The Mathematics Playground of Billiard

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May 2, 2017

0.1 Introduction

Billiard trajectories are trajectories that are based on a billiard table. In this project, our goal is to create a table with dimensions m and n, an angle α , and a point that the ball will start at, which would be (a, b). If the ball goes back to its initial point after trials, we define it as "periodic. The question is: Is the path going to be closed after a given amount of time? Is there going to be an untouched region after an infinite number of bounces? There are multiple billiard table shapes that have been used in this project, for example, rectangles, squares, parallelograms, circles, ellipses, etc. It is fascinating that a game with such simple rules can derive numerous variations. In this paper, we will incorporate quantitative proof with illustrations and discover the mystery of a parallelogram billiard table.

0.2 Rectangular Billiard Trajectory

The main billiard trajectory that has been studied is the rectangular billiard trajectory. There are various methods to understanding these trajectories. An important concept in mathematical billiards is the fact that the angle of incidence is equal to the angle of reflection. This concept is helpful because it can be used to prove theorems and understand paths of billiard balls. Mathematician's focus on the bounces of a billiard ball and the concept that the angle of incidence equals the angle of reflection will give them the knowledge of the angle it bounces off at. From Figure 1, $\angle$a is equal to $\angle$b and $\angle$c is equal to $\angle$d. By using this information, we can continue to investigate the pathway of this trajectory. However, this method is very time consuming and visually not very understandable. Mathematicians found an easier method called

