Analysis: Study Notes

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May 6, 2022

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1 Finite and Infinite Sets

1.1 Basic Definitions

The 'vocabulary' needed for Analysis comes from set theory. So, before going further, what is a set?

Intuitively, a **set** is any collection of objects, called **elements** of the set.

Definition 1.1. Some basic notation:

1. $x \in A$ means that x is an element of A. If x is not an element of A, we write $x \notin A$.

2. The **union** of two sets A and B is denoted by $A \cup B$ and is defined by

$$x \in A \cup B \iff x \in A \text{ or } x \in B$$

3. The **intersection** $A \cap B$ is the set defined by

$$x \in A \cap B \iff x \in A \text{ and } x \in B$$

   If the intersection $A \cap B$ is the empty set $\emptyset$, then these sets are said to be **disjoint**.

4. The **inclusion** $A \subseteq B$ means that every element of A is also an element of B. And we say that A is a **subset** of B, or B **contains** A. If there is some element in B which is not in A, then A is said to be a **proper subset** of B, denoted by $A \subsetneq B$. To say that $A = B$ means that $A \subseteq B$ and $B \subseteq A$, i.e., both sets have exactly the same elements.

5. Given $A \subseteq \mathbf{R}$, the **complement** of A, denoted by A^c is the set of all elements in $\mathbf{R}$ which are not in A.

$$A^c = \{x \in \mathbf{R} : x \notin A\}$$

Analysis is concerned mainly about the construction of Real Numbers and functions between them. The concept of function also comes from set theory. Here, we'll define it and introduce some basic nomenclature to help our discussions.

Definition 1.2 (Function)**.** A **function** from a set A into a set B is a rule or mapping that associates each element $x \in \mathrm{A}$ to a single element of B, denoted by $f : \mathrm{A} \longrightarrow \mathrm{B}$. And the expression $f(x)$ represents the element of B associated with x by f.

The set A is called **domain of** f, and the elements $f(x)$ are called the **values of** f. The set of all values of f is called **range of** f.

Definition 1.3 (Surjection)**.** Let f be a function of A into B. If $\mathrm{E} \subseteq \mathrm{A}$, then $f(\mathrm{E})$ is the set of all elements $f(x)$ such that $x \in \mathrm{E}$ and we call $f(\mathrm{E})$ the **image** of E under f. Clearly, $f(\mathrm{A}) \subseteq \mathrm{B}$. If $f(\mathrm{A}) = \mathrm{B}$, then f maps A **onto** B and is called a **surjective function**. In other words, for all $y \in \mathrm{B}$ there is at least one $x \in \mathrm{A}$ such that $f(x) = y$.

Definition 1.4 (Inverse image)**.** If $\mathrm{C} \subseteq \mathrm{B}$, then $f^{-1}(\mathrm{C})$ denotes the set of all $x \in \mathrm{A}$ such that $f(x) \in \mathrm{C}$, and is called **inverse image of** C **under** f or **pre-image**. If $y \in \mathrm{B}$, then $f^{-1}(y)$ is the set of all $x \in \mathrm{A}$ such that $f(x) = y$.

$$f^{-1}(\mathrm{C}) = \{a \in \mathrm{A} : f(a) \in \mathrm{C}\}$$

Definition 1.5 (Injection)**.** If, for each $y \in \mathrm{B}$, $f^{-1}(y)$ consist of at most one element of A, then f is said to be a **one-to-one** mapping of A into B, also called a **injective function**. I.e., $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Definition 1.6 (Bijection)**.** If there exists a one-to-one mapping of A onto B (i.e. an injective and surjective function), then we can say that A and B can be put in **one-to-one correspondence**, or that A and B have the same **cardinal number**, and the function satisfying this property is called a **bijective function**, and we write $\mathrm{A} \sim \mathrm{B}$. This relation is an **equivalence relation**.

Definition 1.7 (Left inverse)**.** Given $f : \mathrm{A} \longrightarrow \mathrm{B}$ and $g : \mathrm{B} \longrightarrow \mathrm{A}$, we say that g is a **left inverse** of f when $g \circ f = \mathrm{id}_{\mathrm{A}}$.

Proposition 1.1. A function $f : \mathrm{A} \longrightarrow \mathrm{B}$ has a left inverse iff. it is injective.

Definition 1.8 (Right inverse)**.** Given $f : \mathrm{A} \longrightarrow \mathrm{B}$ and $g : \mathrm{B} \longrightarrow \mathrm{A}$, we say that g is a **right inverse** of f when $f \circ g = \mathrm{id}_{\mathrm{B}}$.

Proposition 1.2. A function $f : \mathrm{A} \longrightarrow \mathrm{B}$ has a right inverse iff. it is surjective.

Definition 1.9 (Inverse function)**.** A function $g : \mathrm{B} \longrightarrow \mathrm{A}$ is said to be an **inverse** of $f : \mathrm{A} \longrightarrow \mathrm{B}$ when g is both a left and a right inverse of f.

Proposition 1.3. A function f has an inverse function iff. it is a bijection.

Proof. Follows from Proposition 1.1 and Proposition 1.2. ■

Proposition 1.4. If a function $f : \mathrm{A} \longrightarrow \mathrm{B}$ has an inverse, then the inverse is unique.

Proof. Suppose that $g : \mathrm{B} \longleftrightarrow \mathrm{A}$ and $h : \mathrm{B} \longleftrightarrow \mathrm{A}$ are both inverses of f. Then

$$h = h \circ \mathrm{id}_\mathrm{B} = h \circ (f \circ g) = (h \circ f) \circ g = \mathrm{id}_\mathrm{A} \circ g = g$$

That means that if f has a left inverse h and a right inverse g, then $h = g$ and f has an inverse. ∎

Corollary 1.5. If $f : \mathrm{A} \longrightarrow \mathrm{B}$ and $g : \mathrm{B} \longrightarrow \mathrm{C}$ are bijections, then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

1.2 Natural numbers

The set of natural numbers, denoted by $\mathbf{N}$, is defined by the following axioms, known as **Peano Axioms**.

- A1 There is an injective function $s : \mathbf{N} \longrightarrow \mathbf{N}$ such that the image $s(n)$ is called the **successor** of n.
- A2 There is a unique number $1 \in \mathbf{N}$ such that $1 \neq s(n)$ for all $n \in \mathbf{N}$. I.e., 1 is not the successor of any natural number.
- A3 If $\mathrm{X} \subseteq \mathbf{N}$ is such that $1 \in \mathrm{X}$ and $s(\mathrm{X}) \subseteq \mathrm{X}$, then $\mathrm{X} = \mathbf{N}$. This axiom is known as the **Principle of mathematical induction**.

We will denote $2 = s(1), 3 = s(2) = s(s(1)), \dots, n = s^{n-1}(1) = \underbrace{s \circ s \circ \dots \circ s}_{(n-1)\text{ times}}(1)$.

Definition 1.10 (Addition)**.** The **addition** in $\mathbf{N}$ is defined by the function

$$\begin{aligned} \mathbf{N} \times \mathbf{N} &\longrightarrow \mathbf{N} \\ (m, n) &\mapsto m + n \end{aligned}$$

satisfying the following properties:

1. $m + 1 = s(m)$.
2. $m + s(n) = s(m + n)$.

Theorem 1.6. For all $m, n, p \in \mathbf{N}$:

- $(m+n)+p = m+(n+p)$.
- $m+n = n+m$.
- If $m+p = n+p$, then $m = n$.

Definition 1.11 (Multiplication)**.** The **multiplication** is defined by the function

$$\begin{aligned} \mathbf{N} \times \mathbf{N} &\longrightarrow \mathbf{N} \\ (m,n) &\mapsto m \cdot n \end{aligned}$$

satisfying:

1. $m \cdot 1 = m$.
2. $m \cdot s(n) = m \cdot n + m$.

And we will simply denote $m \cdot n$ by mn.

Theorem 1.7. For all $m,n,p \in \mathbf{N}$,

- $p(m+n) = pm + pn$.
- $(m+n)p = mp + np$.
- $(mn)p = m(np)$.
- $mn = nm$.
- If $mp = np$, then $m = n$.

Given two natural numbers m,n we write $m < n$ if there is a natural number p such that $n = m+p$. Naturally,

Theorem 1.8.

- If $m < n$ and $n < p$, then $m < p$.
- (Trichotomy law) One and only one of the following is true: $m = n$, or $m < n$, or $m > n$.

