# AN INTRODUCTION TO MALLIAVIN CALCULUS

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# INTRODUCTION

# MOTIVATION

- The Itô Representation Theorem states that, for $F \in L^2(\mathbf{P})$, there exists a unique adapted process $f$ such that

$$F = \mathbf{E}[F] + \int_0^T f(t)\, dW(t)$$

- How can we find this process?

# MOTIVATION

- We'll show that

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}[D_t F \mid \mathcal{F}_t]\, dW(t)$$

  which is known as the **Clark-Ocone Formula**.
- To do that, we need to know what is $D_t F$, which begs the question: how can we define a derivative for random variables?
- Notice that this derivative also allows us to study the smoothness of densities of diffusions and the regularity of densities.

# THREE APPROACHES

- **Malliavin**: $\Omega$ as the Wiener space $C_0([0, T])$, with $\omega(0) = 0$, equipped with the uniform topology.
- **Hida**: $\Omega$ as the space $\mathcal{S}'$ of tempered distributions $\omega : S \longrightarrow \mathbf{R}$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing smooth functions on **R**, constructing a probability via the Bochner-Minlos-Sazonov theorem.
- **Wiener-Itô chaos expansion**: the one presented in the following slides.

# THE WIENER-ITÔ CHAOS EXPANSION

We define $\hat{L}^2([0,T]^n)$ as the subspace of $L^2([0,T]^n)$ consisting of symmetric functions.

Now consider the set

$$S_n = \{(t_1, \ldots, t_n) \in [0,T]^n : 0 \leq t_1 \leq \cdots \leq t_n \leq T\}$$

Notice that the set $S_n$ occupies $\frac{1}{n!}$ of the whole box $[0,T]^n$. Thus, if $g \in \hat{L}^2([0,T]^n)$, then $g|_{S_n} \in L^2(S_n)$ and

$$\|g\|^2_{L^2([0,T]^n)} = n! \int_{S_n} g^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = n! \|g\|^2_{L^2(S_n)}$$

**Definition 1.** Let $f$ be a deterministic function defined on $\mathrm{S}_n$ such that $\|f\|^2_{\mathrm{L}^2(\mathrm{S}_n)} < \infty$. We define the *n***-fold iterated Itô integral** as

$$\mathrm{J}_n(f) = \int_0^{\mathrm{T}} \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \ldots, t_n)\, \mathrm{dW}(t_1)\mathrm{dW}(t_2) \cdots \mathrm{dW}(t_{n-1})\mathrm{dW}(t_n)$$

***Remark 2.***

1. *Note that each i-th Itô integral with respect to* $\mathrm{dW}(t_i)$ *is well-defined, since the integrand is an* **F***-adapted stochastic process.*
2. *Furthermore,* $\mathrm{J}_n(f) \in \mathrm{L}^2(\mathbf{P})$*.*

***Theorem 3.*** *For $m, n \in \mathbf{Z}_{>0}$,*

$$\mathbf{E}[\mathrm{J}_m(g)\mathrm{J}_n(h)] = \begin{cases} 0, & n \neq m \\ \langle g, h\rangle_{\mathrm{L}^2(\mathrm{S}_n)}, & n = m \end{cases}$$

*In particular,*

$$\|\mathrm{J}_n(h)\|_{\mathrm{L}^2(\mathbf{P})} = \|h\|_{\mathrm{L}^2(\mathrm{S}_n)}$$

*For $n = 0$ or $m = 0$, we define $\mathrm{J}_0(g) = g$, when $g$ is a constant, and $\langle g, h\rangle_{\mathrm{L}^2(\mathrm{S}_0)} = gh$, when $g$ and $h$ are constants.*

*Proof.* Follows by applying Itô's isometry iteratively in two different cases: $g \in \mathrm{L}^2(\mathrm{S}_m)$ and $h \in \mathrm{L}^2(\mathrm{S}_n)$ with $m < n$, and $g, h \in \mathrm{L}^2(\mathrm{S}_n)$. □

**Definition 4.** Let $g \in \hat{L}^2([0, T]^n)$. Then

$$I_n(g) = \int_{[0,T]^n} g(t_1, \ldots, t_n)\, dW(t_1) \cdots dW(t_n) = n! J_n(g)$$

is also called *n*-**fold iterated Itô integral**.

For $n = 0$, we define

$$I_0(g) = \int_{\mathbf{R}^0} g\, dW^{\otimes 0} = g$$

***Theorem 5.*** *If $\xi_1, \xi_2, \ldots$ are orthonormal functions in $\mathrm{L}^2([0,\mathrm{T}])$, then*

$$\mathrm{I}_n(\xi_1^{\otimes \alpha_1} \hat{\otimes} \cdots \hat{\otimes} \xi_m^{\otimes \alpha_m}) = \prod_{k=1}^{m} h_{\alpha_k}\left(\int_0^{\mathrm{T}} \xi_k(t)\,\mathrm{d}W(t)\right)$$

*with $\alpha_1 + \cdots + \alpha_m = n$, $\alpha_k \in \mathbf{N}_0$ for all $k$, and $\hat{\otimes}$ is the symmetrized tensor product, which is the symmetrization of $f \otimes g$.*

