Problem 2.5. (*) Compute the following Skorohod integrals:

(a) $\int\limits_0^T W(t)\delta W(t)$,

Note that $W(t)$ is $\mathbb{F}$-adapted. Thus, the Skorohod and Itô integrals are the same:

$$\delta(W(t)) = \int_0^T W(t)\,\delta W(t) = \int_0^T W(t)\,dW(t) = \frac{1}{2}\left(W_T^2 - T\right)$$

(b) $\int\limits_0^T \Big(\int\limits_0^T g(s)dW(s)\Big)\delta W(t)$, for a given function $g \in L^2([0,T])$,

First step: write the Wiener-Itô chaos expansion of the integrand

$$\int_0^T g(s)\,dW(s) = I_1\left(g(s)\right), \qquad f_i = 0,\ \forall i \in \mathbb{Z}_{\geq 0} \setminus \{1\}$$

Second step: compute the symmetrization of f_i

$$f_1(t_1, t) = g(t_1) \Rightarrow \tilde{f}_1(t_1, t) = \frac{1}{2}\left[f_1(t_1, t) + f_1(t, t_1)\right] = \frac{1}{2}\left[g(t_1) + g(t)\right]$$

Third step: write the Skorohod integral

$$\delta\left(\int_0^T g(s)\,dW(s)\right) = I_2\left(\tilde{f}_1\right) = 2\int_0^T\int_0^{t_2} \tilde{f}_1(t_1, t_2)\,dW(t_1)\,dW(t_2)$$

$$= 2\int_0^T\int_0^{t_2} \frac{1}{2}\left[g(t_1) + g(t_2)\right] dW(t_1)\,dW(t_2)$$

$$= \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T \int_0^{t_2} g(t_2)\, dW(t_1)\, dW(t_2)$$

$$= \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T g(t_2) \int_0^{t_2} dW(t_1)\, dW(t_2)$$

$$= \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T g(t_2)\, W(t_2)\, dW(t_2)$$

Using

4.3. Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \, .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \, .$$

$$\left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) = \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2)$$
$$+ \int_0^T g(t_2)\, W(t_2)\, dW(t_2) + \int_0^t g(t)\, dt$$

Replacing we obtain

$$\delta\left(\int_0^T g(s)\, dW(s) \right) = \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2)$$

$$+ \left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) - \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) - \int_0^t g(t)\, dt$$

Hence,

$$\int_0^T \left(\int_0^T g(s)\, dW(s) \right) \delta W(t) = \left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) - \int_0^t g(t)\, dt$$

(c) $\int\limits_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed,

First step: write the Wiener-Itô chaos expansion of the integrand

$$W^2(t_0) = I_2(\chi_{[0,t_0]}(t_1,t_2)) + t_0 \qquad \text{(see exercise 1.3c)}$$

Second step: compute the symmetrization of f_i

Since $f_2(t_1,t_2,t) = \chi_{[0,t_0]}(t_1,t_2)$, we have

(1)
$$\begin{aligned}
\tilde{f}_2(t_1,t_2) &= \frac{1}{3}\left[f_2(t_1,t_2,t) + f_2(t_2,t,t_1) + f_2(t_1,t,t_2) \right] \\
&= \frac{1}{3}\left[\chi_{[0,t_0]}(t_1,t_2) + \chi_{[0,t_0]}(t_2,t) + \chi_{[0,t_0]}(t_1,t) \right] \\
&= \chi_{\{t,t_1,t_2<t_0\}} + \frac{1}{3}\left[\chi_{\{t_1,t_2<t_0<t\}} + \chi_{\{t_2,t<t_0<t_1\}} + \chi_{\{t_1,t<t_0<t_2\}} \right]
\end{aligned}$$

Third step: write the Skorohod integral

Using the linearity of the Skorohod integral and the fact that t_0 is $\mathbb{F}$-adapted,

(2)
$$\begin{aligned}
\int_0^T W^2(t_0)\,\delta W(t) &= \int_0^T \left(I_2(\chi_{[0,t_0]}(t_1,t_2)) + t_0 \right) \delta W(t) \\
&= \int_0^T I_2(\chi_{[0,t_0]}(t_1,t_2))\,\delta W(t) + t_0 W(T) \\
&= t_0 W(T) + I_3(\tilde{f}_2)
\end{aligned}$$

