Consensus of Incommensurate-order Fractional Multiagent Systems with a Fixed-length Memory

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Abstract— **This paper is devoted to considering the consensus problem of fractional-order multiagent systems (FMASs) with incommensurate orders and a fixed-length memory. For this purpose, a method is proposed wherein a fractional protocol with a time-varying lower terminal is designed for the FMASs to reach consensus. It is developed based on approximating the system's monodromy matrix that determines the system's stability if its spectral radius locates on the unit circle. The major advantage of the proposed method is its ability to solve FMASs with incommensurate orders and shortened memories. Therefore, it is possible to design fractional protocols that do not suffer from the buffer overflow, which is a common issue for the current ones. The proposed method is examined and verified by means of a test example.**

I. Introduction

In the fast-paced development of science and technology, there has been an increasing interest in studying networks, communication technologies, and the coordination of multiagent systems (MASs) [?], [?], [?], [?], [?], [?], [?]. This demand is because of their potential applications in several diverse areas, such as cooperative robots [?], [?], unmanned air vehicles [?], autonomous underwater vehicles [?], spacecrafts [?], etc. For this purpose, several approaches have been presented for the control of cooperative objectives, such as consensus [?], rendezvous [?], containment [?], or flocking control [?].

The consensus problem for MASs is a basic method for obtaining cooperative control, which means that a group of agents achieve an agreement on their common states through local interactions only. Moreover, consensus protocols are ruled by using shared information among the agents to build the system. Recently, the consensus problem has brought much attention to researchers due to its importance in diverse fields, such as decision-making in control [?], the cooperative control of unmanned air vehicles [?], formation control of multiple robots [?], [?], the design of distributed sensor networks [?], [?], etc. [?], [?], [?], [?], [?]. Consensus problems can be mainly classified into two groups: Leaderless and leader-following consensus problems. In contrast to leaderless consensus problems [?], [?] that do not have a leader, leader-following consensus systems [?], [?] have a virtual leader who determines global goals for the agents. The consensus problem has been drawn from the simplest single-integrator systems [?] to double-integrator systems [?], [?] and to high-order dynamical systems [?], [?]. Most previous studies have considered the consensus problem of dynamical systems governed by ordinary differential equations (ODEs), but many practical engineering systems cannot be properly modeled in the framework of integer-order dynamics. In essence, dynamical systems with memory or hereditary properties cannot be well modeled using ODEs. This difficulty is the main reason for the birth of fractional differential equations (FDEs) that successfully express the nonlocality of anomalous dynamical systems with hereditary properties. In [?], some real applications of FMASs have been provided. For instance, a group of agents working in complex environments, such as macromolecule fluids and porous media, vehicles moving on the sand, muddy road, or grass, high-speed aircraft traveling in rain, dust storm, or snow environment, underwater vehicles operating in lentic lakes with viscous substances.

Fractional differential equations are a generalization of ODEs wherein the orders are relaxed to include fractions. Fractional differential equations have been used in diffusion processes [?], viscoelastic materials [?], economics [?], impact problems [?], continuum [?], statistical mechanics [?]. Moreover, they are classified into two main groups [?]: (i) Commensurate-order FDEs whose fractional orders are multiples of a number, and (ii) incommensurate-order FDEs whose fractional orders are not multiples of any number. Several techniques have been developed to approximate incommensurate-order FDEs with analogous commensurate-order FDEs, often in a larger size, but this is not feasible all the time [?]. The main advantage of using incommensurate-order FDEs in comparison to the commensurate-order ones is their applications in describing physical phenomena governed by Abel-type integrals [?], Volterra integral equations [?], Burgers-type equations [?], etc. In general, studying incommensurate-order FDEs is more rigorous than the commensurate-order ones [?]. On the other hand, FDEs require a large size buffer to save all the past data for obtaining the next state, which can cause the buffer overflow in a longer run. This is probably one of the most common issues of using fractional calculus for practical applications [?]. Consequently, the memory is often shortened using the short-memory principle, wherein the lower terminal of fractional operators is modified to a fixed-length memory.