*Proof.* See, e.g., [Itô51]. □

Using this, it is possible to prove [NN18, p. 64] that

$$\begin{aligned}
\mathrm{I}_n(g^{\otimes n}) &= n! \int_0^{\mathrm{T}} \int_0^{t_n} \cdots \int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\,\mathrm{d}W(t_1)\cdots \mathrm{d}W(t_n) \\
&= \|g\|^n h_n\left(\frac{\int_0^{\mathrm{T}} g(t)\,\mathrm{d}W(t)}{\|g\|}\right)
\end{aligned}$$

***Theorem 6 (The Wiener-Itô Chaos Expansion).*** *Let $\xi$ be an $\mathcal{F}_T$-measurable random variable in $L^2(\mathbf{P})$. There exists a unique sequence $(f_n)$ of functions $f_n \in \hat{L}^2([0,T]^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n)$$

*with convergence in $L^2(\mathbf{P})$. Moreover, we have the following isometry:*

$$\|\xi\|^2_{L^2(\mathbf{P})} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2([0,T]^n)}$$

*Proof.* Our goal is to obtain an orthogonal decomposition of $L^2(\mathbf{P})$. To do that, we show that a certain function $\psi$ is orthogonal to

$$\exp\left(\int_0^T g(t)\,dW(t)\right)$$

which form a total set in $L^2(\mathbf{P})$, implying that $\psi \equiv 0$.

1. By the Itô's Representation Theorem, there exists an **F**-adapted process $\varphi_1(s_1)$, $0 \leq s_1 \leq T$, such that

$$\mathbf{E}\left[\int_0^T \varphi_1^2(s_1)\,ds_1\right] \leq \mathbf{E}[\xi^2]$$

and

$$\xi = \mathbf{E}[\xi] + \int_0^T \varphi_1(s_1)\, dW(s_1)$$

Define $g_0 = \mathbf{E}[\xi]$.

2. For almost all $s_1 \leq T$, we can apply the Itô's Representation Theorem to $\varphi_1(s_1)$ and obtain an $\mathbf{F}$-adapted process $\varphi_2(s_1, s_1)$, $0 \leq s_2 \leq s_1$ such that

$$\mathbf{E}\left[\int_0^{s_1} \varphi_2^2(s_2, s_1)\, ds_2\right] \leq \mathbf{E}[\varphi_1^2(s_1)] < \infty$$

and

$$\varphi_1(s_1) = \mathbf{E}[\varphi_1(s_1)] + \int_0^{s_1} \varphi_2(s_2, s_1)\, dW(s_2)$$

# THE WIENER-ITÔ CHAOS EXPANSION [3]

3. Replacing $\varphi_1(s_1)$ into our expression for $\xi$ yields

$$\xi = g_0 + \int_0^T g_1(s_1)\, dW(s_1) + \int_0^T \int_0^{s_1} \varphi(s_2, s_1)\, dW(s_2)dW(s_1)$$

   where $g_1(s_1) = \mathbf{E}[\varphi_1(s_1)]$.

4. Applying Itô's isometry,

$$\mathbf{E}\left[\left(\int_0^T \int_0^{s_1} \varphi_2(s_2, s_1)\, dW(s_2)dW(s_1)\right)^2\right]$$
$$= \int_0^T \int_0^{s_1} \mathbf{E}[\varphi_2^2(s_2, s_1)]\, ds_2 ds_1 \leq \mathbf{E}[\xi^2]$$

# THE WIENER-ITÔ CHAOS EXPANSION [4]

5. Iterating this procedure $n+1$ times, we obtain a process $\varphi_{n+1}(t_1, \dots, t_{n+1})$, $0 \leq t_1 \leq \cdots \leq t_{n+1} \leq \mathrm{T}$, and $n+1$ deterministic functions $g_0, g_1, \dots, g_n$ (where $g_0 = \mathbf{E}[\xi]$ and $g_k(s_k, s_{k-1}, \dots, s_1) = \mathbf{E}[\varphi_k(s_k, s_{k-1}, \dots, s_1)]$ for $1 \leq k \leq n$) such that

$$\xi = \sum_{k=0}^{n} \mathrm{J}_k(g_k) + \int_{\mathrm{S}_{n+1}} \varphi_{n+1} \, \mathrm{d}\mathrm{W}^{\otimes(n+1)}$$

6. Note that we have a $(n+1)$-fold Iterated Itô Integral

$$\int_{\mathrm{S}_{n+1}} \varphi_{n+1} \, \mathrm{d}\mathrm{W}^{\otimes(n+1)} =: \psi_{n+1}$$

and

$$\mathbf{E}\left[\left(\int_{S_{n+1}} \varphi_{n+1}\, dW^{\otimes(n+1)}\right)^2\right] \leq \mathbf{E}[\xi^2]$$

7. Also remark that the family $\psi_{n+1}$ is bounded in $L^2(\mathbf{P})$ and, from Itô's Isometry,

$$\langle \psi_{n+1}, J_k(f_k)\rangle_{L^2(\mathbf{P})} = 0$$

for $k \leq n$ and $f_k \in L^2([0,T]^k)$.

