma^2}(\sigma^2)^2 \cancel{n\sigma^2}$

$(-\sigma^2)^2 = \sigma^4$

$$MSE(S_2^2) = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2}$$

$$= \frac{\sigma^4}{n^2}(2n-2+1)$$

$$MSE(S_2^2) = \frac{(2n-1)\sigma^4}{n^2}$$

$\therefore$ we get

$$MSE(S_1^2) = \frac{2\sigma^4}{n-1}$$

$$MSE(S_2^2) = \frac{(2n-1)\sigma^4}{n^2}$$

$$\therefore MSE(S_1^2) > MSE(S_2^2)$$

ie. Mean Sq^r Error of S_1^2 is strictly greater than S_2^2

So if we use S_1^2 as an estimator of σ^2 then will incur more error. with MSE as measure/criteria

Criteria to judge our Estimator (T)

Unbiasness: S_1^2 is good compared to S_2^2

MSE: S_2^2 is good compared to S_1^2

But they can't be compared

$N(\mu, \sigma^2) \Rightarrow$ μ: location parameter
σ^2: Scale parameter

for

eg.

an

No. of success in qua product doesn't pa passed test

for estimating { locatn para (μ) → Take MSE as criteria
{ Scale para (σ^2) → Unbiasness criterion

CLASSMATE
Page

① Finding Estimators :-

There are diff methods to find estimators → syll → Max^m likelihood est^m

24/09/01

eg. If population has a Binomial distributn with para (n,p)

Assume that company is producing a product & going through a quality test. You have an idea that quality tests could be 2 or 3. Also if P(quality test pass) = $\frac{1}{2}$ or $\frac{1}{3}$

(find out numerical summary of data & will use that to find estimatn of n and p here)

i.e. You dont know exact n,p But you have an rough idea about them.

Now we have to estimate whether n is 2 or 3 and whether p is $\frac{1}{2}$ or $\frac{1}{3}$ } To estimate

* We ~~judge it~~ based on a single observation.
↳ want to estimate (n,p)

No. of successes in quality test $X=x$	(n,p) (2, $\frac{1}{2}$)	(2, $\frac{1}{3}$)	(3, $\frac{1}{2}$)	(3, $\frac{1}{3}$)	Max Prob
product doesnt pass any test → $X=0$	$\frac{1}{4}$	$\boxed{4/9}$	$\frac{1}{8}$	8/27	4/9
passed one test $X=1$	$\boxed{\frac{1}{2}}$	4/9	3/8	12/27	$\frac{1}{2}$
$X=2$	$\frac{1}{4}$	$\frac{1}{9}$	$\boxed{3/8}$	6/27	3/8
$X=3$	0	0	$\boxed{\frac{1}{8}}$	$\frac{1}{27}$	$\frac{1}{8}$

Now based on obs what can you claim about population distribution B(n,p)

$X \sim Bin(n, p)$

$$P(X = x) = {}^nC_x \, p^x (1-p)^{n-x}$$

0 success.

∴ If X=0 then Max Prob is 4/9 i.e. there is Max^m chance that data is coming from $(n, p) = (2, 1/3)$ distribution.

$\hat{n}$ = estimate for n

$\hat{p}$ = estimate for p → (success probability)

$$(\hat{n}, \hat{p})_{MLE} = \begin{cases} (2, 1/3), & \text{if } X = 0 \\ (2, 1/2), & \text{if } X = 1 \\ (3, 1/2), & \text{if } X = 2 \\ (3, 1/2), & \text{if } X = 3 \end{cases}$$

(columns: n, p)

So here MLE its finding the parameter values (n, p) for which the observed sample (X) is 'having' max^m chance of occurrence

n-sample scenario :

$(X_1, X_2, X_3, \ldots X_n) \sim$ population $\longrightarrow P(\lambda)$ poisson dist^n

eg. no of accidents, no of earthquakes, etc are examples of poisson distribution.

$(X_1, X_2, \dots X_n) \rightarrow P(\lambda)$ $\qquad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

observation

$X_1 = x_1, \quad X_2 = x_2, \dots \quad X_n = x_n$

$\therefore$ $0 < \lambda < \infty$, find the value of λ such that there is max^m propability that the observed sample will match value λ.

eg. $\lambda = 10$ then find ~~max chance~~ from which sample it can come with max^m chance.

i.e. Arg $\sup\limits_{\lambda} \{ \underbrace{P(X_1 = x_1, X_2 = x_2, \dots X_n = x_n)} \}$.

finding parameter λ

i.e. find para value λ for which this probability is max^m.

let $L(\lambda / x_1, x_2 \dots x_n) = L(\lambda / \underline{x})$

$= P(X_1 = x_1, X_2 = x_2 \dots X_n = x_n)$

Samples $\rightarrow$ independent & identical $\quad = P(X_1 = x_1) \cdot P(X_2 = x_2) \dots P(X_n = x_n)$

$$= \frac{e^{-\lambda} \cdot \lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \dots$$

$$= \frac{e^{-n\lambda} \cdot \lambda^{x_1 + x_2 + \dots + x_n}}{\prod_{i=1}^{n} x_i!}$$

$$\therefore L(\lambda / \underset{\sim}{x}) = \frac{e^{-n\lambda} \cdot \lambda^{n\bar{x}}}{\prod_{i=1}^{n} x_i!}$$ // likelihood f^n

$\max^m f(\theta)$

$f'(\theta) = 0 \longrightarrow \theta_0,$

$f''(\theta) \big/_{\theta=\theta_0} < 0$ – $\max^m$ at θ_0

The log likelihood f^n:-

$$\ell(\lambda) = \ln L(\lambda / \underset{\sim}{x})$$

$$= -n\lambda + n\bar{x} \ln \lambda - \ln\left(\prod_{i=1}^{n} x_i\right)$$

$$\frac{d\ell(\lambda)}{d\lambda} = 0 \Rightarrow -n + \frac{n\bar{x}}{\lambda} = 0$$