This idea of order between natural numbers motivates the following theorem.

Theorem 1.9 (Well ordering principle)**.** Every non-empty subset $A \subseteq \mathbf{N}$ has a minimum element. I.e., there is an element $n \in A$ such that $n \leq m$ for all $m \in A$.

Proof. If $1 \in A$, then $1 \leq k$ for all $k \in A$.

Assume $1 \notin A$ and define $J_m = \mathbf{N} \setminus \{1, 2, \ldots, m\}$ for every $m \in \mathbf{N}$.

Consider the following set $B = \{m \in \mathbf{N} : A \subseteq J_m\}$. Since $1 \notin A$, it follows that $A \subseteq J_1$. Hence, $1 \in B$.

Since A is non-empty, there is an element $m \in A$. And since $m \notin J_m$, we have $A \not\subseteq J_m$. Hence, $m \notin B$, which implies that $B \neq \mathbf{N}$.

Finally, since $1 \in B$ and $B \neq \mathbf{N}$, there is $n \in B$ such that $n + 1 \notin B$ by the induction principle. Therefore, $A \not\subseteq J_{n+1}$, and hence $n+1 \leq k$, for all $k \in A$. ■

1.3 Finite and Infinite sets

The following two lemmas will allow us to distinguish between finite and infinite sets.

Lemma 1.10. Let $f : A \longrightarrow B$ be a bijective function. Given $a \in A$ and $b \in B$, there is always a bijective function $g : A \longrightarrow B$ such that $g(a) = b$.

Proof. If $f(a) = b$, take $g = f$.

Diversely, let $f(a) = b_1 \neq b$. Since b is surjective, there is $a_1 \in A$ such that $f(a_1) = b$. Hence, $a \neq a_1$. Otherwise, we would have $f(a) = f(a_1) = b$, which is not possible.

Define $g : A \longrightarrow B$ by $g(a) = b$, $g(a_1) = b_1$, and $g(x) = f(x)$ for all $x \in A$ such that $x \neq a$ and $x \neq a_1$. ■

Lemma 1.11. Let $m, n \in \mathbf{N}$. If there is a bijective function

$$f : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$$

then $m = n$.

Proof by contradiction. Suppose that there is a bijection

$$f : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$$

and $m < n$.

By the previous lemma, there is a bijection $f_1 : \{1,2,\dots,m\} \longrightarrow \{1,2,\dots,n\}$ such that $f_1(1) = 1$.

Applying this lemma again to the restriction

$$f_1|_{\{2,\dots,m\}} : \{2,\dots,m\} \longrightarrow \{2,\dots,n\}$$

we construct a bijection $g : \{2,\dots,m\} \longrightarrow \{2,\dots,n\}$ such that $g(2) = 2$ by defining

$$f_2 : \{1,2,\dots,m\} \longrightarrow \{1,2,\dots,n\}$$

where $f_2(1) = f_1(1) = 1$, and $f_2(k) = g(k)$, for $k = 2,\dots,m$. Notice that f_2 is bijective and $f_2(1) = 1$, and $f_2(2) = 2$.

Applying the same lemma m times in an analogous manner, we obtain a bijective function

$$f_m : \{1,2,\dots,m\} \longrightarrow \{1,2,\dots,n\}$$

such that $f_m(k) = k$ if $k = 1,2,\dots,m$. If $m < n$, then f_m cannot be surjective, which contradicts our hypothesis.

The case $m > n$ is analogous. ■

Definition 1.12 (Finite and Infinite Sets)**.** A set A is said to be **finite** if A is empty or if there is a bijective function $f : \{1,2,\dots,n\} \longrightarrow \mathrm{A}$ for a $n \in \mathbf{N}$.

By the previous lemma, this n is unique. In this case, we say that n is the **number of elements** or **cardinality** of A. If A is not finite, we say that A is **infinite**.

Two sets A and B **have the same cardinality** if there exists a bijective function $f : \mathrm{A} \longrightarrow \mathrm{B}$, which we denote by $\mathrm{A} \sim \mathrm{B}$.

Proposition 1.12. **N** is infinite.

Proof by contradiction. Suppose that **N** is finite. That implies that there is a bijective function $f : \{1,2,\dots,n\} \longrightarrow \mathbf{N}$.

Using the same argument from the proof of the previous lemma, it is possible to construct a bijective function $f_n : \{1,2,\dots,n\} \longrightarrow \mathbf{N}$ such that $f_n(k) = k$ for all $k = 1,2,\dots,n$. This is a contradiction, since $n+1$ is not in the image of f_n. Thus, **N** is infinite. ■

Notation: If $n \in \mathbf{N}$ and $n > 1$, the denote by $n-1$ the unique $m \in \mathbf{N}$ such that $m + 1 = n$.

Proposition 1.13. If A is a finite set, then each subset of A is also finite.

Proof. If A is empty, there's nothing to prove.

Suppose $A \neq \emptyset$. First, we are going to prove for a particular case: if A is a finite set and $a \in A$, then $A \setminus \{a\}$ is finite.

To see that this is the case, notice that by hypothesis there is an one-to-one correspondence (bijection) $f : \{1, 2, \ldots, n\} \longrightarrow A$. By the Lemma 1.10, there is a bijection $g : \{1, 2, \ldots, n\} \longrightarrow A$ such that $g(n) = a$.

If $n = 1$, then $A \setminus \{a\}$ is empty and, hence empty.

If $n > 1$, then the restriction

$$g|_{\{1,\ldots,n-1\}} : \{1, \ldots, n-1\} \longrightarrow A \setminus \{a\}$$

is a bijection and, therefore $A \setminus \{a\}$ is finite and has $n - 1$ elements.

With that particular case, we can prove the proposition by induction of the number n of elements of the set A.

If $n = 1$, then A has a single element a. Let $B \subseteq A$. If $B = A$, then B is finite. If $B \neq A$, then $a \notin B$ and, therefore, $B \subseteq A \setminus \{a\}$. Hence, B is empty.

Induction Hypothesis: suppose that the result is valid for a set with n elements.

Let A be a set with $n + 1$ elements and $B \subseteq A$. If $A = B$, then B is finite. Suppose $B \neq A$. Then there exists $a \in A \setminus B$.

Therefore, $B \subseteq A \setminus \{a\}$ and the previous argument (the particular case) shows that $A \setminus \{a\}$ is finite with n elements. By the induction hypothesis, B is finite. ■

Corollary 1.14. Let $f : A \longrightarrow B$. Then:

- If f is injective and B is finite, then A is finite.
- If f is surjective and A is finite, then B is finite.

Proof. **(a)** By the Proposition 1.13, since $f(A) \subseteq B$, $f(A)$ is also finite. Since $f : A \longrightarrow f(A)$ is bijective, then A is finite.

(b) Exercise. ■

The following two propositions will give tools to discover whether a set is infinite.

Proposition 1.15. A set A is infinite iff. there is an injective function $f : \mathbf{N} \longrightarrow A$.

Proof. ($\Leftarrow$) We are going to show that if there is an injective function $f : \mathbf{N} \longrightarrow A$ then A is infinite.

Suppose, by contradiction, $f : \mathbf{N} \longrightarrow A$ injective, but A is finite. By the Corollary 1.14, $\mathbf{N}$ must be finite, which contradicts the Proposition 1.12.

($\Rightarrow$) Suppose that A is infinite. We are going to construct an injective function $f : \mathbf{N} \longrightarrow A$.

Let $x_1 \in A$ and define $f(1) = x_1$. Clearly, $f : \{1\} \longrightarrow A$ is injective.

Now suppose that $f(1), f(2), \ldots, f(n)$ are defined satisfying that

$$f : \{1, 2, \ldots, n\} \longrightarrow A$$

is injective.

And let $x_{n+1} \in A \setminus \{f(1), \ldots, f(n)\}$ and define $f(n+1) = x_{n+1}$. Clearly, $f : \{1, \ldots, n+1\} \longrightarrow A$ is injective.

This process defines a function $f : \mathbf{N} \longrightarrow A$. To show that f is injective, let $m < n$. By construction, $f(m) \in \{f(1), \ldots, f(n-1)\}$ and, on the other hand, $f(n) \notin \{f(1), \ldots, f(n-1)\}$, which means that $f(m) \neq f(n)$. Hence, f is injective. ■

Proposition 1.16. A set A is infinite iff. there is a bijective function g of A into a proper subset B of A.