8. Compute $\|\xi\|^2_{L^2(\mathbf{P})}$ and notice that $\sum_{k=0}^{\infty} J_k(g_k)$ is convergent in $L^2(\mathbf{P})$. Thus,

$$\langle J_k(f_k), \psi\rangle_{L^2(\mathbf{P})} = 0$$

## THE WIENER-ITÔ CHAOS EXPANSION [6]

9. Using that

$$I_n(g^{\otimes n}) = \|g\|^n h_n\left(\frac{\theta}{\|g\|}\right), \quad \theta = \int_0^T g(t)\, dW(t)$$

and Hermite polynomials, we have

$$\mathbf{E}\left[h_n\left(\frac{\theta}{\|g\|}\right)\psi\right] = 0, \ \mathbf{E}\left[\theta^k\psi\right] = 0, \ \mathbf{E}\left[\exp\theta\cdot\psi\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{E}\left[\theta^k\psi\right] = 0$$

10. Since $\{\exp\theta : g \in L^2([0,T]^n)\}$ is total in $L^2(\mathbf{P})$ (i.e. its linear span is dense), $\psi = 0$. Thus, we obtain

$$\xi = \sum_{k=0}^{\infty} J_k(g_k)$$

and

$$\|\xi\|_{L^2(\mathbf{P})}^2 = \sum_{k=0}^{\infty} \|J_k(g_k)\|_{L^2(\mathbf{P})}^2$$

11. To extend $g_n$ from $S_n$ to $[0,T]^n$, we put

$$g_n(t_1, \dots t_n) = 0, \quad (t_1, \dots t_n) \in [0,T]^n \setminus S_n$$

and define $f_n = \hat{g}_n$, i.e., the symmetrization of $g_n$.

Then,

$$I_n(f_n) = n! J_n(f_n) = n! J_n(\hat{g}_n) = J_n(g_n)$$

and the result follows.

□

# THE MALLIAVIN DERIVATIVE

## THE MALLIAVIN DERIVATIVE

1. Using the Wiener-Itô Chaos Expansion, it is quite natural to define the derivative of a random variable.
2. But for this to make sense, it is necessary to restrict the definition to a suitable context as follows.

THE MALLIAVIN DERIVATIVE

**Definition 7.** Let $F \in L^2(\mathbf{P})$ be $\mathcal{F}_t$-measurable with Wiener-Itô Chaos Expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$.

1. We say that $F \in \mathbf{D}_{1,2}$ if

$$\|F\|^2_{\mathbf{D}_{1,2}} := \sum_{i=0}^{\infty} nn!\|f_n\|^2_{L^2([0,T]^n)} < \infty$$

2. If $F \in \mathbf{D}_{1,2}$, we define the **Malliavin derivative** $D_tF$ of F at time $t$ as

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)), \quad t \in [0, T]$$

in which $I_{n-1}(f_n(\cdot, t))$ is the $(n-1)$-fold iterated integral of $f_n(t_1, \ldots, t_{n-1}, t)$ with respect to the first $n-1$ variables, and $t_n = t$ as a parameter.

The restriction to $\mathbf{D}_{1,2}$ ensures that the Malliavin derivative is well-defined in $L^2$.

***Remark 8.*** *If* $F \in \mathbf{D}_{1,2}$, *then*

$$\|D_\cdot F\|^2_{L^2(\mathbf{P}\times\lambda)} = \mathbf{E}\left[\int_0^T (D_t F)^2 \, dt\right] = \sum_{n=1}^{\infty} \int_0^T n^2 (n-1)! \|f_n(\cdot, t)\|^2_{L^2([0,T]^n)} \, dt$$
$$= \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^n)} = \|F\|^2_{\mathbf{D}_{1,2}} < \infty$$

*Therefore,* $D_\cdot F = D_t F$ *is well-defined in* $L^2(\mathbf{P} \times \lambda)$.

# CLOSABILITY OF THE MALLIAVIN DERIVATIVE [1]

***Theorem 9 (Closability of Malliavin Derivative).*** *Suppose that* $\mathrm{F} \in \mathrm{L}^2(\mathbf{P})$ *and* $\mathrm{F}_k \in \mathbf{D}_{1,2}$, $k = 1, 2, \ldots$, *satisfy*

a) $\mathrm{F}_k \to \mathrm{F}$ *as* $k \to \infty$ *in* $\mathrm{L}^2(\mathbf{P})$,

b) $(\mathrm{D}_t \mathrm{F}_k)_{k=1}^{\infty}$ *converges in* $\mathrm{L}^2(\mathbf{P} \times \lambda)$.

*Then* $\mathrm{F} \in \mathbf{D}_{1,2}$ *and* $\mathrm{D}_t \mathrm{F}_k \to \mathrm{D}_t \mathrm{F}$ *in* $\mathrm{L}^2(\mathbf{P} \times \lambda)$.

*Proof.*

1. Write the Wiener-Itô Expansion of $\mathrm{F}_k$ and F. Let

$$\mathrm{F} = \sum_{n=0}^{\infty} \mathrm{I}_n(f_n), \quad \mathrm{F}_k = \sum_{n=0}^{\infty} \mathrm{I}_n(f_n^{(k)})$$

2. Using (a), we have that $f_n^{(k)} \to f_n$ as $k \to \infty$ in $\mathrm{L}^2(\lambda^n)$ for all $n$.

3. By (b),

$$\sum_{n=1}^{\infty} nn! \|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = \|D_t F_k - D_t F_j\|_{L^2(\mathbf{P}\times\lambda)} \longrightarrow 0, \quad j,k \to \infty$$

4. Using Fatou's Lemma,

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} nn! \|f_n^{(k)} - f_n\|^2_{L^2(\lambda^n)} \leq \lim_{k\to\infty} \liminf_{j\to\infty} \sum_{n=1}^{\infty} nn! \|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = 0$$

5. Thus, $F \in \mathbf{D}_{1,2}$ and $D_t F_k \to D_t F$ in $L^2(\mathbf{P} \times \lambda)$.

□

# CHAIN RULE [1]

In practice, how do we compute the Malliavin derivative? The following deduction will often help.

