MALLIAVIN CALCULUS

Adair Antonio da Silva Neto

December 19, 2023

Contents

Chapter 1

The Wiener-Itô Chaos Expansion

1.1 Iterated Itô Integrals

Consider

- $W(t)$ a one-dimensional Wiener process on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that $W(0) = 0$ a.s.;
- $\mathscr{F}_t$ the σ-algebra generated by $W(s)$, $0 \leq s \leq t$, augmented by the events with probability zero;
- The filtration $\mathbb{F} = \{\mathscr{F}_t : t \in [0, T]\}$.

Definition 1.1.1. A function $g : [0, T]^n \longrightarrow \mathbb{R}$ is **symmetric** if

$$g(t_{\sigma_1}, \dots, t_{\sigma_n}) = g(t_1, \dots, t_n)$$

for all permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ of $(1, 2, \dots, n)$.

We define $\hat{L}^2([0, T]^n)$ as the subspace of $L^2([0, T]^n)$ consisting of symmetric functions.

Now consider the set

$$S_n = \{(t_1, \dots, t_n) \in [0, T]^n : 0 \leq t_1 \leq \cdots \leq t_n \leq T\}$$

Notice that the set S_t occupies $\frac{1}{n!}$ of the whole box $[0, T]^n$. Thus, if $g \in \hat{L}^2([0, T]^n)$, then $g|_{S_n} \in L^2(S_n)$ and

$$\|g\|^2_{L^2([0,T]^n)} = n! \int_{S_n} g^2(t_1, \dots, t_n)\, dt_1 \cdots dt_n = n! \|g\|^2_{L^2(S_n)}$$

Definition 1.1.2. Given a function $f : [0, T]^n \longrightarrow \mathbb{R}$, we define its **symmetrization** $\hat{f}$ as

$$\hat{f}(t_1, \dots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \dots, t_{\sigma_n})$$

in which the sum runs over all permutations σ of $\{1,2,\ldots,n\}$.

Now we are ready to define the n-fold iterated Itô integral.

Definition 1.1.3. Let f be a deterministic function defined on S_n such that $\|f\|^2_{\mathrm{L}^2(\mathrm{S}_n)} < \infty$. We define the **$n$-fold iterated Itô integral** as

$$\mathrm{J}_n(f) = \int_0^{\mathrm{T}} \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1,\ldots,t_n)\,\mathrm{dW}(t_1)\mathrm{dW}(t_2)\cdots\mathrm{dW}(t_{n-1})\mathrm{dW}(t_n)$$

Remark.

1. Note that each i-th Itô integral with respect to $\mathrm{dW}(t_i)$ is well-defined, since the integrand is an $\mathbb{F}$-adapted stochastic process.
2. Furthermore, $\mathrm{J}_n(f) \in \mathrm{L}^2(\mathbb{P})$.

Now, apply Itô's isometry iteratively (we can apply it because $g \in \mathrm{L}^2(\mathrm{S}_m)$ and $h \in \mathrm{L}^2(\mathrm{S}_n)$?). *First case:* If $g \in \mathrm{L}^2(\mathrm{S}_m)$ and $h \in \mathrm{L}^2(\mathrm{S}_n)$ with $m < n$, then

$$\begin{aligned}
\mathbb{E}[\mathrm{J}_m(g)\mathrm{J}_n(h)] &= \mathbb{E}\Bigg[\left(\int_0^{\mathrm{T}} \int_0^{s_m} \cdots \int_0^{s_2} g(s_1,\ldots,s_m)\,\mathrm{dW}(s_1)\cdots\mathrm{dW}(s_m)\right) \\
&\quad \left(\int_0^{\mathrm{T}} \int_0^{s_m} \cdots \int_0^{t_2} h(t_1,\ldots,t_{n-m},s_1,\ldots,s_m)\,\mathrm{dW}(t_1)\cdots\mathrm{dW}(t_{n-m})\mathrm{dW}(s_1)\cdots\mathrm{dW}(s_m)\right)\Bigg] \\
&= \int_0^{\mathrm{T}} \int_0^{s_m} \cdots \int_0^{s_2} g(s_1,s_2,\ldots,s_m) \\
&\quad \mathbb{E}\left[\int_0^{s_1} \cdots \int_0^{t_2} h(t_1,\ldots,t_{n-m},s_1,\ldots,s_m)\,\mathrm{dW}(t_1)\cdots\mathrm{dW}(t_{n-m})\right] \mathrm{d}s_1\cdots\mathrm{d}s_m \\
&= 0
\end{aligned}$$

Second case: If $g, h \in \mathrm{L}^2(\mathrm{S}_n)$, then

$$\mathbb{E}[\mathrm{J}_n(g)\mathrm{J}_n(h)] = \int_0^{\mathrm{T}} \cdots \int_0^{s_2} g(s_1,\ldots,s_n)h(s_1,\ldots,s_n)\,\mathrm{d}s_1\cdots\mathrm{d}s_n = \langle g,h\rangle_{\mathrm{L}^2(\mathrm{S}_n)}$$

in which $\langle g,h\rangle_{\mathrm{L}^2(\mathrm{S}_n)}$ is the inner product of $\mathrm{L}^2(\mathrm{S}_n)$.

