Analysis: Study Notes

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Contents

Disclaimer. This text condenses my lecture and reading notes when taking the undergraduate course in Analysis 1 at the State University of Campinas with professor Lino Grama, to whom I am deeply grateful. There should be typos and some other mistakes, so please read it with caution.

1 Finite and Infinite Sets

1.1 Basic Definitions

The 'vocabulary' needed for Analysis comes from set theory. So, before going further, what is a set?

Intuitively, a **set** is any collection of objects, called **elements** of the set.

Definition 1.1. Some basic notation:

1. $x \in A$ means that x is an element of A. If x is not an element of A, we write $x \notin A$.

2. The **union** of two sets A and B is denoted by $A \cup B$ and is defined by

$$x \in A \cup B \iff x \in A \text{ or } x \in B$$

3. The **intersection** $A \cap B$ is the set defined by

$$x \in A \cap B \iff x \in A \text{ and } x \in B$$

   If the intersection $A \cap B$ is the empty set $\emptyset$, then these sets are said to be **disjoint**.

4. The **inclusion** $A \subseteq B$ means that every element of A is also an element of B. And we say that A is a **subset** of B, or B **contains** A. If there is some element in B which is not in A, then A is said to be a **proper subset** of B, denoted by $A \subsetneq B$. To say that $A = B$ means that $A \subseteq B$ and $B \subseteq A$, i.e., both sets have exactly the same elements.

5. Given $A \subseteq \mathbf{R}$, the **complement** of A, denoted by A^c is the set of all elements in $\mathbf{R}$ which are not in A.

$$A^c = \{x \in \mathbf{R} : x \notin A\}$$

Analysis is concerned mainly with the construction of Real Numbers and functions between them. The concept of function also comes from set theory. Here, we'll define it and introduce some basic nomenclature to help our discussions.

Definition 1.2 (Function)**.** A **function** from a set A into a set B is a rule or mapping that associates each element $x \in \mathrm{A}$ to a single element of B, denoted by $f : \mathrm{A} \longrightarrow \mathrm{B}$. And the expression $f(x)$ represents the element of B associated with x by f.

The set A is called **domain of** f, and the elements $f(x)$ are called the **values of** f. The set of all values of f is called **range of** f.

Definition 1.3 (Surjection)**.** Let f be a function of A into B. If $\mathrm{E} \subseteq \mathrm{A}$, then $f(\mathrm{E})$ is the set of all elements $f(x)$ such that $x \in \mathrm{E}$ and we call $f(\mathrm{E})$ the **image** of E under f. Clearly, $f(\mathrm{A}) \subseteq \mathrm{B}$. If $f(\mathrm{A}) = \mathrm{B}$, then f maps A **onto** B and is called a **surjective function**. In other words, for all $y \in \mathrm{B}$ there is at least one $x \in \mathrm{A}$ such that $f(x) = y$.

Definition 1.4 (Inverse image)**.** If $\mathrm{C} \subseteq \mathrm{B}$, then $f^{-1}(\mathrm{C})$ denotes the set of all $x \in \mathrm{A}$ such that $f(x) \in \mathrm{C}$, and is called **inverse image of** C **under** f or **pre-image**. If $y \in \mathrm{B}$, then $f^{-1}(y)$ is the set of all $x \in \mathrm{A}$ such that $f(x) = y$.

$$f^{-1}(\mathrm{C}) = \{a \in \mathrm{A} : f(a) \in \mathrm{C}\}$$

Definition 1.5 (Injection)**.** If, for each $y \in \mathrm{B}$, $f^{-1}(y)$ consist of at most one element of A, then f is said to be a **one-to-one** mapping of A into B, also called a **injective function**. I.e., $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 1.6 (Bijection)**.** If there exists a one-to-one mapping of A onto B (i.e. an injective and surjective function), then we can say that A and B can be put in **one-to-one correspondence**, or that A and B have the same **cardinal number**, and the function satisfying this property is called a **bijective function**, and we write $\mathrm{A} \sim \mathrm{B}$. This relation is an **equivalence relation**.

Definition 1.7 (Left inverse)**.** Given $f : \mathrm{A} \longrightarrow \mathrm{B}$ and $g : \mathrm{B} \longrightarrow \mathrm{A}$, we say that g is a **left inverse** of f when $g \circ f = \mathrm{id}_{\mathrm{A}}$.

Proposition 1.1. A function $f : \mathrm{A} \longrightarrow \mathrm{B}$ has a left inverse iff. it is injective.

Definition 1.8 (Right inverse)**.** Given $f : \mathrm{A} \longrightarrow \mathrm{B}$ and $g : \mathrm{B} \longrightarrow \mathrm{A}$, we say that g is a **right inverse** of f when $f \circ g = \mathrm{id}_{\mathrm{B}}$.

Proposition 1.2. A function $f : \mathrm{A} \longrightarrow \mathrm{B}$ has a right inverse iff. it is surjective.

Definition 1.9 (Inverse function)**.** A function $g : B \longrightarrow A$ is said to be an **inverse** of $f : A \longrightarrow B$ when g is both a left and a right inverse of f.

Proposition 1.3. A function f has an inverse function iff. it is a bijection.

Proof. Follows from Proposition 1.1 and Proposition 1.2. ■

Proposition 1.4. If a function $f : A \longrightarrow B$ has an inverse, then the inverse is unique.

Proof. Suppose that $g : B \longleftrightarrow A$ and $h : B \longleftrightarrow A$ are both inverses of f. Then

$$h = h \circ \mathrm{id}_B = h \circ (f \circ g) = (h \circ f) \circ g = \mathrm{id}_A \circ g = g$$

That means that if f has a left inverse h and a right inverse g, then $h = g$ and f has an inverse. ■

Corollary 1.5. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are bijections, then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

1.2 Natural numbers

The set of natural numbers, denoted by $\mathbf{N}$, is defined by the following axioms, known as **Peano Axioms**.

A1 There is an injective function $s : \mathbf{N} \longrightarrow \mathbf{N}$ such that the image $s(n)$ is called the **successor** of n.

A2 There is a unique number $1 \in \mathbf{N}$ such that $1 \neq s(n)$ for all $n \in \mathbf{N}$. I.e., 1 is not the successor of any natural number.

A3 If $X \subseteq \mathbf{N}$ is such that $1 \in X$ and $s(X) \subseteq X$, then $X = \mathbf{N}$. This axiom is known as the **Principle of mathematical induction**.

We will denote $2 = s(1), 3 = s(2) = s(s(1)), \ldots, n = s^{n-1}(1) = \underbrace{s \circ s \circ \ldots \circ s}_{(n-1)\text{ times}}(1)$.

Definition 1.10 (Addition)**.** The **addition** in $\mathbf{N}$ is defined by the function

$$\begin{aligned} \mathbf{N} \times \mathbf{N} &\longrightarrow \mathbf{N} \\ (m, n) &\mapsto m + n \end{aligned}$$

satisfying the following properties:

1. $m + 1 = s(m)$.
2. $m + s(n) = s(m + n)$.

Theorem 1.6. For all $m, n, p \in \mathbf{N}$:

- $(m + n) + p = m + (n + p)$.
- $m + n = n + m$.
- If $m + p = n + p$, then $m = n$.

Definition 1.11 (Multiplication)**.** The **multiplication** is defined by the function

$$\begin{aligned} \mathbf{N} \times \mathbf{N} &\longrightarrow \mathbf{N} \\ (m, n) &\mapsto m \cdot n \end{aligned}$$

satisfying:

1. $m \cdot 1 = m$.
2. $m \cdot s(n) = m \cdot n + m$.

And we will simply denote $m \cdot n$ by mn.

Theorem 1.7. For all $m, n, p \in \mathbf{N}$,

- $p(m + n) = pm + pn$.
- $(m + n)p = mp + np$.
- $(mn)p = m(np)$.
- $mn = nm$.
- If $mp = np$, then $m = n$.

Given two natural numbers m, n we write $m < n$ if there is a natural number p such that $n = m + p$. Naturally,

Theorem 1.8.

- If $m < n$ and $n < p$, then $m < p$.
- (Trichotomy law) One and only one of the following is true: $m = n$, or $m < n$, or $m > n$.

This idea of order between natural numbers motivates the following theorem.

Theorem 1.9 (Well ordering principle)**.** Every non-empty subset $A \subseteq \mathbf{N}$ has a minimum element. I.e., there is an element $n \in A$ such that $n \leq m$ for all $m \in A$.

Proof. If $1 \in A$, then $1 \leq k$ for all $k \in A$.

Assume $1 \notin A$ and define $J_m = \mathbf{N} \setminus \{1, 2, \ldots, m\}$ for every $m \in \mathbf{N}$.

Consider the following set $B = \{m \in \mathbf{N} : A \subseteq J_m\}$. Since $1 \notin A$, it follows that $A \subseteq J_1$. Hence, $1 \in B$.

Given that A is non-empty, there is an element $m \in A$. And since $m \notin J_m$, we have $A \not\subset J_m$. Hence, $m \notin B$, which implies that $B \neq \mathbf{N}$.

Finally, since $1 \in B$ and $B \neq \mathbf{N}$, there is $n \in B$ such that $n + 1 \notin B$ by the induction principle. Therefore, $A \not\subset J_{n+1}$, and hence $n+1 \leq k$, for all $k \in A$. ∎

1.3 Finite and Infinite sets

The following two lemmas will allow us to distinguish between finite and infinite sets.

Lemma 1.10. Let $f : A \longrightarrow B$ be a bijective function. Given $a \in A$ and $b \in B$, there is always a bijective function $g : A \longrightarrow B$ such that $g(a) = b$.

Proof. If $f(a) = b$, take $g = f$.

Diversely, let $f(a) = b_1 \neq b$. Since b is surjective, there is $a_1 \in A$ such that $f(a_1) = b$. Hence, $a \neq a_1$. Otherwise, we would have $f(a) = f(a_1) = b$, which is not possible.

Define $g : A \longrightarrow B$ by $g(a) = b$, $g(a_1) = b_1$, and $g(x) = f(x)$ for all $x \in A$ such that $x \neq a$ and $x \neq a_1$. ∎

Lemma 1.11. Let $m, n \in \mathbf{N}$. If there is a bijective function

$$f : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$$

then $m = n$.

Proof by contradiction. Suppose that there is a bijection

$$f : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$$

and $m < n$.

By the previous lemma, there is a bijection $f_1 : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ such that $f_1(1) = 1$.

Applying this lemma again to the restriction

$$f_1|_{\{2,\ldots,m\}} : \{2, \ldots, m\} \longrightarrow \{2, \ldots, n\}$$

we construct a bijection $g : \{2, \ldots, m\} \longrightarrow \{2, \ldots, n\}$ such that $g(2) = 2$ by defining

$$f_2 : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$$

where $f_2(1) = f_1(1) = 1$, and $f_2(k) = g(k)$, for $k = 2, \ldots, m$. Notice that f_2 is bijective and $f_2(1) = 1$, and $f_2(2) = 2$.

Applying the same lemma m times in an analogous manner, we obtain a bijective function

$$f_m : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$$

such that $f_m(k) = k$ if $k = 1, 2, \ldots, m$. If $m < n$, then f_m cannot be surjective, which contradicts our hypothesis.

The case $m > n$ is analogous. ■

Definition 1.12 (Finite and Infinite Sets)**.** A set A is said to be **finite** if A is empty or if there is a bijective function $f : \{1, 2, \ldots, n\} \longrightarrow \mathrm{A}$ for a $n \in \mathbf{N}$.

By the previous lemma, this n is unique. In this case, we say that n is the **number of elements** or **cardinality** of A. If A is not finite, we say that A is **infinite**.

Two sets A and B **have the same cardinality** if there exists a bijective function $f : \mathrm{A} \longrightarrow \mathrm{B}$, which we denote by $\mathrm{A} \sim \mathrm{B}$.

Proposition 1.12. $\mathbf{N}$ is infinite.

Proof by contradiction. Suppose that $\mathbf{N}$ is finite. That implies that there is a bijective function $f : \{1, 2, \ldots, n\} \longrightarrow \mathbf{N}$.

Using the same argument from the proof of the previous lemma, it is possible to construct a bijective function $f_n : \{1, 2, \ldots, n\} \longrightarrow \mathbf{N}$ such that $f_n(k) = k$ for all $k = 1, 2, \ldots, n$. This is a contradiction, since $n + 1$ is not in the image of f_n. Thus, $\mathbf{N}$ is infinite. ■

Notation: If $n \in \mathbf{N}$ and $n > 1$, the denote by $n - 1$ the unique $m \in \mathbf{N}$ such that $m + 1 = n$.

Proposition 1.13. If A is a finite set, then each subset of A is also finite.

Proof. If A is empty, there's nothing to prove.

Suppose $A \neq \emptyset$. First, we are going to prove for a particular case: if A is a finite set and $a \in A$, then $A \setminus \{a\}$ is finite.

To see that this is the case, notice that by hypothesis there is an one-to-one correspondence (bijection) $f : \{1, 2, \ldots, n\} \longrightarrow A$. By the Lemma 1.10, there is a bijection $g : \{1, 2, \ldots, n\} \longrightarrow A$ such that $g(n) = a$.

If $n = 1$, then $A \setminus \{a\}$ is empty and, hence empty.

If $n > 1$, then the restriction

$$g|_{\{1,\ldots,n-1\}} : \{1, \ldots, n-1\} \longrightarrow A \setminus \{a\}$$

is a bijection and, therefore $A \setminus \{a\}$ is finite and has $n-1$ elements.

With that particular case, we can prove the proposition by induction of the number n of elements of the set A.

If $n = 1$, then A has a single element a. Let $B \subseteq A$. If $B = A$, then B is finite. If $B \neq A$, then $a \notin B$ and, therefore, $B \subseteq A \setminus \{a\}$. Hence, B is empty.

Induction Hypothesis: suppose that the result is valid for a set with n elements.

Let A be a set with $n+1$ elements and $B \subseteq A$. If $A = B$, then B is finite. Suppose $B \neq A$. Then there exists $a \in A \setminus B$.

Therefore, $B \subseteq A \setminus \{a\}$ and the previous argument (the particular case) shows that $A \setminus \{a\}$ is finite with n elements. By the induction hypothesis, B is finite. ■

Corollary 1.14. Let $f : A \longrightarrow B$. Then:

- If f is injective and B is finite, then A is finite.
- If f is surjective and A is finite, then B is finite.

Proof. **(a)** By the Proposition 1.13, since $f(A) \subseteq B$, $f(A)$ is also finite. Given that $f : A \longrightarrow f(A)$ is bijective, then A is finite.

(b) Exercise. ■

The following two propositions will give tools to discover whether a set is infinite.

Proposition 1.15. A set A is infinite iff. there is an injective function $f : \mathbf{N} \longrightarrow$ A.

Proof. ($\Leftarrow$) We are going to show that if there is an injective function $f : \mathbf{N} \longrightarrow \mathrm{A}$ then A is infinite.

Suppose, by contradiction, $f : \mathbf{N} \longrightarrow \mathrm{A}$ injective, but A is finite. By the Corollary 1.14, $\mathbf{N}$ must be finite, which contradicts the Proposition 1.12.

($\Rightarrow$) Suppose that A is infinite. We are going to construct an injective function $f : \mathbf{N} \longrightarrow \mathrm{A}$.

Let $x_1 \in \mathrm{A}$ and define $f(1) = x_1$. Clearly, $f : \{1\} \longrightarrow \mathrm{A}$ is injective.

Now suppose that $f(1), f(2), \ldots, f(n)$ are defined satisfying that

$$f : \{1, 2, \ldots, n\} \longrightarrow \mathrm{A}$$

is injective.

And let $x_{n+1} \in \mathrm{A} \setminus \{f(1), \ldots, f(n)\}$ and define $f(n+1) = x_{n+1}$. Clearly, $f : \{1, \ldots, n+1\} \longrightarrow \mathrm{A}$ is injective.

This process defines a function $f : \mathbf{N} \longrightarrow \mathrm{A}$. To show that f is injective, let $m < n$. By construction, $f(m) \in \{f(1), \ldots, f(n-1)\}$ and, on the other hand, $f(n) \notin \{f(1), \ldots, f(n-1)\}$, which means that $f(m) \neq f(n)$. Hence, f is injective. ∎

Proposition 1.16. A set A is infinite iff. there is a bijective function g of A into a proper subset B of A.

Proof. ($\Rightarrow$) Suppose that A is infinite. By the proposition 1.15, there is an injective function $f : \mathbf{N} \longrightarrow \mathrm{A}$.

Let $x_n = f(n)$ for each $n \in \mathbf{N}$, and let $\mathrm{B} = \mathrm{A} \setminus \{x_1\}$. Define $g : \mathrm{A} \longrightarrow \mathrm{B}$ by $g(n) = x_{n+1}$, for each $n \in \mathbf{N}$ and $g(x) = x$, for each $x \in \mathrm{A} \setminus \operatorname{Im} f$.

Exercise. Show that g is an one-to-one correspondence (bijection).

($\Leftarrow$) Let B a proper subset of A and $g : \mathrm{A} \longrightarrow \mathrm{B}$ a bijection.