1. Suppose that $f_n(t_1, \ldots, t_n) = f(t_1) \cdots f(t_n)$.
2. Let us write $\mathrm{D}_t \mathrm{I}_n(f_n)$ using Hermite polynomials

$$\begin{aligned} \mathrm{D}_t \mathrm{I}_n(f_n) &= n \mathrm{I}_{n-1}(f_n(\cdot, t)) \\ &= n \mathrm{I}_{n-1}(f^{\otimes(n-1)}) f(t) \\ &= n \|f\|^{n-1} h_{n-1}\left(\frac{\theta}{\|f\|}\right) f(t) \end{aligned}$$

3. Then, using that $h_n'(x) = n h_{n-1}(x)$,

$$\mathrm{D}_t h_n\left(\frac{\theta}{\|f\|}\right) = h_n'\left(\frac{\theta}{\|f\|}\right) \frac{f(t)}{\|f\|}$$

# CHAIN RULE [2]

4. From here, we extract two useful identities. For $n = 1$, we have

$$D_t \int_0^T f(s)\, dW(s) = f(t)$$

and for $n > 1$, using induction,

$$D_t \left( \int_0^T f(s)\, dW(s) \right)^n = n \left( \int_0^T f(s)\, dW(s) \right)^{n-1} f(t)$$

Let $\mathbf{D}_{1,2}^0$ be the set of $F \in L^2(\mathbf{P})$ whose chaos expansion has only finitely many items.

***Theorem 10 (Product Rule).*** *If* $F_1, F_2 \in \mathbf{D}_{1,2}^0$*, then* $F_1, F_2 \in \mathbf{D}_{1,2}$ *and* $F_1 F_2 \in \mathbf{D}_{1,2}$ *and*

$$D_t(F_1F_2) = F_1 D_t F_2 + F_2 D_t F_1$$

*Proof.* It is clear that $F_1, F_2 \in \mathbf{D}_{1,2}$. To prove the second claim, notice that Gaussian random variables have finite moments.

Let $\{\xi_i\}_{i=1}^{\infty}$ be an orthogonal basis of $L^2([0,T]^n)$. Take $F_k^{(n)}$ as a linear combination of iterated integrals of the tensor product of $\xi_i$.

By the Theorem 5, we have the result for $F_K^{(n)}$. Choose sequence $F_k^{(n)} \to F_k$ and $D_t F_k^{(n)} \to D_t F_k$ by closability. $\square$

***Theorem 11 (Chain Rule).*** *Let* $G \in \mathbf{D}_{1,2}$ *and* $g \in C^1(\mathbf{R})$ *with bounded derivative. Then* $g(G) \in \mathbf{D}_{1,2}$ *and*

$$D_t g(G) = g'(G) D_t G$$

*Proof.* Uses techniques from White Noise Theory, so we skip. $\square$

# CONSEQUENCES

What happens when we take the conditional expectation of

1. The Itô integral of a function in $L^2([0, T])$?
2. The Itô integral of an $\mathbf{F}$-adapted process?
3. An iterated integral?

After answering these questions, we look at what happens when we take the Malliavin derivative of a conditional expectation.

**Definition 12.** Let G be a Borel set in $[0, T]$. We define $\mathcal{F}_G$ to be the completed $\sigma$-algebra generated by all random variables of the form

$$F = \int_0^T \chi_A(t)\, dW(t)$$

for all Borel sets $A \subseteq G$.

## CONSEQUENCES [1]

***Lemma 13.*** *For any $g \in L^2([0,T])$ we have*

$$\mathbf{E}\left[\int_0^T g(t)\, dW(t) \,\middle|\, \mathcal{F}_G\right] = \int_0^T \chi_G(t) g(t)\, dW(t)$$

*Proof.* The first step is to prove that $\int_0^T \chi_G(t)g(t)\, dW(t)$ is $\mathcal{F}_G$-measurable. Since continuous functions are dense in $L^2([0,T])$, assume that $g$ is continuous. Then

$$\int_0^T \chi_G(t)g(t)\, dW(t) = \lim_{\Delta t_i \to 0} \sum_{i=0}^{n} g(t_i) \int_{t_i}^{t_{i+1}} \chi_G(t)\, dW(t)$$

with the limit in $L^2(\mathbf{P})$. And we can take a subsequence which converges almost surely.

# CONSEQUENCES [2]

Now we prove that

$$\mathbf{E}\left[F\int_0^T g(t)\,dW(t)\right] = \mathbf{E}\left[F\int_0^T \chi_G(t)g(t)\,dW(t)\right]$$

in which F is a bounded $\mathcal{F}_G$-measurable random variable. We may assume $F = \int_0^T \chi_A(t)\,dW(t)$ for $A \subseteq G$. Applying Itô Isometry, we have the result. $\square$

***Lemma 14.*** *Let* $G \subseteq [0, T]$ *be a Borel set and* $v(t)$ *be a stochastic process such that*

1. $v(t)$ *is measurable with respect to* $\mathcal{F}_t \cap \mathcal{F}_G = \mathcal{F}_{[0,t] \cap G}$ *for all* $t \in [0, T]$.
2. $\mathbf{E}\left[\int_0^T v^2(t)\,\mathrm{d}t\right] < \infty$.