To the authors' best knowledge, all the current studies on the consensus problem of fractional-order multiagent systems (FMASs) are conducted on commensurate ones and those with infinite buffer sizes [?], [?], [?], [?]. In other words, none can achieve the consensus condition for incommensurate-orders FMASs with a fixed-length memory. Therefore, this paper presents a new technique to design fractional-order protocols for those systems. For this purpose, the monodromy operator is approximated with shifted-Chebyshev polynomials, and then the stability of the proposed method is guaranteed by the use of a developed stability criterion. The efficiency of the proposed method is illustrated in a test example.

This paper is organized as follows. Section II gives some basic definitions and properties for the graph theory and fractional calculus. Section III presents a method for designing a fractional protocol to reach an FMAS in consensus. This section is followed by an illustrative example in Section IV. The conclusion is summarized in Section V.

II. Preliminaries

In this section, the basic concepts, definitions, notations of the graph theory and fractional calculus are given, which are used hereafter in this paper.

Definition 1. Information exchanging among m agents can be modeled by means of a graph topology. Moreover, an mth-order graph G is fully determined by the pair (V, ε), where $V = \{v_i\}_{i=1}^m$ and $\varepsilon = \{e_i\}_{i=1}^m$ denote the set of the vertices and the edge set, respectively. The vertices v_i and v_j are called the *parent* and *child* nodes of the edge $e_{ij} \overset{\text{def}}{=} (v_i, v_j) \in \varepsilon$. A set of all the neighbors for v_i is also defined by $N_i \overset{\text{def}}{=} \{v_j | e_{ji} \in \varepsilon\}$. Each vertex indicates an agent, and the edge e_{ij} illustrates that the ith agent can provide information to the jth agent.

Definition 2. The interaction of agents in a graph can be fully defined by the Laplacian matrix L defined as

$$L \overset{\text{def}}{=} D - A, \tag{1}$$

where D is a diagonal matrix whose diagonal elements are defined by $[D]_{ii} = \sum_{j=1}^m [A]_{ij}$ and it is called *degree matrix* of G, the matrix A is $m \times m$ whose element $[A]_{ij}$ is one if $e_{ij} \in \varepsilon$ and it is called the *adjacency matrix* of the graph G.

Definition 3. The most accepted definition of fractional derivatives for engineering applications is Caputo's definition [?] since Caputo fractional differential equations are subjected to integer-order initial conditions [?]. The left-sided Caputo αth-order fractional-derivative, $\alpha > 0$, for a function $x(t) \in L^1(a,b)$ is defined by [?]

$${}_a^C\mathcal{D}_t^\alpha x(t) \overset{\text{def}}{=} {}_a\mathcal{J}_t^{\lceil \alpha \rceil - \alpha} \left(\mathcal{D}^{\lceil \alpha \rceil} x\right)(t), \quad t \geq a, \tag{2}$$

where a is called the lower terminal indicating the start point of memory, the function $\Gamma(\cdot)$ represents the Gamma function, the operator $\lceil \cdot \rceil$ is the ceiling function, $\mathcal{D}^n(\cdot)$ represents the nth order derivative, and ${}_a\mathcal{J}_t^\alpha(\cdot)$ is the left-sided RL definition of fractional integral operator

$${}_a\mathcal{J}_t^\alpha x(t) \overset{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}\, x(\tau)\, d\tau. \tag{3}$$

III. Consensus of Incommensurate-Order Fractional Multiagent Systems with a Fixed-Length Memory

Consider an FMAS consisting of m agents whose dynamics is explained by the following system of linear FDEs:

$${}_{t_0}^C\mathcal{D}_t^\alpha \mathbf{x}_i(t) = A\mathbf{x}_i(t) + B\mathbf{u}_i(t), \tag{4}$$

for $i = 1, \ldots, m$, where $\mathbf{x}_i(t) = [x_{i_1}, \ldots, x_{i_n}]^T \in \mathbb{R}^n$ and $\mathbf{u}_i(t) \in \mathbb{R}^m$ are the state and control input of the ith agent, respectively, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrices, and $0 < \alpha < 1$. In addition, we consider the following fractional consensus protocol, which changes the problem to an incommensurate one as $0 < \beta < \alpha$ and $\alpha/\beta \notin \mathbb{N}$:

$$\mathbf{u}_i(t) = \sum_{j=1}^n a_{ij} \left(K_p \Delta\mathbf{x}_{ij}(t) + K_D\, {}_{t_0}^C\mathcal{D}_t^\beta [\Delta\mathbf{x}_{ij}(t)]\right), \tag{5}$$

for $i = 1, \ldots, m$, where $\Delta\mathbf{x}_{ij} = \mathbf{x}_j(t) - \mathbf{x}_i(t)$, $K_p \in \mathbb{R}^{m \times n}$ and $K_D \in \mathbb{R}^{m \times n}$ are the proportional and derivative gain matrices, respectively.

It is noticed an infinite size buffer is needed as t goes to infinity because the lower terminal of fractional derivatives in Eqs. (6) and (5) are fixed at t_0. This issue can be tackled by only counting the most recent history of the system in the solution. In other words, the lower terminal of the Caputo differentiation operator sets to be shorter in τ. Namely,

$${}_{t-\tau}^C\mathcal{D}_t^\alpha \mathbf{x}_i(t) = A\mathbf{x}_i(t) + B\mathbf{u}_i(t). \tag{6}$$

This technique is called the *short memory principle*, which yields an error bound

$$\begin{aligned} &\left\| {}_{t_0}^C\mathcal{D}_t^\alpha x(t) - {}_{t-\tau}^C\mathcal{D}_t^\alpha x(t) \right\| \leq \\ &C\left(t^{1-\alpha} - \tau^{1-\alpha}\right) / \Gamma(2-\alpha), \end{aligned} \tag{7}$$

where $C = \sup\limits_{t \in [t_0, t]} |dx(t)/dt|$.

This modification, along with the shortening of the memory, causes all the current techniques developed for the consensus problem of FMASs to fail to obtain the correct solution.

Let assume that $x_{i_j}(t)$ are square-integrable functions for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. So they can be approximated using any orthogonal polynomial set, such as shifted Chebyshev polynomials of the first kind giving the most accuracy under the maximum norm [?]. The space of all shifted Chebyshev polynomials $\{T_i(t)\}_{i=0}^{i=N-1}$ is denoted by $\mathbb{T}_N(\Omega)$, where the n^{th}-degree shifted Chebyshev polynomial of the first kind $T_n(t) : \Omega \mapsto [-1, 1]$, $\Omega = [t_0, t_f]$, is defined by the recurrence relation

$$T_n(t) = 2\left(\zeta(t - t_0) + 1\right) T_{n-1}(t) - T_{n-2}(t), \tag{8}$$

for $n = 2, 3, \ldots$, $\zeta = \frac{-2}{t_f - t_0}$, $T_0(t) = 1$, and $T_1(t) = \zeta(t - t_0) + 1$.

The space $\mathbb{T}_N(\Omega)$ is complete in $\mathcal{L}_w(\Omega)$ with respect to the weight function $w(t) = w\left(\zeta(t - t_0) + 1\right)$ in which $w(t) = 1/\sqrt{1 - t^2}$. That is

$$\langle T_n, T_m \rangle_{\mathcal{L}_w(\Omega)} = c_n \frac{\pi}{2\zeta} \delta_{nm}, \tag{9}$$

where δ_{nm} is the well-known Kronocker delta function, $c_n = 2$ for $n = 0$, and $c_n = 1$ for $n \neq 0$. Therefore, any square-integrable function $\phi(t) \in \mathcal{L}_w(\Omega)$ can be expanded by the interpolation operator $\mathcal{T}_N : \mathcal{L}_w(\Omega) \mapsto \mathbb{T}_N(\Omega)$ as

$$\left(\mathcal{T}_N \phi\right)(t) = \mathbf{T}_N{}^T(t) H_N \boldsymbol{\phi}_d, \tag{10}$$