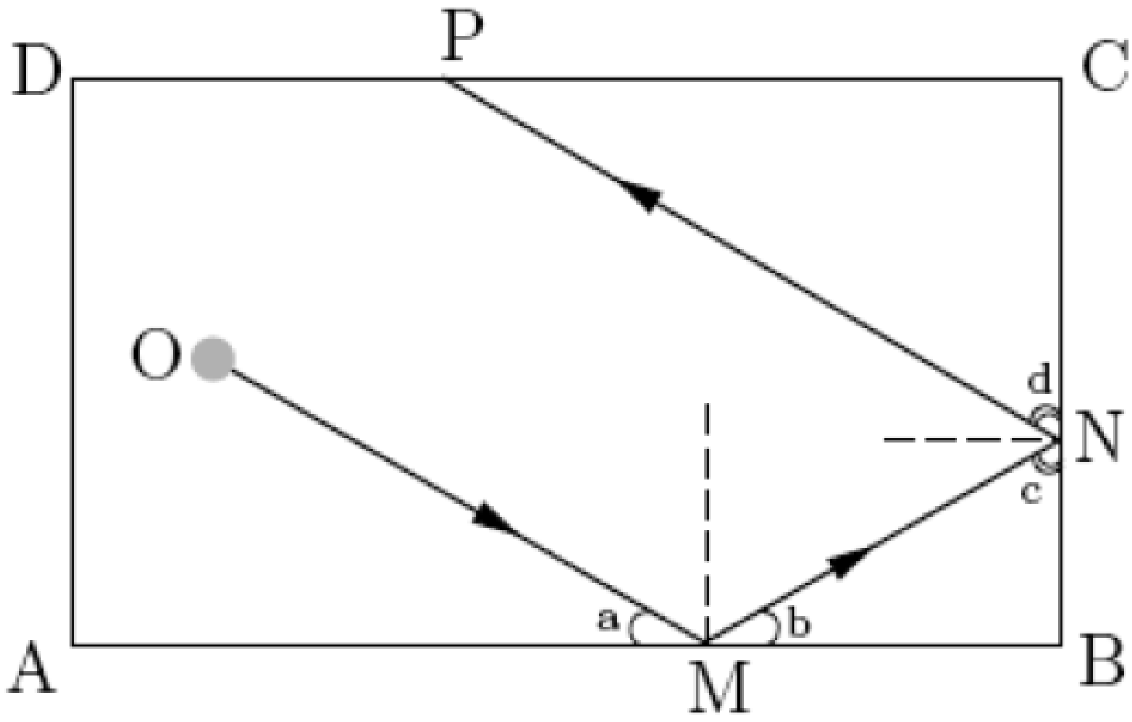

Figure 1: Reflection Rule

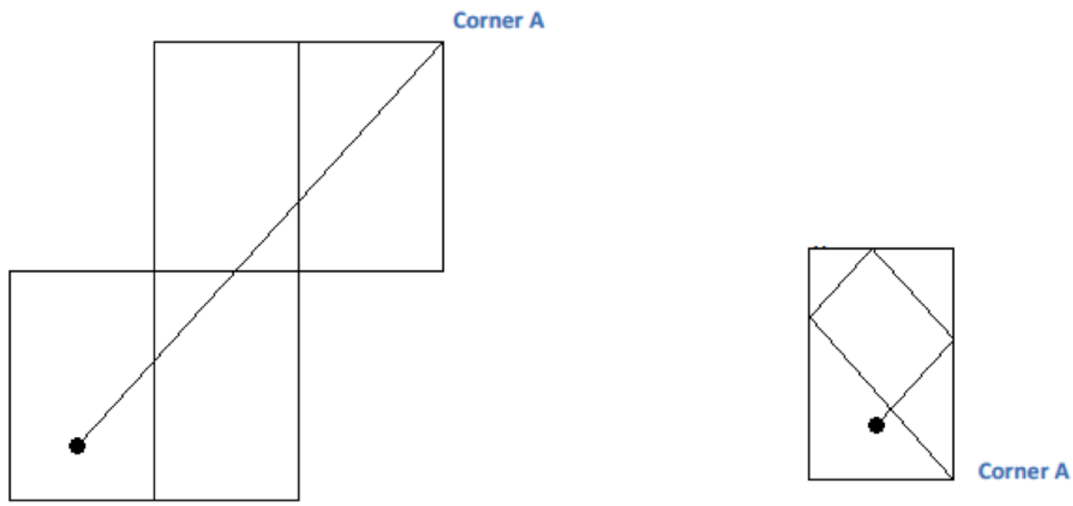

Figure 2: Unfolding method

the unfolding method. This method starts with the original rectangle and then adds another rectangle of the same size on the right side of that and another one above the second one. In Figure 2, it shows three rectangles added to the original rectangle. We can then draw a straight line from the ball in the original rectangle through the other rectangles at the given angle of 45 °(see Figure 2). This pattern of unfolding right and up is continued until the straight line reaches a corner (in this case, Corner A). When the ball reaches the given point we can reflect each line over the x-axis or y-axis, respectively, to project it onto the original rectangle. When put together, we will see the path of the ball when shot at 45 °(see Figure 3). (Miller, 6) This method works for any angle, any shape, and can easily be applied.

We will start with the most trivial paths and angles for square and rectangle billiard tables. There are a few trajectories that the square billiard table has a period of 2. The first one is when we start on one of the four edges of the square and we hit the ball with a 90 angle. This gives us infinitely many periodic trajectories with a

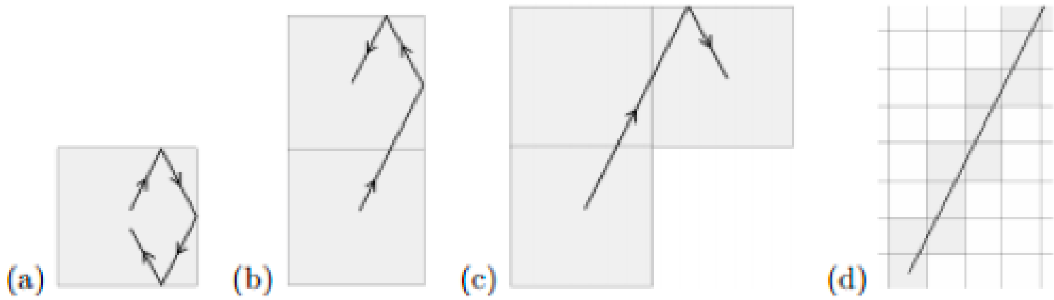

Figure 3: Unfolding Method

period of 2, which are horizontal and vertical lines. (See Figure 1(a)) If we start from one of the 4 corners of a square billiard and hit the ball at a 45°we will get a period of 2 for all 4 cases. These cases will work for all dimensions in the real numbers for a square. In Figure 1(b), the trajectory is periodic with a period 4, which would work for any side of the square at the midpoint with a 45°angle. In Figure 1(c), the trajectory is periodic with a period 6, which would work for the vertical edges at the midpoint with a 30°and also if the ball starts on the horizontal edges at specific points with a 60°angle.