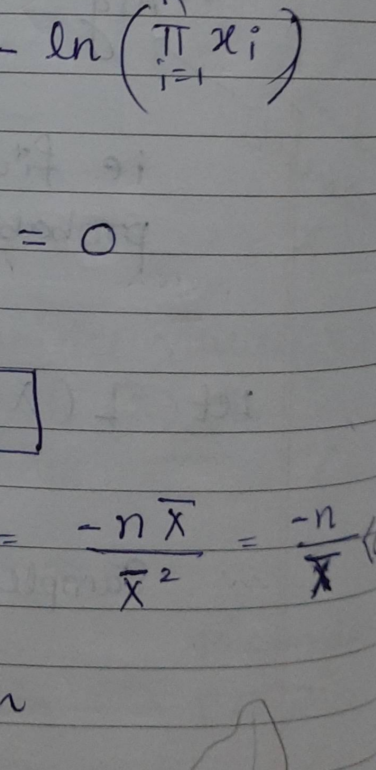
$$\boxed{\lambda = \bar{x}}$$

$$\frac{d^2\ell(\lambda)}{d\lambda^2}\Big/_{\lambda=\bar{x}} = \frac{-n\bar{x}}{\lambda^2}\Big/_{\lambda=\bar{x}} = \frac{-n\bar{x}}{\bar{x}^2} = \frac{-n}{\bar{x}} < 0$$

$\therefore$ $\lambda = \bar{x}$ is local maximum

at $\lambda \to 0$, $L(\lambda / \underset{\sim}{x}) \to 0$

at $\lambda \to \infty$, $L(\lambda / \underset{\sim}{x}) \to 0$



$\lambda = \bar{x}$

$\hat{\lambda} = \bar{x}$ is the pt where f^n $L(\lambda / \underset{\sim}{x})$ is taking the global $\max^m$

$\boxed{\hat{\lambda}_{MLE} = \bar{x}}$ i.e. Sample mean is the pt where f^n is taking max^m.

(check 1st page)

classmate

Date
Page

24_02.

Maximum likelihood of Estimation Method (MLE)

* MLE for $N(\mu, \sigma^2)$ continuous distribution
  1. σ^2 known
  2. μ known
  3. μ, σ^2 unknown

case I : Let σ^2 is known.

likelihood f^n

discrete setup. $L(\theta / x_1, x_2 \ldots x_n) \equiv L(\theta / \underset{\sim}{x}) = P(X_1 = x_1, \ldots X_n = x_n)$

for discrete distribution

continuous setup so PMF will be replaced by Prob Density f^n.

The likelihood f^n is given by. (for continuous f^n)

$$L(\mu / \underset{\sim}{x}) = f_{X_1 X_2 \ldots X_n}(x_1, x_2 \ldots x_n)$$

$\therefore$ Join density can be written as product of marginal densities. iid $X_i \sim X$

& each of them has same distribution

$$= \prod_{i=1}^{n} f_{X_i}(x_i)$$

$f_{X_i} \sim f_X^{(n)}$

$f_X(x) = \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}, \quad -\infty < x_1, x_2 \ldots x_n < \infty$$

density PDF for $N(\mu, \sigma^2)$

$$L(\mu / \underset{\sim}{x}) = \frac{1}{(2\pi)^{n/2} (\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2}.$$

μ : location para

$-\infty < \mu < \infty$

$\therefore$ You find value of μ for which the f^n $L(\mu / \underset{\sim}{x})$ will take maximum value.

Point Estimation:

Maximizat^n of f^n is same as maximizat^n of its log.

The log-likelihood f^n

$$l(\mu) = \ln L(\mu/\underset{\sim}{x}) = \ln\left[\frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}\right]$$

$$= -n/2 \ln 2\pi - n/2 \ln \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Now $\frac{dl(\mu)}{d\mu} = 0$

$$\Rightarrow \frac{-1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\bar{x} = \mu$$

$$\left.\frac{d^2 l(\mu)}{d\mu^2}\right|_{\mu=\bar{x}} = \frac{-n}{\sigma^2} < 0 \quad \text{local max}^m \text{ at } \mu = \bar{x}$$

$-\frac{n}{\sigma^2}$

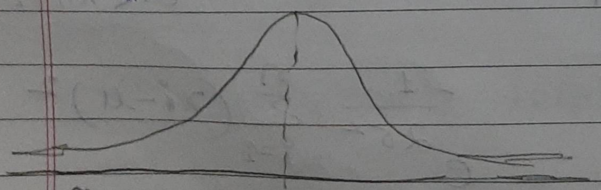
$$\left\{ -\frac{1}{\sigma^2}\frac{d^2}{d\mu^2}\left(x_1-\mu+x_2-\mu+x_3-\mu+\dots+x_n-\mu\right) \right.$$

$$\frac{d^2(\bar{x}-n\mu)}{d\mu^2} = \underline{\underline{-n}}$$

Then $\mu = \bar{x}$ is a point of local maximum

$\therefore$ Check at boundaries $\Rightarrow$ $\mu \to \pm\infty$, then

$$L(\mu/\underset{\sim}{x}) = 0 \quad (\because e^{\pm\infty} = 0)$$



$-\infty$ $\mu = \bar{x}$ $+\infty$

$$\therefore \boxed{\hat{\mu}_{MLE} = \bar{x}}$$

case II : Let μ is known.

ans $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

Now here the likelihood^n fn is given by

$$L(\sigma^2/\underset{\sim}{x}) = \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}.$$

The log - likelihood fn

$$l_n(\sigma^2) = \ln L(\sigma^2/\underset{\sim}{x}) = \ln\left(\quad\right)$$

$$= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Now $\frac{dl(\sigma^2)}{d\sigma^2} = 0$ $\quad \frac{n}{2}\frac{1}{\sigma^2} - \sigma\sum_{i=1}^{n}(x_i-\mu)^2$

$\frac{n}{2\sigma^2} - \sigma\sum_{i=1}^{n}(x_i-\mu)^2$

n $\quad \frac{d^2l(\sigma^2)}{(d\sigma^2)^2} =$

$$\boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Case III: Both μ and σ^2 are unknown.

