**Problem 1.3.** (*) Find the Wiener–Itô chaos expansion of the following random variables:

(a) $\xi = W(t)$, where $t \in [0, T]$ is fixed,

Notice that

$$I_1(1) = \int_0^T dW_t = W(T) - \cancel{W(0)}$$

Since we want only $W(t)$,

$$\int_0^T \chi_{[0,t]}(s)\, dW_s = W(t) \implies \xi = I_1(\chi_{[0,t]}(s))$$

(b) $\xi = \int\limits_0^T g(s) dW(s)$, where $g \in L^2([0, T])$,

$$I_1(g) = \int_0^t g(s)\, dW_s = \xi$$

(c) $\xi = W^2(t)$, where $t \in [0, T]$ is fixed,

Since $h_2(x) = x^2 - 1$ and

$$\mathbb{E}[W^2(T)] = T = I_0$$

we have

$$I_2(1) = 2 \cdot \int_0^T \int_0^{t_2} dW(t_1)\, dW(t_2) = T h_2\left(\frac{\int_0^T 1 \cdot dW(t)}{\sqrt{T}}\right) = T h_2\left(\frac{W(T)}{\sqrt{T}}\right) = W^2(T) - T$$

To restrict to $[0,t]$, we consider

$$2 \cdot \int_0^T \int_0^{t_2} \chi_{[0,T]}(t_1, t_2)\, dW(t_1)\, dW(t_2) = T h_2\left(\frac{\int_0^T \chi_{[0,T]}(s)\, dW_s}{\sqrt{T}}\right)$$

$$= T h_2\left(\frac{W(t)}{\sqrt{T}}\right) = W^2(t) - T = I_2(\chi_{[0,T]}(t_1,t_2))$$

Thus,

$$\xi = W^2(t) = I_2(\chi_{[0,T]}(t_1,t_2)) + I_0$$

(d) $\xi = \exp\{\int_0^T g(s)dW(s)\}$, where $g \in L^2([0,T])$ [*Hint.* Use (1.15).],

By the exercise 1.c, we have

$$\exp\left[\int_0^T g(s)\,dW(s) - \frac{1}{2}\|g\|^2\right] = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\left(\frac{\theta}{\|g\|}\right)$$

$$= \exp\left[\int_0^T g(s)\,dW(s)\right] \cdot \exp\left[-\frac{1}{2}\|g\|^2\right]$$

Thus,

$$\exp\left[\int_0^T g(s)\,dW(s)\right] = \exp\left[\frac{1}{2}\|g\|^2\right] \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\left(\frac{\theta}{\|g\|}\right)$$

$$= \sum_{n=0}^{\infty} I_n[g^{\otimes n}] \frac{\exp(1/2\,\|g\|^2)}{n!}$$

$$= \sum_{n=0}^{\infty} I_n\left[\frac{g^{\otimes n}\exp(1/2\,\|g\|^2)}{n!}\right] = \xi$$

(e) $\xi = \int_0^T g(s)W(s)ds$, where $g \in L^2([0,T])$.

$$\int_0^T g(s)\,W(s)\,ds = \int_0^T g(s)\int_0^s dW(t)\,ds \overset{(*)}{=} \int_0^T\int_t^T g(s)\,ds\,dW(t)$$

Hence,

$$\xi = I_1\left[\int_t^T g(s)\,ds\right]$$

$(*)$ $\chi_{[0,s]}(t) = \chi_{[t,T]}(s)$

First step: use

$$I_n(g^{\otimes n}) = n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\,dW(t_1)\cdots dW(t_n) = \|g\|^n h_n\left(\frac{\int_0^T g(t)\,dW(t)}{\|g\|}\right)$$

Second step: combine with the Wiener-Itô chaos expansion:

**Theorem 1.2.1** (The Wiener-Itô Chaos Expansion). Let $\xi$ be an $\mathscr{F}_T$-measurable random variable in $L^2(\mathbb{P})$. There exists a unique sequence $(f_n)$ of functions $f_n \in \hat{L}^2([0,T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n)$$