Malliavin Calculus with Applications to Finance

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Contents

1 The Wiener-Itô Chaos Expansion **3**

1.1 Iterated Itô Integrals 3

1.2 The Wiener-Itô Chaos Expansion 6

2 The Skorohod Integral **9**

2.1 Construction 9

2.2 Properties 9

2.3 The Skorohod Integral as an Extension of the Itô Integral 10

3 Malliavin Derivative **11**

3.1 Construction 11

3.2 Computation 12

3.2.1 Chain Rule 12

3.2.2 Conditional Expectation 13

3.3 Malliavin Derivative and Skorohod Integral 15

4 Stochastic Integral Representations **19**

4.1 The Clark-Ocone Formula 19

4.2 Under Change of Measure 20

4.3 Portfolio Selection 23

4.4 Sensitivity Analysis and Computation of Greeks 25

5 White Noise, the Wick Product, and Stochastic Integration **29**

5.1 White Noise 29

5.2 Revisiting the Wiener-Itô Chaos Expansion 30

5.3 The Wick Product 33

6 The Hida-Malliavin Derivative on the Integration Space **38**

6.1 Stochastic Gradient 38

6.2 Calculus of the Hida-Malliavin Derivative 39

6.3 Conditional Expectation on $(\mathscr{S})^*$ 41
6.4 Generalized Clark-Ocone 43

7 Application to Finance **47**
7.1 Introduction 47
7.2 Necessary Results 47
7.3 Main Result 49

References **50**

Preface

These notes were written as a part of my Undergraduate Research on ‘Malliavin Calculus with Applications to Finance’. My previous notes, covering the basics of Stochastic Differential Equations up to Applications in Option Pricing, are available at [dSN23].

Chapter 1

The Wiener-Itô Chaos Expansion

1.1 Iterated Itô Integrals

Consider

- $W(t)$ a one-dimensional Wiener process on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ such that $W(0) = 0$ a.s.;
- $\mathscr{F}_t$ the σ-algebra generated by $W(s)$, $0 \le s \le t$, augmented by the events with probability zero;
- The filtration $\mathbf{F} = \{\mathscr{F}_t : t \in [0, T]\}$.

Definition 1.1.1. A function $g : [0,T]^n \longrightarrow \mathbf{R}$ is **symmetric** if

$$g(t_{\sigma_1}, \ldots, t_{\sigma_n}) = g(t_1, \ldots, t_n)$$

for all permutations $\sigma = (\sigma_1, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$.

We define $\hat{L}^2([0,T]^n)$ as the subspace of $L^2([0,T]^n)$ consisting of symmetric functions.

Now consider the set

$$S_n = \{(t_1, \ldots, t_n) \in [0,T]^n : 0 \le t_1 \le \cdots \le t_n \le T\}$$

Notice that the set S_n occupies $\frac{1}{n!}$ of the whole box $[0,T]^n$. Thus, if $g \in \hat{L}^2([0,T]^n)$, then $g|_{S_n} \in L^2(S_n)$ and

$$\|g\|^2_{L^2([0,T]^n)} = n! \int_{S_n} g^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = n! \|g\|^2_{L^2(S_n)}$$

Definition 1.1.2. Given a function $f : [0,T]^n \longrightarrow \mathbf{R}$, we define its **symmetrization** $\hat{f}$ as

$$\hat{f}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \ldots, t_{\sigma_n})$$

in which the sum runs over all permutations σ of $\{1, 2, \ldots, n\}$.

Now we are ready to define the n-fold iterated Itô integral.

Definition 1.1.3. Let f be a deterministic function defined on S_n such that $\|f\|^2_{L^2(S_n)} < \infty$. We define the **$n$-fold iterated Itô integral** as

$$J_n(f) = \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \dots, t_n)\, dW(t_1) dW(t_2) \cdots dW(t_{n-1}) dW(t_n)$$

Remark 1.1.1.
1. Note that each i-th Itô integral with respect to $dW(t_i)$ is well-defined, since the integrand is an $\mathbf{F}$-adapted stochastic process.
2. Furthermore, $J_n(f) \in L^2(\mathbf{P})$.

Now, apply Itô's isometry iteratively.

First case: If $g \in L^2(S_m)$ and $h \in L^2(S_n)$ with $m < n$, then

$$\begin{aligned}
\mathbf{E}[J_m(g)J_n(h)] = \mathbf{E}\Bigg[& \left(\int_0^T \int_0^{s_m} \cdots \int_0^{s_2} g(s_1, \dots, s_m)\, dW(s_1) \cdots dW(s_m) \right) \\
& \left(\int_0^T \int_0^{s_m} \cdots \int_0^{t_2} h(t_1, \dots, t_{n-m}, s_1, \dots, s_m)\, dW(t_1) \cdots dW(t_{n-m}) dW(s_1) \cdots dW(s_m) \right) \Bigg] \\
& = \int_0^T \int_0^{s_m} \cdots \int_0^{s_2} g(s_1, s_2, \dots, s_m) \\
& \mathbf{E}\left[\int_0^{s_1} \cdots \int_0^{t_2} h(t_1, \dots, t_{n-m}, s_1, \dots, s_m)\, dW(t_1) \cdots dW(t_{n-m}) \right] ds_1 \cdots ds_m \\
& = 0
\end{aligned}$$

Second case: If $g, h \in L^2(S_n)$, then

$$\mathbf{E}[J_n(g)J_n(h)] = \int_0^T \cdots \int_0^{s_2} g(s_1, \dots, s_n) h(s_1, \dots, s_n)\, ds_1 \cdots ds_n = \langle g, h \rangle_{L^2(S_n)}$$

in which $\langle g, h \rangle_{L^2(S_n)}$ is the inner product of $L^2(S_n)$.

This proves the following

Proposition 1.1.2. For $m, n \in \mathbf{Z}_{>0}$,

$$\mathbf{E}[J_m(g)J_n(h)] = \begin{cases} 0, & n \neq m \\ \langle g, h \rangle_{L^2(S_n)}, & n = m \end{cases}$$

In particular,

$$\|J_n(h)\|_{L^2(\mathbf{P})} = \|h\|_{L^2(S_n)}$$

For $n = 0$ or $m = 0$, we define $J_0(g) = g$, when g is a constant, and $\langle g, h \rangle_{L^2(S_0)} = gh$, when g and h are constants.

Remark 1.1.3.
1. If $f \in L^2(S_n)$, then $J_n(f) \in L^2(\mathbf{P})$.
2. The n-fold iterated Itô integral is a linear operator.

Definition 1.1.4. Let $g \in \hat{\mathrm{L}}^2([0,\mathrm{T}]^n)$. Then

$$\mathrm{I}_n(g) = \int_{[0,\mathrm{T}]^n} g(t_1,\ldots,t_n)\, \mathrm{dW}(t_1)\cdots\mathrm{dW}(t_n) = n!\mathrm{J}_n(g)$$

is also called **n-fold iterated Itô integral**.

For $n = 0$, we define

$$\mathrm{I}_0(g) = \int_{\mathbf{R}^0} g\, \mathrm{dW}^{\otimes 0} = g$$

Let $x \in \mathbf{R}$ and $n = 0,1,2,\ldots$. Then the **Hermite polynomials** are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right)$$

For example, the first Hermite polynomials are

1. $h_0(x) = 1$,
2. $h_1(x) = x$,
3. $h_2(x) = x^2 - 1$,
4. $h_3(x) = x^3 - 3x$,
5. $h_4(x) = x^4 - 6x^3 + 3$,
6. $h_5(x) = x^5 - 10x^3 + 15x$.

The family of Hermite polynomials constitute an orthogonal basis for $\mathrm{L}^2(\mathbf{R}, \mu(dx))$, in which $\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx$.

Proposition 1.1.4. If $\xi_1, \xi_2, \ldots$ are orthonormal functions in $\mathrm{L}^2([0,\mathrm{T}])$, then

$$\mathrm{I}_n(\xi_1^{\otimes\alpha_1} \hat{\otimes} \cdots \hat{\otimes} \xi_m^{\otimes\alpha_m}) = \prod_{k=1}^{m} h_{\alpha_k}\left(\int_0^{\mathrm{T}} \xi_k(t)\, \mathrm{dW}(t)\right)$$

with $\alpha_1 + \cdots + \alpha_m = n$, $\alpha_k \in \mathbf{N}_0$ for all k, and $\hat{\otimes}$ is the symmetrized tensor product, which is the symmetrization of $f \otimes g$.

Proof. [Itô51] □

See Itô's formula for iterated Itô integral ([Oks13]'s problem 3.7).

Using this, it is possible to prove (see [NN18] p. 64)

$$\mathrm{I}_n(g^{\otimes n}) = n! \int_0^{\mathrm{T}} \int_0^{t_n} \cdots \int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\, \mathrm{dW}(t_1)\cdots\mathrm{dW}(t_n) = \|g\|^n h_n\left(\frac{\int_0^{\mathrm{T}} g(t)\, \mathrm{dW}(t)}{\|g\|}\right)$$

1.2 The Wiener-Itô Chaos Expansion

Theorem 1.2.1 (The Wiener-Itô Chaos Expansion)**.** Let ξ be an $\mathscr{F}_T$-measurable random variable in $L^2(\mathbf{P})$. There exists a unique sequence (f_n) of functions $f_n \in \hat{L}^2([0,T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n)$$

with convergence in $L^2(\mathbf{P})$. Moreover, we have the following isometry:

$$\|\xi\|^2_{L^2(\mathbf{P})} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2([0,T]^n)}$$

Proof. Our goal is to obtain an orthogonal decomposition of $L^2(\mathbf{P})$. To do that, we show that a certain function ψ is orthogonal to

$$\exp\left(\int_0^T g(t)\ dW(t)\right)$$

which form a total set in $L^2(\mathbf{P})$, implying that $\psi \equiv 0$.

1. By the Itô's Representation Theorem, there exists an **F**-adapted process $\varphi_1(s_1)$, $0 \le s_1 \le T$, such that

$$\mathbf{E}\left[\int_0^T \varphi_1^2(s_1)\ ds_1\right] \le \mathbf{E}[\xi^2]$$

and

$$\xi = \mathbf{E}[\xi] + \int_0^T \varphi_1(s_1)\ dW(s_1)$$

Define $g_0 = \mathbf{E}[\xi]$.

2. For almost all $s_1 \le T$, we can apply the Itô's Representation Theorem to $\varphi_1(s_1)$ and obtain an **F**-adapted process $\varphi_2(s_1, s_1)$, $0 \le s_2 \le s_1$ such that

$$\mathbf{E}\left[\int_0^{s_1} \varphi_2^2(s_2, s_1)\ ds_2\right] \le \mathbf{E}[\varphi_1^2(s_1)] < \infty$$

and

$$\varphi_1(s_1) = \mathbf{E}[\varphi_1(s_1)] + \int_0^{s_1} \varphi_2(s_2, s_1)\ dW(s_2)$$

3. Replacing $\varphi_1(s_1)$ into our expression for ξ yields

$$\xi = g_0 + \int_0^T g_1(s_1)\ dW(s_1) + \int_0^T \int_0^{s_1} \varphi(s_2, s_1)\ dW(s_2) dW(s_1)$$

where $g_1(s_1) = \mathbf{E}[\varphi_1(s_1)]$.

4. Applying Itô's isometry,

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T\int_0^{s_1}\varphi_2(s_2,s_1)\,\mathrm{d}W(s_2)\mathrm{d}W(s_1)\right)^2\right]\\&=\int_0^T\int_0^{s_1}\mathbf{E}[\varphi_2^2(s_2,s_1)]\,\mathrm{d}s_2\mathrm{d}s_1\le\mathbf{E}[\xi^2]\end{aligned}\tag{1.1}$$

5. Iterating this procedure $n+1$ times, we obtain a process $\varphi_{n+1}(t_1,\dots,t_{n+1})$, $0\le t_1\le\dots\le t_{n+1}\le T$, and $n+1$ deterministic functions $g_0,g_1,\dots,g_n$ (where $g_0=\mathbf{E}[\xi]$ and $g_k(s_k,s_{k-1},\dots,s_1)=\mathbf{E}[\varphi_k(s_k,s_{k-1},\dots,s_1)]$ for $1\le k\le n$) such that

$$\xi=\sum_{k=0}^n J_k(g_k)+\int_{S_{n+1}}\varphi_{n+1}\,\mathrm{d}W^{\otimes(n+1)}$$

6. Note that we have a $(n+1)$-fold Iterated Itô Integral

$$\int_{S_{n+1}}\varphi_{n+1}\,\mathrm{d}W^{\otimes(n+1)}=:\psi_{n+1}$$

and

$$\mathbf{E}\left[\left(\int_{S_{n+1}}\varphi_{n+1}\,\mathrm{d}W^{\otimes(n+1)}\right)^2\right]\le\mathbf{E}[\xi^2]$$

7. Also remark that the family ψ_{n+1} is bounded in $L^2(\mathbf{P})$ and, from Itô's Isometry,

$$\langle\psi_{n+1},J_k(f_k)\rangle_{L^2(\mathbf{P})}=0$$

for $k\le n$ and $f_k\in L^2([0,T]^k)$

8. Compute $\|\xi\|^2_{L^2(\mathbf{P})}$ and notice that $\sum_{k=0}^\infty J_k(g_k)$ is convergent in $L^2(\mathbf{P})$. Thus,

$$\langle J_k(f_k),\psi\rangle_{L^2(\mathbf{P})}=0$$

9. Using that

$$I_n(g^{\otimes n})=\|g\|^n h_n\left(\frac{\theta}{\|g\|}\right),\quad\theta=\int_0^T g(t)\,\mathrm{d}W(t)$$

and Hermite polynomials, we have

$$\mathbf{E}\left[h_n\left(\frac{\theta}{\|g\|}\right)\psi\right]=0,\ \mathbf{E}\left[\theta^k\psi\right]=0,\ \mathbf{E}\left[\exp\theta\cdot\psi\right]=\sum_{k=0}^\infty\frac{1}{k!}\mathbf{E}\left[\theta^k\psi\right]=0$$

10. Since $\{\exp\theta : g\in L^2([0,T]^n)\}$ is total in $L^2(\mathbf{P})$ (i.e. its linear span is dense) [Oks13, Lemma 4.3.2], $\psi=0$. Thus, we obtain

$$\xi=\sum_{k=0}^\infty J_k(g_k)$$

and

$$\|\xi\|^2_{L^2(\mathbf{P})}=\sum_{k=0}^\infty\|J_k(g_k)\|^2_{L^2(\mathbf{P})}$$

11. To extend g_n from S_n to $[0,T]^n$, we put

$$g_n(t_1,\dots t_n) = 0, \quad (t_1,\dots t_n) \in [0,T]^n \setminus S_n$$

and define $f_n = \hat{g}_n$, i.e., the symmetrization of g_n.

Then,

$$I_n(f_n) = n!J_n(f_n) = n!J_n(\hat{g}_n) = J_n(g_n)$$

and the result follows.

□

Chapter 2

The Skorohod Integral

2.1 Construction

1. Take $u(t)$ a measurable random variable from a stochastic process $u(x,t)$ in $L^2(\mathbf{P})$.

2. Write the Chaos Expansion $u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t})$ for each $t \in [0,T]$. This gives us symmetric functions $f_{n,t}(t_1,\ldots,t_n)$.

3. Since $f_{n,t}(t_1,\ldots,t_n)$ also depends on t, we can write $f_n(t_1,\ldots,t_n,t_{n+1})$, with $t_{n+1} = t$, and take its symmetrization $\hat{f}_n$.

4. Define the **Skorohod Integral** as

$$\delta(u) = \int_0^T u(t)\ \delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n)$$

Remark 2.1.1. $u \in \mathrm{Dom}(\delta)$ iff.

$$\mathbf{E}[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty$$

2.2 Properties

1. The Skorohod integral is a linear operator.

2. The Skorohod integral is additive: we can compute over $(0,t]$ and $(t,T]$ using characteristic functions.

3. Since Itô integrals have zero expectation,

$$\mathbf{E}[\delta(u)] = 0$$

2.3 The Skorohod Integral as an Extension of the Itô Integral

Now, how is the Skorohod Integral related to the Itô integral? The Skorohod integral is an extension of the Itô integral for integrands that, not necessarily, are **F**-adapted. We'll show that if the integrand is **F**-adapted, then they coincide.

