Ex. $m_{T|_{C_T(v)}} = m_{T,v}$

Suponha $V = C_T(v)$.

$$\alpha = \{ \underset{\underset{w_0}{\shortparallel}}{v}, \underset{\underset{w_1}{\shortparallel}}{T(v)}, \dots, \underset{\underset{w_\ell}{\shortparallel}}{T^\ell(v)} \} \text{ base de } V$$

então

$$T(w_0) = w_1, \ T(w_1) = w_2, \dots, T(w_\ell) = -b_0 w_0 - \dots - b_\ell w_\ell$$

$$[T]_\alpha = \begin{pmatrix} 0 & 0 & & -b_0 \\ 1 & 0 & & -b_1 \\ 0 & 1 & \cdots & \vdots \\ \vdots & 0 & & \vdots \\ & \vdots & & \\ 0 & 0 & & -b_\ell \end{pmatrix}$$

matriz de Frobenius / Companheira de T

14/10 MONITORIA PED

RESUMO. FORMAS MULTILINEARES

K - anel comutativo; V - K-mod. livre de posto n; $M^r(V)$ - formas mult.

Se V é K-mod. livre, então V^* também é K-mod. livre

$\alpha = \{v_1, \dots, v_n\}$ base de V $\Rightarrow$ $\alpha^* = \{f_1, \dots, f_n\}$ base de V^* dual a α

$\Lambda^r(V)$ - formas alternadas

Teorema. O conjunto

$$\{ f_{j_1} \otimes \dots \otimes f_{j_r} : J = \{j_1, \dots, j_r\} \subseteq \{1, \dots, n\}^r \}$$

é uma base de $M^r(V)$.

Obs. $(f \otimes g)(v_1, \dots, v_s, v_{s+1}, \dots, v_r) = f(v_1, \dots, v_s) g(v_{s+1}, \dots, v_r)$
onde $f \in M^s(V)$ e $g \in M^r(V)$.

Dem. $\forall v \in V$, $v = f_1(v) v_1 + \dots + f_n(v) v_n$

Tome $u_1, \dots, u_r \in V$. Então

$$f(u_1, \dots, u_r) = \sum_{j_1, \dots, j_r} f_{j_1}(u_1) \cdots f_{j_r}(u_r) f(v_{j_1}, \dots, v_{j_r})$$

$$= \sum_{j_1, \dots, j_r} f(v_{j_1}, \dots, v_{j_r}) f_{j_1} \otimes \dots \otimes f_{j_r}$$

Definimos $\Pi_r : \mathcal{M}^r(V) \longrightarrow \mathcal{F}(V^r, K)$ e, $\forall \sigma \in S_r$,

$$f_\sigma(u_1, \dots, u_r) = f(u_{\sigma(1)}, \dots, u_{\sigma(r)}) \qquad \Pi_r\text{: alternador}$$

$$\Pi_r(f) = \sum_{\sigma \in S_r} (-1)^\sigma f_\sigma$$

$$(f+g)_\sigma(u_1, \dots, u_r) = (f+g)(u_{\sigma(1)}, \dots, u_{\sigma(r)}) = f_\sigma(u_1, \dots, u_r) + g_\sigma(u_1, \dots, u_r)$$

$\Rightarrow \Pi_r$ é linear

$\varphi \in \mathcal{M}^r(V)$ é alternada $\Leftrightarrow \varphi_\tau = (-1)^\tau \varphi$

$$\Pi_r(f)(u_{\tau(1)}, \dots, u_{\tau(r)}) = \sum_{\sigma \in S_r} (-1)^\sigma f_\sigma(u_{\tau(1)}, \dots, u_{\tau(r)})$$

$$= \sum_{\sigma \in S_r} (-1)^\sigma f(u_{\sigma\tau(1)}, \dots, u_{\sigma\tau(r)}) = (-1)^\tau \sum_{\sigma \in S_r} (-1)^{\sigma\tau} f(u_{\sigma\tau(1)}, \dots, u_{\sigma\tau(r)}) \quad (*)$$

Considere $\eta \in S_r$ t.q. $\eta = \sigma\tau$. Assim, como S_r é transitivo,

$$(*) = (-1)^\sigma (\Pi_\sigma f)(u_1, \dots, u_r)$$

Assim, $\Pi_r(\mathcal{M}^r(V)) \subseteq \Lambda^r(V)$.

