

Laboratory Session : April 30, 2024  
Exercises due on : May 13, 2024

### Exercise 1: Bayesian Inference for Poisson model

The number of particles emitted by a radioactive source during a fixed interval of time ( $\Delta t = 10$  s) follows a Poisson distribution on the parameter  $\mu$ . The number of particles observed during consecutive time intervals is: 4, 1, 3, 1, 5 and 3.

- (a) assuming a *positive uniform* prior distribution for the parameter  $\mu$ 
  - determine and draw the posterior distribution for  $\mu$ , given the data
  - evaluate mean, median and variance, both analytically and numerically in R
- (b) assuming a *Gamma* prior such that the expected value is  $\mu = 3$  with a standard deviation  $\sigma = 1$ ,
  - determine and draw the posterior distribution for  $\mu$ , given the data
  - evaluate mean, median and variance, both analytically and numerically in R.
- (c) evaluate a 95% credibility interval for the results obtained with different priors. Compare the result with that obtained using a normal approximation for the posterior distribution, with the same mean and standard deviation

### Exercise 2: Efficiency using Bayesian approach

A researcher A wants to evaluate the efficiency of detector 2 (Det2). For this purpose, he sets up the apparatus shown in the figure 1, where Det2 is sandwiched between Det1 and Det3. Let  $\mathbf{n}$  be the number of signals recorded simultaneously by Det1 and Det3, and  $\mathbf{r}$  be those also recorded by Det2, researcher A obtains  $\mathbf{n} = 500$  and  $\mathbf{r} = 312$ .

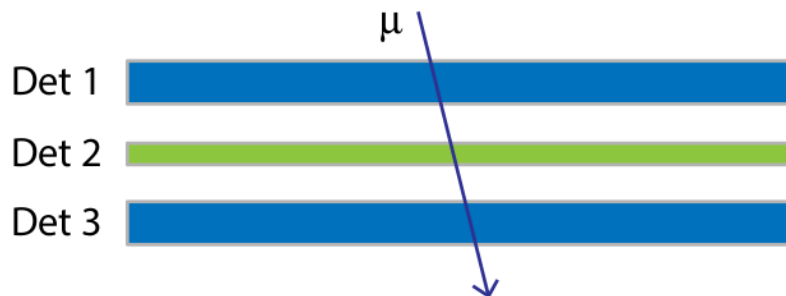


Figure 1:

Assuming a binomial model where  $\mathbf{n}$  is the number of trials and  $\mathbf{r}$  is the number of success out of  $\mathbf{n}$  trials,

- a) Evaluate the *mean* and the *variance* using a Bayesian approach under the hypothesis of:

- uniform prior  $\sim \mathcal{U}(0, 1)$
- Jeffrey's prior  $\sim \text{Beta}(1/2, 1/2)$

b) Plot the posterior distributions for both cases

Taking into account that the same detector has been studied by researcher B, who has performed only  $n = 10$  measurements and has obtained  $r = 10$  signals,

- c) Evaluate the *mean*, the *variance* and the *posterior* distribution using a uniform prior with the results of researcher B.
- d) Repeat the computation of points a) and b) with the data of researcher A using as a prior the posterior obtained from point c).
- e) [Optional] Compute 95% credible interval using the posterior of the previous point d).

### Exercise 3 - Bayesian Inference for Binomial model

- A coin is flipped  $n = 30$  times with the following outcomes:

T, T, T, T, T, H, T, T, H, H, T, T, H, H, H, T, H, T, H, T, H, H, T, H,  
T, H, T, H, H, H

- a) Assuming a flat prior, and a beta prior, plot the likelihood, prior and posterior distributions for the data set.
- b) Evaluate the most probable value for the coin probability  $p$  and, integrating the posterior probability distribution, give an estimate for a 95% credibility interval.
- c) Repeat the same analysis assuming a sequential analysis of the data. Show how the most probable value and the credibility interval change as a function of the number of coin tosses (i.e. from 1 to 30).
- d) Do you get a different result, by analysing the data sequentially with respect to a one-step analysis (i.e. considering all the data as a whole) ?

### Exercise 4 - Poll

A couple of days before an election in which four parties (A,B,C,D) compete, a poll is taken using a sample of 200 voters who express the following preferences 57, 31, 45 and 67 for, respectively, parties A,B,C and D.

Using a Bayesian approach, for all parties

- Calculate the expected percentage of votes and a 68% credibility interval by assuming as prior a
  - uniform prior
  - a prior constructed from the results obtained from another poll conducted the previous week on a sample of 100 voters who expressed the following preferences 32, 14, 26, 28 for, respectively, parties A,B,C and D.
- Sample size to obtain a margin of error less or equal than  $\pm 3\%$  for each party