Nested surds

Solution

For each card below, determine which non-negative values of a, b, c, and d, if any, make the equation true.

We have presented the solutions below in an order that seems sensible, starting with the perhaps more familiar statements and making use of these in some of the subsequent solutions.

(a) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

This is true for all positive values of a and b. It is also true when one or both of a and b is zero.

(b) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

This is true for all positive values of a and b and when $a = 0$. However, when $b = 0$ then both sides of the equation are undefined.

(d) $a\sqrt{b} = \sqrt{ab}$

In the case where one or both of a and b is zero then the equation is true. It is also true if $a = 1$ because then both sides of the equation simplify to $\sqrt{b}$.

However, if we select other positive values such as $a = 2$ and $b = 4$, then $a\sqrt{b} = 2 \times 2 = 4$ whereas $\sqrt{ab} = \sqrt{2 \times 4} = \sqrt{8}$, so the equation is not true.

In more general terms, if we take the original equation and square both sides we get $a^2b = ab$. This equation can only be true when $a^2 = a$ so when $a = 0$ or $a = 1$ as we have already identified.

(e) $\dfrac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}} = 1$

For this to be true we require that $\sqrt{ab} = \sqrt{a} + \sqrt{b}$.

However, it is not easy to find values of a and b that satisfy this requirement. For example, taking $a = 1$ and $b = 4$ gives $\sqrt{ab} = 2$ but $\sqrt{a} + \sqrt{b} = 3$.

It looks as though $a = b = 0$ might be a solution. But, we must remember that the original equation looked like $\dfrac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}} = 1$ and taking a and b as zero would result in a division by zero, which is undefined.

We know that for the equation to be true we require that $\sqrt{ab} = \sqrt{a} + \sqrt{b}$. Subtracting $\sqrt{a}$ and $\sqrt{b}$ from both sides we get

$$\sqrt{ab} - \sqrt{a} - \sqrt{b} = 0.$$

Now, this can be factorised as

$$(\sqrt{a} - 1)(\sqrt{b} - 1) - 1 = 0,$$

and subtracting the negative 1 gives us

$$(\sqrt{a} - 1)(\sqrt{b} - 1) = 1.$$

So if I pick two positive numbers which multiply to 1, say c and $\frac{1}{c}$, then set $\sqrt{a} - 1 = c$ and $\sqrt{b} - 1 = \frac{1}{c}$, it should work!

To confirm that this is the case we should work backwards from these solutions and check that they will work in the original equation.

If $\sqrt{a} - 1 = c$, then $\sqrt{a} = c + 1$ and

$$a = (c+1)^2.$$

Similarly, if $\sqrt{b} - 1 = \frac{1}{c}$ then $\sqrt{b} = 1 + \frac{1}{c}$ and

$$b = \left(1 + \frac{1}{c}\right)^2 = \left(\frac{c+1}{c}\right)^2.$$

Then we have

$$\sqrt{ab} = (c+1)\frac{c+1}{c} = \frac{(c+1)^2}{c},$$

and

$$\sqrt{a} + \sqrt{b} = c + 1 + \frac{c+1}{c} = \frac{c^2 + 2c + 1}{c} = \frac{(c+1)^2}{c}.$$

Returning to the original equation we now have

$$\frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}} = \frac{\left(\frac{(c+1)^2}{c}\right)}{\left(\frac{(c+1)^2}{c}\right)} = 1.$$

So we have demonstrated that there are in fact infinitely many solutions to this equation.

Finally, it is sensible to try it out with some numbers. Let's take $c = 2$ and then from above,

$$a = (c+1)^2 = 9$$

and

$$b = \left(\frac{c+1}{c}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

Substituting these values of a and b into the original equation gives

$$\frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}} = \frac{\left(\frac{9}{2}\right)}{\left(\frac{9}{2}\right)} = 1$$

as required!