$$\left(\hat{\mu}, \hat{\sigma}^2\right)_{MLE} = \left(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$$

$$= \left(\bar{X}, S_2^2\right)$$

$$L(\mu, \sigma^2/\underline{x}) = f(x_1)f(x_2)\cdots f(x_n)$$

$$= \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= f(\mu, \sigma^2)$$

For 2-variable fn $f(x, y)$

① $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

Get $x = x^*$ and $y = y^*$

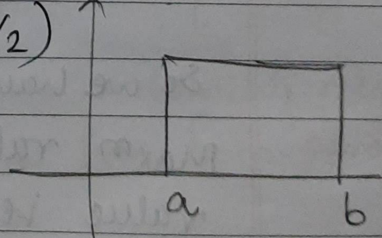
② $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{\substack{x = x^* \\ y = y^*}} < 0$ or $\left.\dfrac{\partial^2 f}{\partial y^2}\right|_{\substack{x = x^* \\ y = y^*}} < 0$

③ $$J = \left.\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}\right|_{\substack{x = x^* \\ y = y^*}} > 0$$

Then x^* and y^* are local maximum

Ex.1) Suppose populn is having uniform distribution. If you are drawing a random sample from it. Estimate para θ.

$(X_1, X_2 \ldots X_n) \sim U(\theta - 1/2, \theta + 1/2)$



→ $L(\theta / \underline{x}) = f\{x_1, x_2 - f_n x_n\}$

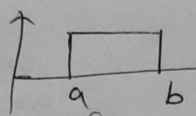
$=$

$f(x) = \dfrac{1}{b-a}$, $a<x<b$

$X \sim$ uniform(a, b)

$f(x) = \dfrac{1}{\theta + \frac{1}{2} - \theta + \frac{1}{2}}$ ←

$f(x) = 1, \quad \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$



$f(x) = \frac{1}{b-a}, \quad a < x < b$

eg. $L(\theta/\underline{x}) = f_{x_1}(x_1) \cdot f_{x_2}(x_2) \ldots f_{x_n}(x_n)$

$= 1 \cdot 1 \ldots 1$

$\theta - \frac{1}{2} < \underbrace{x_1, x_2 \ldots x_n}_{} < \theta + \frac{1}{2}$

when different^n method fails

$\frac{\partial L}{\partial \theta} = 0$ so we cant use this derivative method

we can order this data

$L(\theta/\underline{x}) = 1, \quad \theta - \frac{1}{2} < x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)} < \theta + \frac{1}{2}$

$x_{(1)} = \min\{x_1, x_2 \ldots x_n\}$

$x_{(n)} = \max\{x_1, x_2 \ldots x_n\}$

$= 1, \quad \theta < x_{(1)} + \frac{1}{2}, \quad x_{(n)} - \frac{1}{2} < \theta$

$\therefore$ Range of $\theta \Rightarrow x_{(n)} - \frac{1}{2} < \theta < x_{(1)} + \frac{1}{2}$

So we have to find max of $L(\theta/\underline{x})$ over this range. Max^m value is 1 & every pt is taking same value i.e. Not a unique maximum rather infinite pts where f^n is max^m

$x_{(n)} - \frac{1}{2}$ $\quad$ $x_{(1)} + \frac{1}{2}$

i.e. MLE is not unique, i.e. we dont have unique estimator of θ. Any pt betn $x_{(n)} - \frac{1}{2}$ & $x_{(1)} + \frac{1}{2}$ is MLE point $est^m \to$ single point

classmate

Date
Page

25 - Morning

* Interval Estimation

$\to$ interval along with guarantee (99%, 95% etc)

* Pivot:-

Let $\{X_1, \ldots X_n\}$ Random sample drawn from density fn $\sim f(x/\theta)$ then a random variable $Q(X_1, \ldots X_n, \theta)$ is called the pivot quantity for θ if its distribution is free from parameters.

Examples of pivot quantity

eg. $(X_1, X_2, \ldots X_n) \sim N(\mu, \sigma^2)$

1) $$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \underbrace{N(0,1)}_{\text{free of parameters}}$$

RN Z is fn of Random sample $X_1, X_2 \ldots X_n$ & para μ when σ is known.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

free of parameters (No para involved)

2) $$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$$

This is also free from para so it is example of pivot quantity θ.

3) $$W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

distribution is free from para

pivot quantity for σ^2

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

4) $$W^* = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

pivot for μ when σ known or vice versa normal $\rightarrow$ $\theta = (\mu, \sigma^2)$

Binomial $\rightarrow$ $\theta = (n, p)$

classmate
Date
Page

* Shortest length Confidence Interval :-

Let $\{X_1, X_2, \dots X_n\} \sim f(x|\theta)$ then for a given pivot $Q(\underset{\sim}{X}, \theta)$ suppose that there exists two statistics $L(\underset{\sim}{X})$ & $U(\underset{\sim}{X})$ such that

$\underset{\sim}{X} = (X_1, X_2, \dots X_n)$

$$P(L(\underset{\sim}{X}) \leq \theta \leq U(\underset{\sim}{X})) = 1-\alpha$$

Suppose that $\underset{\sim}{X} = \underset{\sim}{x}$ is observed then the interval $(L(\underset{\sim}{x}), U(\underset{\sim}{x}))$ is called

$\underset{\sim}{X} = (X_1, X_2, \dots X_n)$

$100(1-\alpha)\%$ two sided confidence interval for θ, based on pivot quantity $Q(\underset{\sim}{X}, \theta)$ provided the length of this interval is shorter among all other admissible intervals.