If A is finite, then B is also finite by the Corollary 1.14. Given that g is bijective, then A and B have the same cardinality by the Lemma 1.11, which is a contradiction, since B is a proper subset of A. ∎

1.4 Countable and Uncountable sets

Definition 1.13 (Countable set)**.** A set A is said to be **countable** if A is finite or if there is a bijective function $f : \mathbf{N} \longrightarrow A$. And we say that the elements $f(1), f(2), \ldots$ form an enumeration of A.

Example 1.1.

- $\mathbf{N}$ is countable.
- The set of even numbers is countable.
- $\mathbf{Z}$ is countable. Just take

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even} \end{cases}$$

Similar to what occurred in our discussion about finite and infinite sets, some propositions follow this definition.

Proposition 1.17. Every infinite set contains an infinite and countable subset.

Proof. Let A be an infinite subset. Then, by the Proposition 1.15, there is an injective function $f : \mathbf{N} \longrightarrow A$. Hence, there is a bijection $\mathbf{N} \longrightarrow f(\mathbf{N}) \subseteq A$. ■

Proposition 1.18. If A is countable, then every subset $B \subseteq A$ is also countable.

Proof. If A is finite, then every subset $B \subseteq A$ is also finite (by the Proposition 1.13), thus countable.

Suppose that A is infinite and countable. Then there is a bijection $f : \mathbf{N} \longrightarrow A$. If B is finite, then B is countable. Suppose that B is infinite.

Let $C = f^{-1}(B)$. Then C is an infinite subset of $\mathbf{N}$. We are going to define $g : \mathbf{N} \longrightarrow B$ in an inductive fashion. The following commutative diagram represents the idea.

$$\begin{tikzcd}
C \arrow[r, "f"] & B \subseteq A \\
& \mathbf{N} \arrow[ul] \arrow[u, "g"']
\end{tikzcd}$$

Let n_1 be the smallest element of C and $g(1) = f(n_1)$. Suppose that

$$g(1), g(2), \ldots, g(k)$$

are already defined in order that for each $j = 2, 3, \ldots, k$, $g(j) = f(n_j)$, where n_j is the smallest element of $C \setminus \{n_1, n_2, \ldots, n_{j-1}\}$.

Now let n_{k+1} be the smallest element of $C \setminus \{n_1, n_2, \ldots, n_k\}$ and let $g(k + 1) = f(n_{k+1})$. Given that $n_1 < n_2 < \ldots$, the function g is injective (since f is bijective).

To show that g is surjective, suppose that there is an element $b \in B \setminus g(\mathbf{N})$. Then we would have $f^{-1}(b) \in C \setminus \{n_1, \ldots, n_k\}$ for all $k \in \mathbf{N}$. Hence, $f^{-1}(b) \geq n_{k+1}$ for all $k \in \mathbf{N}$, which is a contradiction since C is infinite.

Therefore, g is bijective.

■

Proposition 1.19. Let $f : A \longrightarrow B$.

(a) If f is injective and B is countable, then A is countable.

(b) If f is surjective and A is countable, then B is countable.

Proof. **Exercise.** ■

Proposition 1.20. $\mathbf{N} \times \mathbf{N}$ is countable.

Idea of the proof. By the Fundamental Theorem of Arithmetic, each $n \in \mathbf{N}$ can be uniquely decomposed in prime factors (modulo permutation). So we can write

$$n = 2^{k-1}(2l - 1)$$

and define

$$\begin{aligned} f : \mathbf{N} &\longrightarrow \mathbf{N} \times \mathbf{N} \\ n &\longmapsto (k_n, l_n) \end{aligned}$$

which is a bijection. ■

Corollary 1.21.

(a) If A and B are countable, then $A \times B$ is countable.

(b) Let A_m be a countable set for all $m \in \mathbf{N}$. Then

$$\bigcup_{m=1}^{\infty} A_m \text{ is countable}$$

Proof. **Exercise.** Hint: Define a bijection $f_m : \mathbf{N} \longrightarrow A_m$ and a function

$$f : \mathbf{N} \times \mathbf{N} \longrightarrow \bigcup_{m=1}^{\infty} A_m$$

where $f(m,n) = f_m(n)$. ■

After talking about countable sets and how **N** is countable, a new question arises: is $\mathscr{P}(\mathbf{N})$ countable?

The **Diagonal Method**, proposed by Cantor, shows that this is not the case: there is not surjective function $f : \mathbf{N} \longrightarrow \mathscr{P}(\mathbf{N})$.

Theorem 1.22 (Cantor's Theorem)**.** For every set X,

$$|X| < |\mathscr{P}(\mathbf{X})|$$

Example 1.2. The set $\mathscr{F} = \{f : \mathbf{N} \longrightarrow \{0,1\}\}$, i.e., the set of all infinite sequences of zeroes and ones, is uncountable.

Finally, we can define the integers and rational numbers as follows:

Definition 1.14 (Integers)**.**

$$\mathbf{Z} = \{n \in \mathbf{N}\} \cup \{0\} \cup \{-n : n \in \mathbf{N}\}$$

Definition 1.15 (Rational numbers)**.**

$$\mathbf{Q} = \{p/q, p, q \in \mathbf{Z}, q \neq 0\}$$

These definitions of finite and countable sets can be summarized in the following definition.

Definition 1.16. For any positive integer n, let J_n be the set containing the integers $1, 2, \ldots, n$, and let J be the set of all positive integers. For any set A, we say that:

1. A is **finite** if $A \sim J_n$ for some n (and the empty set is also considered to be finite).
2. A is **infinite** if A is not finite.
3. A is **countable** if $A \sim J$.
4. A is **uncountable** if A is neither finite nor countable.
5. A is **at most countable** if it is finite or countable.

Evidently, $A \sim B$ iff. A and B contain the same number of elements.

2 Real Numbers

2.1 The Real Field

How to accurately define a real number?

Abbott: “Rational numbers are densely nestled together.”

But rational number system has gaps and the real number system fills these gaps.

Every rational number has a decimal expansion (just use the euclidean algorithm). This expansion can be terminal or infinite and may not be unique.

Proposition 2.1. Every rational number with infinite decimal expansion is periodic, which means that

$$x = q, d_1 \dots d_n d_{n+1} \dots d_k d_{n+1} \dots d_k d_{n+1} \dots = q, d_1 \dots d_n \overline{d_{n+1} d_k}$$

Proof. **Exercise.** ■

Example 2.1. The number $0.010010001\dots$ is not rational.

Proposition 2.2. $\sqrt{2}$ is not rational.

Proof. Suppose that $\sqrt{2}$ is rational. Then, $\sqrt{2} = a/b$, $a, b \in \mathbf{Z}$, $b \neq 0$, and a/b co-prime. Hence,

$$a = \sqrt{2}b \iff a^2 = 2b^2$$

It follows that a^2 is even, hence a is even and is of the form $a = 2k$, $k \in \mathbf{Z}$. Thus,

$$(2k)^2 = 2b^2 \iff 4k^2 = 2b^2 \iff b^2 = 2k^2$$

Which means that b is even.

Therefore, we have a contradiction, since we assumed a/b irreducible. ■

It can be proved that $\sqrt{2}$ can be approximated by a rational number with an error as small as one wants.

To put it another way, given $\varepsilon > 0$, there exists $x \in \mathbf{Q}$ such that

$$|x - \sqrt{2}| < \varepsilon$$

In the general case, for any $r \in \mathbf{R}^+$, an algorithm for decimal expansion can be done as follows.

Algorithm for decimal expansion.

1. First, choose q such that $q \leq r < q+1$.

2. Then d_1 is determined by

$$q + \frac{d_1}{10} \leq r < q + \frac{d_1}{10} + \frac{1}{10}$$

   where d_1 is between 0 and 9 because $q \leq r < q+1$.

3. For the step $n+1$, d_{n+1} is determined by

$$q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_{n+1}}{10^{n+1}} \leq r < q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}}$$

   where d_{n+1} is between 0 and 9 because

$$q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} \leq r < q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} + \frac{1}{10^n}$$

Exercise. Write the expansion of $\sqrt{3}$ and $\sqrt{5}$.

Definition 2.1 (Field)**.** A **field** is a set $\mathbf{F}$ with two closed operations, called addition and multiplication satisfying the field axioms: associativity, commutativity, neutral element, inverse element and distributivity between addition and multiplication.

Proposition 2.3. The axioms for addition imply that:

1. If $x+y = x+z$ then $y = z$ (cancellation law).
2. If $x+y = x$ then $y = 0$ (uniqueness of the neutral element).
3. If $x+y = 0$ then $y = -x$ (uniqueness of the inverse element).
4. $-(-x) = x$.

Proposition 2.4. The axioms for multiplication imply that:

1. If $x \neq 0$ and $xy = xz$ then $y = z$.
2. If $x \neq 0$ and $xy = x$ then $y = 1$.
3. If $x \neq 0$ and $xy = 1$ then $y = x^{-1}$.
4. If $x \neq 0$ and $1/x^{-1} = x$.

Proposition 2.5. For any $x, y, z \in \mathbf{F}$:

1. $0x = 0$.
2. If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
3. $(-x)y = -(xy) = x(-y)$.
4. $(-x)(-y) = xy$.

Definition 2.2 (Order)**.** An **order** on a set S is a relation, denoted by $<$, satisfying:

1. One and only one of the following statements is true: $x < y$, $x = y$, $y < x$.
2. If $x < y$ and $y < z$, then $x < z$.

where $x, y, z \in S$.

Definition 2.3 (Ordered Field)**.** An **ordered field** is a field $\mathbf{F}$ which is also an ordered set, such that for any $x, y, z \in \mathbf{F}$:

1. If $y < z$ then $x + y < x + z$.
2. If $x, y > 0$ then $xy > 0$.

Alternatively, a field $\mathbf{F}$ is said to be an **ordered field** if there is in $\mathbf{F}$ a total order relation $\leq$ such that if $x \leq y$ then $x + z \leq z + y$, $\forall z \in \mathbf{F}$ and $xz \leq yz$, $z \geq 0$. In other words,

- $x \leq x$ (reflexivity).
- If $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetry).
- If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).
- $x \leq y$ or $y \leq x$ (dichotomy law).

The familiar properties for inequalities hold.

Proposition 2.6. Suppose $x, y, z \in \mathbf{F}$, where $\mathbf{F}$ is an ordered field. Then the following statements are true.

1. If $x > 0$ then $-x < 0$, and vice versa.

2. If $x > 0$ and $y < z$ then $xy < xz$.

3. If $x < 0$ and $y < z$ then $xy > xz$.

4. If $x \neq 0$ then $x^2 > 0$.

5. If $0 < x < y$ then $0 < 1/y < 1/x$.

2.2 Upper and Lower Bounds

Definition 2.4 (Upper Bound)**.** Suppose S is an ordered set, and $E \subseteq S$. If there exists $z \in S$ such that $x \leq z$ for every $x \in E$, then E is said **bounded above** and z is called an **upper bound** of E.

Lower Bounds are analogous. And we say that a set E is **bounded** if E is bounded above and bounded below.

Intuitively, an upper bound is a number that is greater or equal than every number in the given set.

Notice that every larger number $z' \geq z$ is also an upper bound of E, but it is not always the case that a number smaller than z is also an upper bound of E.

Definition 2.5 (Least Upper Bound)**.** Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. If there exists $z \in S$ satisfying:

1. z is an upper bound of E.

2. If $y < z$ then y is not an upper bound of E.

Then z is called **least upper bound of** E or the **supremum of** E denoted by $z = \sup E$.

The **Greatest Lower Bound** (or **infimum**) is analogous, taking E bounded below. Denoted by $z = \inf E$.

Intuitively, the second property in the definition states that any other upper bound z' for E is larger than or equal to z.

Notice that the supremum can exist and not be a maximum, but when a maximum exists, it is also the supremum.

Also, every ordered set with a least-upper-bound property also has the greatest-lower-bound property.

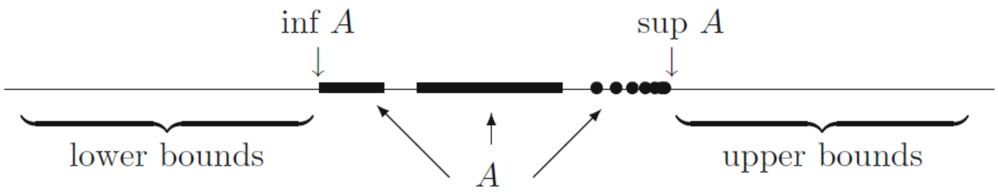

Figure 1: Definition of sup A and inf A [Abb01]

The natural question now is: when does the supremum/infimum exist?

Axiom of Completeness. Every non-empty set of real numbers that is bounded above has a least upper bound.

The following theorem is an important property of the real numbers. It states that no matter how large y is and how small x is, if one keeps adding x to itself, one will eventually overtake y.

Theorem 2.7 (Archimedean Property)**.** Let $x, y \in \mathbf{R}$, and $x > 0$. Then there exists a positive integer n such that $nx > y$.

Proof by contradiction. Suppose that there is $a, b > 0$ such that $na \leq b$, $\forall n \in \mathbf{N}$. We define

$$S = \{n \cdot a : n \in \mathbf{N}\}$$

In this case, b is an upper bound of S. Using the supremum axiom, let $s_0 = \sup S$.

Given that $a > 0$, we have $s_0 < s_0 + a$, hence $s_0 - a < s_0$.

Since s_0 is the least upper bound of S, $s_0 - a$ can not be an upper bound of S. Therefore, $s_0 - a < n_0 a$ for some $n_0 \in \mathbf{N}$.

Hence, $s_0 - a < n_0 a$. Which means that $s_0 < n_0 a + a = (n_0 + 1)a$.

Notice that we obtained $(n_0 + 1)a \in S$ and s_0 is not an upper bound of S, which is a contradiction. ■

As a consequence, for every $a \in \mathbf{R}_+$, there exists $n \in \mathbf{N}$ such that $na > 1$, i.e., $1/n < a$.

The theorem following states that between any two real numbers there is a rational one.

Theorem 2.8 (**Q** is dense in **R**)**.** Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

Proof. Exercise. ∎

Theorem 2.9 (Existence of n^{th} roots)**.** Let x be a positive real number, and let n be a positive integer. Then there is one and only one positive real number z such that $z^n = x$. This number z is written $\sqrt[n]{x}$ or $x^{1/n}$.

In other words, then the set $\mathrm{E} := \{y \in \mathbf{R} : y \geq 0 \wedge y^n \leq x\}$ is non-empty and is also bounded above. In particular, $x^{1/n}$ is a positive real number.

We will end this section with two important propositions about least upper bounds.

Proposition 2.10 (Uniqueness of least upper bound)**.** Let $\mathrm{E} \subseteq \mathbf{R}$. Then E can have at most one least upper bound.

Proof. Let M_1 and M_2 be two least upper bounds. Given that M_1 is a least upper bound and M_2 is an upper bound, then by definition of the least upper bound we have $\mathrm{M}_2 \geq \mathrm{M}_1$. Similarly, since M_2 is a least upper bound and M_1 is an upper bound, then $\mathrm{M}_1 \geq \mathrm{M}_2$. Hence, $\mathrm{M}_1 = \mathrm{M}_2$. ∎

Theorem 2.11 (Existence of least upper bound)**.** Let E be a non-empty subset of **R**. If E has an upper bound, then it must have exactly one least upper bound.

Proof. Let M be an upper bound of E. By the uniqueness of the least upper bound, we know that E has at most one least upper bound. We want to show that E has at least one least upper bound.

Given that E is non-empty, we can choose some $x_0 \in \mathrm{E}$. Let $n \geq 1$ be a positive integer. By the Archimedean property, we can find $\mathrm{K} \in \mathbf{Z}$ such that $\mathrm{K}/n \geq \mathrm{M}$, and hence K/n is also an upper bound for E.

Using the Archimedean property again, there exists $\mathrm{L} \in \mathbf{Z}$ such that $\mathrm{L}/n < x_0$. Since $x_0 \in \mathrm{E}$, L/n is not an upper bound for E. Given that K/n is an upper bound but L/n is not, we have $\mathrm{K} \geq \mathrm{L}$.

With that, we can find an integer $\mathrm{L} < m_n \leq \mathrm{K}$ with the property that m_n/n is an upper bound for E, but $(m_n - 1)/n$ is not. In fact, m_n is unique. This gives a well-defined and unique sequence $m_1, m_2, m_3, \ldots$ of integers with the property above.

Now let $N \geq 1$ be a postive integer, and let $n, n' \geq N$. Since m_n/n is an upper bound for E and $m_{n'} - 1/n'$ is not, we have $m_n/n > m_{n'} - 1/n'$. This implies that

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} > -\frac{1}{n'} \geq -\frac{1}{N}$$

Similarly, since $m_{n'}/n'$ is an upper bound for E and $m_n - 1/n$ is not, we have $m_{n'}/n' > m_n - 1/n$, and hence

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} \leq \frac{1}{n} \leq \frac{1}{N}$$

Putting these two bounds together,

$$\left|\frac{m_n}{n} - \frac{m_{n'}}{n'}\right| \leq \frac{1}{N} \text{ for all } n, n' \geq N \geq 1$$

This implies that $\frac{m_n}{n}$ is a Cauchy sequence. Since they are rational numbers, we can define the real number S as

$$S := \lim_{n\to\infty} \frac{m_n}{n}$$

Hence,

$$S = \lim_{n\to\infty} \frac{m_n - 1}{n}$$

Now we need to show that S is the least upper bound for E.