To prove that, we'll need the following lemma.

Lemma 2.3.1. Suppose that $u(t)$, $t \in [0,\mathrm{T}]$, is a measurable stochastic process such that, for all $t \in [0,\mathrm{T}]$, the random variable $u(t)$ is $\mathscr{F}_\mathrm{T}$-measurable and $\mathbf{E}[u^2(t)] < \infty$. If

$$u(t) = \sum_{n=0}^{\infty} \mathrm{I}_n(f_n(\cdot,t))$$

is its Wiener-Itô chaos expansion, then u is **F**-adapted iff.

$$t < \max\{t_1,\dots,t_n\} \implies f_n(t_1,\dots,t_n,t) = 0 \quad \text{a.e.}$$

with respect to the Lebesgue measure in $[0,\mathrm{T}]^n$.

Proof. Let $g \in \hat{\mathrm{L}}^2([0,\mathrm{T}]^n)$ and compute

$$\mathbf{E}[\mathrm{I}_n(g) \mid \mathscr{F}_t] = \mathrm{I}_n(g(t_1,\dots,t_n)\chi_{\{\max\ t_i<t\}})$$

Note that $u(t)$ is **F**-adapted iff.

$$\mathbf{E}[u(t) \mid \mathscr{F}_t] = u(t)$$

which is equivalent to

$$f_n(t_1,\dots,t_n,t)\chi_{\{\max\ t_i<t\}} = f_n(t_1,\dots,t_n,t)$$

Since the sequence of deterministic functions in the Wiener-Itô chaos expansion is unique, the proof is finished. □

Theorem 2.3.2. Suppose that $u(t)$ is measurable and **F**-adapted and

$$\mathbf{E}\left[\int_0^\mathrm{T} u^2(t)\ \mathrm{d}t\right] < \infty$$

Then $u(t) \in \mathrm{Dom}(\delta)$ and the Itô and Skorohod integrals coincide.

Proof. Write the Chaos expansion of $u(t)$ and apply the Lemma 2.3.1 to simplify.

Using the Lemma again, compute

$$\|\hat{f}_n\|^2_{\mathrm{L}^2([0,\mathrm{T}]^{n+1})} = \frac{1}{n+1}\int_0^\mathrm{T} \|f_n(\cdot,t)\|^2_{\mathrm{L}^2([0,\mathrm{T}]^n)}\ \mathrm{d}t$$

Show that $u \in \mathrm{Dom}(\delta)$ using the Wiener Itô Chaos Expansion and the Remark about Skorohod Integrability:

$$\sum_{n=0}^{\infty}(n+1)!\|\hat{f}_n\|^2_{\mathrm{L}^2([0,\mathrm{T}]^{n+1})} = \mathbf{E}\left[\int_0^\mathrm{T} u^2(t)\ \mathrm{d}t\right] < \infty$$

Finally, by the Chaos Expansion and using symmetrization, compute

$$\int_0^\mathrm{T} u(t)\ \mathrm{dW}(t) = \sum_{n=0}^{\infty}\int_0^\mathrm{T} \mathrm{I}_n(f_n(\cdot,t))\ \mathrm{dW}(t) = \sum_{n=0}^{\infty}\mathrm{I}_{n+1}(\hat{f}_n) = \int_0^\mathrm{T} u(t)\ \delta\mathrm{W}(t)$$

□

Chapter 3

Malliavin Derivative

In this chapter, we construct the Malliavin Derivative, prove some handy results to compute it and finish by presenting the duality between the Malliavin Derivative and the Skorohod Integral.

3.1 Construction

Using the Wiener-Itô Chaos Expansion, it is quite natural to define the derivative of a random variable. But for this to make sense, it is necessary to restrict the definition to a suitable context as follows.

Definition 3.1.1. Let $F \in L^2(\mathbf{P})$ be $\mathscr{F}_t$-measurable with Wiener-Itô Chaos Expansion:

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

1. We say that $F \in \mathbf{D}_{1,2}$ if

$$\|F\|^2_{\mathbf{D}_{1,2}} := \sum_{i=0}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^n)} < \infty \tag{3.1}$$

2. If $F \in \mathbf{D}_{1,2}$, we define the **Malliavin derivative** $D_t F$ of F at time t as

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0, T] \tag{3.2}$$

in which $I_{n-1}(f_n(\cdot, t))$ is the $(n-1)$-fold iterated integral of $f_n(t_1, \dots, t_{n-1}, t)$ with respect to the first $n-1$ variables, and $t_n = t$ as a parameter.

The restriction to $\mathbf{D}_{1,2}$ guarantees that the Malliavin derivative is well-defined in L^2.

Remark 3.1.1. If (3.1) holds, then

$$\|D.F\|^2_{L^2(\mathbf{P}\times\lambda)} = \mathbf{E}\left[\int_0^T (D_t F)^2 \, dt\right] = \sum_{n=1}^{\infty} \int_0^T n^2 (n-1)! \|f_n(\cdot, t)\|^2_{L^2([0,T]^n)} \, dt$$
$$= \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^n)} = \|F\|^2_{\mathbf{D}_{1,2}} < \infty$$

Therefore, $D.F = D_tF$ is well-defined in $L^2(\mathbf{P}\times\lambda)$.

Theorem 3.1.2 (Closability of Malliavin Derivative)**.** Suppose that $F \in L^2(\mathbf{P})$ and $F_k \in \mathbf{D}_{1,2}$, $k = 1, 2, \ldots$, satisfy

a) $F_k \to F$ as $k \to \infty$ in $L^2(\mathbf{P})$,

b) $(D_tF_k)_{k=1}^{\infty}$ converges in $L^2(\mathbf{P}\times\lambda)$.

Then $F \in \mathbf{D}_{1,2}$ and $D_tF_k \to D_tF$ in $L^2(\mathbf{P}\times\lambda)$.

Proof. 1. Write the Wiener-Itô Expansion of F_k and F. Let

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad F_k = \sum_{n=0}^{\infty} I_n(f_n^{(k)})$$

2. Using (a), we have that $f_n^{(k)} \to f_n$ as $k\to\infty$ in $L^2(\lambda^n)$ for all n.

3. By (b),

$$\sum_{n=1}^{\infty} nn!\|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = \|D_tF_k - D_tF_j\|_{L^2(\mathbf{P}\times\lambda)} \longrightarrow 0, \quad j,k\to\infty$$

4. Using Fatou's Lemma,

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^{(k)} - f_n\|^2_{L^2(\lambda^n)} \le \lim_{k\to\infty}\liminf_{j\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = 0$$

5. Thus, $F \in \mathbf{D}_{1,2}$ and $D_tF_k \to D_tF$ in $L^2(\mathbf{P}\times\lambda)$. □

3.2 Computation

In this section, we explore how to compute the Malliavin derivative and some properties.

3.2.1 Chain Rule

Suppose that $f_n(t_1,\ldots,t_n) = f(t_1)\cdots f(t_n)$.

Write $D_tI_n(f_n)$ using Hermite polynomials:

$$\begin{aligned} D_tI_n(f_n) &= nI_{n-1}(f_n(\cdot,t)) \\ &= nI_{n-1}(f^{\otimes(n-1)})f(t) \\ &= n\|f\|^{n-1}h_{n-1}\left(\frac{\theta}{\|f\|}\right)f(t) \end{aligned}$$

Then use that $h_n'(x) = nh_{n-1}(x)$ to get

$$D_th_n\left(\frac{\theta}{\|f\|}\right) = h_n'\left(\frac{\theta}{\|f\|}\right)\frac{f(t)}{\|f\|} \tag{3.3}$$

From here, we extract two useful identities. For $n = 1$, we have

$$D_t \int_0^T f(s)\, dW(s) = f(t)$$

and for $n > 1$, using induction,

$$D_t \left(\int_0^T f(s)\, dW(s) \right)^n = n \left(\int_0^T f(s)\, dW(s) \right)^{n-1} f(t)$$

Let $\mathbf{D}^0_{1,2}$ be the set of $F \in L^2(\mathbf{P})$ whose chaos expansion has only finitely many items.

Theorem 3.2.1 (Product Rule)**.** If $F_1, F_2 \in \mathbf{D}^0_{1,2}$, then $F_1, F_2 \in \mathbf{D}_{1,2}$ and $F_1F_2 \in \mathbf{D}_{1,2}$ and

$$D_t(F_1F_2) = F_1 D_t F_2 + F_2 D_t F_1$$

Proof. It is clear that $F_1, F_2 \in \mathbf{D}_{1,2}$.

To prove the second claim, notice that Gaussian r.v. have finite moments.

Let $\{\xi_i\}_{i=1}^{\infty}$ be an orthogonal basis of $L^2([0,T]^n)$. Take $F_k^{(n)}$ as linear combination of iterated integrals of the tensor product of ξ_i.

By the Proposition 1.1.4, we have the result for $F_K^{(n)}$. Choose sequence that converges $F_k^{(n)} \to F_k$ and $D_t F_k^{(n)} \to D_t F_k$. □

Theorem 3.2.2 (Chain Rule)**.** Let $G \in \mathbf{D}_{1,2}$ and $g \in C^1(\mathbf{R})$ with bounded derivative. Then $g(G) \in \mathbf{D}_{1,2}$ and

$$D_t g(G) = g'(G) D_t G \tag{3.4}$$

For a proof, see 6.1.1.

3.2.2 Conditional Expectation

What happens when we take the conditional expectation of 1. the integral of a function in $L^2([0,T])$, 2. the integral of an $\mathbf{F}$-adapted process, and 3. an iterated integral? After answering these questions, we look at what happens when we take the Malliavin derivative of a conditional expectation.

Definition 3.2.1. Let G be a Borel set in $[0,T]$. We define $\mathscr{F}_G$ to be the completed σ-algebra generated by all random variables of the form

$$F = \int_0^T \chi_A(t)\, dW(t)$$

for all Borel sets $A \subseteq G$.

Lemma 3.2.3. For any $g \in L^2([0,T])$ we have

$$\mathbf{E}\left[\int_0^T g(t)\, dW(t) \,\middle|\, \mathscr{F}_G \right] = \int_0^T \chi_G(t) g(t)\, dW(t)$$

Proof. The first step is to prove that $\int_0^T \chi_G(t)g(t)\,dW(t)$ is $\mathscr{F}_G$-measurable. Since continuous functions are dense in $L^2([0,T])$, assume that g is continuous. Then

$$\int_0^T \chi_G(t)g(t)\,dW(t) = \lim_{\Delta t_i \to 0} \sum_{i=0}^{n} g(t_i) \int_{t_i}^{t_{i+1}} \chi_G(t)\,dW(t)$$

with the limit in $L^2(\mathbf{P})$. And we can take a subsequence which converges almost surely.

Now we prove that

$$\mathbf{E}\left[F \int_0^T g(t)\,dW(t)\right] = \mathbf{E}\left[F \int_0^T \chi_G(t)g(t)\,dW(t)\right]$$

in which F is a bounded $\mathscr{F}_G$-measurable random variable. We may assume $F = \int_0^T \chi_A(t)\,dW(t)$ for $A \subseteq G$. Applying Itô Isometry, we have the result. □

Lemma 3.2.4. Let $G \subseteq [0,T]$ be a Borel set and $v(t)$ be a stochastic process

1. $v(t)$ is measurable with respect to $\mathscr{F}_t \cap \mathscr{F}_G = \mathscr{F}_{[0,t]\cap G}$ for all $t \in [0,T]$.
2. $\mathbf{E}\left[\int_0^T v^2(t)\,dt\right] < \infty$.

Then the following integral is $\mathscr{F}_G$-measurable:

$$\int_G v(t)\,dW(t)$$

Proof. Consider v as an elementary process and integrate. The general case follows from approximation. □

Lemma 3.2.5. Let $u(t)$ be an **F**-adapted stochastic process in $L^2(\mathbf{P} \times \lambda)$. Then

$$\mathbf{E}\left[\int_0^T u(t)\,dW(t)\,\middle|\,\mathscr{F}_G\right] = \int_G \mathbf{E}[u(t) \mid \mathscr{F}_G]\,dW(t)$$

Proof. By the lemma 3.2.4, we have that $\int_G \mathbf{E}[u(t) \mid \mathscr{F}_G]\,dW(t)$ is $\mathscr{F}_G$-measurable.

Our goal is to verify

$$\mathbf{E}\left[F \int_0^T u(t)\,dW(t)\right] = \mathbf{E}\left[F \int_G \mathbf{E}[u(t) \mid \mathscr{F}_G]\,dW(t)\right]$$

for $F = \int_A dW(t)$ and $A \subseteq G$ Borel.

Apply Itô Isometry to both sides of the equation and use a density argument.

□

Proposition 3.2.6. Let $f_n \in \hat{L}^2([0,T]^n)$. Then $\mathbf{E}[I_n(f_n) \mid \mathscr{F}_G] = I_n[f_n \chi_G^{\otimes n}]$.

Proof. By induction on n and the lemma 3.2.5. □

Proposition 3.2.7. If $F \in \mathbf{D}_{1,2}$, then $\mathbf{E}[F \mid \mathscr{F}_G] \in \mathbf{D}_{1,2}$ and

$$D_t\mathbf{E}[F \mid \mathscr{F}_G] = \mathbf{E}[D_tF \mid \mathscr{F}_G]\chi_G(t)$$

Proof. If $F = I_n(f_n)$, then the result follows from the Proposition 3.2.6.

More generally, if $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbf{D}_{1,2}$, we define $F_k = \sum_{n=0}^{k} I_n(f_n)$.

Then $F_k \to F$ in $L^2(\Omega)$ and $D_tF_k \to D_tF$ in $L^2(\mathbf{P}\times\lambda)$ as $k \to \infty$.

Using the previous case, we have $D_t\mathbf{E}[F_k \mid \mathscr{F}_G] = \mathbf{E}[D_tF_k \mid \mathscr{F}_G]\chi_G(t)$ for all k. Taking the limit in $L^2(\mathbf{P}\times\lambda)$, we have the result. □

Corollary 3.2.8. Let $u(s)$ be an $\mathbf{F}$-adapted stochastic process such that $u(s) \in \mathbf{D}_{1,2}$ for all $s \in [0,T]$. Then

1. $D_tu(s)$, $s \in [0,T]$, is $\mathbf{F}$-adapted for all t.
2. $D_tu(s) = 0$ for $t > s$.

Proof. Apply the Proposition 3.2.7 to $D_tu(s)$. □

3.3 Malliavin Derivative and Skorohod Integral

The main question behind this section is how the Malliavin derivative and the Skorohod integral are related. We show the duality between them, an integration by parts formula, the closability of the Skorohod integral, and the Fundamental Theorem of Calculus.

We start by showing that the Malliavin derivative is the adjoint operator of the Skorohod integral.

Theorem 3.3.1 (Duality Formula). Let $F \in \mathbf{D}_{1,2}$ be $\mathscr{F}_T$-measurable and u be a Skorohod integrable stochastic process. Then

$$\mathbf{E}\left[F\int_0^T u(t)\ \delta W(t)\right] = \mathbf{E}\left[\int_0^T u(t)D_tF\ dt\right]$$

Put another way,

$$\langle \delta(u), F\rangle_{L^2(\mathbf{P})} = \langle u, D.F\rangle_{L^2(\mathbf{P}\times\lambda)}$$

Proof. 1. Write the Chaos expansions of F and $u(t)$.

