# Index issues

Solutions

For what values of $x$ and $y$ do these statements make sense?

What are the possible values for the expressions?

(1) $(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

Is the left-hand side the same as the right-hand side?

$(xy)^{\frac{1}{2}} = \sqrt{xy} = \sqrt{x}\sqrt{y} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

It would seem so from the reasoning above, however what happens if $x = -2$ and $y = -8$? $\sqrt{xy}$ will give us an answer of $4$ but $\sqrt{x}\sqrt{y}$ will not be defined.

We can only get a real value from a square root when the number is not negative, so $x \geq 0$ and $y \geq 0$.

The overall value of the expression $(xy)^{\frac{1}{2}}$ (and therefore $x^{\frac{1}{2}}y^{\frac{1}{2}}$) will also be $\geq 0$.

We have said that the value of the expressions $(xy)^{\frac{1}{2}}$ and $x^{\frac{1}{2}}y^{\frac{1}{2}}$ will be $\geq 0$. But will they always be positive? We may think not, as a square root has two solutions: a positive and a negative one.

However, the square root symbol $\sqrt{\phantom{x}}$, and writing something as raised to the power of $\frac{1}{2}$, denotes the *principal* root, which is always the positive root. If we take a square root, as part of rearranging an equation then we may need to write $\pm\sqrt{\phantom{x}}$ or $\pm(\quad)^{\frac{1}{2}}$ depending on the situation.

Can you think of an equation which involves taking a square root where you will always want the positive root only?



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(2) $(xy)^{\frac{5}{3}} = x^{\frac{5}{3}} y^{\frac{5}{3}}$

$$(xy)^{\frac{5}{3}} = \left(\sqrt[3]{xy}\right)^5 = \left(\sqrt[3]{x}\right)^5 \left(\sqrt[3]{y}\right)^5 = x^{\frac{5}{3}} y^{\frac{5}{3}}$$

Is there anywhere this chain of reasoning breaks down? Does it make a difference if we write $\sqrt[3]{x^5}$ instead of $\left(\sqrt[3]{x}\right)^5$?

We can cube root any real value, positive or negative, so $x, y \in \mathbb{R}$. What values can $(xy)^{\frac{5}{3}}$ take?

(3) $(xy)^{\frac{2}{3}} = x^{\frac{2}{3}} y^{\frac{2}{3}}$

$$(xy)^{\frac{2}{3}} = \left(\sqrt[3]{xy}\right)^2 = \left(\sqrt[3]{x}\right)^2 \left(\sqrt[3]{y}\right)^2 = x^{\frac{2}{3}} y^{\frac{2}{3}}$$

We know we can cube root any number, so $x, y \in \mathbb{R}$. The value of $(xy)^{\frac{2}{3}}$ however, is restricted. How is it different from the value of $(xy)^{\frac{5}{3}}$ and why is that?

**Explanation**

A cube root can give a positive or negative answer. Raising a negative to the power of $5$ (or any odd number), will give you a negative answer so $(xy)^{\frac{5}{3}}$ can take any value. This is unlike $(xy)^{\frac{2}{3}}$, as squaring any number will always result in a positive answer. So $(xy)^{\frac{2}{3}}$ is restricted to being greater than or equal to zero.

(4) $(xy)^{-\frac{1}{2}} = x^{-\frac{1}{2}} y^{-\frac{1}{2}}$

This is similar to the first question. It only involves square roots, so we know $x$ and $y$ cannot be negative. What happens to either side of the equation if $x$ or $y = 0$?

(5) $(xy)^{\frac{1}{2}} = x^{\frac{1}{3}}y^{\frac{2}{3}}$

On first glance, we might easily dismiss this as being incorrect. The question that was asked though, is 'for what values of $x$ and $y$ do these statements make sense?' Are there any values of $x$ and $y$ when this equation would be true?

If $x = 0$ or $y = 0$, then the equation is satisfied, but to see if there are any other possibilities we should try to solve this equation.

$$(xy)^{\frac{1}{2}} = x^{\frac{1}{3}}y^{\frac{2}{3}}$$

For the left hand side to be defined, $xy \geq 0$. In addition, $x$ and $y$ cannot be negative. Can you explain why?

We can write

$$x^{\frac{1}{2}}y^{\frac{1}{2}} = x^{\frac{1}{3}}y^{\frac{2}{3}},$$

then divide by $x^{\frac{1}{3}}y^{\frac{1}{2}}$, assuming $x, y \neq 0$,to get

$$x^{\frac{1}{6}} = y^{\frac{1}{6}}.$$

Therefore

$$x = y$$

is a solution for $x, y > 0$, along with $x = 0$ or $y = 0$.

This is perhaps more solutions than we might have first thought. Can you see why this happens? Try some numbers if you are not sure.

  

(6) $(xy)^{-2} = x^2y^2$

Once again, this is not true for all values, but may still be true for certain values of $x$ and $y$.

It can also be written

$$\frac{1}{(xy)^2} = x^2y^2,$$

so we can see that if $x$ or $y$ was zero the left-hand side would not exist, therefore $x, y \neq 0$.

We can multiply by $(xy)^2$

$$1 = x^4y^4,$$

then take the fourth root. This gives us a positive and negative value so we get

$$1 = xy \text{ or } -1 = xy,$$

giving us the solutions

$$x = \frac{1}{y} \text{ and } x = -\frac{1}{y}.$$

(7) $\left(\frac{x}{y}\right)^{-\frac{1}{3}} = x^{-\frac{1}{3}}y^{\frac{1}{3}}$

From the $\left(\frac{x}{y}\right)$ on the left hand side we can see that $y \neq 0$, even though it could be on the right hand side.

On the right hand side we have $x^{-\frac{1}{3}}$ which is $\frac{1}{\sqrt[3]{x}}$ and so $x \neq 0$.

Otherwise we can cube root any value so $x, y \in \mathbb{R}$ except for $x, y \neq 0$ and the value of the expressions can also take any value except $0$.

(8) $\left(\frac{x}{y} - \frac{1}{y}\right)^{\frac{1}{2}} = y^{-\frac{1}{2}}(x-1)^{\frac{1}{2}}$

On the left hand side we require $y \neq 0$ and $\frac{x}{y} - \frac{1}{y} \geq 0$, which implies that $x \geq 1$or $x \leq 1$ depending on whether $y$ is positive or negative.

On the right hand side we have $y^{-\frac{1}{2}}$ which requires $y > 0$, and $(x-1)^{\frac{1}{2}}$ which also requires $x \geq 1$.

Putting these together gives us $y > 0$ and $x \geq 1$ and the value of either side of the equation must be $\geq 0$.

Remember that although square rooting something gives us a positive and negative solution, the notation of 'to the power of $\frac{1}{2}$' denotes the *principal root* which is positive.