Is it possible to hit the ball so that it never repeats its path? It is more difficult to draw a picture of an example of such a non-periodic trajectory because the trajectory never repeats. In fact, it will gradually fill up the table until the picture is a black square (Figure 1 (d)). (Davis, 2)

Consider the simple trajectory in Figure 3 (a) below. When the ball hits the top edge, instead of having it bounce and go, we unfold the table upwards, creating another copy of the table in which the ball can keep going straight (Figure 3 (b)). 1 Now when the trajectory hits the right edge, we do the same thing: we unfold

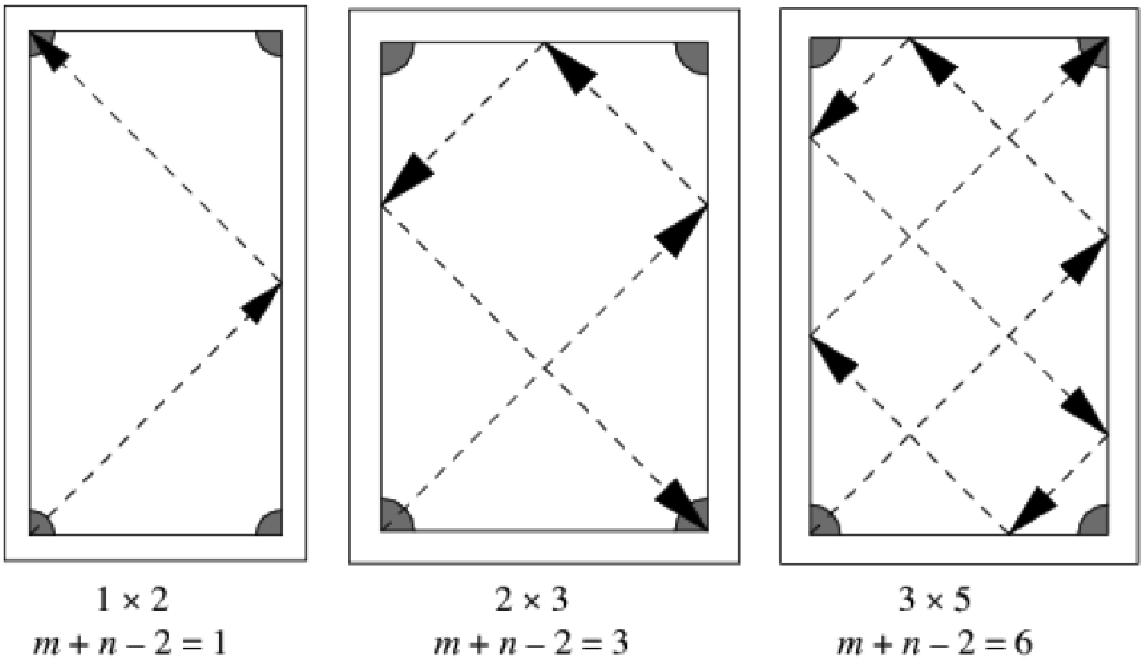

Figure 4: Rectangle Billiards

the table to the right, creating another copy of the table, in which the ball can keep going straight (Figure 3 (c)). We can keep doing this, creating a new square every time the trajectory crosses an edge. In this way, a trajectory on the square table is represented as a line on a piece of graph paper (Davis, 3). This is the unfolding method on a square billiard table. From the square we can interpret the trivial cases to the rectangular shaped billiard table and also find a formula for specific types of dimensions and characteristics.

Given a rectangular billiard table with only corner pockets and sides of integers m which are relatively prime, a ball sent at a 45°from a corner will be pocketed in another corner after $m + n - 2$ bounces. (Steinhaus 1999, p. 63; Gardner 1984, pp. 211-214) This is a very helpful method for figuring out when a ball starts from one corner and ends up in another corner on the table. However, this does not give an exact formula for figuring out if the ball comes back to the starting point. So now, we know how to calculate the period for a rectangular billiard given that the trajectory is closed. However, is the path always going to be closed? The answer is no. For example, we have points (x , y) lie in a rectangle with center at (0 , 0) and vertices at $(\pm a, \pm b)$.

$$(\alpha')^2 = (\frac{dX}{dx})^2 \tag{1}$$

$$(\alpha'')^2 = (\frac{dY}{dy})^2 \tag{2}$$

$$Px = \pm a' \tag{3}$$

$$Py = \pm a'' \tag{4}$$

For the particle to accomplish a periodic path, it has to return to the initial point with the same momentum (speed and direction). This is possible only if

$$T = Nx\,Tx = Ny\,Ty$$

where T is the time period of the orbit, and Nx, Ny are integers. Here, Tx and Ty represent the time in which the particle returns to its starting point with the same speed and direction. This formula is also equivalent to saying that for an orbit to be closed and periodic, $\tan\theta = \frac{Px}{Py}$ must be rational, where θ is the initial shooting angle of the ball? Atreyee 5).

