Problem 1.4. (*) The *Itô representation theorem* states that if $F \in L^2(P)$ is $\mathcal{F}_T$-measurable, then there exists a unique $\mathbb{F}$-adapted process $\varphi = \varphi(t), 0 \leq t \leq T$, such that

$$F = E[F] + \int_0^T \varphi(t) dW(t).$$

This result only provides the *existence* of the integrand φ, but from the point of view of applications it is important also to be able to find the integrand φ more explicitly. This can be achieved, for example, by the *Clark–Ocone formula* (see Chap. 4), which says that, under some suitable conditions,

$$\varphi(t) = E[D_t F | \mathcal{F}_t], \qquad 0 \leq t \leq T,$$

where $D_t F$ is the *Malliavin derivative* of F. We discuss this topic later in the book. However, for certain random variables F it is possible to find φ directly, by using the Itô formula. For example, find φ when

(a) $F = W^2(T)$

First, we recall that

$$\int_0^T W(t)\, dW(t) = \frac{1}{2} W^2(T) - \frac{1}{2} T \quad \text{and} \quad E[W^2(T)] = T$$

$$\boxed{\begin{array}{l} g(t,x) = 1/2\, x^2, \quad Y_t = g(t, W_t) \\ dY_t = \frac{1}{2} dt + W_t\, dW_t \Rightarrow \frac{1}{2} W_T^2 = \frac{1}{2} T + \int_0^T W_t\, dW_t \end{array}}$$

Let $\varphi(t) = 2W(t)$. Then

$$T + 2\int_0^T W(t)\, dW(t) = W^2(T)$$

(b) $F = \exp\{W(T)\}$

Let $U(t) = \exp(W(t) - 1/2\, t)$. Then, by Itô's formula,

$$dU(t) = \left(-\frac{1}{2} U(t) + \frac{1}{2} U(t)\right) dt + U(t)\, dW(t) = U(t)\, dW(t), \quad U(0) = 1$$

Therefore

$$U_T - U_0 = \int_0^T U_s\, dW_s$$

and we have that

$$\exp(W_T - 1/2T) = 1 + \int_0^T \exp(W_s - 1/2s)\,dW_s$$

$$\Longleftrightarrow \quad e^{W_T} = e^{1/2T} + \int_0^T e^{W_s + 1/2(T-s)}\,dW_s$$

i.e., $\varphi(t) = \exp(W_t + 1/2(T-t))$.

(c) $F = \int\limits_0^T W(t)dt$

If $g(t,x) = tx$ and $Y_t = g(t, W_t)$, then

$$dY_t = W_t dt + t\,dW_t \implies TW_T = \int_0^T W_t\,dt + \int_0^T t\,dW_t$$

Thus,

$$\int_0^T W_t\,dt = T\int_0^T dW_t - \int_0^T t\,dW_t = \int_0^T (T-t)\,dW_t$$

Since $E[F] = 0$, $\varphi(t) = T-t$.

(d) $F = W^3(T)$

Let $g(t, x) = 1/3\ x^3$. Then $d(1/3\ W_t^3) = W_t dt + W_t^2 dW_t$.

In the integral form,

$$\frac{1}{3} W_T^3 = \int_0^T W_t dt + \int_0^T W_t^2 dW_t = \int_0^T W_t^2 dW_t + \int_0^T (T-t)\, dW_t$$

Recall that $\mathbb{E}[W_t^3] = 0$. Then

$$W_T^3 = 3\int_0^T (W_t^2 + T - t)\, dW_t$$

(e) $F = \cos W(T)$ [*Hint.* Check that $N(t) := e^{\frac{1}{2}t} \cos W(t)$, $t \in [0,T]$, is a martingale.]

Consider $g(t, x) = e^{1/2t} \cos(x)$, $N_t = g(t, W_t)$. By Itô's formula,

$$dN_t = \left(\frac{1}{2} e^{1/2t} \cos(W_t) - \frac{1}{2} e^{1/2t} \cos(W_t)\right) dt - e^{1/2t} \sin(W_t)\, dW_t$$

$$= - e^{1/2t} \sin(W_t)\, dW_t$$

Hence,

$$e^{1/2t} \cos(W_t) = 1 - \int_0^t e^{1/2s} \sin(W_s)\, dW_s$$

is a martingale and

$$\cos W_T = e^{1/2T} - \int_0^T e^{1/2(t-T)} \sin W_t\, dW_t$$