Markit Analytics Model Framework Mathematical Reference

Product Version: **1.18**

Document Revision: **A**

Date: **March 2017**

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Revision History

Product Version	**Document Revision**	**Key Changes**
1.18	A	Added Multi-Factor Hull-White Interest Rate Model with Real-World Drift
1.17	A	Hybrid Commodity Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates: Added the description of the Monte Carlo simulation in the real-world measure. Added description of historical calibration.
1.16	A	Multi-Factor Hull-White Interest Rate Model: Added description of historical calibration.
1.15	A	New document

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Chapter 1

Overview

This document describes the implementation of the models used in Markit Analytics Model Framework (MFWK). The models listed in the following table are included.

Model Name	**Authors**	**First Version Date**	**Current Version Date**
Credit Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Hull-White Models for Hazard Rates (CR FX IR HW NF). See page 69.	Anastasia Kolodko	June 10, 2016	June 10, 2016
Equity Hybrid Single Factor Hull-White Model (EQ FX IR HW 1F). See page 41.	Dave Peterson, Leslie Ng, Stefano Renzitti	November 20, 2007	November 17, 2015
Hybrid Commodity Credit Equity Inflation Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Black-Karasinski Models for Hazard Rates (CM CR EQ FX IF IR HW NF BK 1F). See page 99.	Anastasia Kolodko	June 10, 2016	June 10, 2016
Hybrid Commodity Credit Equity Inflation Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Hull-White Models for Hazard Rates (CM CR EQ FX IF IR HW NF). See page 103.	Anastasia Kolodko	June 10, 2016	June 10, 2016
Hybrid Commodity Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates (CM FX IR HW NF). See page 79.	Allan Cowan and Anastasia Kolodko	June 10, 2016	December 21, 2016

Model Name	Authors	First Version Date	Current Version Date
Hybrid Credit Equity Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Black-Karasinski Models for Hazard Rates (CR EQ FX IR HW NF BK 1F). See page 85.	Leslie Ng	August 14, 2013	November 20, 2013
Hybrid Equity Inflation Multi-Currency Multi-Factor Hull-White Interest Rate Model (EQ FX IF IR HW NF). See page 61.	Stefano Renzitti	August 25, 2011	December 11, 2013
Hybrid Equity Multi-Currency Multi-Factor Hull-White Interest Rate Model (EQ FX IR HW NF). See page 51.	Leslie Ng	September 29, 2010	November 17, 2015
Multi-Currency Single Factor Hull-White Model (FX IR HW 1F). See page 33.	Dave Peterson, Leslie Ng, Stefano Renzitti	November 20, 2007	November 20, 2007
Multi-Factor Hull-White Interest Rate Model (HW NF). See page 19.	Leslie Ng	June 12, 2009	August 26, 2016
Multi-Factor Hull-White Interest Rate Model with Real-World Drift (HW RW NF). See page 29.	Allan Cowan, Anastasia Kolodko	March 1, 2017	March 1, 2017
Single Factor Hull-White Model (HW 1F). See page 11.	Dave Peterson, Leslie Ng, Stefano Renzitti	November 20, 2007	July 9, 2012
Single Factor Hull-White Model (HW 1F). See page 11.	Dave Peterson, Leslie Ng, Stefano Renzitti	November 20, 2007	July 9, 2012
Wrong Way Risk Model with Deterministic Hazard Rates and FX Jump at Name Default (CR EQ FX IF IR HW NF JFX). See page 91.	Anastasia Kolodko	June 9, 2014	June 9, 2014

Chapter 2

Single Factor Hull-White Model

This chapter describes the implementation of a single factor Hull-White (HW 1F) interest rate model.

2.1 Introduction

The single factor Hull-White (HW 1F) model is an interest rate model where the dynamics of the short rate $r(t)$ is governed by the following equations:

$$dx(t) = -a(t)x(t)dt + \sigma(t)dW, \quad x(0) = 0 \tag{2.1}$$

$$r(t) = x(t) + \phi(t) \tag{2.2}$$

where, W is a Wiener process, $a(t)$ is a mean reversion parameter, $\sigma(t)$ is the short rate volatility and $\phi(t)$ is a function chosen so that the initial discount factor curve is reproduced.

Note that the specification we use for the short rate SDE differs slightly from the one originally proposed by Hull and White:

$$dr(t) = (\theta(t) - a(t)r(t))dt + \sigma(t)dW \tag{2.3}$$

It can be show that the SDE we use is equivalent to the Hull-White SDE. The HW 1F model is also referred to as the extended-Vasicek model or as a Gaussian HJM model.

In our implementation, we take $a(t) = a$ to be a constant, $\sigma(t)$ a piece-wise constant function and $\phi(t)$ a piece-wise continuous function.

2.2 Basic Formulas

This section presents some basic formulas related to the HW 1F Model.

2.2.1 Zero Coupon Bonds

The HW 1F model has the following formula for zero coupons bonds:

$$P(t,T) = e^{A(t,T)-B(t,T)x(t)} \tag{2.4}$$

where,

$$\begin{aligned} A(t,T) &= -\int_t^T \phi(s)ds + \frac{1}{2}U(t,T), && (2.5)\\ U(t,T) &= \int_t^T [\sigma(s)B(s,T)]^2 ds, && (2.6)\\ B(t,T) &= \frac{1-e^{-a(T-t)}}{a} && (2.7)\end{aligned}$$

The parameter $\phi(t)$ is chosen to reproduce the initial discount factor curve $P(0,T)$ at node points $T = T_k$. It can be shown that log-linear interpolation between node points is ensured if we take ϕ to be the following form:

$$\begin{aligned} \phi(T) &= \frac{1}{2}\frac{\partial}{\partial T}\int_0^T [\sigma(s)B(s,T)]^2 ds && (2.8)\\ &\quad + \frac{1}{T_k - T_{k-1}} ln\left(\frac{P(0,T_{k-1})}{P(0,T_k)}\right), \quad T_{k-1} \le T < T_k \end{aligned}$$

When calculating bond prices, the term $A(t,T)$ in Eq.(2.4) requires an integral of $\phi(t)$. Note that the integral on $\phi(t)$ can be broken up into two integrals:

$$\int_t^T \phi(s)ds = \int_0^T \phi(s)ds - \int_0^t \phi(s)ds \quad (2.9)$$

When evaluating these integrals, a double integral arises when the formula for $\phi(t)$ is substituted into the integral. By changing the order of integration, the expression can be simplified resulting in the formula:

$$\begin{aligned} \int_0^T \phi(s)ds &= \frac{1}{2}\int_0^T [\sigma(s)B(s,T)]^2 ds - ln\left(P(0,T_{k-1})\right) && (2.10)\\ &\quad + \frac{T - T_{k-1}}{T_k - T_{k-1}} ln\left(\frac{P(0,T_{k-1})}{P(0,T_k)}\right), \quad T_{k-1} \le T < T_k \end{aligned}$$

2.2.2 Measure Relationships

Under the T forward measure, the Wiener processes are related by:

$$dW^T = \sigma(t)B(t,T)dt + dW \quad (2.11)$$

The SDE under the T forward measure becomes:

$$dx(t) = \left(-a(t)x(t) - \sigma^2(t)B(t,T)\right)dt + \sigma(t)dW^T \quad (2.12)$$

2.3 Monte Carlo Simulation

There exist exact formulas for the simulation of the HW 1F model.

Under the risk neutral measure, integrating Eq.(2.1), the HW 1F model has the solution:

$$x(t) = x(s)e^{-a(t-s)} + e^{-at}\int_s^t \sigma(u)e^{au}dW(u) \quad (2.13)$$

The variance of the x risk factor from time s to t is given by:

$$Var(s,t) = e^{-2at}\int_s^t \sigma^2(u)e^{2au}du \quad (2.14)$$

Under the risk neutral measure, the x risk factor can be simulated using the equation:

$$x(t) = x(s)e^{-a(t-s)} + \sqrt{Var(s,t)}W \tag{2.15}$$

where, W is a random draw from a normal distribution $\mathcal{N}(0,1)$. When simulating under the risk neutral measure, we must also keep track of the numeraire.

$$N(t) = e^{\int_0^t r(\tau)d\tau} = e^{\int_0^t \phi(\tau)d\tau + \int_0^t x(\tau)d\tau} \tag{2.16}$$

We approximate the x integral term using one of either:

$$\int_0^t x(\tau)d\tau = \int_0^s x(\tau)d\tau + x(s)(t-s) \tag{2.17}$$

or

$$\int_0^t x(\tau)d\tau = \int_0^s x(\tau)d\tau + \frac{1}{2}\left[x(s) + x(t)\right](t-s) \tag{2.18}$$

Similarly, under the T forward measure, integrating Eq.(2.12), the HW 1F model has the solution:

$$x(t) = x(s)e^{-a(t-s)} + \mu(s,t,T) + e^{-at}\int_s^t \sigma(u)e^{au}dW^T(u) \tag{2.19}$$

where, $\mu(s,t,T)$ is the drift of the x risk factor from time s to t under the T forward measure.

$$\mu(s,t,T) = -e^{-at}\int_s^t \sigma^2(u)B(u,T)e^{au}du \tag{2.20}$$

Under the T forward measure, the x risk factor can be simulated using the equation:

$$x(t) = x(s)e^{-a(t-s)} + \mu(s,t,T) + \sqrt{Var(s,t)}W \tag{2.21}$$

2.4 Finite Difference Pricing

Applying the Feynman-Kac theorem to the SDE Eq.(2.1), the pricing PDE for the HW 1F model under the risk neutral measure is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 V}{\partial x^2} - ax\frac{\partial V}{\partial x} - (x + \phi(t))V = 0 \tag{2.22}$$

The x risk factor bounds of the PDE state space is set to:

$$x_{bounds} = \Big(-K_{scale}\sqrt{Var(0,t_{end})},\quad K_{scale}\sqrt{Var(0,t_{end})}\Big) \tag{2.23}$$

where, $Var(0,t_{end})$ is the x variance from the valuation date to the last time step t_{end} and K_{scale} is a scale factor (typically around 3) indicating how many standard deviations to include in the risk factor bounds.

Under the T forward measure, the pricing PDE is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 V}{\partial x^2} - \left(ax + B(t,T)\sigma^2(t)\right)\frac{\partial V}{\partial x} = 0 \tag{2.24}$$

The x risk factor bounds under the T forward measure are:

$$\begin{aligned} x^T_{bounds} = \Big(&\mu(0,t_{end},T) - K_{scale}\sqrt{Var(0,t_{end})}, \\ &\mu(0,t_{end},T) + K_{scale}\sqrt{Var(0,t_{end})}\Big) \end{aligned} \tag{2.25}$$

2.5 Calibration

This section describes how to calibrate the HW 1F model and the required formulas.

2.5.1 Bond Option Formula

Following Brigo and Mercurio [3], in the HW 1F model, the price of a European option at time t with strike K and maturity T on a zero coupon bond maturing at time S is given by:

$$PV_{call/put}(t,T,S,X) \quad = \quad \omega\Big(P(t,S)\Phi(\omega h) - KP(t,T)\Phi(\omega(h-\sigma_p))\Big) \tag{2.26}$$

where,

$$\begin{aligned} \sigma_p &= B(T,S)\sqrt{Var(t,T)} \\ h &= \frac{1}{\sigma_p} ln\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma_p}{2} \\ \omega &= \begin{cases} 1, & \text{if call} \\ -1, & \text{if put} \end{cases} \end{aligned} \tag{2.27}$$

2.5.2 Caplet Formula

The price of a caplet/floorlet at time t based on the forward rate from t_1 to t_2 is given by:

$$PV_{floorlet/caplet}(t,t_1,t_2,N,K) = N(1+K\tau)PV_{call/put}(t,t_1,t_2,\frac{1}{1+K\tau}) \tag{2.28}$$

where, N is the notional, K is the strike and τ is the year fraction for the caplet.

The price of caps/floors can be expressed as the sum of individual caplets/floorlets.

2.5.3 Swaption Formula

The price of a receiver/payer swaption at time t maturing at time T on a swap with payment times $\mathcal{T} = \{t_1, \ldots, t_n\}$ is given by:

$$PV_{receiver/payer}(t,T,\mathcal{T},N,K) \quad = \quad N\sum_{i=1}^{n} c_i PV_{call/put}(t,T,t_i,K_i) \tag{2.29}$$

where, K is the strike (fixed rate of the swap), τ_i is the year fraction from t_{i-1} to t_i and:

$$c_i \quad = \quad K\tau_i + \begin{cases} 1, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases} \tag{2.30}$$

$$K_i \quad = \quad e^{A(T,t_i)-B(T,t_i)x^*} \tag{2.31}$$

The K_i term comes from applying Jamshidian's method that allows us express the price of an option on a coupon bearing bond as a combination of zero coupon bond options. The term x^* in the equation for K_i is defined as the root of the equation:

$$\sum_{i=1}^{n} c_i e^{A(T,t_i)-B(T,t_i)x^*} = 1 \tag{2.32}$$

The value of x^* can be obtained using a numerical root finding procedure.

2.5.4 Exact Calibration

We describe a calibration method that exactly reproduces a set of market caplet/swaption prices.

Assume we are given a set of n calibrating instruments and market prices, one per maturity date, and a constant mean reversion parameter a. Let the set of instrument maturity dates $\mathcal{T}_{mat}$ and the corresponding market prices $\mathcal{M}$ be:

$$\begin{aligned} \mathcal{T}_{mat} &= \{T_1, \ldots, T_n\}, \quad 0 < T_i < T_{i+1} && (2.33) \\ \mathcal{M} &= \{M_1, \ldots, M_n\} && (2.34) \end{aligned}$$

We assume that $\sigma(t)$ is piece-wise constant function with values $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ in the intervals $\mathcal{I} = \{(0, T_1), (T_1, T_2), \ldots, (T_{n-1}, T_n)\}$ that correspond to the instrument maturity dates.

For each calibrating instrument, the caplet and swaption formulas previously presented can be used to calculate model prices for a given set of model parameters. In our implementation, we use a different formulation of the swaption formula present by Hunt and Kennedy [6]. In their formulation, the price of a swaption is given by:

$$PV_{receiver/payer}(t, T, \mathcal{T}, N, K) = N \sum_{i=1}^{n} c_i PV_{call/put}(t, T, t_i, K_i) \qquad (2.35)$$

where,

$$\begin{aligned} \phi(t) &= e^{\int_0^t a du} = e^{at} \\ \psi(t) &= \int_0^t \frac{1}{\phi(u)} du = \frac{1 - e^{-at}}{a} \\ \xi(t) &= \int_0^t \sigma^2(u)\phi^2(u) du = \int_0^t \sigma^2(u) e^{2au} du \\ V_0 &= \mathbb{E}_{\mathbb{F}}\Big[\Big(\sum_{j \le J} c_j P(0, S_j) e^{(\psi_T - \psi_{S_j})\hat{W}(\xi_T) - \frac{1}{2}(\psi_T - \psi_{S_j})^2 \xi_T} - P(0,T)K \Big)_+ \Big] \\ V_0 &= \mathbb{E}_{\mathbb{F}}\Big[\sum_{j \le J} c_j P(0, S_j) \left(e^{(\psi_T - \psi_{S_j})\hat{W}(\xi_T) - \frac{1}{2}(\psi_T - \psi_{S_j})^2 \xi_T} - K_j \right)_{\pm} \Big] \\ K_j &= e^{(\psi_T - \psi_{S_j})\hat{\omega} - \frac{1}{2}(\psi_T - \psi_{S_j})^2 \xi_T} \end{aligned}$$

where, $\pm$ is $+$ if payoff is increasing and $-$ if payoff is decreasing in $\hat{W}$.

We use this formulation in calibration because the swaption price can be expressed as a function of the variable $\xi(t)$. Expressing the swaption price as a function of $\xi(t)$ allows for an more efficient implementation. Note the previously provided caplet formula can also be expressed as a function of $\xi(t)$.

Starting with the first instrument maturing at T_1, we determine the value of $\xi(T_1)$ so that the model price $F_1(\xi(T_1))$ matches the market price M_1. This is done by apply a root finding procedure for $\xi(T_1)$ on the equation:

$$M_1 - F_1(\xi(T_1)) = 0 \qquad (2.36)$$

Using the formula for $\xi(t)$, Eq.(2.33), σ_1 can be determined from $\xi(T_1)$.

The calibration procedure then proceeds iteratively determining the value of $\xi(T_i)$ for each instrument by performing a root find on:

$$M_i - F_i(\xi(T_i)) = 0 \qquad (2.37)$$

From the value obtained for $\xi(T_i)$ and the previously determined values of $\{\sigma_1, \ldots, \sigma_{i-1}\}$, σ_i is calculated using the formula for $\xi(t)$.

2.6 Basis Adjustment

Deterministic basis spreads are incorporated into the HW 1F model using an additive spread approach.

Define the Libor rate observed at time t for a period starting at time T_1 and ending at T_2 with accrual factor $\delta_{1,2}$ to be:

$$L(t,T_1,T_2,\delta_{1,2}) \quad = \quad \frac{1}{\delta_{1,2}}\left(\frac{P(t,T_1)}{P(t,T_2)}-1\right) \tag{2.38}$$

In the HW 1F model, the Libor rate $\tilde{L}$ from a prediction curve is defined as the sum of the Libor calculated using the regular discounting curve plus an additional deterministic basis spread term $s^B(T_1,T_2,\delta_{1,2})$:

$$\tilde{L}(t,T_1,T_2,\delta_{1,2}) \quad = \quad L(t,T_1,T_2,\delta_{1,2}) + s^B(T_1,T_2,\delta_{1,2}) \tag{2.39}$$

The additive basis spread term is calculated from the initial discounting and prediction curves $P(0,T)$ and $\tilde{P}(0,T)$:

$$\begin{aligned} s^B(T_1,T_2,\delta_{1,2}) &= \tilde{L}(0,T_1,T_2,\delta_{1,2}) - L(0,T_1,T_2,\delta_{1,2}) \\ &= \frac{1}{\delta_{1,2}}\left(\frac{\tilde{P}(0,T_1)}{\tilde{P}(t,T_2)}-\frac{P(0,T_1)}{P(t,T_2)}\right) \end{aligned} \tag{2.40}$$

2.6.1 Caplet with Basis

With basis adjustment, the caplet formula Eq.(2.28) becomes:

$$PV^B_{floorlet/caplet}(t,t_1,t_2,N,K) = N(1+K'\tau)PV_{call/put}(t,t_1,t_2,\frac{1}{1+K'\tau}) \tag{2.41}$$

where $K' = K - s^B(t_1,t_2,\tau)$.

2.6.2 Swaption with Basis

With basis adjustment, the swaption formula Eq.(2.29) changes only in the definition of the c_i term in Eq.(2.30) which now becomes:

$$c_i \quad = \quad (K - s^B_i)\tau_i + \begin{cases} 1, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases} \tag{2.42}$$

where $s^B_i = s^B(t_{i-1},t_i,\tau_i)$

2.7 Additional Formulas

This section presents addition formulas related to the HW 1F model.

2.7.1 Digital Option Formula

The condition a for digital option with strike K on a Libor $L(T_{fix},T_{start},T_{end})$ fixed at time T_{fix} staring at T_{start} and ending at T_{end} with strike K is:

$$K = \frac{1}{\tau}\left[\frac{P(x,T_{fix},T_{start})}{P(x,T_{fix},T_{end})}-1\right] \tag{2.43}$$

Solving for the value of the stochastic factor x^* which satisifies the above equation gives:

$$x^* = \frac{A(T_{fix},T_{start}) - A(T_{fix},T_{end}) - ln(1+K\tau)}{B(T_{fix},T_{start}) - B(T_{fix},T_{end})} \tag{2.44}$$

The price of a digital option at time t that pays 1 at time T_{pay} if a Libor $L(T_{fix}, T_{start}, T_{end})$ is less that strike K (0 otherwise) is:

$$PV_{Digital}(x(t), t) = P(x(t), t, T_{pay}) E^{T_{pay}}[1_{L(T_1,T_2)<K}] \tag{2.45}$$

$$= P(x(t), t, T_{pay}) \int_{-\infty}^{x^*} \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(u-\mu_x)^2}{2\sigma_x^2}} du \tag{2.46}$$

$$= P(x(t), t, T_{pay}) \Phi\left(\frac{x^* - \mu_x}{\sigma_x}\right) \tag{2.47}$$

where $\Phi(x)$ is the cumulative normal function and:

$$\mu_x = x(t)e^{-a(T_{fix}-t)} - e^{-aT_{fix}} \int_t^{T_{fix}} \sigma^2(u) B(u, T_{pay}) e^{au} du, \tag{2.48}$$

$$\sigma_x = \sqrt{e^{-2aT_{fix}} \int_t^{T_{fix}} [\sigma(u)e^{au}]^2 du} \tag{2.49}$$

$$x^* = \frac{A(T_{fix}, T_{start}) - A(T_{fix}, T_{end}) - ln(1 + K\tau)}{B(T_{fix}, T_{start}) - B(t, T_{end})} \tag{2.50}$$

The digital option formula can be used in pricing Libor based range accruals. Also, the equation for x^* can be used to apply adjustments useful when pricing Libor based barriers.

Chapter 3

Multi-Factor Hull-White Interest Rate Model

This chapter describes issues relating to the implementation of a multi-factor Hull-White interest rate model.

3.1 Introduction

A multi-factor Hull-White (HW NF) model is an interest rate model where the dynamics of the short rate $r(t)$ is governed by the following equations:

$$dx_i(t) = -a_i(t)x_i(t)dt + \eta_i(t)dW_i, \quad x(0) = 0 \tag{3.1}$$

$$r(t) = \phi(t) + \sum_{i=1}^{N} x_i(t), \quad dW_i dW_j = \rho_{i,j}(t)dt \tag{3.2}$$

where, W_i are correlated Brownian motions with correlations $\rho_{i,j}(t)$, $a_i(t)$ are mean reversion parameters and $\eta_i(t)$ are volatility parameters and $\phi(t)$ is a function chosen so that the initial discount factor curve is reproduced.

These models are also referred to as Gaussian short rate models, Gaussian HJM models with separable volatility and for the 2 factor case, the G2++ of Brigo and Mercurio [3].

3.2 Zero Coupon Bond Formulas

A HW NF model has the following closed form solution for zero coupons bonds:

$$P(t,T) = e^{A(t,T) - \sum_{i=1}^{N} B(a_i,t,T)x_i} \tag{3.3}$$

Where,

$$A(t,T) = -\int_t^T \phi(s)ds + \frac{1}{2}V(t,T), \tag{3.4}$$

$$V(t,T) = \sum_{i=1}^{N}\sum_{j=1}^{N}\int_t^T \rho_{i,j}\eta_i(s)\eta_j(s)B(a_i,s,T)B(a_j,s,T)ds, \tag{3.5}$$

$$B(z,t,T) = \frac{1-e^{-z(T-t)}}{z} \tag{3.6}$$

Note that the integral on $\phi(t)$ can be broken up into two integrals:

$$\int_t^T \phi(s)ds = \int_0^T \phi(s)ds - \int_0^t \phi(s)ds \tag{3.7}$$

which each can be evaluated using:

$$\begin{aligned}\int_0^T \phi(s)ds &= \frac{1}{2}V(0,T) - ln\left(P(0,T_{k-1})\right) \\ &\quad + \frac{T-T_{k-1}}{T_k - T_{k-1}} ln\left(\frac{P(0,T_{k-1})}{P(0,T_k)}\right), \quad T_{k-1} \leq T < T_k\end{aligned} \tag{3.8}$$

3.3 Monte Carlo Simulation

Under the T forward measure, the equations become:

$$dx_i = \Big[-a_i x_i - \sum_{j=1}^{N} \rho_{i,j}\eta_i\eta_j B(a_j,t,T)\Big]dt + \eta_i dW_i^T, \quad dW_i^T dW_j^T = \rho_{i,j}dt \tag{3.9}$$

which has exact solutions

$$x_i(t) = x_i(s)e^{-a_i(t-s)} - M_i^T(s,t) + \int_s^t \eta_i e^{-a_i(t-u)} dW_i^T \tag{3.10}$$

with the forward measure drift terms:

$$M_i^T(s,t) = \sum_{j=1}^{N}\int_s^t \rho_{i,j}\eta_i\eta_j B(a_j,u,T)e^{-a_i(t-u)}du \tag{3.11}$$

Under the risk neutral measure, the exact solution is similar but with no additional drift term (i.e. $M_i^T(s,t) = 0$). When simulating under the risk neutral measure, we must also keep track of the numeraire.

$$N(t) = e^{\int_0^t r(\tau)d\tau} = e^{\int_0^t \phi(\tau)d\tau + \int_0^t \sum_{i=1}^N x_i(\tau)d\tau} \tag{3.12}$$

We approximate the x integral term using one of either:

$$\int_0^t \sum_{i=1}^{N} x_i(\tau)d\tau = \int_0^s \sum_{i=1}^{N} x_i(\tau)d\tau + \sum_{i=1}^{N} x_i(s)(t-s) \tag{3.13}$$

or

$$\int_0^t \sum_{i=1}^{N} x_i(\tau)d\tau = \int_0^s \sum_{i=1}^{N} x_i(\tau)d\tau + \frac{1}{2}\sum_{i=1}^{N}\left[x_i(s) + x_i(t)\right](t-s) \tag{3.14}$$

Under the T forward measure, from s to t, the x_i follow a multi-variate normal distribution with probability density function given by:

$$f(s,t) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{N/2}} \exp\left(-\frac{1}{2}(x(t)-\mu)^T|\Sigma|^{-1}(x(t)-\mu)\right) \tag{3.15}$$

where, μ and Σ are the mean and covariance:

$$\mu_i(s,t,T) = x_i(s)e^{-a_i(t-s)} - M_i^T(s,t) \tag{3.16}$$

$$\Sigma_{i,j}(s,t) = \int_s^t \rho_{i,j}\eta_i\eta_j e^{-(a_i+a_j)(t-u)}du \tag{3.17}$$

3.4 Finite Difference Pricing

Applying the Feynman-Kac theorem to the SDE Eq.(3.1), the pricing PDE for the HW NF model under the risk neutral measure is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{i,j}\eta_i(t)\eta_j(t)\frac{\partial^2 V}{\partial x_i \partial x_j} - \sum_{i=1}^{N} a_i x_i \frac{\partial V}{\partial x_i} \qquad (3.18)$$
$$- \left(\sum_{i=1}^{N} x_i + \phi(t)\right) V = 0$$

The risk factor bounds of the PDE state space are set to:

$$x_{i,bounds} = \left(- K_{scale}^{i}\sqrt{\Sigma_{i,i}(0,t_{end})},\quad K_{scale}^{i}\sqrt{\Sigma_{i,i}(0,t_{end})}\right) \qquad (3.19)$$

where, $\Sigma_{i,i}(0,t_{end})$ is the x_i variance from the valuation date to the last time step t_{end} and K_{scale}^{i} are a scale factors (typically around 3) indicating how many standard deviations to include in the risk factor bounds.

Under the T forward measure, the pricing PDE is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{i,j}\eta_i(t)\eta_j(t)\frac{\partial^2 V}{\partial x_i \partial x_j} \qquad (3.20)$$
$$- \sum_{i=1}^{N}\left(a_i x_i + \sum_{j=1}^{N}\rho_{i,j}\eta_i(t)\eta_j(t)B(a_j,t,T)\right)\frac{\partial V}{\partial x_i} = 0$$

The risk factor bounds under the T forward measure are:

$$x_{i,bounds}^{T} = \Big(\mu_i(0,t_{end},T) - K_{scale}^{i}\sqrt{\Sigma_{i,i}(0,t_{end})}, \qquad (3.21)$$
$$\mu_i(0,t_{end},T) + K_{scale}^{i}\sqrt{\Sigma_{i,i}(0,t_{end})}\Big)$$

3.5 Calibration

3.5.1 Caplet Formula

The price of a caplet/floorlet with notional N and strike K on a Libor starting at time T expiring at time S with year fraction τ is given by:

$$V_{caplet/floorlet} = -\omega N' P(0,S)\Phi\left(\omega\left(\frac{ln\left(\frac{NP(0,T)}{N'P(0,S)}\right)}{\Lambda} - \frac{1}{2}\Lambda\right)\right) \qquad (3.22)$$
$$+\omega N P(0,T)\Phi\left(\omega\left(\frac{ln\left(\frac{NP(0,T)}{N'P(0,S)}\right)}{\Lambda} + \frac{1}{2}\Lambda\right)\right) \qquad (3.23)$$

where, $\omega = +1$ for a caplet and $\omega = -1$ for a floorlet and

$$N' = N(1+K\tau) \qquad (3.24)$$
$$\Lambda^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\Big[B(a_i,T,S)B(a_j,T,S) \times \qquad (3.25)$$
$$\int_0^T \rho_{i,j}\eta_i(u)\eta_j(u)e^{-(a_i+a_j)(T-u)}du\Big]$$

3.5.2 European Swaption Approximation

For the 1 and 2 factor cases, exact formulas for European swaption prices are available (see Brigo and Mercurio [3]). In general, the price of a European swaption requires the valuation of an N dimensional integral.

Schrager and Pelsser [9] present approximate formulas for European swaptions. They approximate the swap rate process as a normal random variable by replacing some low variance martingales by their time zero values.

Consider a swap starting at T_0 with n coupons. The payment dates of the swap are $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$. Let δ_i be the accrual factor for the coupon with payment date T_i (i.e. the year fraction between T_{i-1} and T_i).

The swap rate $\kappa_{T_0,n}$ is:

$$\kappa_{T_0,n}(t) = \frac{P(t,T_0) - P(t,T_n)}{A_{T_0,n}(t)} \tag{3.26}$$

where, $A_{T_0,n}(t)$ is the annuity:

$$A_{T_0,n}(t) = \sum_{j=1}^{n} \delta_j P(t,T_j) \tag{3.27}$$

Under the swap measure where the annuity is the numeraire, the SDE for the swap rate is:

$$d\kappa_{T_0,n}(t) = \frac{\partial \kappa_{T_0,n}(t)}{\partial x}\eta(t) \cdot dW^{T_0,n} \tag{3.28}$$

Note that there is no drift term as the swap rate is a martingale under this measure.

The partial derivative of $\kappa_{T_0,n}$ w.r.t. x_i is:

$$\begin{aligned} \frac{\partial \kappa_{T_0,n}(t)}{\partial x_i} &= -B(a_i,t,T_0)\frac{P(t,T_0)}{A_{T_0,n}(t)} + B(a_i,t,T_n)\frac{P(t,T_n)}{A_{T_0,n}(t)} \\ &\quad +\kappa_{T_0,n}(t)\sum_{j=1}^{n}\delta_j B(a_i,t,T_j)\frac{P(t,T_j)}{A_{T_0,n}(t)} \end{aligned} \tag{3.29}$$

They approximate the stochastic $P(t,T_i)/A_{T_0,n}(t)$ terms with their time zero values:

$$\begin{aligned} \frac{\partial \kappa^*_{T_0,n}(t)}{\partial x_i} &= -B(a_i,t,T_0)\frac{P(0,T_0)}{A_{T_0,n}(0)} + B(a_i,t,T_n)\frac{P(0,T_n)}{A_{T_0,n}(0)} \\ &\quad +\kappa_{T_0,n}(0)\sum_{j=1}^{n}\delta_j B(a_i,t,T_j)\frac{P(0,T_j)}{A_{T_0,n}(0)} \\ &= \frac{e^{a_i t}}{a_i}\Big[\frac{1}{A_{T_0,n}(0)}\Big(e^{-a_i T_0}P(0,T_0) - e^{-a_i T_n}P(0,T_n) \\ &\qquad - \kappa_{T_0,n}(0)\sum_{j=1}^{n}\delta_j e^{-a_i T_j}P(0,T_j)\Big)\Big] \\ &= \frac{e^{a_i t}}{a_i} C^{(i)}_{T_0,n} \end{aligned} \tag{3.30}$$

The approximation for the swap rate follows a normal process with the SDE:

$$d\kappa_{T_0,n}(t) = \frac{\partial \kappa^*_{T_0,n}(t)}{\partial x}\eta(t) \cdot dW^{T_0,n} \tag{3.31}$$

The normal volatility of the approximate swap rate process is given by:

$$\sigma_{T_0,n} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^T C^{(i)}_{T_0,n}C^{(j)}_{T_0,n}\rho_{i,j}\eta_i(s)\eta_j(s)\frac{e^{(a_i+a_j)s}}{a_i a_j}ds\right)^{1/2} \tag{3.32}$$

The approximate price for a European swaption is given by:

$$V_{swaption} = A_{T_0,n}(0)\left[-\omega K^*\Phi\left(\frac{-\omega K^*}{\sigma_{T_0,n}}\right) + \sigma_{T_0,n}\phi\left(\frac{K^*}{\sigma_{T_0,n}}\right)\right] \tag{3.33}$$

where, $K^* = K - \kappa_{T_0,n}(0)$ and $\omega = \pm 1$ for a payers and receivers swaption respectively.

3.5.3 Calibration Procedure

For calibration, we use the procedure described in Andreasen [1] for a class of stochastic volatility HJM models. In the HJM formulation, the instantaneous forward rates:

$$f(t,T) = -\frac{\partial lnP(t,T)}{\partial T} \tag{3.34}$$

are governed by the SDE:

$$df(t,T) = \sigma(t,T)'\left(\int_t^T \sigma(t,s)ds\right)dt + \sigma(t,T)'dW(t) \tag{3.35}$$

Separable volatility structures are of the form:

$$\sigma(t,T)' = g(T)'h(t) \tag{3.36}$$

The multi-factor Hull-White model can be expressed as a Gaussian HJM model with separable volatility where:

$$g(T) = g(0,T), \quad h(t) = I_{g(t)}^{-1}\xi(t) \tag{3.37}$$

and

$$g(t,T) = \left(e^{-\int_t^T a_1(u)du}, \ldots, e^{-\int_t^T a_k(u)du}\right)' \tag{3.38}$$

$$I_{g(t)} = \begin{bmatrix} e^{-\int_0^t a_1(u)du} & & & 0 \\ & e^{-\int_0^t a_2(u)du} & & \\ & & \ddots & \\ 0 & & & e^{-\int_0^t a_k(u)du} \end{bmatrix} \tag{3.39}$$

For constant values of $a_1, \ldots, a_k$ and fixed tenors $\tau_1, \ldots, \tau_k$, the SDE for the corresponding instantaneous forward rates is:

$$dF(t) = \Lambda^*(t)R(t)dW(t) + o(dt) \tag{3.40}$$

where,

$$\Lambda^*(t) = \begin{bmatrix} f(0,\tau_1)\lambda_1(t) & & & 0 \\ & f(0,\tau_2)\lambda_2(t) & & \\ & & \ddots & \\ 0 & & & f(0,\tau_k)\lambda_k(t) \end{bmatrix} \tag{3.41}$$

and $\lambda_i(t)$ is the forward rate volatility for $f(t, t+\tau_k)$ and RR' is the instantaneous correlation matrix for the k forward rates.

Under the separable volatility specification, the forward rate SDE is:

$$dF(t) = \Gamma(t)\xi(t)dW(t) + o(dt), \quad \Gamma(t) = \begin{bmatrix} g(t, t+\tau_1)' \\ \vdots \\ g(t, t+\tau_k)' \end{bmatrix}' \tag{3.42}$$

Equating terms Eq.(3.40) and Eq.(3.42) in gives:

$$\xi(t) \quad = \quad \Gamma^{-1}(t)\Lambda^*(t)R(t) \tag{3.43}$$

Let $\Xi(t) = \xi\xi^T$ be the $k \times k$ covariance matrix at time t. The models parameters in Eq.(3.2) can be expressed in terms of $\Xi(t)$ as:

$$\eta_i(t) \quad = \quad \sqrt{\Xi_{i,i}(t)}, \tag{3.44}$$

$$\rho_{i,j}(t) \quad = \quad \frac{\Xi_{i,j}(t)}{\eta_i(t)\eta_j(t)} \tag{3.45}$$

Using a similar approach to Andreasen [1], we use the following calibration procedure. Given fixed values of $a_1, \ldots, a_k$, $\tau_1, \ldots, \tau_k$ and a constant correlation structure RR' (estimated from historical market data), we calibrate the model to swaptions and caps. We assume a piece-wise constant form for the forward rate volatilities:

$$\lambda_i(t) = \tilde{\lambda}_{i,\beta(t)}, \quad \beta(t) = \{j : T_{j-1} < t \leq T_j\} \tag{3.46}$$

for a fixed time grid $0 = T_0 < T_1 < \ldots < T_m$.