0.3 Parallelogram Billiard Trajectory

Our research of rectangle and square shaped billiard tables has given us an idea of how to interpret the pathways and formulas of trajectories. However, we have come to realize that we cannot exactly apply the same results from these trajectories to our parallelogram shaped billiard tables. These examples have shown that to understand a parallelogram shaped billiard table we will have to first understand the more basic shape, which is an equilateral triangular billiard table and from there on we can connect two triangular billiard tables to get a parallelogram. This is one of the methods mathematicians have used to understand the pathways of rectangles from first observing square shaped billiard tables.

In general, there are two types of billiard trajectories we have observed. The first one is easier to calculate pathways for any type of trajectories. It is called the video game billiard trajectory, which is similar to video games from the past. In these video games, generally there exists a ball that you shoot from any point on the table and whenever it hits either the bottom or top of the table it appears from the opposite side, same for the left or right side of the table it appears from the opposite side the ball hits, which is either the left or right side of the table. So, every time a billiard table is folded back towards the first billiard table (like in figure 23), there is no need to translate. The second type is called the standard reflective billiard trajectory, which has more steps in figuring out the pathway of the billiard. In this version, every time a billiard table is folded back towards the first billiard table, there exists a translation that has to be included in calculating the period of this trajectory. So, for every step there has to exist a translation, which will alter the reflection of the billiards pathway.

So when is the path going to be closed in a parallelogram? Would the ball return to the same point after a given period of time? To figure out the general pattern, we will start off with an a : b = 1:1 easy case where (a, b) is the dimensions of the unit parallelogram. Suppose we shoot the ball from (x0, y0) in an a:b parallelogram, in which case our parallelogram angle would be $\alpha and the initial shooting angle would be \beta$. In this picture, k is the distance from the starting point to the right corner of the triangle. By using the unfolding method, our ball reaches (xo + k+ l * a, yo + l * b) after a series of bouncing and reflections, where L is a scaling factor depending on the final position of the ball. In our previous paragraph, we concluded that the trajectory in a rectangle would be periodic if and only if the tangent value of the shooting angle is rational and the same rule also applies to our parallelogram case.

To tackle this problem, we first composed a triangle with point (xo, yo), (xo+k, yo) and (xo + k + la, yo + lb).

To simplify the question, we translate it to the origin and have the following vertices: (0, 0), (k, 0) and (k + l*a, l*b). By using the Cosine Rule, we derived,

$$Cos\beta = \frac{2k^2 + 2kla}{2k * (\sqrt{(k+la)^2 + (lb)^2})} = \frac{(k+la)}{\sqrt{(k+la)^2 + (lb)^2}}$$

Therefore, for any given θ , if there exists k, l, a and b that satisfies the aforementioned equation and gives it a rational value, we can then conclude that there is a closed trajectory path in the parallelogram, otherwise, the path would never close

