Problem 4.4. (*) Suppose we have a market with two investments of type (4.14) and (4.15). Find the initial fortune $V^\theta(0)$ and the number of units $\theta_1(t)$, which must be invested at time t in the risky investment to produce the terminal value $V^\theta(T) = F = W(T)$ when $\rho(t) = \rho > 0$ is constant and the price $S_1(t)$, $t \in [0, T]$, of the risky investment is given by:

(a) $dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t)$; μ, σ constants ($\sigma \neq 0$). This is the case of the geometric Brownian motion.
(b) $dS_1(t) = c dW(t)$; $c \neq 0$ constant.
(c) $dS_1(t) = \mu S_1(t)dt + c dW(t)$; μ, c constants. This is the case of the Ornstein–Uhlenbeck process. [*Hint.* $S_1(t) = e^{\mu t} S_1(0) + c \int_0^T e^{\mu(t-s)} dW(s)$.]

(a) A risk-less asset (e.g., a bond), with price dynamics

$$\begin{cases} dS_0(t) = \rho(t) S_0(t) dt, \\ S_0(0) = 1. \end{cases} \tag{4.14}$$

(b) A risky asset (e.g., a stock), with price dynamics

$$\begin{cases} dS_1(t) = \mu(t) S_1(t) dt + \sigma(t) S_1(t) dW(t) \\ S_1(0) > 0. \end{cases} \tag{4.15}$$

a) $dS_1(t) = \mu S_1(t) dt + \sigma S_1(t) dW(t)$

Recall that the solution to the SDE above is given by

$$S_1(t) = S_1(0) \exp\left(\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right) t\right)$$

(see Example 5.1.5 in the previous research notes).

By definition

$$V^\theta(t) = \theta_0(t) S_0(t) + \theta_1(t) S_1(t)$$

and, by self financing hypothesis,

(1) $$dV^\theta(t) = \theta_0(t) dS_0(t) + \theta_1(t) dS_1(t)$$

Using that

$$\theta_0(t) = \frac{V^\theta(t) - \theta_1(t) S_1(t)}{S_0(t)}$$

and 4.14 in (1) yields

(2) $$dV^\theta(t) = \rho\left(V^\theta(t) - \theta_1(t)S_1(t)\right)dt + \theta_1(t)\,dS_1(t)$$

Replacing $dS_1(t)$,

$$dV^\theta(t) = \left[(\mu-\rho)\theta_1(t)S_1(t) + \rho V^\theta(t)\right]dt + \sigma\theta_1(t)S_1(t)\,dw(t)$$

Let $u(t) = (\mu-\rho)/\sigma$. By Girsanov theorem,

$$dV^\theta(t) = \rho V^\theta(t)\,dt + \sigma\theta_1(t)S_1(t)\,d\tilde{w}(t)$$

where
$$\tilde{w}(t) = w(t) + \int_0^t u(s)\,ds$$

Defining
$$U^\theta(t) = e^{-\rho t}V^\theta(t)$$
we obtain
$$dU^\theta(t) = -\rho e^{-\rho t}V^\theta(t)\,dt + e^{-\rho t}\,dV^\theta(t)$$
$$= e^{-\rho t}\sigma\theta_1(t)S_1(t)\,d\tilde{w}(t)$$

Thus
$$e^{-\rho t}V^\theta(t) = V^\theta(0) + \int_0^t e^{-\rho s}\sigma\theta_1(s)S_1(s)\,d\tilde{w}(s)$$

We now apply Clark-Ocone to $G = e^{-\rho T}F$:

(3) $$G = E_Q[G] + \int_0^T E_Q\left[D_t G - G\int_t^T D_t u(s)\,d\tilde{w}(s) \,\middle|\, \mathcal{F}_t\right]d\tilde{w}(t)$$

Notice that

$$V^{\theta}(0) = \mathbb{E}_{Q}[G] = \mathbb{E}_{Q}\left[e^{-\rho T} W(T)\right]$$

$$= e^{-\rho T}\, \mathbb{E}_{Q}\left[\tilde{W}(T) - \int_0^T \upsilon(s)\,ds\right]$$

$$= e^{-\rho T}\, \mathbb{E}_{Q}\left[\tilde{W}(T) - \upsilon T\right] = \underline{-e^{-\rho T}(\mu-\rho)T/\sigma}$$

Since $\upsilon(t)$ is constant, $D_t \upsilon(s) = 0$ and, by the Chain Rule,

$$D_t G = D_t e^{-\rho T} W(T) = e^{-\rho T}$$

we have

$$\theta_1(t) = e^{\rho t} \sigma^{-1} S_1^{-1}(t)\, \mathbb{E}_{Q}\left[D_t G - G \int_t^T D_t \upsilon(s)\, d\tilde{W}(s) \,\middle|\, \mathcal{F}_t \right]$$

$$= e^{\rho t} \sigma^{-1} S_1^{-1}(t)\, \mathbb{E}_{Q}\left[e^{-\rho T} \mid \mathcal{F}_t \right] = \underline{e^{\rho(t-T)} \sigma^{-1} S_1^{-1}(t)}$$

b) $dS_1(t) = c\,dW(t),\ c \neq 0$

Using (2),

$$dV^\theta(t) = \rho\left(V^\theta(t) - \theta_1(t) S_1(t)\right)dt + c\,\theta_1(t)\,dW(t)$$