Proof. ($\Rightarrow$) Suppose that A is infinite. By the proposition 1.15, there is an injective function $f : \mathbf{N} \longrightarrow A$.

Let $x_n = f(n)$ for each $n \in \mathbf{N}$, and let $B = A \setminus \{x_1\}$. Define $g : A \longrightarrow B$ by $g(n) = x_{n+1}$, for each $n \in \mathbf{N}$ and $g(x) = x$, for each $x \in A \setminus \operatorname{Im} f$.

Exercise. Show that g is an one-to-one correspondence (bijection).

($\Leftarrow$) Let B a proper subset of A and $g : A \longrightarrow B$ a bijection.

If A is finite, then B is also finite by the Corollary 1.14. Since g is bijective, then A and B have the same cardinality by the Lemma 1.11, which is a contradiction, since B is a proper subset of A. ■

1.4 Countable and Uncountable sets

Definition 1.13 (Countable set)**.** A set A is said to be **countable** if A is finite or if there is a bijective function $f : \mathbf{N} \longrightarrow A$. And we say that the elements $f(1), f(2), \ldots$ form an enumeration of A.

Example 1.1.

- $\mathbf{N}$ is countable.
- The set of even numbers is countable.
- $\mathbf{Z}$ is countable. Just take

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even} \end{cases}$$

Similarly to what occured in our discussion about finite and infinite sets, some propositions follow from this definition.

Proposition 1.17. Every infinite set contains an infinite and countable subset.

Proof. Let A be an infinite subset. Then, by the Proposition 1.15, there is an injective function $f : \mathbf{N} \longrightarrow \mathrm{A}$. Hence, there is a bijection $\mathbf{N} \longrightarrow f(\mathbf{N}) \subseteq \mathrm{A}$. ■

Proposition 1.18. If A is countable, then every subset $\mathrm{B} \subseteq \mathrm{A}$ is also countable.

Proof. If A is finite, then every subset $\mathrm{B} \subseteq \mathrm{A}$ is also finite (by the Proposition 1.13), thus countable.

Suppose that A is infinite and countable. Then there is a bijection $f : \mathbf{N} \longrightarrow \mathrm{A}$. If B is finite, then B is countable. Suppose that B is infinite.

Let $\mathrm{C} = f^{-1}(\mathrm{B})$. Then C is an infinite subset of $\mathbf{N}$. We are going to define $g : \mathbf{N} \longrightarrow \mathrm{B}$ in an inductive fashion. The following commutative diagram represents the idea.

$$\begin{tikzcd} \mathrm{C} \arrow[r, "f"] & \mathrm{B} \subseteq \mathrm{A} \\ & \mathbf{N} \arrow[u, "g"'] \arrow[ul] \end{tikzcd}$$

Let n_1 be the smallest element of C and $g(1) = f(n_1)$. Suppose that

$$g(1), g(2), \ldots, g(k)$$

are already defined in order that for each $j = 2, 3, \ldots, k$, $g(j) = f(n_j)$, where n_j is the smallest element of $\mathrm{C} \setminus \{n_1, n_2, \ldots, n_{j-1}\}$.

Now let n_{k+1} be the smallest element of $C \setminus \{n_1, n_2, \dots, n_k\}$ and let $g(k+1) = f(n_{k+1})$. Since $n_1 < n_2 < \dots$, the function g is injective (since f is bijective).

To show that g is surjective, suppose that there is an element $b \in B \setminus g(\mathbf{N})$. Then we would have $f^{-1}(b) \in C \setminus \{n_1, \dots, n_k\}$ for all $k \in \mathbf{N}$. Hence, $f^{-1}(b) \geq n_{k+1}$ for all $k \in \mathbf{N}$, which is a contradiction since C is infinite.

Therefore, g is bijective.

■

Proposition 1.19. Let $f : A \longrightarrow B$.

(a) If f is injective and B is countable, then A is countable.

(b) If f is surjective and A is countable, then B is countable.

Proof. **Exercise.** ■

Proposition 1.20. $\mathbf{N} \times \mathbf{N}$ is countable.

Idea of the proof. By the Fundamental Theorem of Arithmetic, each $n \in \mathbf{N}$ can be uniquely decomposed in prime factors (modulo permutation). So we can write

$$n = 2^{k-1}(2l - 1)$$

and define

$$\begin{aligned} f : \mathbf{N} &\longrightarrow \mathbf{N} \times \mathbf{N} \\ n &\longmapsto (k_n, l_n) \end{aligned}$$

which is a bijection. ■

Corollary 1.21.

(a) If A and B are countable, then $A \times B$ is countable.

(b) Let A_m be a countable set for all $m \in \mathbf{N}$. Then

$$\bigcup_{m=1}^{\infty} A_m \text{ is countable}$$

Proof. **Exercise.** Hint: Define a bijection $f_m : \mathbf{N} \longrightarrow A_m$ and a function

$$f : \mathbf{N} \times \mathbf{N} \longrightarrow \bigcup_{m=1}^{\infty} A_m$$

where $f(m, n) = f_m(n)$. ■

After talking about countable sets and how $\mathbf{N}$ is countable, a new question arises: is $\mathscr{P}(\mathbf{N})$ countable?

The **Diagonal Method**, proposed by Cantor, shows that this is not the case: there is not surjective function $f : \mathbf{N} \longrightarrow \mathscr{P}(\mathbf{N})$.

Theorem 1.22 (Cantor's Theorem)**.** For every set X,

$$|\mathrm{X}| < |\mathscr{P}(\mathbf{X})|$$

Example 1.2. The set $\mathscr{F} = \{f : \mathbf{N} \longrightarrow \{0,1\}\}$, i.e., the set of all infinite sequences of zeroes and ones, is uncountable.

Finally, we can define the integers and rational numbers as follows:

Definition 1.14 (Integers)**.**

$$\mathbf{Z} = \{n \in \mathbf{N}\} \cup \{0\} \cup \{-n : n \in \mathbf{N}\}$$

Definition 1.15 (Rational numbers)**.**

$$\mathbf{Q} = \{p/q, p, q \in \mathbf{Z}, q \neq 0\}$$

These definitions about finite and countable sets can be summarized in the following definition.

Definition 1.16. For any positive integer n, let J_n be the set containing the integers $1, 2, \ldots, n$, and let J be the set of all positive integers. For any set A, we say that:

1. A is **finite** if $\mathrm{A} \sim \mathrm{J}_n$ for some n (and the empty set is also considered to be finite).
2. A is **infinite** if A is not finite.
3. A is **countable** if $\mathrm{A} \sim \mathrm{J}$.
4. A is **uncountable** if A is neither finite nor countable.
5. A is **at most countable** if it is finite or countable.

Evidently, $\mathrm{A} \sim \mathrm{B}$ iff. A and B contain the same number of elements.

2 Real Numbers

2.1 The Real Field

How to accurately define a real number?

Abbott: “Rational numbers are densely nestled together.”

But rational number system has gaps and the real number system fills these gaps.

Every rational number has a decimal expansion (just use the euclidean algorithm). This expansion can be terminal or infinite and may note be unique.

Proposition 2.1. Every rational number with infinite decimal expansion is periodic, which means that

$$x = q, d_1 \dots d_n d_{n+1} \dots d_k d_{n+1} \dots d_k d_{n+1} \dots = q, d_1 \dots d_n \overline{d_{n+1} d_k}$$

Proof. **Exercise.** ■

Example 2.1. The number $0.010010001\dots$ is not rational.

Proposition 2.2. $\sqrt{2}$ is not rational.

Proof. Suppose that $\sqrt{2}$ is rational. Then, $\sqrt{2} = a/b$, $a, b \in \mathbf{Z}$, $b \neq 0$, and a/b co-prime. Hence,

$$a = \sqrt{2}b \iff a^2 = 2b^2$$

It follows that a^2 is even, hence a is even and is of the form $a = 2k$, $k \in \mathbf{Z}$. Thus,

$$(2k)^2 = 2b^2 \iff 4k^2 = 2b^2 \iff b^2 = 2k^2$$

Which means that b is even.

Therefore, we have a contradiction, since we assumed a/b irreducible. ■

It can be proved that $\sqrt{2}$ can be approximated by a rational number with an error as small as one wants.