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad u(t) = \sum_{k=0}^{\infty} I_k(g_k(\cdot,t))$$

2. Replace the expansions in the left hand side, taking the symmetrization $\widetilde{g}_k$ of $g_k(x_1,\ldots,x_n,t)$.

$$
\begin{aligned}
\mathbf{E}\left[F\int_0^T u(t)\ \delta W(t)\right] &= \mathbf{E}\left[\sum_{n=0}^{\infty} I_n(f_n)\int_0^T \sum_{k=0}^{\infty} I_k\big(g_k(\cdot,t)\big)\ \delta W(t)\right] \\
&= \mathbf{E}\left[\sum_{n=0}^{\infty} I_n(f_n)\sum_{k=0}^{\infty} I_{k+1}(\widetilde{g}_k)\right] \\
&= \mathbf{E}\left[\sum_{k=0}^{\infty} I_{k+1}(f_{k+1})I_{k+1}(\widetilde{g}_k)\right] \\
&= \sum_{k=0}^{\infty}(k+1)!\int_{[0,T]^{k+1}} f_{k+1}(x)\widetilde{g}_k(x)\ dx \\
&= \sum_{k=0}^{\infty}(k+1)!\langle f_{k+1},\widetilde{g}_k\rangle_{L^2([0,T]^{k+1})}
\end{aligned}
$$

3. Replace the expansions in the right hand side.

$$
\begin{aligned}
\mathbf{E}\left[\int_0^T u(t)D_tF\ dt\right] &= \mathbf{E}\left[\int_0^T \left(\sum_{k=0}^{\infty} I_k\big(g_k(\cdot,t)\big)\right)\left(\sum_{n=1}^{\infty} nI_{n-1}\big(f_n(\cdot,t)\big)\right)\ dt\right] \\
&= \int_0^T \sum_{k=0}^{\infty}\mathbf{E}\big[(k+1)I_k\big(g_k(\cdot,t)\big)I_k\big(f_{k+1}(\cdot,t)\big)\big]\ dt \\
&= \int_0^T \sum_{k=0}^{\infty}(k+1)k!\langle f_{k+1}(\cdot,t),g_k(\cdot,t)\rangle_{L^2([0,T]^k)}\ dt \\
&= \sum_{k=0}^{\infty}(k+1)!\langle f_{k+1},g_k\rangle_{L^2([0,T]^{k+1})}
\end{aligned}
$$

4. Notice that $\langle f_{k+1},\hat{g}_k\rangle_{L^2([0,T]^{k+1})} = \langle f_{k+1},g_k\rangle_{L^2([0,T]^{k+1})}$.

$$
\begin{aligned}
\langle f_{k+1},\widetilde{g}_k\rangle_{L^2([0,T]^{k+1})} &= \int_0^T \langle f_{k+1}(\cdot,t),\widetilde{g}_k(\cdot,t)\rangle_{L^2([0,T]^k)}\ dt \\
&= \frac{1}{k+1}\sum_{j=1}^{k+1}\int_0^T \langle f_{k+1}(\cdot,t_j),g_k(\cdot,t_j)\rangle_{L^2([0,T]^k)}\ dt_j \\
&= \int_0^T \langle f_{k+1}(\cdot,t),g_k(\cdot,t)\rangle_{L^2([0,T]^k)}\ dt \\
&= \langle f_{k+1},g_k\rangle_{L^2([0,T]^{k+1})}
\end{aligned}
$$

□

Theorem 3.3.2 (Integration by Parts). Let $u(t)$, $t\in[0,T]$ be Skorohod integrable and $F\in\mathbf{D}_{1,2}$ such that $Fu(t)$ is Skorohod integrable. Then

$$
F\int_0^T u(t)\ \delta W(t) = \int_0^T Fu(t)\ \delta W(t) + \int_0^T u(t)D_tF\ dt
$$

Proof. Assume that $F \in \mathbf{D}^0_{1,2}$ and let $G \in \mathbf{D}^0_{1,2}$. Using the Duality Formula and the Product Rule, we have that

$$\begin{aligned}\mathbf{E}\left[G\int_0^T Fu(t)\ \delta W(t)\right] &= \mathbf{E}\left[\int_0^T Fu(t)D_tG\ dt\right] \\ &= \mathbf{E}\left[GF\int_0^T u(t)\ \delta W(t)\right] - \mathbf{E}\left[G\int_0^T u(t)D_tF\ dt\right]\end{aligned}$$

Since $\mathbf{D}^0_{1,2}$ is dense in $L^2(\mathbf{P})$, the result follows. The general case follows by approximation. □

In fact, we can replace the hypothesis that Fu is Skorohod integrable by the existence in $L^2(\mathbf{P})$ of

$$F\int_0^T u(t)\ \delta W(t) \quad \text{and} \quad \int_0^T u(t)D_tF\ dt$$

Theorem 3.3.3 (Closability of the Skorohod Integral)**.** Let (u_n) be a sequence of Skorohod integrable stochastic processes and that

$$\delta(u_n) = \int_0^T u_n(t)\ \delta W(t)$$

converges in $L^2(\mathbf{P})$. And suppose that $\lim_{n\to\infty} u_n = 0$ in $L^2(\mathbf{P}\times\lambda)$.

Then $\lim_{n\to\infty}\delta(u_n) = 0$ in $L^2(\mathbf{P})$.

Proof. Using the Duality Formula,

$$\langle\delta(u_n), F\rangle_{L^2(\mathbf{P})} = \langle u_n, D.F\rangle_{L^2(\mathbf{P}\times\lambda)} \longrightarrow 0$$

as $n\to\infty$. Thus, $\delta(u_N)\to 0$ weakly in $L^2(\mathbf{P})$. Since $(\delta(u_n))$ is convergent in $L^2(\mathbf{P})$, we have that $\delta(u_n)\to 0$ in $L^2(\mathbf{P})$. □

Theorem 3.3.4 (Fundamental Theorem of Calculus)**.** Let $u(s)$ be a stochastic process such that

$$\mathbf{E}\left[\int_0^T u^2(s)\ ds\right] < \infty$$

Also suppose that $u(s)\in\mathbf{D}_{1,2}$ for all $s\in[0,T]$, $D_tu\in\mathrm{Dom}(\delta)$ and that

$$\mathbf{E}\left[\int_0^T (\delta(D_tu))^2\ dt\right] < \infty$$

Then $\int_0^T u(s)\ \delta W(s)$ is well-defined and belongs to $\mathbf{D}_{1,2}$ and

$$D_t\left(\int_0^T u(s)\ \delta W(s)\right) = \int_0^T D_tu(s)\ \delta W(s) + u(t)$$

Proof. We start by proving a simpler case. Suppose that $u(s) = \mathrm{I}_n(f_n(\cdot, s))$, where $f_n(t_1, \ldots, t_n, s)$ is symmetric with respect to $t_1, \ldots, t_n$. Then

$$\int_0^{\mathrm{T}} u(s)\ \delta \mathrm{W}(s) = \mathrm{I}_{n+1}[\tilde{f}_n]$$

where

$$\tilde{f}_n(x_1, \ldots, x_{n+1}) = \frac{1}{n+1}[f_n(\cdot, x_1) + \cdots + f_n(\cdot, x_{n+1})]$$

Thus,

$$\begin{aligned} \mathrm{D}_t\left(\int_0^{\mathrm{T}} u(s)\ \delta \mathrm{W}(s)\right) &= (n+1)\mathrm{I}_n[\tilde{f}_n(\cdot, t)] \\ &= \mathrm{I}_n[f_n(\cdot, x_1) + \cdots + f_n(\cdot, x_n) + f_n(\cdot, t)] \\ &= \mathrm{I}_n[f_n(\cdot, x_1) + \cdots + f_n(\cdot, x_n)] + u(t) \end{aligned} \tag{3.5}$$

Now consider

$$\begin{aligned} \delta(\mathrm{D}_t u) &= \int_0^{\mathrm{T}} \mathrm{D}_t u(s)\ \delta \mathrm{W}(s) \\ &= \int_0^{\mathrm{T}} n\mathrm{I}_{n-1}[f_n(\cdot, t, s)]\ \delta \mathrm{W}(s) \\ &= n\mathrm{I}_n[\hat{f}_n(\cdot, t, \cdot)] \end{aligned} \tag{3.6}$$

where

$$\hat{f}_n = \frac{1}{n}[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)]$$

Using the symmetrization above into (3.6), we obtain

$$\int_0^{\mathrm{T}} \mathrm{D}_t u(s)\ \delta \mathrm{W}(s) = \mathrm{I}_n[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)] \tag{3.7}$$

Combining (3.5) and (3.7), we have

$$\mathrm{D}_t\left(\int_0^{\mathrm{T}} u(s)\ \delta \mathrm{W}(s)\right) = \int_0^{\mathrm{T}} \mathrm{D}_t u(s)\ \delta \mathrm{W}(s) + u(t)$$

For the general case, when $u(s) = \sum_{n=0}^{\infty} \mathrm{I}_n[f_n(\cdot, s)]$, consider $u_m(s) = \sum_{n=0}^{m} \mathrm{I}_n[f_n(\cdot, s)]$. By the previous argument,

$$\mathrm{D}_t(\delta(u_m)) = \delta(\mathrm{D}_t u_m) + u_m(t)$$

for all t.

The result follows by taking $m \to \infty$, which requires some technicalities (see [NØP08]). □

Corollary 3.3.5. Suppose that u satisfies the conditions of the theorem and that $u(s)$ is **F**-adapted. Then

$$\mathrm{D}_t\left(\int_0^{\mathrm{T}} u(s)\ d\mathrm{W}(s)\right) = \int_t^{\mathrm{T}} \mathrm{D}_t u(s)\ d\mathrm{W}(s) + u(t)$$

Proof. Apply the Corollary 3.2.8. □

Chapter 4

Stochastic Integral Representations

Given a random variable $F \in L^2(\mathbf{P})$, how can we represent it as a stochastic integral? The Clark-Ocone formula gives this representation using the Malliavin derivative. After presenting it, we discuss a generalization and, then, two applications in Finance: Portfolio Selection and Sensitivity Analysis.

4.1 The Clark-Ocone Formula

Recalling the Itô Representation Theorem [Oks13, Theorem 4.3.3], the following makes explicit the stochastic process that appears inside the integral.

Theorem 4.1.1 (The Clark-Ocone Formula). Let $F \in \mathbf{D}_{1,2}$ be $\mathscr{F}_T$-measurable. Then

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}[D_t F \mid \mathscr{F}_t]\, dW(t) \tag{4.1}$$

Proof. The idea is to write the Chaos Expansion of F and compute the integral on the right hand side using the proposition 3.2.6.

Let

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

be the Wiener-Itô chaos expansion of F. Then

$$\begin{aligned}
\int_0^T \mathbf{E}[D_t F \mid \mathscr{F}_t]\, dW(t) &= \int_0^T \mathbf{E}\left[D_t \sum_{n=0}^{\infty} I_n(f_n) \,\middle|\, \mathscr{F}_t \right] dW(t) \\
&= \int_0^T \mathbf{E}\left[\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) \,\middle|\, \mathscr{F}_t \right] dW(t) \\
&= \int_0^T \sum_{n=1}^{\infty} n \mathbf{E}[I_{n-1}(f_n(\cdot, t)) \mid \mathscr{F}_t]\, dW(t)
\end{aligned} \tag{4.2}$$

Using 3.2.6, we have

$$
\begin{aligned}
\int_0^T \sum_{n=1}^{\infty} n\mathbf{E}\left[\mathrm{I}_{n-1}(f_n(\cdot,t)) \mid \mathscr{F}_t\right] \mathrm{dW}(t) &= \int_0^T \sum_{n=1}^{\infty} n\mathrm{I}_{n-1}(f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}(\cdot)) \mathrm{dW}(t) \\
&= \int_0^T \sum_{n=1}^{\infty} n(n-1)!\mathrm{J}_{n-1}(f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}(\cdot)) \mathrm{dW}(t) \\
&= \sum_{n=1}^{\infty} n!\mathrm{J}_n(f_n) = \sum_{n=1}^{\infty} \mathrm{I}_n(f_n) \\
&= \sum_{n=0}^{\infty} \mathrm{I}_n(f_n) - \mathrm{I}_0(f_0) = \mathrm{F} - \mathbf{E}[\mathrm{F}]
\end{aligned}
\tag{4.3}
$$

□

As a corollary, we get a simpler proof for the duality formula in the case that u is $\mathbf{F}$-adapted.

Corollary 4.1.2 (Duality Formula)**.** Suppose that $\mathrm{F} \in \mathbf{D}_{1,2}$ is $\mathscr{F}_{\mathrm{T}}$-measurable, u is an $\mathbf{F}$-adapted process, and

$$
\mathbf{E}\left[\int_0^T u^2(t)\, \mathrm{d}t\right] < \infty
$$

Then

$$
\mathbf{E}\left[\mathrm{F}\int_0^T u(t)\, \mathrm{dW}(t)\right] = \mathbf{E}\left[\int_0^T u(t)\mathrm{D}_t\mathrm{F}\, \mathrm{d}t\right] \tag{4.4}
$$

Proof. Replacing F using the Clark-Ocone formula and applying Itô isometry,

$$
\begin{aligned}
\mathbf{E}\left[\mathrm{F}\int_0^T u(t)\, \mathrm{dW}(t)\right] &= \mathbf{E}\left[\left(\mathbf{E}[\mathrm{F}] + \int_0^T \mathbf{E}[\mathrm{D}_t\mathrm{F} \mid \mathscr{F}_t]\, \mathrm{dW}(t)\right)\int_0^T u(t)\, \mathrm{dW}(t)\right] \\
&= \mathbf{E}\left[\int_0^T u(t)\mathbf{E}[\mathrm{D}_t\mathrm{F} \mid \mathscr{F}_t]\, \mathrm{d}t\right] \\
&= \mathbf{E}\left[\int_0^T u(t)\mathrm{D}_t\mathrm{F}\, \mathrm{d}t\right]
\end{aligned}
\tag{4.5}
$$

□

4.2 Under Change of Measure

Before stating the Clark-Ocone formula under change of measure, let us recall the Girsanov theorem and Novikov's criterion. The processes (u_t), (Z_t) and $(\widetilde{\mathrm{W}}_t)$ defined in them will be used throughout the section.

Theorem 4.2.1 (Girsanov)**.** Let (u_t) be an adapted process satisfying

$$
\int_0^T u_s^2\, \mathrm{d}s < \infty \text{ a.s.}
$$

and such that the process (Z_t) given by

$$Z_t = \exp\left(-\int_0^t u_s \, dW_s - \frac{1}{2}\int_0^t u_s^2 \, ds\right)$$

is a martingale.

Then, under the probability $\mathbf{P}^Z$ with density Z_T with respect to $\mathbf{P}$, the process $(\widetilde{W}_t)$ defined by

$$\widetilde{W}_t = W_t + \int_0^t u_s \, ds$$

is an $(\mathscr{F}_t)$-Brownian motion.

Proof. [S+04, Theorem 5.2.3]. □

Remark 4.2.2 (Novikov's criterion)**.** If

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T u_t^2 \, dt\right)\right] < \infty$$

then the (Z_t) in Girsanov theorem is a martingale.

Our goal in this section is to prove the following.

Theorem 4.2.3 (Clark-Ocone Formula Under Change of Measure)**.** Suppose that $F \in \mathbf{D}_{1,2}$ is $\mathscr{F}_T$-measurable, and that the following conditions are met

1. $\mathbf{E}_Q[|F|] < \infty$;
2. $\mathbf{E}_Q\left[\int_0^T |D_t F|^2 \, dt\right] < \infty$;
3. $\mathbf{E}_Q\left[|F| \int_0^T \left(\int_0^T D_t u(s) \, dW(s) + \int_0^T u(s) D_t u(s) \, ds\right)^2 \, dt\right] < \infty$.

Then

$$F = \mathbf{E}_Q[F] + \int_0^T \mathbf{E}_Q\left[\left(D_t F - F\int_t^T D_t u(s) \, d\widetilde{W}(s)\right) \middle| \mathscr{F}_t\right] d\widetilde{W}(t)$$

To prove it, we'll use a couple of lemmas.