Let $x \in E$. Given that m_n/n is an upper bound for E, we have $x \leq m_n/n$ for all $n \geq 1$. Therefore, $x \leq \lim_{n\to\infty} \frac{m_n}{n} = S$. Thus S is an upper bound for E.

Suppose that y is an upper bound for E. Since $(m_n - 1)/n$ is not an upper bound, $y \geq (m_n - 1)/n$. Hence, $y \geq \lim_{n\to\infty} \frac{m_n - 1}{n} = S$. Thus the upper bound S is less than or equal to every upper bound of E, and S is thus a least upper bound of E. ■

3 Sequences of Real Numbers

3.1 Sequences

Definition 3.1 (Sequence)**.** A **sequence** is a function f defined on the set $\mathbf{N}$. If $f(n) = x_n$, $n \in \mathbf{N}$, we denote the sequence f by the symbol $(x_n)_{n=1}^{\infty}$ or, simply, (x_n). The elements x_n are called the **terms** of the sequence.

Intuitively, a sequence of real numbers is a collection of reals

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

If A is a set and if $x_n \in \mathrm{A}$ for all $n \in \mathbf{N}$, then $\{x_n\}$ is said to be a **sequence of elements of** A.

Definition 3.2 (Convergence)**.** A real sequence (x_n) is said to **converge** if there is a point $a \in \mathbf{R}$ satisfying that for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$|x_n - a| < \varepsilon$$

whenever $n > n_0$.

We also say that (x_n) converges to a or that a is the limit of (x_n), and we write $x_n \longrightarrow a$ or $\lim x_n = a$.

If (x_n) does not converge, then we say that the sequence **diverges**.

Definition 3.3 (ε-neighbourhood)**.** An **ε-neighbourhood** of a, or the neighbourhood of center a and radius ε is defined as

$$\mathrm{V}_{\varepsilon}(a) = \{x \in \mathbf{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$$

With this definition, we can understand convergence as follows:

$$x_n \longrightarrow a \text{ if } \forall \varepsilon > 0\, \exists n_0 \in \mathbf{N} \text{ such that } x_n \in \mathrm{V}_{\varepsilon}(a), \forall n > n_0$$

Proposition 3.1 (Uniqueness of the limit)**.** *If a and b are limits of (x_n), then $a = b$.*

Proof. Let $\varepsilon > 0$. Given that a and b are limits of $(x_n$, then there exists n_1, n_2 such that

$$|x_n - a| < \frac{\varepsilon}{2} \text{ for all } n > n_1$$

$$|x_n - b| < \frac{\varepsilon}{2} \text{ for all } n > n_2$$

Now let $n_0 = \max\{n_1, n_2\}$. Then

$$|b-a| = |b-x_n+x_n-a| \leq |x_n-b| + |x_n-a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n > n_0$. Since ε is arbitrarily small, $|b-a| = 0$. Hence, $a = b$. ■

Definition 3.4. A sequence is **upper bounded** if there exists $b \in \mathbf{R}$ such that $x_n \leq b$ for all $n \in \mathbf{N}$.

A sequence is **lower bounded** if there exists $a \in \mathbf{R}$ such that $x_n \geq a$ for all $n \in \mathbf{N}$.

If a sequence is upper bounded and lower bounded, then the sequence is said to be **bounded**. In this case, there exists an $M \geq 0$ such that $|x_n| \leq M$, i.e., $x_n \in (-M, M)$ for all $n \in \mathbf{N}$.

Proposition 3.2. Every convergent sequence is bounded.

Proof. Let $a = \lim x_n$ and take $\varepsilon = 1$. Hence, there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$ we have $x_n \in (a-1, a+1)$.

Therefore, our task is to 'control' the following finite set:

$$I = \{x_1, x_2, \ldots, x_{n_0}\} \cup \{a-1\} \cup \{a+1\}$$

Let b the smallest value of I and c the biggest value of I. Then every term of (x_n) is in the interval $[b, c]$. ■

The contrapositive of this proposition consists in an useful criteria for the study of the convergence of sequences.

Example 3.1. The sequence $1, 2, 3, 4, \ldots$ is not bounded. Hence, it is not convergent.

Example 3.2. The sequence $1, -1, 1, -1, \ldots$ is bounded. However, it is not convergent.

Definition 3.5. A sequence (x_n) is **non-decreasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$. And (x_n) is **non-increasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbf{N}$.

If the sequence is non-decreasing or non-increasing, then it is said to be **monotone**.

Proposition 3.3. If a sequence is non-decreasing and upper bounded, then it is convergent.

Proof. Let (x_n) be a non-decreasing sequence and upper bounded, and let $b = \sup\{x_n : n \in \mathbf{N}\}$, which is valid by the Axiom of Completeness. We're going to show that $x_n \longrightarrow b$.

Given $\varepsilon > 0$, since b is the least upper bound, there is $n_0 \in \mathbf{N}$ such that $b - \varepsilon < x_{n_0}$.

Since the sequence is non-decreasing,

$$b - \varepsilon < x_{n_0} \leq x_n, \ \forall n > n_0$$

Hence,

$$b - \varepsilon < x_n < b + \varepsilon$$

Which means that $|x_n - b| < \varepsilon$, for all $n > n_0$. I.e., $x_n \longrightarrow b$. ■

Analogously, if a sequence is non-increasing and lower bounded, then it is convergent.

Another way of stating the fact above is that every bounded monotone sequence is convergent.

The following proposition states that if a sequence has a positive limit, then, after a finite number of terms, all of its terms will be positive.

Proposition 3.4. If $\lim x_n = a > 0$, then there exists $n_0 \in \mathbf{N}$ such that $x_n > 0$ whenever $n > n_0$.

Proof. Let $\varepsilon = a/2 > 0$. Then $(a - \varepsilon, a + \varepsilon) = (a/2, 3a/2)$. There exists $n_0 \in \mathbf{N}$ such that if $n > n_0$, then $x_n \in (a/2, 3a/2)$, i.e., $x_n > a/2$. Hence, $x_n > 0$ if $n > n_0$. ■

Corollary 3.5. Let (x_n) and (y_n) be convergent sequences. If $x_n \leq y_n$ for all $n \in \mathbf{N}$, then $\lim x_n \leq \lim y_n$.

Proof. If $\lim x_n > \lim y_n$, we would have

$$0 < \lim x_n - \lim y_n = \lim(x_n - y_n)$$

which implies that $x_n - y_n > 0$ for every n sufficiently big. ■

Corollary 3.6. Let (x_n) be a convergent sequence. If $x_n \geq a$ for all n, then $\lim x_n \geq a$.

Lemma 3.7. If $\lim x_n = 0$ and (y_n) is bounded, then $\lim x_n y_n = 0$.

Proof. Since (y_n) is bounded, there exists $c > 0$, such that $|y_n| \leq c$, for all $n \in \mathbf{N}$. And since $\lim x_n = 0$, we can take $|x_n - 0| = |x_n|$ as small as we want.

Hence, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n| < \varepsilon/c$, for all $n > n_0$.

Therefore, for all $n > n_0$,

$$|x_n y_n - 0| = |x_n|\,|y_n| < \frac{\varepsilon}{c} \cdot c = \varepsilon$$

■

Example 3.3.

$$\lim \frac{(-1)^n}{n} = \lim\left((-1)^n \cdot \frac{1}{n}\right) = 0$$

Theorem 3.8 (Squeeze theorem)**.** Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$. If $\lim x_n = \lim z_n = \mathrm{L}$, then $\lim y_n = \mathrm{L}$.

Proof. Given $\varepsilon > 0$, there exists $n_1 \in \mathbf{N}$ such that $|x_n - \mathrm{L}| < \varepsilon$, for all $n > n_1$. Analogously, there exists $n_2 \in \mathbf{N}$ such that $|z_n - \mathrm{L}| < \varepsilon$, for all $n > n_2$.

Let $n_0 = \max\{n_1, n_2\}$. By hypothesis, $x_n \leq y_n \leq z_n$, for all $n \in \mathbf{N}$. Therefore, for $n > n_0$,

$$\mathrm{L} - \varepsilon < x_n \leq y_n \leq z_n < \mathrm{L} + \varepsilon$$

That means that $|y_n - \mathrm{L}| < \varepsilon$ for all $n > n_0$. ■

Example 3.4. Compute $\lim \frac{\sin n}{n}$.

Since $-1 \leq \sin n \leq n$, we have

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

for all $n \in \mathbf{N}$.

But $\lim \frac{-1}{n} = \lim \frac{1}{n} = 0$. Therefore, by the Squeeze theorem,

$$\lim \frac{\sin n}{n} = 0$$

3.2 Subsequences

Definition 3.6 (Subsequence)**.** Let (x_n) be a sequence. A **subsequence** of (x_n) is a sequence of the form $(x_{n_j})_{j=1}^{\infty}$, where n_j is a strictly increasing sequence in $\mathbf{N}$.

Proposition 3.9. If $\lim x_n = x$ then every subsequence of (x_n) converges to x.

Proof. Consider $(x_{n_j})_{j=1}^{\infty}$ a subsequence of (x_n). By hypothesis, $x_n \longrightarrow x$. Therefore, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n - x| < \varepsilon$, for all $n > n_0$.

Given that the sequence of the index n_j is strictly increasing, there is $j_0 \in \mathbf{N}$ such that $n_{j_0} > n_0$. Hence, for all $j > j_0$,

$$|x_{n_j} - x| < \varepsilon$$

■

The contrapositive of this proposition is also an useful criteria for studying convergence.

Example 3.5. The sequence $x_n = (-1)^n$ does not converge, since $x_{2n} \longrightarrow 1$ and $x_{2n-1} \longrightarrow -1$.

Before proving the Bolzano-Weierstrass theorem, which is one of the most important results in Real Analysis, we will prove the following result.

Theorem 3.10 (Nested intervals)**.** Let $([a_n, b_n])_{n=1}^{\infty}$ be a non-increasing sequence of bounded and closed intervals, i.e.,

$$[a_1, b_1] \supset [a_2, b_2] \supset \dots$$

Then, there exists $c \in \mathbf{R}$ which belongs to all intervals $[a_n, b_n]$.

Proof. Since the sequence of intervals is non-increasing, we have

$$a_1 \le a_2 \le a_3 \le \dots \le a_n \le \dots \le b_n \le \dots \le b_2 \le b_1$$

Consider the set $A = \{a_n : n \in \mathbf{N}\}$. Given that A is upper bounded and non-empty, let $c = \sup A$. Then $a_n \le c \le b_n$ for all $n \in \mathbf{N}$. Which means that $c \in [a_n, b_n]$ for all $n \in \mathbf{N}$. ■

Theorem 3.11 (Bolzano-Weierstrass)**.** Every bounded sequence has a convergent subsequence.

Proof. Let (a_n) be a bounded sequence. Then there exists $M > 0$ such that $|a_n| \le M$, for all $n \in \mathbf{N}$.

Divide the interval $[-M, M]$ into the halves $[-M, 0]$ and $[0, M]$. Notice that at least one of the intervals contains infinite terms of the sequence (a_n). Choose the half that contains infinite terms and denote it by I_1. Let a_{n_1} a point of the sequence (a_n) such that $a_{n_1} \in \mathrm{I}_1$.

Divide I_1 into two closed intervals with the same length and let I_2 be the half that contains infinite terms. Choose a_{n_2} from the original sequence with $n_2 > n_1$.

In the general case, define I_k by dividing I_{k-1} into two halves and take I_k to be the half with infinite terms of (a_n). Then select $n_k > n_{k-1} > \ldots > n_2 > n_1$ such that $a_{n_k} \in \mathrm{I}_k$.

Now let us show that the subsequence (a_{n_k}) is convergent. Notice that $\mathrm{I}_1 \supset \mathrm{I}_2 \supset \ldots$ is a sequence of closed and nested intervals. By the nested intervals theorem, there exists $x \in \mathbf{R}$ which belong to I_k, for all $k \in \mathbf{N}$.

Affirmation: (a_{n_k}) converges to x.

Let $\varepsilon > 0$. By construction, the length of I_k is equal to $\mathrm{M}\left(\frac{1}{2}\right)^k$. Using the Lemma 3.7,

$$\lim_{k\to\infty} \mathrm{M}\left(\frac{1}{2}\right)^k = 0$$

Hence, there exists $n_0 \in \mathbf{N}$ such that for all $k > n_0$ the length of I_k is lesser than ε. Since x and a_{n_k} are in I_k, we have that the distance $|a_{n_k} - x| < \varepsilon$ for all $k > n_0$. ■

We now state some important properties of limits, which will ease their computations.

Theorem 3.12. If $\lim a_n = a$ and $\lim b_n = b$, then:

1. $\lim(c \cdot a_n) = c \cdot a$, $\forall c \in \mathbf{R}$.
2. $\lim(a_n + b_n) = a + b$.
3. $\lim a_n b_n = ab$.
4. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ if $b \neq 0$.

Proposition 3.13. A monotone sequence is bounded iff. it has a bounded subsequence.

Proof. Consider $x_{n_j} \leq b$ a bounded subsequence of the non-decreasing sequence (x_n). Then, for all $n \in \mathbf{N}$, there exists $n_k > n$, hence, $x_n \leq x_{n_k} \leq b$. Hence, $x_n \leq b$ for all n. ■

3.3 Cauchy Sequences

Definition 3.7. A sequence (a_n) is said to be a **Cauchy sequence** if for all $\varepsilon > 0$, there is $n_0 \in \mathbf{N}$ such that

$$|a_n - a_m| < \varepsilon$$

for all $n, m > n_0$. Intuitively, this means that after a certain point, every pair of terms are close to each other.

With that definition, our goal is to show that a real sequence is convergent iff. it is a Cauchy sequence. This is known as the **Cauchy criterion**.

Lemma 3.14. Every Cauchy sequence is bounded.

Proof. Let $\varepsilon = 1$. Then, there exists n_0 such that $|x_n - x_m| < 1$ for all $n, m > n_0$. Notice that

$$|x_n| = |x_n - x_{n_0+1} + x_{n_0+1}| \leq |x_n - x_{n_0+1}| + |x_{n_0+1}| < 1 + |x_{n_0+1}|$$

for all $n > n_0$.

Let $c = \max\{|x_1|, |x_2|, \dots, |x_{n_0}|, 1 + |x_{n_0+1}|\}$. Then $|x_n| \leq c$ for all $n \in \mathbf{N}$. ■

Theorem 3.15. A sequence is convergent if, and only if, it is a Cauchy sequence.

Proof. ($\Rightarrow$) Let (x_n) convergent and $\lim x_n = x$. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n - x| < \varepsilon/2$ for all $n > n_0$. Therefore,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall n, m > n_0$$

($\Leftarrow$) Suppose that (x_n) is a Cauchy sequence. Then (x_n) is bounded. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}). Let $x = \lim x_{n_k}$. We're going to show that the original sequence also converges to x.

Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there is $n_0 \in \mathbf{N}$, such that $|x_n - x_m| < \varepsilon/2$ for all $n, m > n_0$.

Now, since $(x_{n_k}) \longrightarrow x$, there exists $n_k > n_0$ such that $|x_{n_k} - x| < \varepsilon/2$ for all $n > n_k$.

Hence, for all $n > n_k$,

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

With Cauchy sequences, Georg Cantor gave another construction of the real numbers **R**.

Definition 3.8 (ε-close Sequences)**.** Consider two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$, and let $\varepsilon > 0$. These two sequences are said to be **ε-close** iff. a_n is ε-close to the sequence b_n, for each $n \in \mathbf{N}$. I.e., $|a_n - b_n| \leq \varepsilon$ for all $n \in \mathbf{N}$.

If there exists an $\mathrm{N} \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq \mathbf{N}$, then the two sequences are said **eventually ε-close sequences**.

Definition 3.9 (Equivalent Sequences)**.** Two sequences are **equivalent** iff. for each rational $\varepsilon > 0$, the sequences are eventually ε-close.

Definition 3.10 (Real numbers)**.** A **real number** is an object of the form

$$\lim_{n\to\infty} a_n$$

where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ are said to be equal iff. $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences.

3.4 Upper and Lower Limits

Definition 3.11. Let (x_n) be a sequence of real numbers.

- We say that (x_n) goes to **infinity** if, for every $\mathrm{N} > 0$, there exists $n_0 \in \mathbf{N}$ such that $x_n > \mathrm{N}$ for all $n > n_0$. We write $x_n \longrightarrow \infty$ or $\lim x_n = \infty$.
- We say that (x_n) goes to **minus infinity** if, given $\mathrm{N} > 0$, there exists $n_0 \in \mathbf{N}$ such that $x_n < -\mathrm{N}$ for all $n > n_0$. We write $x_n \longrightarrow -\infty$ or $\lim x_n = -\infty$.

Definition 3.12. Let (x_n) be a sequence of real numbers. Suppose that (x_n) is upper bounded, i.e., $x_n \leq b$ for all $n \in \mathbf{N}$.

Define

$$b_n := \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} = \sup_{k\geq n}\{x_k\}$$

Notice that (b_n) is non-increasing. If (b_n) is lower bounded, then (b_n) is convergent. If (b_n) is not lower bounded, then $b_n \longrightarrow -\infty$.

In both cases, we define

$$\limsup_{n\to\infty} x_n = \lim_n b_n$$

If (x_n) is not upper bounded, then we define

$$\limsup_{n\to\infty} x_n = \infty$$

Definition 3.13. Let (x_n) be a sequence of real numbers. Suppose that (x_n) is lower bounded, i.e., $x_n \geq a$ for all $n \in \mathbf{N}$.

