Problem 1.1. (*) Let $h_n(x)$, $n = 0, 1, 2, \ldots$, be the Hermite polynomials defined in (1.13).

(a) Prove that

$$\exp\left\{tx - \frac{t^2}{2}\right\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x).$$

[*Hint.* Write $\exp\{tx - \frac{t^2}{2}\} = \exp\{\frac{1}{2}x^2\} \cdot \exp\{-\frac{1}{2}(x-t)^2\}$ and apply the Taylor formula on the last factor.]

Using the hint,

$$\exp(tx - t^2/2) = \exp(1/2\, x^2)\exp(-1/2\,(x-t)^2)$$

$$= e^{1/2 x^2} \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\left[\exp(-1/2\,(x-t)^2)\right]\frac{t^n}{n!}$$

Letting $u = x - t$, we have

$$e^{1/2 x^2} \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\left[\exp(-1/2\,(x-t)^2)\right]\frac{t^n}{n!} = e^{1/2 x^2} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \frac{d^n}{du^n}\left[\exp(-1/2\, u^2)\right]$$

$$= e^{1/2 x^2} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \frac{d^n}{dx^n}\left[\exp(-1/2\, x^2)\right]$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x)$$

(b) Show that if $\lambda > 0$ then

$$\exp\{tx - \frac{t^2\lambda}{2}\} = \sum_{n=0}^{\infty} \frac{t^n \lambda^{\frac{n}{2}}}{n!} h_n(\frac{x}{\sqrt{\lambda}}).$$

Using $u = t\sqrt{\lambda}$ we have

$$\exp\left(tx - \frac{t^2\lambda}{2}\right) = \exp\left(\frac{ux}{\sqrt{\lambda}} - \frac{u^2}{2}\right)$$

By the previous part,

$$\exp\left(\frac{u}{\sqrt{\lambda}}x - \frac{u^2}{2}\right) = \sum_{n=0}^{\infty} \frac{u^n}{n!} h_n\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{n=0}^{\infty} \frac{t^n \lambda^{n/2}}{n!} h_n\left(\frac{x}{\sqrt{\lambda}}\right)$$

(c) Let $g \in L^2([0,T])$. Put

$$\theta = \int_0^T g(s)dW(s).$$

Show that

$$\exp\Big\{\int_0^T g(s)dW(s) - \frac{1}{2}\|g\|^2\Big\} = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n(\frac{\theta}{\|g\|}),$$

where $\|g\| = \|g\|_{L^2([0,T])}$.

Let $t = \|g\|$ and $x = \theta / \|g\|$ and apply item a.

(d) Let $t \in [0,T]$. Show that $\exp\{W(t) - \frac{1}{2}t\} = \sum_{n=0}^{\infty} \frac{t^{n/2}}{n!} h_n(\frac{W(t)}{\sqrt{t}})$.

Apply item b. Use $t = 1$, $\lambda = t$ and $x = W(t)$.

Problem 1.2. Let ξ and ζ be F_T-measurable random variables in $L^2(P)$ with Wiener–Itô chaos expansions $\xi = \sum_{n=0}^{\infty} I_n(f_n)$ and $\zeta = \sum_{n=0}^{\infty} I_n(g_n)$, respectively. Prove that the chaos expansion of the sum $\xi + \zeta = \sum_{n=0}^{\infty} I_n(h_n)$ is such that $h_n = f_n + g_n$ for all $n = 1, 2, ...$

By induction on n. For $n=1$,

$$\sum_{n=0}^{1} I_n(f_n) + \sum_{n=0}^{1} I_n(g_n) = f_0 + \int_{[0,T]} f_1(t_1)\,dW(t_1) + g_0 + \int_{[0,T]} g_1(t_1)\,dW(t_1)$$

$$= (f_0 + g_0) + \int_{[0,T]} (f_1(t_1) + g_1(t_1))\,dW(t_1) = h_0 + \int_{[0,T]} h_1(t_1)\,dW(t_1)$$

Suppose that the identity holds for K. Then, for $K+1$,

$$\sum_{n=0}^{K+1} I_n(f_n) + \sum_{n=0}^{K+1} I_n(g_n) = \sum_{n=0}^{K} I_n(f_n) + \sum_{n=0}^{K} I_n(g_n) + I_{K+1}(f_{K+1}) + I_{K+1}(g_{K+1})$$

$$= \sum_{n=0}^{K} I_n(h_n) + \int_{[0,T]^{K+1}} f_{K+1}(t_1, \dots, t_{K+1})\,dW(t_1)\cdots dW(t_{K+1})$$

$$+ \int_{[0,T]^{K+1}} g_{K+1}(t_1, \dots, t_{K+1})\,dW(t_1)\cdots dW(t_{K+1})$$

$$= \sum_{n=0}^{K} I_n(h_n) + \int_{[0,T]^{K+1}} (f_{K+1} + g_{K+1})(t_1, \dots, t_{K+1})\,dW(t_1)\cdots dW(t_{K+1})$$

$$= \sum_{n=0}^{K} I_n(h_n) + \int_{[0,T]^{K+1}} h_{K+1}(t_1, \dots, t_{K+1})\,dW(t_1)\cdots dW(t_{K+1}) = \sum_{n=0}^{K+1} I_n(h_n)$$

