Problem 6.3. Verify that if $g \in L^2(\mathbb{R})$, $n \in \mathbb{N}$, then

$$W^n(T) \diamond \int_0^T g(t)\delta W(t) = W^n(T) \int_0^T g(t)\delta W(t) - nW^{n-1}(T) \int_0^T g(t)dt.$$

We need to compute $D_\gamma W^n(T)$.

$$D_\gamma W^n(T) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\langle \omega + \varepsilon\gamma, \chi_{[0,T]}\rangle^n - \langle \omega, \chi_{[0,T]}\rangle^n\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\left(\langle \omega, \chi_{[0,T]}\rangle + \varepsilon\langle \gamma, \chi_{[0,T]}\rangle\right)^n - \langle \omega, \chi_{[0,T]}\rangle^n\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\sum_{k=0}^{n} \binom{n}{k} \langle \omega, \chi_{[0,T]}\rangle^{n-k} \cdot \left(\varepsilon\langle \gamma, \chi_{[0,T]}\rangle\right)^k - \langle \omega, \chi_{[0,T]}\rangle^n\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[\langle \omega, \chi_{[0,T]}\rangle^n + n\langle \omega, \chi_{[0,T]}\rangle^{n-1} \varepsilon\langle \gamma, \chi_{[0,T]}\rangle + \cdots +$$

$$\varepsilon^n \langle \gamma, \chi_{[0,T]}\rangle^n - \langle \omega, \chi_{[0,T]}\rangle^n\Big]$$

$$= n\langle \omega, \chi_{[0,T]}\rangle^{n-1} \langle \gamma, \chi_{[0,T]}\rangle = nW^{n-1}(T) \int_0^T \gamma(t)\,dt$$

Thus, $D_t F = nW^{n-1}(T)$. Now, using (6.9),

$$W^n(T) \diamond \int_{\mathbb{R}} g(t)\,dW(t) = W^n(T) \int_{\mathbb{R}} g(t)\,dW(t) - nW^{n-1}(T) \int_{\mathbb{R}} g(t)\,dt$$