Time delay complicates the mathematical analysis of dynamic systems behavior. It is expressed by Delayed Differential Equations (DDE) and adds exponential terms in the Laplace domain leading to infinite complex roots of the characteristic equation [1-roya], [2-roya].

The development of control methods for delayed systems is imperative as time delay is detectable in the responses of most communicational equipment, microprocessors, and electronic circuits. Also, it has been observed in numerous processes such as market pricing [], population growth [], drivers' behavior in traffic [], biological systems [], and quantum scattering processes [].

Prevalence of the time delay phenomena has motivated many researchers to try to tackle the complicated mathematics of the delay systems. Thus, Numerous attempts have been made to develop some conventional control methods such as PID [], LQR [], LQG [], H∞ [], MPC [], and LMI to cover the dynamic systems with time delays. Their efforts resulted in devising tricky methods based on infinite-dimension approximation [], abstract system theory, dual-theory [], maximum principle, discretization approach [], and continuous time approximation viewpoints [].

The linear quadratic Gaussian (LQG) control problem is one of the most fundamental optimal control problems []. For example, the model predictive control, one of the most applicable control methods in recent years, runs the first LQG optimal control input repeatedly. The LQG problem is to find an output feedback law that minimizes the expected value of a quadratic cost function including control error and input values. In the LQG problem, all the states may not be measured, and the measured signals are contaminated by Gaussian white noises. Furthermore, the problem considers that some Gaussian process disturbances can corrupt the response of the dynamic model.

The separation principle plays the most significant role in solving the LQG problem. It simplifies the LQG problem by dividing it into two independently solved problems: the optimal linear quadrative regulatory problem (LQR), and the optimal linear quadratic estimation problem (LQE) []. The LQR is solved by the linear dynamic programming approach using the solution of an inverse Riccati differential equation []. The LQE is solved by the first-order optimization condition using the solution of a forward Riccati differential equation [].

The separation principle no longer holds in presence of the delay time to aggravate the LQG problem for the delayed dynamic systems []. Meanwhile, the LQG problem cannot be divided into the LQG and LQE problems for time delay systems, many researchers try to extend the LQR and LQE for delayed systems separately [].

The abstract system theory converts the delayed state space to the functional state space equations with no delay time. It replaces the finite states of the delayed state-space with functional variables on the delay time range. It means the values of states between the time delay and zeros are considered functional variables (called functional states) for the specific time. The initial values of the functional states are derived from the delayed time state space when the dummy variable is zero. The problem with this method is that the linear operator of the linear functional differential equation (FDE) is not defined explicitly, and the implicit initial condition equation should be solved. Providing that the linear operator of the abstract FDE is defined, an explicit solution for LQR was presented []. The problem of finding explicit feedback law for the control and estimation of the LQG problem is still outstanding to our best knowledge. (Also see []).

The abstract FDE and the initial condition equation have inclined many researchers into opt approximation methods. Discretization-based and polynomial approximation-based approaches have been introduced in this decade. Among all the methods, the Chebyshev polynomials demonstrate their performance by providing highly stable and accurate solutions for the time-delayed systems [,]. In this research, we …..

**Preliminaries:**

In this section, the definitions, theories, and lemma’s supporting the abstract system theory are presented. Then the abstract form of the time delay system and the differential equation of the initial condition are expressed. The abstract FDE and the differential equation of the initial condition are used for Chebyshev discretization in the next section.

**Definition 1.** The space is a Banach vector space over the field with norm if all the norms in are members of the . It means the space is complete to its norm.

**Definition 2.** The function is linear if two conditions are satisfied:

(1). Superposition: for any two vectors in such as (x,y), the following relationship holds:

|  |  |
| --- | --- |
|  | (1) |

(2) Proportionality: for any vector in like x, and any scalar like α, this relationship holds:

|  |  |
| --- | --- |
|  | (2) |

**Definition 3.** The family of one Parameter linear functions T(t) on the Banach space is called a semigroup linear operator on if two conditions are satisfied:

|  |  |
| --- | --- |
|  | (3) |
|  | (4) |

**Definition 4.** ([42, Section 7.1]). The family of one Parameter functions T(t) on the Banach space is called a strongly continuous semigroup, denoted by -semigroup, if it holds:

|  |  |
| --- | --- |
| h > 0, | (5) |

Where in means h approaches zero from above that is h > 0.