To put it another way, given $\varepsilon > 0$, there exists $x \in \mathbf{Q}$ such that

$$|x - \sqrt{2}| < \varepsilon$$

In the general case, for any $r \in \mathbf{R}^+$, an algorithm for decimal expansion can be done as follows.

Algorithm for decimal expansion.

1. First, choose q such that $q \leq r < q+1$.
2. Then d_1 is determined by

$$q + \frac{d_1}{10} \leq r < q + \frac{d_1}{10} + \frac{1}{10}$$

where d_1 is between 0 and 9 because $q \leq r < q+1$.

3. For the step $n+1$, d_{n+1} is determined by

$$q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_{n+1}}{10^{n+1}} \leq r < q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}}$$

where d_{n+1} is between 0 and 9 because

$$q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} \leq r < q + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} + \frac{1}{10^n}$$

Exercise. Write the expansion of $\sqrt{3}$ and $\sqrt{5}$.

Definition 2.1 (Field)**.** A **field** is a set $\mathbf{F}$ with two closed operations, called addition and multiplication satisfying the field axioms: associativity, commutativity, neutral element, inverse element and distributivity between addition and multiplication.

Proposition 2.3. The axioms for addition imply that:

1. If $x+y = x+z$ then $y = z$ (cancellation law).
2. If $x+y = x$ then $y = 0$ (uniqueness of the neutral element).
3. If $x+y = 0$ then $y = -x$ (uniqueness of the inverse element).
4. $-(-x) = x$.

Proposition 2.4. The axioms for multiplication imply that:

1. If $x \neq 0$ and $xy = xz$ then $y = z$.
2. If $x \neq 0$ and $xy = x$ then $y = 1$.
3. If $x \neq 0$ and $xy = 1$ then $y = x^{-1}$.
4. If $x \neq 0$ and $1/x^{-1} = x$.

Proposition 2.5. For any $x, y, z \in \mathbf{F}$:

1. $0x = 0$.
2. If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
3. $(-x)y = -(xy) = x(-y)$.
4. $(-x)(-y) = xy$.

Definition 2.2 (Order)**.** An **order** on a set S is a relation, denoted by $<$, satisfying:

1. One and only one of the following statements is true: $x < y$, $x = y$, $y < x$.
2. If $x < y$ and $y < z$, then $x < z$.

where $x, y, z \in S$.

Definition 2.3 (Ordered Field)**.** An **ordered field** is a field $\mathbf{F}$ which is also an ordered set, such that for any $x, y, z \in \mathbf{F}$:

1. If $y < z$ then $x + y < x + z$.
2. If $x, y > 0$ then $xy > 0$.

Alternatively, a field $\mathbf{F}$ is said to be an **ordered field** if there is in $\mathbf{F}$ a total order relation $\leq$ such that if $x \leq y$ then $x + z \leq z + y$, $\forall z \in \mathbf{F}$ and $xz \leq yz$, $z \geq 0$. In other words,

- $x \leq x$ (reflexivity).
- If $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetry).
- If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).
- $x \leq y$ or $y \leq x$ (dichotomy law).

The familiar properties for inequalities hold.

Proposition 2.6. Suppose $x, y, z \in \mathbf{F}$, where $\mathbf{F}$ is an ordered field. Then the following statements are true.

1. If $x > 0$ then $-x < 0$, and vice versa.

2. If $x > 0$ and $y < z$ then $xy < xz$.
3. If $x < 0$ and $y < z$ then $xy > xz$.
4. If $x \neq 0$ then $x^2 > 0$.
5. If $0 < x < y$ then $0 < 1/y < 1/x$.

2.2 Upper and Lower Bounds

Definition 2.4 (Upper Bound)**.** Suppose S is an ordered set, and $E \subseteq S$. If there exists $z \in S$ such that $x \leq z$ for every $x \in E$, then E is said **bounded above** and z is called an **upper bound** of E.

Lower Bounds are analogous. And we say that a set E is **bounded** if E is bounded above and bounded below.

Intuitively, an upper bound is a number that is greater or equal than every number in the given set.

Notice that every larger number $z' \geq z$ is also an upper bound of E, but it is not always the case that a number smaller than z is also an upper bound of E.

Definition 2.5 (Least Upper Bound)**.** Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. If there exists $z \in S$ satisfying:

1. z is an upper bound of E.
2. If $y < z$ then y is not an upper bound of E.

Then z is called **least upper bound of** E or the **supremum of** E denoted by $z = \sup E$.

The **Greatest Lower Bound** (or **infimum**) is analogous, taking E bounded below. Denoted by $z = \inf E$.

Intuitively, the second property in the definition states that any other upper bound z' for E is larger than or equal to z.

Notice that the supremum can exist and not be a maximum, but when a maximum exists, it is also the supremum.

Also, every ordered set with a least-upper-bound property also has the greatest-lower-bound property.

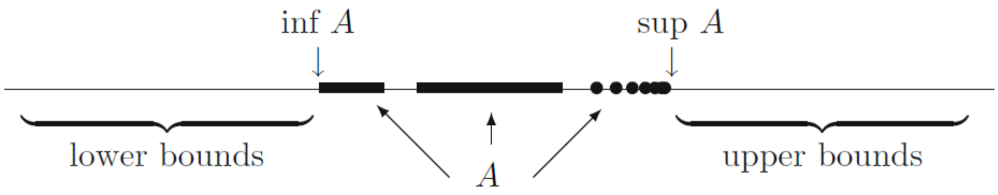

Figure 1: Definition of sup A and inf A (Source: ABBOTT)

The natural question now is: when does the supremum/infimum exist?

Axiom of Completeness. Every non-empty set of real numbers that is bounded above has a least upper bound.

The following theorem is an important property of the real numbers. It states that no matter how large y is and how small x is, if one keeps adding x to itself, one will eventually overtake y.

Theorem 2.7 (Archimedean Property)**.** Let $x, y \in \mathbf{R}$, and $x > 0$. Then there exists a positive integer n such that $nx > y$.

Proof by contradiction. Suppose that there is $a, b > 0$ such that $na \leq b$, $\forall n \in \mathbf{N}$. We define

$$\mathrm{S} = \{n \cdot a : n \in \mathbf{N}\}$$

In this case, b is an upper bound of S. Using the supremum axiom, let $s_0 = \sup \mathrm{S}$.

Since $a > 0$, we have $s_0 < s_0 + a$, hence $s_0 - a < s_0$.

Since s_0 is the least upper bound of S, $s_0 - a$ can not be an upper bound of S. Therefore, $s_0 - a < n_0 a$ for some $n_0 \in \mathbf{N}$.

Hence, $s_0 - a < n_0 a$. Which means that $s_0 < n_0 a + a = (n_0 + 1)a$.

Notice that we obtained $(n_0 + 1)a \in \mathrm{S}$ and s_0 is not an upper bound of S, which is a contradiction. ■

As a consequence, for every $a \in \mathbf{R}_{+}$, there exists $n \in \mathbf{N}$ such that $na > 1$, i.e., $1/n < a$.

The theorem following states that between any two real numbers there is a rational one.

Theorem 2.8 (**Q** is dense in **R**)**.** Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

Proof. Exercise. ■

Theorem 2.9 (Existence of n^{th} roots)**.** Let x be a positive real number, and let n be a positive integer. Then there is one and only one positive real number z such that $z^n = x$. This number z is written $\sqrt[n]{x}$ or $x^{1/n}$.

In other words, then the set $\mathrm{E} := \{y \in \mathbf{R} : y \geq 0 \wedge y^n \leq x\}$ is non-empty and is also bounded above. In particular, $x^{1/n}$ is a positive real number.

We will end this section with two important propositions about least upper bounds.

Proposition 2.10 (Uniqueness of least upper bound)**.** Let $\mathrm{E} \subseteq \mathbf{R}$. Then E can have at most one least upper bound.

Proof. Let M_1 and M_2 be two least upper bounds. Since M_1 is a least upper bound and M_2 is an upper bound, then by definition of the least upper bound we have $\mathrm{M}_2 \geq \mathrm{M}_1$. Similarly, since M_2 is a least upper bound and M_1 is an upper bound, then $\mathrm{M}_1 \geq \mathrm{M}_2$. Hence, $\mathrm{M}_1 = \mathrm{M}_2$. ■

Theorem 2.11 (Existence of least upper bound)**.** Let E be a non-empty subset of **R**. If E has an upper bound, then it must have exactly one least upper bound.