Lemma 4.2.4 (Bayes rule)**.** Let μ and ν be two probability measures on a measurable space $(\Omega, \mathscr{G})$ such that

$$\nu(d\omega) = f(\omega)\mu(d\omega)$$

for some $f \in L^1(\mu)$. Suppose that X is a random variable in the same measurable space such that $X \in L^1(\nu)$ and that $\mathscr{H} \subset \mathscr{G}$ is a σ-algebra. Then

$$\mathbf{E}_\nu[X \mid \mathscr{H}]\mathbf{E}_\mu[f \mid \mathscr{H}] = \mathbf{E}_\mu[fX \mid \mathscr{H}] \tag{4.6}$$

Proof. By the definition of conditional expectation, if $H \in \mathscr{H}$, then

$$\begin{aligned}\int_H \mathbf{E}_\nu[X \mid \mathscr{H}] f d\mu &= \int_H \mathbf{E}_\nu[X \mid \mathscr{H}]\, d\nu = \int_H X\, d\nu \\ &= \int_H X f d\mu = \int_H \mathbf{E}_\mu[fX \mid \mathscr{H}]\, d\mu\end{aligned} \tag{4.7}$$

Since $\mathscr{H} \subset \mathscr{G}$, we know that

$$\mathbf{E}[X \mid \mathscr{H}] = \mathbf{E}[\mathbf{E}[X \mid \mathscr{G}] \mid \mathscr{H}]$$

Using this fact,

$$\begin{aligned}\int_H \mathbf{E}_\nu[X \mid \mathscr{H}] f d\mu &= \mathbf{E}_\mu[\mathbf{E}_\nu[X \mid \mathscr{H}] f \chi_H] \\ &= \mathbf{E}_\mu[\mathbf{E}_\mu[\mathbf{E}_\nu[X \mid \mathscr{H}] f \chi_H \mid \mathscr{H}]] \\ &= \mathbf{E}_\mu[\chi_H \mathbf{E}_\nu[X \mid \mathscr{H}] \mathbf{E}_\mu[f \mid \mathscr{H}]] \\ &= \int_H \mathbf{E}_\nu[X \mid \mathscr{H}] \mathbf{E}_\mu[f \mid \mathscr{H}]\, d\mu\end{aligned} \tag{4.8}$$

Combining both equations, we have the result. □

Corollary 4.2.5. If $G \in L^1(Q)$, then

$$\mathbf{E}_Q[G \mid \mathscr{F}_t] = \frac{\mathbf{E}[Z(T)G \mid \mathscr{F}_t]}{Z(t)} \tag{4.9}$$

Lemma 4.2.6.

$$D_t(Z(T)F) = Z(T)\left[D_t F - F\left(u(t) + \int_t^T D_t u(s)\, d\widetilde{W}(s)\right)\right] \tag{4.10}$$

Proof. Apply the chain rule and the corollary of the fundamental theorem of calculus to $D_t Z(T)$:

$$\begin{aligned}D_t Z(T) &= Z(T)\left[-D_t \int_0^T u(s)\, dW(s) - \frac{1}{2} D_t \int_0^T u^2(s)\, ds\right] \\ &= Z(T)\left[-\int_t^T D_t u(s)\, dW(s) - u(t) - \int_0^T u(s) D_t u(s)\, ds\right] \\ &= Z(T)\left[-\int_t^T D_t u(s)\, d\widetilde{W}(s) - u(t)\right].\end{aligned}$$

□

We're ready to prove Theorem 4.2.3.

Proof of the Theorem 4.2.3. Define $Y(t) = \mathbf{E}_Q[F \mid \mathscr{F}_t]$ and $\Lambda(t) = Z^{-1}(t)$. Notice that

$$\begin{aligned}\Lambda(t) &= \exp\left(\int_0^t u(s)\, dW_s + \frac{1}{2}\int_0^t u^2(s)\, ds\right) \\ &= \exp\left(\int_0^t u(s)\, d\widetilde{W}_s - \frac{1}{2}\int_0^t u^2(s)\, ds\right)\end{aligned}$$

Using the corollary 4.2.5,

$$Y_t = \Lambda(t)\mathbf{E}[Z(T)F \mid \mathscr{F}_t]$$

Applying the Clark-Ocone formula,

$$Y_t = \Lambda(t)\left[\mathbf{E}[\mathbf{E}[Z(T)F \mid \mathscr{F}_t]] + \int_0^T \mathbf{E}[D_s\mathbf{E}[Z(T)F \mid \mathscr{F}_t] \mid \mathscr{F}_s]\, dW(s)\right]$$

Simplifying and using the proposition 3.2.7,

$$Y_t = \Lambda(t)\left[\mathbf{E}[Z(T)F] + \int_0^T \mathbf{E}[D_s(Z(T)F) \mid \mathscr{F}_s]\, dW(s)\right] = \Lambda(t)U(t)$$

where we defined

$$U(t) = \mathbf{E}[Z(T)F] + \int_0^T \mathbf{E}[D_s(Z(T)F) \mid \mathscr{F}_s]\, dW(s)$$

Apply Itô formula to $\Lambda(t)$,

$$d\Lambda(t) = \Lambda(t)u(t)\, d\widetilde{W}(t)$$

By the lemma 4.2.6, using the change of measure and the expression above,

$$\begin{aligned}
dY(t) &= \Lambda(t)\mathbf{E}\left[D_t(Z(T)F) \mid \mathscr{F}_t\right]dW(t) + \Lambda(t)u(t)U(t)d\widetilde{W}(t)\\
&\quad + \Lambda(t)u(t)\mathbf{E}\left[D_t(Z(T)F) \mid \mathscr{F}_t\right]dW(t)d\widetilde{W}(t)\\
&= \Lambda(t)\mathbf{E}\left[D_t(Z(T)F) \mid \mathscr{F}_t\right]d\widetilde{W}(t) + u(t)Y(t)d\widetilde{W}(t)\\
&= \Lambda(t)\Big(\mathbf{E}\left[Z(T)D_tF \mid \mathscr{F}_t\right] - \mathbf{E}\left[Z(T)Fu(t) \mid \mathscr{F}_t\right]\\
&\quad -\mathbf{E}\left[Z(T)F\int_t^T D_tu(s)\, d\widetilde{W}(s) \,\middle|\, \mathscr{F}_t\right]\Big)d\widetilde{W}(t) + u(t)Y(t)d\widetilde{W}(t)
\end{aligned}$$

Since $Y(T) = \mathbf{E}_Q[F \mid \mathscr{F}_T] = F$ and $Y(0) = \mathbf{E}_Q[F \mid \mathscr{F}_0] = \mathbf{E}_Q[F]$, the result follows.

□

4.3 Portfolio Selection

Consider a market consisting of a riskless asset S_0 with

$$\text{riskless asset} \quad \begin{cases} dS_0(t) = \rho(t)S_0(t)\, dt \\ S_0(0) = 1 \end{cases} \tag{4.11}$$

and a risky asset S_1 satisfying

$$\text{risky asset} \quad \begin{cases} dS_1(t) = \mu(t)S_1(t)\, dt + \sigma(t)S_1(t)\, dW(t) \\ S_1(0) > 0 \end{cases} \tag{4.12}$$

where $\rho(t), \mu(t)$, and $\sigma(t) \neq 0$ are $\mathbf{F}$-adapted processes satisfying the following condition

$$\mathbf{E}\left[\int_0^T (|\rho(t)| + |\mu(t)| + \sigma^2(t))\, dt\right] < \infty$$

Let $\theta_0(t)$ and $\theta_1(t)$ denote the number of units of $S_0(t)$ and $S_1(t)$, respectively. Then the value of the portfolio $\theta = (\theta_0, \theta_1)$ is $V^\theta = \theta_0 S_0 + \theta_1 S_1$.

We also suppose that the portfolio is self-financing, i.e.,

$$dV^\theta(t) = \theta_0(t)dS_0(t) + \theta_1(t)dS_1(t) \tag{4.13}$$

Substituting

$$\theta_0(t) = \frac{V^\theta(t) - \theta_1(t)S_1(t)}{S_0(t)}$$

into (4.13) and using (4.11) we have

$$dV^\theta = \rho(t)(V^\theta(t) - \theta_1(t)S_1(t))dt + \theta_1(t)dS_1 \tag{4.14}$$

Replacing (4.12),

$$dV^\theta = [\rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)dW(t) \tag{4.15}$$

Our goal is to find a replicating (hedging) portfolio

$$V^\theta(T) = F, \quad \mathbf{P} - a.s. \tag{4.16}$$

where F is $\mathscr{F}_t$-measurable. For an European call, for example, $F = \max\{S_1 - K, 0\} = (S_1 - K)^+$.

How much do we need to invest at time $t = 0$ and which portfolio $\theta(t)$ should we use? Are V^θ and θ unique?

We consider $(V^\theta(t), \theta_1(t))$ an **F**-adapted process. The equations (4.14) and (4.16) form a **backward stochastic differential equation** (BSDE). To find an explicit solution, we can change the measure and apply Clark-Ocone.

Define

$$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)} \tag{4.17}$$

Using the change of measure as in the last section, we can write

$$\begin{aligned} dV^\theta &= [\rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)d\widetilde{W}(t) \\ &\quad - \sigma(t)\theta_1(t)S_1(t)\sigma^{-1}(t)(\mu(t) - \rho(t))dt \\ &= \rho(t)V^\theta(t)dt + \sigma(t)\theta_1(t)S_1(t)d\widetilde{W}(t) \end{aligned} \tag{4.18}$$

Let

$$U^\theta(t) = e^{-\int_0^t \rho(s)\,ds} V^\theta(t)$$

Then using (4.18),

$$dU^\theta(t) = e^{-\int_0^t \rho(s)\,ds}\sigma(t)\theta_1(t)S_1(t)\,d\widetilde{W}(t)$$

or, equivalently,

$$e^{-\int_0^t \rho(s)\,ds} V^\theta(T) = V^\theta(0) + \int_0^T e^{-\int_0^t \rho(s)\,ds}\sigma(t)\theta_1(t)S_1(t)\,d\widetilde{W}(t) \tag{4.19}$$

Applying the generalized Clark-Ocone formula to

$$G = e^{-\int_0^t \rho(s)\, ds} F$$

we have

$$G = \mathbf{E}_Q[G] + \int_0^T \mathbf{E}_Q\left[\left(D_t G - G \int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \,\middle|\, \mathscr{F}_t\right] d\widetilde{W}(t) \tag{4.20}$$

Comparing (4.19) with (4.20), we have $V^\theta(0) = \mathbf{E}_Q[G]$ by uniqueness, and the replicating portfolio is given by

$$\theta_1(t) = e^{-\int_0^t \rho(s)\, ds} \sigma^{-1}(t) S_1^{-1}(t) \mathbf{E}_Q\left[\left(D_t G - G \int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \,\middle|\, \mathscr{F}_t\right] \tag{4.21}$$

In particular, if ρ and μ are constants, and $\sigma(t) = \sigma \neq 0$, then

$$u(t) = u = \frac{\mu - \rho}{\sigma}$$

is also constant, whence $D_t u = 0$. Then the equation (4.21) simplifies to

$$\theta_1(t) = e^{\rho(t-T)} \sigma^{-1} S_1^{-1}(t) \mathbf{E}_Q[D_t F \mid \mathscr{F}_t] \tag{4.22}$$

4.4 Sensitivity Analysis and Computation of Greeks

We start in a Markovian setting and compute the solution. To find the Δ-hedge, we want to compute the derivative. This is hard using numerical methods, since the function φ may not be smooth or even discontinuous. We use Malliavin calculus instead.

Context Setting

Consider $\rho(t) = \rho$ constant and μ and σ Markovian, i.e., $\mu(t) = \mu(S_1(t))$ and $\sigma(t) = \sigma(S_1(t)) \neq 0$. Our goal is to replicate the payoff $F = \varphi(S_1(T))$, where $\varphi : \mathbf{R} \longrightarrow \mathbf{R}$ is bounded, and find a self-financing portfolio θ and function $f(t,x)$, $x > 0$, such that

$$V^\theta(t) = \theta_0(t) S_0(t) + \theta_1(t) S_1(t) = f(t, S_1(t))$$

Note that $f(T,x) = \varphi(x)$. Applying the Itô formula,

$$dV^\theta(t) = \frac{\partial f}{\partial t}(t, S_1(t))\, dt + \frac{\partial f}{\partial x}(t, S_1(t))\, dS_1(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_1(t)) \sigma^2(S_1(t)) S_1^2(t)\, dt \tag{4.23}$$

By the hypothesis that θ is self-financing,

$$dV^\theta(t) = \theta_0(t) S_0(t) \rho\, dt + \theta_1\, dS_1(t) \tag{4.24}$$

Comparing equations (4.23) and (4.24), we obtain

$$\begin{aligned} &\theta_0 S_0(t)\rho + \theta_1(t) S_1(t) \mu(S_1(t)) = \\ &\qquad \frac{\partial f}{\partial t}(t, S_1(t)) + \frac{\partial f}{\partial x}(t, S_1(t)) S_1(t) \mu(S_1(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_1(t)) \sigma^2(S_1(t)) S_1^2(t) \end{aligned} \tag{4.25}$$

and

$$\theta_1(t)\sigma(S_1(t))S_1(t) = \frac{\partial f}{\partial x}(t, S_1(t))\sigma(S_1(t))S_1(t)$$

whence

$$\theta_1(t) = \frac{\partial f}{\partial x}(t, S_1(t)) \tag{4.26}$$

Replacing (4.26) into (4.25),

$$\left[f(t, S_1(t)) - S_1(t)\frac{\partial f}{\partial x}(t, S_1(t))\right]\rho = \frac{\partial f}{\partial t}(t, S_1(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_1(t))\sigma^2(S_1(t))S_1^2(t)$$

which means that $f(t,x)$ must satisfy the Black-Scholes equation

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) - \rho f(t,x) + \rho x \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)x^2\frac{\partial^2 f}{\partial x^2}(t,x) = 0 \\ f(T,x) = \varphi(x) \end{cases}$$

By Feynman-Kac (see [dSN23, Theorem 6.6.3] or [S+04, Theorem 6.4.3]), the solution is

$$f(t, S_1(t)) = \mathbf{E}^x[e^{-\rho(T-t)}\varphi(X(T-t))]|_{x=S_1(t)} = e^{-\rho(T-t)}\mathbf{E}^x[\varphi(X(T-t))]|_{x=S_1(t)}$$

where $\mathbf{E}^x[\varphi(X(T))]$ denotes $\mathbf{E}[\varphi(X^x(T))]$ and $X(t) = X^x(t)$ is the solution of

$$dX(t) = X(t)[\rho\ dt + \sigma(X(t))\ dW(t)], \quad X(0) = x > 0$$

Computing the Delta hedge

To find the Δ-hedge, we need to compute

$$\frac{\partial f}{\partial x}(t,x) = e^{-\rho(T-t)}\frac{\partial}{\partial x}\mathbf{E}^x[\varphi(X(T-t))] = e^{-\rho(T-t)}\frac{\partial}{\partial x}\mathbf{E}[\varphi(X^x(T-t))]$$

Consider a general Itô diffusion $X^x(t)$, $t \geq 0$, given by

$$dX^x(t) = b(X^x(t))\ dt + \sigma(X^x(t))\ dW(t), \quad X^x(0) = x$$

where $b : \mathbf{R} \longrightarrow \mathbf{R}$ and $\sigma : \mathbf{R} \longrightarrow \mathbf{R}$ are functions in $C^1(\mathbf{R})$ and $\sigma(x) \neq 0$ for all $x \in \mathbf{R}$.

The **first variation process**

$$Y(t) = \frac{\partial}{\partial x}X^x(t)$$

is given by

$$dY(t) = b'(X^x(t))Y(t)\ dt + \sigma'(X^x(t))Y(t)\ dW(t), \quad Y(0) = 1$$

i.e. (see [Kun97]),

$$Y(t) = \exp\left(\int_0^t \left[b'(X^x(u)) - \frac{1}{2}(\sigma'(X^x(u)))^2\right] du + \int_0^t \sigma'(X^x(u))\ dW(u)\right)$$

Fix $T > 0$ and define

$$g(x) = \mathbf{E}^x[\varphi(X(T))]$$

We will use two lemmas to compute the derivative $g'(x)$.

Lemma 4.4.1. The Malliavin Derivative of $X(t)$ is

$$D_s X(t) = Y(t)Y^{-1}(s)\sigma(X(s))\chi_{[0,t]}(s)$$

Proof. Since

$$X(t) = x + \int_0^t b(X(u))\,du + \int_0^t \sigma(X(u))dW(u)$$

By the Fundamental Theorem of Calculus, for $t \geq s$,

$$Z(t) := D_s X(t) = \int_s^t b'(X(u))D_s X(u)\,du + \int_s^t \sigma'(X(u))D_s X(u)\,dW(u) + \sigma(X(s))$$

Then

$$\begin{cases} dZ(t) = b'(X(t))Z(t)\,dt + \sigma'(X(t))Z(t)\,dW(t), & t \geq s \\ Z(s) = \sigma(X(s)) \end{cases}$$

with solution

$$Z(t) = \sigma(X(s)) \exp\left(\int_s^t \left[b'(X(u)) - \frac{1}{2}(\sigma'(X(u)))^2\right] du + \int_s^t \sigma'(X(u))\,dW(u)\right)$$

for $t \geq s$.