The model is calibrated by minimizing a quality of fit function to reproduce market prices for swaptions and caps. The optimization is performed over the values $\tilde{\lambda}_{i,j}$ and piece–wise constant values for $\eta_i(t)$ and $\rho_{i,j}(t)$ are obtained using Eqs.(3.44) and (3.45). Note, in the calibration procedure, an penalty function is added to the quality of fit function to introduce a measure of smoothness to the values of $\lambda_i(t)$ being fitted.

3.5.3.1 PDE Transformation

Using the calibration procedure described above we find often leads to risk factors with correlations very close to 1 or -1. When pricing using finite difference methods, the method for selecting risk factor bounds described in section 3.4 performs poorly as the risk factor distribution is not well described by a rectangular region when correlations are high. This results in poor convergence with increasing number of grid points.

To address this problem, we transform the state space using information from the terminal distribution. Given the mean $\mu_i(0, t_{end}, T)$ and covariance $\Sigma_{i,j}(0, t_{end})$ at the end date, we define the transformed set of variables u:

$$x = M^{-1}u + \mu \tag{3.47}$$

where, x are the original variables, $M^{-1} = \sqrt{\lambda}\xi$, and λ are the eigenvalues of Σ and ξ are the eigenvectors of Σ. Under the T forward measure, the transformed PDE is:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} M_{i,k}M_{j,l}\rho_{k,l}\eta_k(t)\eta_l(t)\frac{\partial^2 V}{\partial u_i \partial u_j} \\ &- \sum_{i=1}^{N}\Big[\sum_{j=1}^{N} M_{i,j}a_j \sum_{k=1}^{N}(M^{-1}_{j,k}u_k + \mu_k) \\ &\qquad + \sum_{j=1}^{N}\sum_{k=1}^{N} M_{i,j}\rho_{j,k}\eta_j(t)\eta_k(t)B(a_k, t, T)\Big]\frac{\partial V}{\partial u_i} = 0 \end{aligned} \tag{3.48}$$

We choose the risk factor bounds under the T forward measure to be:

$$u^T_{i,bounds} \quad = \quad \Big(-K^i_{scale}, K^i_{scale}\Big) \tag{3.49}$$

3.5.4 Alternative Swaption Diagonal Calibration Procedure

We also provide an alternative calibration procedure for the multi-factor Hull-White model in addition to the one previously described. This second calibration procedure calibrates to a swaption diagonal similar to what is usually done for single factor Hull-White models (see Chapter 2 on page 11). However, as the multi-factor Hull-White model has more parameters, some additional inputs are also required.

As before, the swaption diagonal calibration procedure assumes constant values for the mean reversions $a_1, \ldots, a_k$ which are given as explicit inputs. Also required are the $\rho_{i,j}(t)$ correlations between the Brownian drivers dW_i in (3.2). Lastly, we assume the model volatilities are given by:

$$\eta_i(t) = \Upsilon_i(t)\eta_1(t) \tag{3.50}$$

where $\Upsilon_i(t)$ are scale factor terms that represents the ratio of $\eta_i(t)$ to $\eta_1(t)$ (with $\Upsilon_1(t) = 1$). The scale factor terms $\Upsilon_i(t)$ are also explicit inputs to the calibration procedure.

With the additional inputs $\rho_{i,j}(t)$ and $\Upsilon_i(t)$, if we assume $\eta_1(t)$ is piece-wise constant, we can calibrate the value of $\eta_1(t)$ in each interval to match each element of a swaption diagonal. Here, we apply the European swaption approximation described in Chapter 2 on page 11.

In the swaption diagonal calibration procedure, we first convert the swaption market quotes to an equivalent normal volatility. For ATM swaptions, the conversion is analytic and for non-ATM, it requires a root find. Because the European swaption approximation approximates the swap rate as a normal process, calculating the values of $\eta_1(t)$ for each interval to match the equivalent normal volatilities involves only a simple analytic calculation. Thus, the swaption diagonal calibration procedure is quite fast. Although this alternative calibration procedure may be fast, the additional inputs required may be more difficult to estimate compared to what is required for the calibration procedure presented in Chapter 2 on page 11.

3.5.5 Calibration to Historical Data

Using the Andreasen parameterization described in the previous section, we can calibrate a multi-factor Hull-White model so that the dynamics of a given set of benchmark rates are reproduced.
Suppose we specify a set of benchmark tenors $\tau_1, \ldots, \tau_k$ which corresponds to the set of benchmark instantaneous forward rates $f_i(t) = f(t, t + \tau_i)$. The covariance matrix $C_{i,j}$ of the $f_i(t)$ benchmark rates could be obtained using historical time series analysis. Also assume we have specified a set of corresponding of mean reversions $a_1, \ldots, a_k$. We can now obtain the required Hull-White model parameters using Eqs.(3.44)-(3.45) and the transformed covariance matrix $\Xi(t) = \Gamma^{-1}C(\Gamma^{-1})^T$.
In addition to calculating the benchmark covariance matrix, we also have to select values for mean reversions and benchmark tenors. The benchmark tenors should be chosen to be representative points on the yield curve. Andersen and Piterbarg [2] advocate choosing d fixed values for mean reversions and using them for all cases noting that they all should be different since the inverse of the matrix Γ is required to exist. They note that in the multi-factor context, the effect of the mean reversions is to essentially define how volatilities and correlations of non-benchmark rates are obtained from those of the benchmark rates. They suggest a reasonable chose for mean reversion values is to span the interval $[0, 1]$. As an example, for a 4 factor model, they suggest:

$$a = \{0.015, 0.15, 0.3, 1.20\} \tag{3.51}$$

$$\tau = \{6m, 2y, 10y, 30y\} \tag{3.52}$$

3.6 Basis Adjustment

Deterministic basis spreads are incorporated into the HW NF model using an additive spread approach.

Define the Libor rate observed at time t for a period starting at time T_1 and ending at T_2 with accrual factor $\delta_{1,2}$ to be:

$$L(t, T_1, T_2, \delta_{1,2}) = \frac{1}{\delta_{1,2}}\left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right) \tag{3.53}$$

In the HW NF model, the Libor rate $\tilde{L}$ from a prediction curve is defined as the sum of the Libor calculated using the regular discounting curve plus an additional deterministic basis spread term $s^B(T_1, T_2, \delta_{1,2})$:

$$\tilde{L}(t, T_1, T_2, \delta_{1,2}) = L(t, T_1, T_2, \delta_{1,2}) + s^B(T_1, T_2, \delta_{1,2}) \tag{3.54}$$

The additive basis spread term is calculated from the initial discounting and prediction curves $P(0, T)$ and $\tilde{P}(0, T)$:

$$\begin{aligned} s^B(T_1, T_2, \delta_{1,2}) &= \tilde{L}(0, T_1, T_2, \delta_{1,2}) - L(0, T_1, T_2, \delta_{1,2}) \\ &= \frac{1}{\delta_{1,2}} \left(\frac{\tilde{P}(0, T_1)}{\tilde{P}(t, T_2)} - \frac{P(0, T_1)}{P(t, T_2)} \right) \end{aligned} \tag{3.55}$$

3.6.1 Caplet with Basis

With basis adjustment, the caplet formula Eq.(3.22) changes only in that the N' term in Eq.(3.24) now becomes:

$$N' = N\left[1 + (K - s^B(T, S, \tau))\tau\right] \tag{3.56}$$

3.6.2 Swaption with Basis

With basis adjustment, the approximate swaption formula Eq.(3.33) changes only in the definition of the $C^{(i)}_{T_0,n}$ terms in Eq.(3.30) which become:

$$\begin{aligned} C^{(i)}_{T_0,n} = \frac{1}{A_{T_0,n}(0)} \Big(& e^{-a_i T_0} P(0, T_0) - e^{-a_i T_n} P(0, T_n) \\ & - \sum_{j=1}^{n} \left[\kappa_{T_0,n}(0) - s^B_j\right] \delta_j e^{-a_i T_j} P(0, T_j) \Big) \end{aligned} \tag{3.57}$$

where $s^B_i = s^B(T_{i-1}, T_i, \delta_i)$

3.7 Additional Formulas

3.7.1 CMS Spread Option Approximation

Let us consider the dynamics of a swap rate under the T_0 forward measure. Following Dun [4], define $F_{T_0}(t, T)$ to be the T_0 forward bond price:

$$F_{T_0}(t, T) = \frac{P(t, T)}{P(t, T_0)} \tag{3.58}$$

The SDE for $F_{T_0}(t, T)$ under the T_0 forward measure is:

$$\frac{dF_{T_0}(t, T)}{F_{T_0}(t, T)} = -b(t, T_0, T) \cdot dW^{T_0} \tag{3.59}$$

where,

$$b_j(t, T_0, T) = \int_{T_0}^{T_i} \eta_j(t) e^{-a_j(u-t)} du = \eta_j(t) \frac{e^{a_j t}}{a_j} (e^{-a_j T_0} - e^{-a_j T}) \tag{3.60}$$

Consider the quantity:

$$\frac{P(t, T_k)}{\sum_{j=1}^{n} \delta_j P(t, T_j)} = \frac{F_{T_0}(t, T_k)}{\sum_{j=1}^{n} \delta_j F_{T_0}(t, T_j)} \tag{3.61}$$

The SDE for the sum term in the denominator of Eq.(3.61) is given by:

$$d\left(\sum_{j=1}^{n}\delta_j F_{T_0}(t,T_j)\right) = -\sum_{j=1}^{n}\delta_j F_{T_0}(t,T_j)b(t,T_0,T_j)\cdot dW^{T_0} \tag{3.62}$$

Let $F_k = F_{T_0}(t,T_k)$. Applying Ito's lemma to Eq.(3.61) gives:

$$\begin{aligned} d\left(\frac{F_k}{\sum_{j=1}^{n}\delta_j F_j}\right) &= \left(\frac{F_k\sum_{j=1}^{n}\delta_j F_j b(t,T_0,T_j)}{\left(\sum_{j=1}^{n}\delta_j F_j\right)^2} - \frac{F_k b(t,T_0,T_k)}{\sum_{j=1}^{n}\delta_j F_j}\right)\cdot \\ &\quad \left(dW^{T_0} + \frac{\sum_{j=1}^{n}\delta_j F_j b(t,T_0,T_j)}{\sum_{j=1}^{n}\delta_j F_j}dt\right) \end{aligned} \tag{3.63}$$

Since we know the ratio $F_k/(\sum_{j=1}^{n}\delta_j F_j)$ is a martingale, the relationship between the T_0 forward measure and the swap measure with $A_{T_0,n}(t)$ as the numeraire is:

$$dW^{T_0,n} = dW^{T_0} + \frac{\sum_{j=1}^{n}\delta_j F_{T_0}(t,T_j)b(t,T_0,T_j)}{\sum_{j=1}^{n}\delta_j F_{T_0}(t,T_j)}dt \tag{3.64}$$

Substituting this into the swap rate SDE Eq.(3.28) under the swap measure gives:

$$\begin{aligned} d\kappa_{T_0,n}(t) &= \left(\frac{\partial\kappa^*_{T_0,n}(t)}{\partial x}\eta(t)\right)\cdot\frac{\sum_{j=1}^{n}\delta_j F_{T_0}(t,T_j)b(t,T_0,T_j)}{\sum_{j=1}^{n}\delta_j F_{T_0}(t,T_j)}dt \\ &\quad + \frac{\partial\kappa^*_{T_0,n}(t)}{\partial x}\eta(t)\cdot dW^{T_0} \end{aligned} \tag{3.65}$$

Setting the state dependent terms to their time zero values gives the approximation:

$$\begin{aligned} d\kappa_{T_0,n}(t) &= \left(\frac{e^{at}}{a}C_{T_0,n}\eta(t)\right)\cdot\frac{\sum_{j=1}^{n}\delta_j P(0,T_j)b(t,T_0,T_j)}{\sum_{j=1}^{n}\delta_j P(0,T_j)}dt \\ &\quad + \left(\frac{e^{at}}{a}C_{T_0,n}\eta(t)\right)\cdot dW^{T_0} \end{aligned} \tag{3.66}$$

At time T_0 under the T_0 forward, each swap rate is approximately normal with drift:

$$\mu_{T_0,n} = \int_0^{T_0}\left(\frac{e^{au}}{a}C_{T_0,n}\eta(u)\right)\cdot\frac{\sum_{j=1}^{n}\delta_j P(0,T_j)b(u,T_0,T_j)}{\sum_{j=1}^{n}\delta_j P(0,T_j)}du \tag{3.67}$$

which can be simplified to:

$$\begin{aligned} \mu_{T_0,n} &= \sum_{i=1}^{N}\sum_{j=1}^{N}C^{(j)}_{T_0,n}\left(e^{-a_iT_0} - \frac{\sum_{k=1}^{n}\delta_k P(0,T_k)e^{-a_iT_k}}{\sum_{k=1}^{n}\delta_k P(0,T_k)}\right)\times \\ &\quad \int_0^{T_0}\rho_{i,j}\eta_i(u)\eta_j(u)\frac{e^{(a_i+a_j)u}}{a_ia_j}du \end{aligned} \tag{3.68}$$

Now if we consider the spread between two swap rates $S(t) = \kappa_{T_0,n_1}(t) - \kappa_{T_0,n_2}(t)$, because each swap rate is approximately normal, the spread is also approximately normal with mean:

$$\mu_S = \mu_{T_0,n_1} - \mu_{T_0,n_2} \tag{3.69}$$

and variance:

$$\sigma_S^2 = \sigma^2_{T_0,n_1} + \sigma^2_{T_0,n_2} - 2\rho_{S_{1,2}}\sigma_{T_0,n_1}\sigma_{T_0,n_2} \tag{3.70}$$

where ρ_S is the correlation between the two swap rates:

$$\rho_S = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^{T_0} C^{(i)}_{T_0,n_1} C^{(j)}_{T_0,n_2}\rho_{i,j}\eta_i(s)\eta_j(s)\frac{e^{(a_i+a_j)s}}{a_i a_j}ds}{\sigma_{T_0,n_1}\sigma_{T_0,n_2}} \tag{3.71}$$

The approximate price for a European option on the spread $S(T_0)$ is given by:

$$V_{CMSSpread} = P(0,T_0)\left[-\omega K^*\Phi\left(\frac{-\omega K^*}{\sigma_S}\right) + \sigma_S\phi\left(\frac{K^*}{\sigma_S}\right)\right] \tag{3.72}$$

where, $K^* = K - (\kappa_{T_0,n_1}(0) - \kappa_{T_0,n_2}(0) + \mu_S)$ and $\omega = \pm 1$ for a call and a put respectively.

3.7.1.1 CMS Spread Digital Option Approximation

Related to the CMS spread option formula is a formula for digital CMS spread options which can be used for pricing CMS spread range accrual products.

Consider a digital CMS spread option on two swap rates $\kappa_{T_0,n_1}(T_0)$ and $\kappa_{T_0,n_2}(T_0)$ where the payoff occurs at some time $T_p \geq T_0$. Our calculation from before changes as we now use the T_p forward measure instead of the T_0 forward measure. This gives slightly different drift terms:

$$\begin{aligned}\mu^D_{T_0,T_p,n}(t) = \mu_{T_0,n}(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} C^{(j)}_{T_0,n}(t)\left(e^{-a_i T_p} - e^{-a_i T_0}\right) \times \\ \int_0^{T_0}\rho_{i,j}\eta_i(u)\eta_j(u)\frac{e^{(a_i+a_j)u}}{a_i a_j}du\end{aligned} \tag{3.73}$$

The value of a digital CMS spread option at time t that pays \$1 at time T_p if $\kappa_{T_0,n_1}(T_0) - \kappa_{T_0,n_2}(T_0) < K$ is:

$$V^{Digital}_{CMSSpread}(t) = P(t,T_p)\Phi\left(\frac{K^D}{\sigma_S(t)}\right) \tag{3.74}$$

where, $K^D = K - (\kappa_{T_0,n_1}(t) - \kappa_{T_0,n_2}(t) + \mu^D_{T_0,T_p,n_1}(t) - \mu^D_{T_0,T_p,n_2}(t))$,

Chapter 4

Multi-Factor Hull-White Interest Rate Model With Real-World Drift

The chapter descibes an enhancement to the multi-factor Hull-White interest rate model to support a real-world drift to interest rates.

4.1 Introduction

The multi-factor Hull-White model (3.1)-(3.2) simulates the dynamics of the short interest rate in the risk-neutral measure. Here, the function φ is calibrated such that the terms strucutre of interest rates is reproduced by the model. The expected future interest rates of the model therefore follow the market implied forward rates. In the real-world measure, the future rates are expected to be lower than the market implied forward rates due to the risk tolerance of investors; this is known as the market price of risk. Ignoring the market price of risk may lead to over estimates of real-world exposures for ling time horizons.

To capture the market price of risk, the model (3.1)-(3.2) is modified to

$$dx_i(t) = \left(-a_i(t)x_i(t) + \lambda_i(t)\eta_i(t)\right)dt + \eta_i(t)dW_i, \quad x_i(0) = 0 \tag{4.1}$$

$$r(t) = \phi(t) + \sum_{i=1}^{N} x_i(t), \quad dW_i dW_j = \rho_{i,j}(t)dt. \tag{4.2}$$

where $\lambda_i(t)$ is the instantaneous market price of risk corresponding to the ith short rate factor.

4.2 Zero Coupon Bond Formula

The zero coupon bond price is provided in section 3.3. note that the zero coupon bond t-price $P(t,T)$ is measure-independent condition on model state $x(t)$, and hence has the same closed-form representation as in the risk neutral Hull-White model. However, the probability distribution of the random variable $P(t,T)$ is affected by the drift in the simulation of the process $x(t)$.

4.3 Monte Carlo Simulation

Monte Carlo simulation of the random process $x_i(t)$ is performed using the exact representation of the solution to (4.1),

$$x_i(t) = x_i(s)e^{-a_i(t-s)} + M_i^{RW}(s,t) + \int_s^t \eta_i e^{-a_i(t-u)} dW_i \tag{4.3}$$

with the drift terms:

$$M_i^{RW}(s,t) = \int_s^t \lambda_i \eta_i e^{-a_i(t-u)} du. \tag{4.4}$$

4.4 Calibration

The calibration of the market price of risk $\lambda_i(t)$ is performed after the calibration of interest rate volaitlity η and correlation ρ. For the details of the calibration of η in case of single factor model, see section 2.5. For the details of the calibration of η and ρ in case of multi-factor model, see section 3.5.

The market price of risk is calibrated in such a way that the difference between the average forward and short rates calculated by the model matches the difference between the historical average forward and short rate at the pre-specified set of M forward rate tenor points,

$$f_{HistAvg}(t_0, T_i) - r_{HistAvg}(t_0) = f(t_0, T_i) - E_{t_0}[r(T_i)], \quad i = 1, \dots, M. \tag{4.5}$$

Here, $f_{HistAvg}(t, T_i)$ and $r_{HistAvg}(t)$ are forward and short rate average calculated from the historical data; $r(t)$ is the model short rate, and $f(t, T)$ is the model forward rate,

$$f(t, T) = -\frac{\partial}{\partial T} \ln P(t, T), \tag{4.6}$$

where $P(t, T)$ is the price of the zero coupon bond with the maturity T.

Using (3.3) and (4.1)-(4.2), we can write out the following SDE for the forward rate $f(t, T)$,

$$df(t, T) = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \eta_i \eta_j \rho_{ij} e^{-a_i(T-t)} B(a_j, t, T) + \sum_{i=1}^{N} \lambda_i \eta_i e^{-a_i(T-t)} \right) dt + \sum_{i=1}^{N} \eta_i e^{-a_i(T-u)} dW_i, \tag{4.7}$$

from which we get:

$$f(T, T) = f(t_0, T) + C(t_0, T) + RW(t_0, T) + \int_{t_0}^{T} \sum_{i=1}^{N} \eta_i e^{-a_i(T-u)} dW_i. \tag{4.8}$$

Here, $C(t_0, T)$ is the convexity adjustment term,

$$C(t_0, T) = \int_{t_0}^{T} \sum_{i=1}^{N} \sum_{j=1}^{N} \eta_i \eta_j \rho_{ij} e^{-a_i(T-t)} B(a_j, t, T) dt, \tag{4.9}$$

and $RW(t_0, T)$ is the real-world drift adjustment term,

$$RW(t_0, T) = \int_{t_0}^{T} \sum_{i=1}^{N} \lambda_i \eta_i e^{-a_i(T-t)} dt. \tag{4.10}$$

Further we note that

$$E_{t_0}[r(T_i)] = E_{t_0}[f(T_i, T_i)] = f(t_0, T_i) + C(t_0, T_i) + RW(t_0, T_i). \tag{4.11}$$

Finally, plugging (4.11) into (4.5), we get the following relationship to find the market price of risk matching the relationship (4.5) at tenor points $T_1, \dots, T_M$, given historical forward rate average, historical short rate average, short rate volatilities η_i, correlations $\rho_{i,j}$, and mean reversion rates a_i, $i = 1, \dots, N$,

$$f_{HistAvg}(t_0, T_j) - r_{HistAvg}(t_0) = -C(t_0, T_j) - \int_{t_0}^{T_j} \sum_{i=1}^{N} \lambda_i \eta_i e^{-a_i(T_j-t)} dt, \quad j = 1, \dots, M, \tag{4.12}$$

where $C(t_0, T)$ is the convexity adjustment term defined in (4.9). The result is analogous to that of [7].

4.4.1 Calculation of Historical Forward and Short Rate Average

The historical forward and short rate average is calculated from the input zero rate data

$$z_1, \ldots, z_N,$$

where z_i is the zero rate at the historical date t with the maturity $t + \tau_i$. To generate the historical time series for the instantaneous forward rates $f(t, t + \tau_i)$, from the historical zero rates, we interpolate the term structure of the zero rate to the union over all tenor pointsfrom all historical dates, and use the log-linear interpolation on the discount factor curve. Then, we get

$$f_1 := f(t, t + \tau_1) = z_1; \quad f_i := f(t, t + \tau_i) = \frac{\tau_i z_i - \tau_{i-1} z_{i-1}}{\tau_i - \tau_{i-1}}, \quad i = 2, .., M$$

Here, $\tau_1, \ldots, \tau_M$ is the combined set of all tenor points of the zero rates from all historical dates.

The historical short rate time series is defined as the time series of the forward rate corresponding to the smallest tenors among $\tau_1, \ldots, \tau_M$.

4.4.2 Single Factor Model Calibration

In the case of single factor model, the relationship (4.12) has the following form,

$$f_{HistAvg}(t_0, T_i) - r_{HistAvg}(t_0) = -C(t_0, T_i) - \int_{t_0}^{T_i} \lambda \eta e^{-a(T_i - t)} dt. \tag{4.13}$$

In this case, the market price of risk $\lambda(t)$ is calculated as the piece-wise constant function of time in such a way that the equation (4.13) is fulfilled on the set of forward rate tenors $T_1, \ldots, T_M$ specified by a user.

For $t_0 \le t < T_1$, we define:

$$\lambda_1 := \frac{r_{HistAvg}(t_0) - f_{HistAvg}(t_0, T_1) - C(t_0, T_1)}{\int_{t_0}^{T_1} \eta e^{-a(T_1 - t)} dt}.$$

Assume that we have calculated

$$\lambda(t) := \lambda_1, \ t \in [t_0, T_1), \quad \ldots, \quad \lambda(t) := \lambda_k, \ t \in [T_{k-1}, T_k).$$

Then, we define $\lambda(t) := \lambda_{k+1}, \ t \in [T_k, T_{k+1})$ as follows,

$$\lambda_{k+1} := \frac{r_{HistAvg}(t_0) - f_{HistAvg}(t_0, T_{k+1}) - C(t_0, T_{k+1}) - \int_{t_0}^{T_k} \lambda \eta e^{-a_i(T_{k+1} - t)} dt}{\int_{T_k}^{T_{k+1}} \eta e^{-a(T_{k+1} - t)} dt}.$$

4.4.3 Multi-Factor Model Calibration

In case of multi factor model, we assume that the market price of risk λ_i is constnt for each short rate factor x_i.

Let $T_1, \ldots, T_N$ be the benchmark forward rate tenor points corresponding to the N factors of the short rate. Using the relationship (4.12) at the benchmark tenor points, we can write out the following system of N linear equations to find $\lambda_1, \ldots, \lambda_N$,

$$f_{HistAvg}(t_0, T_i) - r_{HistAvg}(t_0) = -C(t_0, T_i) - \sum_{i=1}^{N} \lambda_i \int_{t_0}^{T_i} \eta_i e^{-a_i(T_i - t)} dt, \quad i = 1, \ldots, N. \tag{4.14}$$

Chapter 5

Multi-Currency Single Factor Hull-White Model

This chapter describes the implementation of a multi-currency extension of the single factor Hull-White (HW 1F) model.

5.1 Introduction

In the multi-currency extension of the HW 1F model (XCCY HW 1F), a single currency HW 1F model is used for the interest rates of each currency and the spot foreign exchange (FX) rates between currencies are modelled as lognormal random variables. For more information on the single currency HW 1F model, see Chapter 2 on page 11.

Consider a model with $n+1$ currencies: $\{CCY_0, CCY_1, \ldots, CCY_n\}$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_i, are modelled using a single currency HW 1F model:

$$dx_i(t) = -a_i(t)x_i(t)dt + \sigma_i(t)dW_{x_i}, \quad x_i(0) = 0 \tag{5.1}$$

$$r_i(t) = x_i(t) + \phi_i(t) \tag{5.2}$$

where, for each currency, W_{x_i} is a Wiener process, $a_i(t)$ is a mean reversion parameter, $\sigma_i(t)$ is the short rate volatility and $\phi_i(t)$ is a function chosen so that the initial discount factor curve is reproduced. The SDE for each currency is in the risk neutral measure for that currency.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \ldots, y_n(t)$ where $y_i(t)$ is the spot FX rate between CCY_i and the domestic currency CCY_0. The dynamics of the ith spot FX rate $y_i(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_i(t)}{y_i(t)} = (r_0(t) - r_i(t))\,dt + \nu_i(t)dW_{y_i} \tag{5.3}$$

where, $\nu_i(t)$ is the volatility for FX rate $y_i(t)$ and W_{y_i} is a Wiener process.

Thus, for a XCCY HW 1F model with $n+1$ currencies, there are $2n+1$ risk factors and $2n+1$ driving Wiener processes. We assume constant correlations between short rates and spot FX rates. We denote ρ_{r_i,r_j} to be the correlation between short rates for CCY_i and CCY_j, ρ_{r_i,y_j} to be the correlation between the short rate for CCY_i and spot FX rate j and ρ_{y_i,y_j} to be the correlation between spot FX rates i and j

5.2 Basic Formulas

This section presents some basic formulas related to the XCCY HW 1F Model.

5.2.1 Measure Relationships

The relationship between risk neutral measures in different currencies is given by:

$$dW_{x_i} = \rho_{r_i,r_j}\sigma_j(t)dt + dW_{x_j} \tag{5.4}$$

For measure relationships in the same currency, refer to the document describing the single currency HW 1F model.

5.2.2 System of Equations

Let us define:

$$\xi(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \\ z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}, \quad A = \left(\begin{array}{ccccc|c} -a_0 & & & & & \\ & -a_1 & & & & \\ & & -a_2 & & & 0 \\ & & & \ddots & & \\ & & & & -a_n & \\ 1 & -1 & & & & \\ 1 & & -1 & & & \\ \vdots & & & \ddots & & 0 \\ 1 & & & & -1 & \end{array}\right) \tag{5.5}$$

where, $z_i(t) = ln(y_i(t))$

$$V(t) = diag\ [\ \sigma_0(t), \dots, \sigma_n(t), \nu_1(t), \dots, \nu_n(t)\ ] \tag{5.6}$$

$$dW = \begin{pmatrix} dW_{x_0} \\ dW_{x_1} \\ dW_{x_2} \\ \vdots \\ dW_{x_n} \\ dW_{y_1} \\ dW_{y_2} \\ \vdots \\ dW_{y_n} \end{pmatrix}, \quad f_A(t) = \begin{pmatrix} 0 \\ -\rho_{r_1,y_1}\nu_1(t)\sigma_1(t) \\ -\rho_{r_2,y_2}\nu_2(t)\sigma_2(t) \\ \vdots \\ -\rho_{r_n,y_n}\nu_n(t)\sigma_n(t) \\ \phi_0(t) - \phi_1(t) - \frac{1}{2}\nu_1^2(t) \\ \phi_0(t) - \phi_2(t) - \frac{1}{2}\nu_2^2(t) \\ \vdots \\ \phi_0(t) - \phi_n(t) - \frac{1}{2}\nu_n^2(t) \end{pmatrix} \tag{5.7}$$

$$f_B(t) = \sigma_0(t)\frac{1-e^{-a_0(T-t)}}{a_0}\begin{pmatrix} \sigma_0(t) \\ \rho_{r_0,r_1}\sigma_1(t) \\ \rho_{r_0,r_2}\sigma_2(t) \\ \vdots \\ \rho_{r_0,r_n}\sigma_n(t) \\ \rho_{r_0,y_1}\nu_1(t) \\ \rho_{r_0,y_2}\nu_2(t) \\ \vdots \\ \rho_{r_0,y_n}\nu_n(t) \end{pmatrix}, \quad f(t) = f_A(t) - f_B(t) \tag{5.8}$$

Under the domestic T_N forward measure, the system of SDE's can be written as the vector equation:

$$d\xi(t) = [A\xi(t) + f(t)]\,dt + V(t)dW \tag{5.9}$$

5.3 Monte Carlo Simulation

The vector SDE in Eq.(5.9) admits an exact solution. Given $\xi(s)$, where $s < t$, the random variable $\xi(t)$ is normally distributed with mean $m(t)$ and covariance matrix $C(t)$ where m and C satisfy the ODE's:

$$\frac{dm}{dt} = Am + f(t) \tag{5.10}$$

$$\frac{dC}{dt} = AC + CA^T + V(t)RV(t)^T \tag{5.11}$$

Subject to initial conditions:

$$m(s) = \xi(s) \tag{5.12}$$

$$C(s) = 0 \tag{5.13}$$

where, R is the correlation matrix for the risk factor vector $\xi(t)$:

$$R = \begin{pmatrix} \rho_{r_0,r_0} & \cdots & \rho_{r_0,r_n} & \rho_{r_0,y_1} & \cdots & \rho_{r_0,y_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{r_n,r_0} & \cdots & \rho_{r_n,r_n} & \rho_{r_n,y_1} & \cdots & \rho_{r_n,y_n} \\ \rho_{y_1,r_0} & \cdots & \rho_{y_1,r_n} & \rho_{y_1,y_1} & \cdots & \rho_{y_1,y_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{y_n,r_0} & \cdots & \rho_{y_n,r_n} & \rho_{y_n,y_1} & \cdots & \rho_{y_n,y_n} \end{pmatrix} \tag{5.14}$$

Accordingly, we can simulate the evolution of ξ over any time step $[s,t]$ by first integrating the boundary value problem Eq.(5.10) - Eq.(5.13) over $[s,t]$ then drawing a $2n+1$ dimensional normal variable having mean $m(t)$ and covariance matrix $C(t)$.

In our implementation, we obtain values for $C(t)$ by numerically integrating the ODE for $C(t)$ Eq.(5.11) from time s to time t. Values for $m(t)$ are obtained using the following closed form solutions to the ODE for $m(t)$ Eq.(5.10).

The exact solutions for the x_i mean is given by:

$$m_{x_i}(t) = e^{-a_i(t-s)}\left[\int_s^t e^{a_i(u-s)} f_{x_i}(u)du + m_{x_i}(s)\right] \tag{5.15}$$

The exact solutions for the z_i mean is given by:

$$m_{z_i}(t) = \int_s^t (m_{x_0}(u) - m_{x_i}(u) + f_{z_i}(u))du + m_{z_i}(s) \tag{5.16}$$

Let us write:

$$f(t) = f_C(t) + e^{-a_0(T-t)} f_D(t) \tag{5.17}$$

where,

$$f_C(t) = f_A(t) - f_D(t), \quad f_D(t) = \frac{1}{a_0}\begin{pmatrix} \sigma_0(t) \\ \rho_{r_0,r_1}\sigma_0(t)\sigma_1(t) \\ \rho_{r_0,r_2}\sigma_0(t)\sigma_2(t) \\ \vdots \\ \rho_{r_0,r_n}\sigma_0(t)\sigma_n(t) \\ \rho_{r_0,y_1}\sigma_0(t)\nu_1(t) \\ \rho_{r_0,y_2}\sigma_0(t)\nu_2(t) \\ \vdots \\ \rho_{r_0,y_n}\sigma_0(t)\nu_n(t) \end{pmatrix} \tag{5.18}$$

For piece-wise constant volatilities between times s and t, the formulas for the means becomes:

$$\begin{aligned} m_{x_i}(t) &= \xi_{x_i}(s)e^{-a_i(t-s)} + -f_{C,x_i}(\hat{t})\frac{e^{-a_i(t-s)} - 1}{a_i} && (5.19) \\ &\quad + f_{D,x_i}(\hat{t})e^{-a_0(T-s)}\left(\frac{e^{a_0(t-s)} - e^{-a_i(t-s)}}{a_0 + a_i}\right) \\ m_{y_i}(t) &= I_{x_0}(s,t) - I_{x_i}(s,t) && (5.20) \\ &\quad + \int_s^t (\phi_0(u) - \phi_i(u))\, du - \left(\frac{1}{2}\nu_i^2(\hat{t}) + f_{D,y_i}(\hat{t})\right)(t-s) \\ &\quad + f_{D,y_i}(\hat{t})\frac{e^{-a_0(T-s)}}{a_0}\left(e^{a_0(t-s)} - 1\right) + \xi_{y_i}(s) \end{aligned}$$

where, $\hat{t} = \frac{1}{2}(t+s)$ and

$$\begin{aligned} I_{x_i}(s,t) &= \int_s^t m_{x_i}(u)du && (5.21) \\ &= \xi_{x_i}(s)\frac{e^{-a_i(t-s)}}{-a_i} + f_{C,x_i}(\hat{t})\frac{e^{-a_i(t-s)} - 1 + a_i(t-s)}{a_i^2} \\ &\quad + f_{D,x_i}(\hat{t})\frac{e^{-a_0(T-s)}}{a_0 + a_i}\left(\frac{e^{a_0(t-s)} - 1}{a_0} + \frac{e^{-a_i(t-s)} - 1}{a_i}\right) \end{aligned}$$

5.4 Finite Difference Pricing

We only give the PDE's for the 2 currency case as finite difference methods are no longer tractable for > 4 factors. Note, the PDE's are expressed in terms of the log transformed spot FX rate $z_1(t) = ln(y_1(t))$.

The pricing PDE for a two currency HW 1F model under the risk neutral measure is:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2(t)\frac{\partial^2 V}{\partial x_0^2} + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial x_1^2} + \frac{1}{2}\nu_1^2(t)\frac{\partial^2 V}{\partial z_1^2} \\
&+ \rho_{r_0,r_1}\sigma_0(t)\sigma_1(t)\frac{\partial^2 V}{\partial x_0 \partial x_1} + \rho_{r_0,z_1}\sigma_0(t)\nu_1(t)\frac{\partial^2 V}{\partial x_0 \partial z_1} \\
&+ \rho_{r_1,z_1}\sigma_1(t)\nu_1(t)\frac{\partial^2 V}{\partial x_1 \partial z_1} - a_0 x_0 \frac{\partial V}{\partial x_0} - a_1 x_1 \frac{\partial V}{\partial x_1} \\
&+ \left(\phi_0(t) - \phi_1(t) + x_0(t) - x_1(t) - \frac{1}{2}\nu_1^2(t)\right)\frac{\partial V}{\partial z_1} - (\phi_0(t) + x_0(t))V = 0
\end{aligned} \tag{5.22}$$

The x risk factor risk bounds are generated in the same way as for a single currency HW 1F model.