**Definition 5.** The infinitesimal generator of a -semigroup is defined by:

|  |  |
| --- | --- |
|  | (6) |

Where:

|  |  |
| --- | --- |
|  | (7) |

**Theorem 1.** [] Consider the abstract Cauchy problem as follow:

|  |  |
| --- | --- |
|  | (8) |

subject to −valued initial function If the strongly continuous C0-semigroup one-parameter family of functions can be find so that the infinitesimal generator of equals to A, then the unique solution of the stated Cauchy problem is as follow:

|  |  |
| --- | --- |
|  | (9) |

The theorem (1) can be simply proved by substituting the solution into the FDE.

A simple example can depict the theory 1 more clearly. Consider the special case that the infinitesimal operator is a constant matrix named A. The strongly continuous C0- semi group family functions of is the solution operator as its infinitesimal operator is A.

|  |  |
| --- | --- |
|  | (10) |
|  | (11) |
|  | (12) |

The delayed time differential equation DDE can be expressed as a non-delayed FDE by defining the abstract functional variable. Consider the DDE state space as follow:

|  |  |
| --- | --- |
|  | (13) |

For solving a DDE, the history of the states for is needed. As a result, the initial condition is expressed by a function containing the history of the states in the delay interval before t=0 as follow:

|  |  |
| --- | --- |
|  | (14) |

To remove the time delay from the delayed equation, a functional variable is defined which contains all the values of states from the arbitrary time “t” to the delay time after that “t+ *”*. The dummy variable is the new symbol of time in this range. The abstract functional variable can be defined as Definition (6).

**Definition 6.** The functional state in the Banach space is defined as

|  |  |
| --- | --- |
|  | (15) |

So, the initial condition for a FDE is defined as follows:

|  |  |
| --- | --- |
|  | (16) |

And the initial value is:

|  |  |
| --- | --- |
|  | (17) |

We have a functional variable for any specific time “t”. The equation can be rewritten as:

|  |  |
| --- | --- |
|  | (18) |

Now the above FDE can be defined as a new abstract Cauchy problem.

**Definition 7.**  is defined as Banach space with the sup-norm . It includes all the continuous functions such as the linear bounded function and the initial function The functional differential equation (FDE) is defined as:

And the solution operator is defined as such that:

Based on the

Lemma 1 ([42, Lemma 1.2 (Sec 7)]). The FDE (4) can be rewritten as an abstract Cauchy problem with the C0- semigroup T(t)≥0 of linear and bounded operators on B whose infinitesimal generator A is given by φ 0 (θ) = Aφ(θ), D(A) = {φ(θ) ∈ B: φ 0 (θ) ∈ B, φ 0 (0) = L(φ(θ))}.

Theorem 5. The time-delay system (11) without uncertainties can be written as an abstract Cauchy problem with the C0-semigroup T(t)≥0 of linear and bounded operators on B = C([−τ, 0], R n) whose infinitesimal generator is φ 0 (θ) = Aφ(θ), D(A) = φ(θ) ∈ B: φ 0 (θ) ∈ B, φ 0 (0) = Aφ(0)+ Adφ(−τ ) + B Z 0 −τ κ(θ)φ(θ)dθ . (20) Proof. According to Definition 7, x(t) = xt(0) when −τ ≤ t ≤ 0 as well as the feedback control (19) can be rewritten as u(t) = Z 0 −τ κ(θ)xt(θ)dθ. (21) Thus, the linear functional operator can be defined as L(xt(θ)) := Axt(0) + Adxt(−τ ) + B Z 0 −τ κ(θ)xt(θ)dθ. (22) Now, according to Lemma 1 the infinitesimal generator for the C0−semigroup solution operator of the abstract Cauchy problem with the above linear functional operator is given by Eq. (23). This completes the proof of Theorem 5.

Based on Definition (6) we can write:

Simply we can write:

If the above equation is used in derivative definition, it will result in:

The equation can be rewritten as:

At t=0, the differential equation is:

The above equation is linear and can be solved at t=0.