Proof. Let M be an upper bound of E. By the uniqueness of the least upper bound, we know that E has at most one least upper bound. We want to show that E has at least one least upper bound.

Since E is non-empty, we can choose some $x_0 \in \mathrm{E}$. Let $n \geq 1$ be a positive integer. By the Archimedean property, we can find $\mathrm{K} \in \mathbf{Z}$ such that $\mathrm{K}/n \geq \mathrm{M}$, and hence K/n is also an upper bound for E.

Using the Archimedean property again, there exists $\mathrm{L} \in \mathbf{Z}$ such that $\mathrm{L}/n < x_0$. Since $x_0 \in \mathrm{E}$, L/n is not an upper bound for E. Since K/n is an upper bound but L/n is not, we have $\mathrm{K} \geq \mathrm{L}$.

With that, we can find an integer $\mathrm{L} < m_n \leq \mathrm{K}$ with the property that m_n/n is an upper bound for E, but $(m_n - 1)/n$ is not. In fact, m_n is unique. This gives a well-defined and unique sequence $m_1, m_2, m_3, \ldots$ of integers with the property above.

Now let $N \geq 1$ be a postive integer, and let $n, n' \geq N$. Since m_n/n is an upper bound for E and $m_{n'} - 1/n'$ is not, we have $m_n/n > m_{n'} - 1/n'$. This implies that

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} > -\frac{1}{n'} \geq -\frac{1}{N}$$

Similarly, since $m_{n'}/n'$ is an upper bound for E and $m_n - 1/n$ is not, we have $m_{n'}/n' > m_n - 1/n$, and hence

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} \leq \frac{1}{n} \leq \frac{1}{N}$$

Putting these two bounds together,

$$\left| \frac{m_n}{n} - \frac{m_{n'}}{n'} \right| \leq \frac{1}{N} \text{ for all } n, n' \geq N \geq 1$$

This implies that $\frac{m_n}{n}$ is a Cauchy sequence. Since they are rational numbers, we can define the real number S as

$$S := \lim_{n \to \infty} \frac{m_n}{n}$$

Hence,

$$S = \lim_{n \to \infty} \frac{m_n - 1}{n}$$

Now we need to show that S is the least upper bound for E.

Let $x \in E$. Since m_n/n is an upper bound for E, we have $x \leq m_n/n$ for all $n \geq 1$. Therefore, $x \leq \lim_{n \to \infty} \frac{m_n}{n} = S$. Thus S is an upper bound for E.

Suppose that y is an upper bound for E. Since $(m_n - 1)/n$ is not an upper bound, $y \geq (m_n - 1)/n$. Hence, $y \geq \lim_{n \to \infty} \frac{m_n - 1}{n} = S$. Thus the upper bound S is less than or equal to every upper bound of E, and S is thus a least upper bound of E. ∎

3 Sequences of Real Numbers

3.1 Sequences

Definition 3.1 (Sequence)**.** A **sequence** is a function f defined on the set $\mathbf{N}$. If $f(n) = x_n$, $n \in \mathbf{N}$, we denote the sequence f by the symbol $(x_n)_{n=1}^{\infty}$ or, simply, (x_n). The elements x_n are called the **terms** of the sequence.

Intuitively, a sequence of real numbers is a collection of reals

$$a_m, a_{m+1}, a_{m+2}, \dots$$

If A is a set and if $x_n \in$ A for all $n \in \mathbf{N}$, then $\{x_n\}$ is said to be a **sequence of elements of** A.

Definition 3.2 (Convergence)**.** A real sequence (x_n) is said to **converge** if there is a point $a \in \mathbf{R}$ satisfying that for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$|x_n - a| < \varepsilon$$

whenever $n > n_0$.

We also say that (x_n) converges to a or that a is the limit of (x_n), and we write $x_n \longrightarrow a$ or $\lim x_n = a$.

If (x_n) does not converge, then we say that the sequence **diverges**.

Definition 3.3 (ε-neighbourhood)**.** An **ε-neighbourhood** of a, or the neighbourhood of center a and radius ε is defined as

$$\mathrm{V}_\varepsilon(a) = \{x \in \mathbf{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$$

With this definition, we can understand convergence as follows:

$$x_n \longrightarrow a \text{ if } \forall \varepsilon > 0\, \exists n_0 \in \mathbf{N} \text{ such that } x_n \in \mathrm{V}_\varepsilon(a), \forall n > n_0$$

Proposition 3.1 (Uniqueness of the limit)**.** *If a and b are limits of (x_n), then $a = b$.*

Proof. Let $\varepsilon > 0$. Since a and b are limits of $(x_n$, then there exists n_1, n_2 such that

$$|x_n - a| < \frac{\varepsilon}{2} \text{ for all } n > n_1$$
$$|x_n - b| < \frac{\varepsilon}{2} \text{ for all } n > n_2$$

Now let $n_0 = \max\{n_1, n_2\}$. Then

$$|b-a| = |b-x_n+x_n-a| \leq |x_n-b| + |x_n-a| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

whenever $n > n_0$. Since ε is arbitrarily small, $|b-a| = 0$. Hence, $a = b$. ■

Definition 3.4. A sequence is **upper bounded** if there exists $b \in \mathbf{R}$ such that $x_n \leq b$ for all $n \in \mathbf{N}$.

A sequence is **lower bounded** if there exists $a \in \mathbf{R}$ such that $x_n \geq a$ for all $n \in \mathbf{N}$.

If a sequence is upper bounded and lower bounded, then the sequence is said to be **bounded**. In this case, there exists an $M \geq 0$ such that $|x_n| \leq M$, i.e., $x_n \in (-M, M)$ for all $n \in \mathbf{N}$.

Proposition 3.2. Every convergent sequence is bounded.

Proof. Let $a = \lim x_n$ and take $\varepsilon = 1$. Hence, there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$ we have $x_n \in (a-1, a+1)$.

Therefore, our task is to 'control' the following finite set:

$$I = \{x_1, x_2, \ldots, x_{n_0}\} \cup \{a-1\} \cup \{a+1\}$$

Let b the smallest value of I and c the biggest value of I. Then every term of (x_n) is in the interval $[b, c]$. ■

The contrapositive of this proposition consists in an useful criteria for the study of the convergence of sequences.

Example 3.1. The sequence $1, 2, 3, 4, \ldots$ is not bounded. Hence, it is not convergent.

Example 3.2. The sequence $1, -1, 1, -1, \ldots$ is bounded. However, it is not convergent.

Definition 3.5. A sequence (x_n) is **non-decreasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$. And (x_n) is **non-increasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbf{N}$.

If the sequence is non-decreasing or non-increasing, then it is said to be **monotone**.

Proposition 3.3. If a sequence is non-decreasing and upper bounded, then it is convergent.

Proof. Let (x_n) be a non-decreasing sequence and upper bounded, and let $b = \sup\{x_n : n \in \mathbf{N}\}$, which is valid by the Axiom of Completeness. We're going to show that $x_n \longrightarrow b$.

Given $\varepsilon > 0$, since b is the least upper bound, there is $n_0 \in \mathbf{N}$ such that $b - \varepsilon < x_{n_0}$.

Since the sequence is non-decreasing,

$$b - \varepsilon < x_{n_0} \leq x_n, \; \forall n > n_0$$

Hence,

$$b - \varepsilon < x_n < b + \varepsilon$$

Which means that $|x_n - b| < \varepsilon$, for all $n > n_0$. I.e., $x_n \longrightarrow b$. ■

Analogously, if a sequence is non-increasing and lower bounded, then it is convergent.

Another way of stating the fact above is that every bounded monotone sequence is convergent.

The following proposition states that if a sequence has a positive limit, then, after a finite number of terms, all of its terms will be positive.

Proposition 3.4. If $\lim x_n = a > 0$, then there exists $n_0 \in \mathbf{N}$ such that $x_n > 0$ whenever $n > n_0$.

Proof. Let $\varepsilon = a/2 > 0$. Then $(a - \varepsilon, a + \varepsilon) = (a/2, 3a/2)$. There exists $n_0 \in \mathbf{N}$ such that if $n > n_0$, then $x_n \in (a/2, 3a/2)$, i.e., $x_n > a/2$. Hence, $x_n > 0$ if $n > n_0$. ■

Corollary 3.5. Let (x_n) and (y_n) be convergent sequences. If $x_n \leq y_n$ for all $n \in \mathbf{N}$, then $\lim x_n \leq \lim y_n$.