where $\mathbf{T}_N(t) = [T_0(t), \ldots, T_N(t)]^T$, the column-vector $\boldsymbol{\phi}_d = [\phi(t_{d_0}), \ldots, \phi(t_{d_{N-1}})]^T$ is the discretized values of $\phi(t)$ at all the shifted Chebyshev-Gauss-Lobatto (CGL) points $\mathbf{t}_d$ defined by $t_{d_i} = \frac{1}{\zeta}\left(\cos\left(\frac{k\pi}{N-1}\right) - 1\right) + t_0$ for $k = 0, 1, \ldots, N-1$, and H_N is a constant $N \times N$ matrix defined by

$$H_N = \frac{1}{N} \begin{bmatrix} \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \\ (-1)^1 & 2T_1\left(t_{d_1}\right) & \cdots & 2T_1\left(t_{d_{N-1}}\right) & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (-1)^{N-2} & 2T_{N-1}\left(t_{d_1}\right) & \cdots & 2T_{N-1}\left(t_{d_{N-1}}\right) & 1 \\ \frac{1}{2}(-1)^{N-1} & T_N\left(t_{d_1}\right) & \cdots & T_N\left(t_{d_{N-1}}\right) & \frac{1}{2} \end{bmatrix}. \tag{11}$$

Now, let us define the left-sided elementary $\alpha-$power functions, $\alpha > 0$, as

$$\mathbf{E}_N^{\alpha}(t - t_0) = \left[0, (t - t_0)^{\alpha+1}, \ldots, (t - t_0)^{\alpha+N-1}\right]^T, \tag{12}$$

and for $\alpha = 0$, we have

$$\mathbf{E}_N^{0}(t - t_0) = \left[1, (t - t_0)^{1}, \ldots, (t - t_0)^{N-1}\right]^T. \tag{13}$$

Then the vector shifted Chebyshev polynomial $\mathbf{T}_N(t) = [T_0(t), \ldots, T_{N-1}(t)]^T$ is a linear map of the polynomials $\mathbf{E}_N^0(t - t_0)$ using the $N \times N$ map C_N defined by

$$\mathbf{T}_N(t) = C_N\, B_N\, \mathbf{E}_N^0(t - t_0), \tag{14}$$

where C_N is an $N \times N$ matrix whose first row is $[1, 0, \ldots, 0]$ and the other elements $[C_N]_{nk}$, $k \in \mathbb{N}$ and $n \geq 2$, are defined by

$$[C_N]_{nk} = \begin{cases} \frac{n-1}{2} \frac{(-1)^k 2^{n-2k-1}(n-k-2)!}{k!(n-2k-1)!}, & k \leq \lfloor \frac{n-1}{2} \rfloor, \\ 0, & k > \lfloor \frac{n-1}{2} \rfloor, \end{cases} \tag{15}$$

and the $N \times N$ matrix B_N is determined by using the binomial theorem as

$$[B_N]_{nk} = \begin{cases} \zeta^k \begin{pmatrix} n \\ n-k \end{pmatrix}, & k \leq n, \\ 0, & k > n. \end{cases} \tag{16}$$

Consequently, substituting Eq. (14) into Eq. (10) yields

$$\left(\mathcal{T}_N \phi\right)(x) = \mathbf{E}_N^{0\,T}(x - t_0)\, C_N{}^T\, H_N\, \boldsymbol{\phi}_d. \tag{17}$$

The fractional Chebyshev differentiation matrices of the left-sided Caputo, denoted by D_N^{α} is an $N \times N$ matrix that maps the discretized values of any function $\phi(t)$ to the discretized values of the left-sided α-derivative of $\phi(t)$ at the shifted CGL collocation points $\mathbf{t}_d$. Namely,

$$\mathrm{D}_N^{\alpha}\, \boldsymbol{\phi}_d = [({}_a\mathcal{D}_t^{\alpha}\, \phi)(t_{d_0}), \ldots, ({}_a\mathcal{D}_t^{\alpha}\, \phi)(t_{d_{N-1}})]^T, \tag{18}$$

where t_{d_i}, $i = 0, \ldots, N-1$, are the GLC collocation points.

