Problem 3.5. Use the integration by parts formula to compute the Skorohod integrals in Problem 2.5.

(a) $\int\limits_0^T W(t)\delta W(t)$,

By the integration by parts formula with $u = 1$ and $F = W(t)$,

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= W(t)\,W(T) - T$$

Since

$$D_t F = D_t \int_0^T \chi_{[0,t]}(s)\,dW(s) = \chi_{[0,t]}(t) = 1$$

(b) $\int\limits_0^T \Big(\int\limits_0^T g(s)dW(s)\Big)\delta W(t)$, for a given function $g \in L^2([0,T])$,

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= \left(\int_0^T g(s)\,dW(s)\right) W(T) - \int_0^T g(t)\,dt$$

Since

$$D_t \int_0^T g(s)\,dW(s) = g(t)$$

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed,

Recall that

$$\int_0^{t_0} W(t)\, dW(t) = \frac{1}{2}(W^2(t_0) - t_0) \Rightarrow W^2(t_0) = 2\int_0^{t_0} W(t)\, dW(t) + t_0$$

Hence

$$D_t W^2(t_0) = D_t\left(2\int_0^{t_0} W(s)\, dW(s) + t_0\right) = 2W(t)$$

and

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\, dt$$

$$= \boxed{W^2(t_0)\, W(T) - \int_0^T 2W(t)\, dt}$$

not necessary $\longrightarrow$ Using that $\int_0^t s\,dB_s = tB_t - \int_0^t B_s ds$. Øksendal, Integration by Parts

we can write

$$-2\int_0^T W(t)dt = 2\left(\int_0^T t\, dW(t) - TW(T)\right)$$

Hence,

$$\int_0^T F\,\delta W(t) = W^2(t_0)\, W(T) + 2\left(\int_0^T t\, dW(t) - TW(T)\right)$$

(d) $\int_0^T \exp\{W(T)\}\delta W(t)$ [*Hint*. Use Problem 1.3.],

$F\int_0^T u(t)\,\delta W(t) = \int_0^T F u(t)\,\delta W(t) + \int_0^T u(t) D_t F\,dt$

By the Integration by Parts formula:

$$\int_0^T \exp(W(T))\,\delta W(t) = \exp(W(T))\,W(T) - \int_0^T D_t \exp(W(T))\,dt$$

By the problem 1.3.d, taking $g(s)=1$,

$$\exp\left[\int_0^T g(s)\,dW(s)\right] = \sum_{n=0}^{\infty} I_n\left[\frac{g^{\otimes n}\exp(1/2\|g\|^2)}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\exp(T/2)\, I_n[1]$$

Using the definition of the Malliavin derivative,

$$D_t \exp(W(T)) = \sum_{n=1}^{\infty} n\cdot\frac{1}{(n-1)!}\exp(T/2)\, I_{n-1}[1]$$

$$= \sum_{n=1}^{\infty} \frac{n}{(n-1)!} e^{T/2}\cdot T^{(n-1)/2} h_{n-1}\left(\frac{W(T)}{\sqrt{T}}\right) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e^{T/2}\, T^{(n-1)/2} h_n'\left(\frac{W(T)}{\sqrt{T}}\right)$$

Hence,

$$\int_0^T \exp(W(T))\,\delta W(t) = W(T)\sum_{n=0}^{\infty} \frac{1}{n!} e^{T/2}\, T^{n/2} h_n\left(\frac{W(T)}{\sqrt{T}}\right)$$

$$- \int_0^T \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e^{T/2}\, T^{(n-1)/2} h_n'\left(\frac{W(T)}{\sqrt{T}}\right) dt$$

change $n-1 \to n$, integrate?

(e) $\int_0^T F\delta W(t)$, where $F = \int_0^T g(s)W(s)ds$, with $g \in L^2([0,T])$ [*Hint*. Use Problem 1.3].

(1) $$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

Using the problem 1.3,

(2) $$\int_0^T g(s)W(s)ds = I_1\left[\int_t^T g(s)ds\right]$$

Thus,

(3) $$D_t\int_0^T g(s)W(s)ds = I_0\left[\int_t^T g(s)ds\right] = \int_t^T g(s)ds$$

Using (2) and (3) in (1),

$$\int_0^T F\,\delta W(t) = W(T)\int_0^T g(s)W(s)\,ds - \int_0^T\int_t^T g(s)ds\,dt$$

$$= W(T)\int_0^T\int_t^T g(s)ds\,dW(t) - \int_0^T\int_t^T g(s)ds\,dt$$