Proof. If $\lim x_n > \lim y_n$, we would have

$$0 < \lim x_n - \lim y_n = \lim(x_n - y_n)$$

which implies that $x_n - y_n > 0$ for every n sufficiently big. ■

Corollary 3.6. Let (x_n) be a convergent sequence. If $x_n \geq a$ for all n, then $\lim x_n \geq a$.

Lemma 3.7. If $\lim x_n = 0$ and (y_n) is bounded, then $\lim x_n y_n = 0$.

Proof. Since (y_n) is bounded, there exists $c > 0$, such that $|y_n| \leq c$, for all $n \in \mathbf{N}$. And since $\lim x_n = 0$, we can take $|x_n - 0| = |x_n|$ as small as we want.

Hence, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n| < \varepsilon/c$, for all $n > n_0$.

Therefore, for all $n > n_0$,

$$|x_n y_n - 0| = |x_n|\,|y_n| < \frac{\varepsilon}{c} \cdot c = \varepsilon$$

■

Example 3.3.

$$\lim \frac{(-1)^n}{n} = \lim \left((-1)^n \cdot \frac{1}{n} \right) = 0$$

Theorem 3.8 (Squeeze theorem)**.** Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$. If $\lim x_n = \lim z_n = \mathrm{L}$, then $\lim y_n = \mathrm{L}$.

Proof. Given $\varepsilon > 0$, there exists $n_1 \in \mathbf{N}$ such that $|x_n - \mathrm{L}| < \varepsilon$, for all $n > n_1$. Analogously, there exists $n_2 \in \mathbf{N}$ such that $|z_n - \mathrm{L}| < \varepsilon$, for all $n > n_2$.

Let $n_0 = \max\{n_1, n_2\}$. By hypothesis, $x_n \leq y_n \leq z_n$, for all $n \in \mathbf{N}$. Therefore, for $n > n_0$,

$$\mathrm{L} - \varepsilon < x_n \leq y_n \leq z_n < \mathrm{L} + \varepsilon$$

That means that $|y_n - \mathrm{L}| < \varepsilon$ for all $n > n_0$. ■

Example 3.4. Compute $\lim \frac{\sin n}{n}$.

Since $-1 \leq \sin n \leq n$, we have

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

for all $n \in \mathbf{N}$.

But $\lim \frac{-1}{n} = \lim \frac{1}{n} = 0$. Therefore, by the Squeeze theorem,

$$\lim \frac{\sin n}{n} = 0$$

3.2 Subsequences

Definition 3.6 (Subsequence)**.** Let (x_n) be a sequence. A **subsequence** of (x_n) is a sequence of the form $(x_{n_j})_{j=1}^{\infty}$, where n_j is a strictly increasing sequence in $\mathbf{N}$.

Proposition 3.9. If $\lim x_n = x$ then every subsequence of (x_n) converges to x.

Proof. Consider $(x_{n_j})_{j=1}^{\infty}$ a subsequence of (x_n). By hypothesis, $x_n \longrightarrow x$. Therefore, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n - x| < \varepsilon$, for all $n > n_0$.

Since the sequence of the index n_j is strictly increasing, there is $j_0 \in \mathbf{N}$ such that $n_{j_0} > n_0$. Hence, for all $j > j_0$,

$$|x_{n_j} - x| < \varepsilon$$

■

The contrapositive of this proposition is also an useful criteria for studying convergence.

Example 3.5. The sequence $x_n = (-1)^n$ does not converge, since $x_{2n} \longrightarrow 1$ and $x_{2n-1} \longrightarrow -1$.

Before proving the Bolzano-Weierstrass theorem, which is one of the most important results in Real Analysis, we will prove the following result.

Theorem 3.10 (Nested intervals). Let $([a_n, b_n])_{n=1}^{\infty}$ be a non-increasing sequence of bounded and closed intervals, i.e.,

$$[a_1, b_1] \supset [a_2, b_2] \supset \dots$$

Then, there exists $c \in \mathbf{R}$ which belongs to all intervals $[a_n, b_n]$.

Proof. Since the sequence of intervals is non-increasing, we have

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots \leq b_n \leq \dots \leq b_2 \leq b_1$$

Consider the set $\mathrm{A} = \{a_n : n \in \mathbf{N}\}$. Since A is upper bounded and non-empty, let $c = \sup \mathrm{A}$. Then $a_n \leq c \leq b_n$ for all $n \in \mathbf{N}$. Which means that $c \in [a_n, b_n]$ for all $n \in \mathbf{N}$. ■

Theorem 3.11 (Bolzano-Weierstrass). Every bounded sequence has a convergent subsequence.

Proof. Let (a_n) be a bounded sequence. Then there exists $\mathrm{M} > 0$ such that $|a_n| \leq \mathrm{M}$, for all $n \in \mathbf{N}$.

Divide the interval $[-\mathrm{M}, \mathrm{M}]$ into the halves $[-\mathrm{M}, 0]$ and $[0, \mathrm{M}]$. Notice that at least one of the intervals contains infinite terms of the sequence (a_n). Choose the half that contains infinite terms and denote it by I_1. Let a_{n_1} a point of the sequence (a_n) such that $a_{n_1} \in I_1$.

Divide I_1 into two closed intervals with the same length and let I_2 be the half that contains infinite terms. Choose a_{n_2} from the original sequence with $n_2 > n_1$.

In the general case, define I_k by dividing I_{k-1} into two halves and take I_k to be the half with infinite terms of (a_n). Then select $n_k > n_{k-1} > \ldots > n_2 > n_1$ such that $a_{n_k} \in I_k$.

Now let us show that the subsequence (a_{n_k}) is convergent. Notice that $I_1 \supset I_2 \supset \ldots$ is a sequence of closed and nested intervals. By the nested intervals theorem, there exists $x \in \mathbf{R}$ which belong to I_k, for all $k \in \mathbf{N}$.

Affirmation: (a_{n_k}) converges to x.

Let $\varepsilon > 0$. By construction, the length of I_k is equal to $M\left(\frac{1}{2}\right)^k$. Using the Lemma 3.7,

$$\lim_{k\to\infty} M\left(\frac{1}{2}\right)^k = 0$$

Hence, there exists $n_0 \in \mathbf{N}$ such that for all $k > n_0$ the length of I_k is lesser than ε. Since x and a_{n_k} are in I_k, we have that the distance $|a_{n_k} - x| < \varepsilon$ for all $k > n_0$. ■

We now state some important properties of limits, which will ease their computations.

Theorem 3.12. If $\lim a_n = a$ and $\lim b_n = b$, then:

1. $\lim(c \cdot a_n) = c \cdot a$, $\forall c \in \mathbf{R}$.
2. $\lim(a_n + b_n) = a + b$.
3. $\lim a_n b_n = ab$.
4. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ if $b \neq 0$.

Proposition 3.13. A monotone sequence is bounded iff. it has a bounded subsequence.

Proof. Consider $x_{n_j} \leq b$ a bounded subsequence of the non-decreasing sequence (x_n). Then, for all $n \in \mathbf{N}$, there exists $n_k > n$, hence, $x_n \leq x_{n_k} \leq b$. Hence, $x_n \leq b$ for all n. ■

3.3 Cauchy Sequences

Definition 3.7. A sequence (a_n) is said to be a **Cauchy sequence** if for all $\varepsilon > 0$, there is $n_0 \in \mathbf{N}$ such that

$$|a_n - a_m| < \varepsilon$$

for all $n, m > n_0$. Intuitively, this means that after a certain point, every pair of terms are close to each other.

With that definition, our goal is to show that a real sequence is convergent iff. it is a Cauchy sequence. This is known as the **Cauchy criterion**.

Lemma 3.14. Every Cauchy sequence is bounded.

Proof. Let $\varepsilon = 1$. Then, there exists n_0 such that $|x_n - x_m| < 1$ for all $n, m > n_0$. Notice that

$$|x_n| = |x_n - x_{n_0+1} + x_{n_0+1}| \leq |x_n - x_{n_0+1}| + |x_{n_0+1}| < 1 + |x_{n_0+1}|$$

for all $n > n_0$.

Let $c = \max\{|x_1|, |x_2|, \ldots, |x_{n_0}|, 1 + |x_{n_0+1}|\}$. Then $|x_n| \leq c$ for all $n \in \mathbf{N}$. ■

Theorem 3.15. A sequence is convergent if, and only if, it is a Cauchy sequence.