The z risk factor bounds under the risk neutral measure are:

$$z_{bounds} = \Big(D_z(0,t_{end}) - K^z_{scale}\sqrt{Var_z(0,t_{end})}, \tag{5.23}$$

$$D_z(0,t_{end}) + K^z_{scale}\sqrt{Var_z(0,t_{end})}\Big) \tag{5.24}$$

$$D_z(0,t_{end}) = \int_0^{t_{end}} \left(\phi_0(u) - \phi_1(u) - \frac{1}{2}|\nu_1(u)|^2\right) du \tag{5.25}$$

$$Var_z(0,t_{end}) = \int_0^{t_{end}} |\nu_1(u)|^2 du \tag{5.26}$$

where, t_{end} is the date of the last time step and K^z_{scale} is a scale factor (typically around 3) indicating how many standard deviations to include in the risk factor bounds.

Under the T forward measure, the pricing PDE is:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2(t)\frac{\partial^2 V}{\partial x_0^2} + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial x_1^2} + \frac{1}{2}\nu_1^2(t)\frac{\partial^2 V}{\partial z_1^2} \\
&+ \rho_{r_0,r_1}\sigma_0(t)\sigma_1(t)\frac{\partial^2 V}{\partial x_0 \partial x_1} + \rho_{r_0,z_1}\sigma_0(t)\nu_1(t)\frac{\partial^2 V}{\partial x_0 \partial z_1} \\
&+ \rho_{r_1,z_1}\sigma_1(t)\nu_1(t)\frac{\partial^2 V}{\partial x_1 \partial z_1} - \left(a_0 x_0 + B(a_0,t,T)\sigma_0^2(t)\right)\frac{\partial V}{\partial x_0} \\
&- \left(a_1 x_1 + \rho_{r_0,r_1}B(a_0,t,T)\sigma_0(t)\sigma_1(t) + \rho_{r_1,z_1}\sigma_1(t)\nu_1(t)\right)\frac{\partial V}{\partial x_1} \\
&+ \Big(\phi_0(t) - \phi_1(t) + x_0(t) - x_1(t) \\
&\quad - \rho_{r_0,z_1}B(a_0,t,T)\sigma_0(t)\nu_1(t) - \frac{1}{2}\nu_1^2(t)\Big)\frac{\partial V}{\partial z_1} = 0
\end{aligned} \tag{5.27}$$

where, $B(x,t,T) = \frac{1-e^{-x(T-t)}}{x}$.

The z risk factor bounds under the T forward measure are:

$$z^T_{bounds} = \Big(D_z(0,t_{end}) + \mu_z(0,t_{end},T) - K^z_{scale}\sqrt{Var_z(0,t_{end})}, \tag{5.28}$$

$$D_z(0,t_{end}) + \mu_z(0,t_{end},T) + K^z_{scale}\sqrt{Var_z(0,t_{end})}\Big) \tag{5.29}$$

$$\mu_z(0,t_{end}) = \int_0^{t_{end}} \rho_{r_0,z_1}B(a_0,u,T)\sigma_0(u)\nu_1(u)du \tag{5.30}$$

5.5 Calibration

The calibration of the interest rate components for each currency can be performed separately and independently (refer to the document describing single currency HW 1F model). The calibration of the spot FX rate volatilities can then be performed separately and independently using the interest rate calibration results.

We first provide a formula for FX options and then describe the FX calibration procedure.

5.5.1 FX Option Formula

The price of a European call/put at time t maturing at time T on the spot FX rate $y_i(t)$ is given by:

$$PV_{FXcall/put}(t, T, y_i(t), K) = \omega P(t, T)\Big(F(t)\Phi(\omega d_1) - K\Phi(\omega d_2)\Big) \tag{5.31}$$

where, K is the strike, $\Phi(x)$ is the cumulative normal distribution function and:

$$\begin{aligned}
F(t) &= \frac{P_i(t,T)}{P_0(t,T)} y_i(t), \quad \omega = \begin{cases} 1 & \text{if call} \\ -1 & \text{if put} \end{cases} \\
d_1 &= \frac{ln\left(\frac{F(t)}{K}\right) + \frac{1}{2}\vartheta(t,T,T)}{\sqrt{\vartheta(t,T,T)}}, \quad d_2 = d_1 - \sqrt{\vartheta(t,T,T)} \\
\vartheta(t, t^*, T) &= \int_t^{t^*} \Big(|\nu_i(u)|^2 + B^2(a_0, u, T)|\sigma_0(u)|^2 + B^2(a_i, u, T)|\sigma_i(u)|^2 \\
&\quad +2\rho_{r_0,z_i} B(a_0, u, T)\sigma_0(u) \cdot \nu_i(u) - 2\rho_{r_i,z_i} B(a_i, u, T)\sigma_i(u) \cdot \nu_i(u) \\
&\quad -2\rho_{r_0,r_i} B(a_0, u, T)B(a_i, u, T)\sigma_0(u) \cdot \sigma_i(u)\Big) du
\end{aligned} \tag{5.32}$$

5.5.2 FX Calibration

We describe a calibration method that exactly reproduces a set of market FX option implied volatilities. The calibration procedure is similar to that of the single currency HW 1F model.

For a currency pair, CCY_0 and CCY_i, assume we are given a set of n FX options and market implied volatilities, one per maturity date, and the calibration results for the single currency HW 1F model for the base currency CCY_0 and the foreign currency CCY_i. Note that the FX calibration for each currency pair can be done independently.

Let the set of option maturity dates $\mathcal{T}_{mat}$ and the corresponding market implied volatilities $\mathcal{M}$ be:

$$\mathcal{T}_{mat} = \{T_1, \ldots, T_n\}, \quad 0 < T_k < T_{k+1} \tag{5.33}$$

$$\mathcal{M} = \{M_1, \ldots, M_n\} \tag{5.34}$$

We assume that $\nu_i(t)$ is piece-wise constant function with values $\{\nu_i^1, \nu_i^2, \ldots, \nu_i^n\}$ in the intervals $\mathcal{I} = \{(0, T_1), (T_1, T_2), \ldots, (T_{n-1}, T_n)\}$ that correspond to the FX option maturity dates.

Starting with the first instrument maturing at T_1, we determine the value of ν_i^1 such that the market implied volatility M_1 is reproduced. This can be done using Eq.(5.32) by solving:

$$\vartheta(0, T_1, T_1) = (M_1)^2 T_1 \tag{5.35}$$

The calibration procedure then proceeds iteratively determining each ν_i^k so that M_k is reproduced by solving the equation:

$$\vartheta(0, T_{k-1}, T_k) + \vartheta(T_{k-1}, T_k, T_k) \quad = \quad (M_k)^2 T_k \tag{5.36}$$

The known constants $\nu_i^1, \nu_i^2, \ldots, \nu_i^{k-1}$ determine $\vartheta(0, T_{k-1}, T_k)$ while the unknown ν_i^k only enters the $\vartheta(T_{k-1}, T_k, T_k)$ term. The resulting equation is a quadratic in ν_i^k. If there are no real roots or two negative real roots, we omit that market implied volatility from the calibration procedure. Otherwise, we choose the larger of the two roots (which will be a positive value).

Chapter 6

Equity Hybrid Single Factor Hull-White Model

This chapter describes the implementation of an equity extension of the multi-currency single factor Hull-White (HW 1F) model.

6.1 Introduction

In the equity extension of the multi-currency single factor Hull-White model (Hybrid HW 1F), spot equity prices are modelled as lognormal random variables in a similar way as to how FX rates are modelled.

Consider a multi-currency HW 1F model (XCCY HW 1F) with $n+1$ currencies: $\{CCY_0, CCY_1, \ldots, CCY_n\}$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. Let there also be k equities $q_1(t), \ldots, q_k(t)$ where equity q_j is in currency CCY_{c_j} and c_j is one of the $n+1$ currencies. The dynamics of the j'th spot equity price $q_j(t)$ in the risk neutral measure of the equity currency is governed by:

$$\frac{dq_j(t)}{q_j(t)} = (r_{c_j}(t) - \delta_j(t))dt + \eta_j(t)dW_{q_j} \tag{6.1}$$

where, $r_{c_j}(t)$ is the short rate for currency CCY_{c_j}, $\delta_j(t)$ is a continuously compounded dividend rate, $\eta_j(t)$ is the volatility for equity price $q_j(t)$ and W_{q_j} is a Wiener process.

Also, assume constant correlations between short rates, spot FX rates and equities. We denote ρ_{r_i,x_j} to be the correlation between short rates for CCY_i and CCY_j, ρ_{r_i,y_j} to be the correlation between the short rate for CCY_i and spot FX rate j, ρ_{y_i,y_j} to be the correlation between spot FX rates i and j, ρ_{r_i,q_j} to be the correlation between the short rate for CCY_i and equity j, ρ_{y_i,q_j} to be the correlation between the spot FX rate i and equity j and ρ_{q_i,q_j} to be the correlation between equities i and j.

6.2 Basic Formulas

The formulas for the hybrid equity HW 1F model are similar to the formulas for the multi-currency HW 1F model.

6.2.1 System of Equations

Let us define:

$$\xi(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \\ z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \\ w_1(t) \\ w_2(t) \\ \vdots \\ w_k(t) \end{pmatrix}, \quad A = \left(\begin{array}{c|c} A_X & 0 \\ A_Z & 0 \\ A_W & 0 \end{array}\right) \tag{6.2}$$

where, $r_i(t) = \phi_i(t) + x_i(t)$ are the short rates for each currency, $z_i(t) = ln(y_i(t))$ are the logs of the spot FX rates $y_i(t)$ and $w_i(t) = ln(q_j(t))$ are the logs of the spot equity prices $q_j(t)$.
A_X is a $(n+1) \times (n+1)$ matrix such that:

$$A_X = \begin{pmatrix} -a_0 & & & & \\ & -a_1 & & & \\ & & -a_2 & & \\ & & & \ddots & \\ & & & & -a_n \end{pmatrix} \tag{6.3}$$

A_Z is a $(n+1) \times n$ matrix such that:

$$A_Z = \begin{pmatrix} 1 & -1 & & \\ 1 & & -1 & \\ \vdots & & & \ddots \\ 1 & & & & -1 \end{pmatrix} \tag{6.4}$$

A_W is a $(n+1) \times k$ matrix where $A_{c_j,j} = 1$.

$$V(t) = diag\ [\ \sigma_0(t), \ldots, \sigma_n(t), \nu_1(t), \ldots, \nu_n(t), \eta_1(t), \ldots, \eta_k(t)\ ] \tag{6.5}$$

$$dW = \begin{pmatrix} dW_{x_0} \\ dW_{x_1} \\ dW_{x_2} \\ \vdots \\ dW_{x_n} \\ dW_{y_1} \\ dW_{y_2} \\ \vdots \\ dW_{y_n} \\ dW_{q_1} \\ dW_{q_2} \\ \vdots \\ dW_{q_k} \end{pmatrix}, \quad f_A(t) = \begin{pmatrix} 0 \\ -\rho_{r_1,y_1}\nu_1(t)\sigma_1(t) \\ -\rho_{r_2,y_2}\nu_2(t)\sigma_2(t) \\ \vdots \\ -\rho_{r_n,y_n}\nu_n(t)\sigma_n(t) \\ \phi_0(t) - \phi_1(t) - \frac{1}{2}\nu_1^2(t) \\ \phi_0(t) - \phi_2(t) - \frac{1}{2}\nu_2^2(t) \\ \vdots \\ \phi_0(t) - \phi_n(t) - \frac{1}{2}\nu_n^2(t) \\ \phi_{c_1}(t) - \delta_1(t) - \frac{1}{2}\eta_1^2(t) - \zeta_{q_1}(t) \\ \phi_{c_2}(t) - \delta_2(t) - \frac{1}{2}\eta_2^2(t) - \zeta_{q_2}(t) \\ \vdots \\ \phi_{c_k}(t) - \delta_k(t) - \frac{1}{2}\eta_n^2(t) - \zeta_{q_k}(t) \end{pmatrix} \tag{6.6}$$

$$\zeta_{q_j}(t) = \begin{cases} \rho_{q_j,y_{c_j}}\nu_{c_j}(t)\eta_j(t) & , \quad c_j \neq 0 \\ 0 & , \quad c_j = 0 \end{cases} \tag{6.7}$$

$$f_B(t) = \sigma_0(t)\frac{1-e^{-a_0(T-t)}}{a_0}\begin{pmatrix} \sigma_0(t) \\ \rho_{r_0,r_1}\sigma_1(t) \\ \rho_{r_0,r_2}\sigma_2(t) \\ \vdots \\ \rho_{r_0,r_n}\sigma_n(t) \\ \rho_{r_0,y_1}\nu_1(t) \\ \rho_{r_0,y_2}\nu_2(t) \\ \vdots \\ \rho_{r_0,y_n}\nu_n(t) \\ \rho_{r_0,q_1}\eta_1(t) \\ \rho_{r_0,q_2}\eta_2(t) \\ \vdots \\ \rho_{r_0,q_k}\eta_k(t) \end{pmatrix}, \quad f(t) = f_A(t) - f_B(t) \tag{6.8}$$

Under the domestic T_N forward measure, the system of SDE's can be written as the vector equation:

$$d\xi(t) = [A\xi(t) + f(t)]\,dt + V(t)dW \tag{6.9}$$

6.3 Monte Carlo Simulation

As with the multi-currency case, the vector SDE in Eq.(6.9) admits an exact solution. Given $\xi(s)$, where $s < t$, the random variable $\xi(t)$ is normally distributed with mean $m(t)$ and covariance matrix $C(t)$ where m and C satisfy the ODE's:

$$\frac{dm}{dt} = Am + f(t) \tag{6.10}$$

$$\frac{dC}{dt} = AC + CA^T + V(t)RV(t)^T \tag{6.11}$$

Subject to initial conditions:

$$m(s) = \xi(s) \tag{6.12}$$

$$C(s) = 0 \tag{6.13}$$

where, R is the correlation matrix for the risk factor vector $\xi(t)$:

$$R = \begin{pmatrix} \rho_{r_0,r_0} & \cdots & \rho_{r_0,r_n} & \rho_{r_0,y_1} & \cdots & \rho_{r_0,y_n} & \rho_{r_0,q_1} & \cdots & \rho_{r_0,q_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{r_n,r_0} & \cdots & \rho_{r_n,r_n} & \rho_{r_n,y_1} & \cdots & \rho_{r_n,y_n} & \rho_{r_n,q_1} & \cdots & \rho_{r_n,q_k} \\ \rho_{y_1,r_0} & \cdots & \rho_{y_1,r_n} & \rho_{y_1,y_1} & \cdots & \rho_{y_1,y_n} & \rho_{y_1,q_1} & \cdots & \rho_{y_1,q_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{y_n,r_0} & \cdots & \rho_{y_n,r_n} & \rho_{y_n,y_1} & \cdots & \rho_{y_n,y_n} & \rho_{y_n,q_1} & \cdots & \rho_{y_n,q_k} \\ \rho_{q_1,r_0} & \cdots & \rho_{q_1,r_n} & \rho_{q_1,y_1} & \cdots & \rho_{q_1,y_n} & \rho_{q_1,q_1} & \cdots & \rho_{q_1,q_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{q_k,r_0} & \cdots & \rho_{q_k,r_n} & \rho_{q_k,y_1} & \cdots & \rho_{q_k,y_n} & \rho_{q_k,q_1} & \cdots & \rho_{q_k,q_k} \end{pmatrix} \tag{6.14}$$

The exact solutions for the short rate and FX risk factor means $(m_{x_i}(t), m_{z_i}(t))$ presented in the multi-currency HW 1F model document (see Chapter 5 on page 33) still apply to the hybrid equity LFM 1F model.

The exact solution for the equity risk factor $w_j(t)$ mean is given by:

$$m_{w_j}(t) = \int_s^t (m_{x_{c_j}}(u) + f_{w_j}(u))du + m_{w_j}(s) \tag{6.15}$$

Let us write:

$$f(t) = f_C(t) + e^{-a_0(T-t)} f_D(t) \tag{6.16}$$

where,

$$f_C(t) = f_A(t) - f_D(t), \quad f_D(t) = \frac{1}{a_0} \begin{pmatrix} \sigma_0(t) \\ \rho_{r_0,r_1}\sigma_0(t)\sigma_1(t) \\ \rho_{r_0,r_2}\sigma_0(t)\sigma_2(t) \\ \vdots \\ \rho_{r_0,r_n}\sigma_0(t)\sigma_n(t) \\ \rho_{r_0,y_1}\sigma_0(t)\nu_1(t) \\ \rho_{r_0,y_2}\sigma_0(t)\nu_2(t) \\ \vdots \\ \rho_{r_0,y_n}\sigma_0(t)\nu_n(t) \\ \rho_{r_0,q_1}\sigma_0(t)\eta_1(t) \\ \rho_{r_0,q_2}\sigma_0(t)\eta_2(t) \\ \vdots \\ \rho_{r_0,q_k}\sigma_0(t)\eta_k(t) \end{pmatrix} \tag{6.17}$$

For piece-wise constant volatilities between times s and t, the formula for the equity risk factor mean becomes:

$$\begin{aligned} m_{w_j}(t) &= I_{x_{c_j}}(s,t) + \int_s^t \phi_{c_j}(u)du - \int_s^t \delta_j(u)du \\ &\quad - \left(\frac{1}{2}\eta_j^2(\hat{t}) + \zeta_{q_j}(\hat{t}) + f_{D,w_j}(\hat{t})\right)(t-s) \\ &\quad + f_{D,w_j}(\hat{t})\frac{e^{-a_0(T-s)}}{a_0}\left(e^{a_0(t-s)} - 1\right) + m_{w_j}(s) \end{aligned} \tag{6.18}$$

where, $\hat{t} = \frac{1}{2}(t+s)$ and

$$\begin{aligned} I_{x_i}(s,t) &= \int_s^t m_{x_i}(u)du \\ &= m_{x_i}(s)\frac{e^{-a_i(t-s)}}{-a_i} + f_{C,x_i}(\hat{t})\frac{e^{-a_i(t-s)} - 1 + a_i(t-s)}{a_i^2} \\ &\quad + f_{D,x_i}(\hat{t})\frac{e^{-a_0(T-s)}}{a_0+a_i}\left(\frac{e^{a_0(t-s)} - 1}{a_0} + \frac{e^{-a_i(t-s)} - 1}{a_i}\right) \end{aligned} \tag{6.19}$$

6.4 Finite Difference Pricing

We only give the PDE's for single currency case and the 2 currency case 1 equity case as finite difference methods are no longer tractable for > 4 factors. Note, the PDE's are expressed in terms of the log transformed spot FX rate ($z_1(t) = ln(y_1(t))$) and equity prices ($w_j(t) = ln(q_j(t))$).

The pricing PDE for the two currency single domestic equity case under the domestic T forward measure is:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2(t)\frac{\partial^2 V}{\partial x_0^2} + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial x_1^2} + \frac{1}{2}\nu_1^2(t)\frac{\partial^2 V}{\partial z_1^2} + \frac{1}{2}\eta_1^2(t)\frac{\partial^2 V}{\partial w_1^2} \\
&+ \rho_{r_0,r_1}\sigma_0(t)\sigma_1(t)\frac{\partial^2 V}{\partial x_0 \partial x_1} + \rho_{r_0,z_1}\sigma_0(t)\nu_1(t)\frac{\partial^2 V}{\partial x_0 \partial z_1} \\
&+ \rho_{r_1,z_1}\sigma_1(t)\nu_1(t)\frac{\partial^2 V}{\partial x_1 \partial z_1} - \left(a_0 x_0 + B(a_0,t,T)\sigma_0^2(t)\right)\frac{\partial V}{\partial x_0} \\
&- \Big(a_1 x_1 + \rho_{r_0,r_1}B(a_0,t,T)\sigma_0(t)\sigma_1(t)\Big)\frac{\partial V}{\partial x_1} + \Big(\phi_0(t) - \phi_1(t) \\
&\qquad + x_0(t) - x_1(t) + \rho_{r_0,z_1}B(a_0,t,T)\sigma_0(t)\nu_1(t) - \frac{1}{2}\nu_1^2(t)\Big)\frac{\partial V}{\partial z_1} \\
&+ \rho_{r_0,w_1}\sigma_0(t)\eta_1(t)\frac{\partial^2 V}{\partial x_0 \partial w_1} + \rho_{r_1,w_1}\sigma_1(t)\eta_1(t)\frac{\partial^2 V}{\partial x_1 \partial w_1} \\
&+ \rho_{y_1,w_1}\nu_1(t)\eta_1(t)\frac{\partial^2 V}{\partial z_1 \partial w_1} + \Big(\phi_0(t) + x_0(t) - \delta_1(t) - \frac{1}{2}\eta_1^2(t) \\
&\qquad\qquad - \rho_{r_0,w_1}B(a_0,t,T)\sigma_0(t)\eta_1(t)\Big)\frac{\partial V}{\partial w_1} = 0
\end{aligned} \tag{6.20}$$

The pricing PDE for the two currency single foreign equity case under the domestic T forward measure is:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2(t)\frac{\partial^2 V}{\partial x_0^2} + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial x_1^2} + \frac{1}{2}\nu_1^2(t)\frac{\partial^2 V}{\partial z_1^2} + \frac{1}{2}\eta_1^2(t)\frac{\partial^2 V}{\partial w_1^2} \\
&+ \rho_{r_0,r_1}\sigma_0(t)\sigma_1(t)\frac{\partial^2 V}{\partial x_0 \partial x_1} + \rho_{r_0,z_1}\sigma_0(t)\nu_1(t)\frac{\partial^2 V}{\partial x_0 \partial z_1} \\
&+ \rho_{r_1,z_1}\sigma_1(t)\nu_1(t)\frac{\partial^2 V}{\partial x_1 \partial z_1} - \left(a_0 x_0 + B(a_0,t,T)\sigma_0^2(t)\right)\frac{\partial V}{\partial x_0} \\
&- \Big(a_1 x_1 + \rho_{r_0,r_1}B(a_0,t,T)\sigma_0(t)\sigma_1(t)\Big)\frac{\partial V}{\partial x_1} + \Big(\phi_0(t) - \phi_1(t) \\
&\qquad + x_0(t) - x_1(t) + \rho_{r_0,z_1}B(a_0,t,T)\sigma_0(t)\nu_1(t) - \frac{1}{2}\nu_1^2(t)\Big)\frac{\partial V}{\partial z_1} \\
&+ \rho_{r_0,w_1}\sigma_0(t)\eta_1(t)\frac{\partial^2 V}{\partial x_0 \partial w_1} + \rho_{r_1,w_1}\sigma_1(t)\eta_1(t)\frac{\partial^2 V}{\partial x_1 \partial w_1} \\
&+ \rho_{y_1,w_1}\nu_1(t)\eta_1(t)\frac{\partial^2 V}{\partial z_1 \partial w_1} + \Big(\phi_1(t) + x_1(t) - \delta_1(t) - \frac{1}{2}\eta_1^2(t) \\
&\qquad - \rho_{z_1,w_1}\nu_1(t)\eta_1(t) - \rho_{r_0,w_1}B(a_0,t,T)\sigma_0(t)\eta_1(t)\Big)\frac{\partial V}{\partial w_1} = 0
\end{aligned} \tag{6.21}$$

The pricing PDE for the single currency case under the T forward measure is:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2(t)\frac{\partial^2 V}{\partial x_0^2} - \left(a_0 x_0 + B(a_0,t,T)\sigma_0^2(t)\right)\frac{\partial V}{\partial x_0} \\
&+ \sum_{j=1}^{k}\Big(\phi_0(t) + x_0(t) - \rho_{r_0,w_j}B(a_0,t,T)\sigma_0(t)\eta_j(t) - \delta_j(t) - \frac{1}{2}\eta_j^2(t)\Big)\frac{\partial V}{\partial w_j} \\
&+ \frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{k}\rho_{w_i,w_j}\eta_i(t)\eta_j(t)\frac{\partial^2 V}{\partial w_j \partial w_j} + \sum_{j=1}^{k}\rho_{r_0,w_j}\sigma_0(t)\eta_j(t)\frac{\partial^2 V}{\partial x_0 \partial w_j} = 0
\end{aligned} \tag{6.22}$$

The risk factor risk bounds are generated in a similar manner as for the multi-currency HW 1F model.

6.5 Calibration

The calibration of the equity components is similar to the procedure used to calibrate FX options in the XCCY HW 1F model. The calibration of the equity volatilities can then be performed separately and independently using the interest rate and FX calibration results.

We first describe how we model discrete dividend payments. We then provide a formula for equity options and describe the equity calibration procedure.

6.5.1 Modeling Dividends

The equity hybrid model we have presented allows for a continuous dividend rate as an input. Discrete dividend payments can be approximated using a spikey dividend rate.

Suppose we have an equity q_j in currency c_j with a set of discrete dividend payments $D_1, D_2, \ldots, D_n$ ocurring at times $T_1, T_2, \ldots, T_n$. This stream of dividend payments can be modelled using a piece-wise constant continuous dividend rate with intervals $[0, T_1 - \Delta], [T_1 - \Delta, T_1], [T_1, T_2 - \Delta], [T_2 - \Delta, T_2], \ldots, [T_n - \Delta, T_n]$ where Δ is some small time interval (1 day). The continuous dividend rate $\delta(t)$ takes constant values δ_i when t is within $[T_i - \Delta, T_i]$ and is zero otherwise.

The present value of the change in the stock (equity) over the course of each individual dividend payment period $[T_i - \Delta, T_i]$ must equal the PV of the corresponding dividend; i.e.

$$\mathbb{E}_0^T\left[\frac{S'(T_i-\Delta)-S'(T_i)}{P_0(T_i-\Delta,T)}\right]=\frac{PV_0(\text{dividend paid at }T_i)}{P_0(0,T)} \tag{6.23}$$

where, the expection $\mathbb{E}_0^T$ is in the domestic T forward measure at time 0, $S'(t) = q_j(t)y_{c_j}(t)$ is the currency translated equity value and $P_i(t,T)$ is the value of a zero coupon bond in currency i observed at time t maturing at T.

This condition ensures that the forward prices of the stock, paying the actual discrete dividend stream, is equal to the forward price of a fictitious stock paying the continuous dividend stream.

The RHS of Eq.(6.23) is easy to evaluate:

$$\frac{PV_0(\text{dividend paid at }T_i)}{P_0(0,T)}=y_{c_j}(0)D_i\frac{P_{c_j}(0,T_i)}{P_0(0,T)} \tag{6.24}$$

The LHS of Eq.(6.23) can be show to be:

$$\begin{aligned}\mathbb{E}_0^T\left[\frac{S'(T_i-\Delta)-S'(T_i)}{P_0(T_i-\Delta,T)}\right] &\approx e^{\int_{T_i-\Delta}^{T}\delta(s)ds}[1-e^{-\delta_i\Delta}]\mathbb{E}_0^T\left[\frac{V(T_i-\Delta)}{P_0(T_i-\Delta,T)}\right]\\ &= e^{-\int_0^{T_i-\Delta}\delta(s)ds}[1-e^{-\delta_i\Delta}]\frac{q_j(0)y_{c_j}(0)}{P_0(0,T)}\end{aligned} \tag{6.25}$$

where,

$$V(t)=e^{-\int_t^T\delta(s)ds}S'(t) \tag{6.26}$$

Equating the LHS and the RHS, we have an iterative procedure for calculating the dividend rate heights δ_i from a sequence of discrete dividend dates T_i and amounts D_i.

6.5.2 Equity Option Formula

The price of a European call/put at time t maturing at time T on the equity $q_j(t)$ in equity currency CCY_{c_j} is given by:

$$PV_{Equity\ call/put}(t,T,q_j(t),K)=\omega P_{c_j}(t,T)\Big(F(t)\Phi(\omega d_1)-K\Phi(\omega d_2)\Big) \tag{6.27}$$

where, K is the strike, $\Phi(x)$ is the cumulative normal distribution function and:

$$
\begin{aligned}
F(t) &= \frac{e^{-\int_t^T \delta_j(s)ds}}{P_{c_j}(t,T)} q_j(t), \quad \omega = \begin{cases} 1 & \text{if call} \\ -1 & \text{if put} \end{cases} \\
d_1 &= \frac{\ln\left(\frac{F(t)}{K}\right) + \frac{1}{2}\vartheta(t,T,T)}{\sqrt{\vartheta(t,T,T)}}, \quad d_2 = d_1 - \sqrt{\vartheta(t,T,T)} \\
\vartheta(t,t^*,T) &= \int_t^{t^*} \Big(|\eta_j(u)|^2 + B^2(a_{c_j},u,T)|\sigma_{c_j}(u)|^2 \\
&\qquad + 2\rho_{r_{c_j},q_j} B(a_{c_j},u,T)\sigma_{c_j}(u)\cdot\eta_j(u)\Big) du
\end{aligned} \tag{6.28}
$$

6.5.3 Equity Calibration

The equity calibration procedure is similar to the FX calibration procedure where Eq.(6.28) is now used instead of corresponding FX equation.

Chapter 7

Hybrid Equity Multi-Currency Multi-Factor Hull-White Interest Rate Model

This chapter describes issues relating to the implementation of a hybrid equity multi-currency multi-factor Hull-White interest rate model.

7.1 Introduction

In the hybrid equity multi-currency extension of the multi-factor HW NF model (HW NF), a single currency HW NF model is used for the interest rates of each currency and the spot foreign exchange (FX) rates between currencies and spot equity prices are modelled as lognormal random variables. For more information on the single currency multi-factor HW NF model, see Chapter 3 on page 19.

Consider a model with $n+1$ currencies: $\{CCY_0, CCY_1, \ldots, CCY_n\}$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_i, are modelled using a multi-factor single currency HW NF model:

$$dx_{k,i}(t) = -a_{k,i}(t)x_{k,i}(t)dt + \eta_{k,i}(t)dW^{(k)}_{x_{k,i}}, \quad x_{k,i}(0) = 0 \tag{7.1}$$

$$r_k(t) = \phi_k(t) + \sum_{i=1}^{N_k} x_{k,i}(t), \quad dW^{(k)}_{x_{k,i}} dW^{(k)}_{x_{k,j}} = \rho_{x_{k,i},x_{k,j}}(t)dt \tag{7.2}$$

where, for each currency k, $W^{(k)}_{k,i}$ are Wiener processes in the risk neutral measure for that currency k, $a_{k,i}(t)$ are mean reversion parameters, $\eta_{k,i}(t)$ are volatility parameters and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. In what follows, we will assume time independent mean reversion parameters $a_{k,i}(t) = a_{k,i}$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \ldots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_k(t)}{y_i(t)} = (r_0(t) - r_k(t))\, dt + \eta^{FX}_k(t) dW^{(0)}_{y_k} \tag{7.3}$$

where, $\eta^{FX}_k(t)$ is the volatility for FX rate $y_k(t)$ and $W^{(0)}_{y_k}$ is a Wiener process in the risk neutral measure for that domestic currency 0.

Let there also be m equities $s_0(t), \ldots, s_{m-1}(t)$ where equity s_j is in currency CCY_{c_j} and c_j is one of the $n+1$ currencies. The dynamics of the j'th spot equity price $s_j(t)$ in the risk neutral measure of the equity currency

CCY_{c_j} s governed by:

$$\frac{ds_j(t)}{s_j(t)} = (r_{c_j}(t) - \delta_j(t))dt + \eta_j^{EQ}(t)dW_{s_j}^{(c_j)} \tag{7.4}$$

where, $r_{c_j}(t)$ is the short rate for currency CCY_{c_j}, $\delta_j(t)$ is a continuously compounded dividend rate, $\eta_j^{EQ}(t)$ is the volatility for equity price $s_j(t)$ and $W_{s_j}^{(c_j)}$ is a Wiener process in the risk neutral measure for the equity currency c_j.

Thus, for a hybrid HW NF model with $n+1$ currencies and m equities, there are $N_{All} = N_{IR} + n + m$ risk factors and driving Wiener processes where $N_{IR} = \sum_{k=0}^{n} N_k$ is the number of all interest rate risk factors and N_k is the number of factors for currency k. We denote $\rho_{X,Y}$ to be the correlation between risk factors X and Y, For example, $\rho_{x_{k,i},y_l}$ is the correlation between the ith IR risk factor for CCY_k and spot FX rate l.

7.2 Basic Formulas

This section presents some basic formulas related to the Hybrid HW NF Model.