To construct such a map, the following property for the left-sided Caputo derivative of the power function $(t - a)^{\beta}$, $t > a$, is used (see Property 2.1 in [?]):

$${}_a^C\mathcal{D}_t^{\alpha}\left((t - a)^{\beta}\right) = \begin{cases} 0, & \beta = 0, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t - a)^{\beta-\alpha}, & \beta \neq 0. \end{cases} \tag{19}$$

Moreover, it can be simply shown that the left-sided fractional derivatives of $\mathbf{E}_N^0(t - t_0)$ in the sense of Caputo is

$${}_a\mathcal{D}_t^{\alpha}\left(\mathbf{E}_N^0(t - t_0)\right) = \Lambda_N^{\alpha}\, \mathbf{E}_N^{-\alpha}(t - t_0). \tag{20}$$

Therefore, the function $\phi(t) \in \mathcal{L}_w(\Omega)$ is approximated using the left-sided shifted elementary zero-degree functions given in Eqs. (17) as

$$\begin{aligned} {}_a\mathcal{D}_t^{\alpha}\left(\mathcal{T}_N \phi\right)(t) &= {}_a\mathcal{D}_t^{\alpha}\left(\mathbf{E}_N^T(t - t_0)\right)\, C^T\, H_N\, \boldsymbol{\phi}_d \\ &= \mathbf{E}_N^{-\alpha\,T}(t - t_0)\, \Lambda_N^{\alpha}\, C^T\, H_N\, \boldsymbol{\phi}_d, \end{aligned} \tag{21}$$

which obtains the fractional Chebyshev differentiation matrices of the left-sided Caputo

$$\mathrm{D}_N^{\alpha} = E_N^{\alpha\,T}\, \Lambda_N^{\alpha}\, C_N{}^T\, H_N. \tag{22}$$

Moreover, the matrices C_N is defined in Eq. (14), H_N is defined in Eq. (11), the binomial matrix Λ_N^{α} is a diagonal matrix with elements $[\Lambda]_{kk} = \Gamma(k+1)/\Gamma(k+1-\alpha)$,

$k = 0, \ldots, N-1$, and

$$E_N^{\alpha} = \begin{bmatrix} 0 & (t_{d_0} - t_0)^{1-\alpha} & \cdots & (t_{d_0} - t_0)^{N-1-\alpha} \\ 0 & (t_{d_1} - t_0)^{1-\alpha} & \cdots & (t_{d_1} - t_0)^{N-1-\alpha} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & (t_{d_{N-2}} - t_0)^{1-\alpha} & \cdots & (t_{d_{N-2}} - t_0)^{N-1-\alpha} \\ 0 & (t_{d_{N-1}} - t_0)^{1-\alpha} & \cdots & (t_{d_{N-1}} - t_0)^{N-1-\alpha} \end{bmatrix}. \tag{23}$$

A more numerically robust scheme for constructing fractional Chebyshev differentiation matrices is developed in [?], [?].

Similarly, let $\mathbf{x}_{i_{jd},k}^T$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$, denote discretized solution of Eq. (6) (i.e., $x_{ij}(t)$) at the shifted CGL collocation points $\mathbf{t}_{d,k} \in [k\tau, (k+1)\tau]$. Additionally, we define $\mathbf{Y}_{d,k} = [\mathbf{x}_{id,k}^T, \ldots, \mathbf{x}_{md,k}^T]^T$ where $\mathbf{x}_{id,k} = [\mathbf{x}_{i_{1d},k}^T, \ldots, \mathbf{x}_{i_{nd},k}^T]^T$. Then, the solution of the FMAS (6) can be discretized using Eqs. (17) and (22). That is

$$\mathbb{D}_{nmN} \mathbf{Y}_{d,k} = \mathbb{A}_{nmN} \mathbf{Y}_{d,k}, \tag{24}$$

where

$$\mathbb{D}_{nmN} = I_{nm} \otimes (\bar{I}_N \, \mathrm{D}_N^{\alpha_1} + \bar{J}_N), \tag{25a}$$

