Problem 6.4. (*) Show that the singular noise $\overset{\bullet}{W}$ does not belong to the domain of the Hida–Malliavin derivative as given in (6.8).

Since

$$\overset{\bullet}{W}(t) = \sum_{k} e_k(t) H_{\varepsilon^{(k)}} \in (S)^*$$

we have that

$$D_t \overset{\bullet}{W}(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} e_k(s) e_j(t) H_{\varepsilon^{(k)} - \varepsilon^{(j)}}$$

$$= \sum_{k=1}^{\infty} e_k(s) e_k(t) \quad \text{does not converge in } (S)^*$$

where we used that

$$H_{\varepsilon^{(k)} - \varepsilon^{(j)}} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases}$$

Notice that $D_t \overset{\bullet}{W}(s)$ does not converge in $(S)^*$ since

$$\left\| \sum_{k=1}^{\infty} e_k(s) e_k(t) \right\| \leq \sum_{k=1}^{\infty} \| e_k(s) e_k(t) \| = \sum_{k=1}^{\infty} | e_k(s) | \cdot \| e_k(t) \|$$

and $\{e_k\}$ is a orthonormal basis of $L^2(\mathbb{R})$.