7.2.1 Measure Relationships

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 is given by:

$$dW_{x_{k,i}}^{(0)} = \rho_{x_{k,i},y_k}\eta_k^{FX}(t)dt + dW_{x_{k,i}}^{(k)} \tag{7.5}$$

The relationship between the risk neutral measure and the T forward measure in the same currency is given by:

$$dW_{x_{k,i}}^{(k)} = \sum_{j=0}^{N_k-1} \rho_{x_{k,i},x_{k,j}} B(a_{k,j}, t, T)\eta_{k,j}(t)dt + dW_{x_{k,i}}^{T,(k)} \tag{7.6}$$

where,

$$B(x,t,T) = \frac{1 - e^{-x(T-t)}}{x} \tag{7.7}$$

7.2.2 System of Equations

Under the domestic currency CCY_0 T forward measure, the SDE's become:

$$
\begin{aligned}
dx_{0,i} &= \Big[-a_{0,i}x_{0,i} - \sum_{j=0}^{N_0-1} \rho_{x_{0,i},x_{0,j}}\eta_{0,i}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_{0,i}dW^{T,(0)}_{x_{0,i}} && (7.8)\\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \sum_{j=0}^{N_0-1} \rho_{x_{k,i},x_{0,j}}\eta_{k,i}\eta_{0,j}B(a_{0,j},t,T) \\
&\qquad\qquad - \rho_{x_{k,i},y_k}\eta_{k,i}\eta_k^{FX}\Big]dt + \eta_{k,i}dW^{T,(0)}_{x_{k,i}} && (7.9)\\
\frac{dy_k}{y} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=0}^{N_0-1} x_{0,i} - \sum_{i=0}^{N_k-1} x_{k,i} \\
&\quad - \sum_{j=0}^{N_0-1} \rho_{y_k,x_{0,j}}\eta_k^{FX}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_k^{FX}dW^{T,(0)}_{y_k} && (7.10)\\
\frac{ds_k}{s} &= \Big[\phi_{c_k}(t) + \sum_{j=0}^{N_{c_k}-1} x_{c_k,j} - \delta_k(t) - \mathbf{1}_{[c_k\neq 0]}\rho_{s_k,y_{c_k}}\eta_k^{EQ}\eta_{c_k}^{FX} \\
&\quad - \sum_{j=0}^{N_0-1} \rho_{s_k,x_{0,j}}\eta_k^{EQ}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_k^{EQ}dW^{T,(0)}_{s_k} && (7.11)
\end{aligned}
$$

Now define $z_i(t) = ln(y_i(t))$ and $q_i(t) = ln(s_i(t))$. The SDE's can be written as the system of equations:

$$d\xi(t) = [A\xi(t) + f(t)]\,dt + V(t)dW \qquad (7.12)$$

$$\xi(t) = \begin{pmatrix} x_{0,0}(t) \\ \vdots \\ x_{0,N_0-1}(t) \\ x_{1,0}(t) \\ \vdots \\ x_{1,N_1-1}(t) \\ \vdots \\ x_{n,0}(t) \\ \vdots \\ x_{n,N_n-1}(t) \\ z_1(t) \\ \vdots \\ z_n(t) \\ q_1(t) \\ \vdots \\ q_m(t) \end{pmatrix}, \quad A = \left(\begin{array}{c|c} A_X & 0 \\ A_Z & 0 \\ A_W & 0 \end{array}\right) \tag{7.13}$$

where, A is a $N_{All} \times N_{All}$ matrix and A_X is a $N_{IR} \times N_{IR}$ matrix such that:

$$A_X = \begin{pmatrix} A_X^{(0)} & & & & 0 \\ & A_X^{(1)} & & & \\ & & A_X^{(2)} & & \\ & & & \ddots & \\ 0 & & & & A_X^{(n)} \end{pmatrix} \tag{7.14}$$

and each $A_X^{(k)}$ is a $N_k \times N_k$ matrix of mean reversions such that:

$$A_X^{(k)} = \begin{pmatrix} -a_{k,0} & & & & 0 \\ & -a_{k,1} & & & \\ & & -a_{k,2} & & \\ & & & \ddots & \\ 0 & & & & -a_{k,N_k-1} \end{pmatrix} \tag{7.15}$$

A_Z is a $N_{IR} \times n$ matrix such that $A_{Z,i,j} = 1$ if $0 \le i < N_0$ and $A_{Z,i,j} = -1$ if $0 \le i - \sum_{k=0}^{j-1} N_k < N_j$, $A_{Z,i,j} = 0$ otherwise. Also, A_W is a $N_{IR} \times m$ matrix where $A_{W,i,j} = 1$ if $0 \le i - \sum_{k=0}^{c_j-1} N_k < N_{c_j}$, $A_{W,i,j} = 0$ otherwise. $V(t)$ is the diagonal matrix:

$$\begin{aligned} V(t) \quad = \quad diag\Big[\, & \eta_{0,0}(t), \ldots, \eta_{0,N_0-1}(t), \eta_{1,0}(t), \ldots, \eta_{1,N_1-1}(t), \ldots, \eta_{n,0}(t), \ldots \\ & \ldots, \eta_{n,N_n-1}(t), \eta_1^{FX}(t), \ldots, \eta_k^{FX}(t), \eta_0^{EQ}(t), \ldots, \eta_{m-1}^{EQ}(t) \,\Big] \end{aligned} \tag{7.16}$$

$$dW = \begin{pmatrix} dW_{x_{0,0}}^{T,(0)} \\ \vdots \\ dW_{x_{0,N_0-1}}^{T,(0)} \\ dW_{x_{1,0}}^{T,(0)} \\ \vdots \\ dW_{x_{1,N_1-1}}^{T,(0)} \\ \vdots \\ dW_{x_{n,0}}^{T,(0)} \\ \vdots \\ dW_{x_{n,N_n-1}}^{T,(0)} \\ dW_{y_1}^{T,(0)} \\ \vdots \\ dW_{y_n}^{T,(0)} \\ dW_{s_0}^{T,(0)} \\ \vdots \\ dW_{s_{m-1}}^{T,(0)} \end{pmatrix}, f_A(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\rho_{x_{1,0},y_1}\eta_{1,0}(t)\eta_1^{FX}(t) \\ \vdots \\ -\rho_{x_{1,N_1-1},y_1}\eta_{1,N_1-1}(t)\eta_1^{FX}(t) \\ \vdots \\ -\rho_{x_{n,0},y_1}\eta_{n,0}(t)\eta_n^{FX}(t) \\ \vdots \\ -\rho_{x_{n,N_n-1},y_n}\eta_{n,N_n-1}(t)\eta_n^{FX}(t) \\ \phi_0(t) - \phi_1(t) - \frac{1}{2}(\eta_1^{FX})^2(t) \\ \vdots \\ \phi_0(t) - \phi_n(t) - \frac{1}{2}(\eta_n^{FX})^2(t) \\ \phi_{c_0}(t) - \frac{1}{2}(\eta_0^{EQ})^2 - \zeta_0(t) \\ \vdots \\ \phi_{c_{m-1}}(t) - \frac{1}{2}(\eta_{m-1}^{EQ})^2 - \zeta_{m-1}(t) \end{pmatrix} \tag{7.17}$$

$$\zeta_j(t) = \delta_j(t) + \begin{cases} \rho_{s_j,y_{c_j}}\eta_{c_j}^{FX}(t)\eta_j^{EQ}(t) & , \ c_j \neq 0 \\ 0 & , \ c_j = 0 \end{cases}, \quad f(t) = f_A(t) - f_B(t) \tag{7.18}$$

$$f_B(t) = \sum_{k=0}^{N_0-1} B(a_{0,k}, t, T)\eta_{0,k}(t) \begin{pmatrix} \rho_{x_{0,k},x_{0,0}}\eta_{0,0}(t) \\ \vdots \\ \rho_{x_{0,k},x_{0,N_0-1}}\eta_{0,N_0-1}(t) \\ \rho_{x_{0,k},x_{1,0}}\eta_{1,0}(t) \\ \vdots \\ \rho_{x_{0,k},x_{1,N_1-1}}\eta_{1,N_1-1}(t) \\ \vdots \\ \rho_{x_{0,k},x_{n,0}}\eta_{n,0}(t) \\ \vdots \\ \rho_{x_{0,k},x_{n,N_n-1}}\eta_{n,N_n-1}(t) \\ \rho_{x_{0,k},y_1}\eta_1^{FX}(t) \\ \vdots \\ \rho_{x_{0,k},y_n}\eta_n^{FX}(t) \\ \rho_{x_{0,k},s_0}\eta_0^{EQ}(t) \\ \vdots \\ \rho_{x_{0,k},s_m}\eta_m^{EQ}(t) \end{pmatrix} \tag{7.19}$$

7.3 Monte Carlo Simulation

The methods used to simulate hybrid equity multi-currency single factor Hull-White models can also be applied to the multi-factor case. For more details, see Chapter 5 on page 33 and Chapter 6 on page 41.

7.4 Calibration

We begin by presenting formulas for Vanilla FX options and Equity options in the Hybrid multi-factor Hull-white model. We then discuss procedures for calibrating the model.

7.4.1 FX Option Formula

The price of a European call/put at time t maturing at time T on the spot FX rate $y_i(t)$ is given by:

$$PV_{FXcall/put}(t, T, y_i(t), K) = \omega P(t, T)\Big(F(t)\Phi(\omega d_1) - K\Phi(\omega d_2)\Big) \tag{7.20}$$

where, K is the strike, $\Phi(x)$ is the cumulative normal distribution function and:

$$
\begin{aligned}
F(t) &= \frac{P_i(t,T)}{P_0(t,T)} y_i(t), \quad \omega = \begin{cases} 1 & \text{if call} \\ -1 & \text{if put} \end{cases} \\
d_1 &= \frac{ln\left(\frac{F(t)}{K}\right) + \frac{1}{2}\vartheta(t,T,T)}{\sqrt{\vartheta(t,T,T)}}, \quad d_2 = d_1 - \sqrt{\vartheta(t,T,T)} \\
\vartheta(t,t^*,T) &= \int_t^{t^*} \Big(|\eta_k^{FX}(u)|^2 \\
&+ \sum_{i=0}^{N_0-1} \sum_{j=0}^{N_0-1} \rho_{x_{0,i},x_{0,j}} \eta_{0,i}(u)\eta_{0,j}(u) B(a_{0,i},u,T) B(a_{0,j},u,T) \\
&+ \sum_{i=0}^{N_k-1} \sum_{j=0}^{N_k-1} \rho_{x_{k,i},x_{k,j}} \eta_{k,i}(u)\eta_{k,j}(u) B(a_{k,i},u,T) B(a_{k,j},u,T) \\
&+2 \sum_{i=0}^{N_0-1} \rho_{x_{0,i},y_k} \eta_{0,i}(u)\eta_k^{FX}(u) B(a_{0,i},u,T) \\
&-2 \sum_{i=0}^{N_k-1} \rho_{x_{k,i},y_k} \eta_{k,i}(u)\eta_k^{FX}(u) B(a_{k,i},u,T) \\
&-2 \sum_{i=0}^{N_0-1} \sum_{j=0}^{N_k-1} \rho_{x_{0,i},x_{k,j}} \eta_{0,i}(u)\eta_{k,j}(u) B(a_{0,i},u,T) B(a_{k,j},u,T) \Big) du
\end{aligned} \tag{7.21}
$$

7.4.2 Equity Option Formula

The price of a European call/put at time t maturing at time T on the equity $s_k(t)$ in equity currency CCY_{c_k} is given by:

$$
PV_{Equity\ call/put}(t,T,s_k(t),K) = \omega P_{c_k}(t,T)\Big(F(t)\Phi(\omega d_1) - K\Phi(\omega d_2)\Big) \tag{7.22}
$$

where, K is the strike, $\Phi(x)$ is the cumulative normal distribution function and:

$$
\begin{aligned}
F(t) &= \frac{e^{-\int_t^T \delta_k(u)du}}{P_{c_k}(t,T)} s_k(t), \quad \omega = \begin{cases} 1 & \text{if call} \\ -1 & \text{if put} \end{cases} \\
d_1 &= \frac{\ln\left(\frac{F(t)}{K}\right) + \frac{1}{2}\vartheta(t,T,T)}{\sqrt{\vartheta(t,T,T)}}, \quad d_2 = d_1 - \sqrt{\vartheta(t,T,T)} \\
\vartheta(t,t^*,T) &= \int_t^{t^*} \Big(|\eta_k^{EQ}(u)|^2 \\
&+ \sum_{i=0}^{N_{c_k}-1} \sum_{j=0}^{N_{c_k}-1} \rho_{x_{c_k,i},x_{c_k,j}} \eta_{c_k,i}(u)\eta_{c_k,j}(u) B(a_{c_k,i},u,T) B(a_{c_k,j},u,T) \\
&+2 \sum_{i=0}^{N_{c_k}-1} \rho_{x_{c_k,i},y_{c_k}} \eta_{c_k,i}(u)\eta_k^{EQ}(u) B(a_{c_k,i},u,T) \Big) du
\end{aligned} \tag{7.23}
$$

7.4.3 Calibration Procedure

We begin by looking calibration for a single currency multi-factor Hull-White model. We then extend the procedure to hybrid equity multi-currency model.

7.4.4 Single Currency Calibration

For calibration of a single currency multi-factor Hull-White, we use the procedure described in Andreasen [1] for a class of stochastic volatility HJM models. In the HJM formulation, the instantaneous forward rates:

$$f(t,T) = -\frac{\partial lnP(t,T)}{\partial T} \tag{7.24}$$

are governed by the SDE:

$$df(t,T) = \sigma(t,T)'\left(\int_t^T \sigma(t,s)ds\right)dt + \sigma(t,T)'dW(t) \tag{7.25}$$

Separable volatility structures are of the form:

$$\sigma(t,T)' = g(T)'h(t) \tag{7.26}$$

The multi-factor Hull-White model can be expressed as a Gaussian HJM model with separable volatility where:

$$g(T) = g(0,T), \quad h(t) = I_{g(t)}^{-1}\xi(t) \tag{7.27}$$

and

$$g(t,T) = \left(e^{-\int_t^T a_0(u)du}, \ldots, e^{-\int_t^T a_{N-1}(u)du}\right)' \tag{7.28}$$

$$I_{g(t)} = \begin{bmatrix} e^{-\int_0^t a_0(u)du} & & & 0 \\ & e^{-\int_0^t a_1(u)du} & & \\ & & \ddots & \\ 0 & & & e^{-\int_0^t a_{N-1}(u)du} \end{bmatrix} \tag{7.29}$$

For constant values of $a_0, \ldots, a_{N-1}$ and fixed tenors $\tau_0, \ldots, \tau_{N-1}$, the SDE for the corresponding instantaneous forward rates is:

$$dF(t) = \Lambda^*(t)R(t)dW(t) + o(dt) \tag{7.30}$$

where,

$$\Lambda^*(t) = \begin{bmatrix} f(0,\tau_0)\lambda_0(t) & & & 0 \\ & f(0,\tau_2)\lambda_2(t) & & \\ & & \ddots & \\ 0 & & & f(0,\tau_{N-1})\lambda_{N-1}(t) \end{bmatrix} \tag{7.31}$$

and $\lambda_i(t)$ is the forward rate volatility for $f(t,t+\tau_k)$ and RR' is the instantaneous correlation matrix for the k forward rates.

Under the separable volatility specification, the forward rate SDE is:

$$dF(t) = \Gamma(t)\xi(t)dW(t) + o(dt), \quad \Gamma(t) = \begin{bmatrix} g(t,t+\tau_1)' \\ \vdots \\ g(t,t+\tau_k)' \end{bmatrix}' \tag{7.32}$$

Equating terms Eq.(7.30) and Eq.(7.32) in gives:

$$\xi(t) = \Gamma^{-1}(t)\Lambda^*(t)R(t) \tag{7.33}$$

Let $\Xi(t) = \xi\xi^T$ be the $k \times k$ covariance matrix at time t. The models parameters can be expressed in terms of $\Xi(t)$ as:

$$\eta_i(t) = \sqrt{\Xi_{i,i}(t)}, \tag{7.34}$$

$$\rho_{i,j}(t) = \frac{\Xi_{i,j}(t)}{\eta_i(t)\eta_j(t)} \tag{7.35}$$

Using a similar approach to Andreasen [1], we use the following calibration procedure. Given fixed values of $a_0, \ldots, a_{N-1}$, $\tau_0, \ldots, \tau_{N-1}$ and a constant correlation structure RR' (estimated from historical market data), we calibrate the model to swaptions and caps. We assume a piece-wise constant form for the forward rate volatilities:

$$\lambda_i(t) = \tilde{\lambda}_{i,\beta(t)}, \quad \beta(t) = \{j : T_{j-1} < t \leq T_j\} \tag{7.36}$$

for a fixed time grid $0 = T_0 < T_1 < \ldots < T_m$.

The model is calibrated by minimizing a quality of fit function to reproduce market prices for swaptions and caps. The optimization is performed over the values $\tilde{\lambda}_{i,j}$ and piece–wise constant values for $\eta_i(t)$ and $\rho_{i,j}(t)$ are obtained using Eqs.(7.34) and (7.35). Note, in the calibration procedure, an penalty function is added to the quality of fit function to introduce a measure of smoothness to the values of $\lambda_i(t)$ being fitted.

7.4.5 Hybrid Model Calibration

Given mean reversions $a_{k,i}$, bench mark tenors $\tau_{k,i}$ and correlations between bench mark forward rates $C_{f_{k,i},f_{k,j}}$, we can calibrate each currency $k = 0, \ldots, n$ using the procedure described previously resulting in the forward rate volatilities $\lambda_{k,i}(t)$.

For each FX rate $k = 1, \ldots, n$, we can obtain the FX volatilities $\eta_k^{FX}(t)$ by calibrating to FX options using the single currency calibration results for currencies CCY_0 and CCY_k and additional correlation information between the bench mark rates in the two currencies, $C_{f_{0,i},f_{k,j}}$, and the spot FX rate and the bench mark rates, $C_{y_k,f_{0,i}}$ and $C_{y_k,f_{k,i}}$. Similar to the case of multi-currency single factor Hull-White models, FX options can be calibrated to exactly using an iterative procedure which involves solving quadratic polynomials. For more details, see Chapter 5 on page 33. This procedure can be adapted to the multi-factor case using the FX option formula given in Eq.(7.21).

Similarly, for each equity $k = 0, \ldots, n_{EQ} - 1$, we can obtain the equity volatilities $\eta_k^{EQ}(t)$ by calibrating to equity options using the single currency calibration results for the equity currency CCY_{c_k} and additional correlation information between spot equity price and the bench mark rates $C_{s_k,f_{c_k,i}}$.

7.4.5.1 Obtaining Model Parameters

After performing calibrations for each currency and then calibrations for each FX rate and equity, the calibration results can be combined to give the required model parameters $\eta_{k,i}(t)$, $\eta_k^{FX}(t)$, $\eta_k^{EQ}(t)$ and the correlations $\rho_{X,Y}$ between model risk factors.

Define $\tilde{\Lambda}^*(t)$ as the $N_{All} \times N_{All}$ diagonal matrix:

$$\begin{aligned}\tilde{\Lambda}^*(t) = diag\Big[& f_0(0,\tau_{0,0})\lambda_{0,0}(t), \ldots, f_0(0,\tau_{0,N_0-1})\lambda_{0,N_0-1}(t), \\ & f_1(0,\tau_{1,0})\lambda_{1,0}(t), \ldots, f_1(0,\tau_{1,N_1-1})\lambda_{1,N_1-1}(t), \ldots, \\ & f_n(0,\tau_{n,0})\lambda_{n,0}(t), \ldots, f_n(0,\tau_{n,N_n-1})\lambda_{n,N_n-1}(t), \\ & \eta_1^{FX}(t), \ldots, \eta_n^{FX}(t), \eta_0^{EQ}(t), \ldots, \eta_{m-1}^{EQ}(t) \Big]\end{aligned} \tag{7.37}$$

and $\tilde{\Gamma}(t)$ as the $N_{All} \times N_{All}$ matrix:

$$\tilde{\Gamma}(t) = \begin{pmatrix} \Gamma_0(t) & & & & & & 0 \\ & \Gamma_1(t) & & & & & \\ & & \ddots & & & & \\ & & & \Gamma_n(t) & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \tag{7.38}$$

Similar to the single currency calibration procedure, we calculate:

$$\tilde{\xi}(t) = \tilde{\Gamma}^{-1}(t)\tilde{\Lambda}^*(t)\tilde{R} \tag{7.39}$$

$$\tilde{\Xi}(t) = \tilde{\xi}\tilde{\xi}^T \tag{7.40}$$

$$\tilde{\eta}_i(t) = \sqrt{\tilde{\Xi}_{i,i}(t)} \tag{7.41}$$

$$\tilde{\rho}_{i,j}(t) = \frac{\tilde{\Xi}_{i,j}(t)}{\tilde{\eta}_i(t)\tilde{\eta}_j(t)} \tag{7.42}$$

where now $\tilde{C} = \tilde{R}\tilde{R}'$ is the $N_{All} \times N_{All}$ correlation matrix between the bench mark forward rates in all currencies, FX rates and spot equity prices.

The model parameters $\eta_{k,i}(t)$, $\eta_k^{FX}(t)$, $\eta_k^{EQ}(t)$ and the correlations $\rho_{X,Y}$ between model risk factors can be obtained as entries in $\tilde{\eta}_i(t)$ in $\tilde{\rho}_{i,j}(t)$.

Chapter 8

Hybrid Equity Inflation Multi-Currency Multi-Factor Hull-White Interest Rate Model

This chapter describes issues relating to the implementation of a hybrid equity inflation multi-currency multi-factor Hull-White interest rate model.

8.1 Introduction

In the hybrid equity inflation multi-currency extension of the multi-factor HW NF model (HW NF), a single currency HW NF model is used for the nominal interest rates for each currency. In addition, the spread between the nominal and real interest rates of each currency is also modelled as a single currency HW NF model. The spot foreign exchange (FX) rates between currencies, spot equity prices, and inflation index are modelled as lognormal random variables. For more information on the single currency multi-factor HW NF model, see Chapter 3 on page 19.

Consider a model with $n+1$ currencies: $\{CCY_0, CCY_1, \ldots, CCY_n\}$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_i, are modelled using a multi-factor single currency HW NF model:

$$dx_{k,i}(t) = -a_{k,i}(t)x_{k,i}(t)dt + \eta_{k,i}(t)dW^{(k)}_{x_{k,i}}, \quad x_{k,i}(0) = 0 \tag{8.1}$$

$$r_k(t) = \phi_k(t) + \sum_{i=1}^{N_k} x_{k,i}(t), \quad dW^{(k)}_{x_{k,i}} dW^{(k)}_{x_{k,j}} = \rho_{x_{k,i},x_{k,j}}(t)dt \tag{8.2}$$

where, for each currency k, $W^{(k)}_{k,i}$ are Wiener processes in the risk neutral measure for that currency k, $a_{k,i}(t)$ are mean reversion parameters, $\eta_{k,i}(t)$ are volatility parameters and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. In what follows, we will assume time independent mean reversion parameters $a_{k,i}(t) = a_{k,i}$.

For each currency, CCY_i, there is one basket of goods and services, CCY_i^R . Let $I_i(t)$ denote the price of this basket at time t. The dynamics of the kth spot inflation rate $I_k(t)$ in the risk neutral measure of the kth currency is governed by:

$$\frac{dI_k(t)}{I_k(t)} = u_k(t)dt + \eta^I_k(t)dW^{(k)}_{I_k(t)} \tag{8.3}$$

where, $u_k(t)$ is the spread between the nominal and real short interest rates, $\eta^I_k(t)$ is the volatility for the inflation index $I_k(t)$ and $W^{(k)}_{I_k}$ is a Wiener process in the risk neutral measure for the currency CCY_k.

The spread between the nominal and real short interest rate for each currency, CCY_i, is modelled using a multi-factor single currency HW NF model:

$$dz_{k,i}(t) = -\alpha_{k,i}(t)z_{k,i}(t)dt + \nu_{k,i}(t)dW^{(k)}_{z_{k,i}}, \quad z_{k,i}(0) = 0 \tag{8.4}$$

$$u_k(t) = \theta_k(t) + \sum_{i=1}^{N_k^R} z_{k,i}(t), \quad dW^{(k)}_{z_{k,i}}dW^{(k)}_{z_{k,j}} = \rho_{z_{k,i},z_{k,j}}(t)dt \tag{8.5}$$

$$u_k(t) = r_k(t) - r_k^R(t) \tag{8.6}$$

where, for each currency k, $W^{(k)}_{z_{k,i}}$ is the Wiener processes in the risk neutral measure for currency CCY_k, $\alpha_{k,i}(t)$ are mean reversion parameters, $\nu_{k,i}(t)$ are volatility parameters and $\theta_k(t)$ is a function chosen so that the initial inflation linked discount factor curve is reproduced. In what follows, we will assume time independent mean reversion parameters $\alpha_{k,i}(t) = \alpha_{k,i}$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \dots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_k(t)}{y_i(t)} = (r_0(t) - r_k(t))\,dt + \eta_k^{FX}(t)dW^{(0)}_{y_k} \tag{8.7}$$

where, $\eta_k^{FX}(t)$ is the volatility for FX rate $y_k(t)$ and $W^{(0)}_{y_k}$ is a Wiener process in the risk neutral measure for that domestic currency 0.

Let there also be m equities $s_0(t), \dots, s_{m-1}(t)$ where equity s_j is in currency CCY_{c_j} and c_j is one of the $n+1$ currencies. The dynamics of the j'th spot equity price $s_j(t)$ in the risk neutral measure of the equity currency CCY_{c_j} s governed by:

$$\frac{ds_j(t)}{s_j(t)} = (r_{c_j}(t) - \delta_j(t))dt + \eta_j^{EQ}(t)dW^{(c_j)}_{s_j} \tag{8.8}$$

where, $r_{c_j}(t)$ is the short rate for currency CCY_{c_j}, $\delta_j(t)$ is a continuously compounded dividend rate, $\eta_j^{EQ}(t)$ is the volatility for equity price $s_j(t)$ and $W^{(c_j)}_{s_j}$ is a Wiener process in the risk neutral measure for the equity currency c_j.

Thus, for a hybrid HW NF model with $n+1$ currencies and m equities, there are $N_{All} = N_{IR} + N_{RIR} + 2n + 1 + m$ risk factors and driving Wiener processes where $N_{IR} = \sum_{k=0}^n N_k$ is the number of all interest rate risk factors and N_k is the number of factors for currency k, $N_{RIR} = \sum_{k=0}^n N_k^R$ is the number of all real interest rate risk factors and N_k^R is the number of factors driving the real interest rates for currency k. We denote $\rho_{X,Y}$ to be the correlation between risk factors X and Y, For example, $\rho_{x_{k,i},y_l}$ is the correlation between the ith IR risk factor for CCY_k and spot FX rate l.

8.2 Measure Relationships

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 is given by:

$$dW^{(0)}_{x_{k,i}} = \rho_{x_{k,i},y_j}\eta_j^{FX}(t)dt + dW^{(j)}_{x_{k,i}} \tag{8.9}$$

The relationship between the risk neutral measure and the T forward measure in the same currency is given by:

$$dW^{(k)}_{x_{k,i}} = \sum_{j=0}^{N_k-1} \rho_{x_{k,i},x_{k,j}} B(a_{k,j}, t, T)\eta_{k,j}(t)dt + dW^{T,(k)}_{x_{k,i}} \tag{8.10}$$

where,

$$B(x,t,T) = \frac{1 - e^{-x(T-t)}}{x} \tag{8.11}$$

8.3 Inflation Linked Trades

8.3.1 Distribution of Inflation under Forward Measure

The distribution of $I(S)$ under the $P(t,T)$ numeraire is equal to:

$$\begin{aligned}
\ln(I(S)) &\sim \mathrm{N}\left(m(t,S,T), v^2(t,S)\right) && (8.12)\\
m(t,S,T) &= \ln(I(t)) + \sum_{i=1}^{N^R} B(\alpha_i,t,S)z_i + \int_t^S \theta(u)du - \frac{1}{2}V_{I,I}(t,S) && (8.13)\\
&\quad -V_{u,r}(t,S,T) - V_{r,I}(t,S,T) &&\\
v^2(t,S) &= V_{u,u}(t,S) + 2V_{u,I}(t,S) + V_{I,I}(t,S) && (8.14)
\end{aligned}$$

where,

$$\begin{aligned}
V_{u,u}(t,S) &= \sum_{i=1}^{N^R}\sum_{j=1}^{N^R}\int_t^S \nu_i(s)\nu_j(s)B(\alpha_i,s,S)B(\alpha_j,s,S)\rho_{z_i,z_j}ds && (8.15)\\
V_{u,I}(t,S) &= \sum_{i=1}^{N^R}\int_t^S \nu_i(s)\eta^I(s)B(\alpha_i,s,S)\rho_{z_i,I}ds && (8.16)\\
V_{I,I}(t,S) &= \int_t^S \eta^I(s)^2 ds && (8.17)\\
V_{u,r}(t,S,T) &= \sum_{i=1}^{N^R}\sum_{j=1}^{N}\int_t^S \nu_i(s)\eta_j(s)B(\alpha_i,s,S)B(a_j,s,T)\rho_{z_i,x_j}ds && (8.18)\\
V_{r,I}(t,S,T) &= \sum_{i=1}^{N}\int_t^S \eta_i(s)\eta^I(s)B(a_i,s,T)\rho_{x_i,I}ds && (8.19)\\
B(a,t,T) &= \frac{1-e^{-a(T-t)}}{a} && (8.20)
\end{aligned}$$

8.3.2 Inflation Linked Zero Coupon Bond

An inflation linked zero coupon bond pays $I(S)$ at time T, where $S \leq T$. The value can be found as:

$$P_I(t,S,T) = P(t,T) * \mathbb{E}_t^T\left[I(S)\right] \tag{8.21}$$

where $\mathbb{E}_t^T[\cdot]$ is the expectation operator conditional on all information up to time t using probabilities that ensure the expectation of all payoffs deflated by $P(\cdot,T)$ are martingales. The price of this instrument has the following closed form solution:

$$\begin{aligned}
P_I(t,S,T) &= P(t,T) * P_I^*(t,S,T) && (8.22)\\
P_I^*(t,S,T) &= e^{m(t,S,T)+\frac{1}{2}v^2(t,S)} && (8.23)
\end{aligned}$$

The parameter $\int_t^S \theta(s)ds$ is chosen to reproduce the initial inflation linked discount factor curve $P_I^*(0,S,S)$ at node points $S = T_k$. It can be shown that log-linear interpolation between node points is ensured if we set this integral to:

$$\begin{aligned}
\int_0^S \theta(s)ds &= ln\left(P_I^*(0,T_{k-1},T_{k-1})\right) && (8.24)\\
&\quad + \frac{S-T_{k-1}}{T_k-T_{k-1}} ln\left(\frac{P_I^*(0,T_k,T_k)}{P_I^*(0,T_{k-1},T_{k-1})}\right) &&\\
&\quad - \frac{1}{2}V_{u,u}(0,S) - V_{u,I}(0,S) &&\\
&\quad + V_{u,r}(0,S,S) + V_{r,I}(0,S,S), \quad T_{k-1} \leq S < T_k &&
\end{aligned}$$

The deterministic component for $P_I(t,S,T)$ can be rewritten as:

$$\frac{P_I^*(0,S,S)}{P_I^*(0,t,t)} \exp\left(\frac{1}{2}T_{u,u}(t,S) + T_{u,I}(t,S) + T_{r,I}(t,S,T) + T_{u,r}(t,S,T)\right)$$

The following notations are used:

$$\begin{aligned}
T_{u,u}(t,S) &= V_{u,u}(t,S) - V_{u,u}(0,S) + V_{u,u}(0,t) \\
T_{u,I}(t,S) &= V_{u,I}(t,S) - V_{u,I}(0,S) + V_{u,I}(0,t) \\
T_{r,I}(t,S,T) &= -V_{r,I}(t,S,T) + V_{r,I}(0,S,S) - V_{r,I}(0,t,t) \\
T_{u,r}(t,S,T) &= -V_{u,r}(t,S,T) + V_{u,r}(0,S,S) - V_{u,r}(0,t,t)
\end{aligned}$$

Here $P_I^*(0,t,t)$ and $P_I^*(0,S,S)$ are calculated using log-linear interploation on the initial inflation linked discount factor curve.

Thus $P_I(t,S,T)$ is simply the product of the determinstic and the stochastic component:

$$\exp\left(\sum_{i=1}^{N^R} B(\alpha_i,t,S)z_i\right)$$

8.3.3 Inflation Linked Cap/Floor

An inflation linked zero coupon bond cap/floor pays $\max(w(I(S)-K),0)$ at time T, where $S \le T$, $w \in \{-1,1\}$. The value is equal to:

$$\begin{aligned}
V(t,S,T) &= P(t,T) * w * (P_I^*(t,S,T) * \Phi(w * d_1(t,S,T)) \\
&\qquad - K * \Phi(w * d_2(t,S,T))) && (8.25) \\
d_1(t,S,T) &= \frac{\ln(P_I^*(t,S,T)) - \ln(K) + \frac{1}{2}v^2(t,S)}{v(t,S)} && (8.26) \\
d_2(t,S,T) &= \frac{\ln(P_I^*(t,S,T)) - \ln(K) - \frac{1}{2}v^2(t,S)}{v(t,S)} && (8.27) \\
& && (8.28)
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative normal distribution function.

8.4 Inflation Ratio Linked Trades

8.4.1 Distribution of Inflation Ratio under Forward Measure

The distribution of $I(S)/I(U)$ under the $P(t,T)$ numeraire, where $U \le S \le T$, is equal to:

$$\begin{aligned}
\ln\left(\frac{I(S)}{I(U)}\right) &\sim \mathrm{N}(m_{yoy}(t,S,U,T), v_{yoy}^2(t,S,U)) && (8.29) \\
m_{yoy}(t,S,U,T) &= m(t,S,T) - m(t,U,T) && (8.30) \\
v_{yoy}^2(t,S,U) &= v^2(t,S) + v^2(t,U) - 2c(t,S,U)) && (8.31) \\
c(t,S,U) &= C_{u,u}(t,S,U) + C_{u,I}(t,S,U) && (8.32) \\
&\qquad + C_{I,u}(t,S,U) + C_{I,I}(t,S,U) && (8.33)
\end{aligned}$$

where,

$$
\begin{aligned}
C_{u,u}(t,S,U) &= \sum_{i=1}^{N^R}\sum_{j=1}^{N^R}\int_t^U \nu_i(s)\nu_j(s)B(\alpha_i,s,S)B(\alpha_j,s,U)\rho_{z_i,z_j}ds && (8.34)\\
C_{u,I}(t,S,U) &= \sum_{i=1}^{N^R}\int_t^U \nu_i(s)\eta^I(s)B(\alpha_i,s,S)\rho_{z_i,I}ds && (8.35)\\
C_{I,u}(t,S,U) &= \sum_{i=1}^{N^R}\int_t^U \nu_i(s)\eta^I(s)B(\alpha_i,s,U)\rho_{z_i,I}ds && (8.36)\\
C_{I,I}(t,S,U) &= V_{I,I}(t,U) && (8.37)
\end{aligned}
$$

8.4.2 Year-On-Year Inflation Linked Zero Coupon Bond

A year-on-year inflation linked zero coupon bond pays $I(T_E)/I(T_S)$ at time T_P, where $T_S \le T_E \le T_P$. The price of this instrument has the following closed form solution:

$$
\begin{aligned}
P_{yoy}(t,T_S,T_E,T_P) &= P(t,T_P)*P^*_{yoy}(t,T_S,T_E,T_P) && (8.38)\\
P^*_{yoy}(t,T_S,T_E,T_P) &= e^{m_{yoy}(t,T_S,T_E,T_P)+\frac{1}{2}v^2_{yoy}(t,T_S,T_E)} && (8.39)
\end{aligned}
$$

8.4.3 Year-On-Year Inflation Linked Cap/Floor

An inflation linked zero coupon bond cap/floor pays $\max(w(I(T_E)/I(T_S)-K),0)$ at time T_P, where $T_S \le T_E \le T_P$, $w \in \{-1,1\}$. The value is equal to:

$$
\begin{aligned}
V(t,T_S,T_E,T_P) &= P(t,T_P)*w*(P^*_{yoy}(t,T_S,T_E,T_P)*\Phi(w*d_1(t,T_S,T_E,T_P))\\
&\qquad -K*\Phi(w*d_2(t,T_E,T_S,T_P))) && (8.40)\\
d_1(t,T_S,T_E,T_P) &= \frac{\ln(P^*_{yoy}(t,T_S,T_E,T_P))-\ln(K)+\frac{1}{2}v^2_{yoy}(t,T_S,T_E)}{v_{yoy}(t,T_S,T_E)} && (8.41)\\
d_2(t,T_S,T_E,T_P) &= \frac{\ln(P^*_{yoy}(t,T_S,T_E,T_P))-\ln(K)-\frac{1}{2}v^2_{yoy}(t,T_S,T_E)}{v_{yoy}(t,T_S,T_E)} && (8.42)\\
& && (8.43)
\end{aligned}
$$

8.5 System of Equations

Under the domestic currency CCY_0 T forward measure, the SDE's become:

$$\begin{aligned}
dx_{0,i} &= \Big[-a_{0,i}x_{0,i} - \sum_{j=0}^{N_0-1} \rho_{x_{0,i},x_{0,j}}\eta_{0,i}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_{0,i}dW^{T,(0)}_{x_{0,i}} && (8.44)\\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \sum_{j=0}^{N_0-1} \rho_{x_{k,i},x_{0,j}}\eta_{k,i}\eta_{0,j}B(a_{0,j},t,T) \\
&\qquad - \rho_{x_{k,i},y_k}\eta_{k,i}\eta_k^{FX}\Big]dt + \eta_{k,i}dW^{T,(0)}_{x_{k,i}} && (8.45)\\
\frac{dy_k}{y} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=0}^{N_0-1} x_{0,i} - \sum_{i=0}^{N_k-1} x_{k,i} \\
&\qquad - \sum_{j=0}^{N_0-1} \rho_{y_k,x_{0,j}}\eta_k^{FX}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_k^{FX}dW^{T,(0)}_{y_k} && (8.46)\\
\frac{ds_k}{s} &= \Big[\phi_{c_k}(t) + \sum_{j=0}^{N_{c_k}-1} x_{c_k,j} - \delta_k(t) - \mathbf{1}_{[c_k\neq 0]}\rho_{s_k,y_{c_k}}\eta_k^{EQ}\eta_{c_k}^{FX} \\
&\qquad - \sum_{j=0}^{N_0-1} \rho_{s_k,x_{0,j}}\eta_k^{EQ}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_k^{EQ}dW^{T,(0)}_{s_k}, && (8.47)\\
dz_{k,i} &= \Big[-\alpha_{k,i}z_{k,i} - \mathbf{1}_{[k\neq 0]}\rho_{z_{k,i},y_k}\nu_{k,i}\eta_k^{FX} \\
&\qquad - \sum_{j=0}^{N_0-1} \rho_{z_{k,i},x_{0,j}}\nu_{k,i}\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \nu_{k,i}dW^{T,(0)}_{z_{k,i}} && (8.48)\\
\frac{dI_k}{I} &= \Big[\theta_k(t) + \sum_{i=0}^{N_k^R-1} z_{k,i} - \mathbf{1}_{[k\neq 0]}\rho_{I_k,y_k}\eta^I\eta_k^{FX} \\
&\qquad - \sum_{j=0}^{N_0-1} \rho_{I_k,x_{0,j}}\eta_k^I\eta_{0,j}B(a_{0,j},t,T)\Big]dt + \eta_k^I dW^{T,(0)}_{I_k} && (8.49)
\end{aligned}$$

This system of equations can be solved using the same techniques outlined in Chapter 7 on page 51.