*Then the following integral is* $\mathcal{F}_G$*-measurable:*

$$\int_G v(t)\,\mathrm{d}W(t)$$

*Proof.* Consider $v$ as an elementary process and integrate. The general case follows from approximation. $\square$

## CONSEQUENCES

***Lemma 15.*** *Let* $u(t)$ *be an* $\mathbf{F}$*-adapted stochastic process in* $L^2(\mathbf{P} \times \lambda)$*. Then*

$$\mathbf{E}\left[\int_0^T u(t)\, dW(t) \,\middle|\, \mathcal{F}_G\right] = \int_G \mathbf{E}[u(t) \mid \mathcal{F}_G]\, dW(t)$$

*Proof.* By the lemma 14, we have that $\int_G \mathbf{E}[u(t) \mid \mathcal{F}_G]\, dW(t)$ is $\mathcal{F}_G$-measurable.
Our goal is to verify

$$\mathbf{E}\left[F \int_0^T u(t)\, dW(t)\right] = \mathbf{E}\left[F \int_G \mathbf{E}[u(t) \mid \mathcal{F}_G]\, dW(t)\right]$$

for $F = \int_A dW(t)$ and $A \subseteq G$ Borel. Apply Itô Isometry to both sides of the equation and use a density argument.

□

***Theorem 16.*** *Let* $f_n \in \hat{L}^2([0,T]^n)$. *Then* $\mathbf{E}[I_n(f_n) \mid \mathcal{F}_G] = I_n[f_n \chi_G^{\otimes n}]$.

*Proof.* By induction on $n$ and the lemma 15. $\square$

# CONSEQUENCES

***Theorem 17.*** *If* $F \in \mathbf{D}_{1,2}$*, then* $\mathbf{E}[F \mid \mathcal{F}_G] \in \mathbf{D}_{1,2}$ *and*

$$D_t \mathbf{E}[F \mid \mathcal{F}_G] = \mathbf{E}[D_t F \mid \mathcal{F}_G] \chi_G(t)$$

*Proof.*

- If $F = I_n(f_n)$, then the result follows from the Theorem 16.
- More generally, if $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbf{D}_{1,2}$, we define $F_k = \sum_{n=0}^{k} I_n(f_n)$.
- Then $F_k \to F$ in $L^2(\Omega)$ and $D_t F_k \to D_t F$ in $L^2(\mathbf{P} \times \lambda)$ as $k \to \infty$.
- Using the previous case, we have $D_t \mathbf{E}[F_k \mid \mathcal{F}_G] = \mathbf{E}[D_t F_k \mid \mathcal{F}_G] \chi_G(t)$ for all $k$. Taking the limit in $L^2(\mathbf{P} \times \lambda)$, we have the result.

$\square$

***Corollary 18.*** *Let* $u(s)$ *be an* $\mathbf{F}$*-adapted stochastic process such that* $u(s) \in \mathbf{D}_{1,2}$ *for all* $s \in [0, T]$. *Then*

1. $D_t u(s)$, $s \in [0, T]$, *is* $\mathbf{F}$*-adapted for all* $t$.
2. $D_t u(s) = 0$ *for* $t > s$.

*Proof.* Apply the Theorem 17 to $D_t u(s)$. $\square$

# THE SKOROHOD INTEGRAL

# THE SKOROHOD INTEGRAL

1. The Skorohod integral is an extension of the Itô integral for integrands that, not necessarily, are **F**-adapted. If the integrand is **F**-adapted, then they coincide.
2. **Duality formula**: the Malliavin derivative is the adjoint operator of the Skorohod integral, i.e., $\langle \delta(u), F \rangle_{L^2(\mathbf{P})} = \langle u, D_{\cdot} F \rangle_{L^2(\mathbf{P} \times \lambda)}$, where $\delta(u)$ is the Skorohod integral of $u$.

1. Take $u(t)$ a measurable random variable from a stochastic process $u(x, t)$ in $L^2(\mathbf{P})$.
2. Write the Chaos Expansion $u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t})$ for each $t \in [0, T]$. This gives us symmetric functions $f_{n,t}(t_1, \ldots, t_n)$.
3. Since $f_{n,t}(t_1, \ldots, t_n)$ also depends on $t$, we can write $f_n(t_1, \ldots, t_n, t_{n+1})$, with $t_{n+1} = t$, and take its symmetrization $\hat{f}_n$.
4. Define the **Skorohod Integral** as

$$\delta(u) = \int_0^T u(t)\, \delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n)$$

# THE CLARK-OCONE FORMULA

## THE CLARK-OCONE FORMULA [1]

***Theorem 19 (The Clark-Ocone Formula).*** *Let* $F \in \mathbf{D}_{1,2}$ *be* $\mathcal{F}_T$*-measurable. Then*

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}[D_t F \mid \mathcal{F}_t] \, dW(t)$$

*Proof.* The idea is to write the Chaos Expansion of F and compute the integral on the right hand side.