This proves the following

Proposition 1.1.1. For $m,n \in \mathbb{Z}_{>0}$,

$$\mathbb{E}[\mathrm{J}_m(g)\mathrm{J}_n(h)] = \begin{cases} 0, & n \neq m \\ \langle g,h\rangle_{\mathrm{L}^2(\mathrm{S}_n)}, & n = m \end{cases}$$

In particular,

$$\|\mathrm{J}_n(h)\|_{\mathrm{L}^2(\mathbb{P})} = \|h\|_{\mathrm{L}^2(\mathrm{S}_n)}$$

For $n = 0$ or $m = 0$, we define $\mathrm{J}_0(g) = g$, when g is a constant, and $\langle g,h\rangle_{\mathrm{L}^2(\mathrm{S}_0)} = gh$, when g and h are constants.

Remark. 1. If $f \in \mathrm{L}^2(\mathrm{S}_n)$, then $\mathrm{J}_n(f) \in \mathrm{L}^2(\mathbb{P})$.

2. The n-fold iterated Itô integral is a linear operator.

Definition 1.1.4. Let $g \in \hat{\mathrm{L}}^2([0,\mathrm{T}]^n)$. Then

$$\mathrm{I}_n(g) = \int_{[0,\mathrm{T}]^n} g(t_1, \dots, t_n)\, \mathrm{dW}(t_1) \cdots \mathrm{dW}(t_n) = n!\mathrm{J}_n(g)$$

is also called ***n*-fold iterated Itô integral**.

Let $x \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. Then the **Hermite polynomials** are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right)$$

For example, the first Hermite polynomials are

1. $h_0(x) = 1$,
2. $h_1(x) = x$,
3. $h_2(x) = x^2 - 1$,
4. $h_3(x) = x^3 - 3x$,
5. $h_4(x) = x^4 - 6x^3 + 3$,
6. $h_5(x) = x^5 - 10x^3 + 15x$.

The family of Hermite polynomials constitute an orthogonal basis for $\mathrm{L}^2(\mathbb{R}, \mu(dx))$, in which $\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx$.

Proposition 1.1.2. If $\xi_1, \xi_2, \dots$ are orthonormal functions in $\mathrm{L}^2([0,\mathrm{T}])$, then

$$\mathrm{I}_n(\xi_1^{\otimes \alpha_1} \hat{\otimes} \cdots \hat{\otimes} \xi_m^{\otimes \alpha_m}) = \prod_{k=1}^{m} h_{\alpha_k}\left(\int_0^{\mathrm{T}} \xi_k(t)\mathrm{W}(t)\right)$$

with $\alpha_1 + \cdots + \alpha_m = n$, $\alpha_k \in \mathbb{N}_0$ for all k, and $\hat{\otimes}$ is the symmetrized tensor product, which is the symmetrization of $f \otimes g$.

Proof. [Itô51] □

See Itô's formula for iterated Itô integral ([Oks13]'s problem 3.7).

Using this, it is possible to prove (see [NN18] p. 64)

$$\mathrm{I}_n(g^{\otimes n}) = n! \int_0^{\mathrm{T}} \int_0^{t_n} \cdots \int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\, \mathrm{dW}(t_1)\cdots \mathrm{dW}(t_n) = \|g\|^n h_n\left(\frac{\int_0^{\mathrm{T}} g(t)\, \mathrm{dW}(t)}{\|g\|}\right)$$

1.2 The Wiener-Itô Chaos Expansion

Theorem 1.2.1 (The Wiener-Itô Chaos Expansion)**.** Let ξ be an $\mathscr{F}_T$-measurable random variable in $L^2(\mathbb{P})$. There exists a unique sequence (f_n) of functions $f_n \in \hat{L}^2([0,T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n)$$

with convergence in $L^2(\mathbb{P})$. Moreover, we have the following isometry:

$$\|\xi\|^2_{L^2(\mathbb{P})} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2([0,T]^n)}$$

Proof. The idea is to obtain an orthogonal decomposition of $L^2(\mathbb{P})$. To do that, we show that ψ is orthogonal to $\exp(\theta)$, which form a total set in $L^2(\mathbb{P})$, implying that $\psi \equiv 0$.