8.6 Derived Inflation Indices

If an inflation baskets I^D and real inflation rate curve have the exact same model parameters and driving factors as one of the inflation baskets and curve pairs above, but have different inflation index and real inflation rate curve starting values, then it is more efficient to simulate the core set of inflation indices and real inflation rates, and compute the derived inflation index values and real inflation rates as functions of the core rates, and the initial starting values:

$$I^D(t) = I(t) * \frac{I^D(0)}{I(0)} * \frac{\exp\left(\int_0^t \theta^D(s)ds\right)}{\exp\left(\int_0^t \theta(s)ds\right)} \tag{8.50}$$

where the ratio of the θ integrals only depends on the initial curves, as the volatility terms cancel out. Likewise, we can derive new formulas for the present value of inflation linked notes:

$$P_{I^D}(t,S,T) = P_I(t,S,T) * \frac{I^D(0)}{I(0)} * \frac{\exp\left(\int_0^S \theta^D(s)ds\right)}{\exp\left(\int_0^S \theta(s)ds\right)} \tag{8.51}$$

and year on year payments:

$$P_{I^D YOY}(t,T_S,T_E,T_P) = P_{IYOY}(t,T_S,T_E,T_P) * \frac{\exp\left(\int_0^{T_E} \theta^D(s)ds\right)}{\exp\left(\int_0^{T_E} \theta(s)ds\right)} * \frac{\exp\left(\int_0^{T_S} \theta(s)ds\right)}{\exp\left(\int_0^{T_S} \theta^D(s)ds\right)} \tag{8.52}$$

8.7 Calibration

8.7.1 Calibrating to Inflation Index Options

Assume we are given the inflation rate parameters α_i, ν_i, ρ_{z_i,z_j}, and $\rho_{z_i,I}$ for i and j in $\{0, \ldots, N^R - 1\}$. We can then calibrate the inflation index volatility η using a procedure similar to the FX calibration procedure where Eq.(8.14) is now used instead of the corresponding FX equation.
Rates and Single Factor Hull-White Models for Hazard Rate

Chapter 9

Credit Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Hull-White Models for Hazard Rates

This chapter describes issues relating to the implementation of a credit multi-currency model with the multi-factor Hull-White models for interest rates and single-factor HW model for hazard rates.

9.1 Introduction

In the credit multi-currency multi-factor HW model, a single currency Hull-White model is used for the interest rates for each currency. The spot foreign exchange (FX) rates between currencies are modelled as lognormal random variables. The hazard rates are modelled by a single-factor HW model.

Consider a model with $n+1$ currencies $k = 0, 1, \ldots, n$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_k, is modelled using a multi-factor single currency HW model with N_k factors:

$$\begin{aligned} dx_{k,i}(t) &= -a_{k,i}(t)x_{k,i}(t)dt + \eta_{k,i}(t)dW^{(k)}_{x_{k,i}}, \quad x_{k,i}(0) = 0 \\ r_k(t) &= \phi_k(t) + \sum_{i=1}^{N_k} x_{k,i}(t), \end{aligned} \tag{9.1}$$

where, for each currency k, $W^{(k)}_{x_{k,i}}$ is the Wiener processes in the risk neutral measure for the factor i of that currency, $a_{k,i}(t)$ is the mean reversion parameter, $\eta_{k,i}(t)$ is the volatility parameter and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. For the details of the calculation of the function $\phi_k(t)$, see Chapter 3 on page 19. In what follows, we will assume time independent mean reversion parameters $a_{k,i}(t) = a_{k,i}$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \ldots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_k(t)}{y_k(t)} = (r_0(t) - r_k(t))\, dt + \eta^{FX}_k(t)dW^{(0)}_{y_k}. \tag{9.2}$$

Here, $\eta^{FX}_k(t)$ is the volatility for FX rate $y_k(t)$, $W^{(0)}_{y_k}$ is a Wiener process in the risk neutral measure for the domestic currency CCY_0.

The dynamics of the hazard rate $h_m(t)$ for each credit name $m = 1, \ldots, N_{CR}$ under the domestic risk neutral measure is governed by:

$$\begin{aligned} dz_m(t) &= -\kappa_m z_m(t)dt + \eta_m^{CR}(t)dW_{z_m}^{(0)}(t), \quad z_m(0) = 0 \\ h_m(t) &= \vartheta_m(t) + z_m(t) \end{aligned} \tag{9.3}$$

where κ_m is the mean reversion parameter, η_m^{CR} is the volatility parameter, ϑ_m is the drift adjustment parameter. The drift adjustment parameter ϑ_m is calibrated to reproduce the initial survival probability curve,

$$S_{c,m}(t_0, T) = \mathbb{E}^{T,(c)}\left[\mathbf{1}_{\{\tau_m > T\}}\right] = \mathbb{E}^{T,(c)}\left[e^{-\int_{t_0}^{T} h_m(u)du}\right]. \tag{9.4}$$

Here τ_m is the counterparty default time and $\mathbb{E}^{T,(c)}$ is the expectation under the T-forward measure in the currency of the input CDS curve. The derivation of ϑ_m is provided in Chapter 9.2.

9.2 Survival Probability Calculation. Choice of ϑ.

The section describes the calculation of the survival probability for mth credit name on the time interval $[t, T]$, conditioned on the name hasn't defaulted until t. The survival probability is defined by:

$$S_{c,m}(t, T) = \mathbf{E}_t^{T,(c)}[1_{\tau > T}] = \mathbf{E}_t^{T,(c)}[e^{-\int_t^T h_m(u)du}]. \tag{9.5}$$

Here $\mathbf{E}^{T,(c)}$ is the expectation under the T-forward measure in the currency c. Also, the section describes how the hazard rate drift adjustment ϑ_m in the equation (9.3) is calculated.

9.2.1 Domestic Currency Measure

The survival probability under the domestic currency $S_{0,m}(t, T)$ conditioned on no default happend until t is defined as:

$$S_{0,m}(t, T) = \mathbf{E}_t^{T,(0)}[1_{\tau > T}] = \mathbf{E}_t^{T,(0)}[e^{-\int_t^T h_m(u)du}]. \tag{9.6}$$

Note that the model risk factors are simulated under the risk-neutral measure of the domestic currency. To calculate the survival probability, we have to represent (9.6) in the risk-neutral measure. Using changing numeraire technique, we get the following representation:

$$\begin{aligned} S_{0,m}(t, T) &= \frac{N(t)}{P(t, T)}\mathbf{E}_t^{(0)}\left[\frac{e^{-\int_t^T h_m(u)du}}{N(T)}\right] = \mathbb{E}_t^{(0)}\left[e^{-\int_t^T h_m(u)du + M_m(t,T)}\right], \\ & N(t) = e^{\int_{t_0}^{t} r_0(u)du}. \end{aligned} \tag{9.7}$$

where

$$\begin{aligned} M_m(t, T) &= Cov\left[\int_t^T r_0(u)du, \int_t^T h_m(u)du\right] \\ &= \sum_{i=1}^{N_0} \int_t^T \rho_{x_{0,i},\kappa_m} \eta_{0,i} \eta_m^{CR} B(a_{0,i}, u, T) B(\kappa_m, u, T)du, \\ & B(a, t, T) = \frac{1 - e^{-a(T-t)}}{a}. \end{aligned} \tag{9.8}$$

Further we note that the hazard rate $h(u)$ governed by the equation (9.3) has the following representation for $u > t$,

$$h_m(u) = \vartheta_m(u) + z_m(t)e^{-\kappa_m(u-t)} + \int_t^u e^{-\kappa_m(u-\tau)}\eta_m^{CR}(\tau)dW(\tau) \tag{9.9}$$

Then, $\int_t^T h_m(u)du$ is Gaussian random variable with the mean

$$\mathbb{E}_t^{(0)}\left[\int_t^T h_m(u)du\right] = z_m(t)B(\kappa_m, t, T) + \int_t^T \vartheta_m(s)ds \tag{9.10}$$

and variance

$$Var_t^{(0)}\left[\int_t^T h_m(u)du\right] = \int_t^T (B(\kappa_m, s, T)\eta_m^{CR}(s))^2 ds. \tag{9.11}$$

Plugging (9.8)-(9.11) into (9.7), we get the following representation for the survival probability under the domestic currency measure,

$$S_{0,m}(t,T) = e^{A_0(t,T)-B(\kappa_m,t,T)z_m(t)}, \tag{9.12}$$

where

$$\begin{aligned} A_0(t,T) &= -\int_t^T \vartheta_m(s)ds + \frac{1}{2}\int_t^T (B(\kappa_m, s, T)\eta_m^{CR}(s))^2 ds \\ &+ \sum_{i=1}^{N_0}\int_t^T \rho_{x_{0,i},\kappa_m}\eta_{0,i}\eta_m^{CR} B(a_{0,i}, s, T)B(\kappa_m, s, T)ds. \end{aligned}$$

9.2.2 Foreign Currency Measure

Let c be one of the model foreign currencies. In this section we derive the analytic representation of the susrival probability $S_{c,m}$ for the mth credit name under the T-forward measure of the currency c. The survival probability $S_{c,m}$ conditioned on no default had happened until t is defined as:

$$S_{c,m}(t,T) = \mathbf{E}_t^{T,(c)}[1_{\tau>T}] = \mathbf{E}_t^{T,(c)}[e^{-\int_t^T h_m(s)ds}]. \tag{9.13}$$

Now let us convert the survival probability (9.13) to the risk-neutral measure in the domestic currency. Using changing numeraire technique, we get

$$\begin{aligned} S_{c,m}(t,T) &= \frac{N(t)}{y_c(t)P_c(t,T)}\mathbf{E}_t^{(0)}\left[\frac{e^{-\int_t^T h_m(u)du}y_c(T)}{N(T)}\right] \\ &= \mathbf{E}_t^{(0)}\left[e^{-\int_t^T h_m(u)du+\Omega_{c,m}(t,T)}\right], \\ & N(t) = e^{\int_{t_0}^t r_0(u)du}. \end{aligned} \tag{9.14}$$

where

$$\begin{aligned} \Omega_{c,m}(t,T) &= Cov\left[\int_t^T h_m(u)du, \int_t^T r_c(u)du - \int_t^T \eta_c^{FX}(u)dW_{y_c}^{(0)}(u)\right] \\ &= \sum_{i=1}^{N_c}\int_t^T \rho_{x_{c,i},z_m}\eta_{x_{c,i}}\eta_m^{CR}B(a_{c,i}, s, T)B(\kappa_m, s, T)ds \\ &- \int_t^T \rho_{y_c,z_m}\eta_c^{FX}\eta_m^{CR}B(\kappa_m, s, T)ds. \end{aligned} \tag{9.15}$$

Note that $\int_t^T h_m(u)du$ is the Gaussian random variable with the mean and variance defined by (9.10)-(9.11). Plugging it into (9.14), we get,

$$S_{c,m}(t,T) = e^{A_c(t,T)-B(\kappa_m,t,T)z_m(t)}, \tag{9.16}$$

where

$$\begin{aligned} A_c(t,T) &= -\int_t^T \vartheta_m(s)ds + \frac{1}{2}\int_t^T (B(\kappa_m, s, T)\eta_m^{CR}(s))^2 ds \\ &+ \sum_{i=1}^{N_c}\int_t^T \rho_{x_{c,i},z_m}\eta_{x_{c,i}}\eta_m^{CR} B(a_{c,i}, s, T)B(\kappa_m, s, T)ds \\ &- \int_t^T \rho_{y_c,z_m}\eta_c^{FX}\eta_m^{CR}B(\kappa_m, s, T)ds. \end{aligned}$$

9.2.3 Choice of ϑ.

The drift adjustment term ϑ_m in the equation (9.3) is chosen to reproduce the initial survival probability.

If the initial survival probability is provided in the domestic currency measure, we have

$$\begin{aligned}\ln S_{0,m}(t_0,T) &= -\int_{t_0}^{T}\vartheta_m(s)ds + \frac{1}{2}\int_{t_0}^{T}(B(\kappa_m,s,T)\eta_m^{CR}(s))^2 ds\\ &+\sum_{i=1}^{N_0}\int_{t_0}^{T}\rho_{x_{0,i},\kappa_m}\eta_{0,i}\eta_m^{CR}B(a_{0,i},s,T)B(\kappa_m,s,T)ds.\end{aligned}$$

Assume that the $S_{0,m}(t_0,T)$ is given by the term structure

$$S_m(t_0,T_1),...,S_m(t_0,T_N).$$

and the log-lin interpolation is used between the nodes. Then, to reproduce the survival probablity curve, we define $\vartheta_m(t)$ as follows,

$$\begin{aligned}\int_{t_0}^{T}\vartheta_m(s)ds &= \frac{1}{2}\int_{t_0}^{T}(B(\kappa_m,\tau,T)\eta_m^{CR}(\tau))^2 d\tau \\ &+\sum_{i=1}^{N_0}\int_{t_0}^{T}\rho_{x_{0,i},z_m}\eta_{0,i}\eta_m^{CR}B(a_{0,i},\tau,T)B(\kappa_m,\tau,T)d\tau\\ &-\ln S_m(t_0,t_{k-1}) + \frac{T-T_{k-1}}{T_k-T_{k-1}}\ln\frac{S_m(t_0,T_{k-1})}{S_m(t_0,T_k)},\\ &=: \frac{1}{2}I_h(m;\,t_0,T,T) + I_{r,h}(0,m;\,t_0,T,T)\\ &-\ln S_m(t_0,t_{k-1}) + \frac{T-T_{k-1}}{T_k-T_{k-1}}\ln\frac{S_m(t_0,T_{k-1})}{S_m(t_0,T_k)},\\ &T_{k-1}\le T<T_k.\end{aligned} \tag{9.17}$$

Here, functions I_h and $I_{r,h}$ denote the variance term of the hazard rate integral for the name m and the covariance term of hazard rate and short rate integrals for the name m and currency c:

$$\begin{aligned}I_h(m;\,t,S,T) &:= \int_t^S(B(\kappa_m,\tau,T)\eta_m^{CR}(\tau))^2 d\tau,\\ I_{r,h}(c,m;\,t,S,T) &= \sum_{i=1}^{N_c}\int_t^S\rho_{x_{c,i},z_m}\eta_{c,i}\eta_m^{CR}B(a_{c,i},\tau,T)B(\kappa_m,\tau,T)d\tau,\\ &t\le S\le T.\end{aligned} \tag{9.18}$$

If the initial survival probability is provided in the measure of the currency c, we have

$$\begin{aligned}\ln S_{c,m}(t_0,T) &= -\int_{t_0}^{T}\vartheta_m(s)ds + \frac{1}{2}\int_{t_0}^{T}(B(\kappa_m,s,T)\eta_m^{CR}(s))^2 ds\\ &+\sum_{i=1}^{N_c}\int_{t_0}^{T}\rho_{x_{c,i},\kappa_m}\eta_{c,i}\eta_m^{CR}B(a_{c,i},s,T)B(\kappa_m,s,T)ds\\ &-\int_t^T\rho_{y_c,z_m}\eta_c^{FX}\eta_m^{CR}B(\kappa_m,s,T)ds.\end{aligned}$$

Now we assume that the $S_{c,m}(t_0,T)$ is given by the term structure

$$S_m(t_0,T_1),...,S_m(t_0,T_N).$$

and the log-lin interpolation is used between the nodes. Then, to reproduce the survival probablity curve, we define $\vartheta_m(t)$ as follows,

$$\begin{aligned}
\int_{t_0}^{T} \vartheta_m(s)ds &= \frac{1}{2}\int_{t_0}^{T} (B(\kappa_m,\tau,T)\eta_m^{CR}(\tau))^2 d\tau \\
&\quad + \sum_{i=1}^{N_c}\int_{t_0}^{T} \rho_{x_{c,i},z_m}\eta_{c,i}\eta_m^{CR} B(a_{c,i},\tau,T)B(\kappa_m,\tau,T)d\tau \\
&\quad - \int_{t_0}^{T} \rho_{y_c,z_m}\eta_c^{FX}\eta_m^{CR} B(\kappa_m,s,T)ds, \\
&\quad - \ln S_m(t_0,t_{k-1}) + \frac{T-T_{k-1}}{T_k-T_{k-1}}\ln\frac{S_m(t_0,T_{k-1})}{S_m(t_0,T_k)} \\
&=: \frac{1}{2}I_h(m;\,t_0,T,T) + I_{r,h}(c,m;\,t_0,T,T) - I_{y,h}(c,m;\,t_0,T,T) \\
&\quad - \ln S_m(t_0,t_{k-1}) + \frac{T-T_{k-1}}{T_k-T_{k-1}}\ln\frac{S_m(t_0,T_{k-1})}{S_m(t_0,T_k)} \\
&\quad T_{k-1} \le T < T_k.
\end{aligned} \tag{9.19}$$

Here, I_h and $I_{r,h}$ are specified by the equation (9.18), and $I_{y,h}$ denotes the covariance term between the hazard rate integral for the name m and exchange rate for the currency c,

$$\begin{aligned}
I_{y,h}(c,m;\,t,S,T) &= \int_{t}^{S} \rho_{y_c,z_m}\eta_c^{FX}\eta_m^{CR} B(\kappa_m,s,T)ds, \\
&\quad t \le S \le T.
\end{aligned} \tag{9.20}$$

The function $\vartheta_m(t)$ is obtained by differentiation of the integral above,

$$\begin{aligned}
\vartheta_m(T) &= \frac{\partial}{\partial T}\int_{t_0}^{T}\vartheta_m(s)ds \\
&= \frac{1}{2}\frac{\partial}{\partial T}I_h(m;\,t_0,T,T) + \frac{\partial}{\partial T}I_{r,h}(c,m;\,t_0,T,T) \\
&\quad -1_{c\neq 0}\frac{\partial}{\partial T}I_{y,h}(c,m;\,t_0,T,T) + \frac{1}{T_k-T_{k-1}}\ln\frac{S_m(t_0,T_{k-1})}{S_m(t_0,T_k)} \\
&\quad T_{k-1} \le T < T_k.
\end{aligned} \tag{9.21}$$

9.3 Hazard Rate Volatility Adjustment

The normal distribution of the simulated hazard rates implies the non-zero probability of a path to have the negative values. The probability depends on the mean and the variance of the hazard rate process, and can be quite high for some use cases. The section describes the approach to adjust the hazard rate volatility in order to make the probability of the negative values to be of a certain level.

The hazard rate process $h_m(t)$ specified by (9.9) has the Gaussian distribution with the mean

$$\mathbb{E}^{(0)}\left[h_m(t)\right] = \vartheta_m(t) \tag{9.22}$$

and variance

$$Var^{(0)}\left[h_m(t)\right] = \int_{t_0}^{t} e^{-2\kappa_m(t-\tau)}(\eta_m^{CR})^2(\tau)d\tau =: V_h(m;\,t). \tag{9.23}$$

Then, the probability of the hazard rate to be negative at time t is given by the following equation,

$$\begin{aligned}
Prob\left\{h_m(t) < 0\right\} &= \Phi\left(-\frac{\mathbb{E}^{(0)}\left[h_m(t)\right]}{\sqrt{Var^{(0)}\left[h_m(t)\right]}}\right) \\
&=: \Phi\left(-\frac{\vartheta_m(t)}{\sqrt{V_h(m;\,t)}}\right),
\end{aligned} \tag{9.24}$$

where Φ is the cumulative function of the standard Gaussian distribution, and the function ϑ_m is provided by the equation (9.21).

In this section, we describe the recursive procedure to build the adjusted hazard rate volatility $\tilde{\eta}_m^{CR}$ in such a way that

$$\Phi\left(-\frac{\vartheta_m(t)}{\sqrt{V_h(m;\,t)}}\right) \le \varepsilon, \tag{9.25}$$

or, equivalently,

$$\frac{\vartheta_m(t)}{\sqrt{V_h(m;\,t)}} \ge \delta; \quad \delta = -\sqrt{2}erf^{-1}(2\varepsilon - 1). \tag{9.26}$$

In what follows, we assume that the volatility of the short rates, exchange rates, and hazard rates η_c, η_c^{FX}, η_m^{CR} are piecewise constant, left-continuous functions. Let us denote

$$t_0 \le t_1 \le t_2 \le \cdots \le t_n$$

the jump dates of η_c, η_c^{FX}, η_m^{CR}, combined with the jump dates of the input survival probability curve $S_{c,m}(t)$. Further, let us denote

$$\begin{aligned} &\eta_{i,k} := \eta_{c,i}(t_k), \quad \eta_k^{FX} := \eta_m^{FX}(t_k), \quad \eta_k^{CR} := \eta_m^{CR}(t_k) \\ &k = 0, \ldots, n-1, \quad i = 1, \ldots N_c. \end{aligned} \tag{9.27}$$

Now let us assume that for $k \ge 0$, the adjusted hazard rate volatilities $\tilde{\eta}_0^{CR}$,...,$\tilde{\eta}_{k-1}^{CR}$ are defined in such a way that

$$inf_{t_0 \le t < t_k} \frac{\vartheta_m(t)}{\sqrt{V_h(m;\,t)}} \ge -\sqrt{2}erf^{-1}(2\varepsilon - 1). \tag{9.28}$$

For $t_k \le t < t_{k+1}$, we have the following representation for the terms in the numerator and denominator of the equation (9.26),

$$\begin{aligned} \frac{\partial}{\partial t} I_h(m;\,t_0, t, t) &= \sum_{j=0}^{k-1} \frac{\partial}{\partial t} I_h(m;\,t_j, t_{j+1}, t) + \frac{\partial}{\partial t} I_h(m;\,t_k, t, t) \\ &= \frac{1}{\kappa_m^2} \sum_{j=0}^{k-1} (\eta_j^{CR})^2 \left(2e^{-\kappa_m t}(e^{\kappa_m t_{j+1}} - e^{\kappa_m t_j}) \right. \\ &\quad \left. -e^{-2\kappa_m t}(e^{2\kappa_m t_{j+1}} - e^{2\kappa_m t_j})\right) \\ &\quad + \frac{(\eta_k^{CR})^2}{\kappa_m^2} \left(1 - 2e^{-\kappa_m t} e^{\kappa_m t_k} + e^{-2\kappa_m t} e^{2\kappa_m t_k}\right) \\ &=: A_h(t) + (\eta_k^{CR})^2 B_h(t); \end{aligned} \tag{9.29}$$

$$\begin{aligned} \frac{\partial}{\partial t} I_{r,h}(c, m;\,t_0, t, t) &= \sum_{j=0}^{k-1} \frac{\partial}{\partial t} I_{r,h}(c, m;\,t_j, t_{j+1}, t) + \frac{\partial}{\partial t} I_{r,h}(c, m;\,t_k, t, t) \\ &= \frac{1}{\kappa_m} \sum_{j=0}^{k-1} \eta_j^{CR} \sum_{i=1}^{N_c} \frac{\eta_{i,j} \rho_{x_{c,i}, z_m}}{a_i} \left(e^{-\kappa_m t}(e^{\kappa_m t_{j+1}} - e^{\kappa_m t_j}) \right. \\ &\quad + e^{-a_i t}(e^{a_i t_{j+1}} - e^{a_i t_j}) \\ &\quad \left. - e^{-(\kappa_m + a_i)t}(e^{(\kappa_m + a_i)t_{j+1}} - e^{(\kappa_m + a_i)t_j})\right) \\ &+ \frac{\eta_k^{CR}}{\kappa_m} \sum_{i=1}^{N_c} \left(1 - e^{-\kappa_m t} e^{\kappa_m t_k} - e^{-a_i t} e^{a_i t_k} + e^{-2\kappa_m t} e^{2\kappa_m t_k}\right) \\ &=: A_{r,h}(t) + \eta_k^{CR} B_{r,h}(t); \end{aligned} \tag{9.30}$$

$$
\begin{aligned}
\frac{\partial}{\partial t} I_{y,h}(c,m;\, t_0,t,t) &= \sum_{j=0}^{k-1} \frac{\partial}{\partial t} I_{y,h}(c,m;\, t_j,t_{j+1},t) + \frac{\partial}{\partial t} I_{y,h}(c,m;\, t_k,t,t) \\
&= \frac{1}{\kappa_m} \sum_{j=0}^{k-1} \eta_j^{CR} \eta_j^{FX} \rho_{y_c,z_m} e^{-\kappa_m t} (e^{\kappa_m t_{j+1}} - e^{\kappa_m t_j}) \\
&\quad + \frac{\eta_k^{CR}}{\kappa_m} \left(1 - e^{-\kappa_m t} e^{\kappa_m t_k}\right) \\
&=: A_{y,h}(t) + \eta_k^{CR} B_{y,h}(t); \qquad (9.31)
\end{aligned}
$$

$$
\begin{aligned}
V_h(m;\, t) &= \frac{1}{2\kappa_m} \sum_{j=0}^{k-1} (\eta_j^{CR})^2 e^{-2\kappa_m t} (e^{2\kappa_m t_{j+1}} - e^{2\kappa_m t_j}) \qquad (9.32) \\
&\quad + \frac{(\eta_k^{CR})^2}{2\kappa_m} (1 - e^{-2\kappa_m (t - t_k)}) \\
&=: A_V(t) + (\eta_k^{CR})^2 B_V(t)
\end{aligned}
$$

Plugging (9.29)-(9.32) into (9.26) we get the following inequality on the interval $t_k \leq t < t_{k+1}$:

$$
\inf_{t_k \leq t < t_{k+1}} \frac{A_\vartheta(t) + (\eta_k^{CR})^2 B_{\vartheta,1}(t) + \eta_k^{CR} B_{\vartheta,2}(t)}{\sqrt{A_V(t) + (\eta_k^{CR})^2 B_V(t)}} \geq \delta. \qquad (9.33)
$$

where

$$
\begin{aligned}
A_\vartheta(t) &= \frac{1}{t_{k+1} - t_k} \ln \frac{S_m(t_0, t_k)}{S_m(t_0, t_{k+1})} + 0.5 A_h(t) + A_{r,h}(t) - 1_{c \neq 0} A_{y,h}(t), \\
B_{\vartheta,1}(t) &= B_h(t) \\
B_{\vartheta,2}(t) &= B_{r,h}(t) - 1_{c \neq 0} B_h(t).
\end{aligned}
$$

Assuming zero correlations between hazard rate and interest rate and exchange rate, we get:

$$
\inf_{t_k \leq t < t_{k+1}} \frac{A_\vartheta(t) + (\eta_k^{CR})^2 B_{\vartheta,1}(t) + \eta_k^{CR} B_{\vartheta,2}(t)}{\sqrt{A_V(t) + (\eta_k^{CR})^2 B_V(t)}} \geq \frac{A_\vartheta(t_k)}{\sqrt{A_V(t_{k+1}) + (\eta_k^{CR})^2 B_V(t_{k+1})}},
$$

from which we define $\tilde{\eta}_k^{CR}$ as:

$$
(\tilde{\eta}_k^{CR})^2 \quad := \quad \frac{(\frac{A_\vartheta(t_k)}{\delta})^2 - A_V(t_{k+1})}{B_V(t_{k+1})}. \qquad (9.34)
$$

9.4 Measure Relationships

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 is given by:

$$
dW_{x_{k,i}}^{(0)} = \rho_{x_{k,i},y_j} \eta_j^{FX}(t) dt + dW_{x_{k,i}}^{(j)} \qquad (9.35)
$$

9.5 System of Equations

Under the domestic currency CCY_0 risk neutral measure, the SDE's become:

$$
\begin{aligned}
dx_{0,i} &= -a_{0,i}x_{0,i}dt + \eta_{0,i}dW^{(0)}_{x_{0,i}} \\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \rho_{x_{k,i},y_k}\eta_{k,i}\eta^{FX}_k\Big]dt + \eta_{k,i}dW^{(0)}_{x_{k,i}} \\
\frac{dy_k}{y_k} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=1}^{N_0} x_{0,i} - \sum_{i=1}^{N_k} x_{k,i}\Big]dt + \eta^{FX}_k dW^{(0)}_{y_k} \\
dz_m &= -\kappa_m z_m dt + \eta^{CR}_m dW^{(0)}_{z_m}
\end{aligned}
\tag{9.36}
$$

The system of the equations (9.36) can be solved using the same technique as outlined in Chapter 7 on page 51.

9.6 Monte Carlo Simulation

For the details of the Monte Carlo simulation of the SDE (9.36), see see Chapter 3 on page 19 and Chapter 5 on page 33. In addition, for each simulation step, we keep track of the integral of $z_m(t)$ which is required for the calculation of the integral of the hazard rate, and of default indicator for each counterparty.

At each simulation date t_N, the integral of $z_m(t)$ is approximated using one of the following approximation formulas:

$$\int_{t_0}^{t_N} z(s)ds \approx \sum_{n=1}^{N} z_m(t_{n-1}) \cdot (t_n - t_{n-1}) \tag{9.37}$$

or

$$\int_{t_0}^{t_N} z(s)ds \approx \sum_{n=1}^{N} \frac{z_m(t_{n-1}) + z_m(t_n)}{2}(t_n - t_{n-1}). \tag{9.38}$$

Then, the hazard rate integral at each simulation date t_N is calculated as:

$$\int_{t_0}^{t_N} h_m(s)ds = \int_{t_0}^{t_N} \vartheta_m(s)ds + \int_{t_0}^{t_N} z(s)ds. \tag{9.39}$$

The default indicator for the mth counterparty

$$I_m(t) := 1_{\tau_m > t} \tag{9.40}$$

is updated using the following rule:

$$I_m(t_{n+1}) = I_m(t_n)1_{\tau_m \notin [t_{n-1},t_n]}, \tag{9.41}$$

where $\tau_m \in [t_{n-1}, t_n]$ with the probability

$$\int_{t_{n-1}}^{t_n} h_m(s)ds = \int_{t_0}^{t_n} h_m(s)ds - \int_{t_0}^{t_{n-1}} h_m(s)ds \tag{9.42}$$

The Gaussian dynamics of the hazard rate $h_m(t)$ implies the non-zero probability of the integral $\int_{t_{n-1}}^{t_n} h_m(s)ds$ being negative. To avoid the negative default probability, the integral is replaced with $\max\{\int_{t_{n-1}}^{t_n} h_m(s)ds, 0\}$ in the simulation:

$$1_{\tau_m \notin [t_{n-1},t_n]} = 1_{\max\{\int_{t_{n-1}}^{t_n} h_m(s)ds, 0\} < U}$$

where U is a random draw from a uniform distribution independent of the driving Brownians in (9.36).

9.7 Model Calibration

As the first step, the calibration of the short rate volatility is performed separately, for each of the model currency $CCY_0, \ldots, CCY_n$. The calibration of a single-currency IR model as described in Chapter 2 on page 11.

Next, the calibration of the exchange rate volatility is performed separately, for each of the model foreign currency $CCY_1, \ldots, CCY_n$. The calibration of the exchange rate volatility is described in Chapter 5 on page 33.

For each model counterparty name, the initial survival probability curve is bootstrapped from the input CDS curve, and the drift adjustment $\int_{t_0}^{t} \vartheta_m(s)ds$ is calculated as described in the Chapter 9.2.3. The mean reversion parameter and volatility of the hazard rate in (9.3) are the input model parameters. The input hazard rate volaitlity is adjusted to keep the probability of hazard rate to go negative below the requested level, as described in the Chapter 9.3.

Chapter 10

Hybrid Commodity Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rate

This chapter describes issues relating to the implementation of a hybrid commodity multi-currency model with multi-factor Hull-White models for interest rates.

10.1 Introduction

In the commodity extension of the multi-currency multi-factor HW NF model (HW NF), a single currency HW NF model is used for the interest rates for each currency. The spot foreign exchange (FX) rates between currencies and spot commodity prices are modelled as lognormal random variables. Commodity futures prices are calculated from the simulated commodity spot. The spread between the interest rate and convenience yield of each commodity is modelled as a single currency HW NF model.

Consider a model with $n+1$ currencies $k = 0, 1, \dots, n$. Let CCY_0 be the domestic currency and $CCY_1, \dots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_k, are modelled using a multi-factor single currency HW NF model with N_k factors:

$$\begin{aligned} dx_{k,i}(t) &= -a_{k,i}(t)x_{k,i}(t)dt + \eta_{k,i}(t)dW^{(k)}_{x_{k,i}}, \quad x_{k,i}(0) = 0 \\ r_k(t) &= \phi_k(t) + \sum_{i=1}^{N_k} x_{k,i}(t), \quad dW^{(k)}_{x_{k,i}} dW^{(k)}_{x_{k,j}} = \rho_{x_{k,i},x_{k,j}}(t)dt \end{aligned} \tag{10.1}$$

where, for each currency k, $W^{(k)}_{k,i}$ are Wiener processes in the risk neutral measure for that currency k, $a_{k,i}(t)$ are mean reversion parameters, $\eta_{k,i}(t)$ are volatility parameters and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. For the details of the calculation of the function $\phi_k(t)$, see Chapter 3 on page 19. In what follows, we will assume time independent mean reversion parameters $a_{k,i}(t) = a_{k,i}$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \dots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_k(t)}{y_k(t)} = (r_0(t) - r_k(t))\,dt + \eta^{FX}_k(t)dW^{(0)}_{y_k} \tag{10.2}$$

where, $\eta^{FX}_k(t)$ is the volatility for FX rate $y_k(t)$ and $W^{(0)}_{y_k}$ is a Wiener process in the risk neutral measure for that domestic currency CCY_0.

Let there also be N_{CM} commodities. For kth commodity, we assume that the dynamics of the spot price process $p_k(t)$ is governed by the following SDE under the risk neutral measure of the commodity currency c_k,

$$\begin{aligned} \frac{dp_k(t)}{p_k(t)} &= \lambda_k(t)dt + \eta_k^{cm}(t)dW_{p_k}^{(c_k)}, \\ \lambda_k(t) &= r_{c_k}(t) - r_k^c(t). \end{aligned} \tag{10.3}$$

Here, $\lambda_k(t)$ is the stochastic spread between the short interest rate for the currency c_k and convenience yield of the kth commdity $r_k^c(t)$, $\eta_k^{cm}(t)$ is the volatility for the commodity spot price process, and $W_{p_k}^{(c_k)}$ is a Wiener process in the risk neutral measure for the currency c_k.

The initial spot commodity price $p_k(t_0)$ is taken to be the price of the closest maturity contract on the initial forward curve for the commodity k.

The spread $\lambda_k(t)$ between the short interest rate for the currency c_k and convenience yield of the kth commdity is modelled using a multi-factor mean reversion SDE, similar to the multi-factor single currency HW model:

$$d\omega_{k,i}(t) = -\gamma_{k,i}(t)\omega_{k,i}(t)dt + \sigma_{k,i}(t)dW_{\omega_{k,i}}^{(c_k)}, \quad \omega_{k,i}(0) = 0 \tag{10.4}$$

$$\lambda_k(t) = \zeta_k(t) + \sum_{i=1}^{N_k^{cm}} \omega_{k,i}(t), \quad dW_{\omega_{k,i}}^{(c_k)} dW_{\omega_{k,j}}^{(c_k)} = \rho_{\omega_{k,i}\omega_{k,j}}(t)dt$$

Here, $W_{\omega_{k,j}}^{(c_k)}$ is the Wiener processes in the risk neutral measure for currency c_k, $\gamma_{k,i}(t)$ are mean reversion parameters, $\sigma_{k,i}(t)$ are volatility parameters and ζ_k is a function chosen so that the initial commodity futures curve is reproduced. For the details of the calculation of ζ_k, see section 10.2.2. In what follows, we will assume time independent mean reversion parameters $\gamma_{k,i}(t) = \gamma_{k,i}$.