Let

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

be the Wiener-Itô chaos expansion of F.

Then

$$\begin{aligned}
\int_0^T \mathbf{E}[D_t F \mid \mathcal{F}_t]\, dW(t) &= \int_0^T \mathbf{E}\left[ D_t \sum_{n=0}^{\infty} I_n(f_n) \,\middle|\, \mathcal{F}_t \right] dW(t) \\
&= \int_0^T \mathbf{E}\left[ \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) \,\middle|\, \mathcal{F}_t \right] dW(t) \\
&= \int_0^T \sum_{n=1}^{\infty} n\mathbf{E}\left[ I_{n-1}(f_n(\cdot, t)) \mid \mathcal{F}_t \right] dW(t)
\end{aligned}$$

Using that $\mathbf{E}[I_n(f_n) \mid \mathcal{F}_G] = I_n[f_n \chi_G^{\otimes n}]$ (Theorem 16), we have

# THE CLARK-OCONE FORMULA [3]

$$
\begin{aligned}
\int_0^T \sum_{n=1}^{\infty} n\mathbf{E}\left[\mathrm{I}_{n-1}(f_n(\cdot,t)) \mid \mathcal{F}_t\right] \mathrm{dW}(t) &= \int_0^T \sum_{n=1}^{\infty} n\mathrm{I}_{n-1}(f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}(\cdot))\, \mathrm{dW}(t) \\
&= \int_0^T \sum_{n=1}^{\infty} n(n-1)!\mathrm{J}_{n-1}(f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}(\cdot))\, \mathrm{dW}(t) \\
&= \sum_{n=1}^{\infty} n!\mathrm{J}_n(f_n) = \sum_{n=1}^{\infty} \mathrm{I}_n(f_n) \\
&= \sum_{n=0}^{\infty} \mathrm{I}_n(f_n) - \mathrm{I}_0(f_0) = \mathrm{F} - \mathbf{E}[\mathrm{F}]
\end{aligned}
$$

$\square$

## WHAT IF WE CHANGE THE MEASURE?

Before stating the Clark-Ocone formula under change of measure, let us recall the Girsanov theorem and Novikov's criterion. The processes $(u_t)$, $(Z_t)$ and $(\widetilde{W}_t)$ defined in them will be used throughout the section.

***Theorem 20 (Girsanov).*** *Let* $(u_t)$ *be an adapted process satisfying* $\int_0^T u_s^2 \, \mathrm{ds} < \infty$ *a.s. and such that the process* $(Z_t)$ *given by*

$$Z_t = \exp\left( - \int_0^t u_s \, \mathrm{d}W_s - \frac{1}{2} \int_0^t u_s^2 \, \mathrm{ds} \right)$$

*is a martingale.*

*Then, under the probability* $Q = \mathbf{P}^Z$ *with density* $Z_T$ *with respect to* $\mathbf{P}$*, the process* $(\widetilde{W}_t)$ *defined by*

$$\widetilde{W}_t = W_t + \int_0^t u_s \, \mathrm{ds}$$

*is an* $(\mathcal{F}_t)$*-Brownian motion.*

*Proof.* See, e.g., [S+04, Theorem 5.2.3]. □

# WHAT IF WE CHANGE THE MEASURE?

***Remark 21 (Novikov's criterion).*** *If*

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T u_t^2\,\mathrm{d}t\right)\right] < \infty$$

*then the* $(Z_t)$ *in Girsanov theorem is a martingale.*

# WHAT IF WE CHANGE THE MEASURE?

***Theorem 22 (Clark-Ocone Formula Under Change of Measure).*** *Suppose that* $F \in \mathbf{D}_{1,2}$ *is* $\mathcal{F}_T$*-measurable, and that the following conditions are met*

1. $\mathbf{E}_Q[|F|] < \infty$;
2. $\mathbf{E}_Q\left[\int_0^T |D_tF|^2\, dt\right] < \infty$;
3. $\mathbf{E}_Q\left[|F| \int_0^T \left(\int_0^T D_t u(s)\, dW(s) + \int_0^T u(s) D_t u(s)\, ds\right)^2 dt\right] < \infty$.

*Then*

$$F = \mathbf{E}_Q[F] + \int_0^T \mathbf{E}_Q\left[\left(D_tF - F\int_t^T D_t u(s)\, d\widetilde{W}(s)\right) \middle| \mathcal{F}_t\right] d\widetilde{W}(t)$$

To prove this resut, we'll need the Bayes Rule and a lemma.

