Problem 3.5. Use the integration by parts formula to compute the Skorohod integrals in Problem 2.5.

(a) $\int_0^T W(t)\delta W(t)$,

By the integration by parts formula with $u = 1$ and $F = W(t)$,

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= W(t)\,W(T) - T$$

Since

$$D_t F = D_t \int_0^T \chi_{[0,t]}(s)\,dW(s) = \chi_{[0,t]}(t) = 1$$

(b) $\int_0^T \left(\int_0^T g(s)dW(s)\right)\delta W(t)$, for a given function $g \in L^2([0,T])$,

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= \left(\int_0^T g(s)\,dW(s)\right) W(T) - \int_0^T g(t)\,dt$$

Since

$$D_t \int_0^T g(s)\,dW(s) = g(t)$$

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed,

Recall that

$$\int_0^{t_0} W(t)\, dW(t) = \frac{1}{2}\left(W^2(t_0) - t_0\right) \Rightarrow W^2(t_0) = 2\int_0^{t_0} W(t)\, dW(t) + t_0$$

Hence

$$D_t W^2(t_0) = D_t\left(2\int_0^{t_0} W(s)\, dW(s) + t_0\right) = 2W(t)$$

and

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\, dt$$

$$= W^2(t_0)\,W(T) - \int_0^T 2W(t)\, dt$$

Using that $\int_0^t s\,dB_s = tB_t - \int_0^t B_s ds$. Øksendal, Integration by Parts

we can write

$$-2\int_0^T W(t)dt = 2\left(\int_0^T t\, dW(t) - TW(T)\right)$$

Hence,

$$\int_0^T F\,\delta W(t) = W^2(t_0)\,W(T) + 2\left(\int_0^T t\, dW(t) - TW(T)\right)$$

(d) $\int\limits_0^T \exp\{W(T)\}\delta W(t)$ [*Hint*. Use Problem 1.3.],

(e) $\int_0^T F\delta W(t)$, where $F = \int_0^T g(s)W(s)ds$, with $g \in L^2([0,T])$ [*Hint*. Use Problem 1.3].

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\, dt$$

$$= W(T)\int_0^T g(s)W(s)\,ds -$$

Since

$$\int_0^T g(s)\,W(s)\,ds = \int_0^T\int_t^T g(s)\,ds\,dW(t)$$

$$= \int_t^T g(s)\,ds\int_0^T dW(t) \cancelto{0}{- \int_0^T D_t\int_t^T g(s)\,ds\;\delta W(t)}$$

$$= W(T)\int_t^T g(s)\,ds$$

$$D_t\int_0^T g(s)\,W(s)\,ds =$$