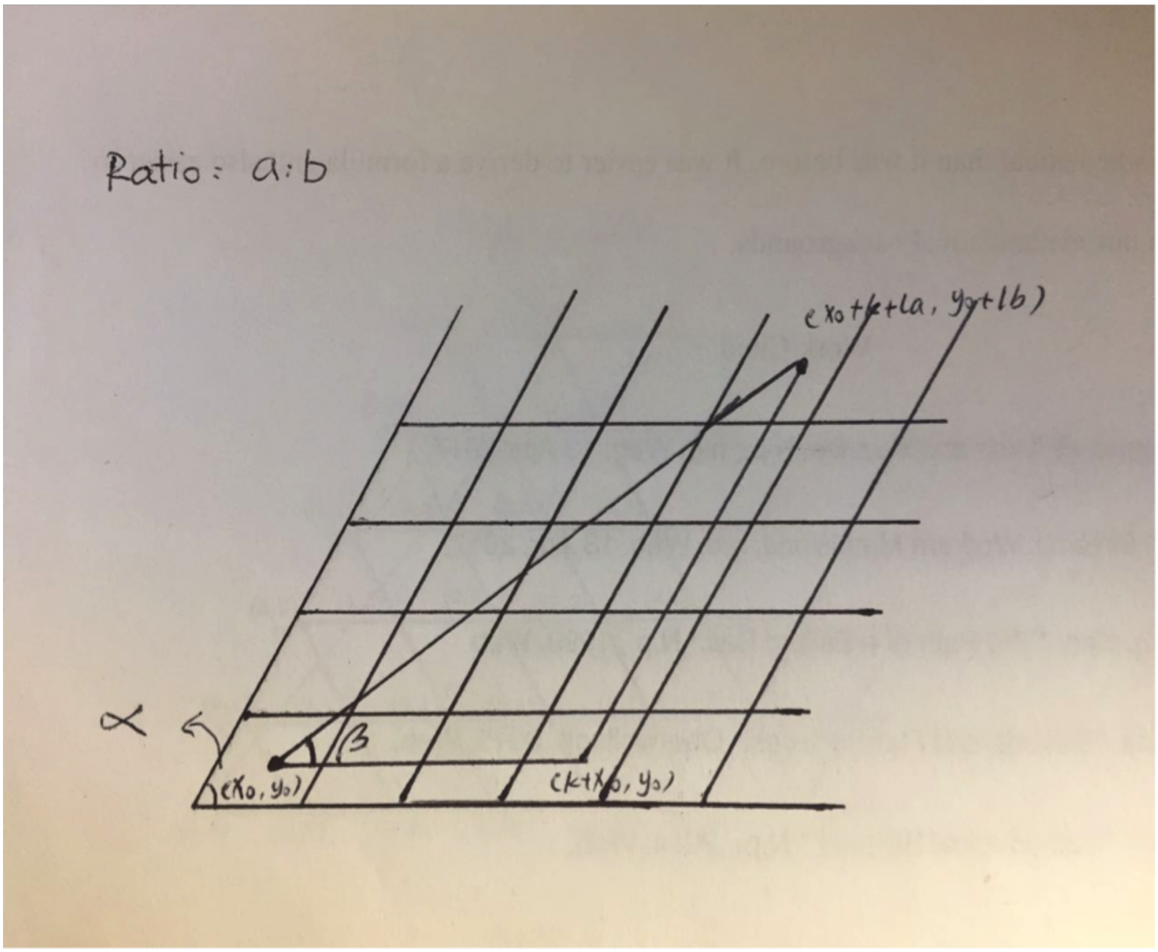

Figure 5: Parallelogram Billiards

up. The same conclusion also works when a and b have different ratios. In this case, we create another variable called s, which will be an integer multiplied by b and L will be an integer multiplied by a. With translation, the vertices then become (0,0), (k,0) and (k+ la, sb) and

$$Cos\beta = \frac{2k^2 + 2kla}{2k(\sqrt{(k+la)^2 + (sb)^2})} = \frac{(k+la)}{\sqrt{(k+la)^2 + (sb)^2}}$$

where the launching angle would be arccos β. Therefore, with any dimensions, angles and initial points for parallelograms, if for a given θ, there exists constants k, l, s, a and b to make the equation hold , we can say the orbit is closed.

Since we already are already able to determine the condition for closed trajectory, oue next step is to figure out the period. To solve this question, let's first take a look at a special case - a diamond shaped billiard table with a 60°angle, which is composed by equilateral triangles. Suppose the ball is shot from (0,0) with an initial angle β . By using the same unfolding technique, the path could be easily visualized. In figure-6, we introduced natural coordinates, so that the lower left corner is at (0.0) and the upper right is (1,4). If we are given an end point (x,y), then by mathematics induction we are able to conclude that the period for equilateral billiard board is going to be 2(x+y)-2. This formula is for equilateral triangles and we know that parallelograms are two equilateral triangles put together. So, if we want to interpret it for parallelograms, we only need to divide our period formula by half. However, if they are not relatively prime, then we would divide the period of an equilateral triangle by the greatest common denominator and then divide it in half to get the