$$\mathbb{A}_{nmN} = \left(I_m \otimes A - L \otimes BK_p\right) \otimes \bar{I}_N - (L \otimes BK_d) \otimes \bar{I}_N \, \mathrm{D}_N^{\beta}, \tag{25b}$$

the $N \times N$ matrix I_N represents the identity matrix, $\bar{I}_N$ stands for a $N \times N$ identity matrix whose first element is replaced by zero, $\bar{J}_N$ is a $N \times N$ matrix of zeros whose first element is replaced by one, J_N is a $N \times N$ matrix of zeros whose first element is replaced by zero, the operator $\otimes$ denotes the Kronecker product, and $\mathbf{Y}_{d,0} = \mathbf{Y}_0 \otimes J_0$ with J_0 represents an $N \times 1$ matrix of $[0, 0, 0, \ldots, 0, 1]^T$.

It is worthy to note that the discretized matrices are modified by employing the matrices $\bar{I}_N$, $\bar{J}_N$, and J_0 such that the solution continuity is assured at the initial conditions and at the boundaries $t_k = k\tau$ for $k = 1, 2, 3, \ldots$.

The monodromy matrix of the discretized system (24), denoted by M_d, is also defined as

$$M_d = (\mathbb{D}_{nmN} - \mathbb{A}_{nmN})^{-1} (I_{nm} \otimes J_N), \tag{26}$$

such that

$$\mathbf{Y}_{d,k} = M_d^k \mathbf{Y}_{d,0}. \tag{27}$$

Theorem 1. *The MAS (6) asymptotically reaches to consensus using the protocol (5) with the fixed-length memory of τ if and only if the spectral radius of its monodromy matrix (26) for a large enough number of collocation points lies inside the unit circle.*

Proof. Since M_d maps the initial condition, i.e. $\mathbf{Y}_{d0}$, to the solution in the next interval, the solution in the $(k+1)$th interval is given by Eq. (27). Applying the induced norm $\|(\cdot)\|$ on Eq. (27) results in

$$\|\mathbf{Y}_{d,k+1}\| \leq \left\|M_d^k\right\| \|\mathbf{Y}_{d,0}\| \leq \|\rho\left(M_d\right)\|^k \|\mathbf{Y}_{d,0}\|, \tag{28}$$

where $\rho\left(M_d\right)$ is the spectral radius of M_d. If $\rho\left(M_{d,\tau}\right) < 1$, then the solution converges to zero. Considering that the system is linear along the fact that the approximated solution converges to the actual solution using a large enough number of collocation points (see [?]) completes the proof. □

Employing Theorem 1 for the consensus problem of the FMAS (6) with the fixed-length memory results in obtaining a nonlinear optimization problem, where the spectral radius of the monodromy matrix is minimized to locate in the unit circle. This optimization problem with inequality constraints can be solved by any well-known optimization method that satisfies the Karush–Kuhn–Tucker (KKT) conditions.

IV. Numerical Examples

In this section, the feasibility and effectiveness of the proposed method are assessed by means of an illustrative example.

Inspiring by the example given in [?], we design a test problem including a group of six agents as shown in Fig. 1, whose Laplacian matrix is

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}. \tag{29}$$

They are governed by the fractional double-integrator dynamics

$${}_{t-\tau}\mathcal{D}_t^{\alpha} \mathbf{x}_i(t) = A\mathbf{x}_i(t) + B\mathbf{u}_i(t), \tag{30}$$

which has a fixed-memory length $\tau = 50$ seconds and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{31}$$

In addition, the protocol is given in Eq. (5) where $K_p = [k_{p_1}, k_{p_2}]$ and $K_d = [k_{d_1}, k_{d_2}]$, and $0 < \beta < \alpha$. The initial condition is also

$$\mathbf{y}(0) = [-3, -3, 4, 2, 6, -3, -6, -8, 5, -6, -2, 1]^T. \tag{32}$$

Figure 2 shows the uncontrolled states and their α-order derivative. It is shown that all the states are diverging, and they do not reach any consensus.