Define

$$a_n := \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\} = \inf_{k\geq n}\{x_k\}$$

Notice that (a_n) is non-decreasing. If (a_n) is upper bounded, then (a_n) is convergent. If (a_n) is not upper bounded, then $a_n \longrightarrow \infty$.

In both cases, we define

$$\liminf_{n\to\infty} x_n = \lim_n a_n$$

If (x_n) is not lower bounded, then we define

$$\liminf_{n\to\infty} x_n = -\infty$$

Example 3.6. Consider $x_n = (-1)^n$, $n \in \mathbf{N}$. Then $b_1 = 1, b_2 = 1, \ldots, b_n = 1$. Hence, $\limsup x_n = \lim b_n = 1$.

And $a_n = -1$, for all $n \in \mathbf{N}$. Hence, $\liminf x_n = \lim a_n = -1$.

Example 3.7. Let $x_n = \frac{(-1)^n}{n}$. Then $\limsup x_n = \lim b_n = 0$, and $\liminf x_n = \lim a_n = 0$.

Example 3.8. Let $x_n = (-1)^n n$. Then $\limsup x_n = \infty$, and $\liminf x_n = -\infty$.

Remark 3.16. $a_n \leq b_n$ and therefore $\liminf x_n \leq \limsup x_n$.

Theorem 3.17. Let (x_n) be a sequence. Then

$$\lim x_n = \mathrm{L} \iff \limsup x_n = \liminf x_n = \mathrm{L}$$

Proof. Suppose that $\mathrm{L} \in \mathbf{R}$.

($\Rightarrow$) Suppose that $\lim x_n = \mathrm{L}$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\mathrm{L} - \varepsilon < x_n < \mathrm{L} + \varepsilon$, for all $n > n_0$.

Notice that

$$b_n = \sup_{k \geq n}\{x_k\} \leq \mathrm{L} + \varepsilon,\ \forall n > n_0$$

On the other hand,

$$a_n = \inf_{k \geq n}\{x_k\} \geq \mathrm{L} - \varepsilon,\ \forall n > n_0$$

Then,

$$\mathrm{L} - \varepsilon \leq a_n \leq b_n \leq \mathrm{L} + \varepsilon,\ \forall n > n_0$$

Hence,

$$\limsup x_n = \liminf x_n = \mathrm{L}$$

($\Leftarrow$) By hypothesis, $\lim a_n = \lim b_n = \mathrm{L}$. Let $\varepsilon > 0$. Since (b_n) is non-increasing and converges to L, there exists $n_1 \in \mathbf{N}$ such that

$$\sup_{k \geq n_1}\{x_k\} = b_{n_1} < \mathrm{L} + \varepsilon$$

Analogously, (a_n) is non-decreasing and converges to L. Then there exists $n_2 \in \mathbf{N}$ such that

$$\inf_{k \geq n_2}\{x_k\} = a_{n_2} > \mathrm{L} - \varepsilon$$

Let $n_0 = \max\{n_1, n_2\}$. Then $\mathrm{L} - \varepsilon < x_n < \mathrm{L} + \varepsilon$, for all $n > n_0$.

Therefore, $x_n \longrightarrow \mathrm{L}$.

If $\lim x_n = \infty$: **Exercise.** ∎

Theorem 3.18. Let (x_n) a bounded sequence of real numbers. And let $a = \liminf x_n$ and $b = \limsup x_n$.

Then there exists a subsequence $(x_{n_k}) \longrightarrow a$ and a subsequence $(x_{n_j}) \longrightarrow b$.

Proof. **Exercise.** ∎

3.5 Infinite Series

In this section, we are going to discuss the meaning of infinite sums like

$$\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}+\ldots=1$$

and give meaning to this equality.

This infinite sum is called a **series**, and the main interest in the study of series is whether or not the series converges. To compute to which value it converges is often a cumbersome task and will not be our primary concern.

Definition 3.14. Let (a_n) be a sequence. An **infinite series** is defined as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

The sequence (s_m), defined as

$$s_m = a_1 + a_2 + \ldots + a_m$$

is called the **sequence of partial sums**.

And the series $\sum_{n=1}^{\infty}$ **converges** to A if the sequence (s_m) converges to A, and we write $\sum_{n=1}^{\infty} = \mathrm{A}$.

Example 3.9 (Harmonic Series)**.** Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Since the sequence of partial sums

$$s_m = 1 + \frac{1}{2} + \ldots + \frac{1}{m}$$

is increasing and unbounded, this series does not converge.

Theorem 3.19. If $\sum_{n=1}^{\infty} a_n = \mathrm{A}$ and $\sum_{n=1}^{\infty} b_n = \mathrm{B}$, then

1. $\sum_{n=1}^{\infty} c a_n = c\mathrm{A}$ for all $c \in \mathbf{R}$.

2. $\sum_{n=1}^{\infty} (a_n + b_n) = \mathrm{A} + \mathrm{B}$.

Theorem 3.20 (Cauchy Criterion for Series)**.** The series $\sum_{n=1}^{\infty} a_n$ converges if, and only if, for every $\varepsilon > 0$, there is $\mathrm{N} \in \mathbf{N}$ such that whenever $n > m > \mathrm{N}$ it follows that

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \ldots + a_n| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$$

Then, using Cauchy criterion for sequences, the result follows immediately. ■

Theorem 3.21. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim(a_n) = 0$.

Proof. Take $n = m + 1$ in the previous theorem. ■

Notice that the converse of this theorem is not valid. Consider, for example, the harmonic series.

Theorem 3.22 (Comparison Test)**.** Suppose that (a_k) and (b_k) are sequences such that $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let $n > m$. Then $|s_n^a - s_m^a| = |a_{m+1} + \ldots + a_n|$ and $|s_n^b - s_m^b| = |b_{m+1} + \ldots + b_n$. And we have that

$$|a_{m+1} + \ldots + a_n| \leq |b_{m+1} + \ldots + b_n|$$

Now, if $\sum_{k=1}^{\infty} b_k$ converges, then given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $n > m \geq n_0$, we have $|b_{m+1} + \ldots + b_n| < \varepsilon$. And, therefore, $|a_{m+1} + \ldots + a_n| < \varepsilon$ and $\sum_{k=1}^{\infty} a_k$ converges. ■

Example 3.10 (Geometric Series)**.** A series is called **geometric** if it is of the form

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \ldots$$

If $x = 1$ and $a \neq 0$, then the series diverges. For $x \neq 1$, since

$$(1-x)(1+x+x^2+x^3+\ldots+x^{m-1}) = 1-x^m$$

we can write

$$s_m = a + ax + ax^2 + \ldots + ax^{m-1} = \frac{a(1-x^m)}{1-x}$$

Hence,

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

if, and only if, $|x| < 1$.

Theorem 3.23 (Cauchy Condensation Test)**.** Suppose that (b_n) is decreasing and satisfies $b_n > 0$ for all $n \in \mathbf{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_16 + \ldots$$

converges.

Example 3.11. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if, and only if, $p > 1$.

If $p \leq 0$, then $\lim \frac{1}{n^p} \neq 0$. Therefore, the series diverges.

If $p > 0$, using Cauchy Condensation Test, we can evaluate

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}$$

Taking $x = 2^{1-p}$ into the geometric series, the series will converge if and only if $1-p < 0$, as we wanted.

Theorem 3.24. If $\sum_{k=1}^{\infty} a_k = \mathrm{A}$ and $\sum_{k=1}^{\infty} b_k = \mathrm{B}$, then

(i) $\sum_{k=1}^{\infty} ca_k = c\mathrm{A}$ for all $c \in \mathbf{R}$.

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = \mathrm{A} + \mathrm{B}$.

Proof. (i) Let $t_m = ca_1 + ca_2 + \ldots + ca_m = c(a_1 + \ldots + a_m) = cs_m$. Then

$$\lim t_m = \lim cs_m = c \lim s_m = c\mathrm{A}$$

(ii) **Exercise.** ■

Theorem 3.25 (Absolute Convergence Test)**.** If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

Proof. Since $\sum_{k=1}^{\infty} |a_k|$ converges, by the Cauchy Criterion for Series, given an $\varepsilon > 0$, there exists an $n_0 \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \ldots + |a_n| < \varepsilon$$

for all $n > m \geq n_0$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \ldots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \ldots + |a_n| < \varepsilon$$

By the sufficiency of the Cauchy Criterion, $\sum_{k=1}^{\infty} a_k$ also converges. ■

Notice that the converse is false. Consider the **alternating harmonic series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Theorem 3.26 (Alternating Series Test)**.** Let (x_k) be a sequence such that

(i) $x_1 \geq x_2 \geq \ldots \geq 0$.

(ii) $\lim x_k = 0$.

Then, the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

Proof. First, we're going to consider the partial sums of odd indexes. Notice that

$$s_{2n+1} = x_1 - x_2 + x_3 - x_4 + \ldots + x_{2n-1} - x_{2n} + x_{2n+1} = s_{2n-1} - (x_{2n} - x_{2n-1}) \leq s_{2n-1}$$

I.e., s_{2n-1} is non-increasing. On the other hand,

$$s_{2n-1} = (x_1 - x_2) + \ldots + (x_{2n-3} - x_{2n-2}) + x_{2n-1}$$

Which means that $s_{2n-1} \geq 0$. Hence, (s_{2n-1}) converges.

Now, we're going to consider the partial sums of even indexes. Notice that

$$s_{2n+2} = x_1 - x_2 + x_3 - x_4 + \ldots + x_{2n-1} - x_{2n} + x_{2n+1} - x_{2n+2} = s_2 n + (x_{2n+1} - x_{2n+2}) gleq s_{2n}$$

Which means that s_{2n} is non-decreasing. On the other hand,

$$s_{2n} = x_1 - (x_2 - x_3) - \ldots - (x_{2n-2} - x_{2n-1}) - x_{2n}$$

I.e., $s_{2n} \le x_1$, for all $n \in \mathbf{N}$. Hence, (s_{2n}) converges.

Let $\mathrm{L} = \lim s_{2n-1}$ and $\mathrm{M} = \lim s_{2n}$. Then,

$$\lim(s_{2n} - s_{2n-1}) = \lim(-x_n) = 0$$

and, therefore, $\mathrm{M} - \mathrm{L} = 0$. Hence, $\lim s_{2n} = \lim s_{2n-1} = \mathrm{L}$.

Given $\varepsilon > 0$, there exists $n_1 \in \mathbf{N}$ such that $|s_{2n-1} - \mathrm{L}| < \varepsilon$ for all $n \ge n_1$. And there exists $n_2 \in \mathbf{N}$ such that $|s_{2n} - \mathrm{L}| < \varepsilon$ for all $n \ge n_2$.

Fix $k_1 = 2n_1 + 1$ and $k_2 = 2n_2$. Let $k_0 = \max\{k_1, k_2\}$. Then,

$$|s_k - \mathrm{L}| < \varepsilon$$

for all $k > k_0$.

Hence, $\lim s_k = \mathrm{L}$ and the series converges. ■

Definition 3.15. If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say that $\sum_{k=1}^{\infty} a_k$ **converges absolutely**. If $\sum_{k=1}^{\infty} a_k$ converges but the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ does not converge, then we say that $\sum_{k=1}^{\infty} a_k$ **converges conditionally**.

Lemma 3.27. Let $\sum b_n$ be an absolutely convergent series with $b_n \neq 0$ for all $n \in \mathbf{N}$ If the sequence $\left(\frac{a_n}{b_n}\right)$ is bounded, then $\sum a_n$ converges absolutely.

Proof. Suppose that $\left(\frac{a_n}{b_n}\right)$ is bounded, i.e., there exists $c \in \mathbf{R}$ such that for all $n \in \mathbf{N}$,

$$\left|\frac{a_n}{b_n}\right| \le c$$

Hence, $|a_n| \le c|b_n|$ for all $n \in \mathbf{N}$. By the Comparison Test, $\sum |a_n|$ converges. Therefore, $\sum a_n$ converges absolutely. ■

Example 3.12. Does

$$\sum \frac{1}{n^2 - 3n + 1}$$

converges?

Consider $\sum \frac{1}{n^2}$, which is absolutely convergent. And notice that

$$\frac{\frac{1}{n^2-3n+1}}{\frac{1}{n^2}} = \frac{n^2}{n^2 - 3n + 1}$$

is bounded by 1. Therefore, the series converges.

Theorem 3.28 (Ratio Test (D'Alambert))**.** Let $a_n \neq 0$ for all $n \in \mathbf{N}$. If there exists $c \in \mathbf{R}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq c < 1$$

for sufficiently large n, then the series $\sum a_n$ converges absolutely.

Proof. Suppose $0 < c < 1$ such that for n sufficiently large, we have

$$\left|\frac{a_{n+1}}{a_n}\right| \leq c = \frac{c^{n+1}}{c^n} \iff \frac{|a_{n+1}|}{c^{n+1}} \leq \frac{|a_n|}{c^n}$$

Hence, the sequence of non-negative numbers $\frac{|a_n|}{c^n}$ is non-increasing for n sufficiently large and, therefore, bounded.

Since the series $\sum c^n$ converges absolutely (geometric series), by the previous lemma, $\sum a_n$ converges absolutely. ∎

Remark 3.29. In general, we compute

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \mathrm{L}$$

- If $\mathrm{L} < 1$, the series converges.
- If $\mathrm{L} > 1$, the series diverges. In this case, $|a_{n+1}| > |a_n|$ and hence the general term does not converges to zero.
- If $\mathrm{L} = 1$, the test implies nothing about the convergence of the series. Consider, for example, $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$. Both limits are 1, but only the first series converges.

Example 3.13. Does the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges?

By the ratio test,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{(n+1)^{(n+1)}}\frac{n^n}{n!} = \frac{(n+1)}{(n+1)^{(n+1)}}n^n = (n+1)^{-n}n^n \\ &= \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} < 1\end{aligned}$$

Hence, the series converges.

Theorem 3.30 (Root Test (Cauchy))**.** If there exists $c \in \mathbf{R}$ such that $\sqrt[n]{|a_n|} \leq c < 1$ for all n sufficiently large, then the series $\sum a_n$ converges absolutely.

Proof. Suppose that $\sqrt[n]{|a_n|} \leq c < 1$. Then $|a_n| \leq c^n$ for all n sufficiently large. Since $\sum c^n$ converges, it follows from the comparison test that $\sum |a_n|$ converges. ∎

Remark 3.31. In general, we compute $\lim \sqrt[n]{|a_n|} = \mathrm{L}$ and the previous remark also applies.

Example 3.14. Does the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

converges?

Notice that

$$\lim \sqrt[n]{\frac{n}{2^n}} = \lim \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{1}{2} < 1$$

Hence, the series converges.

Exercise. Show that $\lim \sqrt[n]{n} = 1$.

3.6 Rearrangements

Intuitively, a rearrangement of a series is a permutation of its terms into another order.

For example, we know that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = p \neq 0$$

In fact, as we will prove later in this text, $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$. Hence, $p = \ln(2)$.

The question that motivates this section is: Does changing the order of the terms change the sum?

Notice that

$$\begin{aligned}
p &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\dots \\
&= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\dots \\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\dots = \frac{1}{2}p
\end{aligned}$$

Hence,

$$p = \frac{p}{2} \underset{p\neq 0}{\Longrightarrow} 1 = 2$$

which is a contradiction.

This example motivates the following definition.

Definition 3.16 (Rearrangement)**.** Let $\sum_{k=1}^{\infty} a_k$ be a series. We say that $\sum_{k=1}^{\infty} b_k$ is a **rearrangement** of $\sum_{k=1}^{\infty} a_k$ if there's a bijection $f : \mathbf{N} \longrightarrow \mathbf{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbf{N}$.

Theorem 3.32. If a series converges absolutely, then any rearrangement of the series converges to the same limit.

Proof. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely to $\mathrm{A} \in \mathbf{R}$, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Define

$$s_n = \sum_{k=1}^{n} a_k \text{ and } t_m = \sum_{k=1}^{m} b_k$$

We want to show that $\lim t_m = \mathrm{A}$.

Let $\varepsilon > 0$. By hypothesis, $\lim s_n = \mathrm{A}$. Hence, choose $\mathrm{N}_1 \in \mathbf{N}$ such that

$$|s_n - \mathrm{A}| < \frac{\varepsilon}{2}$$

for all $n > \mathrm{N}_1$. Since the convergence is absolute, we can choose $\mathrm{N}_2 \in \mathbf{N}$ such that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\varepsilon}{2}$$

for all $n > m > \mathrm{N}_2$.

Notice that $sum_{k=1}^{\infty} |a_k|$ converges, but not necessarily to A.

Now, let $\mathrm{N} = \max\{\mathrm{N}_1, \mathrm{N}_2\}$. We will consider the part of the rearranged series where the terms $\{a_1, a_2, \ldots, a_\mathrm{N}\}$ does not appear. In order to do that, we choose

$$\mathrm{M} = \max\{f(k) : 1 \leq k \leq \mathrm{N}\}$$

where f is the function that rearranges the terms.

Since f is injective, $\mathrm{N} \leq \mathrm{M}$. Let $n \geq \mathrm{M}$. Then,

$$\{a_1, a_2, \ldots, a_\mathrm{N}\} \subseteq \{b_1, b_2, \ldots, b_m\}$$

This means that we moved so far in the rearranged series so that all terms $a_1, \ldots, a_\mathrm{N}$ were included.