Hence,

$$Z(t) = \sigma(X(s))Y(t)Y^{-1}(s), \quad t \geq s$$

□

Lemma 4.4.2. Let $a(t)$, $t \in [0,T]$, be a deterministic function that integrates to 1. Then

$$Y(T) = \int_0^T D_s X(T)a(s)\sigma^{-1}(X(s))Y(s)\,ds \tag{4.27}$$

Proof. Applying the previous lemma with $t = T$,

$$Y(T) = Z(T)Y(s)\sigma(X(s))^{-1}, \quad s \in [0,T]$$

Hence

$$Y(T) = \int_0^T Y(T)a(s)\,ds = \int_0^T D_s X(T)a(s)\sigma^{-1}(X(s))Y(s)\,ds$$

□

Theorem 4.4.3. Consider a as in the Lemma 4.4.2. Then

$$g'(x) = \mathbf{E}^x\left[\varphi(X(T))\int_0^T a(t)\sigma^{-1}(X(t))Y(t)\,dW(t)\right] \tag{4.28}$$

The random variable

$$\pi^{\Delta} = \int_0^T a(t)\sigma^{-1}(X(t))Y(t)\,dW(t)$$

is called **Malliavin weight**.

Proof. 1. Suppose that φ is smooth with bounded derivative. Then using the Lemma 4.4.2, Chain Rule, and the Duality Formula,

$$
\begin{aligned}
g'(x) &= \mathbf{E}\left[\varphi'\left(X^x(T)\right)\frac{d}{dx}X^x(T)\right] = \mathbf{E}\left[\varphi'\left(X^x(T)\right)Y(T)\right] \\
&= \mathbf{E}^x\left[\int_0^T \varphi'(X(T))D_sX(T)a(s)\sigma^{-1}X(s)Y(s)\,ds\right] \\
&= \mathbf{E}^x\left[\int_0^T D_s\left(\varphi(X(T))a(s)\sigma^{-1}X(s)\right)Y(s)\,ds\right] \\
&= \mathbf{E}^x\left[\varphi(X(T))\int_0^T a(s)\sigma^{-1}(X(s))Y(s)\,dW(s)\right]
\end{aligned}
$$

2. General case: we approximate φ pointwise boundedly a.e. with respect to the Lebesgue measure on $[0,T]$ by smooth functions φ_m with bounded derivative. Define

$$g_m(x) = \mathbf{E}^x[\varphi_m(X(T))]$$

Using the previous case,

$$g_m'(x) = \mathbf{E}^x\left[\varphi_m(X(T))\int_0^T a(s)\sigma^{-1}(X(s))Y(s)\,dW(s)\right]$$

Hence the limit

$$\lim_{m\to\infty} g_m'(x) = \mathbf{E}\left[\varphi(X(T))\int_0^T a(s)\sigma^{-1}(X(s))Y(s)\,dW(s)\right] =: h(x)$$

is pointwise boundedly in x. Thus

$$g_m(x) = g_m(0) + \int_0^x g_m'(t)\,dt \longrightarrow g_m(0) + \int_0^x h(t)\,dt$$

and

$$g(x) = \lim_{m\to\infty} g_m(x) = g(0) + \int_0^x g(t)\,dt$$

whence $g'(x) = h(x)$. $\square$

Chapter 5

White Noise, the Wick Product, and Stochastic Integration

In this chapter, we formalize the White Noise, and, using it, we can construct the Brownian motion. After that, we obtain a useful rewriting of the Wiener-Itô Chaos Expansion.

An important concept here is that of Wick Product. We present the construction, some properties, the Hermite transform, and how it relates to iterated integrals and Skorohod integration.

5.1 White Noise

Consider $\mathscr{S}(\mathbf{R}^d)$ the **Schwartz space**, i.e., the space of smooth $\mathscr{C}^\infty(\mathbf{R}^d)$ functions whose derivatives are rapidly decreasing. This is a Fréchet space[1] under the seminorms:

$$\|f\|_{\mathrm{K},\alpha} = \sup_{x\in\mathbf{R}^d} \left\{(1-|x|^k)\,|\partial^\alpha f(x)|\right\}$$

where $\mathrm{K}\in\mathbf{N}_0$, and $\alpha = (\alpha_1,\ldots,\alpha_d)$ is a multi-index with each $\alpha_1\in\mathbf{N}_0$, and

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial\alpha_1\cdots\partial\alpha_d} f(x), \quad |\alpha| = \alpha_1+\cdots+\alpha_d$$

And let $\mathscr{S}'(\mathbf{R}^d)$ be its dual, i.e., the space of **tempered distributions**. Using the weak-$*$ topology[2], we denote by $\mathscr{B}$ the family of Borel subsets of this space.

For $\omega\in\mathscr{S}'$ and $\varphi\in\mathscr{S}$, we use

$$\omega(\varphi) = \langle\omega,\varphi\rangle$$

to denote the action of ω in φ.

Example 5.1.1. If ω is a measure μ on $\mathbf{R}^d$, then

$$\langle\omega,\varphi\rangle = \int_{\mathbf{R}^d}\varphi(x)\;\mathrm{d}\mu(x)$$

[1] Fréchet spaces generalize Banach spaces. A Fréchet space is a topological vector space that is locally convex, and the topology is induced by a complete invariant metric d (i.e., it is metrizable and complete).

[2] The smallest topology that makes every $x\in\mathrm{X}$ continuous on X^*.

Now, let us consider the one-dimensional case. By the Bochner-Minlos-Sazonov theorem ([GV64][Theorem 2, Page 155]), there exists a probability measure $\mathbf{P}$ on $(\mathscr{S}', \mathscr{B})$ such that

$$\int_{\mathscr{S}'} e^{i\langle\omega,\varphi\rangle}\, \mathbf{P}(d\omega) = e^{\frac{-1}{2}\|\varphi\|^2}, \quad \varphi \in \mathscr{S}$$

where we used the norm $\|\varphi\| = \|\varphi\|_{L^2(\mathbf{R})}$.

The measure $\mathbf{P}$ obtained above is called the **white noise probability measure** and the space $(\mathscr{S}', \mathscr{B}, \mathbf{P})$ is called the **white noise probability space**.

Definition 5.1.1 (Smoothed white noise process)**.** We define the (smoothed) white noise process as the measurable map

$$\begin{aligned} w : \mathscr{S} \times \mathscr{S}' &\longrightarrow \mathbf{R} \\ (\omega, \varphi) &\longmapsto \langle\omega, \varphi\rangle \end{aligned}$$

We'll also use an alternative notation $w_\varphi = w(\varphi, \omega)$.

Using the smoothed white noise process w_φ, we can construct a Wiener process $W(t)$ for $t \in \mathbf{R}$ as follows:

1. Notice that we have the next isometry[3]

$$\int_{\mathscr{S}'} \langle\omega, \varphi\rangle^2\, \mathbf{P}(d\omega) = \mathbf{E}[w_\varphi^2] = \|\varphi\|^2 \tag{5.1}$$

2. Use the previous step to extend $\langle\omega, \psi\rangle$ to arbitrary functions $\psi \in L^2(\mathbf{R})$, by taking $\langle\omega, \psi\rangle = \lim\langle\omega, \varphi_n\rangle$, where $\varphi_n \in \mathscr{S}$ and $\varphi_n \to \psi$ in $L^2(\mathbf{R})$.

3. Define $\widetilde{W}(t, \omega) = \langle\omega, \chi_{[0,t]}\rangle$ using the step two.

4. Use Kolmogorov's continuity theorem to obtain a continuous version $W(t)$ of $\widetilde{W}(t)$, i.e., $\mathbf{P}(W(t) = \widetilde{W}(t)) = 1$.

Notice that there is a relationship between the smoothed white noise w_φ and the Wiener process $W(t)$:

$$w_\varphi(\omega) = \int_{\mathbf{R}} \varphi(t)\, dW(t)$$

5.2 Revisiting the Wiener-Itô Chaos Expansion

Recalling the definition of the Hermite polynomial, in this section we find an orthogonal basis for $L^2(\mathbf{P})$, which gives us another version of the Wiener-Itô Chaos Expansion.

To start, we need some definitions.

Definition 5.2.1. The kth **Hermite function** is

$$e_k(x) = \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{k-1}(\sqrt{2}x)$$

To simplify our notation, let

$$\theta_k(\omega) = \langle\omega, e_k\rangle = \int_{\mathbf{R}} e_k(x)\, dW(x)$$

[3]This is Itô's isometry

and, finally,

$$\mathrm{H}_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j(\omega)), \quad \mathrm{H}_0 = 1$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathscr{I}$, $\alpha \neq 0$, and $\mathscr{I}$ is the set of finite multi-indices α.

An Itô result states that

$$\mathrm{I}_m(e^{\hat{\otimes}\alpha}) = \mathrm{H}_\alpha \tag{5.2}$$

Example 5.2.1. If $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the kth coordinate, then

$$\mathrm{H}_{\varepsilon^{(k)}}(\omega) = h_1(\theta_k(\omega)) = \langle \omega, e_k \rangle$$

Remark 5.2.1. The family $\{e_k\}$ constitute an orthonormal basis for $\mathrm{L}^2(\mathbf{R})$ and $e_k \in \mathscr{S}(\mathbf{R})$ for all k. Moreover, the family of $\mathrm{H}_\alpha(\omega)$ forms an orthogonal basis for $\mathrm{L}^2(\mathbf{P})$.

Using the latter fact, we have a different version of Wiener-Itô Chaos Expansion.

Theorem 5.2.2 (Wiener-Itô Chaos Expansion 2)**.** The family $\mathrm{H}_\alpha(\omega)_{\alpha \in \mathscr{I}}$ forms an orthogonal basis of $\mathrm{L}^2(\mathbf{P})$, i.e., for all $\mathscr{F}_\mathrm{T}$-measurable $\mathrm{X} \in \mathrm{L}^2(\mathbf{P})$, there exist uniquely determined $c_\alpha \in \mathbf{R}$ such that

$$\mathrm{X} = \sum_{\alpha \in \mathscr{I}} c_\alpha \mathrm{H}_\alpha$$

Furthermore,

$$\|\mathrm{X}\|^2_{\mathrm{L}^2(\mathbf{P})} = \sum_{\alpha \in \mathscr{I}} \alpha! c_\alpha^2$$

where $\alpha! = \alpha_1! \cdots \alpha_m!$.

Proof. See, e.g., [HØUZ96][Theorem 2.2.4]. □

Example 5.2.2 (Chaos Expansion of the Wiener process)**.**

$$\begin{aligned}
\mathrm{W}(t) &= \int_{\mathbf{R}} \chi_{[0,t]}(s)\, d\mathrm{W}(s) \\
&= \int_{\mathbf{R}} \sum_k \langle \chi_{[0,t]}(s), e_k \rangle_{\mathrm{L}^2(\mathbf{R})} e_k(s)\, d\mathrm{W}(s) \\
&= \sum_k \int_0^t e_k(y)\, dy \int_{\mathbf{R}} e_k(s)\, d\mathrm{W}(s) \\
&= \sum_k \int_0^t e_k(y)\, dy\, \mathrm{H}_{\varepsilon^{(k)}}
\end{aligned}$$

How does the current version of the Wiener-Itô Chaos Expansion compare with the previous one? It is possible to connect these two expansions using the following 'translation':

$$f_n = \sum_{\alpha \in \mathscr{I} : |\alpha| = m} c_\alpha e_1^{\otimes \alpha_1} \hat{\otimes} \cdots \hat{\otimes} e_m^{\otimes \alpha_m} \tag{5.3}$$

Hida Stochastic Test Functions and Stochastic Distributions, Kondratiev Spaces

Now we define some useful terminology for growth conditions. For a more in-depth approach, refer to [HØUZ96].

We'll need the following notation.

$$(2\mathbf{N})^{\alpha} = \prod_{j=1}^{m} (2j)^{\alpha_j}$$

Definition 5.2.2 (Hida test function and distribution spaces, Kondratiev Spaces).

1. A function $f = \sum_{\alpha \in \mathcal{J}} a_\alpha \mathrm{H}_\alpha \in \mathrm{L}^2(\mathbf{P})$ is in the **Hida test function Hilbert space** $(\mathscr{S})_k$, for $k \in \mathbf{R}$, if

$$||f||_k^2 = \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha^2 (2\mathbf{N})^{\alpha k} < \infty$$

The **Hida test function space** $(\mathscr{S})$ is the space

$$(\mathscr{S}) = \bigcap_{k \in \mathbf{R}} (\mathscr{S})_k$$

equipped with the projective topology.

2. A formal sum $\mathrm{F} = \sum_{a\mathcal{J}} b_\alpha \mathrm{H}_\alpha$ belongs to the **Hida distribution Hilbert space** $(\mathscr{S})_{-q}$ is

$$||\mathrm{F}||_{-q}^2 = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 (2\mathbf{N})^{-\alpha q} < \infty$$

The **Hida distribution space** $(\mathscr{S})^*$ is the space

$$(\mathscr{S})^* = \bigcup_{q \in \mathbf{R}} (\mathscr{S})_{-q}$$

equipped with the inductive topology.

3. If $\mathrm{F} = \sum_{\alpha \in \mathcal{J}} b_\alpha \mathrm{H}_\alpha \in (\mathscr{S})^*$, the **generalized expectation** of F is

$$\mathbf{E}[\mathrm{F}] = b_0$$

4. The **Kondratiev test function space** $(\mathrm{S})_1$ is the projective limit of Hilbert spaces $(\mathrm{S})_{1,q}$, $q \geq 0$. For $q \geq 0$, $(\mathrm{S})_{-1,q}$ consists of the formal sums

$$f = \sum_{\alpha \in \mathrm{J}} a_\alpha \mathrm{H}_\alpha$$

such that

$$||f||_{1,q}^2 = \sum_{\alpha \in \mathrm{J}} a_\alpha^2 (2\mathbf{N})^{\alpha q} < \infty$$

5. The **Kondratiev distribution space** $(\mathrm{S})_{-1}$ is defined as the inductive limit of the Hilbert spaces $(\mathrm{S})_{-1,q}$, $q \geq 0$. For $q \geq 0$, $(\mathrm{S})_{-1,q}$ consists of the chaos expansions

$$\mathrm{F} = \sum_{\alpha \in \mathrm{J}} c_\alpha \mathrm{H}_\alpha$$

such that

$$||\mathrm{F}||_{-1,q}^2 = \sum_{\alpha \in \mathrm{J}} c_\alpha^2 (2\mathbf{N})^{-\alpha q} < \infty$$

Remark 5.2.3. First, notice that $(\mathcal{S})^*$ is the dual of $(\mathcal{S})$, and $(S)_{-1}$ is the dual of $(S)_1$.

The action of F on f (as defined above) is given by

$$\langle F, f\rangle = \sum_{\alpha} \alpha! a_\alpha b_\alpha$$

Furthermore, we have the inclusions

$$(\mathcal{S}) \subset (\mathcal{S})_k \subset L^2(\mathbf{P}) \subset (\mathcal{S})_{-q} \subset (\mathcal{S})^*, \quad \forall\, k, q$$

and

$$(S)_1 \hookrightarrow (S) \hookrightarrow L^2(S) \hookrightarrow (S)^* \hookrightarrow (S)_{-1}$$

Example 5.2.3. The smoothed white noise $w_\varphi \in (\mathcal{S})$ if $\varphi \in \mathcal{S}(\mathbf{R})$.

Example 5.2.4. The **singular (pointwise) white noise** $\overset{\bullet}{W}(t)$ is defined as

$$\overset{\bullet}{W}(t) = \sum_{k} e_k(t) H_{\varepsilon(k)}$$

Using the isometry (5.1),

$$\frac{d}{dt} W(t) = \frac{d}{dt} \sum_k \left(\int_0^t e_k(y)\, dy \right) H_{\varepsilon(k)} = \overset{\bullet}{W}(t)$$

with derivative in $(\mathcal{S})^*$.

5.3 The Wick Product

With the previous version of the Wiener-Itô Chaos Expansion, we define the Wick product in this section, which generalizes the ordinary pointwise product of deterministic calculus. It is the only product that is specified for the singular white noise, it is related to Itô and Skorohod integrals, can be used to obtain strong solutions to SDEs, and is used in quantum field theory for renormalization.

Construction

Using the Wiener-Itô Chaos Expansion, we can define a multiplication between $(\mathcal{S})$ and $(\mathcal{S})^*$ in a natural way. We take the chaos expansions of the elements and combine them by multiplying the scalars and summing the indices on H.