Define

$$u(t) = -\frac{\rho S_1(t)}{c}$$

By Girsanov,

$$dV^\theta(t) = \rho V^\theta(t)\,dt + c\,\theta_1(t)\,d\tilde{W}(t)$$

Again, we let

$$U^\theta(t) = e^{-\rho t} V^\theta(t)$$

and obtain

$$dU^\theta(t) = -\rho e^{-\rho t} V^\theta(t)\,dt + e^{-\rho t}\,dV^\theta(t)$$

$$= c e^{-\rho t}\theta_1(t)\,d\tilde{W}(t)$$

Thus

$$e^{-\rho t} V^\theta(t) = V^\theta(0) + \int_0^t c e^{-\rho s}\theta_1(s)\,d\tilde{W}(s)$$

By (3) we have

$$V^{\theta}(0) = \mathbb{E}_Q[G] = \mathbb{E}_Q[e^{-\rho T} W(T)]$$

$$= e^{-\rho T} \mathbb{E}_Q[\tilde{W}(T) - \int_0^T u(s)\,ds]$$

$$= e^{-\rho T} \mathbb{E}_Q\left[\tilde{W}(T) + \frac{\rho}{c}\int_0^T S_1(s)\,ds\right] \overset{?}{=} 0$$

Also,

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[D_t G - G\int_t^T D_t u(s)\, d\tilde{W}(s) \,\middle|\, \mathcal{F}_t\right]$$

Now, since $D_t G = e^{-\rho T}$ and

$$D_t - \frac{\rho S_1(s)}{c} = -\frac{\rho}{c} D_t S_1(s) = -\frac{\rho}{c} D_t\left[S_1(0) + \int_0^s c\, dW\right] = -\rho$$

we have

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[e^{-\rho T} + e^{-\rho T} W(T)\int_t^T \rho\, d\tilde{W}(s) \,\middle|\, \mathcal{F}_t\right]$$

$$= c^{-1} \mathbb{E}_Q\left[1 + \rho\, W(T)[\tilde{W}(T) - \tilde{W}(t)] \,\middle|\, \mathcal{F}_t\right] = \frac{1}{c}$$

c) $dS_1(t) = \mu S_1(t)\,dt + c\,dW(t)$

By (2),

$$dV^\theta(t) = \rho\left(V^\theta(t) - \theta_1(t)S_1(t)\right)dt + \theta_1(t)\,dS_1(t)$$

$$= \left[\rho V^\theta(t) + (\mu-\rho)\theta_1(t)S_1(t)\right]dt + c\theta_1(t)\,dW(t)$$

We apply Girsanov with

$$u(t) = \frac{(\mu-\rho)S_1(t)}{c}$$

and get

$$dV^\theta(t) = \rho V^\theta(t)\,dt + c\theta_1(t)\,d\tilde{W}(t)$$

Again, let $U^\theta(t) = e^{-\rho t}V^\theta(t)$. Then

$$dU^\theta(t) = -\rho e^{-\rho t}V^\theta(t)\,dt + e^{-\rho t}\,dV^\theta(t)$$

$$= ce^{-\rho t}\theta_1(t)\,d\tilde{W}(t)$$

Thus,

$$e^{-\rho t}V^\theta(t) = V^\theta(0) + \int_0^t ce^{-\rho s}\theta_1(s)\,d\tilde{W}(s)$$

Using (3),

$$V^\theta(0) = E_Q[G] = E_Q\left[e^{-\rho T}W(T)\right] = e^{-\rho T}E_Q\left[\tilde{W}(T) - \int_0^T u(s)\,ds\right]$$

$$= e^{-\rho T}E_Q\left[\tilde{W}(T) - \frac{(\mu-\rho)}{c}\int_0^T S_1(s)\,ds\right] \overset{?}{=} -e^{-\rho T}\frac{(\mu-\rho)}{c}e^{\mu T}S_1(0)$$

Also,

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[D_t G - G \int_t^T D_t u(s)\, d\tilde{w}(s) \,\middle|\, \mathcal{F}_t \right]$$