$\underset{\sim}{x} = (x_1, x_2, \dots x_n)$

1) find pivot quantity $Q(\underset{\sim}{X}, \theta)$

2) find $L(\underset{\sim}{X})$ & $U(\underset{\sim}{X})$ s.t. $P(L(\underset{\sim}{X}) \leq \theta \leq U(\underset{\sim}{X})) = 1-\alpha$

then $(L(\underset{\sim}{x}), U(\underset{\sim}{x}))$ is called $100(1-\alpha)\%$ two sided confid interval

$\alpha = 0.05$ i.e. 95% chance that θ is lying betn $L(\underset{\sim}{x})$ & $U(\underset{\sim}{x})$

* Confidence interval : $N(\mu, \sigma^2)$

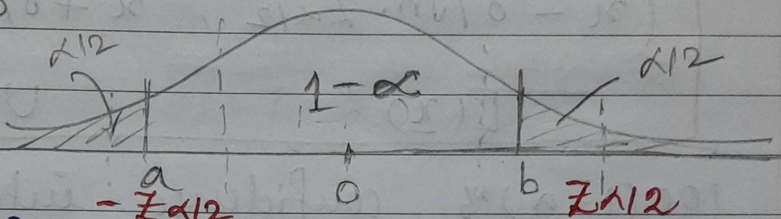
Case I :- σ^2 is known. $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ pivot

Target: Find two numbers a and b such that

$P(a < Z < b) = 1-\alpha$ and $(b-a)$ is Shortest

Z is Std ND

Two sided confidence interval



choose a & b such that Area under the curve is (1-α).

choose two pts a & b s.t. (b-a) is shortest. ND is symm about zero, if we take equal lengths on both sides of 0 to cover (1-α) area then that will be shortest length

Therefore if we choose $a = -Z_{\alpha/2}$ & $b = Z_{\alpha/2}$ then $(b-a)$ length will be shortest & AOC $= (1-\alpha)$

$\therefore \; a = -Z_{\alpha/2}, \; b = Z_{\alpha/2}$

$P(-Z_{\alpha/2} < Z < Z_{\alpha/2}) = 1-\alpha$

$P\left(-Z_{\alpha/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right) = 1-\alpha$

$P\left(-Z_{\alpha/2} \cdot \sigma/\sqrt{n} < \bar{X}-\mu < Z_{\alpha/2} \cdot \sigma/\sqrt{n}\right) = 1-\alpha$

$$P\left(Z_{\alpha/2} \cdot \sigma/\sqrt{n} > -\overline{X} + \mu > -Z_{\alpha/2} \cdot \sigma/\sqrt{n} \right) = 1-\alpha$$

$$P\left(\underbrace{\overline{X} + Z_{\alpha/2} \cdot \sigma/\sqrt{n}}_{U} > \mu > \underbrace{\overline{X} - Z_{\alpha/2} \cdot \sigma/\sqrt{n}}_{L} \right) = 1-\alpha$$

If $X_1 = x_1 \ldots X_n = x_n$ then

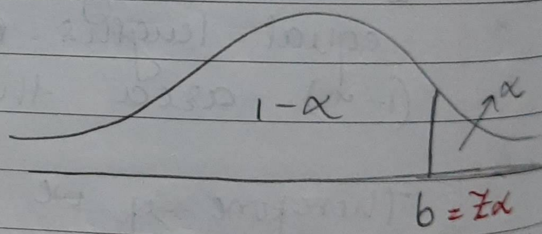
$$\boxed{\left(\underbrace{\overline{x} - \sigma/\sqrt{n} \cdot Z_{\alpha/2}}_{L(\underline{x})} ,\ \underbrace{\overline{x} + \sigma/\sqrt{n} \cdot Z_{\alpha/2}}_{U(\underline{x})} \right)}$$ is called

$100(1-\alpha)\%$ confidence interval for μ based on pivot, $Z = \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$

(2)

* **One Sided Interval** $\Rightarrow$

① find a number b s.t.



$P(-\infty < Z < b) = 1-\alpha$

Upper confidence interval

$\Rightarrow P(-\infty < Z < Z_\alpha) = 1-\alpha$

$\Rightarrow P\left(-\infty < \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} < Z_\alpha\right) = 1-\alpha$

$\Rightarrow P\left(-\infty < \overline{X} - \mu < Z_\alpha \cdot \sigma/\sqrt{n}\right) = 1-\alpha$

$P\left(\infty > -\overline{X} + \mu > -Z_\alpha \cdot \sigma/\sqrt{n}\right) = 1-\alpha$

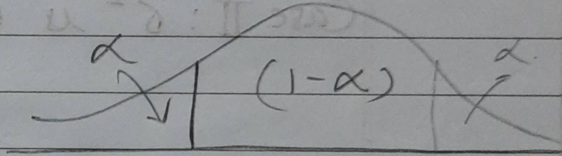
$P\left(\infty > \mu > \overline{X} - Z_\alpha \cdot \sigma/\sqrt{n}\right) = 1-\alpha$ *

$$\therefore \boxed{P\left(\bar{X} - Z_\alpha \cdot \sigma/\sqrt{n} < \mu < \infty\right) = 1-\alpha}$$

If $\underset{\sim}{X} = \underset{\sim}{x}$ is observed then

$\left(\bar{x} - Z_\alpha \cdot \sigma/\sqrt{n}, \infty\right)$ is called $100(1-\alpha)\%$ upper confidence interval for μ, based on the pivot $Z = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

(2) find a number 'a' s.t



$P(a < Z < \infty) = 1-\alpha.$

$a = Z_{1-\alpha}$ Z_α

$-Z_\alpha$

$P\left(a < \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \infty\right) = 1-\alpha.$

Lower confidence interval

$P\left(-Z_\alpha \cdot \sigma/\sqrt{n} < \bar{X}-\mu < \infty\right) = 1-\alpha$

$P\left(Z_\alpha \cdot \sigma/\sqrt{n} > \mu-\bar{X} > -\infty\right) = 1-\alpha$

$P\left(\bar{X} + Z_\alpha \cdot \sigma/\sqrt{n} > \mu > -\infty\right) = 1-\alpha.$

$$\boxed{P\left(-\infty < \mu < \bar{X} + Z_\alpha \cdot \sigma/\sqrt{n}\right) = 1-\alpha}$$

If $\underset{\sim}{x} = \underset{\sim}{x}$ is obs.