10.2 Commodity Futures and Forward Price Calculation. Choice of ζ.

Let us denote $F_{fut}(t,T)$ the t-price of the commodity futures contract, and $F_{fwd}(t,T)$ the t-price of the commodity forward contract with the maturity T, for a k-th commodity. For the simplicity of the notations, we skip the index k. By the definition of the futures and forward contract price, we have:

$$F_{fut}(t,T) := E_t^{(c)}[p(T)], \tag{10.5}$$

$$F_{fwd}(t,T) := E_t^{(c),T}[p(T)], \tag{10.6}$$

where $p(T)$ is the spot commodity price and $E^{(c),T}$ denotes the expected value under the T-forward measure in the commodity currency c. In this section, we derive the analytic representation of the commodity futures and forward price, and the function $\zeta(t)$ used in the spot commodity equation (10.4).

10.2.1 Commodity Futures Price

The commodity futures t-price under the risk-neutral measure of the commodity currency c can be represented as follows,

$$F_{fut}(t,T) := \exp\left\{ E_t^{(c)}[\ln p(T)] + \frac{1}{2} Var_t^{(c)}[\ln p(T)] \right\}, \tag{10.7}$$

as the spot price $p(T)$ has a lognormal distribution under the risk-neutral measure of the commodity currency c.

Now let us calculate the variance and the expectation for the $\ln p(T)$ at time t. From the equation (10.3) we have,

$$\begin{aligned} \ln p(T) &= \ln p(t) + \int_t^T \left(\zeta(u) + \sum_{i=1}^{N^{cm}} \omega_i(u) - \frac{1}{2}(\eta^{cm})^2(u) \right) du \\ &\quad + \int_t^T \eta^{cm}(u) dW_p^{(c)}(u). \end{aligned} \tag{10.8}$$

Here the stochastic spread drivers $\omega_i(u)$, $u > t$ are governed by the equation (10.4), and can be represented as follows:

$$\omega_i(u) = \omega_i(t)e^{-\gamma_i(u-t)} + \int_t^u \sigma_i(\tau)e^{-\gamma_i(u-\tau)}dW^{(c)}_{\omega_i}(\tau). \tag{10.9}$$

Integrating equation (10.9) and changing the order of integration, we get

$$\begin{aligned}\int_t^T \omega_i(u)du &= \omega_i(t)B(\gamma_i,t,T) + \int_t^T \sigma_i(u)B(\gamma_i,u,T)dW^{(c)}_{\omega_i}(u), \\ B(a,u,T) &= \frac{1-e^{-a(T-u)}}{a}.\end{aligned} \tag{10.10}$$

Plugging (10.10) into (10.8), we get the following representation for the expectation and variance of the spot commodity price $p(T)$,

$$E^{(c)}_t[\ln p(T)] = \ln p(t) + \int_t^T \zeta(u)du - \frac{1}{2}\int_t^T (\eta^{cm})^2(u)du + \sum_{i=1}^{N^{cm}} \omega_i(t)B(\gamma_i,t,T);$$

$$\begin{aligned}Var^{(c)}_t[\ln p(T)] &= \int_t^T (\eta^{cm})^2(u)du + 2\sum_{i=1}^{N^{cm}}\int_t^T \rho_{p,\omega_i}\eta^{cm}(u)\sigma_i(u)B(\gamma_i,u,T)du \\ &+ \sum_{i,j=1}^{N^{cm}}\int_t^T \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i,u,T)\sigma_j(u)B(\gamma_j,u,T)du.\end{aligned} \tag{10.11}$$

Finally, we get the folowing analytic representation for the commodity futures price,

$$\begin{aligned}F_{fut}(t,T) \;&:=\; p(t)\exp\left\{\int_t^T \zeta(u)du + \sum_{i=1}^{N^{cm}}\omega_i(t)B(\gamma_i,t,T)\right. \\ &+ \sum_{i=1}^{N^{cm}}\int_t^T \rho_{p,\omega_i}\eta^{cm}(u)\sigma_i(u)B(\gamma_i,u,T)du \\ &\left.+\frac{1}{2}\sum_{i,j=1}^{N^{cm}}\int_t^T \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i,u,T)\sigma_j(u)B(\gamma_j,u,T)du\right\}.\end{aligned} \tag{10.12}$$

10.2.2 Choice of ζ

The drift adjustment term ζ in the equation (10.3) is chosen to reproduce the initial commodity futures price curve, $F_{fut}(t_0,T)$. From (10.12) we have,

$$\begin{aligned}\ln F_{fut}(t_0,T) &= \ln p(t_0) + \int_{t_0}^T \zeta(u)du \\ &+ \sum_{i=1}^{N^{cm}}\int_{t_0}^T \rho_{p,\omega_i}\eta^{cm}(u)\sigma_i(u)B(\gamma_i,u,T)du \\ &+\frac{1}{2}\sum_{i,j=1}^{N^{cm}}\int_{t_0}^T \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i,u,T)\sigma_j(u)B(\gamma_j,u,T)du.\end{aligned} \tag{10.13}$$

To get equity in (10.13), we take

$$\begin{aligned}\int_{t_0}^T \zeta(u)du \;&:=\; \ln F_{fut}(t_0,T) - \ln p(t_0) \\ &- \sum_{i=1}^{N^{cm}}\int_{t_0}^T \rho_{p,\omega_i}\eta^{cm}(u)\sigma_i(u)B(\gamma_i,u,T)du \\ &-\frac{1}{2}\sum_{i,j=1}^{N^{cm}}\int_{t_0}^T \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i,u,T)\sigma_j(u)B(\gamma_j,u,T)du.\end{aligned} \tag{10.14}$$

Here, $F_{fut}(t_0, T)$ is calculated by interpolating into the initial commodity futures curve.
Note that

$$\begin{aligned}
\int_t^T \zeta(u)du \quad := \quad & \int_{t_0}^T \zeta(u)du - \int_{t_0}^t \zeta(u)du \\
& = \ln F_{fut}(t_0, T) - \ln F_{fut}(t_0, t) \\
& - \sum_{i=1}^{N^{cm}} \int_{t_0}^T \rho_{p,\omega_i} \eta^{cm}(u)\sigma_i(u)B(\gamma_i, u, T)du \\
& + \sum_{i=1}^{N^{cm}} \int_{t_0}^t \rho_{p,\omega_i} \eta^{cm}(u)\sigma_i(u)B(\gamma_i, u, t)du \\
& - \frac{1}{2}\sum_{i,j=1}^{N^{cm}} \int_{t_0}^T \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i, u, T)\sigma_j(u)B(\gamma_j, u, T)du. \\
& + \frac{1}{2}\sum_{i,j=1}^{N^{cm}} \int_{t_0}^t \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i, u, t)\sigma_j(u)B(\gamma_j, u, t)du.
\end{aligned} \tag{10.15}$$

Then, plugging (10.15) into (10.12), we get the following analytic representation for the commodity futures price,

$$\begin{aligned}
F_{fut}(t, T) \quad := \quad & p(t)\frac{F_{fut}(t_0, T)}{F_{fut}(t_0, t)} \exp\left\{ \sum_{i=1}^{N^{cm}} \omega_i(t)B(\gamma_i, t, T) \right. \\
& -I_{\omega,p}(t_0, t, T) + I_{\omega,p}(t_0, t, t) \\
& \left. -\frac{1}{2}I_{\omega,\omega}(t_0, t, T) + \frac{1}{2}I_{\omega,\omega}(t_0, t, t) \right\},
\end{aligned} \tag{10.16}$$

where, $I_{\omega,p}(t_0, t_1, t_2)$ is the covariance integral term between stochastic commodity spread and commodity spot price,

$$I_{\omega,p}(t_0, t_1, t_2) \quad = \quad \sum_{i=1}^{N^{cm}} \int_{t_0}^{t_1} \rho_{p,\omega_i}\eta^{cm}(u)\sigma_i(u)B(\gamma_i, u, t_2)du;$$

$I_{\omega,\omega}(t_0, t_1, t_2)$ is the covariance integral term between factors of the stochastic commodity spread,

$$I_{\omega,\omega}(t_0, t_1, t_2) \quad = \quad \sum_{i,j=1}^{N^{cm}} \int_{t_0}^{t_1} \rho_{\omega_i,\omega_j}\sigma_i(u)B(\gamma_i, u, t_2)\sigma_j(u)B(\gamma_j, u, t_2)du.$$

10.2.3 Commodity Forward Price Calculation.

The commodity forward t-price under the risk-neutral measure of the commodity currency c can be represented as follows,

$$\begin{aligned}
F_{fwd}(t, T) \quad := \quad & E_t^{(c),T}[p(T)] = \frac{N_c(t)}{P_c(t, T)} E_t^{(c)}\left[\frac{p(T)}{N_c(T)}\right] \\
= \quad & F_{fut}(t, T)e^{\Omega_c(t,T)},
\end{aligned} \tag{10.17}$$

where

$$\begin{aligned}
\Omega_c(t, T) \quad = \quad & Cov_t\left[\ln p(T), -\int_t^T r_c(s)ds\right] \\
= \quad & -\sum_{i=1}^{N_c} \int_t^T \rho_{p,x_{c,i}} B(a_{c,i}, s, T)\eta_{c,i}(s)\eta^{cm}(s)ds \\
& -\sum_{i=1}^{N_c}\sum_{j=1}^{N^{cm}} \int_t^T \rho_{\omega_j,x_{c,i}} B(a_{c,i}, s, T)B(\gamma_j, s, T)\eta_{c,i}(s)\sigma_j(s)ds.
\end{aligned} \tag{10.18}$$

10.3 Measure Relationships

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 for a model stochastic risk factor R is given by:

$$dW_R^{(0)} = \rho_{R,y_j}\eta^R(t)\eta_j^{FX}(t)dt + dW_R^{(j)}. \tag{10.19}$$

Here, $\eta^R(t)$ is the diffusive volatility of the risk factor R, and ρ_{R,y_j} is the correlation between the Brownian increments governing risk factor R and the exchange rate of the currency CCY_j.

10.4 Monte Carlo Simulation

Under the domestic currency CCY_0 risk neutral measure, the SDE's become:

$$\begin{aligned}
dx_{0,i} &= -a_{0,i}x_{0,i}dt + \eta_{0,i}dW_{x_{0,i}}^{(0)} \\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \rho_{x_{k,i},y_k}\eta_{k,i}\eta_k^{FX}\Big]dt + \eta_{k,i}dW_{x_{k,i}}^{(0)} \\
\frac{dy_k}{y} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=0}^{N_0-1} x_{0,i} - \sum_{i=0}^{N_k-1} x_{k,i}\Big]dt + \eta_k^{FX}dW_{y_k}^{(0)} \\
d\omega_{k,i} &= \Big[-\gamma_{k,i}\omega_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{\omega_{k,i},y_{c_k}}\sigma_{k,i}\eta_{c_k}^{FX}\Big]dt + \sigma_{k,i}dW_{\omega_{k,i}}^{(0)} \\
\frac{dp_k}{p_k} &= \Big[\zeta_k(t) + \sum_{i=1}^{N_k^{cm}} \omega_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{p_k,y_{c_k}}\eta_k^{cm}\eta_{c_k}^{FX}\Big]dt + \eta_k^{cm}(t)dW_{p_k}^{(0)}
\end{aligned} \tag{10.20}$$

The system of the equations (10.20) can be solved using the same technique as outlined in Chapter 7 on page 51.

10.5 Model Calibration

As the first step, the calibration of the short rate volatility is performed separately, for each of the model currency CCY_0,...,CCY_n. The calibration of a single-currency IR model as described in Chapter 2 on page 11.

Next, the calibration of the exchange rate volatility is performed separately, for each of the model foreign currency CCY_1,...,CCY_n. The calibration of the exchange rate volatility is described in Chapter 5 on page 33.

For each commodity, the calibration of the spot price volatility $\eta^{cm}(t)$ is performed independently to the market implied ATM futures option volatility. Assume that for a kth model commodity, the implied ATM volatility

$$s_1, \ldots, s_M$$

is available for the option expiry dates

$$T_1 < \cdots < T_M,$$

where $t_0 < T_1$ is the calibration date. The market commodity futures price corresponding to the pair (T_k, s_k) is calculated from the futures price by Black formula:

$$\begin{aligned}
V_k(t_0) &= P_c(t_0,T_k)[F_{fut}(t_0,T_k)N(d_1) - KN(d_2)], \\
d_1 &= \frac{\ln\left(\frac{F_{fut}(t_0,T_k)}{K}\right) + 0.5s_k^2T_k}{s_k\sqrt{T_k}} \\
d_2 &= d_1 - s_k\sqrt{T_k}.
\end{aligned}$$

In our calibration, we build $\eta_1^{cm},\dots,\eta_M^{cm}$ recursively to match the volatility of the futures price process, simulated by the model at the time $T_1,\dots,T_M$, with the implied market volatilities $s_1,\dots,s_M$. We assume the piecewise-constant dynamics of $\eta^{cm}(t)$, where

$$\eta^{cm}(t) = \eta_k^{cm}, \quad T_{k-1} \le t < T_k.$$

Further we assume, that the convenience spread process is governed by a single-factor SDE.

From the equation (10.11) and (10.16), we have:

$$\begin{aligned} Var\left[\ln F_{fut}(T_k,T_k)\right] &= Var\left[\ln p_k(T_k)\right] \\ &= \int_{t_0}^{T_k} (\eta^{cm})^2(u)du + 2\int_{t_0}^{T_k} \rho_{p,\omega}\eta^{cm}(u)\sigma(u)B(\gamma,u,T_k)du \\ &\quad + \int_{t_0}^{T_k} \sigma^2(u)B^2(\gamma,u,T_k)du. \end{aligned} \tag{10.21}$$

Equating the righthand side to $s_k^2(T_k - t_0)$ gives us the following recursive procedure.

For $k = 1$, we solve the quadratic equation to get η_1^{cm}:

$$\begin{aligned} (\eta_1^{cm})^2(T_1 - t_0) \quad &+ \quad 2\eta_1^{cm}\rho_{p,\omega}\int_{t_0}^{T_1} \sigma(u)B(\gamma,u,T_1)du \\ &+ \int_{t_0}^{T_1} \sigma^2(u)B^2(\gamma,u,T_1)du \\ &= \quad s_1^2(T_1 - t_0). \end{aligned} \tag{10.22}$$

Assume we have $\eta_1^{cm},\dots,\eta_{k-1}^{cm}$. Then, we solve the quadratic equation to get η_k^{cm}:

$$\begin{aligned} (\eta_k^{cm})^2(T_k - T_{k-1}) \quad &+ \quad 2\eta_k^{cm}\rho_{p,\omega}\int_{T_{k-1}}^{T_k} \sigma(u)B(\gamma,u,T_k)du \\ &+ \int_{t_0}^{T_{k-1}} (\eta^{cm})^2(u)du + 2\int_{t_0}^{T_{k-1}} \rho_{p,\omega}\eta^{cm}(u)\sigma(u)B(\gamma,u,T_k)du \\ &+ \int_{t_0}^{T_k} \sigma^2(u)B^2(\gamma,u,T_k)du \\ &= \quad s_k^2(T_k - t_0). \end{aligned} \tag{10.23}$$

The drift adjustment ζ in the convenience spread SDE (10.4) is calculated as described in the section 10.2.2.

Chapter 11

Hybrid Credit Equity Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Black-Karasinski Models for Hazard Rates

This chapter describes issues relating to the implementation of a hybrid credit equity multi-currency model with multi-factor Hull-White models for interest rates and single factor Black-Karasinski models for hazard rates.

11.1 Introduction

Consider a model with $n+1$ currencies $k = 0, 1, \ldots, n$ each described by a multi-factor Hull-White interest rate model with risk factors $x_{k,i}$, n lognormal foreign exchange (FX) rates between currencies y_k and N_{EQ} lognormal equities s_j, $j = 1, \ldots, N_{EQ}$. Let $k = 0$ be the domestic currency and $k = 1, \ldots, n$ be the n foreign currencies. Under the domestic risk neutral measure, the dynamics are governed by the SDE's:

$$\begin{aligned}
dx_{0,i} &= -a_{0,i}x_{0,i}dt + \eta_{0,i}dW_{x_{0,i}} \\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \rho_{x_{k,i},y_k}\eta_{k,i}\eta_k^{FX}\Big]dt + \eta_{k,i}dW_{x_{k,i}} \\
\frac{dy_k}{y} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=0}^{N_0-1} x_{0,i} - \sum_{i=0}^{N_k-1} x_{k,i}\Big]dt + \eta_k^{FX}dW_{y_k} \\
\frac{ds_j}{s} &= \Big[\phi_{c_j}(t) + \sum_{i=0}^{N_{c_j}-1} x_{c_j,i} - \delta_j(t) - \mathbf{1}_{[c_j \neq 0]}\rho_{s_j,y_{c_j}}\eta_j^{EQ}\eta_{c_j}^{FX}\Big]dt + \eta_j^{EQ}dW_{s_j}
\end{aligned} \tag{11.1}$$

where N_k is the number of interest rate risk factors for k^{th} currency and c_j is the currency for the j^{th} equity.

For more information regarding the model, see Chapter 7 on page 51.

To model credit, assume the dynamics of the hazard rate for each credit name $m = 1, \ldots, N_{CR}$ under the domestic risk neutral measure is described by lognormal Black-Karasinski type model:

$$\begin{aligned}
dz_m(t) &= -\kappa_m z_m(t)dt + \eta_m^{CR}(t)dW_{z_m}(t), \quad z_m(0) = 0 \\
h_m(t) &= h_m(0)e^{\vartheta_m(t)+\xi_m(t)+z_m(t)}
\end{aligned} \tag{11.2}$$

where

$$\vartheta_m(t) = \ln\left(h_m(0,t)/h_m(0)\right), \quad h_m(0,T) = -\frac{\partial \ln S_m(0,T)}{\partial T}, \quad h_m(0) = h_m(0,0) \tag{11.3}$$

The initial survival probabilities $S_m(0,T)$ are given by

$$S_m(0,T) = \mathbb{E}^T\left[\mathbf{1}_{\{\tau_m > T\}}\right] = \mathbb{E}^T\left[e^{-\int_0^T h_m(u)du}\right] \tag{11.4}$$

where τ_m is the counterparty default time and $\mathbb{E}^T$ is the expectation under the domestic T forward measure. Initial survival probabilities are analogous to discount factors while the intensity process is analogous to a short rate. Note that when the hazard rates are uncorrelated with domestic interest rates, the survival probabilities under different forward measures are the same in the surival probabilities under the risk neutral measure.

Using a lognormal short rate model (11.2) for hazard rates allows us to achieve higher credit spread volatilities while ensuring that rates remain positive which is not the case for Gaussian or CIR-type processes. However, lognormal short rate models do not possess simple analytic formulas for zero coupon bonds. Thus, numerical methods are required to calibrate the model and to obtain values analogous to bond prices.

11.2 Credit Model Calibration

We now describe procedures to calibrate the credit model. For the remainder of this section, we consider only a single credit name and drop the m subscript notation used to described multiple credit names. We assume that the hazard rate mean reversion κ, volatility $\eta^{CR}(t)$ and correlations with the domestic interest rate $\rho_{x_{0,i},z}(t)$ are user specified parameters.

Given a term structure of initial counterparty survival probabilities $S(0,T)$ in (11.4), we can use a numerical tree-based procedure described in [8] to calibrate the $\xi(t)$ parameter in (11.2) to reproduce the initial survival probabilities. However, we must first extract survival probabilities from credit default swap quotes. The details behind this extraction step are described below. Complications arise due to correlations between interest rate state variables and the counterparty default intensity process.

11.2.1 Credit Default Swaps

Consider a credit default swap (CDS) where party A agrees to pay party B a fixed coupon rate R_f with dates $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$ and year fractions α_i in exchange for a protection amount of L_{GD} in case of default.[1] The value of the CDS to party B at time 0 is then

$$\begin{aligned} CDS(0) &= N(0)\mathbb{E}\Big[R_f \frac{\mathbf{1}_{\{\tau < T_n\}}}{N(\tau)}\alpha(\tau) \\ &\quad + R_f \sum_{i=0}^{n-1} \frac{\mathbf{1}_{\{\tau \geq T_{i+1}\}}}{N(T_{i+1})}\alpha_i - L_{GD}\frac{\mathbf{1}_{\{\tau \leq T_n\}}}{N(\tau)}\Big] \end{aligned} \tag{11.5}$$

where $\alpha(\tau)$ is the year fraction from $T_{\beta(\tau)-1}$ to τ (accrued interest) and $\beta(t)$ to be the index of the first reset on or after t; *i.e.*

$$\beta(t) = \{i : T_{i-1} < t \leq T_i\} \tag{11.6}$$

and $N(t)$ is the value of the numeraire at time t.[2] Using the time discretization $0 = t_0 < t_1 < \cdots < t_l = T_n$, the CDS price (11.5) is approximately

$$\begin{aligned} CDS(0) &\approx R_f \sum_{j=0}^{l-1} DIP(0,t_j,t_{j+1})\alpha(t_j) \\ &\quad + R_f \sum_{i=0}^{n-1} \tilde{P}(0,T_{i+1})\alpha_i - L_{GD}\sum_{j=0}^{l-1} DIP(0,t_j,t_{j+1}) \end{aligned} \tag{11.7}$$

[1] When the CDS coupon schedule is generated using IMM dates, the beginning of the first coupon period can occur prior to the valuation date ($T_0 < 0$) where the protection buyer pays the full coupon amount at T_1 and the protection seller pays a rebate amount at $t = 0$ equal to the accrued interest between $t = T_0$ and $t = 0$, $R_f\alpha^*$, where α^* is the year fraction for $t = T_0$ to $t = 0$.

[2] The above formulas are valid for any expectation-numeraire pair. For more details on CDS see [3].

where $\tilde{P}(0,T)$ is the value of a defaultable bond maturing at time T; *i.e.*

$$\tilde{P}(0,T) = N(0)\mathbb{E}\left[\frac{\mathbf{1}_{\{\tau \geq T\}}}{N(T)}\right] = N(0)\mathbb{E}\left[\frac{e^{-\int_0^T h(s)ds}}{N(T)}\right] \tag{11.8}$$

and $DIP(0,t_1,t_2)$ is defined as

$$DIP(0,t_1,t_2) = N(0)\mathbb{E}\left[\frac{\mathbf{1}_{\{t_1<\tau\leq t_2\}}}{N(t_1)}\right]. \tag{11.9}$$

We refer to the latter quantity as the "default-in-period term" for the remainder of the paper. We will use a time discretization $0 = t_0 < t_1 < \cdots < t_l = T_n$ that corresponds to the the union of coupon schedule dates T_i with the mid points of each coupon period $T_i^\dagger = (T_{i-1} + T_i)/2$.

For small time steps $t_2 - t_1 \ll 1$, the default-in-period term is approximately

$$\begin{aligned} DIP(0,t_1,t_2) &\approx P(0,t_1)\tilde{D}(0,t_1)(t_2-t_1), \\ \tilde{D}(0,T) &= \frac{N(0)}{P(0,T)}\mathbb{E}\left[\frac{h(T)e^{-\int_0^T h(s)ds}}{N(T)}\right]. \end{aligned} \tag{11.10}$$

Evaluating the expectation in (11.8) using the T forward measure, the value of a defaultable bond simplifies to:

$$\tilde{P}(0,T) = P(0,T)S(0,T) \tag{11.11}$$

If we assume counterparty default is independent of movements of the yield curve, $\tilde{D}(0,T)$ further simplifies to

$$\tilde{D}(0,T) = h(0,T)S(0,T) \tag{11.12}$$

Under this independence assumption, $S(0,T)$ (and thus $h(0,T)$) can be obtained from CDS quotes and the initial discount factor curve $P(0,T)$ via a bootstrapping procedure. Now, having obtained $S(0,T)$, we can calibrate the stochastic intensity model as described previously.

11.2.1.1 Calibration to CDS

The procedure we have just described only allows us to calibrate the credit model to defaultable bond prices. However, in practice, we would like to calibrate to CDS quotes instead. As shown in (11.7), the value of a CDS contract depends on both defaultable bond prices and default-in-period terms. Since the default-in-period terms are sensitive to correlations between interest rate and counterparty credit drivers, the bootstrapping procedure used to obtain defaultable bond prices from CDS quotes needs to account for this correlation.

One way to calibrate the credit model to CDS quotes is to embed the tree procedure directly into the CDS bootstrapping routine. The default-in-period terms could be calculated numerically using the tree:

$$\tilde{D}(0,t_i) = \mathbb{E}^{t_i}\left[h(t_i)e^{-\int_0^{t_i} h(s)ds}\right] \tag{11.13}$$

However, this would be slow because CDS bootstrapping routines typically use numerical root finding procedures and a tree calculation would have to be performed during each iteration of the root find. The following approach is more efficient.

First, obtain a curve of initial survival probabilities $S(0,T)$ from CDS quotes by using a bootstrapping routine that assumes independence between default and interest rates. Next, apply the tree as described earlier to calibrate a correlated credit model to $S(0,T)$. The model will reproduce defaultable bond prices, but unless rates/credit correlation is zero, it will fail to reproduce the original CDS quotes. Now, using the model above, calculate $\tilde{D}(0,t_i)$ from the tree using (11.13) and define the ratio:

$$\tilde{R}(0,t_i) = \frac{\tilde{D}(0,t_i)}{h(0,t_i)S(0,t_i)} \tag{11.14}$$

Now, apply a modified CDS bootstrapping routine to obtain a new set of values of $S(0,T)$ where defaultable bonds are calculated using (11.11) but now the default-in-period terms are calculated using

$\tilde{D}(0,T) = h(0,T)S(0,T)\tilde{R}(0,T)$ using the previously calculated values for $\tilde{R}(0,T)$. Finally, recalibrate the credit model using these new values of $S(0,T)$. This procedure can be repeated several times.

The ratio $\tilde{R}(0,T)$ is a correction factor for the $\tilde{D}(0,T)$ term that accounts for correlation. Numerical tests suggest that $\tilde{R}(0,T)$ depends mostly on correlations and volatilities and is not as sensitive to the values of $S(0,T)$. Consequently, CDS quotes can be recovered after only a few iterations. The procedure is quite efficient because it does not explicitly embed the tree calculation in the bootstrapping procedure. For more details of the calibration procedure and some numerical results, see [8] which describes how the calculations can be performed efficiently using a one dimensional tree.

11.2.2 Foreign Currency Denominated CDS

The calibration procedure previously described assumes the CDS are denominated in the domestic currency of the model (11.1). If we want to calibrate the credit model to CDS denominated in a foreign currency $k \neq 0$, we can apply the same procedure as before but now take the risk neutral measure in (11.2) to be in the CDS currency rather than the domestic currency of the model (11.1). The calibrated parameters $\vartheta^{(k)}(t)$ and $\xi^{(k)}(t)$ are now with respect to the risk neutral measure in the foreign currency k. Values for $\vartheta(t)$ and $\xi(t)$ under the domestic risk neutral measure can be obtained from $\vartheta^{(k)}(t)$ and $\xi^{(k)}(t)$ using a numerical tree based procedure taking into account the required quanto drift adjustment (see [8]).

Note that the model and calibration procedure we have described is limited in that it only allows us to calibrate each credit name to a set of CDS qutoes all denominated in a single currency. A more difficult problem is to calibrate the model to reproduce several CDS denominated in different currencies. Assume we have a credit model that has be calibrated to a set of CDS in the domestic currency ($k = 0$). The values of CDS denominated in a foreign currency are affected by the correlations between hazard rates and foreign interest rates and FX rates. In theory, one could to calibrate the foreign interest rate–credit and FX–credit correlations in the model, $\rho_{x_{k,i},z}$ and $\rho_{y_k,z}$, to fit market quotes for foreign CDS. However, as discussed by [5], correlated diffusion processes alone may not be sufficient to generate values for foreign denominated CDS close to what is quoted in the markets. To address this, [5] propose an additional mechanism where an additional jump occurs in the FX rate at the time of default. The jump corresponds to a devaluation of the currency caused by the default.

11.3 Monte Carlo Simulation

We provide methods to simulate this hybrid credit equity multi-currency multi-factor model under the risk neutral measure of the domestic currency. For more details, see Chapter 7 on page 51, Chapter 5 on page 33, and Chapter 6 on page 41.

In addition to simulating the risk factors, we also need to simulate the integrated hazard rate. This is done in a manner similar to how we simulate the numeraire in the risk neutral measure. Consider, for the sake of illustration, the expression (11.8) for the value of a defaultable bond; *i.e.*

$$\tilde{P}(0,T) = N(0)\mathbb{E}\left[\frac{e^{-\int_0^T h(s)ds}}{N(T)}\right].$$

We obtain a Monte Carlo estimate of $\tilde{P}(0,T)$ by simulating the integral $\int_0^T h(s)ds$ jointly with the future numeraire value $N(T)$. Specifically,

$$\tilde{P}(0,T) \approx N(0) \cdot \frac{1}{\Omega}\sum_{\omega=1}^{\Omega} \frac{e^{-\int_0^T h(\omega,s)ds}}{N(\omega,T)},$$

where $\omega \in \{1, 2, \cdots, \Omega\}$ is a simulation index. Given time steps $s < t$, we can approximate the hazard rate integral appearing here by using one of the standard approximations

$$\int_0^T h(\omega,s)ds \approx \sum_{i=0}^{N-1} h(\omega,t_i)(t_{i+1} - t_i), \tag{11.15}$$

or

$$\int_0^T h(\omega, s)ds \approx \sum_{i=0}^{N-1} \frac{h(\omega, t_{i+1}) + h(\omega, t_i)}{2}(t_{i+1} - t_i), \tag{11.16}$$

where the simulation time steps t_i satisfy $0 = t_0 < t_1 < \cdots < t_N = T$. Similar expressions hold for the default–in–period contributions (11.9) to the CDS price and in far more general credit pricing contexts, such as portfolio–level CVA.

11.3.1 Explicit Default Simulation

Recall the alternative expression for the value of a defaultable bond in (11.8); *i.e.*

$$\tilde{P}(0,T) = N(0)\mathbb{E}\left[\frac{\mathbf{1}_{\{\tau \geq T\}}}{N(T)}\right].$$

According to this equation, instead of simulating hazard rates, we could instead simulate default times to obtain the following alternative Monte Carlo approximation to the defaultable bond price; *i.e.*

$$\tilde{P}(0,T) \approx N(0) \cdot \frac{1}{\Omega} \sum_{\omega=1}^{\Omega} \frac{\mathbf{1}_{\{\tau(\omega) \geq T\}}}{N(\omega, T)},$$

where as before, $\omega \in \{1, 2, \cdots, \Omega\}$ is a simulation index. This is implemented as follows. For each simulation path, we keep track of a default indicator function:

$$I(t) = \mathbf{1}_{\{\tau > t\}} = \begin{cases} 1 & , \quad \text{if no default has occurred} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{11.17}$$

with initial value $I(0) = 1$.

Over the simulation time step $t_i \to t_{i+1}$, we update the default indicator using the rule

$$I(t_{n+1}) = I(t_n)\mathbf{1}_{\{\gamma_m(t_n, t_{n+1}) < U\ \}} \tag{11.18}$$

where $\gamma(s,t) = \int_s^t h(u)du$ is the probability of default occuring between s and t and U is a random draw from a uniform distribution independent of the driving Brownians in (11.1) and (11.2).[3]

11.4 Future Value Calculations for CDS

Both Potential Future Exposure (PFE) and Credit Valuation Adjustment (CVA) calculations required pricing trades at future points in time. Thus, to apply the model described in this document for PFE and CVA calculations on CDS trades, we need the ability to price CDS trades at future points in time.

The tree based procedure described in [8] for calibrating the model can also be applied to calculate future values of CDS trades. Similar to (11.7), the value of a CDS at some future time $t > 0$ conditional that it has not defaulted prior to t can be approximated as:

$$\begin{aligned} CDS_{\tau > t}(t) \quad \approx \quad & R_f \sum_{j=0}^{l-1} DIP(t, t_j, t_{j+1})(t_j - T_{\beta(t_j)-1}) \\ & + R_f \sum_{i=\beta(t)}^{n-1} \tilde{P}(t, T_{i+1})\alpha_i - L_{GD} \sum_{j=0}^{l-1} DIP(t, t_j, t_{j+1}) \end{aligned} \tag{11.19}$$

[3] The hazard rate process h is driven by Brownian motions that are correlated with the Brownian drivers of all the other economic state variables. These drivers, in aggregate, generate an information filtration called the *background filtration* $\mathcal{F}_t, t \geq 0$. The default indicator process, together with all other economic state variables, generates a larger filtration $\mathcal{G}_t, t \geq 0$. If we simulate default times, our model lives in this larger filtration. If we instead simulate the hazard rate process, our model lives in the smaller background filtration.

for the time discretization $t = t_0 < t_1 < \cdots < t_l = T_n$ where

$$\tilde{P}(t,T) = P(t,T)S(t,T), \quad S(t,T) = \mathbb{E}^T\left[e^{-\int_t^T h(s)ds}\right] \tag{11.20}$$

$$DIP(t,t_1,t_2) = P(t,t_1)\tilde{D}(t,t_1)(t_2-t_1), \quad \tilde{D}(t,t_1) = \mathbb{E}^{t_1}\left[h(t_1)e^{-\int_t^{t_1} h(s)ds}\right] \tag{11.21}$$

The $S(t,T)$ and $\tilde{D}(t,t_1)$ terms in (11.20) and (11.21) are conditional that default has not occurred prior to t and can be calculated numerically using a one dimensional tree.

Taking into account the default indicator, the future value of a CDS at time t is just:

$$CDS(t) = \mathbf{1}_{\{\tau>t\}}CDS_{\tau>t}(t) \tag{11.22}$$

When future values are used to calculate expectations (such as CVA), we could also replace the default indicator with an integrated hazard rate term:

$$CDS(t) = e^{-\int_0^t h(s)ds}CDS_{\tau>t}(t) \tag{11.23}$$

This avoids the need for explicit default simulation and can be more computationally efficient requiring fewer Monte Carlo paths. For more information, see [3].

Chapter 12

Wrong Way Risk Model with Deterministic Hazard Rates and FX Jump at Name Default

This chapter describes issues relating to the implementation of a hybrid credit equity inflation multi-currency model with the deterministic hazard rates, one-factor Hull-White models for interest rates and the FX jump at name default.

12.1 Introduction

In the hybrid credit equity inflation multi-currency extension of the single factor HW model (HW 1F), a single currency Hull-White model is used for the nominal interest rates for each currency. In addition, the spread between the nominal and real interest rates of each currency is also modelled as a single currency Hull-White model. The spot foreign exchange (FX) rates between currencies, spot equity prices, and inflation index are modelled as lognormal random variables. The FX rates have jumps at name defaults.

For more information on the single currency multi-factor HW 1F model, see Chapter 2 on page 11. For more information on the inflation extension of the hybrid equity multi-currency HW model, see Chapter 8 on page 61.

Consider a model with $n+1$ currencies $k = 0, 1, \ldots, n$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_i, are modelled using a one-factor single currency HW model:

$$\begin{aligned} dx_k(t) &= -a_k(t)x_k(t)dt + \eta_k(t)dW_{x_k}^{(k)}, \quad x_k(0) = 0 \\ r_k(t) &= \phi_k(t) + x_k(t), \end{aligned} \tag{12.1}$$

where, for each currency k, $W_k^{(k)}$ is the Wiener processes in the risk neutral measure for that currency k, $a_k(t)$ is the mean reversion parameter, $\eta_k(t)$ is the volatility parameter and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. In what follows, we will assume time independent mean reversion parameters $a_k(t) = a_k$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \ldots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. Further we assume that for each FX rate $y_k(t)$, there are N_k counterparies $name_1$, ..., $name_{N_k}$, whose default causes the devaluation jump of the FX rate of size $J_{k,1}, \ldots, J_{k,N_k}$:

$$y_k^+(\tau_i) = (1 + J_{k,i}(\tau_i))y_k^-(\tau_i), \quad i = 1, \ldots N_k, \tag{12.2}$$

where τ_i is the default time of the counterparty $name_i$.