Definition 5.3.1 (Wick Product)**.** For

$$X = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^* \quad \text{and} \quad Y = \sum_\beta b_\beta H_\beta \in (\mathcal{S})^*$$

we define the **Wick product** of X and Y as

$$X \diamond Y = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) H_\gamma$$

The Wick product is commutative, associative, and distributive (concerning addition). It is also closed in the following sense. If $X, Y \in (\mathscr{S})^*$, then their product is also in $(\mathscr{S})^*$. And if $X, Y \in (\mathscr{S})$, then $X \diamond Y \in (\mathscr{S})$.

We define the **Wick exponential** naturally as

$$\exp^{\diamond} X = \sum_{n=0}^{\infty} \frac{X^{\diamond n}}{n!}$$

An important property is that the expected value of the Wick product is simply the product of the expected values, i.e.,

$$\mathbf{E}[X \diamond Y] = \mathbf{E}[X]\mathbf{E}[Y]$$

By induction, it follows that

$$\mathbf{E}[\exp^{\diamond} X] = \exp \mathbf{E}[X]$$

Hermite Transform

Now, we introduce a tool to transform elements $X \in (\mathscr{S})_{-1}$ into deterministic functions.

Definition 5.3.2 (Hermite transform)**.** Let

$$X = \sum_{\alpha} c_\alpha H_\alpha \in (\mathscr{S})_{-1}$$

The **Hermite transform** of X, denoted by $\mathscr{H}X$ or $\widetilde{X}$, is

$$\mathscr{H}X = \sum_{\alpha} c_\alpha z^\alpha \in \mathbf{C}$$

where $z = (z_1, z_2, \ldots)$ is a $\mathbf{C}$-valued sequence.

Theorem 5.3.1 (Absolute convergence of the Hermite transform)**.** The **Hermite transform** is absolutely convergent on the infinite-dimensional neighborhood

$$\mathbf{K}_q(R) = \left\{ (z_1, z_2, \ldots) : \sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbf{N})^{q\alpha} < R^2 \right\}$$

for some $0 < q \le R < \infty$.

Proof.

$$\begin{aligned}
|\widetilde{X}(z)| &\le \sum_{\alpha} |c_\alpha| \, |z|^\alpha \\
&\le \left(\sum_{\alpha} |c_\alpha|^2 \alpha! (2\mathbf{N})^{-\alpha q} \right)^{\frac{1}{2}} \left(\sum_{\alpha} |z^\alpha|^2 (2\mathbf{N})^{\alpha q} \right)^{\frac{1}{2}} \\
&\le \left(\sum_{\alpha} |c_\alpha|^2 (2\mathbf{N})^{-\alpha q} \right)^{\frac{1}{2}} \left(\sum_{\alpha} |z^\alpha|^2 (2\mathbf{N})^{\alpha q} \right)^{\frac{1}{2}} \\
&= \|X\|_{-1,q} \left(\sum_{\alpha} |z^\alpha|^2 (2\mathbf{N})^{\alpha q} \right)^{\frac{1}{2}}
\end{aligned}$$

□

Theorem 5.3.2 (Characterization for $(\mathscr{S})_{-1}$)**.**

1. Let $X = \sum_\alpha c_\alpha H_\alpha \in (\mathscr{S})_{-1}$. Then there exists some $q, M_q < \infty$ such that
$$|\tilde{X}(z)| \leq \sum_\alpha |c_\alpha|\,|z|^\alpha \leq M_q \left(\sum_\alpha |c_\alpha|^2 (2\mathbf{N})^{\alpha q}\right)^{\frac{1}{2}}$$
for all $z \in \mathbf{C}_c^{\mathbf{N}}$. In particular, $\tilde{X}$ is a bounded (analytic) function on $\mathbf{K}_q(R)$ for all $R < \infty$.

2. Consider $f(z) = \sum_\alpha b_\alpha z^\alpha$ for coefficients $b_\alpha \in \mathbf{R}$ and $z = (z_1, z_2, \ldots) \in \mathbf{C}_c^{\mathbf{N}}$ and suppose that
$$\sum_\alpha |b_\alpha|\,|z|^\alpha < \infty$$
for all $z \in \mathbf{K}_q(R)$ for some $q < \infty$ and $R > 0$.
If we also suppose that $\sup_{z \in \mathbf{K}_q(R)} |f(z)| < \infty$, then there exists a unique distribution $X \in (S)_{-1}$ such that $\tilde{X}(z) = f(z)$ for all $z \in \mathbf{C}_c^{\mathbf{N}}$, and X has the representation
$$X = \sum_\alpha b_\alpha H_\alpha.$$

Proof. See, e.g., [HØUZ96][Theorem 2.6.11]. □

Definition 5.3.3 (Wick version)**.** Let $f : U \longrightarrow \mathbf{C}$ be an analytic function in the neighbordhood $U \subseteq \mathbf{C}$ of $\zeta_0 = \mathbf{E}[X]$, where $X \in (\mathscr{S})_{-1}$. Then we can write
$$f(z) = \sum_{k \geq 0} a_k (z - \zeta_0)^k$$

Assuming that $a_k \in \mathbf{R}$, the **Wick version** of f applied to X, which we denote by $f^\diamond(X)$, is the unique element $Y \in (\mathscr{S})_{-1}$ such that
$$\tilde{Y}(z) = f(\tilde{X}(z))$$
on $\mathbf{K}_q(R)$ for some $q < \infty$ and $R > 0$.

The existence of the Wick version of analytic functions follows from the theorem 5.3.2.

Theorem 5.3.3 (Wick Chain Rule)**.** Let
$$X : \mathbf{R} \longrightarrow (\mathscr{S})_{-1}$$
be continuously differentiable and $f : \mathbf{C} \longrightarrow \mathbf{C}$ analytic on all $\mathbf{C}$ be such that $f(\mathbf{R}) \subseteq \mathbf{R}$. Then,
$$\frac{d}{dt} f^\diamond(X(t)) = (f')^\diamond(X(t)) \diamond \frac{d}{dt} X(t)$$

Proof. See [HØUZ96]. □

Wick Product and Iterated Integrals

Definition 5.3.4 (Spaces $\mathscr{G}$ and $\mathscr{G}^*$). 1. Let $\lambda \in \mathbf{R}$. The space $\mathscr{G}_\lambda$ consists of the formal expansions

$$X = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n \, dW^{\otimes n}$$

such that

$$\|X\|_{\mathscr{G}_\lambda} = \left(\sum_{n=0}^{\infty} n! e^{2\lambda n} \|f_n\|^2_{L^2(\mathbf{R}^n)} \right)^{1/2} < \infty$$

For each $\lambda \in \mathbf{R}$, the space $\mathscr{G}_\lambda$ is a Hilbert space with the inner product

$$\langle X, Y \rangle_{\mathscr{G}_\lambda} = \sum_{n=0}^{\infty} n! e^{2\lambda n} \langle f_n, g_n \rangle_{L^2(\mathbf{R}^n)}$$

where $Y = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} g_n \, dW^{\otimes n}$.

Note that if $\lambda_1 < \lambda_2$, then $G_{\lambda_2} \subseteq G_{\lambda_1}$. We define

$$\mathscr{G} = \bigcap_{\lambda \in \mathbf{R}} G_\lambda = \bigcap_{\lambda > 0} G_\lambda$$

with the projective limit topology.

2. We define the **space of stochastic distributions** $\mathscr{G}^*$ as the dual of $\mathscr{G}$, i.e.,

$$\mathscr{G}^* = \bigcup_{\lambda \in \mathbf{R}} \mathscr{G}_\lambda = \bigcup_{\lambda < 0} \mathscr{G}_\lambda$$

with the inductive limit topology.

Using the spaces $\mathscr{G}$ and $\mathscr{G}^*$, we have the following result.

Theorem 5.3.4. Suppose $X = \sum_{n=0}^{\infty} I_n(f_n) \in \mathscr{G}^*$ and $Y = \sum_{m=0}^{\infty} I_m(g_m) \in \mathscr{G}^*$. Then we can write the Wick product of X and Y as

$$X \diamond Y = \sum_{n,m=0}^{\infty} I_{n+m}(f_n \widehat{\otimes} g_m) = \sum_{k=0}^{\infty} \left(\sum_{n+m=k} I_k(f_n \widehat{\otimes} g_m) \right).$$

From this result, it follows that, for $g \in L^2(\mathbf{R})$,

$$\left(\int_{\mathbf{R}} g(t) \, dW(t) \right)^{\diamond n} = I_n(g^{\otimes n})$$

Wick Product and Skorohod Integration

Definition 5.3.5 ($(\mathscr{S})^*$-integrable). A function $Y : \mathbf{R} \longrightarrow (\mathscr{S})^*$ is $(\mathscr{S})^*$-integrable if

$$\langle Y(t), f \rangle \in L^1(\mathbf{R}), \quad \forall f \in (\mathscr{S})$$

The $(\mathscr{S})^*$-integral of Y, $\int_{\mathbf{R}} Y(t) \, dt$ is the unique element in $(\mathscr{S})^*$ such that

$$\left\langle \int_{\mathbf{R}} Y(t) \, dt, f \right\rangle = \int_{\mathbf{R}} \langle Y(t), f \rangle \, dt, \quad \forall f \in (\mathscr{S})$$

Theorem 5.3.5. If $\mathrm{Y}(t)$ is Skorohod integrable, then $\mathrm{Y}(t)\diamond \overset{\bullet}{\mathrm{W}}$ is $(\mathscr{S})^*$-integrable and

$$\int_{\mathbf{R}} \mathrm{Y}(t)\ \delta \mathrm{W}(t) = \int_{\mathbf{R}} \mathrm{Y}(t)\diamond \overset{\bullet}{\mathrm{W}}\ \ dt \tag{5.4}$$

Proof. The idea of the proof is to write the Chaos expansion of $\mathrm{Y}(t)$, replace it on both sides of the desired equation, and compare.

Since $\mathrm{L}^2(\mathbf{P}) \subseteq (\mathscr{S})^*$, we can write

$$\mathrm{Y}(t) = \sum_{\alpha\in\mathscr{I}} c_\alpha(t)\mathrm{H}_\alpha = \sum_{n=0}^{\infty} \mathrm{I}_n(f_n(\cdot,t))$$

The right-hand side of the equation equals

$$\int_{\mathbf{R}} \left(\sum_{\alpha\in\mathscr{I}} c_\alpha(t)\mathrm{H}_\alpha\right) \diamond \left(\sum_k e_k(t)\mathrm{H}_{\varepsilon^{(k)}}\right)\ dt = \sum_{\alpha,k} \langle c_\alpha, e_k\rangle_{\mathrm{L}^2(\mathbf{R})} \mathrm{H}_{\alpha+\varepsilon^{(k)}}$$

While the left-hand side is

$$\begin{aligned}
\int_{\mathbf{R}} \mathrm{Y}(t)\delta \mathrm{W}(t) &= \int_{\mathbf{R}} \sum_{n=0}^{\infty} \mathrm{I}_n(f_n(\cdot,t))\ \delta \mathrm{W}(t)\\
&= \sum_{n=0}^{\infty} \int_{\mathbf{R}} \mathrm{I}_n\left(\sum_{|\alpha|=n} c_\alpha(t) e^{\hat{\otimes}\alpha}\right)\ \delta \mathrm{W}(t)\\
&= \sum_{n=0}^{\infty} \int_{\mathbf{R}} \mathrm{I}_n\left(\sum_{|\alpha|=n} \sum_{k=1}^{\infty} \langle c_\alpha, e_k\rangle_{\mathrm{L}^2(\mathbf{R})} e_k(t) e^{\hat{\otimes}\alpha}\right)\ \delta \mathrm{W}(t)\\
&= \sum_{n=0}^{\infty} \mathrm{I}_{n+1}\left(\sum_{|\alpha|=n} \sum_{k=1}^{\infty} \langle c_\alpha, e_k\rangle_{\mathrm{L}^2(\mathbf{R})} (e^{\hat{\otimes}\alpha} \hat{\otimes} e_k)\right)\\
&= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \sum_{k=1}^{\infty} \langle c_\alpha, e_k\rangle_{\mathrm{L}^2(\mathbf{R})} \mathrm{I}_{n+1}(e^{\hat{\otimes}(\alpha+\epsilon^{(k)})})\\
&= \sum_{\alpha,k} \langle c_\alpha, e_k\rangle_{\mathrm{L}^2(\mathbf{R})} \mathrm{H}_{\alpha+\epsilon^{(k)}}.
\end{aligned}$$

where we used (5.2) and (5.3). □

This theorem motivates the following.

Definition 5.3.6 (Generalized Skorohod Integral)**.** Let Y be an $(\mathscr{S})^*$-valued process satisfying

$$\int_{\mathbf{R}} \mathrm{Y}(t)\diamond \overset{\bullet}{\mathrm{W}}(t)\ dt \in (\mathscr{S})^*$$

This integral is called the **generalized Skorohod integral** of Y.

Chapter 6

The Hida-Malliavin Derivative on the Integration Space

In this chapter, we present the Hida-Malliavin derivative, and, using it, we prove the Clark-Ocone formula for $L^2(\mathbf{P})$. Thorough this chapter, our space $\Omega = \mathscr{S}'(\mathbf{R})$.

6.1 Stochastic Gradient

Definition 6.1.1. 1. Let $F \in L^2(\mathbf{P})$ and $\gamma \in L^2(\mathbf{R})$. The **directional derivative** of F in $(\mathscr{S})^*$ (respectively, in $L^2(\mathbf{P})$) in the direction γ is

$$D_\gamma F = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon}$$

when the limit exists.

2. Let $\psi : \mathscr{R} \longrightarrow (\mathscr{S})^*$ (resp. to $L^2(\mathbf{P})$) such that $\int_{\mathbf{R}} \psi(t)\gamma(t)\ dt$ converges in $\mathscr{S}^*$ (resp. in $L^2(\mathbf{P})$). We say that F is **Hida-Malliavin differentiable** if

$$D_\gamma F = \int_{\mathbf{R}} \psi(t)\gamma(t)\ dt$$

and we write $\psi(t) = D_t F$ to denote the **Hida-Malliavin derivative** or the **stochastic gradient** of F at t.

Example 6.1.1. For $F(\omega) = \langle \omega, f\rangle = \int_{\mathbf{R}} f(t)\ dW(t), f \in L^2(\mathbf{R})$. Then

$$D_\gamma F = \frac{\langle \omega + \varepsilon\gamma, f\rangle - \langle \omega, f\rangle}{\varepsilon} = \langle \gamma, f\rangle = \int_{\mathbf{R}} f(t)\gamma(t)\ dt$$

Thus, F is Hida-Malliavin differentiable and

$$D_t\left(\int_{\mathbf{R}} f(t)\ dW(t)\right) = f(t)$$

This is an alternative (and shorter) computation of what we did on 3.2.1.

Theorem 6.1.1 (Chain Rule). Suppose that $F_1, \ldots, F_m \in L^2(\mathbf{P})$ are Hida-Malliavin differentiable in $L^2(\mathbf{P})$, $\varphi \in C^1(\mathbf{R}^m)$, $D_t F_i \in L^2(\mathbf{P})$ for all $t \in \mathbf{R}$, and $\frac{\partial \varphi}{\partial x_i}(F) D.F_i \in L^2(\mathbf{P} \times \lambda)$. Then $\varphi(F)$ is Hida-Malliavin differentiable and

$$D_t \varphi(F) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F) D_t F_i$$

Proof. We prove the result for $\varphi \in C^1(\mathbf{R})$. If $\gamma \in L^2(\mathbf{R})$, then

$$\begin{aligned} D_\gamma(\varphi(F)) &= \lim_{\varepsilon \to 0} \frac{\varphi(F(\omega + \varepsilon\gamma)) - \varphi(F(\omega))}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varphi(F(\omega) + \varepsilon D_\gamma F) - \varphi(F(\omega))}{\varepsilon} \\ &= \frac{\varphi'(F(\omega))\varepsilon D_\gamma F}{\varepsilon} = \varphi'(F) D_\gamma F = \int_{\mathbf{R}} \varphi'(F) D_t F \gamma(t)\, dt. \end{aligned}$$

Hence, $\varphi(F)$ is Hida-Malliavin differentiable and $D_t(\varphi(F)) = \varphi'(F) D_t F$. □

It coincides with the Malliavin derivative defined previously on $\mathbf{D}_{1,2}$. This proves Theorem 3.2.2.