Notice that we can take the limit because $\xi, \zeta \in L^2(\mathbb{P})$. More explicitly, let

$$y_K = \sum_{i=1}^{K} I_1(h_1)$$

We want to show $\|y_m - y_n\|_2 \longrightarrow 0$

$$\|y_m - y_n\|_2 = \left\| \sum_{i=n+1}^{m} I_i(h_i) \right\| = \left\| \sum_{i=n+1}^{m} I_i(f_i) + \sum_{i=n+1}^{m} I_i(g_i) \right\|$$

$$\leq \sum_{i=n+1}^{\infty} \| I_i(f_i) \| + \sum_{i=n+1}^{\infty} \| I_i(g_i) \| \longrightarrow 0$$

$\blacksquare$

Problem 1.3. **(*)** Find the Wiener–Itô chaos expansion of the following random variables:

(a) $\xi = W(t)$, where $t \in [0,T]$ is fixed,

Notice that

$$I_1(1) = \int_0^T dW_t = W(T) - \cancelto{0}{W(0)}$$

Since we want only $W(t)$,

$$\int_0^T \chi_{[0,t]}(s)\, dW_s = W(t) \implies \xi = I_1(\chi_{[0,t]}(s))$$

(b) $\xi = \int\limits_0^T g(s)dW(s)$, where $g \in L^2([0,T])$,

$$I_1(g) = \int_0^T g(s)\, dW_s = \xi$$

(c) $\xi = W^2(t)$, where $t \in [0,T]$ is fixed,

Since $h_2(x) = x^2 - 1$ and

$$\mathbb{E}[W^2(T)] = T = I_0$$

we have

$$I_2(1) = 2 \cdot \int_0^T \int_0^{t_2} dW(t_1)\, dW(t_2) = T h_2\left(\frac{\int_0^T 1 \cdot dW(t)}{\sqrt{T}}\right) = T h_2\left(\frac{W(T)}{\sqrt{T}}\right) = W^2(T) - T$$

To restrict to $[0,t]$, we consider

$$2 \cdot \int_0^T \int_0^{t_2} \chi_{[0,T]}(t_1,t_2)\, dW(t_1)\, dW(t_2) = T h_2\left(\frac{\int_0^T \chi_{[0,T]}(s)\, dW_s}{\sqrt{T}}\right)$$

$$= T h_2\left(\frac{W(t)}{\sqrt{T}}\right) = W^2(t) - T = I_2(\chi_{[0,T]}(t_1,t_2))$$

Thus,

$$\xi = W^2(t) = I_2(\chi_{[0,T]}(t_1,t_2)) + I_0$$

(d) $\xi = \exp\{\int_0^T g(s)dW(s)\}$, where $g \in L^2([0,T])$ [*Hint.* Use (1.15).],

By the exercise 1.c, we have

$$\exp\left[\int_0^T g(s)\,dW(s) - \frac{1}{2}\|g\|^2\right] = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\left(\frac{\theta}{\|g\|}\right)$$

$$= \exp\left[\int_0^T g(s)\,dW(s)\right] \cdot \exp\left[-\frac{1}{2}\|g\|^2\right]$$

Thus,

$$\exp\left[\int_0^T g(s)\,dW(s)\right] = \exp\left[\frac{1}{2}\|g\|^2\right] \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\left(\frac{\theta}{\|g\|}\right)$$

$$= \sum_{n=0}^{\infty} I_n[g^{\otimes n}] \frac{\exp(1/2\,\|g\|^2)}{n!}$$

$$= \sum_{n=0}^{\infty} I_n\left[\frac{g^{\otimes n}\exp(1/2\,\|g\|^2)}{n!}\right] = \xi$$

(e) $\xi = \int_0^T g(s)W(s)ds$, where $g \in L^2([0,T])$.

$$\int_0^T g(s)\,W(s)\,ds = \int_0^T g(s)\int_0^s dW(t)\,ds \overset{(*)}{=} \int_0^T\int_t^T g(s)\,ds\,dW(t)$$

