

# How to Solve Ordinary Differential Equations

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# 1 First Order Linear

## 1.1 Integrating Factor

Applies to  $y'(t) + p(t)y(t) = q(t)$ .

First step: Let  $\mu(t) = e^{\int p(t)dt}$  be the integrating factor.

Second step: Multiply the ODE by this factor and write  $[\mu(t)y]' = q_1(t)$ .

Third step: Integrate both sides to obtain  $y(t)$ .

## 1.2 Separate Equations

Applies to

$$g(y)\frac{dy}{dt} = f(t)$$

First step: Isolate the factors of  $y$  and  $t$  to obtain the format above.

Second step: Integrate both sides to obtain  $y(t)$ .

$$\int g(y)dy = \int f(t)dt$$

# 2 First Order Non-Linear

## 2.1 Separate Equations

Same as **above**.

## 2.2 Substitution Method

There are two main cases here. The first is when the DE does not depend on  $y$ , like  $ay'(t) = f(t)$ . Then we set  $v(t) = y'(t)$ , thus obtaining  $av(t) = f(t)$ .

The second case is when the DE does not depend on  $t$ . We set  $v(y(t)) = y'(t)$ .

We'll see more details and applications on [second order DEs](#).

## 2.3 Homogenous Equations

Applies to

$$y' = f\left(\frac{y}{t}\right)$$

First step: let  $v(t) = \frac{y(t)}{t}$ . Then  $tv(t) = y(t)$ .

Second step: differentiate to obtain  $v(t) + tv'(t) = y'(t)$ .

Third step: let  $y'(t) = f(v)$ . Then use the method of separate equations.

## 2.4 Bernoulli Equations

Applies to

$$y' + p(t)y = f(t)y^n$$

Notice that by dividing the equation by  $y^n$  we obtain

$$y^{-n}y' + p(t)yy^{-n} = f(t)$$

It's natural then to substitute  $v = y^{1-n}$ . Which gives us  $v' = (1 - n)y^{-n}y' \iff \frac{1}{(1-n)}v' = y^{-n}y'$ .

Just multiplying by  $(1 - n)$  we get  $v' + (1 - n)p(t)v = (1 - n)f(t)$  which is linear.

## 2.5 Ricatti Equations

Applies to

$$y' + p(t)y + q(t)y^2 = f(t)$$

given a solution  $y_1(t)$ .

First step: let  $y_2(t) = z + y_1(t)$  where  $z = \frac{1}{v}$ .

Then compute  $\frac{dy_2(t)}{dt}$  and substitute in the equation.

## 2.6 Exact Equations

Applies to

$$M(x, y)dx + N(x, y)dy = 0$$

where  $M(x, y) = \frac{\partial F}{\partial x}$  and  $N(x, y) = \frac{\partial F}{\partial y}$ .

Note that it only works if  $F$  is a conservative field. Then, the first step is to check if  $M_y = N_x$ , which is called exactness criterion. If the equality holds, then we must find  $F(x, y) = C$ .

So we integrate  $M$  or  $N$  in respect to  $dx$  or  $dy$  respectively, find the  $g(y)$  or  $g(x)$  to write  $F(x, y) = C$ .

If the equation is not exact, we may find an integrating factor  $\mu$  that only depends on  $x$  or  $y$ , which gives us the following criteria:

$$\frac{N_x - M_y}{M} \text{ depends only on } y \implies \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

$$\frac{M_y - N_x}{N} \text{ depends only on } x \implies \mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

### 3 Higher Order Linear, Homogeneous, Constant Coefficients

#### 3.1 Abel's Formula

$$W(x, y) = W_0 e^{-\int p(t) dt}$$

#### 3.2 General Case

We take a possible solution  $y(x) = e^{rx}$  and substitute into the equation. Then  $e^{rx}$  can be factored out giving us a polynomial equation.

The first case, when we obtain  $n$  distinct roots, is immediate:  $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ .

If there are repeated roots, then the second solution (assuming  $n = 2$ ) will be  $y = c_1 e^{r_1 x} + c_2 x e^{r_2 x}$ .

If the roots are complex, then we write  $y_1 = e^{(a+bi)x}$ ,  $y_2 = e^{(a-bi)x}$ . Using Euler's Formula and the fact that the real and imaginary parts are solutions of the DE, we have  $y = e^{at}(\cos bx + \sin bx)$ .

#### 3.3 Reduction of Order

There are three main cases. The first two we already saw in First Order DEs.

The first is when the DE does not depend on  $y$ , like  $ay'(t) = f(t)$ . Then we set  $v(t) = y'(t)$  and  $v'(t) = y''(t)$ .

The second case is when the DE does not depend on  $t$ . We set  $v(y(t)) = y'(t)$ . Taking the derivative,  $v'(y(t))y'(t) = y''(t) \iff v'(y)v(y) = y''$ .