Thus, $(t_m - s_\mathrm{N})$ consists of the values such that their absolute value appear in $\sum_{k=\mathrm{N}+1}^{\infty} |a_k|$. Our choice of N_2 guarantees that $|t_m - s_\mathrm{N}| < \varepsilon/2$. Then,

$$\begin{aligned} |t_m - \mathrm{A}| &= |t_m - s_\mathrm{N} + s_\mathrm{N} - \mathrm{A}| \\ &\leq |t_m - s_\mathrm{N}| + |s_\mathrm{N} - \mathrm{A}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

whenever $m > \mathrm{M} \geq \mathrm{N}_2$. ∎

Exercise. Show that

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = +\infty \text{ and } \sum_{n=1}^{\infty} \frac{-1}{2n} = -\infty$$

The following theorem states that changing conveniently the order of the terms of a conditionally convergent series, it is possible to make its sum to be equal to any given real number.

Theorem 3.33 (Riemann). Let $\sum a_n$ be a conditionally convergent series and $-\infty \leq a \leq b \leq +\infty$. Then there exists a rearrangement $\sum b_n$, with partial sums t_n such that

$$\liminf_{n\to\infty} t_n = a \text{ and } \limsup_{n\to\infty} t_n = b$$

Idea of the proof. Let $\sum a_n$ be a conditionally convergent series. Fix a real number $c > 0$.

We start with the sum of the positive terms p_n of $\sum a_n$ in their original order until the moment that, when we sum a_n, the sum goes beyond c for the first time. Notice that this is possible because $\sum p_n = +\infty$.

To this sum, we add the negative terms, also in their natural order, stopping when we sum a_{n_2} such that this sum is lower than c.

Repeating this algorithm, we obtain a new series whose terms are the same ones from $\sum a_n$, but in a different order.

Notice that $|s_{n_{k-1}} - c| < a_{n_k}$, where the change of sign happened. Since $\lim a_n = 0$, this new series converges. ∎

4 Basic Topology of R

In this section, we are going to go deeper in our previous study of sets, discussing open and closed sets, compact sets, and perfect sets.

4.1 Open and Closed Sets

Definition 4.1 (Interior point)**.** Let O be a subset of **R**. A point x is an **interior point** of O if there is an ε-neighbourhood of x contained in O, i.e., $V_\varepsilon(x) \subseteq O$

Definition 4.2 (Open set)**.** Let $O \subseteq \mathbf{R}$. Then O is said to be **open** if every point $x \in O$ is an interior point of O.

Example 4.1.

1. **R** is open.
2. $\emptyset$ is open (vacuously true).
3. The open interval (c,d) is open. To see that, take $x \in (c,d)$ and take *varepsilon* $= \min\{x-c, d-x\}$. Then $V_\varepsilon(x) \subseteq (c,d)$. Does this proof remains valid for $(c,d]$?

Theorem 4.1.

a) The union of any collection of open sets is open.

b) The intersection of a finite collection of open sets is open.

Proof. **a)** Let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets. Define $O = \bigcup_{\lambda\in\Lambda} O_\lambda$ and $a \in O$. Then $a \in O_{\lambda'}$ for some $\lambda' \in \Lambda$. Since $O_{\lambda'}$ is open, then there exists $\varepsilon > 0$ such that $V_\varepsilon(a) \subseteq O_{\lambda'} \subseteq O$.

b) Let $\{O_1, \ldots, O_N\}$ be a finite collection of open sets. Define $O = \bigcap_{k=1}^{N}$ and $a \in O$. Then $a \in O_k$ for all $k = 1, \ldots, N$. Since each O_k is open, there exists $\varepsilon_k > 0$ such that $V_\varepsilon(a) \subseteq O_k$. Take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_N\}$. Then

$$V_\varepsilon(a) \subseteq V_{\varepsilon_k}(a) \subseteq O_k$$

for all $k = 1, \ldots, N$. Hence, $V_\varepsilon(a) \subseteq O$. ∎

Example 4.2. Let $I_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$, $n \in \mathbf{N}$. Then $\bigcap_{n\in\mathbf{N}} I_n = \{0\}$ is not open.

Definition 4.3 (Limit and isolated points)**.** A point x is a **limit point** (or **accumulation points**) of a set A if every ε-neighbourhood of x intersects the set A in some point other than x.

A point $a \in \mathrm{A}$ is an **isolated point** if a is not a limit point of A.

Theorem 4.2. A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbf{N}$.

Proof. ($\Rightarrow$) Suppose that x is a limit point of A. We are going to construct a sequence such that $a_n \longrightarrow x$.

Since x a limit point, for each $n \in \mathbf{N}$ we can choose an element of the set $\mathrm{V}_{1/n}(x) \cap \mathrm{A}$ other than x, which we will call a_n.

Given $\varepsilon > 0$, take $\mathrm{N} \in \mathbf{N}$ such that $1/\mathrm{N} < \varepsilon$. Then, for all $n \geq \mathrm{N}$.

$$|a_n - x| < \frac{1}{\mathrm{N}} < \varepsilon$$

i.e., $a_n \longrightarrow x$.

($\Leftarrow$) Assume that $\lim a_n = x$, where a_n is an element of A other than x. Let $\mathrm{V}_\varepsilon(x)$ be arbitrary. By the definition of convergence, there exists a term $a_\mathrm{N} \in \mathrm{V}_\varepsilon(x)$. ■

Notice that a limit point may not belong to the set A.

Definition 4.4 (Closed set)**.** A set $\mathrm{F} \subseteq \mathrm{R}$ is **closed** if F contains all of its limit points.

In Analysis, to be 'closed' is with respect to the limiting operation. A closed set is a set where convergent sequences within the set have limits that are also in the set.

Example 4.3. Consider

$$\mathrm{A} = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\}$$

Each point of A is isolated. In fact, given $1/n \in \mathrm{A}$, choose $\varepsilon = \frac{1}{n} - \frac{1}{n+1}$. Then

$$\mathrm{V}_\varepsilon(1/n) \cap \mathrm{A} = \left\{ \frac{1}{n} \right\}$$

Then, $1/n$ is isolated.

Although all of the points of A are isolated, the set does have one limit point, 0. Hence, A is not closed. However, the set $\mathrm{F} = \mathrm{A} \cup \{0\}$ is closed and is called the **closure** of A.

Example 4.4. The set $[c,d] = \{x \in \mathbf{R} : c \leq x \leq d\}$ is closed.

In fact, let x be a limit point of $[c,d]$. Then, by the Theorem 4.2, there exists $(x_n) \subseteq [c,d]$ such that $x_n \to x$. We're going to prove that $x \in [c,d]$. Since $c \leq x_n \leq d$, for all $n \in \mathbf{N}$, then $c \leq \lim x_n \leq d$. Hence, $[c,d]$ is closed.

Example 4.5. Consider the set of rational numbers $\mathbf{Q} \subseteq \mathbf{R}$. The set of limit points of $\mathbf{Q}$ is $\mathbf{R}$.

Let $y \in \mathbf{R}$. Then consider the neighbourhood $\mathrm{V}_\varepsilon(y) = (y-\varepsilon, y+\varepsilon)$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists $r \neq y$ such that $r \in \mathrm{V}_\varepsilon(y)$. Thus, y is a limit point of $\mathbf{Q}$.

With the last example, we can restate the density of $\mathbf{Q}$ in $\mathbf{R}$.

Theorem 4.3 (Density of $\mathbf{Q}$ in $\mathbf{R}$)**.** Given $y \in \mathbf{R}$, there exists a sequence of rational numbers that converges to y.

Definition 4.5 (Closure)**.** Given a set $\mathrm{A} \subseteq \mathbf{R}$, let L be the set of all limit points of A. The **closure** of A is defined as $\bar{\mathrm{A}} = \mathrm{A} \cup \mathrm{L}$.

Notice that a set $\mathrm{F} \subseteq \mathbf{R}$ is closed if, and only if, every Cauchy sequence contained in $\mathbf{R}$ which has a limit, the limit is also in F.

Example 4.6. 1. If $\mathrm{A} = (a,b)$, then $\bar{\mathrm{A}} = [a,b]$.

2. If $\mathrm{A} = \mathbf{Q}$, then $\bar{\mathrm{A}} = \mathbf{R}$.

3. If $\mathrm{A} = [a,b]$, then $\bar{\mathrm{A}} = [a,b]$.

Theorem 4.4. For any $\mathrm{A} \subseteq \mathbf{R}$, the closure $\bar{\mathrm{A}}$ is a closed set and is the smallest closed set containing A.

Proof. Let L denote the set of limit points. Then $\bar{\mathrm{A}} = \mathrm{A} \cup \mathrm{L}$ contains all limit points of A.

However, any closed set containing A must contain L. Hence, $\bar{\mathrm{A}}$ is the smallest closed set containing A. ■

Definition 4.6 (Complement)**.** Let $\mathrm{O} \subseteq \mathbf{R}$. Then the **complement** of O is defined as

$$\mathrm{O}^c = \{x \in \mathbf{R} : x \notin \mathrm{O}\}$$

An important relationship between open and closed sets is given by the following theorem.

Theorem 4.5. A set O is open if, and only if, O^c is closed.

Proof. Let $O \subseteq \mathbf{R}$ be an open set. To show that O^c is closed, we need to prove that O^c contains all of its limit points. If x is a limit point of O^c, then every neighbourhood of x contains a point of O^c. Now, if $x \in O$, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq O$, which is not possible. Hence, $x \in O^c$.

Reciprocally, assume that O^c is closed and let $x \in O$. Since O^c is closed, x is not a limit point of O^c. By the definition of limit point, there exists $\varepsilon > 0$ such that $V_\varepsilon(x)$ does not intersect O^c, i.e., $V_\varepsilon(x) \subseteq O$, and, hence, O is open. ■

Corollary 4.6. F is closed if, and only if, F^c is open.

Proof. Follows from the previous proof, observing that $(A^c)^c = A$. ■

Theorem 4.7.

a) The union of a finite collection of closed sets is closed.

b) The intersection of any collection of closed sets is closed.

Proof. Using DeMorgan's laws, for a collection $\{E_\lambda : \lambda \in \Lambda\}$, we have

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \text{ and } \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$$

and the result follows immediately by the Theorem 4.1. ■

4.2 Compact Sets

Compact sets are generalization of closed intervals.

Definition 4.7 (Compactness)**.** A set $K \subseteq \mathbf{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Example 4.7. The interval $[c, d]$ is compact. That can be seen by using Bolzano-Weierstrass Theorem, noticing that a closed interval is a closed set.

Generalizing this example, we can obtain an equivalent definition for compactness.

Theorem 4.8 (Heine-Borel)**.** A set $K \subseteq \mathbf{R}$ is compact if, and only if, it is closed and bounded.

Proof. Suppose that K is compact. First, we'll assume that K is not bounded. Then, there exists $x_1 \in K$ such that $|x_1| > 1$, and $x_2 \in K$ such that $|x_2| > 2$, and, for all $n \in \mathbf{N}$ we can produce $x_n \in K$ such that $|x_n| > n$.

Since we assumed that K is compact, (x_n) must have a convergent subsequence (x_{n_k}). And since (x_{n_k}) is not bounded, (x_{n_k}) does not converge. Hence, by contradiction, K must be bounded.

Now, to show that K is closed, let $x = \lim x_n$, with $(x_n) \subseteq K$. We're going to show that $x \in K$. Since K is compact, (x_n) has a convergent subsequence (x_{n_k}). Since K is compact and $(x_n) \to x$, we obtain $x \in K$. Hence, K is closed.

To prove the converse statement, suppose that K is bounded and closed and $(x_n) \subseteq K$. By the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence $(x_{n_k}) \subseteq K$. Since K is closed, $\lim x_{n_k} = x \in K$. Hence, K is compact. ■

Theorem 4.9 (Nested Compact Set Property). If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ is a nested sequence of non-empty compact sets, then the intersection is non-empty.

$$\bigcap_{n=1}^{\infty} \neq \emptyset$$

Proof. For each $n \in \mathbf{N}$, let $x_n \in K_n$. Then $(x_n) \subseteq K_1$ and, since K_1 is compact, then there exists a subsequence $(x_{n_k}) \subseteq K_1$ such that $\lim x_{n_k} = x \in K_1$.

Notice that $x \in K_n$ for every $n \in \mathbf{N}$. Let $n_0 \in \mathbf{N}$. Then, the terms of the sequence (x_n) are in k_{n_0}, for all $n \geq n_0$. Ignoring a finite number of terms $n_k < n_0$, the same sequence (x_{n_k}) is also contained in K_{n_0} and converges to $x = \lim x_{n_k}$ in K_{n_0}. Since n_0 is arbitrary, x is in the intersection of all K_n, as we wanted. ■

A third way of defining compact sets is in terms of open covers and finite subcovers.

Definition 4.8 (Open cover). Let $A \subseteq \mathbf{R}$. An **open cover** for A is a collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$, possibly infinite, whose union contains the set A. I.e.,

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

Given an open cover for A, a **finite subcover** is a finite subcollection of the original collection of open sets, whose union still completely contains A. I.e.,

$$A \subseteq O_{\lambda_1} \cup O_{\lambda_2} \cup \ldots \cup O_{\lambda_n}$$

Example 4.8. Consider the open interval $A = (0,1)$. For each point $x \in (0,1)$ let $O_x = \left(\frac{x}{2}, 1\right)$. Then the infinite collection $\{O_x : x \in (0,1)\}$ is an open cover of $(0,1)$.

Notice that is not possible to find a finite subcover. Assume that a finite subcollection $\{O_{x_1}, O_{x_2}, \dots, O_{x_n}\}$ covers A.

Let $x' = \min\{x_1, x_2, \dots, x_n\}$. Now notice that the real number y, such that $0 < y < \frac{x'}{2}$ is not contained in the union $\bigcup_{i=1}^{n} O_{x_i}$.

Example 4.9. Consider the closed interval $B = [0,1]$. If $x \in [0,1]$, then the sets $O_x = \left(\frac{x}{2}, 1\right)$ are an open cover of $(0,1)$. However, the end points must also be covered. Fix $\varepsilon > 0$ and let $O_0 = (-\varepsilon, \varepsilon)$ and $0_1 = (1-\varepsilon, 1+\varepsilon)$. Then, the collection

$$\{O_0, O_1, O_x\}$$

is an open cover for $[0,1]$.

Now it is possible to find a finite subcover. Choose x' such that $x'/2 < \varepsilon$. Then $\{O_0, O_1, O_x'\}$ is a finite subcover for $[0,1]$.

Theorem 4.10 (Borel-Lebesgue)**.** Let $K \subseteq \mathbf{R}$. Then the following statements are equivalent:

(i). K is compact.

(ii). K is closed and bounded.

(iii). Every open cover of K has a finite subcover.

Proof. We already proved the first two equivalences (see the proof of the Theorem 4.8).

First, we're going to show that (ii) $\implies$ (iii). Consider an open cover $[a,b] \subseteq O_{\lambda\in\Lambda} O_\lambda$ of the compact interval $[a,b]$. Suppose, by contradiction, that $\mathfrak{C} = (O_\lambda)_{\lambda\in\Lambda}$ doesn't admit a finite subcover.

Take the mean point of $[a,b]$ and decompose $[a,b]$ into two intervals of length $(b-a)/2$. At least one of this intervals doesn't admit a finite subcover, that we'll call $[a_1,b_1]$. Repeating this argument, we obtain a decreasing sequence of intervals

$$[a,b] \supseteq [a_1,b_1] \supseteq [a_2,b_2] \supseteq \dots$$

where $[a_n,b_n]$ has length $(b-a)/2^n$ and no interval $[a_n,b_n]$ is contained in a finite union of open sets O_λ.

By the nested interval theorem, there exists $c \in \mathbf{R}$ such that $c \in [a_n, b_n]$ for all $n \in \mathbf{N}$. In particular, $c \in [a, b]$.

By the definition of open cover, there exists $\lambda \in \Lambda$ such that $c \in O_\lambda$. Since O_λ is open, we have $(c-\varepsilon, c+\varepsilon) \subseteq O_\lambda$, for $\varepsilon > 0$. Taking $n \in \mathbf{N}$ such that

$$\frac{b-a}{2^n} < \varepsilon$$

we have $c \in [a_n, b_n] \subseteq (c-\varepsilon, c+\varepsilon)$ and $[a_n, b_n] \subseteq O_\lambda$. Hence, $[a_n, b_n]$ can be covered by only one open set, which is a contradiction.

In the general case, consider

$$K \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

where K is a compact set. Take an interval $[a, b]$ that contains K. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of $[a, b]$, and add a new open $O_{\lambda_0} = \mathbf{R} \setminus K$.

Thus, we obtained a new cover of $[a, b]$, from which we extract, from the already proven part, a finite subcover

$$[a, b] \subseteq O_{\lambda_0} \cup \subseteq O_{\lambda_1} \cup \ldots \cup \subseteq O_{\lambda_n}$$

Since no point of K is in O_{λ_0}, we have $K \subseteq\subseteq O_{\lambda_1} \cup \ldots \cup \subseteq O_{\lambda_n}$.