1. Apply Itô's Representation Theorem twice.
2. Use Itô's Isometry and then Itô's Representation Theorem again.
3. Iterate this procedure $n+1$ times, we obtain a process $\varphi_{n+1}(t_1,\dots,t_{n+1})$, $0 \le t_1 \le \dots \le t_{n+1} \le T$, and $n+1$ deterministic functions $g_0, g_1, \dots, g_n$ (where $g_0 = \mathbb{E}[\xi]$ and $g_k(s_k, s_{k-1},\dots,s_1) = \mathbb{E}[\varphi_k(s_k,s_{k-1},\dots,s_1)]$ for $1 \le k \le n$) such that

$$\xi = \sum_{k=0}^{n} J_k(g_k) + \int_{S_{n+1}} \varphi_{n+1}\, dW^{\otimes(n+1)}$$

4. Note that we have a $(n+1)$-fold Iterated Itô Integral

$$\int_{S_{n+1}} \varphi_{n+1}\, dW^{\otimes(n+1)} =: \psi_{n+1}$$

and

$$\mathbb{E}\left[\left(\int_{S_{n+1}} \varphi_{n+1}\, dW^{\otimes(n+1)}\right)^2\right] \le \mathbb{E}[\xi^2]$$

5. Also remark that the family ψ_{n+1} is bounded in $L^2(\mathbb{P})$ and, from Itô's Isometry,

$$\langle \psi_{n+1}, J_k(f_k)\rangle_{L^2(\mathbb{P})} = 0$$

for $k \le n$ and $f_k \in L^2([0,T]^k)$

6. Compute $\|\xi\|^2_{L^2(\mathbb{P})}$ and notice that $\sum_{k=0}^{\infty} J_k(g_k)$ is convergent in $L^2(\mathbb{P})$. Thus,

$$\langle J_k(f_k), \psi\rangle_{L^2(\mathbb{P})} = 0$$

7. Using that

$$I_n(g^{\otimes n}) = \|g\|^n h_n\left(\frac{\theta}{\|g\|}\right), \quad \theta = \int_0^T g(t)\, dW(t)$$

and Hermite polynomials, we have

$$\mathbb{E}\left[h_n\left(\frac{\theta}{\|g\|}\right)\psi\right] = 0, \quad \mathbb{E}\left[\theta^k \psi\right] = 0, \quad \mathbb{E}\left[\exp\theta \cdot \psi\right] = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{E}\left[\theta^k \psi\right] = 0$$

8. Since $\{\exp\theta : g \in \mathrm{L}^2([0,\mathrm{T}]^n)\}$ is total in $\mathrm{L}^2(\mathbb{P})$ (i.e. its linear span is dense) [Oks13, Lemma 4.3.2], $\psi = 0$. Thus, we obtain

$$\xi = \sum_{k=0}^{\infty} \mathrm{J}_k(g_k)$$

and

$$\|\xi\|^2_{\mathrm{L}^2(\mathbb{P})} = \sum_{k=0}^{\infty} \|\mathrm{J}_k(g_k)\|^2_{\mathrm{L}^2(\mathbb{P})}$$

9. To extend g_n from S_n to $[0,\mathrm{T}]^n$, we put

$$g_n(t_1, \dots t_n) = 0, \quad (t_1, \dots t_n) \in [0,\mathrm{T}]^n \setminus \mathrm{S}_n$$

and define $f_n = \hat{g}_n$, i.e., the symmetrization of g_n.

Then,

$$\mathrm{I}_n(f_n) = n!\mathrm{J}_n(f_n) = n!\mathrm{J}_n(\hat{g}_n) = \mathrm{J}_n(g_n)$$

and the result follows. □

Chapter 2

The Skorohod Integral

2.1 Construction

Take $u(t)$ measurable random variable (from a stochastic process $u(x,t)$) in $L^2(\mathbb{P})$.

Write the Chaos Expansion $u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t})$ for each $t \in [0,T]$. This gives us symmetric functions $f_{n,t}(t_1,\dots,t_n)$.

Since $f_{n,t}(t_1,\dots,t_n)$ also depends on t, we can write $f_n(t_1,\dots,t_n,t_{n+1})$, with $t_{n+1} = t$, and take its symmetrization $\hat{f}_n$.