Next, we use $N = 50$ collocation points to approximate the monodromy matrix. The cost function for the optimization problem is also defined as

$$\min_{\beta, K_p, K_d} \quad \mathcal{J} = \rho(M_d), \tag{33}$$

where $\rho(M_d)$ is the spectral radius of the monodromy matrix. This optimization problem is solved by

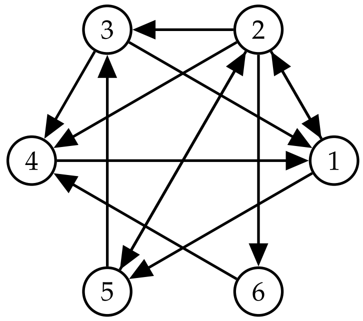

Fig. 1. Interaction graph among six agents with fractional-order dynamics and a fixed-length memory of $\tau = 50$ seconds.

Fig. 2. (a) The uncontrolled state and (b) the α-order derivative of the uncontrolled states when $\alpha = 0.5$ and $\tau = 50$.

employing the `fmincon` in MATLAB with a trust-region method, which is based on interior-point techniques. All the options for this optimization algorithm are left as the `default` adopted by MATLAB. The elements of the gain matrices $K_p = [k_{p_1}, k_{p_2}]$ and $K_d = [k_{d_1}, k_{d_2}]$ are bounded between -100 and 100, and the fractional-order of β is bounded between 0 and α. The initial guess is also set to zeros for the control gains as well as $\beta_0 = \alpha/2$. Figure 3-(a) shows the agent states that reach a consensus. One can see that the reason for the linear growth of the states is due to the common agreement on a nonzero value among the α-order derivative of the states, which is shown in Fig. 3-(b). It is possible to modify the protocol (5) such that all the states reach to the desired values, but this is saved for the journal version of this study.

Table I shows the obtained optimal gains, optimal fractional-orders, and the cost function values for different $\alpha = \{0.1, 0.2, \ldots, 1\}$. It is shown that the spectral radius of the monodromy matrix is less than one for all the orders. It is noticed that increasing the order α increases the order β and decreases the spectral radius. In other words, the system becomes more stable by increasing the fractional-order of the dynamics.

V. Conclusion

In this paper, the consensus problem of fractional-order multiagent systems with incommensurate orders and a fixed-length memory was investigated. A framework was proposed to overcome the main burden of fractional-order systems, which is the demand for a large buffer size for computation and including incommensurate orders. Moreover, the current methods addressing the consensus problem of fractional-order multiagent systems fail to stabilize them when the orders are incommensurate or the memory is fixed at a finite length. The method was developed by approximating the system's monodromy matrix using an operational matrix of fractional differentiation. It was shown that the consensus problem is transformed into a nonlinear programming problem, which can be solved by any conventional algorithm. A numerical example, including a group of six agents, was presented to illustrate the effectiveness and advantages of the proposed method.

(a)(b)

Fig. 3. (a) The controlled states and (b) the α-order derivative of the controlled states when $\alpha = 0.5$ and $\tau = 50$.

TABLE I

THE OBTAINED OPTIMAL CONTROL GAINS AND FRACTIONAL ORDERS FOR DIFFERENT VALUES OF α WHEN $N = 50$.

α	β	k_{p_1}	k_{p_2}	k_{d_1}	k_{d_2}	$\mathcal{J}$
0.10	0.056	3.318	0.759	0.603	-0.181	0.232
0.20	0.130	2.614	0.370	0.697	-0.148	0.173
0.30	0.185	1.879	0.841	1.162	0.511	0.196
0.40	0.275	1.352	1.001	0.964	0.724	0.162
0.50	0.348	1.047	0.895	0.996	0.813	0.151
0.60	0.471	0.954	0.917	0.961	1.060	0.079
0.70	0.500	1.272	0.897	0.866	0.978	0.041
0.80	0.587	1.031	1.027	0.799	1.051	0.029
0.90	0.658	1.312	0.603	0.775	1.229	0.010
1.00	0.744	1.039	0.886	0.694	1.285	0.010