(iii) $\implies$ (ii) Assume that any open cover has a finite subcover. First, we're going to show that K is bounded. For each $x \in K$, take $O_x = (x-1, x+1)$. The open cover $\{O_x : x \in K\}$ has a finite subcover $\{O_{x_1}, \ldots, O_{x_n}\}$. Hence, K is contained in a finite union of bounded sets. I.e., K is bounded.

Now, we're going to show that K is closed. Suppose, by contradiction, that (y_n) is a Cauchy sequence contained in K with $\lim y_n = y$. To show that K is closed, we must show that $y \in K$. So we assume $y \notin K$.

Let O_x the interval with center in x and ray $|x-y|/2$. By hypothesis, the open cover $\{O_x : x \in K\}$ has a finite subcover $\{O_{x_1}, \ldots, O_{x_n}\}$. Set

$$\varepsilon_0 = \min\left\{\frac{|x_i - y|}{2} : 1 \le i \le n\right\}$$

Since $(y_n) \to y$, we can find $N \in \mathbf{N}$ such that $|y_n - y| < \varepsilon_0$. But then

$$y_N \notin \bigcup_{i=1}^{n} O_{x_i}$$

and the finite subcover does not cover all of K, which is a contradiction. Hence, $y \in K$ and K is closed. ■

4.3 The Cantor Set

Let C_0 be the closed interval $[0,1]$. Define C_1 to be the C_0 without the open middle third.

$$C_1 = [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

In a similar manner, define

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

Inductively, each set C_n consists of 2^n intervals of length $1/3^n$. We define the **Cantor set** C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

That is,

$$C = [0,1] \setminus \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots\right]$$

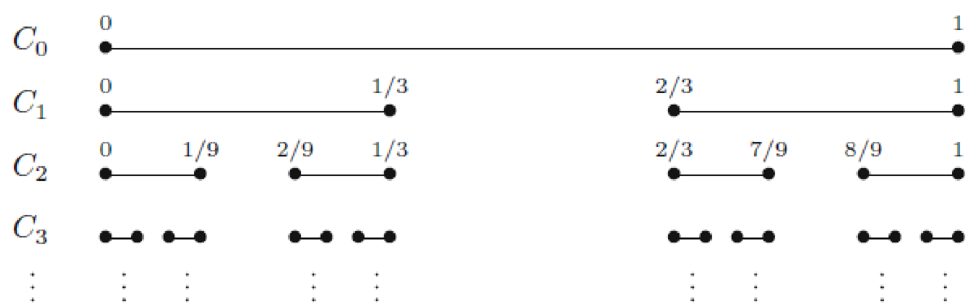

Figure 2: The Cantor set [Abb01]

The question now is: what remains in C?

Since $0 \in C_n$, for all $n \in \mathbf{N}$, clearly $0 \in C$. By the same argument, $1 \in C$. And if y is the endpoint of a closed interval in C_n, then y is also an endpoint of one of the intervals of C_{n+1}. Hence, $y \in C_n$ for all $n \in \mathbf{N}$, i.e., $y \in C$. Also notice that the endpoints are rational numbers of the form $m/3^n$.

It is reasonable that the length of C is 1 minus the total removed length:

$$\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \ldots + \left(2^{n-1}\frac{1}{3^n}\right) + \ldots = \frac{1}{3}\sum \frac{2^{n-1}}{3^{n-1}} = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = \frac{1}{3}3 = 1$$

Hence, the Cantor set has zero length.

We're going to show that C is uncountable by constructing a bijection between C and the set of sequences (x_n), where $x_i \in \{0,1\}$ for all $i \in \mathbf{N}$.

Let $x \in \mathrm{C}$. Then, $x \in \mathrm{C}_1$. We define

$$x_1 = \begin{cases} 1, & \text{if } x \text{ is on the right side in } \mathrm{C}_1 \\ 0, & \text{if } x \text{ is on the left side} \end{cases}$$

Repeating this process, we obtain a sequence of zeros and ones that corresponds to a point in the Cantor set. Since the set of sequences of zeros and ones is uncountable, then C is also uncountable.

Another fact about the Cantor set is that C has an empty interior.

$$\mathrm{int}\mathrm{C} = \{p \in \mathrm{C} : \exists \varepsilon > 0 \text{ such that } (p-\varepsilon, p+\varepsilon) \subseteq \mathrm{C}\}$$

To show that, let $p \in \mathrm{C}$. Then given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\frac{1}{3^{n_0}} < \varepsilon$.

Definition 4.9 (Perfect sets)**.** A set $\mathrm{P} \subseteq \mathbf{R}$ is **perfect** if it is closed and doesn't contain isolated points, i.e., every point of P is a limit point.

Example 4.10. Non-degenerated closed intervals $[a,b]$ are perfect.

In fact, the Cantor set is perfect.

First, to show that C is closed, remember that each C_n is a finite union of closed intervals. Therefore, C_n is closed. Since C is an arbitrary intersection of closed sets, C is closed.

Now, we need to show that C contains no isolated point. Let $x \in \mathrm{C}$. Let us construct a sequence (x_n) of points of C, $x_n \neq x$ for all $n \in \mathbf{N}$, and $x_n \to x$. Remember that the endpoints of each C_n are contained in C.

Since $x \in \mathrm{C}_1$, there exists $x_1 \in \mathrm{C}_1 \cap \mathrm{C}$ such that $x_1 \neq x$ and $|x - x_1| \leq \frac{1}{3}$. Analogously, for each $n \in \mathbf{N}$, there exists $x_n \in \mathrm{C}_n \cap \mathrm{C}$, $x_n \neq x$, such that $|x - x_n| \leq 1/3^n$. Hence, $x_n \to x$.

Exercise. Show that a non-empty perfect set is uncountable.

5 Continuity

In this section, our goal is to study the limit of a function and the concept of continuity.

5.1 Functional Limit

Definition 5.1 (Functional Limit)**.** Let $f : X \longrightarrow \mathbf{R}$ be a real valued function defined on $X \subseteq \mathbf{R}$, and let a be a limit point of X. We say that f goes to L as x goes to a, or simply

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ it follows that

$$|f(x) - L| < \varepsilon$$

The definition of continuity can also be stated in terms of sequences.

Theorem 5.1. Let $f : X \subseteq \mathbf{R} \longrightarrow \mathbf{R}$, and a be a limit point of X. Then

$$\lim_{x \to a} f(x) = L \iff \lim_{n \to \infty} f(a_n) = L$$

for every sequence (a_n) in X such that $a_n \neq a$ and $\lim_{n \to \infty} a_n = a$.

Proof. Suppose that $\lim_{x \to a} f(x) = L$, and let (a_n) be a sequence in X such that $a_n \neq a$ and $\lim_{n \to \infty} a_n = a$.

Given $\varepsilon > 0$, then there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ if $x \in X$ and $0 < |x - a| < \delta$. Also, there exists $n_0 \in \mathbf{N}$ such that $n > n_0$ implies $0 < |a_n - a| < \delta$. Therefore, for $n > n_0$, we have

$$|f(a_n) - L| < \varepsilon$$

as desired.

Now assume that $\lim_{x \to a} f(x) = L$ is false. Then, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in X$, depending on δ, for which $|f(x) - L| \geq \varepsilon$ but $0 < |x - a| < \delta$.

Thus, taking $\delta_n = 1/n$, we find a sequence of points $a_n \in V_{\delta_n}(a)$, $(a_n) \to a$ and $a_n \neq a$ for all $n \in \mathbf{N}$, where the sequence $f(a_n) \notin V_\varepsilon(L)$, i.e., $f(a_n)$ does not converge to L. ∎

Corollary 5.2. If f has a limit, then this limit is unique.

Proof. Follows immediately from the theorem above and the fact that the limit of a sequence is unique. ■

Corollary 5.3. Let f and g be real valued functions defined on a domain $X \subseteq \mathbf{R}$, and assume

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

for some limit point a of X. Then,

1. $\lim_{x\to a} f(x) + g(x) = L + M$.
2. $\lim_{x\to a} f(x) \cdot g(x) = LM$.
3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, if $M \neq 0$.
4. $\lim_{x\to a} kf(x) = kL$ for all $k \in \mathbf{R}$.

Corollary 5.4 (Divergence Criterion for Functional Limits)**.** Let $f : A \longrightarrow \mathbf{R}$ and a be a limit point of X. If there exists two sequences (x_n) and (y_n) in X with $x_n \neq a$ and $y_n \neq a$ and

$$\lim x_n = c = \lim y_n \text{ but } \lim f(x_n) \neq \lim f(y_n)$$

then the limit $\lim_{x\to a} f(x)$ does not exist.

5.2 Continuous Functions

The concept of continuity will help us avoid 'unbroken curves' or functions with 'jumps' and 'holes'.

Definition 5.2 (Continuity)**.** A function $f : X \longrightarrow \mathbf{R}$ is **continuous at a point** $c \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon$$

for all $x \in \mathbf{R}$ for which $|x - c| < \delta$.

If f is continuous at every point of X, then f is said to be **continuous on** X.

One of the differences from this definition to the definition of functional limit is that we require c to be in the domain of f.

An immediate result from this definition is that functions are continuous at isolated points of their domains.

And if c is a limit point of X, then our definition of continuity can be rewritten as $\lim_{x \longrightarrow c} f(x) = f(c)$.

Example 5.1.

- The constant function in **R** is continuous.
- $f(x) = x$ is continuous at **R**.

Theorem 5.5. Assume $f : \mathrm{I} \longrightarrow \mathbf{R}$ and $g : \mathrm{I} \longrightarrow \mathbf{R}$ are continuous at a point $c \in \mathrm{I}$. Then,

1. $kf(x)$ is continuous at c for all $k \in \mathbf{R}$.
2. $f(x) + g(x)$ is continuous at c.
3. $f(x)g(x)$ is continuous at c.
4. $f(x)/g(x)$ is continuous at c, provided that the quotient is defined.

Proof. Let $(x_n) \in \mathrm{I}$ such that $x_n \longrightarrow c$. Then we have $\lim f(x_n) = f(a)$ and $\lim g(x_n) = g(a)$. Then,

$$\begin{aligned}\lim[f + g(x_n)] = \lim[f(x_n) + g(x_n)] = \lim f(x_n) + \lim g(x_n) \\ = f(a) + g(a) = f + g(a)\end{aligned}$$

For the division,

$$\begin{aligned}\lim \frac{f}{g}(x_n) = \lim \frac{f(x_n)}{g(x_n)} = \frac{\lim f(x_n)}{\lim g(x_n)} \\ = \frac{f(a)}{g(a)} = \frac{f}{g}(a)\end{aligned}$$

And similarly for the product. ■

This definition can be extended naturally for the whole interval I.

Example 5.2.

- $P(x) = a_n x^n + \ldots + a_1 x + a_0$ is continuous.
- If p and q are polynomials, then p/q is continuous at the intervals in which $q \neq 0$.

Theorem 5.6 (Composition of Continuous Functions)**.** Let $f : A \longrightarrow \mathbf{R}$ and $g : B \longrightarrow \mathbf{R}$ be functions such that the range $f(A)$ is contained in the domain B, i.e., $f(A) \subseteq B$.

If f is continuous at $c \in A$, and g is continuous at $f(c) \in B$, then the composition $g \circ f = g(f(x))$ is continuous at c.

Proof. Consider a sequence $(x_n) \subseteq A$ such that $(x_n) \longrightarrow c$. Since f is continuous at c, we have $f(x_n) \longrightarrow f(c)$.

Since g is continuous at $f(c)$, we have $g(f(x_n)) \longrightarrow g(f(c))$.

Hence, $g \circ f$ is continuous at c. ■

Theorem 5.7. A mapping $f : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous if and only if the inverse image $f^{-1}(A)$ is open for every $A \subseteq \mathbf{R}$.

Proof. Suppose that f is continuous on $\mathbf{R}$ and $A \subseteq \mathbf{R}$ be an open set. We will show that $f^{-1}(A)$ is an open set.

For each $a \in f^{-1}(A)$, we have $f(a) \in A$. Since A is open, there exists $\varepsilon > 0$ such that $V_\varepsilon(f(a)) \subseteq A$. Since f is continuous at a, for this ε we associate a $\delta > 0$ such that

$$f(V_\delta(a)) \subseteq V_\varepsilon(f(a)) \subseteq A$$

Which means that $V_\delta(a) \subseteq f^{-1}(A)$ and, therefore, $f^{-1}(A)$ is open.

Conversely, suppose that $f^{-1}(A)$ is open for every open set $A \subseteq \mathbf{R}$. Let $\varepsilon > 0$, let $A' = V_\varepsilon(f(a))$ is open in $\mathbf{R}$ and contains $f(a)$. By hypothesis, $A = f^{-1}(A')$ is open. Hence, there exists $\delta > 0$ such that

$$V_\delta(a) \subseteq A \text{ and } f(V_\delta(a)) \subseteq V_\varepsilon(f(a))$$

Therefore, f is continuous. ■

Lemma 5.8. Let $f : I \longrightarrow \mathbf{R}$ be a continuous function in $a \in I$. If $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in I \cap (a - \delta, a + \delta)$.

Proof. Since $f(a) > 0$, let $\varepsilon = f(a)/a$. Since f is continuous at a, there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$\begin{aligned} |f(x) - f(a)| < \varepsilon &\implies f(a) - \varepsilon < f(x) < f(a) + \varepsilon \\ &\implies 0 < f(a) - \frac{f(a)}{2} < f(x) < f(a) + \frac{f(a)}{2} \end{aligned}$$

whenever $x \in I \cap V_\delta(a)$.

And the proof is analogous for $f(a) < 0$. ■

Theorem 5.9 (Bolzano). Let $f : [a, b] \longrightarrow \mathbf{R}$ be a continuous function such that $f(a)f(b) < 0$. Then there exists $c \in [a, b]$ such that $f(c) = 0$.

Proof. Suppose, without loss of generality, that $f(a) < 0$ and $f(b) > 0$. We will consider the set

$$A = \{x \in [a, b] : f(x) < 0\}$$

Since $A \subseteq [a, b]$, A is bounded. And since $a \in A$, $A \neq \emptyset$. Let $c = \sup A$. We will show that $a < c$.

And since $f(a) < 0$, there exists $\delta_1 > 0$ such that $a + \delta_1 < b$ and $f(x) < 0$ for all $x \in [a, a + \delta_1)$. hence, $c \geq a + \delta_1 > a$.

Now we will show that $f(c) \leq 0$ and $c < b$. By the definition of supremum, there exists a sequence $(a_n) \subseteq A$ such that $a_n \longrightarrow c$. Since f is continuous $f(a_n) \longrightarrow f(c)$.

Since $f(a_n) < 0$ for all $n \in \mathbf{N}$, it follows that $f(c) \leq 0$. In particular, $c < b$ and, therefore, $a < c < b$.

Suppose that $f(c) < 0$. Then there exists $\delta_2 > 0$ such that $(c - \delta_2, c + \delta_2) \subseteq [a, b]$ and $f(x) < 0$, for all $x \in (c - \delta_2, c + \delta_2)$.

Hence, $V_{\delta_2}(c) \subseteq A$ and c is not the supremum of A, which contradicts our hypothesis that $c = \sup A$. Hence, $f(c) = 0$. ■

Theorem 5.10 (Intermediate Value Theorem). Let $f : [a, b] \longrightarrow \mathbf{R}$ be a continuous function. Then, for each γ between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = \gamma$.

Proof. Suppose that $f(a) < \gamma < f(b)$. We define $g(x) = f(x) - \gamma$.

Since g is continuous, we have $g(a) = f(a) - \gamma < 0$. And $g(b) = f(b)\gamma > 0$.

By the Bolzano's theorem, there exists $c \in (a, b)$ such that $g(c) = 0$, i.e., $f(c) - \gamma = 0$. Hence, $f(c) = \gamma$. ■

Definition 5.3 (Bounded functions). Let $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$.

- f is **upper bounded** if there exists $B \in \mathbf{R}$ such that $f(x) \leq B$, for all $x \in I$.
- f is **lower bounded** if there exists $A \in \mathbf{R}$ such that $A \leq f(x)$, for all $x \in I$.
- f is **bounded** if f is both upper bounded and lower bounded.

Theorem 5.11. Every continuous functions $f : [a, b] \longrightarrow \mathbf{R}$ is bounded.

Proof. First, we are going to show that f is upper bounded. Suppose that this is not the case. Then, there would be a sequence $(x_n) \subseteq [a, b]$ such that $f(x_n) > n$, for all $n \in \mathbf{N}$.

Since $[a, b]$ is compact, there exists a subsequence $(x_{n_k}) \subseteq (x_n)$ which converges to $x \in [a, b]$. And since f is continuous, $f(x_{n_k})$ converges to $f(x)$. In particular, $f(x_{n_k})$ is bounded. But this contradicts our hypothesis that $f(x_{n_k}) > n_k$ for all $k \in \mathbf{N}$.

Analogously, we can proof that f is lower bounded. ∎

Theorem 5.12 (Weierstrass)**.** Every continuous function $f : [a, b] \longrightarrow \mathbf{R}$ has a maximum and minimum value, i.e., there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d), \; \forall x \in [a, b]$$

Proof. We already proved that f is bounded, i.e., $f([a, b])$ is bounded.

Let $\beta = \sup\{f([a, b])\}$ and $\alpha = \inf\{f([a, b])\}$.