Define the **Skorohod Integral** as

$$\delta(u) = \int_0^T u(t)\ \delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n)$$

Remark. $u \in \mathrm{Dom}(\delta)$ iff.

$$\mathbb{E}[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty$$

2.2 Properties

1. The Skorohod integral is a linear operator.

2. The Skorohod integral is additive: we can compute over $(0,t]$ and $(t,T]$ using characteristic functions.

3. Since Itô integrals have zero expectation,

$$\mathbb{E}[\delta(u)] = 0$$

2.3 The Skorohod Integral as an Extension of the Itô Integral

Now, how is the Skorohod Integral related to the Itô integral? The Skorohod integral is an extension of the Itô integral for integrands that, not necessarily, are $\mathbb{F}$-adapted. We'll show that if the integrand is $\mathbb{F}$-adapted, then they coincide.

To prove that, we'll need the following lemma.

Lemma 2.3.1. Suppose that $u(t)$, $t \in [0, \mathrm{T}]$, is a measurable stochastic process such that, for all $t \in [0, \mathrm{T}]$, the random variable $u(t)$ is $\mathscr{F}_{\mathrm{T}}$-measurable and $\mathbb{E}[u^2(t)] < \infty$. If

$$u(t) = \sum_{n=0}^{\infty} \mathrm{I}_n(f_n(\cdot, t))$$

is its Wiener-Itô chaos expansion, then u is $\mathbb{F}$-adapted iff.

$$t < \max\{t_1, \dots, t_n\} \implies f_n(t_1, \dots, t_n, t) = 0 \quad \text{a.e.}$$

with respect to the Lebesgue measure in $[0, \mathrm{T}]^n$.

Proof. Let $g \in \hat{\mathrm{L}}^2([0, \mathrm{T}]^n)$ and compute

$$\mathbb{E}[\mathrm{I}_n(g) \mid \mathscr{F}_t] = \mathrm{I}_n(g(t_1, \dots, t_n)\chi_{\{\max\ t_i < t\}})$$

Note that $u(t)$ is $\mathbb{F}$-adapted iff.

$$\mathbb{E}[u(t) \mid \mathscr{F}_t] = u(t)$$

which is equivalent to

$$f_n(t_1, \dots, t_n, t)\chi_{\{\max\ t_i < t\}} = f_n(t_1, \dots, t_n, t)$$

Since the sequence of deterministic functions in the Wiener-Itô chaos expansion is unique, the proof is finished. □

Theorem 2.3.2. Suppose that $u(t)$ is measurable and $\mathbb{F}$-adapted and

$$\mathbb{E}\left[\int_0^{\mathrm{T}} u^2(t)\ \mathrm{d}t\right] < \infty$$

Then $u(t) \in \mathrm{Dom}(\delta)$ and the Itô and Skorohod integrals coincide.

Proof. Write the Chaos expansion of $u(t)$ and apply the Lemma 2.3.1 to simplify.

Using the Lemma again, compute

$$\|\hat{f}_n\|^2_{\mathrm{L}^2([0,\mathrm{T}]^{n+1})} = \frac{1}{n+1}\int_0^{\mathrm{T}} \|f_n(\cdot, t)\|^2_{\mathrm{L}^2([0,\mathrm{T}]^n)}\ \mathrm{d}t$$

Show that $u \in \mathrm{Dom}(\delta)$ using the Wiener Itô Chaos Expansion and the Remark about

Skorohod Integrability:

$$\sum_{n=0}^{\infty}(n+1)!\|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} = \mathbb{E}\left[\int_0^T u^2(t)\ dt\right] < \infty$$

Finally, by the Chaos Expansion and using symmetrization, compute

$$\int_0^T u(t)\ dW(t) = \sum_{n=0}^{\infty}\int_0^T I_n(f_n(\cdot,t))\ dW(t) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n) = \int_0^T u(t)\ \delta W(t)$$

□

Chapter 3

Malliavin Derivative

In this chapter, we construct the Malliavin Derivative, prove some handy results to compute it and finish by presenting the duality between the Malliavin Derivative and the Skorohod Integral.

3.1 Construction

Using the Wiener-Itô Chaos Expansion, it is quite natural to define the derivative of a random variable. But for this to make sense, it is necessary to restrict the definition to a suitable context as follows.