The following possibilities are supported:

- $y_k(t)$ has jump at default of each of the conterparties $name_1$, ..., $name_{N_k}$. In this case, the FX rate paths may have multiple jumps. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\begin{aligned} \frac{d\tilde{y}_k(t)}{\tilde{y}_k(t)} &= (r_0(t) - r_k(t))\,dt + \eta_k^{FX}(t)dW_{\tilde{y}_k}^{(0)} \\ y_k(t) &= \prod_{i=1}^{N_k}(1 + I_{\tau_i \le t} J_{k,i}(\tau_i))\tilde{y}_k(t)e^{-\sum_{i=1}^{N_k}\int_{t_0}^{\min\{\tau_i,t\}} h_i(s)J_{k,i}(s)ds}. \end{aligned} \tag{12.3}$$

- $y_k(t)$ has single jump at first default of the conterparties $name_1$, ..., $name_{N_k}$. In this case, the dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\begin{aligned} \frac{d\tilde{y}_k(t)}{\tilde{y}_k(t)} &= (r_0(t) - r_k(t))\,dt + \eta_k^{FX}(t)dW_{\tilde{y}_k}^{(0)} \\ y_k(t) &= (1 + I_{\tau_* \le t} J_{k,*}(\tau_*))\tilde{y}_k(t)e^{-\sum_{i=1}^{N_k}\int_{t_0}^{\min\{\tau_*,t\}} h_i(s)J_{k,i}(s)ds}, \\ \tau_* &= \min\{\tau_1, ..., \tau_{N_k}\}, \\ J_{k,*} &= \{J_{k,i}|\tau_i = \tau_*\}. \end{aligned} \tag{12.4}$$

Here, $\eta_k^{FX}(t)$ is the volatility for FX rate $y_k(t)$, $W_{y_k}^{(0)}$ is a Wiener process in the risk neutral measure for the domestic currency CCY_0, $h_i(t)$ is the deterministic hazard rate for the counterparty $name_i$ in the domestic currency.

The survival probabilities $S_i(0,T)$ for the counterparty $name_i$ is given by

$$S_i(0,T) = \mathbb{E}^T\left[\mathbf{1}_{\{\tau_i > T\}}\right] = \mathbb{E}^T\left[e^{-\int_0^T h_i(u)du}\right] \tag{12.5}$$

where $\mathbb{E}^T$ is the expectation under the domestic T forward measure. Note that when the hazard rates are deterministic, the survival probabilities under different forward measures are the same as the surival probabilities under the risk neutral measure.

Let there also be N_{EQ} equities $s_0(t), \ldots, s_{N_{EQ}-1}(t)$ where equity s_j is in currency CCY_{c_j} and c_j is one of the $n+1$ currencies. The dynamics of the j'th spot equity price $s_j(t)$ in the risk neutral measure of the equity currency CCY_{c_j} s governed by:

$$\frac{ds_j(t)}{s_j(t)} = (r_{c_j}(t) - \delta_j(t))dt + \eta_j^{EQ}(t)dW_{s_j}^{(c_j)} \tag{12.6}$$

where, $r_{c_j}(t)$ is the short rate for currency CCY_{c_j}, $\delta_j(t)$ is a continuously compounded dividend rate, $\eta_j^{EQ}(t)$ is the volatility for equity price $s_j(t)$ and $W_{s_j}^{(c_j)}$ is a Wiener process in the risk neutral measure for the equity currency c_j.

In addition, there are N_I spot inflation rates $I_1(t), ..., I_{N_I}(t)$ where the kth spot inflation rate $I_k(t)$ is for the currency c_k^I. The dynamics of the rate $I_k(t)$ in the risk neutral measure of the currency c_k^I is governed by:

$$\frac{dI_k(t)}{I_k(t)} = u_k(t)dt + \eta_k^I(t)dW_{I_k}^{(c_k^I)}, \tag{12.7}$$

where, $u_k(t)$ is the spread between the nominal and real short interest rates, $\eta_k^I(t)$ is the volatility for the inflation index $I_k(t)$ and $W_{I_k}^{(c_k^I)}$ is a Wiener process in the risk neutral measure for the currency c_k^I.

The spread between the nominal and real short interest rate for the currency c_k^I is modelled using a one-factor HW model:

$$\begin{aligned} dw_k(t) &= -\alpha_k(t)w_k(t)dt + \nu_k(t)dW_{w_k}^{(c_k^I)}, \quad w_k(0) = 0 \\ u_k(t) &= \theta_k(t) + w_k(t), \quad dW_{w_k}^{(k)} \\ u_k(t) &= r_k(t) - r_k^R(t) \end{aligned} \tag{12.8}$$

where for each k, $W_{w_k}^{(c_k^I)}$ is the Wiener processes in the risk neutral measure for currency c_k^I, $\alpha_k(t)$ is the mean reversion parameter, $\nu_k(t)$ is the volatility parameter and $\theta_k(t)$ is a function chosen so that the initial inflation linked discount factor curve is reproduced. In what follows, we will assume time independent mean reversion parameters $\alpha_k(t) = \alpha_k$.

12.2 Measure Relationships

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 is given by:

$$dW_{x_k}^{(0)} \quad = \quad \rho_{x_k,y_j}\eta_j^{FX}(t)dt + dW_{x_k}^{(j)} \tag{12.9}$$

12.3 System of Equations

Under the domestic currency CCY_0 risk neutral measure, the SDE's become:

$$\begin{aligned}
dx_0 &= -a_0 x_0 dt + \eta_0 dW_{x_0}^{(0)} \qquad (12.10)\\
dx_k &= \Big[- a_k x_k - \rho_{x_k,y_k}\eta_k\eta_k^{FX}\Big]dt + \eta_k dW_{x_k}^{(0)}\\
\frac{d\tilde{y}_k}{\tilde{y}} &= \Big[\phi_0(t) - \phi_k(t) + x_0 - x_k\Big]dt + \eta_k^{FX} dW_{y_k}^{(0)}\\
\frac{ds_j}{s} &= \Big[\phi_{c_j}(t) + x_{c_j} - \delta_j(t) - \mathbf{1}_{[c_j\neq 0]}\rho_{s_j,y_{c_j}}\eta_j^{EQ}\eta_{c_j}^{FX}\Big]dt + \eta_j^{EQ} dW_{s_j}^{(0)}\\
dw_l &= \Big[- \alpha_l w_l - \mathbf{1}_{[c_l^I\neq 0]}\rho_{w_l,y_{c_l^I}}\nu_l\eta_{c_l^I}^{FX}\Big]dt + \nu_l dW_{w_l}^{(0)}\\
\frac{dI_l}{I_l} &= \Big[\theta_l(t) + w_k - \mathbf{1}_{[c_l^I\neq 0]}\rho_{I_l,y_{c_l^I}}\eta_l^I\eta_{c_l^I}^{FX}\Big]dt + \eta_k^I(t) dW_{I_l}^{(0)}
\end{aligned}$$

The exchange rate $y(t)$ is computed from $\tilde{y}(t)$ by adding the jump and the compensator terms, via (12.3) or (12.4).

12.4 Monte Carlo Simulation

The system of the equations (12.10) can be solved using the same technique as outlined in Chapter 6 on page 41. In addition, for each simulation step, we keep track of a default indicator for each counterparty, FX jump term and FX compensator integral for each exchange rate, for each counterparty.

The default indicator for the counterparty $name_i$

$$I_i(t) := I_{\tau_i > t} \tag{12.11}$$

is updated using the following rule:

$$I_i(t_{n+1}) = I_i(t_n) 1_{\int_{t_{n-1}}^{t_n} h_i(s)ds < U}, \tag{12.12}$$

where U is a random draw from a uniform distribution independent of the driving Brownians in (12.10).

The FX jump term $\mathbb{J}_{k,i}(t)$ for the currency CCY_k, for the counterparty $name_i$ is updated only for those scenarios, where the counterparty has defaulted over the current simulation step: $I_i(t_n) = 1$ and $I_i(t_{n+1}) = 0$, according to the following rule:

$$\mathbb{J}_{k,i}(t_{n+1}) = \mathbb{J}_{k,i}(t_n)(1 + J_{k,i}(t_{n+1})) \tag{12.13}$$

For the model with single FX jump at first name default, we update the jump term only for those FX rate scenarios, which have no jump before t_n.

The FX compensator integral term $\mathbb{C}_{k,i}(t)$ for the currency CCY_k, for the counterparty $name_i$ is updated only for those scenarios, where the counterpary has not defaulted: $I_i(t_n) = 1$, according to the following rule:

$$\mathbb{C}_{k,i}(t_{n+1}) = \mathbb{C}_{k,i}(t_n) + \int_{t_n}^{t_{n+1}} h_i(s)J_{k,i}(s)ds \tag{12.14}$$

For the model with single FX jump at first name default, we update the compensator term only for those FX rate scenarios, which have no jump before t_n.

The exchange rate $y(t)$ is computed from $\tilde{y}(t)$ by adding the jump and the compensator integral terms, via (12.3) or (12.4).

12.5 Calibration

The current section describes the model calibration in case when FX rates have multiple jumps at counterpary default.

The calibraiton of the IR componenet for each currency is performed separately and independently. Then, the calibration of the spot FX rate volatility, FX jumps, spot equity volaitlity, and inflation index volatility is performed using the intsrest rate calibration restuls.

For more information on single currency HW 1F model calibration, refer to Chapter 2 on page 11. For more information on equity calibration, refer to Chapter 6 on page 41.

12.5.1 FX Jump Calibration

The jump calibration for each currency is performed independently for each FX rate, for each counterpary name, as described below.

Let us consider the calibration of the jumps for the FX rate y_k. Assume that it depends on the model counterparties $name_1$, ..., $name_{N_k}$. Without loss of generality, we consider the calibration of the jumps caused by the counterparty $name_1$.

The calibration reproduces the price of the foreign defaultable bond for the currency CCY_k,

$$\begin{aligned} P_k^{def}(t_0, T) &= \tilde{y}_k(t_0)P_k(t_0, T)\mathbb{E}_k^T\left[I_{\tau_1 > T}\right] \\ &= \tilde{y}_k(t_0)P_k(t_0, T)e^{-\int_{t_0}^T h_{k,1}(s)ds}, \end{aligned} \tag{12.15}$$

where $\mathbb{E}_k^T$ is the expectation under the T forward measure of the currency CCY_k, and $h_{k,1}(t)$ is the hazard rate of the counterparty $name_1$ in the currency CCY_k.

Under the domestic T-forward measure, the price of the foreign defaultable bond $P_k^{def}(t_0, T)$ is given by:

$$P_k^{def}(t_0, T) = P(t_0, T)\mathbb{E}_0^T[y_k(T)I_{\tau_1 > T}]. \tag{12.16}$$

Using the representation (12.3), we get

$$\begin{aligned} y_k(T)I_{\tau_1 > T} &= \prod_{i=2}^{N_k}(1 + I_{\tau_i \le t}J_{k,i}(\tau_i))e^{-\int_{t_0}^{\min\{\tau_i, T\}} h_i(s)J_{k,i}(s)ds}\tilde{y}_k(t) \\ &\times \quad e^{-\int_{t_0}^T h_1(s)J_{k,1}(s)ds}I_{\tau_1 > T}. \end{aligned} \tag{12.17}$$

Here, the following random variables are independent,

$$[\tilde{y}_k(t)], \quad [(1 + I_{\tau_i \le t}J_{k,i}(\tau_i))e^{-\int_{t_0}^{\min\{\tau_i, T\}} h_i(s)J_{k,i}(s)ds}]_{i=2}^{N_k}, \quad [I_{\tau_1 > T}]. \tag{12.18}$$

Plugging this repesentation in (12.16), we get

$$\begin{aligned} P_k^{def}(t_0,T) &= P(t_0,T)\mathbb{E}^T[\tilde{y}_k(T)] \\ &\times \prod_{i=2}^{N_k}\mathbb{E}^T[(1+I_{\tau_i\le T}J_{k,i}(\tau_i))e^{-\int_{t_0}^{\min\{\tau_i,T\}}h_i(s)J_{k,i}(s)ds}] \\ &\times e^{-\int_{t_0}^T h_1(s)J_{k,1}(s)ds}\mathbb{E}^T[I_{\tau_1>T}] \\ &= \tilde{y}_k(t_0)P_k(t_0,T)e^{-\int_{t_0}^T h_1(s)(1+J_{k,1}(s))ds}. \end{aligned} \tag{12.19}$$

Equating (12.15) and (12.19), we get the following relationship to find the jump term structure of the FX rate y_k at the default of counterparty $name_1$,

$$h_{k,1}(t) = h_1(t)(1+J_{k,1}(t)).$$

Analogously, we get the relationship for the orther names,

$$h_{k,i}(t) = h_i(t)(1+J_{k,i}(t)), \quad i=2,...,N_k.$$

Here, $h_{k,i}(t)$ is the hazard rate of the counterparty $name_i$ in the currency CCY_k, and $h_i(t)$ is the hazard rate of the counterparty $name_i$ in the domestic currency.

12.5.2 FX Volatility Calibration

The calibration of the FX volatilities to reproduce the market FX option prices is performed after the calibration of the FX jumps, separately and independently for all FX rates. In this section, we first write out the formula for the FX option which takes into accounte the jumps of the FX rate, and then describe the calibration procedure.

12.5.2.1 FX Option Price

Let us write out the price of the European call/put at time t_0 maturing at time T on the spot FX rate $y_k(t)$. Assume that the FX rate $y_k(t)$ depends on the model counterparties $name_1$, ..., $name_{N_k}$, each of which causes the FX rate jump at default. The price of the FX option can be represented as the sum of the prices condition on all possible combintation of the defaults of the affecting counterparties:

$$\begin{gathered} \tau_{i_1}\le T,\dots,\tau_{i_p}\le T,\tau_{j_1}>T,\dots,\tau_{j_q}>T; \\ i_1<\dots<i_p<j_1<\dots<j_q; \qquad i_p+j_q=N_k. \end{gathered}$$

We have,

$$C(t_0,T,\tilde{y}_k(t_0),\vartheta_k,K) = \sum_{i_1,\dots,i_p,j_1,\dots,j_q} C_{i_1,\dots,i_p,j_1,\dots,j_q}(t_0,T,\tilde{y}_k(t_0),\vartheta_k,K), \tag{12.20}$$

where

$$\begin{aligned} C_{i_1,\dots,i_p,j_1,\dots,j_q}(t_0,T,y_k(t_0),\vartheta_k,K) &= \\ = \prod_{l=1}^q e^{-\int_{t_0}^T h_{j_l}(s)ds}\int_{t_0}^T\cdots\int_{t_0}^T du_{i_1}\dots du_{i_p}\prod_{m=1}^p h_{i_m}(u_{i_m})e^{-\int_{t_0}^{u_{i_m}}h_{i_m}(s)ds} \\ \times\ \omega P(t,T)\Big(\tilde{F}_*(t_0,u_{i_1},\dots,u_{i_p})\Phi(\omega d_1)-K\Phi(\omega d_2)\Big). \end{aligned}$$

Here, $\tilde{F}_*$ is the forward FX rate condition on the default event,

$$\tau_{i_1}=u_{i_1},\dots,\tau_{i_p}=u_{i_p},\tau_{j_1}>T,\dots,\tau_{j_q}>T:$$

$$\begin{aligned}
\tilde{F}_*(t_0, u_{i_1}, \ldots, u_{i_q}) &= \frac{P_k(t_0,T)}{P(t_0,T)} \tilde{y}(t_0) \prod_{m=1}^{p} (1 + J_{i_m}(u_{i_m})) e^{-\int_{t_0}^{u_{i_m}} J_{i_m}(s) h_{i_m}(s) ds} \\
&\times \prod_{l=1}^{q} e^{-\int_{t_0}^{T} J_{j_l}(s) h_{j_l}(s) ds};
\end{aligned}$$

K is the strike, $\Phi(x)$ is the cumulative normal distribution function and:

$$\begin{aligned}
\omega &= \begin{cases} 1 & \text{if call} \\ -1 & \text{if put} \end{cases} \\
d_1 &= \frac{ln\left(\frac{\tilde{F}_*(t_0, u_{i_1}, \ldots, u_{i_p})}{K}\right) + \frac{1}{2}\vartheta_k}{\sqrt{\vartheta_k}}, \quad d_2 = d_1 - \sqrt{\vartheta_k};
\end{aligned}$$

$\vartheta_k = \vartheta_k(t_0, T)$ is the squared integrated imlpied FX volatility,

$$\begin{aligned}
\vartheta_k(t_0,T) &= \int_{t_0}^{T} \Big(|\eta_k^{FX}(u)|^2 + B^2(a_0,u,T)|\eta_0(u)|^2 + B^2(a_k,u,T)|\eta_k(u)|^2 \\
&+ 2\rho_{r_0,y_k} B(a_0,u,T)\eta_0(u) \cdot \eta_k^{FX}(u) - 2\rho_{r_k,y_k} B(a_k,u,T)\eta_i(u) \cdot \eta_k^{FX}(u) \\
&- 2\rho_{r_0,r_k} B(a_0,u,T) B(a_k,u,T)\eta_0(u) \cdot \eta_k(u) \Big) du, \qquad (12.21) \\
B(a,t,T) &= \frac{1 - e^{-a(T-t)}}{a}.
\end{aligned}$$

12.5.2.2 FX Volatility Calibration Procedure

For a currency pair, CCY_0 and CCY_k, assume we are given a set of n FX options, one per maturity date, and the calibration results for the single currency HW 1F model for the base currency CCY_0 and the foreign currency CCY_k.

Let the set of option maturity dates $\mathcal{T}_{mat}$ and the corresponding market FX options $\mathcal{F}$ be:

$$\begin{aligned}
\mathcal{T}_{mat} &= \{T_1, \ldots, T_n\}, \quad 0 < T_i < T_{i+1} \qquad (12.22) \\
\mathcal{F} &= \{F_1, \ldots, F_n\}.
\end{aligned}$$

The calibration of the FX volatility is performed in two steps. First, we compute

$$\vartheta_{k,1} := \vartheta_k(t_0, T_1), \ldots, \vartheta_{k,n} := \vartheta_k(t_0, T_n) \qquad (12.23)$$

sepaartely for each option maturity. For this, we solve n independent 1D minimization problems

$$|C(t_0, T_i, y_k(t_0), \vartheta_{k,i}, K) - F_i| \to_{\vartheta_{k,i}} \min, \quad i = 1, \ldots, n. \qquad (12.24)$$

The procedure provides the implied volatility term structure

$$\begin{aligned}
\mathcal{T}_{mat} &= \{T_1, \ldots, T_n\}, \qquad (12.25) \\
\vartheta_k &= \{\vartheta_{k,1}, \ldots, \vartheta_{k,n}\}.
\end{aligned}$$

Next, we compute the spot FX volaitlity η_k^{FX} using the relationship (12.21), as described in Chapter 5 on page 33.

12.6 Variance Reduced CVA Calculation

Consider the discretized representation of the unilateral CVA over the interval $[t_0, T]$, with the time discretization $t_0 < t_1 < \ldots, t_N = T$. Without loss of generality, we assume in what follows that the CVA is written on the counterparty $name_1$.

$$CVA^{UL}(t_0, T) = \mathbb{E}\Big[\sum_{i=1}^{N}(1-R_C)DF(t_0,t_i)PE(t_i)I_{\tau_1\in(t_{i-1},t_i]}\Big] \tag{12.26}$$

The goal of the variance reduced approach is to replace the default indicator $I_{\tau_1\in(t_{i-1},t_i]}$ in (12.26) with the expected value

$$\mathbb{E}\Big[I_{\tau_1\in(t_{i-1},t_i]}|\mathcal{F}_{t_{i-1}}\Big] = e^{-\int_{t_0}^{t_{i-1}} h_1(s)ds} - e^{-\int_{t_0}^{t_i} h_1(s)ds}, \tag{12.27}$$

which typically has significantly smaller variance. The following two cases are possible:

1. $PE(t)$ does not depend on the default event of $name_1$, i.e., $PE(t)$ either does not depend on the FX rates, or it depends on FX rate, which is not affected by the default of $name_1$. In this case, we can write out the variance reduced representation using the tower property of the expectation,

$$\begin{aligned} & CVA^{UL}(t_0,T) = \\ = \;& \mathbb{E}\Big[\sum_{i=1}^{N}(1-R_C)DF(t_0,t_i)PE(t_i)I_{\tau_1\in(t_{i-1},t_i]}\Big] \\ = \;& \mathbb{E}\Big[\sum_{i=1}^{N}(1-R_C)DF(t_0,t_i)PE(t_i)(e^{-\int_{t_0}^{t_{i-1}} h_1(s)ds} - e^{-\int_{t_0}^{t_i} h_1(s)ds})\Big]. \end{aligned} \tag{12.28}$$

2. $PE(t)$ depends on the FX rate $y_k(t)$, which has a jump at the default of $name_1$.

Below we write out the variance reduced representation for the CVA in the second case.

12.6.1 FX rate with multiple jumps

Let us consider the case when the FX rate $y_k(t)$ has multiple jumps at default of the counterparties $name_1$. $\ldots$, $name_{N_k}$. Using the representation (12.3), we can write out any function $f(\cdot, y_k(t_m))$ of the exchange rate $y_k(t_m)$ as follows,

$$f(\cdot, y_k(t_m))I_{\tau_1\in(t_{m-1},t_m]} \;=\; f(\cdot, y_{k,1}(t_m))I_{\tau_1\in(t_{m-1},t_m]} + o(|t_m - t_{m-1}|). \tag{12.29}$$

where $y_{k,1}(t_m)$ is the spot FX rate condition on $name_1$ defaults at t_m,

$$\begin{aligned} y_{k,1}(t_m) \;&=\; \tilde{y}_k(t_m)(1+J_{k,1}(t_m))e^{-\int_{t_0}^{t_m} h_1(s)J_{k,1}(s)ds} \\ &\times \prod_{i=2}^{N_k}(1+I_{\tau_i\le t_m}J_{k,i}(\tau_i))e^{-\int_{t_0}^{\min\{\tau_i,t_m\}} h_i(s)J_{k,i}(s)ds}. \end{aligned} \tag{12.30}$$

Note that the random variables

$$\tilde{y}_k(t_m), \quad \big[(1+I_{\tau_i\le t_m}J_{k,i}(\tau_i))e^{-\int_{t_0}^{\min\{\tau_i,t_m\}} h_i(s)J_{k,i}(s)ds}\big]_{i=2}^{N_k}$$

and $I_{\tau_1\in(t_{i-1},t_i]}$ are independent. Hence, we can approximate the CVA^{UL} as follows,

$$\begin{aligned} & CVA^{UL}(t_0,T) = \\ = \;& \mathbb{E}\Big[\sum_{i=1}^{N}(1-R_C)DF(t_0,t_i)PE(t_i, y_{k,1}(t_i))I_{\tau_1\in(t_{i-1},t_i]}\Big] \\ \approx \;& \mathbb{E}\Big[\sum_{i=1}^{N}(1-R_C)DF(t_0,t_i)PE(t_i, y_{k,1}(t_i))(e^{-\int_{t_0}^{t_{i-1}} h_1(s)ds} - e^{-\int_{t_0}^{t_i} h_1(s)ds})\Big]. \end{aligned} \tag{12.31}$$

12.6.2 FX rate with single jump at first default

Now let us consider the case when the FX rate $y_k(t)$ has the single jump at the first default among the counterparties $name_1. \ldots, name_{N_k}$. Using the representation (12.4), we can write out any function $f(\cdot, y_k(t_m))$ of the exchange rate $y_k(t_m)$ as follows,

$$f(\cdot, y_k(t_m))I_{\tau_1 \in (t_{m-1}, t_m]} \quad = \quad f(\cdot, y_{k,1}(t_m))I_{\tau_1 \in (t_{m-1}, t_m]} + o(|t_m - t_{m-1}|). \tag{12.32}$$

where where $y_{k,1}(t_m)$ is the spot FX rate condition on $name_1$ defaults at t_m,

$$\begin{aligned} y_{k,1}(t_m) &= \tilde{y}_k(t_m)(1 + I_{\tau_{*_1} > t_m} J_{k,1}(t_m)) \\ &\times (1 + I_{\tau_{*_1} \le t_{m-1}} J_{k,*_1}(\tau_{*_1}))e^{-\sum_{i=1}^{N_k} \int_{t_0}^{\min\{\tau_{*_1}, t_m\}} h_i(s) J_{k,i}(s) ds}, \\ \tau_{*_1} &= \min\{\{\tau_1, \ldots, \tau_{N_k}\} \backslash \tau_1\}, \\ J_{k,*_1} &= \{J_{k,i} | \tau_i = \tau_{*,1}\}. \end{aligned} \tag{12.33}$$

Note that the random variables

$$\begin{array}{ccc} \tilde{y}_k(t_m), & (1 + I_{\tau_{*_1} > t_m} J_{k,1}(t_m)), & (1 + I_{\tau_{*_1} \le t_{m-1}} J_{k,*_1}(\tau_{*_1})), \\ e^{-\sum_{i=1}^{N_k} \int_{t_0}^{\min\{\tau_{*_1}, t_m\}} h_i(s) J_{k,i}(s) ds} & \text{and} & I_{\tau_1 \in (t_{m-1}, t_m]}. \end{array}$$

are independent. Hence, we can approximate the CVA^{UL} as follows,

$$\begin{aligned} & CVA^{UL}(t_0, T) = \\ = \quad & \mathbb{E}\Big[\sum_{i=1}^{N} (1 - R_C) DF(t_0, t_i) PE(t_i, y_{k,1}(t_i)) I_{\tau_1 \in (t_{i-1}, t_i]}\Big] \\ \approx \quad & \mathbb{E}\Big[\sum_{i=1}^{N} (1 - R_C) DF(t_0, t_i) PE(t_i, y_{k,1}(t_i))(e^{-\int_{t_0}^{t_{i-1}} h_1(s) ds} - e^{-\int_{t_0}^{t_i} h_1(s) ds})\Big]. \end{aligned} \tag{12.34}$$

Chapter 13

Hybrid Commodity Credit Equity Inflation Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Black-Karasinski Models for Hazard Rates

This chapter describes issues relating to the implementation of a hybrid commodity credit equity inflation multi-currency model with multi-factor Hull-White models for interest rates and single factor Black-Karasinski models for hazard rates.

13.1 Introduction

In the hybrid commodity credit equity inflation multi-currency extension of the multi-factor HW NF model (HW NF), a single currency HW NF model is used for the nominal interest rates for each currency. In addition, the spread between the nominal and real interest rates of each currency is also modelled as a single currency HW NF model. The spot foreign exchange (FX) rates between currencies, spot equity prices, spot commodity prices, and inflation index are modelled as lognormal random variables. For credit modelling, the dynamics of the hazard rate for each credit name under the domestic risk neutral measure is modelled by the lognormal Black-Karasinski type model. Commodity futures prices are calculated from the simulated commodity spot. The spread between the interest rate and convenience yield of each commodity is modelled as a single currency HW NF model.

Consider a model with $n+1$ currencies $k = 0, 1, \ldots, n$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_k, are modelled using a multi-factor single currency HW NF model with N_k factors:

$$\begin{aligned} dx_{k,i}(t) &= -a_{k,i}(t)x_{k,i}(t)dt + \eta_{k,i}(t)dW^{(k)}_{x_{k,i}}, \quad x_{k,i}(0) = 0 \\ r_k(t) &= \phi_k(t) + \sum_{i=1}^{N_k} x_{k,i}(t), \quad dW^{(k)}_{x_{k,i}} dW^{(k)}_{x_{k,j}} = \rho_{x_{k,i},x_{k,j}}(t)dt \end{aligned} \tag{13.1}$$

where, for each currency k, $W^{(k)}_{k,i}$ are Wiener processes in the risk neutral measure for that currency k, $a_{k,i}(t)$ are mean reversion parameters, $\eta_{k,i}(t)$ are volatility parameters and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. For the details of the calculation of the function $\phi_k(t)$, see Chapter 3 on page 19. In what follows, we will assume time independent mean reversion parameters $a_{k,i}(t) = a_{k,i}$.

In addition, there are N_I spot inflation rates $I_1(t), ..., I_{N_I}(t)$ where the kth spot inflation rate $I_k(t)$ is for the currency c_k. The dynamics of the rate $I_k(t)$ in the risk neutral measure of the currency c_k is governed by:

$$\frac{dI_k(t)}{I_k(t)} = u_k(t)dt + \eta_k^I(t)dW_{I_k}^{(c_k)}, \tag{13.2}$$

where, $u_k(t)$ is the spread between the nominal and real short interest rates, $\eta_k^I(t)$ is the volatility for the inflation index $I_k(t)$ and $W_{I_k}^{(c_k)}$ is a Wiener process in the risk neutral measure for the currency c_k.

The spread between the nominal and real short interest rate for the currency c_k is modelled using a multi-factor single currency HW NF model:

$$\begin{aligned} dw_{k,i}(t) &= -\alpha_{k,i}(t)w_{k,i}(t)dt + \nu_{k,i}(t)dW_{w_{k,i}}^{(c_k)}, \quad w_{k,i}(0) = 0 \\ u_k(t) &= \theta_k(t) + \sum_{i=1}^{N_k^R} w_{k,i}(t), \quad dW_{w_{k,i}}^{(k)} dW_{w_{k,j}}^{(c_k)} = \rho_{w_{k,i},w_{k,j}}(t)dt \\ u_k(t) &= r_k(t) - r_k^R(t) \end{aligned} \tag{13.3}$$

where for each k, $W_{w_{k,i}}^{(c_k)}$ is the Wiener processes in the risk neutral measure for currency c_k, $\alpha_{k,i}(t)$ are mean reversion parameters, $\nu_{k,i}(t)$ are volatility parameters and $\theta_k(t)$ is a function chosen so that the initial inflation linked discount factor curve is reproduced. For the details of the calculation of the function $\theta_k(t)$, see Chapter 8 on page 61 . In what follows, we will assume time independent mean reversion parameters $\alpha_{k,i}(t) = \alpha_{k,i}$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \ldots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_k(t)}{y_k(t)} = (r_0(t) - r_k(t))\,dt + \eta_k^{FX}(t)dW_{y_k}^{(0)} \tag{13.4}$$

where, $\eta_k^{FX}(t)$ is the volatility for FX rate $y_k(t)$ and $W_{y_k}^{(0)}$ is a Wiener process in the risk neutral measure for that domestic currency CCY_0.

Let there also be N_{EQ} equities and N_{CM} commodities.

Let us denote $s_1(t), \ldots, s_{N_{EQ}}(t)$ the spot prices of the equities, where s_j is in currency c_j which is one of the model $n+1$ currencies. The dynamics of the jth spot equity price $s_j(t)$ in the risk neutral measure of the equity currency c_j is governed by:

$$\frac{ds_j(t)}{s_j(t)} = (r_{c_j}(t) - \delta_j(t))dt + \eta_j^{EQ}(t)dW_{s_j}^{(c_j)} \tag{13.5}$$

where, $r_{c_j}(t)$ is the short rate for currency c_j, $\delta_j(t)$ is a continuously compounded dividend rate, $\eta_j^{EQ}(t)$ is the volatility for equity price $s_j(t)$ and $W_{s_j}^{(c_j)}$ is a Wiener process in the risk neutral measure for the equity currency c_j.

For kth commodity, we assume that the dynamics of the spot price process $p_k(t)$ is governed by the following SDE under the risk neutral measure of the commodity currency c_k,

$$\begin{aligned} \frac{dp_k(t)}{p_k(t)} &= \lambda_k(t)dt + \eta_k^{cm}(t)dW_{p_k}^{(c_k)}, \\ \lambda_k(t) &= r_{c_k}(t) - r_k^c(t). \end{aligned} \tag{13.6}$$

Here, $\lambda_k(t)$ is the stochastic spread between the short interest rate for the currency c_k and convenience yield of the kth commdity $r_k^c(t)$, $\eta_k^{cm}(t)$ is the volatility for the commodity spot price process, and $W_{p_k}^{(c_k)}$ is a Wiener process in the risk neutral measure for the currency c_k.

The initial spot commodity price $p_k(t_0)$ is taken to be the price of the closest maturity contract on the initial forward curve for the commodity k.

The spread $\lambda_k(t)$ between the short interest rate for the currency c_k and convenience yield of the kth commdity is modelled using a multi-factor mean reversion SDE, similar to the multi-factor single currency HW model:

$$\begin{aligned} d\omega_{k,i}(t) &= -\gamma_{k,i}(t)\omega_{k,i}(t)dt + \sigma_{k,i}(t)dW^{(c_k)}_{\omega_{k,i}}, \quad \omega_{k,i}(0) = 0 \\ \lambda_k(t) &= \zeta_k(t) + \sum_{i=1}^{N_k^{cm}} \omega_{k,i}(t), \quad dW^{(c_k)}_{\omega_{k,i}} dW^{(c_k)}_{\omega_{k,j}} = \rho_{\omega_{k,i}\omega_{k,j}}(t)dt \end{aligned} \tag{13.7}$$

Here, $W^{(c_k)}_{\omega_{k,j}}$ is the Wiener processes in the risk neutral measure for currency c_k, $\gamma_{k,i}(t)$ are mean reversion parameters, $\sigma_{k,i}(t)$ are volatility parameters and ζ_k is a function chosen so that the initial commodity futures curve is reproduced. For the details of the calculation of the function ζ_k and forward and futures commodity price, see Chapter 10 on page 79. In what follows, we will assume time independent mean reversion parameters $\gamma_{k,i}(t) = \gamma_{k,i}$.

Finally we assume that the dynamics of the hazard rate for each credit name $m = 1, \ldots, N_{CR}$ under the domestic risk neutral measure is governed by:

$$\begin{aligned} dz_m(t) &= -\kappa_m z_m(t)dt + \eta_m^{cr}(t)dW^{(0)}_{z_m}(t), \quad z_m(0) = 0 \\ h_m(t) &= h_m(0)e^{\vartheta_m(t)+\xi_m(t)+z_m(t)} \end{aligned} \tag{13.8}$$

where κ_m are the mean reversion parameters, η_m^{cr} are the volatility parameters, ξ_m are the drift adjustment parameters, calibrated to reproduce the initial CDS price, and

$$\vartheta_m(t) = \ln\left(h_m(0,t)/h_m(0)\right), \quad h_m(0,T) = -\frac{\partial \ln S_m(0,T)}{\partial T}, \quad h_m(0) = h_m(0,0). \tag{13.9}$$

The initial survival probabilities $S_m(0,T)$ are given by

$$S_m(0,T) = \mathbb{E}^T\left[\mathbf{1}_{\{\tau_m > T\}}\right] = \mathbb{E}^T\left[e^{-\int_0^T h_m(u)du}\right] \tag{13.10}$$

where τ_m is the counterparty default time and $\mathbb{E}^T$ is the expectation under the domestic T forward measure. Initial survival probabilities are analogous to discount factors while the intensity process is analogous to a short rate. Note that when the hazard rates are uncorrelated with domestic interest rates, the survival probabilities under different forward measures are the same in the surival probabilities under the risk neutral measure. For the details of the hazard rate simulation, see Chapter 11 on page 85.

13.2 Measure Relationships

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 for a model stochastic risk factor R is given by:

$$dW_R^{(0)} = \rho_{R,y_j}\eta^R(t)\eta_j^{FX}(t)dt + dW_R^{(j)}. \tag{13.11}$$

Here, $\eta^R(t)$ is the diffusive volatility of the risk factor R, and ρ_{R,y_j} is the correlation between the Brownian increments governing risk factor R and the exchange rate of the currency CCY_j.