***Theorem 23 (Bayes Rule).*** *If* $G \in L^1(Q)$, *then*

$$\mathbf{E}_Q[G \mid \mathcal{F}_t] = \frac{\mathbf{E}[Z(T)G \mid \mathcal{F}_t]}{Z(t)}$$

*Proof.* Let $A \in \mathcal{F}_t$. Then

$$\begin{aligned}\mathbf{E}_Q\left[\chi_A \frac{\mathbf{E}[Z(T)G \mid \mathcal{F}_t]}{Z(t)}\right] &= \mathbf{E}[\chi_A \mathbf{E}[Z(T)G \mid \mathcal{F}_t]] \\ &= \mathbf{E}[\chi_A Z(T)G] = \mathbf{E}_Q[\chi_A G]\end{aligned}$$

□

***Lemma 24.***

$$D_t(Z(T)F) = Z(T)\left[D_tF - F\left(u(t) + \int_t^T D_t u(s)\, d\widetilde{W}(s)\right)\right]$$

*Proof.* We use the following result

$$D_t\left(\int_0^T u(s)\, dW(s)\right) = \int_t^T D_t u(s)\, dW(s) + u(t)$$

Applying this fact and the chain rule to $D_tZ(T)$ yields

# WHAT IF WE CHANGE THE MEASURE? [2]

$$
\begin{aligned}
D_t Z(T) &= Z(T)\left[-D_t \int_0^T u(s)\,dW(s) - \frac{1}{2} D_t \int_0^T u^2(s)\,ds\right] \\
&= Z(T)\left[-\int_t^T D_t u(s)\,dW(s) - u(t) - \int_0^T u(s) D_t u(s)\,ds\right] \\
&= Z(T)\left[-\int_t^T D_t u(s)\,d\widetilde{W}(s) - u(t)\right]
\end{aligned}
$$

□

# WHAT IF WE CHANGE THE MEASURE? [1]

Now we're ready to prove Theorem 22.

*Proof.* Define $Y(t) = \mathbf{E}_Q[F \mid \mathcal{F}_t]$ and $\Lambda(t) = Z^{-1}(t)$. Notice that

$$\begin{aligned} \Lambda(t) &= \exp\left(\int_0^t u(s)\, dW_s + \frac{1}{2}\int_0^t u^2(s)\, ds\right) \\ &= \exp\left(\int_0^t u(s)\, d\widetilde{W}_s - \frac{1}{2}\int_0^t u^2(s)\, ds\right) \end{aligned}$$

Using the corollary 23,

$$Y_t = \Lambda(t)\mathbf{E}[Z(T)F \mid \mathcal{F}_t]$$

## WHAT IF WE CHANGE THE MEASURE? [2]

Applying the Clark-Ocone formula,

$$Y_t = \Lambda(t)\left[\mathbf{E}[\mathbf{E}[Z(T)F \mid \mathcal{F}_t]] + \int_0^T \mathbf{E}[D_s\mathbf{E}[Z(T)F \mid \mathcal{F}_t] \mid \mathcal{F}_s]\, dW(s)\right]$$

Simplifying and using the proposition 17,

$$Y_t = \Lambda(t)\left[\mathbf{E}[Z(T)F] + \int_0^T \mathbf{E}[D_s(Z(T)F) \mid \mathcal{F}_s]\, dW(s)\right] = \Lambda(t)U(t)$$

where we defined

$$U(t) = \mathbf{E}[Z(T)F] + \int_0^T \mathbf{E}[D_s(Z(T)F) \mid \mathcal{F}_s]\, dW(s)$$

# WHAT IF WE CHANGE THE MEASURE? [3]

Apply Itô formula to $\Lambda(t)$,

$$d\Lambda(t) = \Lambda(t)u(t)\, d\widetilde{W}(t)$$

By the lemma 24, using the change of measure and the expression above,

$$\begin{aligned}
dY(t) &= \Lambda(t)\mathbf{E}\left[D_t(Z(T)F) \mid \mathcal{F}_t\right] dW(t) + \Lambda(t)u(t)U(t)d\widetilde{W}(t) \\
&\quad + \Lambda(t)u(t)\mathbf{E}\left[D_t(Z(T)F) \mid \mathcal{F}_t\right] dW(t)d\widetilde{W}(t) \\
&= \Lambda(t)\mathbf{E}\left[D_t(Z(T)F) \mid \mathcal{F}_t\right] d\widetilde{W}(t) + u(t)Y(t)d\widetilde{W}(t) \\
&= \Lambda(t)\left(\mathbf{E}\left[Z(T)D_tF \mid \mathcal{F}_t\right] - \mathbf{E}\left[Z(T)Fu(t) \mid \mathcal{F}_t\right]\right. \\
&\quad \left. -\mathbf{E}\left[Z(T)F\int_t^T D_tu(s)\, d\widetilde{W}(s) \,\middle|\, \mathcal{F}_t\right]\right) d\widetilde{W}(t) + u(t)Y(t)d\widetilde{W}(t)
\end{aligned}$$

## WHAT IF WE CHANGE THE MEASURE? [4]

Since $Y(T) = \mathbf{E}_Q[F \mid \mathcal{F}_T] = F$ and $Y(0) = \mathbf{E}_Q[F \mid \mathcal{F}_0] = \mathbf{E}_Q[F]$, the result follows.

$\square$

# EXTENSIONS

- Here we proved the Clark-Ocone formula only for $\mathbf{D}_{1,2}$.
- It is possible to extend this result to $L^2(\mathbf{P})$ and $\mathcal{G}^*$.
- Nevertheless, to do that we need White Noise Theory and the Hida-Malliavin derivative, where we work in $\Omega = \mathcal{S}'(\mathbf{R})$.
- For a reference, see chapters five and six of [NØP08].