$\left(-\infty, \bar{x} + Z_\alpha \cdot \sigma/\sqrt{n}\right)$ is called $100(1-\alpha)\%$ lower confidence interval for μ, where $Z = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

$P(-\infty < \mu < u)$

classmate

Date

Page

② one-sided interval

lower confidence interval $\Rightarrow \left(-\infty,\ \bar{x} + s/\sqrt{n} \cdot t_{\alpha, n-1}\right)$

Upper confidence interval $\Rightarrow \left(\bar{x} - s/\sqrt{n} \cdot t_{\alpha, n-1},\ \infty\right)$

problem sheet ⑪ $n = 20$, avg val of nicotin content (mean) $\bar{x} = 1.2\,mg$

a) compute 99% two-sided confidence interval for mean nicotin cont of cig. if std deviatⁿ $\sigma = 0.2\,mg$ $(N(\mu, \sigma^2))$

$\Rightarrow$ $n = 20$, $\bar{x} = 1.2$, $\sigma^2 = (0.2)^2 = 0.04$

$\therefore$ σ^2 is known,

~~Two sided 99% confidence interval~~ Two-sided 99% confidence interval for μ based on

$1-\alpha$ $\quad Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ is given by

$$\left(\bar{x} - Z_{\alpha/2} \cdot \sigma/\sqrt{n},\ \bar{x} + Z_{\alpha/2} \cdot \sigma/\sqrt{n}\right)$$

$\Rightarrow$ $100(1-\alpha)\% = 99\%$

if $(1-\alpha) = 0.99$ then we get 99% confi int

i.e. $\alpha = 1 - 0.99$

$\boxed{\alpha = 0.01}$

search for this in table & then

$Z_{0.005} = \Phi(0.995) = 2.58$ ← from $(x+a)$

$(\bar{x} - Z_{\alpha/2} \cdot \sigma/\sqrt{n},\ \bar{x} + Z_{\alpha/2} \cdot \sigma/\sqrt{n})$

$\left(1.2 - 2.58 \times \frac{0.2}{\sqrt{20}},\ 1.2 + 2.58 \times \frac{0.2}{\sqrt{20}}\right)$

$\Rightarrow (1.084, 1.315)$

i.e. there is 99% chance that mean nicotine content of cigarettes lies betn 1.084 to 1.315.

b) compute value 'c' which can assert w "with 99% confidence" that c is larger than the mean notine content of a cigarette

μ is smaller than c

$P(\mu < c)$

ie find value c such that its value is ~~less~~ larger than mean nicotine content (μ) with 99% conf

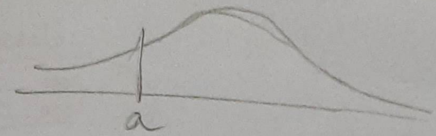
$P(\mu < c) = 0.99$

$\Rightarrow P(-\infty < \mu < c) = 0.99$

$\Rightarrow (-\infty, c)$ with 99% (lower confidence interval)

Ex.2)

a)



$(-\infty, \bar{x} + z_\alpha \sigma/\sqrt{n})$

$\therefore$ $c = \bar{x} + z_\alpha \cdot \sigma/\sqrt{n}$.

For $z_\alpha = z_{0.01} = \Phi(0.99) = 2.33$

$$c = 1.2 + 2.33 \times \frac{0.2}{\sqrt{20}} = 1.304.$$

i.e. there is 99% chance that mean nicotine content in a cigerette will be atmost 1.304.

Ex.2) If population variance(σ^2) is not known (σ^2)
If sample variance $s^2 = 0.04$ then what will be two sided confidence interval.

→ a) Then 99% two-sided confidence int for μ based on pivot quantity $T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ is

given by ~~$(\bar{x} -$~~ $(\bar{x} - S/\sqrt{n} \cdot t_{\alpha/2, n-1}, \bar{x} + S/\sqrt{n} \cdot t_{\alpha/2, n-1})$

$t_{\alpha/2, n-1} = t_{0.005, 19} = 2.861$

$s^2 = 0.04$ $\therefore$ ~~s = 0.02~~ $\Rightarrow$ $s = 0.2$

$$\left(1.2 - \frac{0.2}{\sqrt{20}} \times 2.861,\ 1.2 + 2.861 \times \frac{0.2}{\sqrt{20}}\right)$$

$\Rightarrow$ $(1.072, 1.327)$

99% Two sided confidence interval

ⓐ for known popl^n variance (σ^2) $\Rightarrow$ $(1.084, 1.315)$

ⓑ for unknown popl^n var (σ^2) $\Rightarrow$ $(1.072, 1.327)$

So the interval with known variance is slighty shorter than the interval with unknown variance.
This is cz extra info ie. σ^2 was provided. so better approximation

b) value of 'c' for when σ^2 is unknown

$P(\mu < C)$

$P(-\infty < \mu < C) = 0.99$

here $c = \bar{x} + t_{\cancel{\alpha/2}, n-1} \cdot s/\sqrt{n}$

$= 1.2 + \cancel{2.861} \; 2.539 \times \frac{0.2}{\sqrt{20}}$ $\quad$ ~~$t_{0.005,19}=$~~ $t_{\alpha, n-1} = t_{0.01,19} = 2.539$

$= \cancel{1.327}$

$C = 1.313$

i.e.

there is 99% chance ~~chance~~ that mean nicotine cont of cig will be atmost 1.313

Ex.15) Capacities of 10 batteries
140, 136, 150, 144, 148, 152, 138, 141, 143, 151

a) Estimate populn var σ^2. - point estimation

→ $n = 10$ $N(\mu, \sigma^2)$ 1000

$\bar{X} = \frac{1443}{10} = \cancel{14.43} = 144.3$

Unknown variance σ^2, when μ is Unknown

a) case III, Both μ & σ^2 unknown

$$\hat{\sigma}^2_{MLE} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

~~X = 14.43~~

$$\boxed{\bar{X} = 144.3}$$

$$\hat{\sigma}^2_{MLE} = \frac{1}{10} \Big(4.3^2 + 8.3^2 + \overset{(0.3)^2}{5.7^2} + 3.7^2 + 7.7^2 + (-6.3)^2 + (-3.3)^2 + (-1.3)^2 + (6.7)^2 \Big) +$$

$$\boxed{\hat{\sigma}^2_{MLE} = 29.01}$$

b) Compute 99% two-sided confidence interval for σ^2.