Definition 3.1.1. Let $F \in L^2(\mathbb{P})$ be $\mathscr{F}_t$-measurable with Wiener-Itô Chaos Expansion:

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

1. We say that $F \in \mathbb{D}_{1,2}$ if

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{i=0}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^n)} < \infty \tag{3.1}$$

2. If $F \in \mathbb{D}_{1,2}$, we define the **Malliavin derivative** $D_t F$ of F at time t as

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0, T] \tag{3.2}$$

in which $I_{n-1}(f_n(\cdot, t))$ is the $(n-1)$-fold iterated integral of $f_n(t_1, \ldots, t_{n-1}, t)$ with respect to the first $n-1$ variables, and $t_n = t$ as a parameter.

The restriction to $\mathbb{D}_{1,2}$ guarantees that the Malliavin derivative is well-defined in L^2.

Remark. If (3.1) holds, then

$$\|D.F\|^2_{L^2(\mathbb{P}\times\lambda)} = \mathbb{E}\left[\int_0^T (D_tF)^2\,dt\right] = \sum_{n=1}^{\infty}\int_0^T n^2(n-1)!\|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}\,dt$$
$$= \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2([0,T]^n)} = \|F\|^2_{\mathbb{D}_{1,2}} < \infty$$

Therefore, $D.F = D_tF$ is well-defined in $L^2(\mathbb{P}\times\lambda)$.

Theorem 3.1.1 (Closability of Malliavin Derivative). Suppose that $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}_{1,2}$, $k = 1,2,\ldots$, satisfy

a) $F_k \to F$ as $k \to \infty$ in $L^2(\mathbb{P})$,

b) $(D_tF_k)_{k=1}^{\infty}$ converges in $L^2(\mathbb{P}\times\lambda)$.

Then $F \in \mathbb{D}_{1,2}$ and $D_tF_k \to D_tF$ in $L^2(\mathbb{P}\times\lambda)$.

Proof. 1. Write the Wiener-Itô Expansion of F_k and F. Let

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad F_k = \sum_{n=0}^{\infty} I_n(f_n^{(k)})$$

2. Using (a), we have that $f_n^{(k)} \to f_n$ as $k \to \infty$ in $L^2(\lambda^n)$ for all n.

3. By (b),

$$\sum_{n=1}^{\infty} nn!\|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = \|D_tF_k - D_tF_j\|_{L^2(\mathbb{P}\times\lambda)} \longrightarrow 0, \quad j,k\to\infty$$

4. Using Fatou's Lemma,

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^{(k)} - f_n\|^2_{L^2(\lambda^n)} \le \lim_{k\to\infty}\liminf_{j\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = 0$$

5. Thus, $F \in \mathbb{D}_{1,2}$ and $D_tF_k \to D_tF$ in $L^2(\mathbb{P}\times\lambda)$. □

3.2 Computation

In this section, we explore how to compute the Malliavin derivative and some properties.

3.2.1 Chain Rule

Suppose that $f_n(t_1,\ldots,t_n) = f(t_1)\cdots f(t_n)$.

Write $D_t I_n(f_n)$ using Hermite polynomials:

$$\begin{aligned} D_t I_n(f_n) &= n I_{n-1}(f_n(\cdot, t)) \\ &= n I_{n-1}(f^{\otimes(n-1)}) f(t) \\ &= n \|f\|^{n-1} h_{n-1}\left(\frac{\theta}{\|f\|}\right) f(t) \end{aligned}$$

Then use that $h_n'(x) = n h_{n-1}(x)$ to get

$$D_t h_n\left(\frac{\theta}{\|f\|}\right) = h_n'\left(\frac{\theta}{\|f\|}\right)\frac{f(t)}{\|f\|} \tag{3.3}$$

From here, we extract two useful identities. For $n = 1$, we

$$D_t \int_0^T f(s)\, dW(s) = f(t)$$

and for $n > 1$, using induction,

$$D_t \left(\int_0^T f(s)\, dW(s)\right)^n = n\left(\int_0^T f(s)\, dW(s)\right)^{n-1} f(t)$$

Let $\mathbb{D}^0_{1,2}$ be the set of $F \in L^2(\mathbb{P})$ whose chaos expansion has only finitely many items.

Theorem 3.2.1 (Product Rule)**.** If $F_1, F_2 \in \mathbb{D}^0_{1,2}$, then $F_1, F_2 \in \mathbb{D}_{1,2}$ and $F_1F_2 \in \mathbb{D}_{1,2}$ and

$$D_t(F_1F_2) = F_1 D_t F_2 + F_2 D_t F_1$$

Proof. It is clear that $F_1, F_2 \in \mathbb{D}_{1,2}$.

To prove the second claim, notice that Gaussian r.v. have finite moments.

