Problem 5.7. (a) Let $f \in L^2(\mathbb{R})$ be deterministic. Prove that

$$\frac{d}{dt}\int_{-\infty}^{t} f(s)dW(s) = f(t)\dot{W}(t) \quad \text{in} \quad (\mathcal{S}^*).$$

Using (5.70) and the fact that f is deterministic,

$$\int_{-\infty}^{t} f(s)\, dW(s) = \int_{-\infty}^{t} f(s) \diamond \dot{W}(s)\, ds = \int_{-\infty}^{t} f(s)\, \dot{W}(s)\, ds$$

Thus,

$$\frac{d}{dt}\int_{-\infty}^{t} f(s)\, dW(s) = \int_{-\infty}^{t} \frac{d}{ds}\left(f(s)\, \dot{W}(s)\right) ds$$

$$= f(t)\, \dot{W}(t)$$

(b) Let u be a Skorohod integrable process. Prove that

$$\frac{d}{dt}\int_{-\infty}^{t} u(s)\delta W(s) = u(t) \diamond \dot{W}(t) \quad \text{in} \quad (\mathcal{S}^*).$$

Follows from approximation. Let (φ_n) be a sequence of elementary functions such that $\varphi_n \uparrow f$. By the previous result and monotone convergence,

$$\frac{d}{dt}\int_{-\infty}^{t} u(s)\, \delta W(s) = \int_{-\infty}^{t} \frac{d}{ds}\left(u(s) \diamond \dot{W}(s)\right) ds = u(t) \diamond \dot{W}(t)$$