13.3 Monte Carlo Simulation

Under the domestic currency CCY_0 risk neutral measure, the SDE's become:

$$
\begin{aligned}
dx_{0,i} &= -a_{0,i}x_{0,i}dt + \eta_{0,i}dW^{(0)}_{x_{0,i}} \\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \rho_{x_{k,i},y_k}\eta_{k,i}\eta_k^{FX}\Big]dt + \eta_{k,i}dW^{(0)}_{x_{k,i}} \\
\frac{dy_k}{y} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=0}^{N_0-1} x_{0,i} - \sum_{i=0}^{N_k-1} x_{k,i}\Big]dt + \eta_k^{FX}dW^{(0)}_{y_k} \\
\frac{ds_j}{s} &= \Big[\phi_{c_j}(t) + \sum_{i=0}^{N_{c_j}-1} x_{c_j,i} - \delta_j(t) - \mathbf{1}_{[c_j\neq 0]}\rho_{s_j,y_{c_j}}\eta_j^{EQ}\eta_{c_j}^{FX}\Big]dt + \eta_j^{EQ}dW^{(0)}_{s_j} \\
dz_m &= -\kappa_m z_m dt + \eta_m^{CR}dW^{(0)}_{z_m} \\
dw_{k,i} &= \Big[-\alpha_{k,i}w_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{w_{k,i},y_{c_k}}\nu_{k,i}\eta_{c_k}^{FX}\Big]dt + \nu_{k,i}dW^{(0)}_{w_{k,i}} \\
\frac{dI_k}{I_k} &= \Big[\theta_k(t) + \sum_{i=1}^{N_k^R} w_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{I_k,y_{c_k}}\eta_k^I\eta_{c_k}^{FX}\Big]dt + \eta_k^I(t)dW^{(0)}_{I_k} \\
d\omega_{k,i} &= \Big[-\gamma_{k,i}\omega_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{\omega_{k,i},y_{c_k}}\sigma_{k,i}\eta_{c_k}^{FX}\Big]dt + \sigma_{k,i}dW^{(0)}_{\omega_{k,i}} \\
\frac{dp_k}{p_k} &= \Big[\zeta_k(t) + \sum_{i=1}^{N_k^{cm}} \omega_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{p_k,y_{c_k}}\eta_k^{cm}\eta_{c_k}^{FX}\Big]dt + \eta_k^{cm}(t)dW^{(0)}_{p_k}
\end{aligned}
\tag{13.12}
$$

The system of the equations (13.12) can be solved using the same technique as outlined in Chapter 7 on page 51.

For the details of the Monte Carlo simulation of the SDE (13.12), see Chapter 3 on page 19 and Chapter 5 on page 33. For the details of the default indicator simulation, see Chapter 11 on page 85.

Chapter 14

Hybrid Commodity Credit Equity Inflation Multi-Currency Model with Multi-Factor Hull-White Models for Interest Rates and Single Factor Hull-White Models for Hazard Rates

This chapter describes issues relating to the implementation of a hybrid commodity credit equity inflation multi-currency model with the multi-factor Hull-White models for interest rates and single-factor HW model for hazard rates.

14.1 Introduction

In the hybrid commodity credit equity inflation multi-currency multi-factor HW model, a single currency Hull-White model is used for the nominal interest rates for each currency. In addition, the spread between the nominal and real interest rates of each currency is also modelled as a single currency Hull-White model. The inflation indices, spot foreign exchange (FX) rates between currencies, spot equity prices, and spot commodity prices are modelled as lognormal random variables. For credit modelling, the dynamics of the hazard rate for each credit name under the domestic risk-neutlal measure is modelled by a single-factor HW model. Commodity futures prices are calculated from the simulated commodity spot price. The spread between interest rate and convenience yield for each commodity is modelled by a multi-factor HW model.
Consider a model with $n+1$ currencies $k = 0, 1, \ldots, n$. Let CCY_0 be the domestic currency and $CCY_1, \ldots, CCY_n$ be the n foreign currencies. The interest rates for each currency, CCY_k, is modelled using a multi-factor single currency HW model with N_k factors:

$$\begin{aligned} dx_{k,i}(t) &= -a_{k,i}(t)x_{k,i}(t)dt + \eta_{k,i}(t)dW^{(k)}_{x_{k,i}}, \quad x_{k,i}(0) = 0 \qquad (14.1) \\ r_k(t) &= \phi_k(t) + \sum_{i=1}^{N_k} x_{k,i}(t), \end{aligned}$$

where, for each currency k, $W^{(k)}_{x_{k,i}}$ is the Wiener processes in the risk neutral measure for the factor i of that currency, $a_{k,i}(t)$ is the mean reversion parameter, $\eta_{k,i}(t)$ is the volatility parameter and $\phi_k(t)$ is a function chosen so that the initial discount factor curve is reproduced. For the details of the calculation of the function $\phi_k(t)$, see see Chapter 3 on page 19. In what follows, we will assume time independent mean reversion parameters $a_{k,i}(t) = a_{k,i}$.

In addition, there are N_I spot inflation rates $I_1(t), ..., I_{N_I}(t)$ where the kth spot inflation rate $I_k(t)$ is for the currency c_k. The dynamics of the rate $I_k(t)$ in the risk neutral measure of the currency c_k is governed by:

$$\frac{dI_k(t)}{I_k(t)} = u_k(t)dt + \eta_k^I(t)dW_{I_k}^{(c_k)}, \tag{14.2}$$

where, $u_k(t)$ is the spread between the nominal and real short interest rates, $\eta_k^I(t)$ is the volatility for the inflation index $I_k(t)$ and $W_{I_k}^{(c_k)}$ is a Wiener process in the risk neutral measure for the currency c_k.

The spread between the nominal and real short interest rate for the currency c_k is modelled using a multifactor HW model, with N_k^I factors:

$$\begin{aligned} dw_{k,i}(t) &= -\alpha_{k,i}(t)w_{k,i}(t)dt + \nu_{k,i}(t)dW_{w_{k,i}}^{(c_k)}, \quad w_{k,i}(0) = 0 \\ u_k(t) &= \theta_k(t) + \sum_{i=1}^{N_k^I} w_{k,i}(t), \\ u_k(t) &= r_{c_k}(t) - r_k^R(t) \end{aligned} \tag{14.3}$$

where for each k and i, $W_{w_{k,i}}^{(c_k)}$ is the Wiener processes in the risk neutral measure for currency c_k, $\alpha_{k,i}(t)$ is the mean reversion parameter, $\nu_{k,i}(t)$ is the volatility parameter and $\theta_k(t)$ is a function chosen so that the initial inflation linked discount factor curve is reproduced. For the details of the calculation of the function $\theta_k(t)$, see Chapter 8 on page 61 . In what follows, we will assume time independent mean reversion parameters $\alpha_{k,i}(t) = \alpha_{k,i}$.

For $n+1$ currencies, there are also n relevant spot FX rates $y_1(t), \ldots, y_n(t)$ where $y_k(t)$ is the spot FX rate between CCY_k and the domestic currency CCY_0. The dynamics of the kth spot FX rate $y_k(t)$ in the domestic risk neutral measure is governed by:

$$\frac{dy_k(t)}{y_k(t)} = (r_0(t) - r_k(t))\, dt + \eta_k^{FX}(t)dW_{y_k}^{(0)}. \tag{14.4}$$

Here, $\eta_k^{FX}(t)$ is the volatility for FX rate $y_k(t)$, $W_{y_k}^{(0)}$ is a Wiener process in the risk neutral measure for the domestic currency CCY_0.

Let there also be N_{EQ} equities and N_{CM} commodities.

Let us denote $s_1(t), \ldots, s_{N_{EQ}}(t)$ the spot prices of equities, where equity s_j is in currency c_j, which is one of the model $n+1$ currencies. The dynamics of the jth spot equity price $s_j(t)$ in the risk neutral measure of the equity currency c_j is governed by:

$$\frac{ds_j(t)}{s_j(t)} = (r_{c_j}(t) - \delta_j(t))dt + \eta_j^{EQ}(t)dW_{s_j}^{(c_j)}, \tag{14.5}$$

where, $r_{c_j}(t)$ is the short rate for currency c_j, $\delta_j(t)$ is a continuously compounded dividend rate, $\eta_j^{EQ}(t)$ is the volatility for spot equity price $s_j(t)$ and $W_{s_j}^{(c_j)}$ is a Wiener process in the risk neutral measure for the equity currency c_j.

For kth commodity, we assume that the dynamics of the spot price process $p_k(t)$ is governed by the following SDE under the risk neutral measure of the commodity currency c_k,

$$\begin{aligned} \frac{dp_k(t)}{p_k(t)} &= \lambda_k(t)dt + \eta_k^{cm}(t)dW_{p_k}^{(c_k)}, \\ \lambda_k(t) &= r_{c_k}(t) - r_k^c(t). \end{aligned} \tag{14.6}$$

Here, $\lambda_k(t)$ is the stochastic spread between the short interest rate for the currency c_k and convenience yield of the kth commdity $r_k^c(t)$, $\eta_k^{cm}(t)$ is the volatility for the commodity spot price process, and $W_{p_k}^{(c_k)}$ is a Wiener process in the risk neutral measure for the currency c_k.

The initial spot commodity price $p_k(t_0)$ is taken to be the price of the closest maturity contract on the initial forward curve for the commodity k.

The spread $\lambda_k(t)$ between the short interest rate for the currency c_k and convenience yield of the kth commdity is modelled using a multi-factor mean reversion SDE, similar to the multi-factor single currency HW model:

$$\begin{aligned} d\omega_{k,i}(t) &= -\gamma_{k,i}(t)\omega_{k,i}(t)dt + \sigma_{k,i}(t)dW^{(c_k)}_{\omega_{k,i}}, \quad \omega_{k,i}(0) = 0 \\ \lambda_k(t) &= \zeta_k(t) + \sum_{i=1}^{N_k^{cm}} \omega_{k,i}(t), \quad dW^{(c_k)}_{\omega_{k,i}} dW^{(c_k)}_{\omega_{k,j}} = \rho_{\omega_{k,i}\omega_{k,j}}(t)dt \end{aligned} \tag{14.7}$$

Here, $W^{(c_k)}_{\omega_{k,j}}$ is the Wiener processes in the risk neutral measure for currency c_k, $\gamma_{k,i}(t)$ are mean reversion parameters, $\sigma_{k,i}(t)$ are volatility parameters and ζ_k is a function chosen so that the initial commodity futures curve is reproduced. For the details of the calculation of the function ζ_k and forward and futures commodity price, see Chapter 10 on page 79. In what follows, we will assume time independent mean reversion parameters $\gamma_{k,i}(t) = \gamma_{k,i}$.

The dynamics of the hazard rate $h_m(t)$ for each credit name $m = 1, \ldots, N_{CR}$ under the domestic risk neutral measure is governed by:

$$\begin{aligned} dz_m(t) &= -\kappa_m z_m(t)dt + \eta_m^{CR}(t)dW^{(0)}_{z_m}(t), \quad z_m(0) = 0 \\ h_m(t) &= \vartheta_m(t) + z_m(t), \end{aligned} \tag{14.8}$$

where κ_m is the mean reversion parameter, η_m^{CR} is the volatility parameter, ϑ_m is the drift adjustment parameter. The drift adjustment parameter ϑ_m is calibrated to reproduce the initial survival probability curve,

$$S_{0,m}(t_0, T) = \mathbb{E}^{T,(0)}\left[\mathbf{1}_{\{\tau_m > T\}}\right] = \mathbb{E}^{T,(0)}\left[e^{-\int_{t_0}^T h_m(u)du}\right]. \tag{14.9}$$

Here τ_m is the counterparty default time and $\mathbb{E}^{T,(0)}$ is the expectation under the T-forward measure in the model domestic currency. For the details of the calculation of the function ϑ_m and the survival probability, see Chapter 9 on page 69.

14.2 System of Equations in the Risk Neutral Measure of the Domestic Currency

The relationship between risk neutral measure in currency CCY_j to the risk neutral measure in the domestic currency CCY_0 for a model stochastic risk factor R is given by:

$$dW_R^{(0)} = \rho_{R,y_j}\eta^R(t)\eta_j^{FX}(t)dt + dW_R^{(j)}. \tag{14.10}$$

Here, $\eta^R(t)$ is the diffusive volatility of the risk factor R, and ρ_{R,y_j} is the correlation between the Brownian increments governing risk factor R and the exchange rate of the currency CCY_j.

Then, the model SDE's under the domestic currency CCY_0 risk neutral measure become:

$$
\begin{aligned}
dx_{0,i} &= -a_{0,i}x_{0,i}dt + \eta_{0,i}dW^{(0)}_{x_{0,i}} \\
dx_{k,i} &= \Big[-a_{k,i}x_{k,i} - \rho_{x_{k,i},y_k}\eta_{k,i}\eta^{FX}_k\Big]dt + \eta_{k,i}dW^{(0)}_{x_{k,i}} \\
\frac{dy_k}{y_k} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=1}^{N_0} x_{0,i} - \sum_{i=1}^{N_k} x_{k,i}\Big]dt + \eta^{FX}_k dW^{(0)}_{y_k} \\
dz_m &= -\kappa_m z_m dt + \eta^{CR}_m dW^{(0)}_{z_m} \\
\frac{ds_j}{s} &= \Big[\phi_{c_j}(t) + \sum_{i=1}^{N_{c_j}} x_{c_j,i} - \delta_j(t) - \mathbf{1}_{[c_j\neq 0]}\rho_{s_j,y_{c_j}}\eta^{EQ}_j\eta^{FX}_{c_j}\Big]dt + \eta^{EQ}_j dW^{(0)}_{s_j} \\
dw_{l,i} &= \Big[-\alpha_{l,i}w_{l,i} - \mathbf{1}_{[c_l\neq 0]}\rho_{w_{l,i},y_{c_l}}\nu_{l,i}\eta^{FX}_{c_l}\Big]dt + \nu_{l,i}dW^{(0)}_{w_l} \\
\frac{dI_l}{I_l} &= \Big[\theta_l(t) + \sum_{i=1}^{N^I_l} w_{l,i} - \mathbf{1}_{[c_l\neq 0]}\rho_{I_l,y_{c_l}}\eta^I_l\eta^{FX}_{c_l}\Big]dt + \eta^I_l(t)dW^{(0)}_{I_l} \\
d\omega_{k,i} &= \Big[-\gamma_{k,i}\omega_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{\omega_{k,i},y_{c_k}}\sigma_{k,i}\eta^{FX}_{c_k}\Big]dt + \sigma_{k,i}dW^{(0)}_{\omega_{k,i}} \\
\frac{dp_k}{p_k} &= \Big[\zeta_k(t) + \sum_{i=1}^{N^{cm}_k} \omega_{k,i} - \mathbf{1}_{[c_k\neq 0]}\rho_{p_k,y_{c_k}}\eta^{cm}_k\eta^{FX}_{c_k}\Big]dt + \eta^{cm}_k(t)dW^{(0)}_{p_k}
\end{aligned} \tag{14.11}
$$

The system of the equations (14.11) can be solved using the same technique as outlined in Chapter 7 on page 51.

For the details of the Monte Carlo simulation of the SDE (14.11), see Chapter 3 on page 19 and Chapter 5 on page 33. For the details of the default indicator simulation, see Chapter 9 on page 69.

14.3 System of Equations in the Real-World Measure

In the real-world measure the future interest rates are expected to be lower than the market implied forward rates due to the risk tolerance of investors; this is known as the market price of risk. Ignoring the market price of risk may lead to over estimates of real-world exposures for long time horizons. Therefore real-world measures such as PFE and CCR capital may be overestimated. To capture the market price of risk we assume the follwing dynamics for short rate drivers $x_{k,i}$ under the real-world measure:

$$
dx_{k,i} = \Big[-a_{k,i}x_{k,i} - \lambda_{k,i}\eta_{k,i}\Big]dt + \eta_{k,i}dW^{RW}_{x_{k,i}} \tag{14.12}
$$

wkere $\lambda_{k,i}$ is the instantaneous price of risk.

For the other model factors, we assume the following generic dynamics under the real-world measure:

$$
\begin{aligned}
\frac{dy_k}{y_k} &= \Big[\phi_0(t) - \phi_k(t) + \sum_{i=1}^{N_0} x_{0,i} - \sum_{i=1}^{N_k} x_{k,i} - \lambda_k^{FX}\eta_k^{FX}\Big]dt + \eta_k^{FX} dW_{y_k}^{RW} \\
dz_m &= \Big[-\kappa_m z_m - \lambda_m^{CR}\eta_m^{CR}\Big]dt + \eta_m^{CR} dW_{z_m}^{RW} \\
\frac{ds_j}{s} &= \Big[\phi_{c_j}(t) + \sum_{i=1}^{N_{c_j}} x_{c_j,i} - \delta_j(t) - \lambda_j^{EQ}\eta_j^{EQ}\Big]dt + \eta_j^{EQ} dW_{s_j}^{RW} \\
dw_{l,i} &= \Big[-\alpha_{l,i} w_{l,i} - \lambda_{l,i}^{IRR}\nu_{l,i}\Big]dt + \nu_{l,i} dW_{w_l}^{RW} \\
\frac{dI_l}{I_l} &= \Big[\theta_l(t) + \sum_{i=1}^{N_l^I} w_{l,i} - \lambda_l^I \eta_l^I\Big]dt + \eta_l^I(t) dW_{I_l}^{RW} \\
d\omega_{k,i} &= \Big[-\gamma_{k,i}\omega_{k,i} - \lambda_{k,i}^{cs}\sigma_{k,i}\Big]dt + \sigma_{k,i} dW_{\omega_{k,i}}^{RW} \\
\frac{dp_k}{p_k} &= \Big[\zeta_k(t) + \sum_{i=1}^{N_k^{cm}} \omega_{k,i} - \lambda_k^{cm}\eta_k^{cm}(t)\Big]dt + \eta_k^{cm}(t) dW_{p_k}^{RW}.
\end{aligned}
\tag{14.13}
$$

Here,

$$
\Lambda(t) = \Big(\lambda_{k,i}(t), \ldots, \lambda_k^{FX}(t), \ldots, \lambda_j^{EQ}(t), \ldots, \lambda_{l,i}^{IRR}(t), \ldots, \lambda_l^I(t), \ldots, \lambda_{k,i}^{cs}(t), \ldots, \lambda_k^{cm}(t)\Big)'
$$

is the generic instanteaneous model price of risk. In the current version of the model we assume that the real-world dirft is equal to the risk neutral drift for all risk factors except IR short rate,

$$
\begin{aligned}
\lambda_k^{FX} &= \lambda_k^{CR} = 0, \\
\lambda_j^{EQ} &= \mathbf{1}_{[c_j \neq 0]}\rho_{s_j, y_{c_j}}\eta_{c_j}^{FX}, \\
\lambda_{l,i}^{IRR} &= \mathbf{1}_{[c_l \neq 0]}\rho_{w_{l,i}, y_{c_l}}\eta_{c_l}^{FX}, \\
\lambda_l^I &= \mathbf{1}_{[c_l \neq 0]}\rho_{I_l, y_{c_l}}\eta_{c_l}^{FX}, \\
\lambda_{l,i}^{cs} &= \mathbf{1}_{[c_k \neq 0]}\rho_{\omega_{k,i}, y_{c_k}}\eta_{c_k}^{FX}, \\
\lambda_l^{cm} &= \mathbf{1}_{[c_k \neq 0]}\rho_{p_k, y_{c_k}}\eta_{c_k}^{FX}.
\end{aligned}
$$

The system of the equations (14.13) can be solved using the same technique as outlined in Chapter 7 on page 51.

For the details of the Monte Carlo simulation of the SDE (14.13), see Chapter 4 on page 29 and Chapter 5 on page 33. For the details of the default indicator simulation, see Chapter 9 on page 69.

14.4 Model Calibration

This section describes the calibration of the model volatilities:

- short interest rate volatilities $\eta_{k,i}$
- spot exchange rate volatilities η_k^{FX}
- spot equity volatilities η_j^{EQ}
- hazard rate volatilities η_m^{CR}
- inflation spread volatilities $\nu_{l,i}$
- spot inflation index volatilities η_l^I

- commodity convenience spread volatilities $\sigma_{k,i}$
- spot commodity volatilities η_k^{cm}

and the correlations between model factor increments ρ. In what follows, we denote η the combined vector of all model volatilities,

$$\eta := \Big(\ldots, \eta_{k,i}, \ldots, \eta_k^{FX}, \ldots, \eta_j^{EQ}, \ldots, \eta_m^{CR}, \ldots, \nu_{l,i}, \ldots, \eta_l^{I}, \ldots, \sigma_{k,i}, \ldots, \eta_k^{cm}, \ldots\Big). \tag{14.14}$$

The calibration of market price of risk is performed after the calculation of the volatility vector η and the correlation matrix ρ. The market price of risk is calculated independently for each currency. If the IR short rate model has single factor, the approach described in 4.4.2 is used. If the IR short rate model has multiple factors, the approach described in 4.4.3 is used

14.4.1 Historical Calibration

The historical calibration of the model allows to reproduce the historically calculated covariance between the stochastic time increments of the following risk factors:

- Instantaneous forward interest rates for the currencies CCY_0,...,CCY_n, for the set of benchmark tenors:

$$f_k(t, t+\tau_{k,i}) := -\frac{\partial}{\partial T} \ln P_k(t,T)|_{T=t+\tau_{k,i}}, \quad k = 1, \ldots, n \tag{14.15}$$

  where $P_k(t,T)$ is the zero coupon bond for the currency CCY_k with the maturity date T, and $\tau_{k,i}$, $i = 1,, \ldots, N_k$ is the set of benchmark tenors for the currency CCY_k.

- Log-spot exchange rates for the foreign currencies: y_k, $k = 1, \ldots, n$.
- Log-spot equity prices for N_{EQ} model equities: s_j, $j = 1, \ldots, N_{EQ}$.
- Instantaneous forward hazard rates for N_{CR} model names, for the benchmark tenor:

$$f_m^{CR}(t, t+\tau_m^{CR}) := -\frac{\partial}{\partial T} \ln S_m(t,T)|_{T=t+\tau_m^{CR}}, \quad k = 1, \ldots, N_{CR} \tag{14.16}$$

  where $S_m(t,T)$ is the survival probability of the mth model name with the maturity date T, and τ_m^{CR} is the benchmark tenor for the mth model name.

- Instantaneous forward inflation rate for N_I model inflation indexes, for the set of benchmark tenors:

$$f_l^I(t, t+\tau_{l,i}^I) := -\frac{\partial}{\partial T} \ln P_l^I(t,T)|_{t+\tau_{l,i}^I}, \quad P_l^I(t,T) = \frac{P^r(t,T)}{P(t,T)}, \quad l = 1, \ldots, N_I, \tag{14.17}$$

  where $P(t,T)$ is the zero bond price related to the nominal rate of the inflation index currency, $P^r(t,T)$ is the price of the zero bond price related to the real rate, and $\tau_{l,i}^I$, $i = 1,, \ldots, N_l^I$ is the set of benchmark tenors for the forwardx inflation rate corresponding to the lth model inflation index.

- Log-spot inflation index for N_I model inflation indexes: $I_1, \ldots, I_{N_I}$.
- Instantaneous forward commodity convenience spread for N_{CR} model commodities, for the set of commodity benchmark tenors,

$$f_k^{cs}(t, t+\tau_{k,i}^{cs}) := -\frac{\partial}{\partial T} \ln P_k^{cs}(t,T)|_{t+\tau_{k,i}^{cs}}, \quad P_k^{cs}(t,T) = \frac{F_k(t,T)}{p_k(t)}, \quad k = 1, \ldots, N_{CM}, \tag{14.18}$$

  where $F_k(t,T)$ is the price of the futures for kth commodity with the maturity T, $p_k(t)$ is the spot price of the kth commodity, and $\tau_{k,i}^{cs}$, $i = 1,, \ldots, N_k^{CM}$ is the set of benchmark tenors for the convenience spread of the kth model commodity.

- Log-spot commodity prices for N_{CM} model commodities: p_k, $k = 1, \ldots, N_{CM}$.

Let us denote F the vector of all model factors defined above,

$$F(t) \;=\; \left(\ldots f_k(t,t+\tau_{k,i}),\ldots \ln y_k(t),\ldots \ln s_j(t),\ldots f_m^{CR}(t,t+\tau_m),\ldots f_l^I(t,t+\tau_{l,i}),\ldots \ln I_l(t),\right.$$
$$\left.\ldots f_k^{cs}(t,t+\tau_{k,i}),\ldots \ln p_k(t),\ldots\right)^T \qquad (14.19)$$

and C the covariance between the stochastic time incremenets of F. Now we describe how to calculate the model volatilities (14.14) and the model correlation ρ from the input covariance matrix C, assuming that the mean reversion rates are provided for all of the benchmark tenors of forward interest rate, forward hazard rate, forward inflation rate, and forward commodity convenience spread.

The dynamics of F can be defined by the following SDE,

$$dF(t) = (\ldots)\,dt + \Sigma dW(t), \qquad (14.20)$$

where $dW(t)$ is the vector of independent Brownian increments under some measure, and Σ is a covariance matrix decomposition such that $C = \Sigma\Sigma^T$. On the other hand, it can be shown from (14.11) and the definitions (14.15)-(14.18), that the dynamics of F can be represented as follows,

$$dF(t) = (\ldots)\,dt + \Gamma\xi dW(t), \qquad (14.21)$$

where $dW(t)$ is the vector of independent Brownian increments under the same measure as in (14.20); ξ is a decomposition of the covariance of the increments in (14.11); Γ is the mean reversion adjustment term which is the block-diagonal matrix

$$\Gamma = diag\left(\ldots,\Gamma_k,\ldots,\Gamma_k^{FX},\ldots,\Gamma_j^{EQ},\ldots,\Gamma_m^{CR},\ldots,\Gamma_l^{IS},\ldots,\Gamma_l^{I},\ldots,\Gamma_k^{cs},\ldots,\Gamma_k^{cm},\ldots\right); \qquad (14.22)$$

where

$$\Gamma_k = \left(e^{-a_{k,i}\tau_j}\right)_{i,j=1}^{N_k}, \quad \Gamma_k^{FX} = \Gamma_j^{EQ} = \Gamma_l^{I} = \Gamma_k^{cm} = (1), \quad \Gamma_m^{CR} = \left(e^{-\kappa_m\tau_m^{CR}}\right),$$

$$\Gamma_l^{IS} = \left(e^{-\alpha_{l,i}\tau_j^I}\right)_{i,j=1}^{N_l^I}, \quad \Gamma_k^{cm} = \left(e^{-\gamma_{k,i}\tau_j^{cm}}\right)_{i,j=1}^{N_k^{cm}}.$$

Equating the coefficients of the diffusion terms in (14.20) and (14.21), we get the following representation for the model covariance Ξ from the input covariance C:

$$\Xi := \xi\xi^T = \Gamma^{-1}C\left(\Gamma^{-1}\right)^T, \qquad (14.23)$$

from which we can get the model volatility η and correlation ρ,

$$\eta = \sqrt{\Xi_{i,i}}; \quad \rho = \left(\frac{\Xi_{i,j}}{\eta_i\eta_j}\right). \qquad (14.24)$$

In what follows, we describe how the input covariance matrix C is generated from the historical data.

14.4.1.1 Input Historical Data

The input to the model historical calibration are the time series of the historical data provided at the set of historical dates t_0,...,t_M. The following data are required for model historial calibration:

- Zero rate yield data for the currencies CCY_0,, ..., CCY_n,

$$z_{k,1},\ldots,z_{k,N_k}, \quad k = 0,\ldots,n, \qquad (14.25)$$

  where $z_{k,i}$ is the zero rate of the currency k at the historical date t with the maturity $t+\tau_i$;

- Spot exchange rate data for the foreign currencies, y_k, $k = 1,\ldots,n$;

- Spot equity data for N_{EQ} model equities, s_j, $j = 1, \ldots, N_{EQ}$;
- Survival probability data for N_{CR} model names,

$$S_{m,1}, \ldots, S_{m,N_m^{CR}}, \quad m = 1, \ldots, N_{CR}, \tag{14.26}$$

where $S_{m,i}$ is the survival probability for the mth model name at the historical date t with the maturity date $t + \tau_i^{CR}$;

- Inflation rate data for N_I model inflation indexes,

$$z_{l,1}^I, \ldots, z_{l,N_l^I}^I, \quad l = 1, \ldots, N_I, \tag{14.27}$$

where $z_{l,i}^I$ is the difference between the nominal and real zero rates of the inflation index currency with the maturity τ_i;

- Spot inflation index data for N_I model inflation indexes, I_k, $k = 1, \ldots, N_I$;
- Commodity forward data for N_{CM} model commodities,

$$F_{k,1}, \ldots F_{k,N_k^{CM}}, \quad k = 1, \ldots, N_{CM}, \tag{14.28}$$

where $F_{k,i}$ is the commodity forward or futures price of the kth model commodity at the historical date t with the maturity date $t + \tau_i^{CM}$.

14.4.1.2 Historical Covariance Calculation

The first step in the covariance calculation is to compute the historical time series for instantaneous forward interest rate, forward hazard rate, forward inflation rate, and for the commodity convenience spread, from the input historical data.

To generate the historical time series for the instantaneous forward rates f_k, $k = 0, \ldots, n$ (14.15) from the historical zero rates (14.25) for a currency CCY_k, we intepolrate the term structure of the zero rate to the union of all tenor points for the currency CCY_k from all historical dates, and use the log-linear interpolation on the discount factor curve. Then, we get

$$f_{k,1} := f_k(t, t+\tau_1) = z_{k,1}; \quad f_{k,i} := f_k(t, t+\tau_i) = \frac{\tau_i z_{k,i} - \tau_{i-1} z_{k,i-1}}{\tau_i - \tau_{i-1}}, \quad i = 2, \ldots, M_k. \tag{14.29}$$

Here, $\tau_1, \ldots, \tau_{M_k}$ is the combined set of all tenor points of the zero rate for the currency CCY_k from all historical dates. After that, we take the time series for those tenors, which are the closest to the model benchmark tenors.

For the instantaneous forward hazard rates f_m^{CR}, $m = 1, \ldots, N_{CR}$ (14.16), we use the log-linear interpolation on the survival probability term structure, and interpolate survival probability term structure defined in (14.26) to the union of all tenor points for the mth name from all historical dates. Then, we get

$$f_{m,1}^{CR} := f_m^{CR}(t, t+\tau_1) = \frac{\ln S_{m,1}}{\tau_1}; \quad f_{m,i}^{CR} := f_m^{CR}(t, t+\tau_i) = \frac{1}{\tau_i - \tau_{i-1}} \ln \frac{S_{m,i}}{S_{m,i-1}}, \quad i = 2, \ldots, M_m^{CR}, \tag{14.30}$$

Here, $\tau_1, \ldots, \tau_{M_m^{CR}}$ is the combined set of all tenor points of the survival probability for the mth model credit name from all historical dates. After that, we take the time series for those tenors, which are the closest to the model benchmark tenors.

To calculate the time series for the instantaneous forward inflation rates f_l^I, $i = 0, \ldots, N_I$ (14.17) from the historical inflation rates (14.27), we use the same approach as the one described above for the nominal rates,

$$f_{l,1}^I := f_l^I(t, t+\tau_1) = z_{l,1}^I; \quad f_{l,i}^I := f_l^I(t, t+\tau_i) = \frac{\tau_i z_{l,i}^I - \tau_{i-1} z_{l,i-1}^I}{\tau_i - \tau_{i-1}}, \quad i = 2, \ldots, M_l^I, \tag{14.31}$$

Here, $f^I_{l,i}$ is the instantaneous forward inflation rate corresponding to the lth model inflation index; $\tau_1, \ldots, \tau_{M^I_l}$ is the combined set of all tenor points of the input inflation rate term structure for the lth model inflation index from all historical dates. After that, we take the time series for those tenors, which are the closest to the model benchmark tenors.

To calculate commodity instantaneous forward spread f^{cs}_k, $k = 1, \ldots, N_{CM}$ (14.18) we intepolate the historical data (14.28) corresponding to the kth model commodity to the union of all forward dates in kth commodity forward curve from all historical dates, and use log-linear interpolation on commodity convenience spread. Then, we get

$$f^{cs}_{k,1} = \frac{\ln P^{cs}_{k,1}}{\tau_1}; \quad f^{cs}_{k,i} = \frac{1}{\tau_i - \tau_{i-1}} \ln \frac{P^{cs}_{k,i}}{P^{cs}_{k,i-1}}, \quad i = 2, \ldots, M^{CM}_k, \tag{14.32}$$

Here $P^{cs}_{k,i}$ is the discretization of the convenience spread obtained from the historical data (14.28) as follows,

$$P^{cs}_{k,i} = \frac{F_k(t, t+\tau_i)}{F_k(t, t+\tau_1)}, \quad i = 1, \ldots, M^{CM}_k, \tag{14.33}$$

and $\tau_1, \ldots, \tau_{M^{CM}_k}$ is the combined set of all forward dates of the forward commodity curve for the kth model commodity from all historical dates. After that, we take the time series for those tenors τ_i, which are the closest to the model benchmark tenors.

The second step in the historical covariance calculation is to generate the time increments of the historical series for all of the risk factors,

$$\begin{aligned} \Delta \;=\; & \Bigg(\ldots, f_{k,i}(t_m) - f_{k,i}(t_{m-1}), \ldots, \ln \frac{y_k(t_m)}{y_k(t_{m-1})}, \ldots, \ln \frac{s_j(t_m)}{s_j(t_{m-1})}, \ldots, f^{CR}_{s,i}(t_m) - f^{CR}_{s,i}(t_{m-1}), \ldots, \\ & f^I_{l,i}(t_m) - f^I_{l,i}(t_{m-1}), \ldots \ln \frac{I_l(t_m)}{I_l(t_{m-1})}, \ldots, f^{cs}_{k,i}(t_m) - f^{cs}_{k,i}(t_{m-1}), \ldots \ln \frac{p_k(t_m)}{p_k(t_{m-1})}, \ldots \Bigg)^M_{m=1}. \end{aligned} \tag{14.34}$$

Then, the covariance between the risk factor increments is estimated as follows,

$$C_{i,j} \approx \frac{1}{M} \sum_{m=1}^{M} \left((\Delta_i - \bar{\Delta}_i)(\Delta_j, -\bar{\Delta}_j)\right), \quad \bar{\Delta}_i = \frac{1}{M} \sum_{m=1}^{M} \Delta_i. \tag{14.35}$$

14.4.2 Combined Historical-Implied Calibration

The combined historical-implied calibration approach allows to calibrate some of the model volatilities to the implied market data. This approach is supported only in case when the single factor model is used for all short interest rates, inflation rates, and commodity convenience spread simulation.

As the first step, the model correlation and volatilities are calculated from the historical data. Next, the implied calibration of the short rate volatility is performed separately for each of the model currency. The implied calibration of a single-currency IR model is described in Chapter 2 on page 11.

Next, the implied calibration of the exchange rate volatility is performed separately, for each of the model foreign currency $CCY_1, \ldots, CCY_n$. The implied calibration of the exchange rate volatility is described in Chapter 5 on page 33.

After the short rate and exchange rate calibration is performed, the implied calibration of equitiy, inflation, and commodity spot volatility is performed independently for each quantity. For the details of the equity calibration, see Chapter 6 on page 41. For the details of the inflation index calibration, see Chapter 8 on page 61. For the details of the commodity calibration, see Chapter 10 on page 79.

Chapter 15

Bibliography

[1] Andreasen, J., Back to the future, *Risk*, September, pp.104–109, 2005.

[2] Andersen, L., Piterbarg, V., Interest Rate Modeling, Atlantic Financial Press, 2010.

[3] Brigo, D., Mercurio, F., Interest Rate Models: Theory and Practice: With Smile, Inflation and Credit, Springer-Verlag, 2006.

[4] Dun, T., Numerical studies on the lognormal interest rate model, PhD Thesis, University of Sydney, 2000.

[5] Ehlers, P. and Schonbucher, P., The Influence of FX Risk on Credit Spreads, Working paper, 2006.

[6] Hunt, P., Kennedy, J., Financial Derivatives in Theory and Practice, Wiley, 2004.

[7] Hull, J., Sokol, A., White, A., Short Rate Joint Measures Models, *Risk*, Octoberr, pp.59–63, 2014.

[8] Ng, L., Numerical Procedures for a Wrong Way Risk Model with Lognormal Hazard Rates and Gaussian Interest Rates, Working Paper, 2013.

[9] Schrager, D., Pelsser, A., Pricing Swaptions and Coupon Bond Options in Affine Term Structure Models, Working paper, 2004.