AN APPLICATION OF MALLIAVIN CALCULUS TO FINANCE

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Introduction

Consider a market consisting of a riskless asset S_0 with

$$\text{riskless asset} \quad \begin{cases} dS_0(t) = \rho(t)S_0(t)\ dt \\ S_0(0) = 1 \end{cases}$$

and a risky asset S_1 satisfying

$$\text{risky asset} \quad \begin{cases} dS_1(t) = \mu(t)S_1(t)\ dt + \sigma(t)S_1(t)\ dW(t) \\ S_1(0) > 0 \end{cases}$$

where $\rho(t), \mu(t)$, and $\sigma(t) \neq 0$ are $\mathbf{F}$-adapted processes satisfying the following condition

$$\mathbf{E}\left[\int_0^T (|\rho(t)| + |\mu(t)| + \sigma^2(t))\ dt\right] < \infty$$

Let $\theta_0(t)$ and $\theta_1(t)$ denote the number of units of $S_0(t)$ and $S_1(t)$, respectively. Then the value of the portfolio $\theta = (\theta_0, \theta_1)$ is $V^\theta = \theta_0 S_0 + \theta_1 S_1$, which we suppose to be self-financing.

Our goal is to find a replicating (hedging) portfolio $V^\theta(T) = F$, $\mathbf{P}$-a.s., where F is $\mathscr{F}_t$-measurable, for a digital option, which has a payoff at maturity

$$F = \mathbf{1}_{[K,\infty)}(W(T))$$

where K is the exercise price.

Metodology

Using the generalized Clark-Ocone formula [Oku10, Theorem 3.11], we find that the initial value $V^\theta(0) = \mathbf{E}_Q[G]$, where $G = \exp\left(-\int_0^t \rho(s)\ ds\right)$, by uniqueness, and the replicating portfolio is given by

$$\theta_1(t) = e^{-\int_0^t \rho(s)\ ds}\sigma^{-1}(t)S_1^{-1}(t) \; \mathbf{E}_Q\left[\left(D_t G - G\int_t^T D_t u(s)\ d\widetilde{W}(s)\right) \middle| \mathscr{F}_t\right]$$

where D_t is the Malliavin derivative, and $u(t) = \frac{\mu(t)-\rho(t)}{\sigma(t)}$ is the Girsanov function [NØP08, Section 6.5].

In particular, if ρ and μ are constants, and $\sigma(t) = \sigma \neq 0$, then $u(t) = u = \frac{\mu-\rho}{\sigma}$ is also constant, whence $D_t u = 0$. Then $\theta_1(t)$ simplifies to

$$\theta_1(t) = e^{\rho(t-T)}\sigma^{-1}S_1^{-1}(t)\mathbf{E}_Q[D_t F \mid \mathscr{F}_t]$$

We aim to compute the conditional expectation $\mathbf{E}_Q[D_t F \mid \mathscr{F}_t]$. To do that, we need the following concept.

Definition (Donsker delta function)**.** Let $Y : \Omega \longrightarrow \mathbf{R}$, $Y \in \mathscr{G}^*$. The continuous function $\delta_Y(\cdot) : \mathbf{R} \longrightarrow \mathscr{G}^*$ is the **Donsker delta function** of Y if it has the property that

$$\int_{\mathbf{R}} f(y)\delta_Y(y)\ dy = f(Y) \quad \text{a.s.}$$

for all measurable $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that the integral converges.

Results and Discussion

Theorem. Suppose that ρ is constant, and $u(t)$ is deterministic and satisfying $\mathbf{E}[u^2(t)] < \infty$. Then the replicating portfolio for hedging $\mathbf{1}_{[K,\infty)}(W(T))$ is

$$\theta_1(t) = e^{-\rho(T-t)}(2\pi(T-t))^{-1/2}\sigma^{-1}(t)S_1^{-1}(t)\exp\left(-\frac{(K-W(t))^2}{2(T-t)}\right)$$

Proof. First, notice that $F = \mathbf{1}_{[K,\infty)}(W(T)) \in L^2(\mathbf{P})$. Thus, we can use our expression for $\theta_1(t)$. Now we compute $\mathbf{E}_Q[D_t F \mid \mathscr{F}_t]$ using the Donsker delta function by taking $f(y) = \mathbf{1}_{[K,\infty)}(y)$, and $Y(T) = W(T)$. By the [AØU01, Theorem 4.4],

$$\mathbf{1}_{[K,\infty)}(W(T)) = \int_K^\infty (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(y-W(T))^{\diamond 2}}{2T}\right) dy$$

By the Chain Rule for the Wick product [NØP08, Theorem 6.13],

$$\begin{aligned} D_t(\mathbf{1}_{[K,\infty)}(W(T))) &= \int_K^\infty (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(y-W(T))^{\diamond 2}}{2T}\right) \diamond \frac{(y-W(T))}{2T}\ dy \\ &= (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(K-W(T))^{\diamond 2}}{2T}\right) \end{aligned}$$

Denoting by $\hat{\diamond}$ the Wick product with respect to the probability measure Q, then since $\hat{\diamond} = \diamond$ [HØ03, Lemma 3.21], we have

$$\begin{aligned} &\mathbf{E}[D_t(\mathbf{1}_{[K,\infty)}(W(T))) \mid \mathscr{F}_t] \\ &= \mathbf{E}_Q\left[(2\pi T)^{-1/2}\exp^{\hat{\diamond}}\left(-\frac{(K-W(T))^{\hat{\diamond} 2}}{2T}\right) \middle| \mathscr{F}_t\right] \\ &= (2\pi T)^{-1/2}\mathbf{E}_Q\left[\exp^{\hat{\diamond}}\left(-\frac{(K-\widetilde{W}(T)+\int_0^T u(s)\ ds)^{\hat{\diamond} 2}}{2T}\right) \middle| \mathscr{F}_t\right] \\ &= (2\pi T)^{-1/2}\exp\left(-\frac{(K-W(t))^2}{2(T-t)}\right) \end{aligned}$$

□

References

[AØU01] Knut Aase, Bernt Øksendal, and Jan Ubøe. Using the donsker delta function to compute hedging strategies. *Potential Analysis*, 14(4):351–374, 2001.

[HØ03] Yaozhong Hu and Bernt Øksendal. Fractional white noise calculus and applications to finance. *Infinite dimensional analysis, quantum probability and related topics*, 6(01):1–32, 2003.

[NØP08] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin calculus for Lévy processes with applications to finance*. Springer, 2008.

[Oku10] Yeliz Yolcu Okur. White noise generalization of the clark-ocone formula under change of measure. *Stochastic analysis and applications*, 28(6):1106–1121, 2010.