Estimate σ^2 when μ is unknown

Two sided 99% confidence interval for σ^2 when μ unknown

~~then pivot~~, is given by

based on ↗ $W = \frac{(n-1) s^2}{\sigma^2} \sim \chi^2_{n-1}$

is given by

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,\,n-1}},\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\,n-1}}\right),\quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$1-\alpha = 0.99$

$\boxed{\alpha = 0.01}$

$\chi^2_{0.005,9} = 23.589$

$$s^2 = \frac{1}{9}\sum_{i=1}^{n}(x_i - 144.3)^2 = 29.04$$

$\boxed{s^2 = 32.23}$

$\chi^2_{0.995,\,9} = 1.735$
$(1-\alpha/2,\ n-1)$

$\chi^2_{0.005,\,9} = 23.589$
$(\alpha/2,\ n-1)$

s^2 only Aarti !!

$$\left(\frac{9\times(32.23)}{23.589},\ \frac{9\times(32.23)}{1.735}\right)$$

$= (~~896.8216~~,\ ~~5421.936~~)$

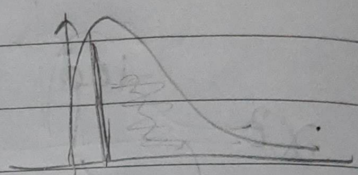
$= (12.29,\ 167.18)$

dis m is from pa

c) Compute value v that enables us to state, with 90% confidence, σ^2 is less than v.

→ $P(0 < \sigma^2 < v) = 0.9$

lower confidence interv



a

$\chi^2_{1-\alpha, n-1}$

$(0, v)$

$$v = \frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}}$$

$$= \frac{9 \times 32.23}{3.325}$$

$\cancel{\chi^2_{1-0.9, 9} = \chi^2_{0.1, 9}} =$

$(1-\alpha) = 0.9$

$\alpha = 0.1$

$\chi^2_{1-\alpha, n-1} = \chi^2_{0.9, 9} = \cancel{87.239}\ 3.325$

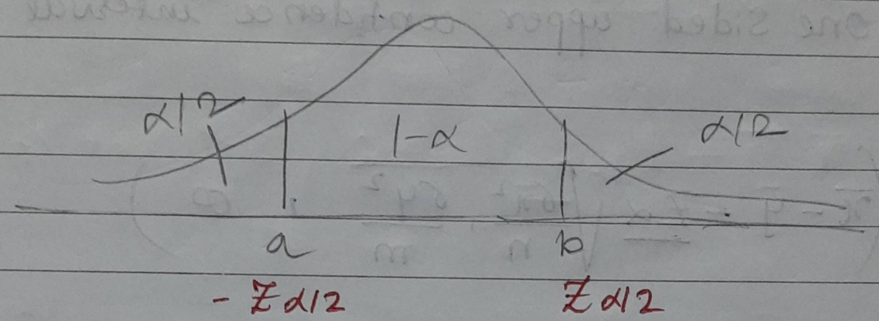
$v = 87.239$

diff betn means of popln

Estimate $(\mu_x - \mu_y)$ when

Case I: σ_x^2 and σ_y^2 are known

pivot, $$Z = \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{n}}} \sim N(0,1)$$



$P(a < Z < b) = 1 - \alpha$ & length $(b-a)$ is shortest

$$P(-Z_{\alpha/2} < Z < Z_{\alpha/2}) = 1 - \alpha.$$

$$P\left(-Z_{\alpha/2} < \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} < Z_{\alpha/2}\right) = 1 - \alpha.$$

$$P\left(\bar{X} - \bar{Y} - Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}} < \mu_x - \mu_y < \bar{X} - \bar{Y} + Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}\right) = 1 - \alpha.$$

If $\underset{\sim}{X} = x$ and $\underset{\sim}{Y} = y$ then $100(1-\alpha)\%$

Confidence interval for $\mu_x - \mu_y$ based on Z is given by

* Case I $\Rightarrow$ σ_x^2 & σ_y^2 are known & unequal.

1) Two sided confidence interval

$$\left(\bar{x}-\bar{y}-Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}+\frac{\sigma_y^2}{m}}\ ,\ \bar{x}-\bar{y}+Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}+\frac{\sigma_y^2}{m}}\right)$$

2) One sided upper confidence interval for $\mu_x-\mu_y$

$$\left(\bar{x}-\bar{y}-Z_{\alpha}\sqrt{\frac{\sigma_x^2}{n},\frac{\sigma_y^2}{m}}\ ,\ \infty\right)$$

3) One sided lower confidence interval for $\mu_x-\mu_y$

$$\left(-\infty,\ \bar{x}-\bar{y}+Z_{\alpha}\cdot\sqrt{\frac{\sigma_x^2}{n}+\frac{\sigma_y^2}{m}}\right)$$

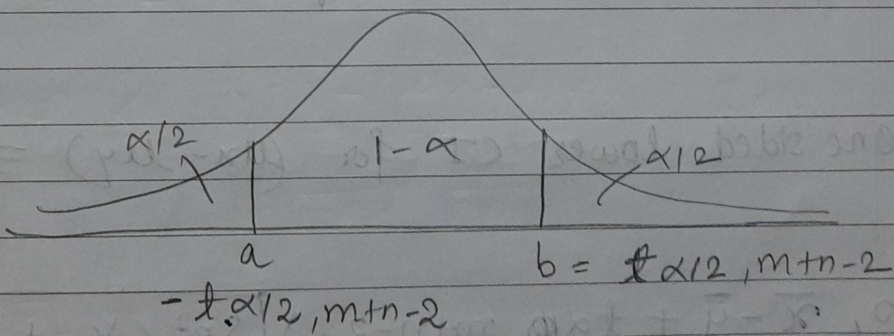
Case II :- σ_x^2 & σ_y^2 are unknown and equal

pivot, $T = \dfrac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{S_p^2(1/m + 1/n)}} \sim t_{m+n-2}$.