Definition of the Malliavin derivative in terms of t and the Chaos Expansion.

Definition 6.1.2. $F = \sum_{\alpha \in \mathscr{I}} c_\alpha H_\alpha \in (\mathscr{S})^*$, we define $D_t F$ as the Malliavin derivative of F as

$$D_t F = \sum_{\alpha \in \mathscr{I}} \sum_{k=1}^{\infty} c_\alpha \alpha_k e_k(t) H_{\alpha - \varepsilon^{(k)}}$$

whenever the series converges in $(\mathscr{S})^*$. We denote by $\mathrm{Dom}(D_t)$ the set of all F for which the series converges.

6.2 Calculus of the Hida-Malliavin Derivative

Theorem 6.2.1. If $g \in L^2(\mathbf{R})$ and $F \in \mathbf{D}_{1,2}$, then

$$F \diamond \int_{\mathbf{R}} g(t)\, dW(t) = F \int_{\mathbf{R}} g(t)\, dW(t) - \int_{\mathbf{R}} g(t) D_t F\, dt$$

Proof. **First step:** Define

$$G_y = \exp^{\diamond}\left(y \int_{\mathbf{R}} g(t)\, dW(t) \right) = \exp\left(y \int_{\mathbf{R}} g(t)\, dW(t) - \frac{1}{2} y^2 \|g\|^2 \right)$$

and

$$F = \exp^{\diamond}\left(\int_{\mathbf{R}} f(t)\, dW(t) \right)$$

where $f \in L^2(\mathbf{R})$.

Step two: Compute $F \diamond G_y$.

$$\begin{aligned}
F \diamond G_y &= \exp^\diamond \left\{ \int_{\mathbf{R}} f(t)\ dW(t) \right\} \diamond \exp^\diamond \left\{ y \int_{\mathbf{R}} g(t)\ dW(t) \right\} \\
&= \exp^\diamond \left\{ \int_{\mathbf{R}} [f(t) + yg(t)]\ dW(t) \right\} \\
&= \exp \left\{ \int_{\mathbf{R}} [f(t) + yg(t)]\ dW(t) - \frac{1}{2} \|f + yg\|^2 \right\} \\
&= \exp^\diamond \left\{ \int_{\mathbf{R}} f(t)\ dW(t) \right\} \exp^\diamond \left\{ \int_{\mathbf{R}} yg(t)\ dW(t) \right\} \exp\{-y\langle f, g\rangle\} \\
&= FG_y \exp\{-y\langle f, g\rangle\}.
\end{aligned}$$

Step three: Differentiate both sides with respect to y and compare.

$$F \diamond \left(G_y \diamond \int_{\mathbf{R}} g(t)\ dW(t) \right) = FG_y \exp\{-y\langle f, g\rangle\} \left(\int_{\mathbf{R}} g(t)\ dW(t) - \langle f, g\rangle \right)$$

Step four: Put $y = 0$.

$$\begin{aligned}
F \diamond \int_{\mathbf{R}} g(t)\ dW(t) &= F \int_{\mathbf{R}} g(t)\ dW(t) - F \int_{\mathbf{R}} f(t)g(t)\ dt \\
&= F \int_{\mathbf{R}} g(t)\ dW(t) - \int_{\mathbf{R}} g(t) D_t F\ dt
\end{aligned}$$

The result follows since linear combinations of functions F are dense in $\mathbf{D}_{1,2}$. □

Applying Girsanov and the previous theorem, we obtain the following.

Theorem 6.2.2 (Integration by parts). If $G, X \in \mathbf{D}_{1,2}$ and $\gamma \in L^2(\mathbf{R})$, then

$$\mathbf{E}[XD_\gamma G] = \mathbf{E}[XG\langle \omega, \gamma\rangle] - \mathbf{E}\left[G \int_{\mathbf{R}} \gamma(t) D_t X\ dt \right] \tag{6.1}$$

Now, using integration by parts, we have the closability of the Hida-Malliavin derivative.

Theorem 6.2.3 (Closability). Suppose that $G, G_n \in \mathbf{D}_{1,2}$, for $n \in \mathbf{N}$, that $\lim_n G_n = G$ in $L^2(\mathbf{P})$, and that the sequence (DG_n) converges in $L^2(\mathbf{P} \times \lambda)$. Then $\lim_n DG_n = DG$ in $L^2(\mathbf{P} \times \lambda)$.

Theorem 6.2.4 (Wick Chain Rule).

1. If $F, G \in \mathbf{D}_{1,2}$ and $F \diamond G \in \mathbf{D}_{1,2}$, then

$$D_t(F \diamond G) = F \diamond D_t G + D_t F \diamond G$$

2. IF $F \in \mathbf{D}_{1,2}$ and $F^{\diamond n} \in \mathbf{D}_{1,2}$, then

$$D_t(F^{\diamond n}) = nF^{\diamond(n-1)} \diamond D_t F$$

3. If $F \in \mathbf{D}_{1,2}$ is Malliavin differentiable and $\exp^\diamond F \in \mathbf{D}_{1,2}$, then

$$D_t \exp^\diamond F = \exp^\diamond F \diamond D_t F$$

Proof. 1. By the Theorem 6.2.3, it suffices to prove for

$$\mathrm{F} = \exp^{\diamond}(\langle \omega, f\rangle), \quad \mathrm{G} = \exp^{\diamond}(\langle \omega, g\rangle)$$

where $f, g \in \mathrm{L}^2(\mathbf{R})$.

Applying the Chain Rule,

$$\begin{aligned}
\mathrm{D}_t(\mathrm{F} \diamond \mathrm{G}) &= \mathrm{D}_t\left(\exp^{\diamond}(\langle \omega, f\rangle) \diamond \exp^{\diamond}(\langle \omega, g\rangle)\right) \\
&= \mathrm{D}_t \exp^{\diamond}(\langle \omega, f+g\rangle) = \mathrm{D}_t\left(\exp\left(\langle \omega, f+g\rangle - \frac{1}{2}\|f+g\|^2_{\mathrm{L}^2(\mathbf{R})}\right)\right) \\
&= \exp\left(\langle \omega, f+g\rangle - \frac{1}{2}\|f+g\|^2_{\mathrm{L}^2(\mathbf{R})}\right)(f(t)+g(t)) \\
&= \exp^{\diamond}(\langle \omega, f+g\rangle)(f(t)+g(t)).
\end{aligned}$$

Using it again,

$$\begin{aligned}
\mathrm{D}_t\mathrm{F} \diamond \mathrm{G} + \mathrm{F} \diamond \mathrm{D}_t\mathrm{G} = {} & \mathrm{D}_t\left(\exp\left(\langle \omega, f\rangle - \frac{1}{2}\|f\|^2_{\mathrm{L}^2(\mathbf{R})}\right)\right) \diamond \mathrm{G} + \mathrm{F} \diamond \mathrm{D}_t\left(\exp\left(\langle \omega, g\rangle - \frac{1}{2}\|g\|^2_{\mathrm{L}^2(\mathbf{R})}\right)\right) \\
= {} & \exp\left(\langle \omega, f\rangle - \frac{1}{2}\|f\|^2_{\mathrm{L}^2(\mathbf{R})}\right) f(t) \diamond \mathrm{G} + \mathrm{F} \diamond \exp\left(\langle \omega, g\rangle - \frac{1}{2}\|g\|^2_{\mathrm{L}^2(\mathbf{R})}\right) g(t) \\
= {} & (\mathrm{F} \diamond \mathrm{G})(f(t)+g(t)) \\
= {} & \exp^{\diamond}(\langle \omega, f+g\rangle)(f(t)+g(t)).
\end{aligned}$$

2. Solved in the exercises.

3. Follows from the previous item and the closability of the Hida-Malliavin derivative.

□

Using these results, we can prove some fundamental properties of the Skorohod integral, such as integration by parts, the duality formula, and the Skorohod isometry. See [NØP08].

6.3 Conditional Expectation on $(\mathscr{S})^*$

Definition 6.3.1. For $\mathrm{F} = \sum_{\alpha \in \mathscr{I}} c_\alpha \mathrm{H}_\alpha \in (\mathscr{S})^*$, we define the **generalized conditional expectation** of F with respect to $\mathscr{F}_t$ by

$$\mathbf{E}[\mathrm{F} \mid \mathscr{F}_t] = \sum_{\alpha \in \mathscr{I}} c_\alpha \mathbf{E}[\mathrm{H}_\alpha \mid \mathscr{F}_t] \tag{6.2}$$

when the series converges in $(\mathscr{S})^*$.

With this definition, we compute the conditional expectation of the Wick product, exponential, singular white noise, and Skorohod integral.

Lemma 6.3.1. If $\mathrm{F}, \mathrm{G}, \mathbf{E}[\mathrm{F} \mid \mathscr{F}_t]$, and $\mathbf{E}[\mathrm{G} \mid \mathscr{F}_t]$ belong to $(\mathscr{S})^*$, then

$$\mathbf{E}[\mathrm{F} \diamond \mathrm{G} \mid \mathscr{F}_t] = \mathbf{E}[\mathrm{F} \mid \mathscr{F}_t] \diamond \mathbf{E}[\mathrm{G} \mid \mathscr{F}_t]$$

Proof. Suppose that $\mathrm{F} = \mathrm{I}_n(f_n)$ and $\mathrm{G} = \mathrm{I}_m(g_m)$ for $f_n \in \hat{\mathrm{L}}^2(\mathbf{R}^n)$ and $g_m \in \hat{\mathrm{L}}^2(\mathbf{R}^m)$. Then

$$\begin{aligned}
\mathbf{E}[\mathrm{F} \diamond \mathrm{G} \mid \mathscr{F}_t] &= \mathbf{E}[\mathrm{I}_n(f_n) \diamond \mathrm{I}_m(g_m) \mid \mathscr{F}_t] \\
&= \mathbf{E}[\mathrm{I}_{n+m}(f_n \hat{\otimes} g_m) \mid \mathscr{F}_t] \\
&= \mathrm{I}_{n+m}(f_n \hat{\otimes} g_m(x_1,\dots,x_n,y_1,\dots,y_m)\mathbf{1}_{[0,t]} \max_{i,j}\{x_i,y_j\}) \\
&= \mathrm{I}_n(f_n(x_1,\dots,x_n)\mathbf{1}_{[0,t]} \max_i x_i) \diamond \mathrm{I}_m(g_m(y_1,\dots,y_m)\mathbf{1}_{[0,t]} \max_j y_j) \\
&= \mathbf{E}[\mathrm{F} \mid \mathscr{F}_t] \diamond \mathbf{E}[\mathrm{G} \mid \mathscr{F}_t]
\end{aligned}$$

□

Corollary 6.3.2. If F and G are as in the previous Lemma and $\mathrm{F}, \mathrm{G} \in \mathrm{L}^1(\mathbf{P})$, then

$$\mathbf{E}[\mathrm{F} \diamond \mathrm{G}] = \mathbf{E}[\mathrm{F}] \cdot \mathbf{E}[\mathrm{G}]$$

Corollary 6.3.3. Suppose that $\mathrm{F}, \exp^\diamond \mathrm{F}, \mathbf{E}[\mathrm{F} \mid \mathscr{F}_t], \exp^\diamond \mathbf{E}[\mathrm{F} \mid \mathscr{F}_t] \in (\mathscr{S})^*$. Then

$$\mathbf{E}[\exp^\diamond \mathrm{F} \mid \mathscr{F}_t] = \exp^\diamond \mathbf{E}[\mathrm{F} \mid \mathscr{F}_t]$$

If $\mathrm{F} \in \mathrm{L}^1(\mathbf{P})$, then

$$\mathbf{E}[\exp^\diamond \mathrm{F}] = \exp \mathbf{E}[\mathrm{F}]$$

Lemma 6.3.4.

$$\mathbf{E}[\overset{\bullet}{\mathrm{W}}(s) \mid \mathscr{F}_t] = \overset{\bullet}{\mathrm{W}}(s)\mathbf{1}_{[0,t]}(s)$$

Proof.

$$\begin{aligned}
\mathbf{E}[\overset{\bullet}{\mathrm{W}}(s) \mid \mathscr{F}_t] &= \mathbf{E}\left[\sum_{i=1}^{\infty} e_i(s)\mathrm{H}_{\varepsilon^{(i)}} \,\middle|\, \mathscr{F}_t\right] \\
&= \sum_{i=1}^{\infty} e_i(s)\mathbf{E}[\mathrm{I}_1(e_i) \mid \mathscr{F}_t] \\
&= \sum_{i=1}^{\infty} e_i(s)\mathrm{I}_1(e_i\mathbf{1}_{[0,t]}) \\
&= \sum_{i=1}^{\infty} e_i(s)\mathrm{I}_1\left(\sum_{j=1}^{\infty}\langle e_i\mathbf{1}_{[0,t]}, e_j\rangle_{\mathrm{L}^2(\mathbf{R})} e_j\right) \\
&= \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}\langle e_i\mathbf{1}_{[0,t]}, e_j\rangle_{\mathrm{L}^2(\mathbf{R})} e_i(s)\right]\mathrm{I}_1(e_j) \\
&= \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}\langle e_j\mathbf{1}_{[0,t]}, e_i\rangle_{\mathrm{L}^2(\mathbf{R})} e_i(s)\right]\mathrm{I}_1(e_j) \\
&= \sum_{j=1}^{\infty} e_j(s)\mathbf{1}_{[0,t]}(s)\mathrm{I}_1(e_j) = \overset{\bullet}{\mathrm{W}}(s)\mathbf{1}_{[0,t]}(s)
\end{aligned}$$

□

Theorem 6.3.5. If $\mathrm{Y}(s)$ is Skorohod integrable and $\mathbf{E}[\mathrm{Y}(s) \mid \mathscr{F}_t] \in (\mathscr{S})^*$ for all $s \in \mathbf{R}$, then

$$\mathbf{E}\left[\int_{\mathbf{R}} \mathrm{Y}(s)\,\delta\mathrm{W}(s) \,\middle|\, \mathscr{F}_t\right] = \int_0^t \mathbf{E}[\mathrm{Y}(s) \mid \mathscr{F}_t]\,\delta\mathrm{W}(s)$$

Proof. By the theorem 5.3.5,

$$\mathbf{E}\left[\int_{\mathbf{R}} Y(s)\ \delta W(s)\ \middle|\ \mathscr{F}_t\right] = \mathbf{E}\left[\int_{\mathbf{R}} Y(s) \diamond \overset{\bullet}{W}(s)\ ds\ \middle|\ \mathscr{F}_t\right]$$

Applying the lemma 6.3.1,

$$\mathbf{E}\left[\int_{\mathbf{R}} Y(s) \diamond \overset{\bullet}{W}(s)\ ds\ \middle|\ \mathscr{F}_t\right] = \int_{\mathbf{R}} \mathbf{E}[Y(s)\mid\mathscr{F}_t] \diamond \mathbf{E}[\overset{\bullet}{W}(s)\mid\mathscr{F}_t]\ ds$$

Now, using lemma 6.3.4,

$$\int_{\mathbf{R}} \mathbf{E}[Y(s)\mid\mathscr{F}_t] \diamond \mathbf{E}[\overset{\bullet}{W}(s)\mid\mathscr{F}_t]\ ds = \int_{\mathbf{R}} \mathbf{E}[Y(s)\mid\mathscr{F}_t] \diamond \overset{\bullet}{W}(s)\mathbf{1}_{[0,t]}(s)\ ds$$

Using the theorem 5.3.5 again, we have the result. □

Corollary 6.3.6. Under the conditions of the theorem,

$$\mathbf{E}\left[\int_t^{\infty} Y(s)\ \delta W(s)\ \middle|\ \mathscr{F}_t\right] = 0$$

It is easier to handle the conditional expectation on the space $\mathscr{G}^*$. See [NØP08].

6.4 Generalized Clark-Ocone

Now we can generalize Clark-Ocone, extending from $\mathbf{D}_{1,2}$ to $L^2(\mathbf{P})$. Before that, let us fix the following notation:

$$X_i = \int_{\mathbf{R}} e_i(s)\ dW(s), \quad X_i^{(t)} = \int_0^t e_i(s)\ dW(s)$$

and

$$X = (X_1, X_2, \ldots), \quad X^{(t)} = (X_1^{(t)}, X_2^{(t)}, \ldots)$$

Rewriting the Chain Rule and the Wick Chain Rule, we obtain the next lemma.

