Problem 1.5. **(*)** This exercise is based on [107]. Suppose the function F of Problem 1.4 has the form

$$F = f(X(T)),$$

where $X = X(t)$, $t \in [0, T]$, is an Itô diffusion given by

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t); \qquad X(0) = x \in \mathbb{R}.$$

Here $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are given Lipschitz continuous functions of at most linear growth, so there exists a unique strong solution $X(t) = X^x(t)$, $t \in [0, T]$. Then there is a useful formula for the process φ in the Itô representation theorem. This formula is achieved as follows. If g is a real function such that

$$E[|g(X^x(t))|] < \infty,$$

then we define

$$u(t, x) := P_t g(x) := E[g(X^x(t))], \qquad t \in [0, T], \quad x \in \mathbb{R}.$$

Suppose that there exists $\delta > 0$ such that

$$|\sigma(x)| \geq \delta \quad \text{for all } x \in \mathbb{R}. \tag{1.28}$$

Then $u(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}$$

(this is the Kolmogorov backward equation, see, for example, [74, Volume 1, Theorem 5.11, p. 162 and Volume 2, Theorem 13.18, p. 53], [176, Theorem 8.1] for details on this issue).

(a) Use the Itô formula for the process

$$Y(t) = g(t, X(t)), \quad t \in [0, T], \qquad \text{with} \quad g(t, x) = P_{T-t}f(x)$$

to show that

$$f(X(T)) = P_T f(x) + \int_0^T \left[\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)\right]_{|\xi = X(t)} dW(t), \tag{1.29}$$

for all $f \in C^2(\mathbb{R})$. In other words, with the notation of Problem 1.4, we have shown that if $F = f(X(T))$, then

$$E[F] = P_T f(x) \quad \text{and} \quad \varphi(t) = \left[\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)\right]_{|\xi = X(t)}. \tag{1.30}$$

Note that P_t is the transition probability. P_0 is the identity matrix.
Also,

$$g(T, X_T) = P_0 f(X(T)) = f(X(T))$$

By Itô's formula,

$$dY_t = \partial_t g(t,x)\, dt + \partial_x g(t,x)\, dX_t + \frac{1}{2}\partial_{x^2} g(t,x)[dX_t, dX_t]$$

$$= \underbrace{\cancel{\left(\partial_t g(t,x) + b(x)\partial_x g(t,x) + \frac{1}{2}\sigma^2(x)\partial_{x^2} g(t,x)\right)}}_{0} dt + \sigma(x)\,\partial_x g(t,x)\, dW_t$$

Hence,

$$Y_T - Y_0 = \int_0^T \sigma(x)\, \partial_x g(t,x)\, dW_t$$

and

$$\boxed{Y_T = P_T f(x) + \int_0^T \sigma(\xi)\, \partial_\xi P_{T-t} f(\xi)\, dW_t}$$

(b) Use (1.30) to compute $E[F]$ and find φ in the Itô representation of the following random variables:
(b.1) $F = W^2(T)$
(b.2) $F = W^3(T)$
(b.3) $F = X(T)$, where $X(t)$, $t \in [0,T]$, is the geometric Brownian motion, that is,

$$dX(t) = \rho X(t)dt + \alpha X(t)dW(t); \quad X(0) = x \in \mathbb{R} \qquad (\rho, \alpha \text{ constants}).$$

(b.1) Goal: write $W^2(T) = P_T f(x) + \int_0^T \varphi(t)\, dW_t$

Let $\xi = x + W_t$. By Itô's formula,

$$d(W_T^2) = dt + 2W_t dW_t$$

Since $f(\xi) = \xi^2$, we have

$$P_T f(\xi) = \mathbb{E}[f(W_T^\xi)] = \mathbb{E}^\xi[W_T^2] = T + \xi^2$$

And

$$\varphi(t) = \frac{\partial}{\partial \xi} P_{T-t} f(\xi) = \frac{\partial}{\partial \xi}[\xi^2 + T] = 2\xi = 2(W_T + x)$$

(b.2) Let $f(t,\xi)=\xi^3$ and $X_t = x+W_t$. Then

$$P_T f(\xi) = \mathbb{E}^{\xi}[X_T^3] = \mathbb{E}^{\xi}[x^3+3x^2W_t+3xW_t^2+W_t^3] = \xi^3+3\xi T$$

we have

$$\mathbb{E}[F] = P_T f(x) = x^3+3xT$$

and

$$\varphi(t) = \frac{\partial}{\partial \xi} P_{T-t} f(\xi) = \frac{\partial}{\partial \xi}\left[\xi^3+3(T-t)\xi\right] = 3\xi^2+3(T-t)$$

$$= 3(x+W_t)^2+3(T-t)$$

(b.3) Given that $f(\xi)=\xi$, we have

GBM: $\mathbb{E}[X_t] = X_0 e^{\mu t}$

$$P_T f(\xi) = \mathbb{E}[f(X_T^{\xi})] = \mathbb{E}^{\xi}[X_T] = \xi e^{\rho T} \Rightarrow \mathbb{E}[F] = P_T f(x) = x e^{\rho T}$$

and

$$\varphi(t) = \alpha\xi \frac{\partial}{\partial \xi} P_{T-t} f(\xi) = \alpha\xi \frac{\partial}{\partial \xi}\left[\xi e^{\rho(T-t)}\right] = \alpha X_t e^{\rho(T-t)}$$

(c) Extend formula (1.30) to the case when $X(t) \in \mathbb{R}^n$, $t \in [0,T]$, and $f : \mathbb{R}^n \to \mathbb{R}$. In this case, condition (1.28) must be replaced by the *uniform ellipticity* condition

$$\eta^T \sigma^T(x)\sigma(x)\eta \geq \delta|\eta|^2 \quad \text{for all } x \in \mathbb{R}^n, \eta \in \mathbb{R}^n, \tag{1.31}$$

where $\sigma^T(x)$ denotes the transposed of the $m \times n$-matrix $\sigma(x)$.

Using Itô's formula,

$$dY = \frac{\partial g}{\partial t}(t, X_t)\,dt + \sum_i \frac{\partial g}{\partial x_i}(t, X_t)\,dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j}(t, X_t)\,dX_i\,dX_j$$

$$= \cancel{\left(\frac{\partial g}{\partial t} + \sum_i b(X_t)\frac{\partial g}{\partial x_i} + \frac{1}{2}\sum_{i,j} (\sigma\sigma^T)(X_t)\frac{\partial^2 g}{\partial x_i \partial x_j}\right)} + \sum_i \sigma^T(X_t)\frac{\partial g}{\partial x_i}\,dW_i(t)$$

Hence,

$$dY(T) = \sigma^T(X_t)\nabla P_{T-t}f(X_t)\,dW_t$$

and

$$Y(T) = P_T f(x) + \int_0^T \sigma^T(x)\nabla P_{T-t}f(x)\,dW_t$$

□