Hence,

$$\xi = I_1\left[\int_t^T g(s)\,ds\right]$$

(*) $\chi_{[0,s]}(t) = \chi_{[t,T]}(s)$

First step: use

$$I_n(g^{\otimes n}) = n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\,dW(t_1)\cdots dW(t_n) = \|g\|^n h_n\left(\frac{\int_0^T g(t)\,dW(t)}{\|g\|}\right)$$

Second step: combine with the Wiener-Itô chaos expansion:

Theorem 1.2.1 (The Wiener-Itô Chaos Expansion). Let ξ be an $\mathscr{F}_T$-measurable random variable in $L^2(\mathbb{P})$. There exists a unique sequence (f_n) of functions $f_n \in \hat{L}^2([0,T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n)$$

Problem 1.4. (*) The *Itô representation theorem* states that if $F \in L^2(P)$ is $\mathcal{F}_T$-measurable, then there exists a unique $\mathbb{F}$-adapted process $\varphi = \varphi(t), 0 \leq t \leq T$, such that

$$F = E[F] + \int_0^T \varphi(t) dW(t).$$

This result only provides the *existence* of the integrand φ, but from the point of view of applications it is important also to be able to find the integrand φ more explicitly. This can be achieved, for example, by the *Clark–Ocone formula* (see Chap. 4), which says that, under some suitable conditions,

$$\varphi(t) = E[D_t F | \mathcal{F}_t], \qquad 0 \leq t \leq T,$$

where $D_t F$ is the *Malliavin derivative* of F. We discuss this topic later in the book. However, for certain random variables F it is possible to find φ directly, by using the Itô formula. For example, find φ when

(a) $F = W^2(T)$

First, we recall that

$$\int_0^T W(t)\, dW(t) = \frac{1}{2} W^2(T) - \frac{1}{2} T \quad \text{and} \quad E[W^2(T)] = T$$

$$\boxed{\begin{array}{l} g(t,x) = 1/2\, x^2, \quad Y_t = g(t, W_t) \\ dY_t = \frac{1}{2} dt + W_t\, dW_t \Rightarrow \frac{1}{2} W_T^2 = \frac{1}{2} T + \int_0^T W_t\, dW_t \end{array}}$$

Let $\varphi(t) = 2W(t)$. Then

$$T + 2\int_0^T W(t)\, dW(t) = W^2(T)$$

(b) $F = \exp\{W(T)\}$

Let $U(t) = \exp(W(t) - 1/2 t)$. Then, by Itô's formula,

$$dU(t) = \left(-\frac{1}{2} U(t) + \frac{1}{2} U(t) \right) dt + U(t)\, dW(t) = U(t)\, dW(t), \quad U(0) = 1$$

Therefore

$$U_T - U_0 = \int_0^T U_s\, dW_s$$

and we have that

$$\exp(W_T - 1/2T) = 1 + \int_0^T \exp(W_s - 1/2s)\,dW_s$$

$$\Longleftrightarrow \quad e^{W_T} = e^{1/2T} + \int_0^T e^{W_s + 1/2(T-s)}\,dW_s$$

i.e., $\varphi(t) = \exp(W_t + 1/2(T-t))$.

(c) $F = \int\limits_0^T W(t)dt$

If $g(t,x) = tx$ and $Y_t = g(t, W_t)$, then

$$dY_t = W_t dt + t\,dW_t \implies TW_T = \int_0^T W_t\,dt + \int_0^T t\,dW_t$$

Thus,

$$\int_0^T W_t dt = T\int_0^T dW_t - \int_0^T t\,dW_t = \int_0^T (T-t)\,dW_t$$

Since $E[F]=0$, $\varphi(t) = T-t$.

(d) $F = W^3(T)$

Let $g(t,x) = 1/3\, x^3$. Then $d(1/3\, W_t^3) = W_t dt + W_t^2 dW_t$.

In the integral form,

$$\frac{1}{3} W_T^3 = \int_0^T W_t dt + \int_0^T W_t^2 dW_t = \int_0^T W_t^2 dW_t + \int_0^T (T-t)\, dW_t$$

Recall that $\mathbb{E}[W_t^3] = 0$. Then

$$W_T^3 = 3\int_0^T (W_t^2 + T - t)\, dW_t$$

(e) $F = \cos W(T)$ [*Hint.* Check that $N(t) := e^{\frac{1}{2}t}\cos W(t)$, $t \in [0,T]$, is a martingale.]