$S_p^2 = \dfrac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}$, $\quad S_x^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$S_y^2 = \dfrac{1}{m-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

find two no's a & b s.t. the interval length (b-a) is shortest & covers (1-α)



$P(a < T < b) = (1-\alpha)$

$P\left(-t_{\alpha/2, m+n-2} < \dfrac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{S_p^2(1/m + 1/n)}} < t_{\alpha/2, m+n-2}\right)$

$P\left(\bar{X} - \bar{Y} - t_{\alpha/2, m+n-2} \cdot \sqrt{S_p^2(1/m + 1/n)} < \mu_x - \mu_y < \bar{X} - \bar{Y} + \dfrac{t_{\alpha/2, m+n-2}}{\sqrt{S_p^2(1/m + 1/n)}}\right)$

Point estimation:

Case II: σ_x^2 & σ_y^2 are unknown & equal

$\underset{\sim}{X} = \underset{\sim}{x}$ and $\underset{\sim}{Y} = \underset{\sim}{y}$ then $100(1-\alpha)\%$ CI for $(\mu_x - \mu_y)$ based on pivot quantity T is given by

① Two sided CI ⇒

$$\left(\bar{x} - \bar{y} - t_{\alpha/2, m+n-2} \cdot \sqrt{S_p^2 \left(\tfrac{1}{m} + \tfrac{1}{n}\right)}, \right.$$

$\alpha/2$

$$\left. \bar{x} - \bar{y} + t_{\alpha/2, m+n-2} \sqrt{S_p^2 \left(\tfrac{1}{m} + \tfrac{1}{n}\right)} \right)$$

② One sided upper CI for $(\mu_x - \mu_y)$ ⇒

(α)

$$\left(\bar{x} - \bar{y} - t_{\alpha, m+n-2} \cdot \sqrt{S_p^2 \left(\tfrac{1}{m} + \tfrac{1}{n}\right)}, \infty \right)$$

③ One sided lower CI for $(\mu_x - \mu_y)$ ⇒

(α)

$$\left(-\infty, \bar{x} - \bar{y} + t_{\alpha, m+n-2} \cdot \sqrt{S_p^2 \left(\tfrac{1}{m} + \tfrac{1}{n}\right)} \right)$$

Case III :- σ_x^2 & σ_y^2 are unknown & unequal.

pivot $T^* = \dfrac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\dfrac{S_x^2}{n} + \dfrac{S_y^2}{m}}} \approx t(\nu)$

— doesn't have exact distributn so appx by $t(\nu)$

t-distn with 'ν' dOF

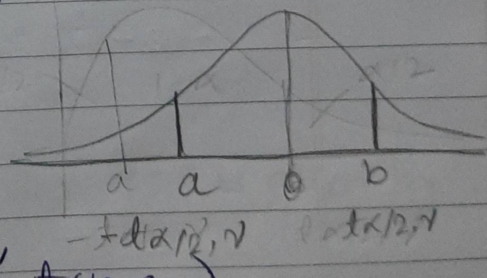
$$\nu = \left[\frac{1}{n-1} \left(\frac{R}{R+1} \right)^2 + \frac{1}{m-1} \left(\frac{1}{R+1} \right)^2 \right]^{-1}$$

$$R = \frac{S_x^2 / n}{S_y^2 / m}. \qquad S_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$S_y = \frac{1}{m} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

If $\nu \geq 30$ then T^* can be approximated by std ND.

Then $100(1-\alpha)\%$ CI for $(\mu_x - \mu_y)$ is given based on T is given by



$P(-a < T < b) = 1 - \alpha$

$$P\left(-t_{\alpha/2,\nu} < \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{S_x^2/n + S_y^2/m}} < t_{\alpha/2,\nu} \right) = 1 - \alpha$$

$$P\left(\bar{X} - \bar{Y} - t_{\alpha/2,\nu} \cdot \sqrt{\frac{S_x^2}{n}, \frac{S_y^2}{m}} < \mu_x - \mu_y < \bar{X} - \bar{Y} + t_{\alpha/2,\nu} \sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}} \right.$$

Case III ⇒ σ_x^2 & σ_y^2 are unknown & unequal

If $X = x$ & $Y = y$ then two sided CI

(1) Two sided CI ⇒

$$\left(\bar{x}-\bar{y}-t_{\alpha/2,\nu}\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}},\ \bar{x}-\bar{y}+t_{\alpha/2,\nu}\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}\right)$$

$(\alpha/2)$

(2) One sided Upper CI for $(\mu_x-\mu_y)$ ⇒

(α)

$$\left(\bar{x}-\bar{y}-t_{\alpha,\nu}\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}},\ \infty\right)$$

(3) One sided lower CI for $(\mu_x-\mu_y)$ ⇒

$$\left(-\infty,\ \bar{x}-\bar{y}+t_{\alpha,\nu}\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}\right)$$

Ex. 18) Two machines M I and M II

	M1	M2
	$n = 36$	$m = 64$
sample mean,	$\bar{x} = 120$	$\bar{y} = 130$
sample var,	$S_x^2 = 4$	$S_y^2 = 5$

X = weight of M1 $\sim N(\mu_1, \sigma_1^2)$ $\quad$ Y = wt of M2 $\sim N(\mu_2, \sigma_2^2)$

a) find 99% confidence interval for $\mu_1 - \mu_2$ when $\sigma_1 = \sigma_2$.