Proof. ($\Rightarrow$) Let (x_n) convergent and $\lim x_n = x$. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|x_n - x| < \varepsilon/2$ for all $n > n_0$. Therefore,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall n, m > n_0$$

($\Leftarrow$) Suppose that (x_n) is a Cauchy sequence. Then (x_n) is bounded. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}). Let $x = \lim x_{n_k}$. We're going to show that the original sequence also converges to x.

Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there is $n_0 \in \mathbf{N}$, such that $|x_n - x_m| < \varepsilon/2$ for all $n, m > n_0$.

Now, since $(x_{n_k}) \longrightarrow x$, there exists $n_k > n_0$ such that $|x_{n_k} - x| < \varepsilon/2$ for all $n > n_k$.

Hence, for all $n > n_k$,

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

With Cauchy sequences, Georg Cantor gave another construction of the real numbers $\mathbf{R}$.

Definition 3.8 (ε-close Sequences)**.** Consider two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$, and let $\varepsilon > 0$. These two sequences are said to be **ε-close** iff. a_n is ε-close to the sequence b_n, for each $n \in \mathbf{N}$. I.e., $|a_n - b_n| \leq \varepsilon$ for all $n \in \mathbf{N}$.

If there exists an $\mathrm{N} \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq \mathbf{N}$, then the two sequences are said **eventually ε-close sequences**.

Definition 3.9 (Equivalent Sequences)**.** Two sequences are **equivalent** iff. for each rational $\varepsilon > 0$, the sequences are eventually ε-close.

Definition 3.10 (Real numbers)**.** A **real number** is an object of the form

$$\lim_{n\to\infty} a_n$$

where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ are said to be equal iff. $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences.

3.4 Upper and Lower Limits

Definition 3.11. Let (x_n) be a sequence of real numbers.

- We say that (x_n) goes to **infinity** if, for every $\mathrm{N} > 0$, there exists $n_0 \in \mathbf{N}$ such that $x_n > \mathrm{N}$ for all $n > n_0$. We write $x_n \longrightarrow \infty$ or $\lim x_n = \infty$.
- We say that (x_n) goes to **minus infinity** if, given $\mathrm{N} > 0$, there exists $n_0 \in \mathbf{N}$ such that $x_n < -\mathrm{N}$ for all $n > n_0$. We write $x_n \longrightarrow -\infty$ or $\lim x_n = -\infty$.

Definition 3.12. Let (x_n) be a sequence of real numbers. Suppose that (x_n) is upper bounded, i.e., $x_n \leq b$ for all $n \in \mathbf{N}$.

Define

$$b_n := \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} = \sup_{k\geq n}\{x_k\}$$

Notice that (b_n) is non-increasing. If (b_n) is lower bounded, then (b_n) is convergent. If (b_n) is not lower bounded, then $b_n \longrightarrow -\infty$.

In both cases, we define

$$\limsup_{n\to\infty} x_n = \lim_n b_n$$

If (x_n) is not upper bounded, then we define

$$\limsup_{n\to\infty} x_n = \infty$$

Definition 3.13. Let (x_n) be a sequence of real numbers. Suppose that (x_n) is lower bounded, i.e., $x_n \geq a$ for all $n \in \mathbf{N}$.

Define

$$a_n := \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\} = \inf_{k\geq n}\{x_k\}$$

Notice that (a_n) is non-decreasing. If (a_n) is upper bounded, then (a_n) is convergent. If (a_n) is not upper bounded, then $a_n \longrightarrow \infty$.

In both cases, we define

$$\limsup_{n\to\infty} x_n = \lim_n a_n$$

If (x_n) is not lower bounded, then we define

$$\liminf_{n\to\infty} x_n = -\infty$$

Example 3.6. Consider $x_n = (-1)^n$, $n \in \mathbf{N}$. Then $b_1 = 1, b_2 = 1, \ldots, b_n = 1$. Hence, $\limsup x_n = \lim b_n = 1$.

And $a_n = -1$, for all $n \in \mathbf{N}$. Hence, $\liminf x_n = \lim a_n = -1$.

Example 3.7. Let $x_n = \frac{(-1)^n}{n}$. Then $\limsup x_n = \lim b_n = 0$, and $\liminf x_n = \lim a_n = 0$.

Example 3.8. Let $x_n = (-1)^n n$. Then $\limsup x_n = \infty$, and $\liminf x_n = -\infty$.

Remark 3.16. $a_n \leq b_n$ and therefore $\liminf x_n \leq \limsup x_n$.

Theorem 3.17. Let (x_n) be a sequence. Then

$$\lim x_n = L \iff \limsup x_n = \liminf x_n = L$$

Proof. Suppose that $L \in \mathbf{R}$.

($\Rightarrow$) Suppose that $\lim x_n = L$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $L - \varepsilon < x_n < L + \varepsilon$, for all $n > n_0$.

Notice that

$$b_n = \sup_{k \geq n} \{x_k\} \leq L + \varepsilon, \; \forall n > n_0$$

On the other hand,

$$a_n = \inf_{k \geq n} \{x_k\} \geq L - \varepsilon, \; \forall n > n_0$$

Then,

$$L - \varepsilon \leq a_n \leq b_n \leq L + \varepsilon, \; \forall n > n_0$$

Hence,

$$\limsup x_n = \liminf x_n = L$$

($\Leftarrow$) By hypothesis, $\lim a_n = \lim b_n = L$. Let $\varepsilon > 0$. Since (b_n) is non-increasing and converges to L, there exists $n_1 \in \mathbf{N}$ such that

$$\sup_{k \geq n_1} \{x_k\} = b_{n_1} < L + \varepsilon$$

Analogously, (a_n) is non-decreasing and converges to L. Then there exists $n_2 \in \mathbf{N}$ such that

$$\inf_{k \geq n_2} \{x_k\} = a_{n_2} > L - \varepsilon$$

Let $n_0 = \max\{n_1, n_2\}$. Then $L - \varepsilon < x_n < L + \varepsilon$, for all $n > n_0$.

Therefore, $x_n \longrightarrow L$.

If $\lim x_n = \infty$: **Exercise.** ■

Theorem 3.18. Let (x_n) a bounded sequence of real numbers. And let $a = \liminf x_n$ and $b = \limsup x_n$.

Then there exists a subsequence $(x_{n_k}) \longrightarrow a$ and a subsequence $(x_{n_j}) \longrightarrow b$.

Proof. **Exercise.** ■

3.5 Infinite Series

Definition 3.14. Let (a_n) be a sequence. An **infinite series** is defined as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The sequence (s_m), defined as

$$s_m = a_1 + a_2 + \dots + a_m$$

is called the **sequence of partial sums**.

And the series $\sum_{n=1}^{\infty}$ **converges** to A if the sequence (s_m) converges to A, and we write $\sum_{n=1}^{\infty} = \mathrm{A}$.

Example 3.9 (Harmonic Series)**.** Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Since the sequence of partial sums

$$s_m = 1 + \frac{1}{2} + \dots + \frac{1}{m}$$

is increasing and unbounded, this series does not converge.

Theorem 3.19. If $\sum_{n=1}^{\infty} a_n = \mathrm{A}$ and $\sum_{n=1}^{\infty} b_n = \mathrm{B}$, then

1. $\sum_{n=1}^{\infty} ca_n = c\mathrm{A}$ for all $c \in \mathbf{R}$.
2. $\sum_{n=1}^{\infty} (a_n + b_n) = \mathrm{A} + \mathrm{B}$.

Theorem 3.20 (Cauchy Criterion for Series)**.** The series $\sum_{n=1}^{\infty} a_n$ converges if, and only if, for every $\varepsilon > 0$, there is $\mathrm{N} \in \mathbf{N}$ such that whenever $n > m > \mathrm{N}$ it follows that

$$|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \dots + a_n| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$$

Then, using Cauchy criterion for sequences, the result follows immediately. ∎

Theorem 3.21. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim(a_n) = 0$.

Proof. Take $n = m + 1$ in the previous theorem. ■

Notice that the converse of this theorem is not valid. Consider, for example, the harmonic series.

