Problem 5.1. Prove equation (5.4). [*Hint.* First consider step functions ϕ of the form $\phi(t) = \sum_i e_i \chi_{(a_i, a_{i+1}]}(t), \quad t \in \mathbb{R}$.]

$$w_\phi(\omega) = \int_{\mathbb{R}} \phi(t) dW(t, \omega), \quad \omega \in \Omega, \qquad \phi \in L^2(\mathbb{R}), \tag{5.4}$$

Let

$$\varphi(t) = \sum_i e_i \chi_{[a_i, a_{i+1}]}(t)$$

Then, since $\tilde{W}(t) = \langle \omega, \chi_{[0,t]} \rangle$,

$$w_\varphi(\omega) = \langle \omega, \varphi \rangle = \left\langle \omega, \sum_i e_i \chi_{[a_i, a_{i+1}]}(t) \right\rangle$$

$$= \sum_i e_i \langle \omega, \chi_{[a_i, a_{i+1}]} \rangle$$

$$= \sum_i e_i [W(a_{i+1}) - W(a_i)]$$

$$= \int_{\mathbb{R}} \varphi(t) \, dW(t)$$

Hence, the result holds for step functions. Since these functions are dense in the space of Itô integrable functions, we have the result.