Problem 3.4. Use the integration by parts formula (Theorem 3.15) to compute the Skorohod integrals

$$\int_0^T F\delta W(t),$$

for the random variables F given in Problem 3.2 and in Problem 3.3.

Recall that

Theorem 3.15. Integration by parts. *Let $u(t)$, $t \in [0,T]$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then*

$$F\int_0^T u(t)\delta W(t) = \int_0^T Fu(t)\delta W(t) + \int_0^T u(t)D_tF dt. \qquad (3.21)$$

(a) $F = W(T)$.

$$\int_0^T F\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\, dt$$

$$= F\cdot W(T) - \int_0^T 1\, dt$$

$$= W^2(T) - T$$

(b) $F = \int\limits_0^T s^2 dW(s)$.

$$\int_0^T\int_0^T s^2 dW(s)\,\delta W(t) = \int_0^T s^2 dW(s)\int_0^T \delta W(t) - \int_0^T t^2 dt$$

$$= W(T)\left[T^2 W(T) - \int_0^T W(s)2s\,ds\right] - \frac{T^3}{3}$$

where we used

Theorem 4.1.5 (Integration by parts). *Suppose $f(s,\omega) = f(s)$ only depends on s and that f is continuous and of bounded variation in $[0,t]$. Then*

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s\,.$$

to obtain that $\displaystyle\int_0^T s^2 dW(s) = T^2 W(T) - \int_0^T W(s)2s\,ds$

(c) $F = \int\limits_0^T \int\limits_0^{t_2} \cos(t_1 + t_2) dW(t_1) dW(t_2)$.

$$\int_0^T F \delta W(t) = F W(T) - \int_0^T \int_0^T \cos(t_1 + t_2)\, dW(t_1) dt$$

(d) $F = 3W(s_0)W^2(t_0) + \log(1 + W^2(s_0))$, for given $s_0, t_0 \in [0, T]$.

$$\int_0^T F \delta W(t) = F W(T)$$

$$- \int_0^T \left(\left[3W^2(t_0) + \frac{2W(s_0)}{1 + W^2(s_0)} \right] \chi_{[0,s_0]}(t) + 6W(s_0)W(t_0)\chi_{[0,t_0]}(t) \right) dt$$

(e) $F = \int\limits_0^T W(t_0) \delta W(t)$, for a given $t_0 \in [0, T]$. [*Hint*. Use Problem 2.5 (b).]

$$\int_0^T F \delta W(t) = F \int_0^T \delta W(t) - \int_0^T D_t F\, dt$$

$$= W(T) \int_0^T W(T)\chi_{[0,t_0]}(s)\, dW(s) - \int_0^T \left(W(t_0) + W(T)\chi_{[0,t_0]}(t) \right) dt$$

$$= W^2(T)W(t_0) - W(t_0)T - W(T)t_0$$

$$F = e^G \quad \text{with} \quad G = \int_0^T g(s)dW(s), \quad g \in L^2([0,T]),$$

$$\int_0^T F\,\delta W(t) = W(T)\,e^G - \int_0^T e^G g(t)\,dt$$

$$= e^G\left(W(T) - \int_0^T g(t)\,dt\right)$$

$F = e^G$ with $G = W(t_0)$

$$\int_0^T F\,\delta W(t) = W(T)\,e^G - \int_0^T e^G \chi_{[0,t_0]}(t)\,dt = e^G\left(W(T) - t_0\right)$$