Let $\{\xi_i\}_{i=1}^{\infty}$ be an orthogonal basis of $L^2([0,T]^n)$. Take $F_k^{(n)}$ as linear combination of iterated itegrals of the tensor product of ξ_i.

By the Proposition 1.1.2, we have the result for $F_K^{(n)}$. Choose sequence that converges $F_k^{(n)} \to F_k$ and $D_t F_k^{(n)} \to D_t F_k$. □

Theorem 3.2.2 (Chain Rule)**.** Let $G \in \mathbb{D}_{1,2}$ and $g \in C^1(\mathbb{R})$ with bounded derivative. Then $g(G) \in \mathbb{D}_{1,2}$ and

$$D_t g(G) = g'(G) D_t G \tag{3.4}$$

3.2.2 Conditional Expectation

What happens when we take the conditional expectation of 1. the integral of a function in $L^2([0,T])$, 2. the integral of an $\mathbb{F}$-adapted process, and 3. an iterated integral? After answering these questions, we look at what happens when we take the derivative of a conditional expectation.

Definition 3.2.1. Let G be a Borel set in $[0,T]$. We define $\mathscr{F}_G$ to be the completed σ-algebra generated by all random variables of the form

$$F = \int_0^T \chi_A(t)\, dW(t)$$

for all Borel sets $A \subseteq G$.

Lemma 3.2.3. For any $g \in L^2([0,T])$ we have

$$\mathbb{E}\left[\int_0^T g(t)\, dW(t) \,\middle|\, \mathscr{F}_G\right] = \int_0^T \chi_G(t) g(t)\, dW(t)$$

Proof. The first step is to prove that $\int_0^T \chi_G(t) g(t)\, dW(t)$ is $\mathscr{F}_G$-measurable. Since continuous functions are dense in $L^2([0,T])$, assume that g is continuous. Then

$$\int_0^T \chi_G(t) g(t)\, dW(t) = \lim_{\Delta t_i \to 0} \sum_{i=0}^{n} g(t_i) \int_{t_i}^{t_{i+1}} \chi_G(t)\, dW(t)$$

with the limit in $L^2(\mathbb{P})$. And we can take a subsequence which converges almost surely.

Now we prove that

$$\mathbb{E}\left[F \int_0^T g(t)\, dW(t)\right] = \mathbb{E}\left[F \int_0^T \chi_G(t) g(t)\, dW(t)\right]$$

in which F is a bounded $\mathscr{F}_G$-measurable random variable. We may assume $F = \int_0^T \chi_A(t)\, dW(t)$ for $A \subseteq G$. Applying Itô Isometry, we have the result. □

Lemma 3.2.4. Let $G \subseteq [0,T]$ be a Borel set and $v(t)$ be a stochastic process

1. $v(t)$ is measurable with respect to $\mathscr{F}_t \cap \mathscr{F}_G = \mathscr{F}_{[0,t]\cap G}$ for all $t \in [0,T]$.
2. $\mathbb{E}\left[\int_0^T v^2(t)\, dt\right] < \infty$.

Then the following integral is $\mathscr{F}_G$-measurable:

$$\int_G v(t)\, dW(t)$$

Proof. Consider v as an elementary process and integrate. The general case follows from approximation. □

Lemma 3.2.5. Let $u(t)$ be an $\mathbb{F}$-adapted stochastic process in $L^2(\mathbb{P} \times \lambda)$. Then

$$\mathbb{E}\left[\int_0^T u(t)\, dW(t) \,\middle|\, \mathscr{F}_G\right] = \int_G \mathbb{E}[u(t) \mid \mathscr{F}_G]\, dW(t)$$

Proof. By the lemma 3.2.4, we have that $\int_G \mathbb{E}[u(t) \mid \mathscr{F}_G]\, dW(t)$ is $\mathscr{F}_G$-measurable.

Our goal is to verify

$$\mathbb{E}\left[F \int_0^T u(t)\, dW(t) \right] = \mathbb{E}\left[F \int_G \mathbb{E}[u(t) \mid \mathscr{F}_G]\, dW(t) \right]$$

for $F = \int_A dW(t)$ and $A \subseteq G$ Borel.

Apply Itô Isometry to both sides of the equation and use a density argument. □

Proposition 3.2.6. Let $f_n \in \hat{L}^2([0,T]^n)$. Then $\mathbb{E}[I_n(f_n) \mid \mathscr{F}_G] = I_n[f_n \chi_G^{\otimes n}]$.