Seja $f \in \Lambda^r(V)$. Então

$$\Pi_r(f) = \sum_{\sigma \in S_r} (-1)^\sigma f_\sigma = \sum_{\sigma \in S_r} (-1)^\sigma (-1)^\sigma f$$

$$= \left(\sum_{\sigma \in S_r} (-1)^\sigma (-1)^\sigma \right) f = r! f$$

- Se $\operatorname{char} K = 0$ e $r! \in K$ é invertível, $r! = \underbrace{1 + \dots + 1}_{r!} = r! \cdot 1$

- Se K é corpo,

$$\pi_1 = \frac{1}{r!} \Pi_r \qquad \begin{pmatrix} \pi_1(f) = f \\ \forall f \in \Lambda^r(V) \end{pmatrix} \sim \text{fixa formas alternadas}$$

Lembre que

$$\#\{(i_1, \dots, i_r) \subseteq \{1, \dots, n\}^r : i_1 < \dots < i_r\} = \binom{n}{r}$$

Considere $f \in M^r(V) \cap \Lambda^r(V)$

$$f = \sum_J f(v_{j_1}, \dots, v_{j_r}) f_{j_1} \otimes \dots \otimes f_{j_r}$$

A priori, $f \neq 0$ sempre que se analisar um conjunto da forma

$$J = \{j_1, \dots, j_r\}, \text{ com } j_1 \neq \dots \neq j_r$$

Vamos começar considerando em J a ordenação $j_1 < \dots < j_r$. Defina

$S = \{j_{\sigma(i)}, \sigma \in S_r\}$ e $f_J = \sum_{\sigma \in S_r} f(v_{j_{\sigma(1)}}, \dots, v_{j_{\sigma(r)}}) f_{j_{\sigma(1)}} \otimes \dots \otimes f_{j_{\sigma(r)}}$.

Portanto,

$$f = \sum_{\substack{J = \{j_1, \dots, j_r\} \\ j_1 < \dots < j_r}} f_J \text{ e } f_J = f(v_{j_1}, \dots, v_{j_r}) \sum_{\sigma \in S_r} (-1)^\sigma f_{j_{\sigma(1)}} \otimes \dots \otimes f_{j_{\sigma(r)}}$$

$$\pi_r(f_{j_1} \otimes \dots \otimes f_{j_r}) = \sum_{\sigma \in S_r} (-1)^\sigma (f_{j_1} \otimes \dots \otimes f_{j_r})_\sigma = \sum_{\sigma \in S_r} (-1)^\sigma (f_{j_{\sigma(1)}} \otimes \dots f_{j_{\sigma(r)}})$$

$$(f_{j_1} \otimes \dots \otimes f_{j_r})_\sigma (v_1, \dots, v_r) = f_{j_1} \otimes \dots \otimes f_{j_r}(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = f_{j_1}(v_{\sigma(1)}) \dots f_{j_r}(v_{\sigma(r)})$$

Consequentemente,

$$f_J = f(v_{j_1}, \dots, v_{j_r}) D_J = f(v_{j_1}, \dots, v_{j_r}) \pi_r(f_{j_1} \otimes \dots \otimes f_{j_r})$$

e

$$\{\pi_r(f_{j_1} \otimes \dots \otimes f_{j_r}) : j_1 < \dots < j_r\} \text{ é base de } \Lambda^r(V).$$

Se "$\dim V$" $= n$, "$\dim \Lambda^r(V)$" $= \binom{n}{n} = 1$ e dados $T \in End(V)$ e $f \in \Lambda^n(V)$, existe $c \in K$ tal que

$$f(u_1, \dots, u_n) = c f(Tu_1, \dots, Tu_n), \quad c := \det T$$

Defina $f_T \in Hom(V^n, K)$, $f_T(x_1, \dots, x_n) = f(Tx_1, \dots, Tx_n)$

$$\Lambda^n(V) = \text{span}\{\underbrace{\pi_r(f_1 \otimes \dots \otimes f_n)}_{m}\}$$

Em particular, $m_T = cm$, $c \in K$. Como $f \in \Lambda^n(V)$, então $f = am$. E

$$f_T = (am)_T = am_T = acm = c.am = cf$$

Considere $T \in End(V)$ e $\alpha = \{v_1, \ldots, v_n\}$ base de V.

$A = [T]_\alpha$. Se $\{f_1, \ldots, f_n\}$ é base dual a α, $a_{ij} = f_j(T(v_i))$

$$\Pi_r(f_1 \otimes \cdots \otimes f_n)(Tv_1, \ldots, Tv_n) = \sum_{\sigma \in S_r} (-1)^\sigma f_1 \otimes \cdots \otimes f_n (Tv_{\sigma(1)}, \ldots, Tv_{\sigma(n)})$$

$$= \sum_{\sigma \in S_r} (-1)^\sigma a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

Pelo resultado anterior,

$$(\det T)\Pi(f_1 \otimes \cdots \otimes f_n)(v_1, \ldots, v_n) = \Pi(f_1 \otimes \cdots \otimes f_n)(Tv_1, \ldots, Tv_n) = \det A$$

$\therefore \det A = \det T$

e a definição anterior está bem construída.