Lemma 6.4.1. $P(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in $x \in \mathbf{R}^n$. Then

$$D_t P(X) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(X) e_i(t) = \sum_\alpha c_\alpha \sum_i \alpha_i X^{\alpha - \varepsilon^{(i)}} e_i(t)$$

and

$$D_t P^\diamond(X) = \sum_{i=1}^n \left(\frac{\partial P}{\partial x_i}\right)^\diamond (X) e_i(t) = \sum_\alpha c_\alpha \sum_i \alpha_i X^{\diamond(\alpha - \varepsilon^{(i)})} e_i(t)$$

Lemma 6.4.2 (Chain Rule)**.** Let $P(x)$ be as in the previous lemma, $X^{(t)} = (X_1^{(t)}, \ldots, X_n^{(t)})$, for $t \geq 0$.

Then

$$\frac{d}{dt} P^\diamond(X^{(t)}) = \sum_{j=1}^n \left(\frac{\partial P}{\partial x_i}\right)^\diamond (X^{(t)}) \diamond e_j(t) \overset{\bullet}{W}(t) \in (\mathscr{S})^*$$

Proof. Follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e_j(s)\,\mathrm{dW}(s) = e_j(t)\,\overset{\bullet}{\mathrm{W}}(t)$$

and the Wick Chain Rule. □

First, we prove it for polynomials, and then, with the help of a lemma, we prove it for $\mathrm{L}^2(\mathbf{P})$.

Lemma 6.4.3 (Clark-Ocone for Polynomials)**.** $\mathrm{F} \in \mathscr{G}^*$ be $\mathscr{F}_\mathrm{T}$-measurable and $\mathrm{F} = \mathrm{P}^\diamond(\mathrm{X})$ for some polynomial $\mathrm{P}(x)$. Then $\mathrm{F} = \mathrm{P}^\diamond(\mathrm{X}^{(\mathrm{T})})$ and

$$\mathrm{F} = \mathbf{E}[\mathrm{F}] + \int_0^\mathrm{T} \mathbf{E}[\mathrm{D}_t\mathrm{F} \mid \mathscr{F}_t]\,\mathrm{dW}(t)$$

Proof. Since F is $\mathscr{F}_\mathrm{T}$-measurable,

$$\mathrm{F} = \mathbf{E}[\mathrm{F} \mid \mathscr{F}_\mathrm{T}] = \mathrm{P}^\diamond(\mathbf{E}[\mathrm{X} \mid \mathscr{F}_\mathrm{T}]) = \mathrm{P}^\diamond(\mathrm{X}^{(\mathrm{T})})$$

Using the lemmas 6.4.1, 6.4.2, and 6.3.1, we have that

$$\begin{aligned}
\int_0^\mathrm{T} \mathbf{E}[\mathrm{D}_t\mathrm{F} \mid \mathscr{F}_t]\,\mathrm{dW}(t) &= \int_0^\mathrm{T} \mathbf{E}\left[\sum_{j=1}^n \left(\frac{\partial \mathrm{P}}{\partial x_j}\right)^\diamond (\mathrm{X}) e_j(t) \,\middle|\, \mathscr{F}_t\right] \mathrm{dW}(t)\\
&= \int_0^\mathrm{T} \sum_{j=i}^n \left(\frac{\partial \mathrm{P}}{\partial x_j}\right)^\diamond \left(\mathrm{X}^{(t)}\right) e_j(t) \diamond \overset{\bullet}{\mathrm{W}}(t)dt\\
&= \int_0^\mathrm{T} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}^\diamond(\mathrm{X}^{(t)})\,\mathrm{d}t = \mathrm{P}^\diamond(\mathrm{X}^{(\mathrm{T})}) - \mathrm{P}^\diamond(\mathrm{X}^{(0)})\\
&= \mathrm{F} - \mathrm{P}^\diamond(0) = \mathrm{F} - \mathbf{E}[\mathrm{F}],
\end{aligned}$$

□

Lemma 6.4.4. Let $\mathrm{F} \in \mathscr{G}^*$, and suppose that

1. For all $q \in \mathbf{N}$,
$$\int_\mathbf{R} \|\mathrm{D}_t\mathrm{F}\|^2_{\mathscr{G}_{-q-1}}\,\mathrm{d}t \le e^{2q}\|\mathrm{F}\|^2_{\mathscr{G}_{-q}}$$
2. $\mathrm{D}_t\mathrm{F} \in \mathscr{G}^*$, for a.a. $t \ge 0$.
3. $\mathrm{F}_n \in \mathscr{G}^*, n \in \mathbf{N}$, and $\mathrm{F}_n \longrightarrow \mathrm{F}$ in $\mathscr{G}^*$.

Then there exists a subsequence $\mathrm{F}_{n_k}, k \in \mathbf{N}$, such that

$$\mathrm{D}_t\mathrm{F}_{n_k} \longrightarrow \mathrm{D}_t\mathrm{F} \quad \text{in } \mathscr{G}^*,\ \text{for a.a. } t \ge 0$$

and

$$\mathscr{E}\left[\mathrm{D}_t\mathrm{F}_{n_k} \mid \mathscr{F}_t\right] \longrightarrow \mathscr{E}[\mathrm{D}_t\mathrm{F} \mid \mathscr{F}_t],\ \text{in } \mathscr{G}^* \text{ for a.a. } t \ge 0$$

Proof. See, e.g., [HØUZ96]. □

Theorem 6.4.5 (Clark-Ocone for $L^2(\mathbf{P})$)**.** Let $F \in L^2(\mathbf{P})$ be $\mathscr{F}_T$-measurable. Then the process

$$(t,\omega) \longrightarrow \mathbf{E}[D_t F \mid \mathscr{F}_t](\omega)$$

is in $L^2(\mathbf{P} \times \lambda)$, and

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}[D_t F \mid \mathscr{F}_t] \, dW(t)$$

Proof. **First step: Write the Chaos expansions of** F **and define** F_n**.** Let

$$F = \sum_{\alpha \in \mathscr{I}} c_\alpha H_\alpha \quad \text{and define} \quad F_n = \sum_{\alpha \in \mathscr{I}_n} c_\alpha H_\alpha$$

where $\mathscr{I}_n = \{\alpha \in \mathscr{I} : |\alpha| \le n, l(\alpha) \le n\}$ and $l(\alpha) = \max\{i : \alpha_i \ne 0\}$ is the length of α.

Second step: Apply the result for polynomials. By the 6.4.3, for all n,

$$F_n = \mathbf{E}[F_n] + \int_0^T \mathbf{E}[D_t F_n \mid \mathscr{F}_t] \, dW(t)$$

Third step: Use Itô Representation Theorem. There exists a unique $\mathbf{F}$-adapted process $u(t)$, $t \in [0,T]$, such that

$$\mathbf{E}\left[\int_0^T u^2(t) \, dt\right] < \infty \quad \text{and} \quad F = \mathbf{E}[F] + \int_0^T u(t) \, dW(t)$$

Fourth step: Take $F_n \to F$ **in** $L^2(\mathbf{P})$**.** Using the previous couple of equations, as $n \to \infty$, we have

$$\mathbf{E}\left[\int_0^T \left(\mathbf{E}[D_t F_n \mid \mathscr{F}_t] - u(t)\right)^2 dt\right] = \mathbf{E}[(F_n - F - \mathbf{E}[F_n] + \mathbf{E}[F])^2] \longrightarrow 0$$

whence follows that $\mathbf{E}[D_t F_n \mid \mathscr{F}_t] \longrightarrow u$ in $L^2(\mathbf{P} \times \lambda)$.

Fifth step: Apply the convergence lemma. Using 6.4.4, we know that for some subsequence F_{n_k},

$$\mathbf{E}[D_t F_{n_k} \mid \mathscr{F}_t] \longrightarrow \mathbf{E}[D_t F \mid \mathscr{F}_t]$$

as $k \to \infty$ in $\mathscr{G}^*$.

Hence,

$$u(t) = \mathbf{E}[D_t F \mid \mathscr{F}_t], \quad \mathbf{P}\text{-a.e.}$$

□

Theorem 6.4.6 (Clark-Ocone for $L^2(\mathbf{P})$ Under Change of Measure)**.** Suppose that $F \in L^2(\mathbf{P})$ is $\mathscr{F}_T$-measurable, and that the following conditions are met

1. $\mathbf{E}_Q[|F|] < \infty$;
2. $\mathbf{E}_Q\left[\int_0^T |D_t F|^2 \, dt\right] < \infty$;
3. $\mathbf{E}_Q\left[|F| \int_0^T \left(\int_0^T D_t u(s) \, dW(s) + \int_0^T u(s) D_t u(s) \, ds\right)^2 dt\right] < \infty$.

Then

$$F = \mathbf{E}_Q[F] + \int_0^T \mathbf{E}_Q\left[\left(D_t F - F\int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \,\middle|\, \mathscr{F}_t\right] d\widetilde{W}(t)$$

where $\widetilde{W}(t)$ is a Brownian motion under the measure Q and $D_t F \in \mathscr{G}^*$ is the Hida-Malliavin derivative.

Proof. Analogous to the the case for $\mathbf{D}_{1,2}$. See, e.g. [Oku10]. □

As a final remark, extending further to $\mathscr{G}^*$ is possible. See [NØP08].

Chapter 7

Application to Finance

7.1 Introduction

Getting back to the context of the section Portfolio Selection, we saw that, for a contingent claim with payoff F (where $F \in L^2(\mathbf{P})$, is $\mathscr{F}_T$-measurable and bounded below), the replicating portfolio is given by (4.21). In particular, if ρ is constant, and μ and σ are deterministic, then we have (4.22), which we reproduce here for the reader's ease.

$$\theta_1(t) = e^{-\rho(T-t)}\sigma^{-1}(t)S_1^{-1}(t)\mathbf{E}_Q[D_tF \mid \mathscr{F}_t]$$

In this chapter, we consider a digital option, which has a payoff at maturity

$$F = \mathbf{1}_{[K,\infty)}(W(T))$$

where K is the exercise price. We aim to compute the conditional expectation $\mathbf{E}_Q[D_tF \mid \mathscr{F}_t]$.

7.2 Necessary Results

To do that, we need the following concept.

Definition 7.2.1 (Donsker delta function)**.** Let $Y : \Omega \longrightarrow \mathbf{R}$, $Y \in \mathscr{G}^*$. The continuous function

$$\delta_Y(\cdot) : \mathscr{R} \longrightarrow \mathscr{G}^*$$

is the **Donsker delta function** of Y if it has the property that

$$\int_{\mathbf{R}} f(y)\delta_Y(y)\,dy = f(Y) \quad \text{a.s.}$$

for all measurable $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that the integral converges.

Theorem 7.2.1. Suppose that

1. $\alpha : [0,T] \longrightarrow \mathbf{R}^n$ is a deterministic function such that $\|\alpha\|^2 = \int_0^T \alpha^2(s)\,ds < \infty$.

2. $\varphi : [0,T] \longrightarrow \mathbf{R}^{n\times n}$ is a deterministic function such that $\|\varphi\|^2 = \sum_{i,j=1}^n \int_0^T \varphi_{ij}^2(s)\,ds < \infty$.

3. $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is bounded.

Define, for $t \in [0, T]$,

$$Y(t) = \int_0^t \varphi(s)\, dB(s) + \int_0^t \varphi(s)\alpha(s)\, ds$$

Then

$$f(Y(T)) = V_0 + \int_0^T u(t,\omega) \diamond (\alpha(t) + W(t))\, dt$$

where $|A| = \det A$ and A is the inverse of the covariance matrix of Y,

$$V_0 = (2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y) \exp\left(-\frac{1}{2}y^T A y\right) dy$$

and

$$u(t,\omega) = (2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y) \exp^{\diamond}\left(-\frac{1}{2}(y - Y(t))^T \diamond A(y - Y(t))\right) \diamond ((y - Y(t))^T A\varphi(t))\, dy$$

Proof. Refer to [AØU01, Theorem 4.4]. □

The next result is a simpler version of [HØ03, Lemma 3.21].

Theorem 7.2.2. Let $\diamond_{\mathbf{P}}$ and $\diamond_Q$ denote the Wick product for the probability measures $\mathbf{P}$ and Q respectively, and u as in the Girsanov Theorem. Then, $F \diamond_{\mathbf{P}} G = F \diamond_Q G$.

Proof. Let $F = \exp^{\diamond}\left(\int_0^\infty f(t)\, dW(t)\right)$ and $G = \exp^{\diamond}\left(\int_0^\infty g(t)\, dW(t)\right)$. Then,

$$F \diamond_{\mathbf{P}} G = \exp\left(\int_0^\infty (f(s) + g(s))\, dW(s) - \frac{1}{2}\|f + g\|^2\right)$$

Applying Girsanov to F and G,

$$F = \exp^{\diamond}\left(\int_0^\infty f(s)\, d\widetilde{W}(s) - \langle f, u\rangle_{L^2}\right), \quad \text{and} \quad G = \exp^{\diamond}\left(\int_0^\infty g(s)\, d\widetilde{W}(s) - \langle g, u\rangle_{L^2}\right)$$

Computing the product,

$$\begin{aligned} F \diamond_Q G &= \exp\left(\int_0^\infty (f(s) + g(s))\, d\widetilde{W}(s) - \frac{1}{2}\|f + g\|^2 - \langle f + g, u\rangle_{L^2}\right) \\ &= \exp\left(\int_0^\infty (f(s) + g(s))\, dW(s) - \frac{1}{2}\|f + g\|^2\right) \end{aligned}$$

Hence, $F \diamond_{\mathbf{P}} G = F \diamond_Q G$ for exponential functions. By density, we have the result. □

7.3 Main Result

Theorem 7.3.1. Suppose that ρ is constant, and $u(t)$, as defined in (4.17), is deterministic and satisfying $\mathbf{E}[u^2(t)] < \infty$. Then the replicating portfolio for hedging $\mathbf{1}_{[K,\infty)}(W(T))$ is

$$\theta_1(t) = e^{-\rho(T-t)}(2\pi(T-t))^{-1/2}\sigma^{-1}(t)S_1^{-1}(t)\exp\left(-\frac{(K-W(t))^2}{2(T-t)}\right) \tag{7.1}$$

Proof. First, notice that $F = \mathbf{1}_{[K,\infty)}(W(T)) \in L^2(\mathbf{P})$. Thus, we can use (4.22).

Now we compute $\mathbf{E}_Q[D_tF \mid \mathscr{F}_t]$ using the Donsker delta function by taking $f(y) = \mathbf{1}_{[K,\infty)}(y)$, and $Y(T) = W(T)$. By the Theorem 7.2.1,

$$\mathbf{1}_{[K,\infty)}(W(T)) = \int_K^\infty (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(y-W(T))^{\diamond 2}}{2T}\right) dy$$

By the Chain Rule for the Wick product,

$$\begin{aligned} D_t(\mathbf{1}_{[K,\infty)}(W(T))) &= \int_K^\infty (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(y-W(T))^{\diamond 2}}{2T}\right)\diamond\frac{(y-W(T))}{2T}\,dy \\ &= (2\pi T)^{-1/2}\exp^\diamond\left(-\frac{(K-W(T))^{\diamond 2}}{2T}\right) \end{aligned}$$

Denoting by $\hat{\diamond}$ the Wick product with respect to the probability measure Q, then since $\hat{\diamond} = \diamond$ (Theorem 7.2.2), we have

$$\begin{aligned} \mathbf{E}[D_t(\mathbf{1}_{[K,\infty)}(W(T))) \mid \mathscr{F}_t] &= \mathbf{E}_Q\left[(2\pi T)^{-1/2}\exp^{\hat{\diamond}}\left(-\frac{(K-W(T))^{\hat{\diamond}2}}{2T}\right)\middle|\,\mathscr{F}_t\right] \\ &= (2\pi T)^{-1/2}\mathbf{E}_Q\left[\exp^{\hat{\diamond}}\left(-\frac{(K-\widetilde{W}(T)+\int_0^T u(s)\,ds)^{\hat{\diamond}2}}{2T}\right)\middle|\,\mathscr{F}_t\right] \\ &= (2\pi T)^{-1/2}\exp\left(-\frac{(K-W(t))^2}{2(T-t)}\right) \end{aligned}$$

$\square$

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