Problem 4.1. (*) Recall the *Girsanov theorem* (see, e.g., [178, Theorem 8.26]). Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form

$$dY(t) = \beta(t)dt + \gamma(t)dW(t), \qquad t \le T,$$

where $\beta(t) \in \mathbb{R}^n$, $\gamma(t) \in \mathbb{R}^{n\times m}$, and $t \in [0,T]$, are $\mathbb{F}$-adapted and $W(t)$, $t \in [0,T]$, is an m-dimensional Wiener process. Suppose there exist $\mathbb{F}$-adapted processes $u(t) \in \mathbb{R}^m$ and $\alpha(t) \in \mathbb{R}^n$, $t \in [0,T]$, such that

$$\gamma(t)u(t) = \beta(t) - \alpha(t)$$

and such that the Novikov condition

$$E\Big[\exp\Big\{\frac{1}{2}\int_0^T u^2(s)ds\Big\}\Big] < \infty$$

holds. Put

$$Z(t) = \exp\Big\{-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u^2(s)ds\Big\}, \qquad t \le T,$$

and define a measure Q on $\mathcal{F}_T$ by

$$dQ = Z(T)dP.$$

Then

$$\widetilde{W}(t) := \int_0^t u(s)ds + W(t), \qquad 0 \le t \le T$$

is a Wiener process with respect to Q, and in terms of $\widetilde{W}$ the process Y has the stochastic integral representation

$$dY(t) = \alpha(t)dt + \gamma(t)d\widetilde{W}(t).$$

(a) Show that $\widetilde{W}$ is an $\mathbb{F}$-martingale with respect to Q. [*Hint.* Apply Itô formula to $Y(t) := Z(t)\widetilde{W}(t)$.]

By the product rule,

(1) $$dY(t) = Z(t)\,d\tilde{W}(t) + \tilde{W}(t)\,dZ(t) + d[Z,\tilde{W}](t)$$

Using

Corollary 4.8. *Let Q and Z be as in (4.3) and (4.4) respectively. Suppose $G \in L^1(Q)$. Then*

$$E_Q[G|\mathcal{F}_t] = \frac{E[Z(T)G|\mathcal{F}_t]}{Z(t)}. \tag{4.10}$$

we have that

(2) $$E_Q[\tilde{W}(s)|\mathcal{F}_t] = Z^{-1}(t)\,\mathbb{E}[Z(T)\tilde{W}(s)|\mathcal{F}_t]$$

$$= Z^{-1}(t)\,\mathbb{E}\big[E[Z(T)\tilde{W}(s)|\mathcal{F}_s]\,|\mathcal{F}_t\big] = Z^{-1}(t)\,\mathbb{E}[\tilde{W}(s)Z(s)\,|\mathcal{F}_t]$$

$Z(t)$ is an exp. martingale

Applying Itô's formula to $Z(t)$,

(3)
$$dZ(t) = Z(t)\frac{1}{2}\upsilon^2(t)\,dt - Z(t)\upsilon(t)\,dw(t) - \frac{1}{2}\upsilon^2(t)Z(t)\,dt$$
$$= -\upsilon(t)Z(t)\,dw(t)$$

With this, we can compute

(4)
$$dZ(t)\,d\tilde{w}(t) = -\upsilon(t)Z(t)\,dt$$

Replacing (2), (3) and (4) in (1) yields

$$dY(t) = Z(t)\left[dw(t) + \upsilon(t)dt\right] - \tilde{w}(t)\upsilon(t)Z(t)\,dw(t) - \upsilon(t)Z(t)\,dt$$
$$= Z(t)\,dw(t) - \upsilon(t)\tilde{w}(t)Z(t)\,dw(t)$$
$$= Z(t)\left[1 - \upsilon(t)\tilde{w}(t)\right]dw(t)$$

It follows that Y_t is a martingale w.r.t. $\mathbb{P}$. Using (2) again,

$$\mathbb{E}_{\mathbb{Q}}[\tilde{w}(s)\,|\,\mathcal{F}_t] = Z^{-1}(t)\,\mathbb{E}[\tilde{w}(s)Z(s)\,|\,\mathcal{F}_t]$$
$$= Z^{-1}(t)\,\mathbb{E}[Y(s)\,|\,\mathcal{F}_t]$$
$$= Z^{-1}(t)Y(t) = \tilde{w}(t)$$

whence $\tilde{w}(t)$ is a martingale w.r.t. $\mathbb{Q}$.

(b) Suppose $X(t) = at + W(t) \in \mathbb{R}$, $t \leq T$, where $a \in \mathbb{R}$ is a constant. Find a probability measure Q on $\mathcal{F}_T$ such that X is a Wiener process with respect to Q.

Let

$$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)} = a$$

Then

$$X(t) = at + W(t) = at + \tilde{W}(t) - \int_0^t a\,ds = \tilde{W}(t)$$

Hence, $X(t)$ is a martingale w.r.t. Q.

(c) Let $a, b, c \neq 0$ be real constants and define

$$dY(t) = bY(t)dt + cY(t)dW(t).$$

Find a probability measure Q and a Wiener process $\widetilde{W}$ with respect to Q such that

$$dY(t) = aY(t)dt + cY(t)d\widetilde{W}(t).$$

$$u(t) = \frac{(b-a)}{c} \qquad \tilde{W}(t) = W(t) + \int_0^t u(s)\,ds$$

$$dY(t) = bY(t)dt - \cancel{c}Y(t)\frac{(b-a)}{\cancel{c}}dt + cY(t)d\tilde{W}(t)$$

$$= aY(t)dt + cY(t)d\tilde{W}(t)$$