Proof. By induction on n and the lemma 3.2.5. □

Proposition 3.2.7. If $F \in \mathbb{D}_{1,2}$, then $\mathbb{E}[F \mid \mathscr{F}_G] \in \mathbb{D}_{1,2}$ and

$$D_t \mathbb{E}[F \mid \mathscr{F}_G] = \mathbb{E}[D_t F \mid \mathscr{F}_G] \chi_G(t)$$

Proof. If $F = I_n(f_n)$, then the result follows from the Proposition 3.2.6.

More generally, if $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{D}_{1,2}$, we define $F_k = \sum_{n=0}^{k} I_n(f_n)$.

Then $F_k \to F$ in $L^2(\Omega)$ and $D_t F_k \to D_t F$ in $L^2(\mathbb{P} \times \lambda)$ as $k \to \infty$.

Using the previous case, we have $D_t \mathbb{E}[F_k \mid \mathscr{F}_G] = \mathbb{E}[D_t F_k \mid \mathscr{F}_G] \chi_G(t)$ for all k. Taking the limit in $L^2(\mathbb{P} \times \lambda)$, we have the result. □

Corollary 3.2.8. Let $u(s)$ be an $\mathbb{F}$-adapted stochastic process such that $u(s) \in \mathbb{D}_{1,2}$ for all $s \in [0,T]$. Then

1. $D_t u(s)$, $s \in [0,T]$, is $\mathbb{F}$-adapted for all t.
2. $D_t u(s) = 0$ for $t > s$.

Proof. Apply the Proposition 3.2.7 to $D_t u(s)$. □

3.3 Malliavin Derivative and Skorohod Integral

The main question behind this section is how the Malliavin derivative and the Skorohod integral are related. We show the duality between them, an integration by parts formula, the closability of the Skorohod integral, and the Fundamental Theorem of Calculus.

We start by showing that the Malliavin derivative is the adjoint operator of the Skorohod integral.

Theorem 3.3.1 (Duality Formula)**.** Let $F \in \mathbb{D}_{1,2}$ be $\mathscr{F}_T$-measurable and u be a Skorohod integrable stochastic process. Then

$$\mathbb{E}\left[F\int_0^T u(t)\ \delta W(t)\right] = \mathbb{E}\left[\int_0^T u(t) D_t F\ dt\right]$$

Put another way,

$$\langle \delta(u), F\rangle_{L^2(\mathbb{P})} = \langle u, D.F\rangle_{L^2(\mathbb{P}\times\lambda)}$$

Proof.
1. Write the Chaos expansions of F and $u(t)$.
2. Replace the expansions in both sides of the equation.
3. Notice that $\langle f_{k+1}, \hat{g}_k\rangle_{L^2([0,T]^{k+1})} = \langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}$.

□

Theorem 3.3.2 (Integration by Parts)**.** Let $u(t)$, $t \in [0,T]$ be Skorohod integrable and $F \in \mathbb{D}_{1,2}$ such that $Fu(t)$ is Skorohod integrable. Then

$$F\int_0^T u(t)\ \delta W(t) = \int_0^T Fu(t)\ \delta W(t) + \int_0^T u(t) D_t F\ dt$$

Proof. Assume that $F \in \mathbb{D}^0_{1,2}$ and let $G \in \mathbb{D}^0_{1,2}$. Using the Duality Formula and the Product Rule, we have that

$$\begin{aligned}\mathbb{E}\left[G\int_0^T Fu(t)\ \delta W(t)\right] &= \mathbb{E}\left[\int_0^T Fu(t) D_t G\ dt\right] \\ &= \mathbb{E}\left[GF\int_0^T u(t)\ \delta W(t)\right] - \mathbb{E}\left[G\int_0^T u(t) D_t F\ dt\right]\end{aligned}$$

Since $\mathbb{D}^0_{1,2}$ is dense in $L^2(\mathbb{P})$, the result follows. The general case follows by approximation. □

In fact, we can replace the hypothesis that Fu is Skorohod integrable by the existence in $L^2(\mathbb{P})$ of

$$F\int_0^T u(t)\ \delta W(t) \quad \text{and} \quad \int_0^T u(t) D_t F\ dt$$

Theorem 3.3.3 (Closability of the Skorohod Integral)**.** Let (u_n) be a sequence of Skorohod integrable stochastic processes and that

$$\delta(u_n) = \int_0^T u_n(t)\ \delta W(t)$$

converges in $L^2(\mathbb{P})$. And suppose that $\lim_{n\to\infty} u_n = 0$ in $L^2(\mathbb{P}\times\lambda)$.

Then $\lim_{n\to\infty} \delta(u_n) = 0$ in $L^2(\mathbb{P})$.