# FINANCIAL APPLICATION

## FINANCIAL APPLICATION

Consider a market consisting of a riskless asset $S_0$ with

$$\text{riskless asset} \quad \begin{cases} dS_0(t) = \rho(t)S_0(t)\,dt \\ S_0(0) = 1 \end{cases} \tag{1}$$

and a risky asset $S_1$ satisfying

$$\text{risky asset} \quad \begin{cases} dS_1(t) = \mu(t)S_1(t)\,dt + \sigma(t)S_1(t)\,dW(t) \\ S_1(0) > 0 \end{cases} \tag{2}$$

where $\rho(t)$, $\mu(t)$, and $\sigma(t) \neq 0$ are $\mathbf{F}$-adapted processes.

We also suppose that they satisfy the following condition

$$\mathbf{E}\left[\int_0^{\mathrm{T}} (|\rho(t)| + |\mu(t)| + \sigma^2(t))\, \mathrm{d}t\right] < \infty$$

Let $\theta_0(t)$ and $\theta_1(t)$ denote the number of units of $\mathrm{S}_0(t)$ and $\mathrm{S}_1(t)$, respectively. Then the value of the portfolio $\theta = (\theta_0, \theta_1)$ is $\mathrm{V}^\theta = \theta_0 \mathrm{S}_0 + \theta_1 \mathrm{S}_1$. We also suppose that the portfolio is self-financing, i.e.,

$$\mathrm{dV}^\theta(t) = \theta_0(t)\mathrm{dS}_0(t) + \theta_1(t)\mathrm{dS}_1(t) \tag{3}$$

Substituting

$$\theta_0(t) = \frac{V^\theta(t) - \theta_1(t)S_1(t)}{S_0(t)}$$

into (3) and using (1) we have

$$dV^\theta = \rho(t)(V^\theta(t) - \theta_1(t)S_1(t))dt + \theta_1(t)dS_1 \tag{4}$$

Replacing (2),

$$dV^\theta = [\rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)dW(t) \tag{5}$$

## FINANCIAL APPLICATION

- Our goal is to find a replicating (hedging) portfolio

$$V^{\theta}(T) = F, \quad \mathbf{P} - a.s. \tag{6}$$

  where F is $\mathcal{F}_t$-measurable. For an European call, for example, $F = \max\{S_1 - K, 0\} = (S_1 - K)^+$.

- How much do we need to invest at time $t = 0$ and which portfolio $\theta(t)$ should we use? Are $V^{\theta}$ and $\theta$ unique?
- We consider $(V^{\theta}(t), \theta_1(t))$ an $\mathbf{F}$-adapted process. The equations (4) and (6) form a **backward stochastic differential equation** (BSDE). To find an explicit solution, we can change the measure and apply Clark-Ocone.

Define

$$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)}$$

Using the change of measure, we can write

$$\begin{aligned} dV^\theta &= [\rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)d\widetilde{W}(t) \\ &\quad - \sigma(t)\theta_1(t)S_1(t)\sigma^{-1}(t)(\mu(t) - \rho(t))dt \\ &= \rho(t)V^\theta(t)dt + \sigma(t)\theta_1(t)S_1(t)d\widetilde{W}(t) \end{aligned} \tag{7}$$

Let

$$\mathrm{U}^{\theta}(t) = e^{-\int_0^t \rho(s)\,\mathrm{d}s}\mathrm{V}^{\theta}(t)$$

Then using (7),

$$\mathrm{dU}^{\theta}(t) = e^{-\int_0^t \rho(s)\,\mathrm{d}s}\sigma(t)\theta_1(t)\mathrm{S}_1(t)\,\mathrm{d}\widetilde{\mathrm{W}}(t)$$

or, equivalently,

$$e^{-\int_0^t \rho(s)\,\mathrm{d}s}\mathrm{V}^{\theta}(\mathrm{T}) = \mathrm{V}^{\theta}(0) + \int_0^{\mathrm{T}} e^{-\int_0^t \rho(s)\,\mathrm{d}s}\sigma(t)\theta_1(t)\mathrm{S}_1(t)\,\mathrm{d}\widetilde{\mathrm{W}}(t) \tag{8}$$

Applying the generalized Clark-Ocone formula to

$$G = e^{-\int_0^t \rho(s)\, \mathrm{d}s} F$$

we have

$$G = \mathbf{E}_Q[G] + \int_0^T \mathbf{E}_Q\left[\left(D_t G - G\int_t^T D_t u(s)\, \mathrm{d}\widetilde{W}(s)\right) \,\middle|\, \mathcal{F}_t\right] \mathrm{d}\widetilde{W}(t) \tag{9}$$

Comparing (8) with (9), we have $V^{\theta}(0) = \mathbf{E}_Q[G]$ by uniqueness, and the replicating portfolio is given by

$$\theta_1(t) = e^{-\int_0^t \rho(s)\, \mathrm{d}s} \sigma^{-1}(t) S_1^{-1}(t) \mathbf{E}_Q\left[\left(D_t G - G\int_t^T D_t u(s)\, \mathrm{d}\widetilde{W}(s)\right) \,\middle|\, \mathcal{F}_t\right] \tag{10}$$

In particular, if $\rho$ and $\mu$ are constants, and $\sigma(t) = \sigma \neq 0$, then

$$u(t) = u = \frac{\mu - \rho}{\sigma}$$

is also constant, whence $D_t u = 0$. Then the equation (10) simplifies to

$$\theta_1(t) = e^{\rho(t-T)}\sigma^{-1}S_1^{-1}(t)\mathbf{E}_Q[D_t F \mid \mathcal{F}_t]$$

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