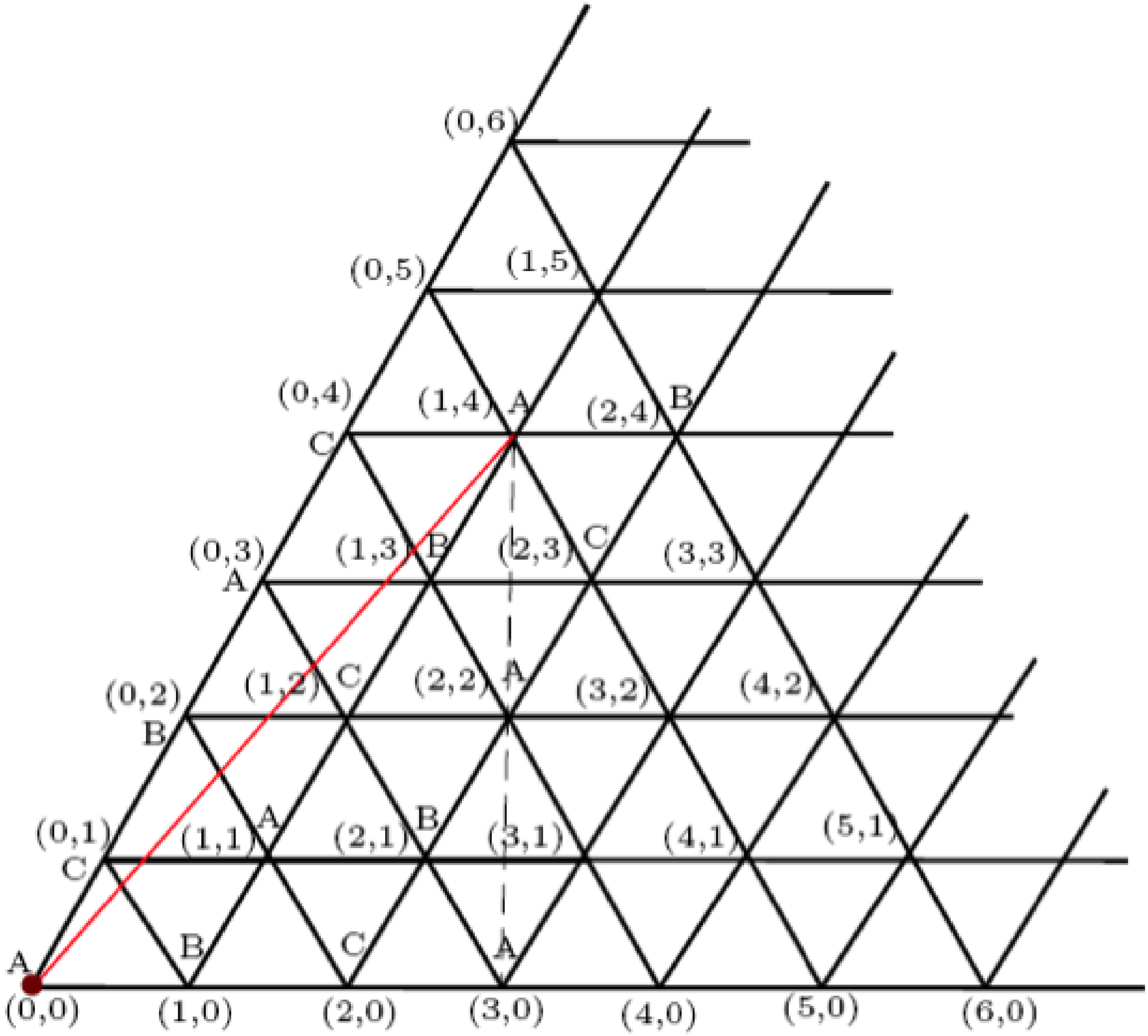

Figure 6: Parallelogram Billiards with 60 °

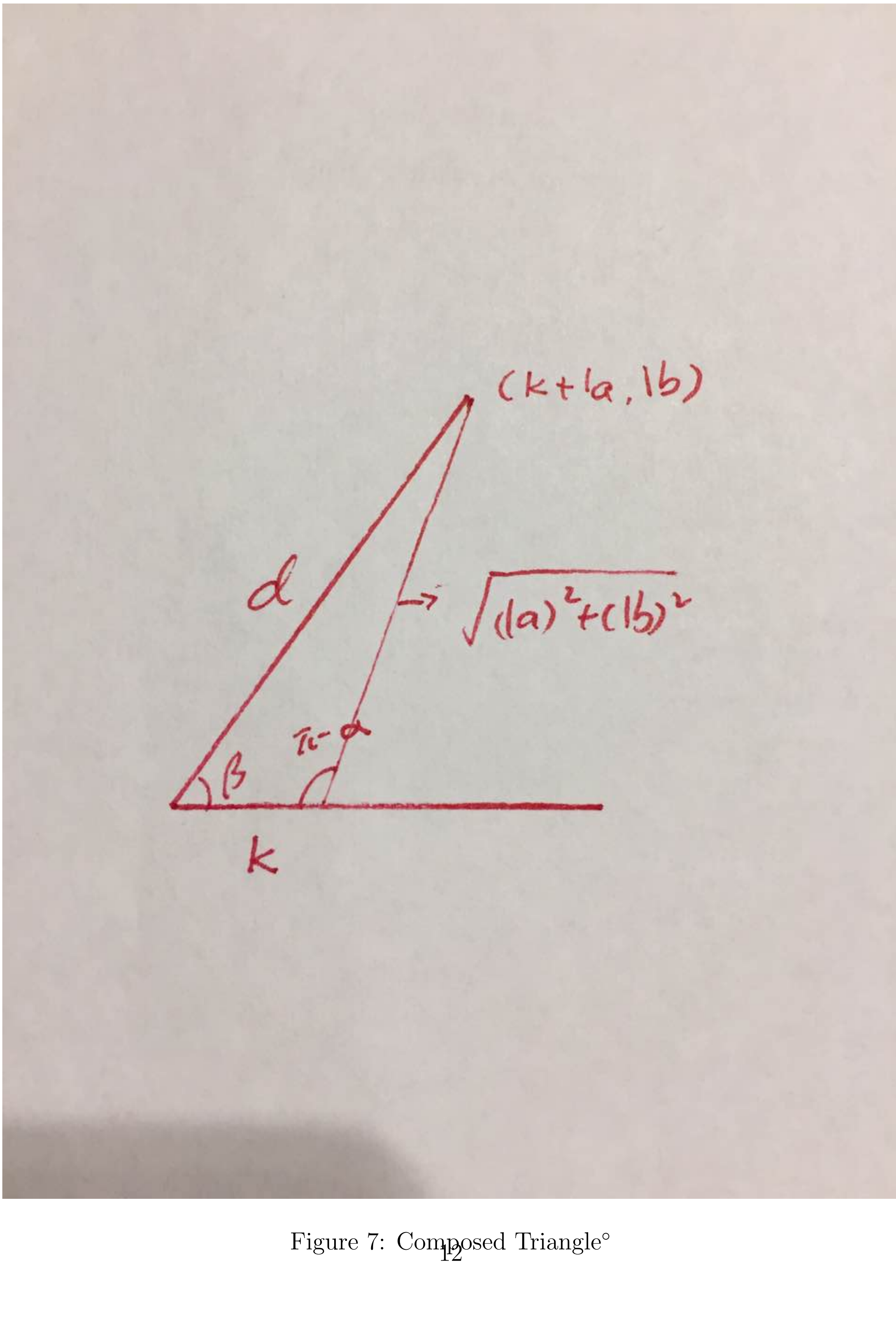

Figure 7: Composed Triangle°

period of a parallelogram. This gives us (x+y)-1 for the period of a parallelogram shaped billiard table. Also in the 60°case, our launching angle will be

$$\alpha = \arctan\frac{Py}{Px} = \arctan(\frac{\sqrt{3}y}{2(x+y/2)}) = \arctan(\frac{\sqrt{3}y}{2x+y}) \tag{5}$$

$$withPy = tan60^{\circ}/2y = \sqrt{3}/2y \tag{6}$$

$$Px = x + ((y * \sqrt{3})/2) * \sqrt{3} = x + \frac{y}{2} \tag{7}$$

$$tan\alpha = \frac{\sqrt{3}y}{2(x+y/2)} = \frac{\sqrt{3}y}{2x+y} \tag{8}$$

where Py and Px is the height and width of the triangle we composed, respectively.

In this case, we calculate our period as (x+y) - 1 for the video game version. However, for the standard reflective billiard version or period would be twice the period of the video game version. So, the standard reflective billiard version has period 2 (x+y) - 2 in this example.

In order for this equation to hold, there needs to exist a countable number of solutions for k, l, s, a, b as integers and a given shooting angle beta. However, on this parallelogram billiard table there exists uncountable combinations of (x, y), so that the cosine value of beta fails to have a closed orbit. In order for this equation to hold, there needs to exist a countable number of solutions for k, l, s, a, b as integers and a given shooting angle β . However, on this parallelogram billiard table there exists uncountable combinations of (x, y), so that the cosine value of β fails to have