Now, since $D_t G = e^{-\rho T}$ and

$$D_t\left[\frac{(\mu-\rho) S_1(s)}{c}\right] = \frac{(\mu-\rho)}{c} D_t S_1(s) = (\mu-\rho) e^{\mu(s-t)} \chi_{[0,s]}(t)$$

we have

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[e^{-\rho T} - e^{-\rho T} w(T) \int_t^T (\mu-\rho) e^{\mu(s-t)}\, d\tilde{w}(s) \,\middle|\, \mathcal{F}_t \right]$$

$$= e^{\rho(t-T)} c^{-1} \left(1 - (\mu-\rho) \mathbb{E}_Q\left[w(T) \int_t^T e^{\mu(s-t)}\, d\tilde{w}(s) \,\middle|\, \mathcal{F}_t \right]\right)$$

Since

$$d\tilde{w}(t) = dw(t) + \frac{(\mu-\rho) S_1(t)}{c}\, dt$$

$$= dw(t) + \frac{(\mu-\rho)}{c}\left[e^{\mu t} S(0) + c \int_0^t e^{\mu(t-r)}\, dw(r)\right] dt$$

we have

$$e^{-\mu t} d\tilde{w}(t) = e^{-\mu t} dw(t) + \left[\frac{(\mu-\rho)}{c} S(0) + (\mu-\rho) \int_0^t e^{-\mu r}\, dw(r)\right] dt$$

Defining $X(t) = \int_0^t e^{-\mu r}\, dw(r), \quad \tilde{X}(t) = \int_0^t e^{-\mu r}\, d\tilde{w}(r)$

we rewrite the previous expression as

$$d\tilde{X}(t) = dX(t) + \frac{(\mu-\rho)}{c} S(0)\, dt + (\mu-\rho) X(t)\, dt$$

$$\Rightarrow d\left(e^{(\mu-\rho)t} X(t)\right) = e^{(\mu-\rho)t}\, d\tilde{X}(t) - \frac{(\mu-\rho)}{c} S(0)\, e^{(\mu-\rho)t}\, dt$$

$$\Rightarrow X(t) = e^{(\rho-\mu)t} \int_0^t e^{-\rho s}\, d\tilde{w}(s) - \frac{(\mu-\rho)}{c} S(0)\, e^{(\rho-\mu)t} \int_0^t e^{(\mu-\rho)s}\, ds$$

$$= e^{(\rho-\mu)t} \int_0^t e^{-\rho s}\, d\tilde{w}(s) - \frac{S(0)}{c}\left(1 - e^{(\rho-\mu)t}\right)$$

Thus,

$$e^{-\mu t}\, dw(t) = e^{(\rho-\mu)t} e^{-\rho t}\, d\tilde{w}(t) + (\rho-\mu) e^{(\rho-\mu)t} \int_0^t e^{-\rho s}\, d\tilde{w}(s)\, dt + \frac{S(0)}{c} (\rho-\mu)\, e^{(\rho-\mu)t}\, dt$$

and

$$dw(t) = d\tilde{w}(t) + (\rho-\mu) e^{\rho t} \int_0^t e^{-\rho s}\, d\tilde{w}(s)\, dt + \frac{S(0)}{c} (\rho-\mu)\, e^{\rho t}\, dt$$

$$\Rightarrow w(T) = \tilde{w}(T) + (\rho-\mu) \int_0^T e^{\rho s} \int_0^s e^{-\rho r}\, d\tilde{w}(r)\, ds + \frac{S(0)}{\rho c} (\rho-\mu)\left(e^{\rho T} - 1\right)$$

Replacing into $\theta_1(t)$,

$$\theta_1(t) = e^{\rho(t-T)} c^{-1} \Bigg(1 - (\mu-\rho) \mathbb{E}_Q \left[\tilde{\omega}(T) \int_t^T e^{\mu(s-t)} \, d\tilde{\omega}(s) \,\middle|\, \mathcal{F}_t \right]$$

$$+ (\mu-\rho)^2 \mathbb{E}_Q \left[\int_0^T e^{\rho s} \int_0^s e^{-\rho r} \, d\tilde{\omega}(r) \, ds \int_t^T e^{\mu(s-t)} \, d\tilde{\omega}(s) \,\middle|\, \mathcal{F}_t \right]$$

$$= e^{\rho(t-T)} c^{-1} \Bigg(1 - (\mu-\rho) \int_t^T e^{\mu(s-t)} \, ds$$

$$+ (\mu-\rho)^2 \int_0^T e^{\rho s} \mathbb{E}_Q \left[\int_0^s e^{-\rho r} \, d\tilde{\omega}(r) \int_t^T e^{\mu(r-t)} \, d\tilde{\omega}(r) \,\middle|\, \mathcal{F}_t \right] ds \Bigg)$$

$$= e^{\rho(t-T)} c^{-1} \left(1 - \frac{(\mu-\rho)}{\mu} \left(e^{\mu(T-t)} - 1 \right) + (\mu-\rho)^2 \int_t^T e^{\rho r} \int_t^s e^{-\rho r + \mu(r-t)} \, dr \, ds \right)$$

$$= e^{\rho(t-T)} c^{-1} \left(1 - \frac{(\mu-\rho)}{\rho} \left(e^{\rho(T-t)} - 1 \right) \right)$$