(f) $\sqrt{a} - \sqrt{b} = \sqrt{a-b}$

When we have $b = 0$ then both sides of the equation will simplify to $\sqrt{a}$ and the equation holds. However, if we have $a = 0$ and $b \neq 0$, then the equation becomes $-\sqrt{b} = \sqrt{-b}$ and, since there is no real solution to $\sqrt{-b}$, the equation is not true.

If we try other positive values for a and b we find that the equation is not always true, for instance when $a = 4$ and $b = 1$ then $\sqrt{a} - \sqrt{b} = 1$ but $\sqrt{a-b} = \sqrt{3}$.

When we let $b > a$ we also run into problems as $\sqrt{a-b}$ will have no real solutions.

We need to identify exactly when this equation is true. If we take the original equation and square both sides we get

$$a+b-2\sqrt{ab}=a-b.$$

Rearranging, this gives

$$b=\sqrt{ab}$$

and squaring both sides to get rid of the square root gives

$$b^2=ab.$$

So we require that $a=b$.

Substituting this back into the original equation to check shows that $a=b$ gives

$$\sqrt{a}-\sqrt{b}=0$$

and

$$\sqrt{a-b}=0,$$

so $a=b$ is indeed a solution.

The equation is true when $a=b$ and for any value of a when $b=0$.

(g) $\sqrt{a}+\sqrt{b}=\sqrt{a+b+\sqrt{4ab}}$

In the case where one or both of a and b is zero the equation is true.

If we take $a=1$ and $b=4$ then $\sqrt{a}+\sqrt{b}=\sqrt{1}+\sqrt{4}=3$. Also, $\sqrt{a+b+\sqrt{4ab}}=\sqrt{1+4+\sqrt{4\times1\times4}}=\sqrt{9}=3$. So the equation is true.

Taking another example, say $a=4$ and $b=9$ we have that $\sqrt{a}+\sqrt{b}=\sqrt{4}+\sqrt{9}=5$ and $\sqrt{a+b+\sqrt{4ab}}=\sqrt{4+9+\sqrt{4\times4\times9}}=\sqrt{25}=5$. So the equation is still true.

So far we have selected square numbers for a and b because the expressions are much easier to simplify and hence demonstrate whether or not they are equal. If we are to be sure that the equation is true for all positive values of a and b then we need to prove it.

Starting with the left-hand side of the equation, squaring gives

$$(\sqrt{a}+\sqrt{b})^2=a+b+2\sqrt{ab}.$$

This is already looking promising. Writing $2\sqrt{ab}$ as $\sqrt{4ab}$ gives us

$$(\sqrt{a}+\sqrt{b})^2=a+b+\sqrt{4ab}.$$

Taking the square root of both sides now gives us the desired result

$$\sqrt{a} + \sqrt{b} = \sqrt{a + b + \sqrt{4ab}}.$$

So this equation is true for all non-negative values of a and b.

(i) $\sqrt{5+2\sqrt{6}} = \sqrt{a}+\sqrt{b}$

This question is essentially the same as the one above and we can use the same approach. Writing the left-hand side in the form $\sqrt{a+b+\sqrt{4ab}}$ gives

$$\sqrt{5+2\sqrt{6}} = \sqrt{5+\sqrt{4\times 6}}.$$

Comparing this to $\sqrt{a+b+\sqrt{4ab}}$ we have that $a+b=5$ and $ab=6$.

By inspection we can quickly see that $a=2$ and $b=3$ or vice versa. Therefore

$$\sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}.$$

(c) $\sqrt{23-6\sqrt{6-4\sqrt{2}}} = \sqrt{a}+\sqrt{b}$

This is a more complicated-looking version of the previous two problems. Our strategy worked for both of those so let's apply it again.

Let's start with the final part of this: $\sqrt{6-4\sqrt{2}}$.

Notice that there is a subtle difference between this and the previous questions. In the previous questions we made use of the expression $\sqrt{a+b+\sqrt{4ab}}$ but this time we need $\sqrt{a+b-\sqrt{4ab}}$.

