Problem 1.2. Let ξ and ζ be F_T-measurable random variables in $L^2(P)$ with Wiener–Itô chaos expansions $\xi = \sum_{n=0}^{\infty} I_n(f_n)$ and $\zeta = \sum_{n=0}^{\infty} I_n(g_n)$, respectively. Prove that the chaos expansion of the sum $\xi + \zeta = \sum_{n=0}^{\infty} I_n(h_n)$ is such that $h_n = f_n + g_n$ for all $n = 1, 2, \ldots$

By inducton on n. For $n=1$,

$$\sum_{n=0}^{1} I_n(f_n) + \sum_{n=0}^{1} I_n(g_n) = f_0 + \int_{[0,T]} f_1(t_1)\, dW(t_1) + g_0 + \int_{[0,T]} g_1(t_1)\, dW(t_1)$$

$$= (f_0 + g_0) + \int_{[0,T]} (f_1(t_1) + g_1(t_1))\, dW(t_1) = h_0 + \int_{[0,T]} h_1(t_1)\, dW(t_1)$$

Suppose that the identity holds for K. Then, for $K+1$,

$$\sum_{n=0}^{K+1} I_n(f_n) + \sum_{n=0}^{K+1} I_n(g_n) = \sum_{n=0}^{K} I_n(f_n) + \sum_{n=0}^{K} I_n(g_n) + I_{K+1}(f_{K+1}) + I_{K+1}(g_{K+1})$$

$$= \sum_{n=0}^{K} I_n(h_n) + \int_{[0,T]^{K+1}} f_{K+1}(t_1, \ldots, t_{K+1})\, dW(t_1) \cdots dW(t_{K+1})$$

$$+ \int_{[0,T]^{K+1}} g_{K+1}(t_1, \ldots, t_{K+1})\, dW(t_1) \cdots dW(t_{K+1})$$

$$= \sum_{n=0}^{K} I_n(h_n) + \int_{[0,T]^{K+1}} (f_{K+1} + g_{K+1})(t_1, \ldots, t_{K+1})\, dW(t_1) \cdots dW(t_{K+1})$$

$$= \sum_{n=0}^{K} I_n(h_n) + \int_{[0,T]^{K+1}} h_{K+1}(t_1, \ldots, t_{K+1})\, dW(t_1) \cdots dW(t_{K+1}) = \sum_{n=0}^{K+1} I_n(h_n)$$

Notice that we can take the limit because $\xi, \zeta \in L^2(\mathbb{P})$. More explictly, let

$$y_K = \sum_{i=1}^{K} I_i(h_1)$$

We want to show $\|y_m - y_n\|_2 \longrightarrow 0$

$$\|y_m - y_n\|_2 = \left\| \sum_{i=n+1}^{m} I_i(h_i) \right\| = \left\| \sum_{i=n+1}^{m} I_i(f_i) + \sum_{i=n+1}^{m} I_i(g_i) \right\|$$

$$\leq \sum_{i=n+1}^{\infty} \| I_i(f_i) \| + \sum_{i=n+1}^{\infty} \| I_i(g_i) \| \longrightarrow 0$$

$\blacksquare$