→ Estimate $(\mu_1 - \mu_2)$ when variances are unknown and equal.

pivot, $T^* = \dfrac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_p^2 (1/n + 1/m)}} \sim t_{m+n-1}$

$\therefore (1-\alpha) = 0.99$

$\alpha = 0.01$

$\therefore$ $100(1-\alpha)\%$ CI for $(\mu_1 - \mu_2)$ for T^* is given by

$$\left(\bar{x} - \bar{y} - t_{\alpha/2, m+n-2} \cdot \sqrt{S_p^2 (1/n + 1/m)},\ \bar{x} - \bar{y} + t_{\alpha/2, m+n-2} \cdot \sqrt{S_p^2 (1/n + 1/m)}\right)$$

$t_{\alpha/2, m+n-2} = t_{0.005,\ 36+64-2}$

$= t_{0.005, 98}$

$= 2.576$

$$S_p^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}$$

$$S_p^2 = \frac{35 \times 4 + 63 \times 5}{98}$$

$S_p^2 = 4.6428$

$\therefore \left(\bar{x} - \bar{y} - t_{\alpha/2, m+n-2}\sqrt{S_p^2(1/m + 1/n)},\ \bar{x} - \bar{y} + t_{\alpha/2, m+n-2}\sqrt{S_p^2(1/m + 1/n)}\right)$

$\bar{x} - \bar{y} = -10$

$$\left(-10 - 2.576 \times \underbrace{\sqrt{4.643 \times \left(\frac{1}{36} + \frac{1}{64}\right)}}_{0.4489},\ -10 + 2.576 \times 0.4489\right)$$

$\Rightarrow (-10 - 1.15,\ -10 + 1.15) \Rightarrow (-11.15,\ -8.84)$

pop^n var $= \sigma^2$ sample var $= S^2$

SD $= \sigma$ sample SD $= S$

classmate Date ____ Page ____

b) Find 99% confidence int for $\mu_1 - \mu_2$ where it is known in advance that populat^n variances are 4 & 5 resp.

→ $\sigma_1^2 = 4$, $\sigma_2^2 = 5$

Estimate $(\mu_1 - \mu_2)$ ~~for~~ when variances are known

case I.

$$Z = \frac{\bar{x} - \bar{y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1).$$

$$\left(\bar{x} - \bar{y} - Z_{\alpha/2} \cdot \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}, \ \bar{x} - \bar{y} + Z_{\alpha/2} \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}\right) = (1-\alpha)$$

$1 - \alpha = 0.99 \Rightarrow \boxed{\alpha = 0.01}$

$Z_{\alpha/2} = Z_{0.005} = \Phi(0.995) = 2.58$

$$\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}} = \sqrt{\frac{4}{36} + \frac{5}{64}} = 0.4350$$

$\bar{x} - \bar{y} = 120 - 130 = -10.$

$(-10 - 2.58 \times 0.4350, \ -10 + 2.58 \times 0.4350)$

$\Rightarrow$ $(-11.122, -8.877)$

Q. find 99% CI for $\mu_1-\mu_2$ when there is no info provided for population variances

→ Estimate $\mu_1-\mu_2$ when variances unknown and unequal

$$T^* = \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}} \sim t(\nu)$$

$100(1-\alpha)\%$ CI is given by

$$\left(\bar{x}-\bar{y}-t_{\alpha/2,\nu}\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}},\ \bar{x}-\bar{y}+t_{\alpha/2,\nu}\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}\right)$$

$$\nu = \left[\frac{1}{n-1}\left(\frac{R}{R+1}\right)^2+\frac{1}{m-1}\left(\frac{1}{1+R}\right)^2\right]^{-1}$$

$$R = \frac{S_x^2/n}{S_y^2/m} = \frac{4/36}{5/64} = 1.422$$

$$\nu = \left[\frac{1}{35}\times\left(\frac{1.422}{2.422}\right)^2+\frac{1}{63}\left(\frac{1}{2.422}\right)^2\right]^{-1} =$$

$= 0.009848+$

$\boxed{\nu = 79.65}$

as $\nu \geq 30$ so t-distn can be approximated by std ND $N(\mu,\sigma^2)$

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1)$$

$$\left(\bar{x} - \bar{y} - Z_{\alpha/2} \cdot \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}},\ \bar{x} - \bar{y} + Z_{\alpha/2} \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}} \right)$$

$\alpha = 0.01$

$Z_{\alpha/2} = Z_{0.005} = \Phi(0.995) = 2.58$

$(-10 - 2.58 \times 0.435,\ -10 + 2.58 \times 0.435)$

$\Rightarrow (-11.122,\ -8.877)$ ✓

Book. Ex. 44) Burning times in seconds of floating smoke pots of two diff types

Type I		Type II	
481	572	526	537
506	561	571	582
527	501	556	605
661	487	542	558
501	524	491	578

find a 99% CI for the mean diff in burning times assuming normality with unknown but equal variances.

→

(1) σ_x^2 & σ_y^2 are unknown but equal.

$$T = \frac{\bar{X}-\bar{Y}-(\mu_x-\mu_y)}{\sqrt{S_p^2\left(\frac{1}{n}+\frac{1}{m}\right)}} \sim t_{m+n-2}$$

$$S_p^2 = \frac{(n-1)S_x^2+(m-1)S_y^2}{n+m-2}$$

μ_1 - mean burning time of type I pot

μ_2 - " type II pot

Then the CI interval is given by

$$\left(\bar{x}-\bar{y}-t_{\alpha/2,\,n+m-2}\cdot\sqrt{S_p^2\left(\frac{1}{n}+\frac{1}{m}\right)},\ \bar{x}-\bar{y}+t_{\alpha/2,\,m+n-2}\sqrt{S_p^2\left(\frac{1}{n}+\frac{1}{m}\right)}\right)$$

$\overline{X} = 532.1$ $\overline{Y} = 548.6$

$S_p^2 = \sim 9 \times$

$$S_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{1}{9} ($$

$$= \cancel{2639.49} \sim 2932.76.$$

$$S_y^2 = \frac{1}{9} \sum_{i=1} (y_i - \bar{y})^2$$

$$= \frac{1}{9} \times 10724.4$$

$$= 1191.6.$$