The third case is when we know a solution  $y_1(t)$ . Then we take as a candidate a  $v(t)y_1(t) = y_2(t)$ . We then differentiate twice and replace it in the equation setting  $v' = w$ .

## 4 Higher Order Linear, Non-Homogeneous, Constant Coefficients

### 4.1 Undetermined Coefficients

We know that the solution is the sum of the complementary solution (of the corresponding homogeneous function) with the particular solution. The idea is to write the particular solution  $y_p$  as something similar to the right hand side.

$f(x)$	$y_p$
$ae^{rx}$	$x^s Ae^{rx}$
$a_n x^n + \dots + a_1 x + a_0$	$x^s (A_n x^n + \dots + A_0)$
$a \cos x + b \sin x$	$x^s (A \cos x + B \sin x)$
$e^{rx} P_n(x)$	$x^s e^{rx} (A_n x^n + \dots + A_0)$
$e^{rx} (a \cos kx + b \sin kx)$	$x^s (Ae^{rx} \cos kx + Be^{rx} \sin kx)$
$P_n(x)(a \cos kx + b \sin kx)$	$x^s [(A_n x^n + \dots + A_0) \cos kx + (B_n x^n + \dots + B_0) \sin kx]$

where we choose  $s$  to be the least integer that eliminates repetition in the solution.

If the right hand side cannot be written as given above, then we should use the method of variation of parameters below.

## 5 Higher Order Linear, Homogeneous, Variable Coefficients

## 5.1 Euler-Cauchy Equation

Applies to

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y^{(1)} + a_0 y = 0$$

Here we have an equidimensional equation, where the degree of the  $n$ th power is equal to the order of the derivative. A possible solution is a power of  $x$ . Thus  $y(x) = x^r$ .

We then differentiate  $y$  until the desired order and replace it in the equation. By factoring  $x^r$  out of the equation, we obtain a characteristic equation.

If we have repeated roots, by reduction of order, we obtain that the next solution is  $C \ln x$  times the given solution. Then,  $y = c_1 x^r + c_2 x^r \ln x$  (for  $n = 2$ ).

## 5.2 Variation of Parameters

Applies to

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y^{(1)} + p_0(x)y = f(x)$$

Given that the coefficients and the right hand side  $f(x)$  are continuous in the interval  $I$  (due to the Existence and Uniqueness Theorem), the first step is to solve the complementary equation.

The idea is to replace the constants with functions  $u_1(x), \dots, u_n(x)$ .

Then we compute the derivatives using that, since we have one degree of freedom, we can assume that  $u'_1 y_1 + u'_2 y_2 = 0$ .

We can now replace the particular solution into the equation and we should solve



$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{cases}$$

Notice that the determinant of the coefficients is the wronskian  $W(y_1, y_2)$ .

By Cranmer's Rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} \text{ and } u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W}$$

Integrating, we obtain  $u_1$  and  $u_2$ .

## 6 Series Solutions of Second Order Equations

We will work here with equations of the form

$$y'' + P(x)y' + Q(x)y = 0$$

If  $P(x)$  and  $Q(x)$  are analytic in  $x = a$ , then  $x = a$  is said ordinary point. Otherwise, it is said singular point.

### 6.1 Ordinary Points

We're looking for solutions like

$$y(x) = \sum_{k=0}^{\infty} c_k x^k$$

Hence

$$y'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1}$$

and

$$y'(x) = \sum_{k=2}^{\infty} c_k k(k-1)x^{k-2}$$

We then replace those series in the equation, rearrange the terms and use the identity principle to find the recurrence relation. Using this relation, we find the coefficients  $c_k$  and we're ready to write our linearly independent solutions.

## 6.2 Regular Singular Points

If

$$\lim_{x \rightarrow a} (x-a)P(x)$$

and

$$\lim_{x \rightarrow a} (x-a)^2 Q(x)$$

are both finite, then  $x = a$  is said a regular singular point.

Remembering Euler-Cauchy equation, a possible solution is of the form

$$y(x) = x^r \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+r}$$

Which gives us

$$y'(x) = \sum_{k=0}^{\infty} c_k (k+r) x^{k+r-1}$$

and

$$y''(x) = \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-2}$$

Replacing those series in the equation and using the identity principle, we'll find an equation of the form

$$r(r-1) + p_0 r + q_0 = 0$$

where  $p_0 = P(0)$  and  $q_0 = Q(0)$ . This equation is called **indicial equation**.

Let  $r_1$  and  $r_2$  be the roots of the indicial equation and  $r_1 > r_2$ . If  $r_1 - r_2 \notin \mathbb{Z}^+ \cup \{0\}$ , then we can obtain two linearly independent solutions

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+r_1} \text{ and } y_2(x) = \sum_{k=0}^{\infty} b_k x^{k+r_2}$$