Problem 6.1. Prove $L^2(P) \subseteq Dom(D_t) \subseteq (\mathcal{S})^*$. See Remark 6.6.

It is immediate from the definition that $Dom(D_t) \subseteq (S)^*$.

If $F \in L^2(\mathbb{P})$, then $F \in (S)^*$. Hence, there exists q_0 such that

$$\|F\|^2_{-q_0} = \sum_{\alpha \in J} \alpha! c_\alpha^2 (2\mathbb{N})^{-\alpha q_0} < \infty \Rightarrow \alpha! c_\alpha^2 (2\mathbb{N})^{-\alpha q_0} \leq J, \ \forall \alpha \in J$$

Our goal is to show that

$$D_t F = \sum_{\alpha \in J} \sum_{k=1}^{\infty} c_\alpha \alpha_k e_k(t) H_{\alpha - \varepsilon^{(k)}} \in (S)^*$$

i.e., we need to check that there exists q_1 such that

$$(\alpha - \varepsilon^{(k)})! \, c_\alpha^2 \alpha_k^2 e_k^2 (2\mathbb{N})^{-(\alpha - \varepsilon^{(k)})(q_0 + q_1)} \leq J$$

Since $\{e_k(t)\}$ is bounded,

$$(\alpha - \varepsilon^{(k)})! \, c_\alpha^2 \alpha_k^2 e_k^2 (2\mathbb{N})^{-(\alpha - \varepsilon^{(k)})(q_0 + q_1)}$$

$$\leq C_1 (\alpha - \varepsilon^{(k)})! \, c_\alpha^2 \alpha_k^2 (2\mathbb{N})^{-(\alpha - \varepsilon^{(k)})(q_0 + q_1)}$$

$$= C_1 \alpha! \frac{(\alpha_k - 1)}{\alpha_k} c_\alpha^2 \alpha_k^2 (2\mathbb{N})^{-q_0(\alpha - \varepsilon^{(k)})} (2\mathbb{N})^{-q_1(\alpha - \varepsilon^{(k)})}$$

$$= C_1 \alpha! \, c_\alpha^2 (2\mathbb{N})^{-q_0(\alpha - \varepsilon^{(k)})} (\alpha_k - 1)\alpha_k (2\mathbb{N})^{-q_1(\alpha - \varepsilon^{(k)})}$$

$$\leq C_1 \alpha! \, c_\alpha^2 (2\mathbb{N})^{-q_0 \alpha} (2k)^{q_0} \left[(\alpha_k - 1)(2(k-1))^{-q_1 \alpha_{k-1}}\right]\left[\alpha_k (2k)^{-q_1(\alpha_k - 1)}\right]$$

$$\leq C_1 \alpha! \, c_\alpha^2 \left[(\alpha_k - 1) 2^{-q_1 \alpha_{k-1}}\right]\left[\alpha_k (2k)^{-q_1(\alpha_k - 1) + q_0}\right] \chi_{\alpha_k > 1} \leq J$$