Proof. Using the Duality Formula,

$$\langle \delta(u_n), F\rangle_{L^2(\mathbb{P})} = \langle u_n, D.F\rangle_{L^2(\mathbb{P}\times\lambda)} \longrightarrow 0$$

as $n \to \infty$. Thus, $\delta(u_N) \to 0$ weakly in $L^2(\mathbb{P})$. Since $(\delta(u_n))$ is convergent in $L^2(\mathbb{P})$, we have that $\delta(u_n) \to 0$ in $L^2(\mathbb{P})$. □

Theorem 3.3.4 (Fundamental Theorem of Calculus)**.** Let $u(s)$ be a stochastic process such that

$$\mathbb{E}\left[\int_0^T u^2(s)\ ds\right] < \infty$$

Also suppose that $u(s) \in \mathbb{D}_{1,2}$ for all $s \in [0,T]$, $D_t u \in \mathrm{Dom}(\delta)$ and that

$$\mathbb{E}\left[\int_0^T (\delta(D_t u))^2\ dt\right] < \infty$$

Then $\int_0^T u(s)\ \delta W(s)$ is well-defined and belongs to $\mathbb{D}_{1,2}$ and

$$D_t\left(\int_0^T u(s)\ \delta W(s)\right) = \int_0^T D_t u(s)\ \delta W(s) + u(t)$$

Proof. We start by proving a simpler case. Suppose that $u(s) = I_n(f_n(\cdot,s))$, where $f_n(t_1,\dots,t_n,s)$ is symmetric with respect to $t_1,\dots,t_n$. Then

$$\int_0^T u(s)\ \delta W(s) = I_{n+1}[\tilde{f}_n]$$

where

$$\tilde{f}_n(x_1,\dots,x_{n+1}) = \frac{1}{n+1}\left[f_n(\cdot,x_1) + \cdots + f_n(\cdot,x_{n+1})\right]$$

Thus,

$$\begin{aligned} D_t\left(\int_0^T u(s)\ \delta W(s)\right) &= (n+1) I_n[\tilde{f}_n(\cdot,t)] \\ &= I_n\left[f_n(\cdot,x_1) + \cdots + f_n(\cdot,x_n) + f_n(\cdot,t)\right] \\ &= I_n\left[f_n(\cdot,x_1) + \cdots + f_n(\cdot,x_n)\right] + u(t) \end{aligned} \tag{3.5}$$

Now consider

$$\begin{aligned} \delta(D_t u) &= \int_0^T D_t u(s)\ \delta W(s) \\ &= \int_0^T n I_{n-1}[f_n(\cdot,t,s)]\ \delta W(s) \\ &= n I_n[\hat{f}_n(\cdot,t,\cdot)] \end{aligned} \tag{3.6}$$

where

$$\hat{f}_n = \frac{1}{n}\left[f_n(t,\cdot,x_1) + \cdots + f_n(t,\cdot,x_n)\right]$$

Using the symmetrization above into (3.6), we obtain

$$\int_0^T D_t u(s)\ \delta W(s) = I_n\left[f_n(t,\cdot,x_1) + \cdots + f_n(t,\cdot,x_n)\right] \tag{3.7}$$

Combining (3.5) and (3.7), we have

$$D_t\left(\int_0^T u(s)\ \delta W(s)\right) = \int_0^T D_t u(s)\ \delta W(s) + u(t)$$

For the general case, when $u(s) = \sum_{n=0}^{\infty} I_n[f_n(\cdot,s)]$, consider $u_m(s) = \sum_{n=0}^{m} I_n[f_n(\cdot,s)]$. By the previous argument,

$$D_t(\delta(u_m)) = \delta(D_t u_m) + u_m(t)$$

for all t.

The result follows by taking $m \to \infty$, which requires some technicalities (see [NØP08]). □

Corollary 3.3.5. Suppose that u satisfies the conditions of the theorem and that $u(s)$ is $\mathbb{F}$-adapted. Then

$$D_t\left(\int_0^T u(s)\ dW(s)\right) = \int_t^T D_t u(s)\ dW(s) + u(t)$$

Proof. Apply the Corollary 3.2.8. □

Bibliography

[Itô51] Kiyosi Itô. Multiple wiener integral. *Journal of the Mathematical Society of Japan*, 3(1):157–169, 1951.

[NN18] David Nualart and Eulalia Nualart. *Introduction to Malliavin calculus*, volume 9. Cambridge University Press, 2018.

[NØP08] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin calculus for Lévy processes with applications to finance*. Springer, 2008.

[Oks13] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.