Problem 3.1. Let ξ, ζ be orthonormal functions in $L^2([0,T])$. Using the properties of Hermite polynomials compute directly the following:

(a) $I_1(\xi)I_2(\zeta^{\otimes 2})$
(b) $I_3(\xi\hat{\otimes}\zeta^{\otimes 2})$
(c) $D_t I_3(\xi\hat{\otimes}\zeta^{\otimes 2})$ [*Hint.* Use (1.14), (3.5)–(3.9)].

Using the chain rule compute:

(d) $D_t(I_1(\xi)I_2(\zeta^{\otimes 2}))$.

Compare the results in (c) and (d).

$h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = x^2 - 1,\ h_3(x) = x^3 - 3x,$
$h_4(x) = x^4 - 6x^2 + 3,\ h_5(x) = x^5 - 10x^3 + 15x, \ldots.$

a)

$$I_1(\xi) = \int_0^T \xi(t)\,dW(t)$$

$$I_2(\zeta^{\otimes 2}) = \|\zeta^{\otimes 2}\|^2 h_2\left(\frac{\int_0^T \zeta^{\otimes 2}\,dW(t)}{\|\zeta^{\otimes 2}\|}\right)$$

Since $\|\zeta^{\otimes 2}\| = 1$ and $h_2(x) = x^2 - 1$,

$$I_2(\zeta^{\otimes 2}) = \left(\int_0^T \zeta^{\otimes 2}\,dW(t)\right)^2 - 1$$

b) We use the following result

Proposition 1.8. *If $\xi_1, \xi_2, \ldots$ are orthonormal functions in $L^2([0,T])$, we have that*

$$I_n(\xi_1^{\otimes\alpha_1}\hat{\otimes}\cdots\hat{\otimes}\xi_m^{\otimes\alpha_m}) = \prod_{k=1}^m h_{\alpha_k}\Big(\int_0^T \xi_k(t)W(t)\Big), \tag{1.14}$$

with $\alpha_1+\cdots+\alpha_m = n$. Here $\otimes$ denotes the tensor power *and $\alpha_k \in \{0,1,2,\ldots\}$ for all k.*

Then

$$I_3(\xi\hat{\otimes}\zeta^{\otimes 2}) = h_1\left(\int_0^T \xi(t)\,dW(t)\right) h_2\left(\int_0^T \zeta(t)\,dW(t)\right)$$

$$= \left(\int_0^T \xi(t)\,dW(t)\right)\left(\left(\int_0^T \zeta(t)\,dW(t)\right)^2 - 1\right)$$

c) Notice that

$$D_t I_3(\xi \hat{\otimes} \zeta^{\otimes 2}) = 3 \| \xi \hat{\otimes} \zeta^{\otimes 2} \|^2 h_2 \left(\frac{\int_0^T \xi \hat{\otimes} \zeta^{\otimes 2} \, dW(t)}{\| \xi \hat{\otimes} \zeta^{\otimes 2} \|} \right) \xi \hat{\otimes} \zeta^{\otimes 2}(t)$$

$$= 3 \| \xi \hat{\otimes} \zeta^{\otimes 2} \|^2 \left(\left(\frac{\int_0^T \xi \hat{\otimes} \zeta^{\otimes 2} \, dW(t)}{\| \xi \hat{\otimes} \zeta^{\otimes 2} \|} \right)^2 - 1 \right) \xi \hat{\otimes} \zeta^{\otimes 2}(t)$$

$$= 3 \, \xi \hat{\otimes} \zeta^{\otimes 2}(t) \left[\left(\int_0^T \xi \hat{\otimes} \zeta^{\otimes 2} \, dW(t) \right)^2 - \| \xi \hat{\otimes} \zeta^{\otimes 2} \|^2 \right]$$

d) On the other hand,

$$D_t \left(\int_0^T \xi(t) \, dW(t) \right) \left(\left(\int_0^T \zeta(t) \, dW(t) \right)^2 - 1 \right)$$

$$= \left(\left(\int_0^T \zeta(t) \, dW(t) \right)^2 - 1 \right) D_t \left(\int_0^T \xi(t) \, dW(t) \right)$$

$$+ \left(\int_0^T \xi(t) \, dW(t) \right) D_t \left(\left(\int_0^T \zeta(t) \, dW(t) \right)^2 - 1 \right)$$

$$= \left(\left(\int_0^T \zeta(t) \, dW(t) \right)^2 - 1 \right) \xi(t)$$

$$+ \left(\int_0^T \xi(t) \, dW(t) \right) \cdot 2 \zeta(t) \int_0^T \zeta(t) \, dW(t)$$