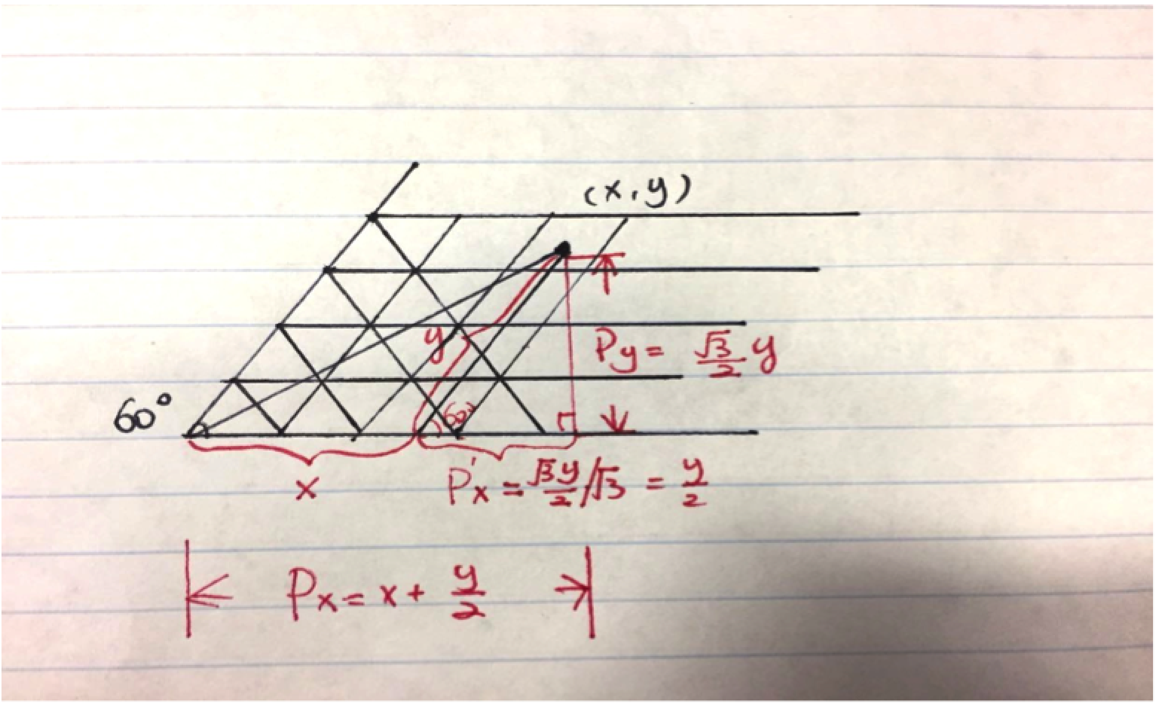

Figure 8: Parallelogram Billiards with 60 °

a closed orbit. For example, if k is 3, l is 2, a and b is 1, and beta is 60 °we get the cosine of our values of k, l, a, b and put it in our formula for cosine we get .599. However, cos 60°is .5. So the coordinates (x, y) which in this case are (k+la, lb) = (5, 2). Another example with unequal dimensions is if k is 5, l is 3, s is 4, a is 2, b is 3, and β is 30 degrees. From these values of k, l, a, b, and our cosine formula we get cos β to be .780. However, cosi 30°is .866. So the coordinates (x, y) which in this case is (k+la, sb) is (11, 12). From these examples, we have showed two points that are in the uncountable combinations of (x, y).

0.4 Conclusion

The history of billiards go back to the 15th century, which means we have gained a lot of knowledge about the game and also mathematical interpretations of billiards. Mathematicians are still working on different shapes of billiard tables and their pathways. These suggest that this is a very in-depth project and also can be used in various research. In this paper, we have researched and discovered how to find a closed and unclosed pathway for rectangular, equilateral triangular and parallelogram shaped billiards. Also we have found the period of equilateral triangular and parallelogram shaped billiards in which case we know the end coordinates. However, there are various problems about billiard trajectories and specifically parallelogram shaped billiard trajectories that are still unsolved, some of which we have not mentioned in this paper. First we did not observe when would a pathway not hit specific orbits like specific shapes or specific regions on the billiard table. Another problem would be when the orbit of our parallelogram shaped billiard table does not pass general points like (x0, y0). This case is for when we know the exact angles and dimensions of the parallelogram. The main reason we chose this project involves this quote from Tabachnikov. ?Billiards is not a single mathematical theory? it is rather a mathematicians playground where various methods and approaches are tested and honed?, said by a Russian Mathematician, Serge Tabachnikov.[1] That is the beauty of billiards. It is both chaotic and periodic, uncertain but informative. It is composed with only one ball and one cue stick but the simple game has intrigued several generations of people to investigate the patterns in different ways and inspired them to make contributions to the science world. I believe the same billiard ball mystery will keep motivating our generation, no matter what education level we have, striking sparkles of curiosity.

0.5 Work Cited

"Billiards." Billiards. Wolfram MathWorld, n.d. Web. 13 Apr. 2017.

Duisenberg, Ken. "The Path of a Billiard Ball." N.p., 1999. Web.

Davis, Diana. "Billiards and Flat Surfaces." Oberwolfach, 2015. Web.

Miller, Emily. "Mathematical Billiards." N.p., 2014. Web.

Radu, Remus. Billiards and Puzzles. N.p., n.d. Web. 13 Apr. 2017.

Sinha, Atreyee, and Bijan Bagchi. "A Classification of Classical Billiard Trajectories." (n.d.): n. pag. Department of Applied Mathematics, University of Calcutta, 22 Aug. 2009. Web.

Tabachnikov, Serge. "Geometry and Billiards." The Student Mathematical Library (2005): n. pag. Web. 30 Mar. 2017.