By the definition of the supremum, there exists a sequence $(x_n) \subseteq [a, b]$ such that $f(x_n) \longrightarrow \beta$. Since (x_n) is bounded, there exists a subsequence (x_{n_k}) which converges to a point $d \in [a, b]$.

Now, since the function f is continuous, we have $f(x_{n_k}) \longrightarrow f(d)$. Given that $f(x_{n_k})$ is a subsequence of $f(x_n)$, we have $f(x_{n_k}) \longrightarrow \beta$, and hence $\beta = f(d) = \sup\{f([a, b])\}$.

In an analogous way, it is proved that there exists $c \in [a, b]$ such that $f(c) = \alpha = \inf\{f([a, b])\}$. Therefore,

$$f(c) \leq f(x) \leq f(d), \; \forall x \in [a, b]$$

∎

Example 5.3. $f : (0, 1] \longrightarrow \mathbf{R}$, where $f(x) = \frac{1}{x}$ does not have a maximum value.

The next theorem is an immediate consequence of the previous two theorems. However, we will present an alternative and elucidating proof.

Theorem 5.13. If $X \subseteq \mathbf{R}$ is compact and f is continuous, then $f(X)$ is compact.

Proof. Suppose that $f : X \longrightarrow Y$ is a continuous function and X is compact.

Let $\{O_\lambda\}$ be an open cover of $f(X)$. Since f is continuous, each set $f^{-1}(O_\lambda)$ is open. And since X is compact, there are finitely many indices, $\lambda_1, \ldots, \lambda_n$, such that

$$X \subseteq f^{-1}(O_{\lambda_1}) \cup \ldots \cup f^{-1}(O_{\lambda_n})$$

Given the fact that $f(f^{-1}(Z)) \subseteq Z$ for every $Z \subseteq Y$, we have

$$f(X) \subseteq O_{\lambda_1} \cup \ldots \cup O_{\lambda_n}$$

which completes the proof. ∎

5.3 Uniform Continuity

Let $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$, and $x_o \in I$. Remember the definition of continuity. When does δ depends on $\delta(\varepsilon, x_0)$ and when does it only depends on ε?

Consider the following example.

Example 5.4. Let $f(x) = 3x + 1$. We have that f is continuous at $c \in \mathbf{R}$.

$$|f(x) - f(c)| = |3x + 1 - (3c + 1)| = 3|x - c|$$

Hence, given $\varepsilon > 0$, choose $\delta = \varepsilon/3$.

$$|x - c| < \delta \implies |f(x) - f(c)| = 3|x - c| < 3\frac{\varepsilon}{3} = \varepsilon$$

And notice that δ only depends on ε.

Example 5.5. Consider $f(x) = x^2$, which is continuous at $c \in \mathbf{R}$.

$$|f(x) - f(c)| = |x^2 - c^2| = |(x - c)(x + c)| = |x - c|\,|x + c|$$

It is necessary, in some way, to 'control' the term $|x - c|\,|x + c|$. To do that let, let $\varepsilon > 0$ be given.

If we control $|x - c|$, we control the product. Suppose that $|x - c| < 1$. Then,

$$|x + c| = |x - c + 2c| \leq |x - c| + |2c| < 1 + 2|c|$$

If we also have $|x-c| < \frac{\varepsilon}{1+2|c|}$, then

$$|x-c|\,|x+c| < \frac{\varepsilon}{1+2|c|}(1+2|c|) = \varepsilon$$

and $|f(x)-f(c)| < \varepsilon$.

Taking $\delta = \min\left\{1, \frac{\varepsilon}{1+2|c|}\right\}$, given $x \in \mathbf{R}$ we have that

$$|x-c| < \delta \implies |f(x)-f(c)| < \varepsilon$$

This example motivates the succeeding definition.

Definition 5.4 (Uniform continuity)**.** Let $f : \mathrm{I} \subseteq \mathbf{R} \longrightarrow \mathbf{R}$. We say that f is **uniformly continuous** if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in \mathrm{I}$, and $|x-y| > \delta$, then $|f(x)-f(y)| < \varepsilon$.

Please take notice on some of the differences between continuity and uniform continuity. Uniform continuity is defined on a set, whereas continuity can be defined at a single point.

The biggest difference is the fact that in uniform continuity, the δ in the definition depends only on ε, and not on any point c in the interval. To say it in another way, if the function is uniformly continuous, we can find a single number $\delta > 0$ which will suffice to prove continuity for all points c of the interval.

Definition 5.5 (Lipschitz)**.** Let $f : \mathrm{I} \subseteq \mathbf{R} \longrightarrow \mathbf{R}$. We say that f is **Lipschtiz** if there exists $\mathrm{M} > 0$, such that

$$|f(x)-f(y)| \leq \mathrm{M}|x-y| \ \forall x, y \in \mathrm{I}$$

Example 5.6.

$$|\sin(x) - \sin(y)| \leq |x-y|$$

Theorem 5.14. Every Lipschitz function is uniformly continuous.

Proof. Let f be a Lipschitz function. By definition, there exists $\mathrm{M} > 0$ such that

$$|f(x)-f(y)| \leq \mathrm{M}|x-y| \ \forall x, y \in \mathrm{I}$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/\mathrm{M}$, which depends only on ε.

If $|x-y| < \delta$, we have

$$|f(x)-f(y)| \leq \mathrm{M}|x-y| < \mathrm{M}\delta = \mathrm{M}\frac{\varepsilon}{\mathrm{M}} = \varepsilon$$

∎

Theorem 5.15 (Sequential Criterion for Absence of Uniform Continuity)**.** Let $f : \mathrm{I} \subseteq \mathbf{R} \longrightarrow \mathbf{R}$. Then f is uniformly continuous if and only if for all pairs of sequences (x_n) and (y_n) in I satisfying $x_n - y_n \longrightarrow 0$, we have that $f(x_n) - f(y_n) \longrightarrow 0$.

Proof. ($\Rightarrow$). Let (x_n), (y_n) be sequences in I with $x_n - y_n \longrightarrow 0$. We are going to show that $f(x_n) - f(y_n) \longrightarrow 0$.

Let $\varepsilon > 0$ be given. Since f is uniformly continuous by hypothesis, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|x_n - y_n| < \delta \implies |f(x_n) - f(y_n)| < \varepsilon$$

Due to the fact that $x_n - y_n \longrightarrow 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n - y_n| < \delta$ whenever $n > n_0$. Therefore, for all $n > n_0$, we have that

$$|f(x_n) - f(y_n)| < \varepsilon \text{ and } f(x_n) - f(y_n) \longrightarrow 0$$

($\Leftarrow$). Suppose that f is not uniformly continuous. We are going to construct squences (x_n) and (y_n) in I with $x_n - y_n \longleftarrow 0$ but $f(x_n) - f(y_n) \not\longrightarrow 0$.

If f does not satisfy the definition of uniform continuity, then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x_\delta, y_\delta \in \mathrm{I}$ with $|x_\delta - y_\delta| < \delta$ but $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$.

Then, for $\varepsilon > 0$ (it exists!), we take $\delta = 1/n$, $n \in \mathbf{N}$. In this way, we have $|x_n - y_n| < 1/n$ and $x_n - y_n \longrightarrow 0$.

Because of our hypothesis, we must have $f(x_n) - f(y_n) \longrightarrow 0$. However, $|f(x_n) - f(y_n)| \geq \varepsilon > 0$, for all $n \in \mathbf{N}$, and $f(x_n) - f(y_n) \not\longrightarrow 0$, which is a contradiction. ∎

Example 5.7. Show that $f : (0,1) \longrightarrow \mathbf{R}, f(x) = 1/x$ is not uniformly continuous.

Consider $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$. Then

$$x_n - y_n = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \longrightarrow 0$$

However,

$$f(x_n) - f(y_n) = n - 2n = -n \not\longrightarrow 0$$

Theorem 5.16 (Heine–Cantor)**.** Let K be a compact set. Then, every continuous function $f : \mathrm{K} \longrightarrow \mathbf{R}$ is uniformly continuous.

Proof. Suppose that f is not uniformly continuous. Then, there exists $\varepsilon > 0$ and $(x_n), (y_n) \subseteq [a,b]$ such that $x_n - y_n \longrightarrow$ and

$$|f(x_n) - f(y_n)| \geq \varepsilon \; \forall n \in \mathbf{N}$$

Given that $(x_n) \subseteq [a,b]$ is a bounded sequence, (x_n) has a convergent subsequence $(x_{n_k})_{k \in \mathbf{N}}$, i.e., $x_{n_k} \longrightarrow x_0$ and $x_0 \in [a,b]$.

Now, since f is continuous at x_0, $f(x_{n_k}) \longrightarrow f(x_0)$.

On the other hand, because $x_n - y_n \longrightarrow 0$, $x_{n_k} - y_{n_k} \longrightarrow 0$. Beyond that fact, since $x_{n_k} \longrightarrow x_0$, also $y_{n_k} \longrightarrow x_0$. Indeed, $y_{n_k} = x_{n_k} - (x_{n_k} - y_{n_k}) \longrightarrow x_0 - 0 = x_0$.

Using the fact that f is continuous at x_0, $f(y_{n_k}) \longrightarrow f(x_0)$.

Therefore,

$$f(x_{n_k}) - f(y_{n_k}) \longrightarrow f(x_0) - f(x_0) = 0$$

which contradicts our hypothesis that $|f(x_n) - f(y_n)| \geq \varepsilon$.

Hence, f is uniformly continuous. ■

6 Differentiation

6.1 The Derivative

Definition 6.1 (Derivative)**.** Let $f : (a,b) \longrightarrow \mathbf{R}$ and $x_0 \in (a,b)$. For each $x \in (a,b), x \neq x_0$, we consider the line passing through $(x_0, f(x_0))$ and $(x, f(x))$. The slope of the line is

$$\frac{f(x) - f(x_0)}{x - x_0}$$

We say that f is **differentiable** at x_0 if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

and f' is called the **derivative** of f.

An alternative and useful notation can be obtained by taking $h = x - x_0$, i.e., $x = x_0 + h$. If $x \to x_0$, then $h \to 0$. Hence,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

In this course, we will use 'differentiable' and 'derivable' to mean the same thing. This is not true, however, in more general cases.

Example 6.1. The derivative of any constant is zero. The derivative of $f(x) = x$ is $f'(x) = 1$ and if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. These results follow immediately from the definition.

Example 6.2. Let $f(x) = |x|$. If $x > 0$, then $f(x) = x$ and hence $f'(x) = 1$.

If $x < 0$, then $f(x) = -x$ and hence $f'(x) = -1$.

And if $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) + f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

which does not exist.

Theorem 6.1. Let f be defined on (a,b) and $x_0 \in (a,b)$. If f is differentiable at x_0, then f is continuous at x_0.

Proof. As $x \to x_0$, we have

$$\begin{aligned}\lim_{x\to x_0}(f(x)-f(x_0)) &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right]\\ &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right]\lim_{x\to x_0}(x-x_0)\\ &= f'(x_0)\cdot 0 = 0\end{aligned}$$

Now notice that

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\left[(f(x)-f(x_0))+f(x_0)\right]\\ &= \lim_{x\to x_0}[f(x)-f(x_0)] + \lim_{x\to x_0} f(x_0) = 0 + f(x_0)\\ &= f(x_0)\end{aligned}$$

i.e., f is continuous at x_0. ∎

Theorem 6.2. Suppose that f and g are defined on (a,b) and differentiable at $x_0 \in (a,b)$. Then

- $f+g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0)+g'(x_0)$.
- fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0)+f(x_0)g'(x_0)$.
- f/g is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$$

Proof. We are going to prove the second result. The other ones are analogous.

$$\begin{aligned}(fg)'(x_0) &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right]\\ &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}g(x)\right]+\lim_{x\to x_0}\left[f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right]\\ &= f'(x_0)g(x_0)+f(x_0)g'(x_0)\end{aligned}$$

∎

Theorem 6.3 (Chain Rule)**.** Let I and J be open intervals and let

$$f : \mathrm{I} \longrightarrow \mathbf{R} \text{ and } g : \mathrm{J} \longrightarrow \mathbf{R}$$

be functions such that $f(\mathrm{I}) \subseteq \mathrm{J}$, i.e., g is defined on an interval that contains the range of f.

And now let $a \in \mathrm{I}$. If f is differentiable at a and if g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. We're going to define some auxiliary functions $\varphi : \mathrm{I} \longrightarrow \mathbf{R}$, where $\varphi(a) = 0$ and

$$\varphi(x) = \frac{f(x) - f(a)}{x - a} - f'(a) \text{ if } x \neq a$$

and also $\psi : \mathrm{J} \longrightarrow \mathbf{R}$ such that $\psi(f(a)) = 0$ and

$$\psi(y) = \frac{g(y) - g(f(a))}{y - f(a)} - g'(f(a)) \text{ if } y \neq f(a)$$

Notice that

$$\begin{aligned} g(f(x)) - g(f(a)) &= [\psi(f(x)) + g'(f(a))] \cdot [f(x) - f(a)] \\ &= [g'(f(a)) + \psi(f(x))] \cdot [f'(a) + \varphi(x)](x - a) \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} &= \lim_{x \to a} ([g'(f(a)) + \psi(f(x))] \cdot [f'(a) + \varphi(x)]) \\ &= g'(f(a)) \cdot f'(a) \end{aligned}$$

■

Lemma 6.4. Let $f : [a, b] \longrightarrow \mathbf{R}$ continuous and injective satisfying $f(a) < f(b)$. Then

1. f is strictly increasing.
2. The inverse function $f^{-1} : [f(a), f(b)] \longrightarrow [a, b]$ is continuous.

Proof. **Exercise.** ■

Theorem 6.5 (Inverse Function)**.** Let $f : \mathrm{I} \longleftarrow \mathbf{R}$, continuous and strictly increasing (or strictly decreasing). Suppose that f is differentiable at a point $a \in \mathrm{I}$ and that $f'(a) \neq 0$. Then its inverse function is differentiable at $b = f(a)$ and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Proof. Let $\mathrm{J} = f(\mathrm{I})$. By the Intermediate Value Theorem, J is an interval. By the previous lemma, $f^{-1} : \mathrm{J} \longrightarrow \mathrm{I}$ is continuous. Then,

$$(f^{-1})'(b) = \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)} = \frac{1}{f'(a)}$$

■

6.2 Mean Value Theorems

Definition 6.2. Let $\mathrm{I} \subseteq \mathbf{R}$ open, $f : \mathrm{I} \longrightarrow \mathbf{R}$ and $a \in \mathrm{I}$.

- f has a **local maximum** at a if there exists $\delta > 0$ such that $f(x) \leq f(a)$ for all $x \in (a - \delta, a + \delta) \subseteq \mathrm{I}$.
- f has a **local minima** at a if there exists $\delta > 0$ such that $f(x) \geq f(a)$ for all $x \in (a - \delta, a + \delta) \subseteq \mathrm{I}$.

Theorem 6.6. Consider an open interval $\mathrm{I} \subseteq \mathbf{R}$, $f : \mathrm{I} \longrightarrow \mathbf{R}$ and f differentiable at $a \in \mathrm{I}$. If f has a local maximum or local minima at a, then $f'(a) = 0$.

Proof. Suppose, without loss of generality, that f has a local maxima at a and let $\delta > 0$ such that $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta)$. Then,

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0$$

and

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0$$

Since f is differentiable at a, both limits are equal. Therefore,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0$$

■

Theorem 6.7 (Rolle)**.** Let $f : [a,b] \longrightarrow \mathbf{R}$ continuous in $[a,b]$ and differentiable in (a,b) satisfying $f(a) = f(b)$. Then, there exists $c \in (a,b)$ such that $f'(c) = 0$.

Proof. By Weierstrass' theorem, f has a maximum and minimum value. That means that there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a,b]$. If f is constant in $[a,b]$, then $f'(x) = 0$ for all $x \in [a,b]$.

Suppose that f is not constant. Then $f(x_1) < f(x_2)$. Since $f(a) = f(b)$, it follows that $x_1 \in (a,b)$ or $x_2 \in (a,b)$. If $x_1 \in (a,b)$, from the previous theorem we have $f'(x_1) = 0$. If $x_2 \in (a,b)$, we have $f'(x_2) = 0$. ∎

The idea behind this theorem is that the tangent line will be, at some point, parallel to the x-axis.

Theorem 6.8 (Generalized Mean Value)**.** If f and g are continuous functions on $[a,b]$ and differentiable in (a,b), then there exists $c \in (a,b)$ at which

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

Proof. Let $h : [a,b] \longrightarrow \mathbf{R}$ defined as

$$h(x) = [f(x) - f(a)][g(b) - g(a)] - [f(b) - f(a)][g(x) - g(a)]$$

Since h is continuous on $[a,b]$ and differentiable in (a,b), and $h(a) = 0 = h(b)$, by the Rolle's theorem, there exists $c \in (a,b)$ such that

$$h'(c) = f'(c)[g(b) - g(a)] - [f(b) - f(a)]g'(c) = 0$$

∎

Theorem 6.9 (Mean Value)**.** If f is a continuous function on $[a,b]$ and differentiable in (a,b), then there exists $c \in (a,b)$ at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Take $g(x) = x$ in the Generalized Mean Value Theorem. ∎

Corollary 6.10. Suppose that f is differentiable in (a,b).

1. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.

2. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

3. If $f'(x) \le 0$ for all $x \in (a,b)$, then f is monotonically decreasing.

Proof. 1. Suppose that $f'(x) \ge 0$ for all $x \in (a,b)$ and let $x_1 < x_2$ both in (a,b). By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \ge 0$$

Given that $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) \ge 0$. I.e., $f(x_2) \ge f(x_1)$.

2. Suppose that $x, y \in (a,b)$ and $x < y$. By the Mean Value Theorem, there exists $c \in [x,y]$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} = 0$$

Hence, $f(y) = f(x)$. Since x and y are arbitrary, $f(x) = k$.

3. Analogous to 1. ■

Corollary 6.11. If f and g are differentiable function on the interval (a,b) and $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x) = g(x) + k$.

Proof. Define $h(x) = f(x) - g(x)$. Then

$$h'(x) = f'(x) - g'(x) = 0 \iff h(x) = k$$

I.e.,

$$h(x) = f(x) - g(x) = k \iff f(x) = g(x) + k$$

■

Theorem 6.12. If $f : [a,b] \longrightarrow \mathbf{R}$ is continuous on $[a,b]$ and differentiable in (a,b) with $|f'(x)| \le \mathrm{M}$ for all $x \in (a,b)$ and $\mathrm{M} \in \mathbf{R}$. Then f is Lipschtiz on $[a,b]$.

Proof. By the Mean Value Theorem,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \le \mathrm{M}$$

for all $c \in (x,y)$. Therefore,

$$|f(x) - f(y)| \le \mathrm{M}|x - y|$$

■

6.3 L'Hospital's Rule

We are going to present the L'Hospital's Rule in two separate theorems in order to better specify their conditions.

Theorem 6.13. Let $f, g : (a, x_0) \cup (x_0, b) \longrightarrow \mathbf{R}$ be differentiable functions such that

1. $\lim_{x \to x_0} f(x) = 0$ and $\lim_{x \to x_0} g(x) = 0$.
2. $g'(x) \neq 0$ and $g(x) \neq 0$ for all x in the domain of definition.

Then

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \mathrm{L} \in [-\infty, \infty] \implies \lim_{x \to x_0} \frac{f(x)}{g(x)} = \mathrm{L}$$

Proof. **Step 1.** We extend f and g to the whole interval (a, b) defining

$$f(x_0) = \lim_{x \to x_0} f(x) = 0 \text{ and } g(x_0) = \lim_{x \to x_0} g(x) = 0$$

Notice that with these definitions, f and g are continuous on (a, b).

Step 2. Given that $f(x_0) = g(x_0) = 0$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)}$$

for all $x \neq x_0$.

Let $x > x_0$. By the Generalized Mean Value Theorem, there exists $y_x \in (x_0, x)$ such that

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(y_x)}{g'(y_x)}$$

If $x < x_0$, there exists $y_x \in (x, x_0)$ such that the same result holds. In both cases, if $x \neq x_0$ in (a, b), then

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f(x)}{g(x)} = \frac{f'(y_x)}{g'(y_x)}$$

for some y_x between x and x_0.

Step 3. Let $\varepsilon > 0$. Since $\lim_{z \to x_0} f'(z)/g'(z) = \mathrm{L}$, there exists $\delta > 0$ such that if $z \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, then

$$\left| \frac{f'(z)}{g'(z)} - \mathrm{L} \right| < \varepsilon$$

For the same δ, let $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. By the step 2, there exists y_x between x_0 and x such that

$$\frac{f(x)}{g(x)} = \frac{f'(y_x)}{g'(y_x)}$$

Hence, $y_x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, which implies

$$\left|\frac{f'(y_x)}{g'(y_x)} - \mathrm{L}\right| < \varepsilon$$

Therefore, for this $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, we obtain

$$\left|\frac{f(x)}{g(x)} - \mathrm{L}\right| < \varepsilon$$

Given that ε is arbitrary,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \mathrm{L}$$

■

Theorem 6.14. Let $f, g : (a, \infty) \longrightarrow \mathbf{R}$ be differentiable functions such that

1. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.
2. $g'(x) \neq 0$ for all x in the domain of definition.

Then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \mathrm{L} \in [-\infty, \infty] \implies \lim_{x\to\infty} \frac{f(x)}{g(x)} = \mathrm{L}$$

Proof. Without loss of generality, we're going to suppose $a > 0$. Let $\varphi, \psi : (0, 1/a) \longrightarrow \mathbf{R}$ defined as

$$\varphi(t) = f\left(\frac{1}{t}\right) \text{ and } \psi(t) = g\left(\frac{1}{t}\right)$$

By the Chain Rule, φ and ψ are differentiable. Moreover,

$$\lim_{t\longrightarrow 0^+} \varphi(t) = \lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{t\longrightarrow 0^+} \psi(t) = \lim_{x\to\infty} g(x) = 0$$

Therefore,

- $\psi'(t) = g'\left(\frac{1}{t}\right) \cdot \left(\frac{-1}{t^2}\right) \neq 0$ for all $t \in (0, 1/a)$.
- $\lim_{t\to 0^+} \frac{\varphi'(t)}{\psi'(t)} = \lim_{t\to 0^+} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = L$.

I.e.,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

$\blacksquare$

Example 6.3. Generalize the L'Hospital's Rule for the next cases:

1. $\lim_{x\to x_0} |f(x)| = \infty$ and $\lim_{x\to x_0} |g(x)| = \infty$.
2. $\lim_{x\to\infty} |f(x)| = \infty$ and $\lim_{x\to\infty} |g(x)| = \infty$.

Pay attention to the fact that every hypothesis is necessary and the reciprocal is not valid.

Theorem 6.15. $f : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous iff. for all $X \subseteq \mathbf{R}$, we have $f(\overline{X}) \subseteq \overline{f(X)}$.

Proof. **Exercise.** $\blacksquare$

This theorem is equivalent to $f(\lim x_n) = \lim f(x_n)$.

Definition 6.3 (Contraction)**.** If $f : \mathbf{R} \longrightarrow \mathbf{R}$, $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for every $x, y \in \mathbf{R}$, then f is said to be a **contraction** of $\mathbf{R}$ into $\mathbf{R}$.

6.4 Taylor's Theorem

Definition 6.4. Let $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$. Then f is said to be of **class** C^k if the derivatives $f', f'', \ldots, f^{(k)}$ exist and are continuous on I.

We define C^0 to be the class of continuous functions.

Our goal in this section is to, given f of class C^n, to construct a polynomial $P_n(x)$ such that

$$P_n(x_0) = f(x_0) \text{ and } P_n^{(k)}(x_0) = f^{(k)}(x_0) \text{ for } 1 \leq k \leq n$$

Theorem 6.16 (Taylor's Theorem)**.** Let $n \in \mathbf{N}$ and $\mathrm{I} = (a,b)$. Suppose that $f : \mathrm{I} \longrightarrow \mathbf{R}$ is of class C^n on I and $f^{(n+1)}$ exists in the interval (a,b).

If $x_0 \in \mathrm{I}$, then for any other $x \in \mathrm{I}$, there exists a point c between x and x_0 such that

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots \\ + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \end{aligned}$$

Proof. Let $x, x_0 \in \mathrm{I}$ and J be the closed interval with endpoints x_0 and x. We define the function $\mathrm{F} : \mathrm{J} \longrightarrow \mathbf{R}$ as

$$\mathrm{F}(t) = f(x) - f(t) - f'(t)(x-t) - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n$$

Notice that

$$\mathrm{F}'(t) = 0 - f'(t) - [(-1)f'(t) + (x-t)f''(t)] - \dots = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Now define $\mathrm{G} : \mathrm{J} \longrightarrow \mathbf{R}$ as

$$\mathrm{G}(t) = \mathrm{F}(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} \mathrm{F}(x_0)$$

and notice that

$$\mathrm{G}(x_0) = \mathrm{F}(x_0) - \mathrm{F}(x_0) = 0 \text{ and } \mathrm{G}(x) = \mathrm{F}(x) = 0$$

More than that, G is continuous on J, since F is continuous (being the sum and product of continuous functions).

By Rolle's theorem, there exists c between x_0 and x such that $\mathrm{G}'(c) = 0$ and

$$\mathrm{G}'(c) = \mathrm{F}'(c) + \frac{(n+1)}{(x-x_0)}\left(\frac{x-c}{x-x_0}\right)^n \mathrm{F}(x_0)$$

Therefore,

$$\begin{aligned} \mathrm{F}(x_0) &= -\mathrm{F}'(c)\frac{1}{(n+1)}\frac{(x-x_0)^{n+1}}{(x-c)^n} = \frac{1}{(n+1)}\frac{(x-x_0)^{n+1}}{(x-c)^n}\frac{(x-c)^n}{n!}f^{(n+1)}(c) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \end{aligned}$$

Then $f(x) = \mathrm{P}_n(x) + \mathrm{R}_n(x)$, where $\mathrm{R}_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the **Lagrange remainder** and P_n is the **Taylor polynomial** of f at x_0 with degree n. ∎

Theorem 6.17. Let I be an interval, x_0 an interior point of I and $n \geq 2$. Suppose that f is of class C^n in a neighbourhood of x_0 and that $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Then,

1. If n is even and $f^{(n)}(x_0) > 0$, then f as a local minimum at x_0.
2. If n is even and $f^{(n)}(x_0) < 0$, then f as a local maximum at x_0.
3. If n is odd, then f does not have a maximum or minimum at x_0.

Proof. By Taylor's theorem in x_0, we can find for some $x \in \mathrm{I}$

$$f(x) = \mathrm{P}_{n-1}(x) + \mathrm{R}_{n-1}(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n$$

where c is a point between x_0 and x.

Given that $f^{(n)}$ is continuous, if $f^{(n)}(x_0) \neq 0$, then there exists an interval U which contains x_0 such that $f^{(n)}(x)$ has the same sign as $f^{(n)}(x_0)$ for each $x \in \mathrm{U}$.

If $x \in \mathrm{U}$, then the point c is also in U and $f^{(n)}(c)$ and $f^{(n)}(x_0)$ have the same sign.

1. If n is even and $f^{(n)}(x_0) > 0$, then for each $x \in \mathrm{U}$ we have $f^{(n)}(c) > 0$ and $(x - x_0)^n \leq 0$ and, therefore, $\mathrm{R}_{n-1}(x) \geq 0$. Hence, $f(x) \geq f(x_0)$ for $x \in \mathrm{U}$ and f has a local minimum at x_0.

2. and **3.** **Exercise.** ∎

6.5 Convex Functions

Definition 6.5 (Convex functions)**.** A function $f : (a, b) \longrightarrow \mathbf{R}$ is **convex** if

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

whenever $x_1, x_2 \in (a, b)$ and $0 \leq \lambda \leq 1$.

Lemma 6.18.

$$f''(a) = \lim_{h \to 0} \frac{f(a + h) - f(a) + f(a - h)}{h^2}$$

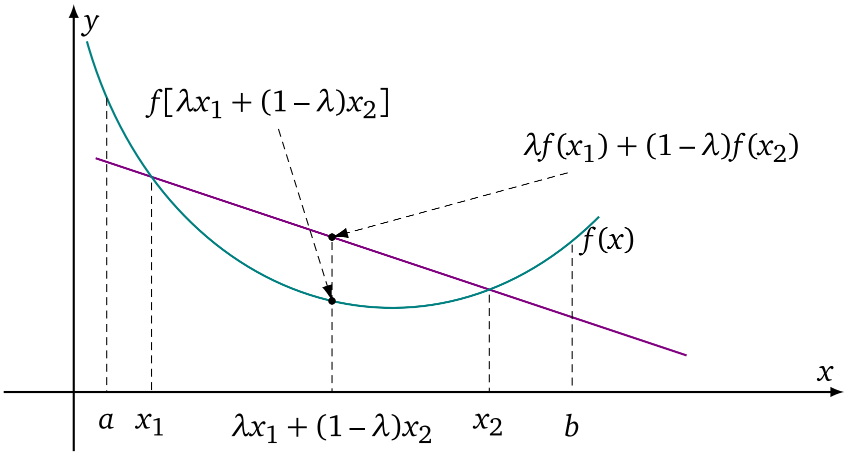

Figure 3: Convex function (Source: [jpa17])

Proof. **Exercise.** ∎

Theorem 6.19. Let I be an open interval and $f : \mathrm{I} \longrightarrow \mathbf{R}$ of class C^2 on I. Then f if convex if, and only if, $f''(x) \geq 0$ for every $x \in \mathrm{I}$.

Proof. ($\Rightarrow$) Given $a \in \mathrm{I}$, let h be such that $a+h$ and $a-h$ are in I. Then

$$a = \frac{1}{2}((a+h)+(a-h))$$

and since f is convex on I,

$$f(a) = f\left(\frac{1}{2}(a+h) + \frac{1}{2}(a-h)\right) \leq \frac{1}{2}f(a+h) + \frac{1}{2}f(a-h)$$

Hence, $f(a+h) - 2f(a) + f(a-h) \geq 0$. Since $h^2 > 0$ for $h \neq 0$, we have

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - f(a) + f(a-h)}{h^2} \geq 0$$

($\Leftarrow$) Here, we use Taylor's theorem. Let x_1 and x_2 be points in I and let $0 < t < 1$ and $x_0 = (1-t)x_1 + tx_2$. Applying Taylor on f at x_0, we obtain a point c_1 between x_0 and x_1 such that

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2}f''(c_1)(x_1 - x_0)^2$$

Analogously, there exists c_2 between x_0 and x_2 such that

$$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2}f''(c_2)(x_2 - x_0)^2$$

Given that $f''(x) \geq 0$, then the term

$$R = \frac{1}{2}(1-t)f''(c_1)(x_1-x_0)^2 + \frac{1}{2}tf''(c_2)(x_2-x_0)^2 \geq 0$$

Therefore,

$$\begin{aligned}(1-t)f(x_1) + tf(x_2) &= f(x_0) + f'(x_0)((1-t)x_1 + tx_2 - x_0) \\ &\quad + \frac{1}{2}(1-t)f''(c_1)(x_1-x_0)^2 + \frac{1}{2}tf''(c_2)(x_2-x_0)^2 \\ &= f(x_0) + R \geq f(x_0) = f((1-t)x_1 + tx_2)\end{aligned}$$

■

6.6 Newton's Method

The goal in this section is to find solutions to $f(x) = 0$. To do that, we're going to suppose f of class C^2 on an interval $I \subseteq \mathbf{R}$ and $f'(x) \neq 0$ for all $x \in I$.

It is known that the tangent line to $(x_0, y_0) \in \text{graph}(f)$ is given by

$$y - y_0 = f'(x_0)(x - x_0)$$

Taking $y = 0$, we have $-y_0 = f'(x_0)x - f'(x_0)x_0$. Hence,

$$x = \frac{f'(x_0)x_0 - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)} =: x_1$$

And then we also define

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \ldots, x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If $x_n \longrightarrow a$, taking $n \longrightarrow \infty$, we have

$$a = a - \frac{f(a)}{f'(a)} \implies f(a) = 0$$

Defining

$$N(x) = x - \frac{f(x)}{f'(x)}$$

and differentiating

$$\begin{aligned}
N'(x) &= 1 - \frac{[f'(x)\cdot f'(x)] - f(x)\cdot f''(x)}{[f'(x)]^2} \\
&= \frac{[f'(x)]^2}{[f'(x)]^2} - \frac{[f'(x)]^2 - f(x)\cdot f''(x)}{[f'(x)]^2} \\
&= \frac{f(x)\cdot f''(x)}{[f'(x)]^2}
\end{aligned}$$

If $N(a) = a$, then $f(a) = 0$. That means that fixed points of N give the zeroes of f.

Finally, if $f(a) = 0$, then

$$N'(a) = \frac{f(a)\cdot f''(a)}{[f'(a)]^2} = 0$$

Hence, for values of x close enough to a, N is a contraction and, therefore, has a fixed point. In fact, a is an attractor of N.

Example 6.4. Compute an approximation for $\sqrt[p]{c}$, $c > 0$ and $p \in \mathbf{N}$.

Define $f(x) = x^p - c$ over the interval $[\sqrt[p]{c}, \infty)$. Then

$$f'(x) = px^{p-1} \text{ and } N(x) = x - \frac{x^p - c}{px^{p-1}}$$

Simplifying,

$$N(x) = \frac{1}{p}\frac{x^{p-1}(px - x + c/x^{p-1})}{x^{p-1}} = \frac{1}{p}[(p-1)x + c/x^{p-1}]$$

and differentiating

$$N'(x) = \frac{p-1}{p}\left[1 - \frac{c}{x^p}\right]$$

Hence, $0 \le N'(x) \le \frac{p-1}{p}$, i.e., N is a contraction and, therefore, has a fixed point.

Example 6.5. Let $f(x) = x - x^3$. Then

$$N(x) = x - \frac{x - x^3}{1 - 3x^2} = \frac{-2x^3}{1 - 3x^2}$$

If $x_0 = \sqrt{5}/5$, then

$$f(x_0) = \frac{4\sqrt{5}}{25} = y_0 \implies y = \frac{2x}{5} + \frac{2\sqrt{5}}{25} = 0 \iff x_1 = \frac{-\sqrt{5}}{5}$$

Repeating the process,

$$y = \frac{2x}{5} - \frac{2\sqrt{5}}{25} = 0 \iff x_2 = \frac{\sqrt{5}}{5}$$

Hence, N does not converge for x_0.

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