Theorem 3.22 (Comparison Test)**.** Suppose that (a_k) and (b_k) are sequences such that $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Let $n > m$. Then $|s_n^a - s_m^a| = |a_{m+1} + \ldots + a_n|$ and $|s_n^b - s_m^b| = |b_{m+1} + \ldots + b_n$. And we have that

$$|a_{m+1} + \ldots + a_n| \leq |b_{m+1} + \ldots + b_n|$$

Now, if $\sum_{k=1}^{\infty} b_k$ converges, then given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $n > m \geq n_0$, we have $|b_{m+1}+\ldots+b_n| < \varepsilon$. And, therefore, $|a_{m+1}+\ldots+a_n| < \varepsilon$ and $\sum_{k=1}^{\infty} a_k$ converges. ■

Example 3.10 (Geometric Series)**.** A series is called **geometric** if it is of the form

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \ldots$$

If $x = 1$ and $a \neq 0$, then the series diverges. For $x \neq 1$, since

$$(1-x)(1+x+x^2+x^3+\ldots+x^{m-1}) = 1-x^m$$

we can write

$$s_m = a + ax + ax^2 + \ldots + ax^{m-1} = \frac{a(1-x^m)}{1-x}$$

Hence,

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

if, and only if, $|x| < 1$.

Theorem 3.23 (Cauchy Condensation Test)**.** Suppose that (b_n) is decreasing and satisfies $b_n > 0$ for all $n \in \mathbf{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_16 + \dots$$

converges.

Example 3.11. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if, and only if, $p > 1$.

If $p \le 0$, then $\lim \frac{1}{n^p} \neq 0$. Therefore, the series diverges.

If $p > 0$, using Cauchy Condensation Test, we can evaluate

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}$$

Taking $x = 2^{1-p}$ into the geometric series, the series will converge if and only if $1 - p < 0$, as we wanted.

Theorem 3.24. If $\sum_{k=1}^{\infty} a_k = \mathrm{A}$ and $\sum_{k=1}^{\infty} b_k = \mathrm{B}$, then

(i) $\sum_{k=1}^{\infty} ca_k = c\mathrm{A}$ for all $c \in \mathbf{R}$.

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = \mathrm{A} + \mathrm{B}$.

Proof. (i) Let $t_m = ca_1 + ca_2 + \dots + ca_m = c(a_1 + \dots + a_m) = cs_m$. Then

$$\lim t_m = \lim cs_m = c \lim s_m = c\mathrm{A}$$

(ii) **Exercise.** ■

Theorem 3.25 (Absolute Convergence Test)**.** If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

Proof. Since $\sum_{k=1}^{\infty} |a_k|$ converges, by the Cauchy Criterion for Series, given an $\varepsilon > 0$, there exists an $n_0 \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$$

for all $n > m \ge n_0$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \dots + a_n| \le |a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$$

By the sufficiency of the Cauchy Criterion, $\sum_{k=1}^{\infty} a_k$ also converges. ■

Notice that the converse is false. Consider the **alternating harmonic series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Theorem 3.26 (Alternating Series Test)**.** Let (x_k) be a sequence such that

(i) $x_1 \geq x_2 \geq \ldots \geq 0$.

(ii) $\lim x_k = 0$.

Then, the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

Proof. First, we're going to consider the partial sums of odd indexes. Notice that

$$s_{2n+1} = x_1 - x_2 + x_3 - x_4 + \ldots + x_{2n-1} - x_{2n} + x_{2n+1} = s_{2n-1} - (x_{2n} - x_{2n-1}) \leq s_{2n-1}$$

I.e., s_{2n-1} is non-increasing. On the other hand,

$$s_{2n-1} = (x_1 - x_2) + \ldots + (x_{2n-3} - x_{2n-2}) + x_{2n-1}$$

Which means that $s_{2n-1} \geq 0$. Hence, (s_{2n-1}) converges.

Now, we're going to consider the partial sums of even indexes. Notice that

$$s_{2n+2} = x_1 - x_2 + x_3 - x_4 + \ldots + x_{2n-1} - x_{2n} + x_{2n+1} - x_{2n+2} = s_2 n + (x_{2n+1} - x_{2n+2}) gleq s_{2n}$$

Which means that s_{2n} is non-decreasing. On the other hand,

$$s_{2n} = x_1 - (x_2 - x_3) - \ldots - (x_{2n-2} - x_{2n-1}) - x_{2n}$$

I.e., $s_{2n} \leq x_1$, for all $n \in \mathbf{N}$. Hence, (s_{2n}) converges.

Let $\mathrm{L} = \lim s_{2n-1}$ and $\mathrm{M} = \lim s_{2n}$. Then,

$$\lim(s_{2n} - s_{2n-1}) = \lim(-x_n) = 0$$

and, therefore, $\mathrm{M} - \mathrm{L} = 0$. Hence, $\lim s_{2n} = \lim s_{2n-1} = \mathrm{L}$.

Given $\varepsilon > 0$, there exists $n_1 \in \mathbf{N}$ such that $|s_{2n-1} - \mathrm{L}| < \varepsilon$ for all $n \geq n_1$. And there exists $n_2 \in \mathbf{N}$ such that $|s_{2n} - \mathrm{L}| < \varepsilon$ for all $n \geq n_2$.

Fix $k_1 = 2n_1 + 1$ and $k_2 = 2n_2$. Let $k_0 = \max\{k_1, k_2\}$. Then,

$$|s_k - \mathrm{L}| < \varepsilon$$

for all $k > k_0$.

Hence, $\lim s_k = \mathrm{L}$ and the series converges. ∎

Definition 3.15. If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say that $\sum_{k=1}^{\infty} a_k$ **converges absolutely**. If $\sum_{k=1}^{\infty} a_k$ converges but the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ does not converge, then we say that $\sum_{k=1}^{\infty} a_k$ **converges conditionally**.

Lemma 3.27. Let $\sum b_n$ be an absolutely convergent series with $b_n \neq 0$ for all $n \in \mathbf{N}$ If the sequence $\left(\frac{a_n}{b_n}\right)$ is bounded, then $\sum a_n$ converges absolutely.

Proof. Suppose that $\left(\frac{a_n}{b_n}\right)$ is bounded, i.e., there exists $c \in \mathbf{R}$ such that for all $n \in \mathbf{N}$,

$$\left|\frac{a_n}{b_n}\right| \leq c$$

Hence, $|a_n| \leq c|b_n|$ for all $n \in \mathbf{N}$. By the Comparison Test, $\sum |a_n|$ converges. Therefore, $\sum a_n$ converges absolutely. ■

Example 3.12. Does

$$\sum \frac{1}{n^2 - 3n + 1}$$

converges?

Consider $\sum \frac{1}{n^2}$, which is absolutely convergent. And notice that

$$\frac{\frac{1}{n^2-3n+1}}{\frac{1}{n^2}} = \frac{n^2}{n^2 - 3n + 1}$$

is bounded by 1. Therefore, the series converges.

Theorem 3.28 (Ratio Test (D'Alambert))**.** Let $a_n \neq 0$ for all $n \in \mathbf{N}$. If there exists $c \in \mathbf{R}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq c < 1$$

for sufficiently large n, then the series $\sum a_n$ converges absolutely.

Proof. Suppose $0 < c < 1$ such that for n sufficiently large, we have

$$\left|\frac{a_{n+1}}{a_n}\right| \leq c = \frac{c^{n+1}}{c^n} \iff \frac{|a_{n+1}|}{c^{n+1}} \leq \frac{|a_n|}{c^n}$$

Hence, the sequence of non-negative numbers $\frac{|a_n|}{c^n}$ is non-increasing for n sufficiently large and, therefore, bounded.

Since the series $\sum c^n$ converges absolutely (geometric series), by the previous lemma, $\sum a_n$ converges absolutely. ■

Remark 3.29. In general, we compute

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \mathrm{L}$$

- If $\mathrm{L} < 1$, the series converges.
- If $\mathrm{L} > 1$, the series diverges. In this case, $|a_{n+1}| > |a_n|$ and hence the general term does not converges to zero.
- If $\mathrm{L} = 1$, the test implies nothing about the convergence of the series. Consider, for example, $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$. Both limits are 1, but only the first series converges.

Example 3.13. Does the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges?

By the ratio test,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{(n+1)^{(n+1)}} \frac{n^n}{n!} = \frac{(n+1)}{(n+1)^{(n+1)}} n^n = (n+1)^{-n} n^n \\ &= \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \underset{n\to\infty}{\longrightarrow} \frac{1}{e} < 1\end{aligned}$$

Hence, the series converges.