Using (1) to compute $I_3(\tilde{f}_2)$,

(3) $$I_3(\tilde{f}_2) = 6J_3(\tilde{f}_2) = 6\int_0^T\int_0^{t_3}\int_0^{t_2} \tilde{f}_2(t_1,t_2,t)\, dW(t_1)\, dW(t_2)\, dW(t_3)$$

$$= 6\int_0^T\int_0^{t_3}\int_0^{t_2} \chi_{\{t,t_1,t_2<t_0\}}\, dW(t_1)\, dW(t_2)\, dW(t_3)$$

$$+ 6\int_0^T\int_0^{t_3}\int_0^{t_2} \frac{1}{3}\chi_{\{t_1,t_2<t_0<t_3\}}\, dW(t_1)\, dW(t_2)\, dW(t_3)$$

$$\overset{(*)}{=} t_0^{3/2}\, h_3\left(\frac{W(t_0)}{\sqrt{t_0}}\right) + 2\underbrace{\int_{t_0}^T\int_0^{t_0}\int_0^{t_2}}_{?} dW(t_1)\, dW(t_2)\, dW(t_3)$$

Since $h^3(x) = x^3 - 3x$, (3) equals

(4) $$t_0^{3/2}\left(\frac{W^3(t_0)}{t_0^{3/2}} - \frac{3W(t_0)}{\sqrt{t_0}}\right) + 2\int_{t_0}^T \frac{1}{2}(W^2(t_0) - t_0)\, dW(t_3)$$

Replacing (4) into (2),

$$\int_0^T W^2(t_0)\, \delta W(t) = t_0 W(T) + W^3(t_0) - 3t_0 W(t_0)$$

$$+ (W^2(t_0) - t_0)(W(T) - W(t_0))$$

Thus,

$$\boxed{\int_0^T W^2(t_0)\, \delta W(t) = W^2(t_0) W(T) - 2t_0 W(t_0)}$$

(d) $\int_0^T \exp\{W(T)\}\delta W(t)$ [*Hint*. Use Problem 1.3.],

By the problem 1.3.d,

$$\exp\left[\int_0^T g(s)\,dW(s)\right] = \sum_{n=0}^{\infty} I_n\left[\frac{g^{\otimes n}\exp(1/2\,\|g\|^2)}{n!}\right]$$

Thus, taking $g(s)=1$,

$$\int_0^T \exp(W(T))\,\delta W(t) = \int_0^T \sum_{n=0}^{\infty} \frac{1}{n!}\exp(T/2)\, I_n[1]\,\delta W(t)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\exp(T/2)\, I_{n+1}[1] \overset{(*)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} e^{T/2}\, T^{\frac{(n+1)}{2}}\, h_{n+1}\left(\frac{W(T)}{\sqrt{T}}\right)$$

(*)

$$I_n(g^{\otimes n}) = n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\,dW(t_1)\cdots dW(t_n) = \|g\|^n h_n\left(\frac{\int_0^T g(t)\,dW(t)}{\|g\|}\right)$$

(e) $\int_0^T F\delta W(t)$, where $F = \int_0^T g(s)W(s)ds$, with $g \in L^2([0,T])$ [*Hint*. Use Problem 1.3].

By the exercise 1.3.e, $\int_0^T g(s)\,W(s)\,ds = I_1\left[\int_t^T g(s)\,ds\right]$

Letting $f_1(t_1,t) = \int_t^T g(s)\,ds$, we have

$$\tilde{f}_1(t_1,t_2) = \frac{1}{2}\left[f_1(t_1,t_2) + f_1(t_2,t_1)\right] = \frac{1}{2}\left[\int_{t_2}^T g(s)\,ds + \int_{t_1}^T g(s)\,ds\right]$$

Thus, the Skorohod Integral is

$$\int_0^T \int_0^T g(s) W(s) ds \, \delta W(t) = I_2[\tilde{f}_1]$$

$$= \int_0^T \int_0^{t_2} \left[\int_{t_2}^T g(s) ds + \int_{t_1}^T g(s) ds \right] dW(t_1) dW(t_2)$$

$$= \int_0^T \int_0^{t_2} \int_{t_2}^T g(s) ds \, dW(t_1) dW(t_2) + \int_0^T \int_0^{t_2} \int_{t_1}^T g(s) ds \, dW(t_1) dW(t_2)$$

$$= \int_0^T \int_0^{t_2} dW(t_1) \int_{t_2}^T g(s) ds \, dW(t_2) + \int_0^T \int_0^{t_2} \int_{t_1}^T g(s) ds \, dW(t_1) dW(t_2)$$

$$= \int_0^T 2W(t_2) \int_{t_2}^T g(s) ds \, dW(t_2) + \int_0^T \int_0^{t_2} g(s) W(s) ds \, dW(t_2)$$

Since

$$\int_0^T \int_0^{t_2} \int_{t_1}^T g(s) ds \, dW(t_1) dW(t_2) =$$

$$\int_0^T W(t_2) \int_{t_2}^T g(s) ds \, dW(t_2) + \int_0^T \int_0^{t_2} g(s) W(s) ds \, dW(t_2)$$