If we square this expression we get

$$a+b-\sqrt{4ab} = a+b-2\sqrt{ab}.$$

This is the result of squaring $\sqrt{a}-\sqrt{b}$ so we can write

$$\sqrt{a+b-\sqrt{4ab}} = \sqrt{a}-\sqrt{b}$$

(assuming that $a \geq b$).

We will try to write $\sqrt{6-4\sqrt{2}}$ in the form $\sqrt{a}-\sqrt{b}$ in order to simplify the expression.

Using the approach from above, $\sqrt{6-4\sqrt{2}} = \sqrt{6-\sqrt{4\times 8}}$ and comparing this to $\sqrt{a+b-\sqrt{4ab}}$ tells us that $a+b=6$ and that $ab=8$. By inspection we can see that $a=2$ and $b=4$ or vice versa. This time however, the order matters as we need to remember that $\sqrt{6-4\sqrt{2}}$ is non-negative.

Therefore we take $a=4$ and $b=2$, showing that $\sqrt{6-4\sqrt{2}}=\sqrt{4}-\sqrt{2}=2-\sqrt{2}$.

Substituting this into the original nested surd gives

$$\sqrt{23-6\sqrt{6-4\sqrt{2}}}=\sqrt{23-6(2-\sqrt{2})}.$$

We can then simplify the right-hand side by expanding the brackets to give

$$\sqrt{23-6\sqrt{6-4\sqrt{2}}}=\sqrt{23-12+6\sqrt{2}}=\sqrt{11+6\sqrt{2}}.$$

We now need to repeat the process to write $\sqrt{11+6\sqrt{2}}$ as $\sqrt{a}+\sqrt{b}$.

Writing $6\sqrt{2}$ in the form $\sqrt{4ab}$ we get

$$6\sqrt{2}=\sqrt{72}=\sqrt{4\times 18}.$$

Substituting this back into $\sqrt{11+6\sqrt{2}}$ gives

$$\sqrt{11+\sqrt{4\times 18}}.$$

Comparing this to $\sqrt{a+b+\sqrt{4ab}}$ tells us that $a+b=11$ and $ab=18$. By inspection we can see that $a=9$ and $b=2$ or vice versa.

Therefore

$$\sqrt{11+\sqrt{4\times 18}}=\sqrt{9}+\sqrt{2}=3+\sqrt{2},$$

and overall we have

$$\sqrt{23-6\sqrt{6-4\sqrt{2}}}=3+\sqrt{2}.$$

So the equation is true when $a=9$ and $b=2$.

(h) $\dfrac{\sqrt{a}+b}{\sqrt{c}+d} = (\sqrt{a}+b)(\sqrt{c}-d)$

As with the other equations, let's start by considering zero values. If $a = b = 0$ then the equation is true. If one of a and b is zero then it is less clear whether or not the equation is true – it depends on the values of c and d too. If $c = d = 0$ then we get a zero divisor on the left-hand side.

Taking a numerical example such as $a = 1$, $b = 2$, $c = 4$ and $d = 3$ we find that

$$\frac{\sqrt{a}+b}{\sqrt{c}+d} = \frac{3}{5}.$$

However,

$$(\sqrt{a}+b)(\sqrt{c}-d) = 3 \times -1 = -3.$$

So we have shown that the equation is not true for all positive values of a, b, c and d.

We now need to consider whether there are any positive values for which the equation does hold. If we multiply both sides of the equation by $\sqrt{c}+d$ we get

$$\sqrt{a}+b = (\sqrt{a}+b)(\sqrt{c}-d)(\sqrt{c}+d).$$

Dealing only with the case where $\sqrt{a}+b \neq 0$, we can divide through by $\sqrt{a}+b$ giving

$$1 = (\sqrt{c}-d)(\sqrt{c}+d) = c - d^2.$$

We have used the difference of two squares here.

We now have a required relationship between c and d,

$$c = d^2 + 1.$$

So the equation is true either when $a = b = 0$ with c and d not both 0, or when $c = d^2 + 1$.