Consider $g(t,x) = e^{1/2t}\cos(x)$, $N_t = g(t, W_t)$. By Itô's formula,

$$dN_t = \left(\frac{1}{2} e^{1/2t}\cos(W_t) - \frac{1}{2} e^{1/2t}\cos(W_t)\right) dt - e^{1/2t}\sin(W_t)\, dW_t$$

$$= -e^{1/2t}\sin(W_t)\, dW_t$$

Hence,

$$e^{1/2t}\cos(W_t) = 1 - \int_0^t e^{1/2s}\sin(W_s)\, dW_s$$

is a martingale and

$$\cos W_T = e^{1/2T} - \int_0^T e^{1/2(t-T)} \sin W_t\, dW_t$$

Problem 1.5. (*) This exercise is based on [107]. Suppose the function F of Problem 1.4 has the form

$$F = f(X(T)),$$

where $X = X(t)$, $t \in [0, T]$, is an Itô diffusion given by

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t); \qquad X(0) = x \in \mathbb{R}.$$

Here $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are given Lipschitz continuous functions of at most linear growth, so there exists a unique strong solution $X(t) = X^x(t)$, $t \in [0, T]$. Then there is a useful formula for the process φ in the Itô representation theorem. This formula is achieved as follows. If g is a real function such that

$$E[|g(X^x(t))|] < \infty,$$

then we define

$$u(t, x) := P_t g(x) := E[g(X^x(t))], \qquad t \in [0, T], \quad x \in \mathbb{R}.$$

Suppose that there exists $\delta > 0$ such that

$$|\sigma(x)| \geq \delta \quad \text{ for all } x \in \mathbb{R}. \tag{1.28}$$

Then $u(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}$$

(this is the Kolmogorov backward equation, see, for example, [74, Volume 1, Theorem 5.11, p. 162 and Volume 2, Theorem 13.18, p. 53], [176, Theorem 8.1] for details on this issue).

(a) Use the Itô formula for the process

$$Y(t) = g(t, X(t)), \quad t \in [0, T], \qquad \text{with} \quad g(t, x) = P_{T-t}f(x)$$

to show that

$$f(X(T)) = P_T f(x) + \int_0^T \left[\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)\right]_{|\xi = X(t)} dW(t), \tag{1.29}$$

for all $f \in C^2(\mathbb{R})$. In other words, with the notation of Problem 1.4, we have shown that if $F = f(X(T))$, then

$$E[F] = P_T f(x) \quad \text{and} \quad \varphi(t) = \left[\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)\right]_{|\xi = X(t)}. \tag{1.30}$$

Note that P_t is the transition probability. P_0 is the identity matrix. Also,

$$g(T, X_T) = P_0 f(X(T)) = f(X(T))$$

By Itô's formula,

$$dY_t = \partial_t g(t,x)\, dt + \partial_x g(t,x)\, dX_t + \frac{1}{2}\partial_{x^2} g(t,x)[dX_t, dX_t]$$

$$= \cancelto{0}{\left(\partial_t g(t,x) + b(x)\partial_x g(t,x) + \frac{1}{2}\sigma^2(x)\partial_{x^2} g(t,x)\right)} dt + \sigma(x)\partial_x g(t,x)\, dW_t$$

Hence,

$$Y_T - Y_0 = \int_0^T \sigma(x)\partial_x g(t,x)\, dW_t$$

and

$$\boxed{Y_T = P_T f(x) + \int_0^T \sigma(\xi)\, \partial_\xi P_{T-t} f(\xi)\, dW_t}$$

(b) Use (1.30) to compute $E[F]$ and find φ in the Itô representation of the following random variables:
(b.1) $F = W^2(T)$
(b.2) $F = W^3(T)$
(b.3) $F = X(T)$, where $X(t)$, $t \in [0,T]$, is the geometric Brownian motion, that is,

$$dX(t) = \rho X(t)dt + \alpha X(t)dW(t); \quad X(0) = x \in \mathbb{R} \qquad (\rho, \alpha \text{ constants}).$$

(b.1) Goal: write $W^2(T) = P_T f(x) + \int_0^T \varphi(t)\, dW_t$

Let $\xi = x + W_t$. By Itô's formula,

$$d(W_T^2) = dt + 2W_t dW_t$$

Since $f(\xi) = \xi^2$, we have

$$P_T f(\xi) = \mathbb{E}[f(W_T^\xi)] = \mathbb{E}^\xi[W_T^2] = T + \xi^2$$

And

$$\varphi(t) = \frac{\partial}{\partial \xi} P_{T-t} f(\xi) = \frac{\partial}{\partial \xi}[\xi^2 + T] = 2\xi = 2(W_T + x)$$

(b.2) Let $f(t,\xi) = \xi^3$ and $X_t = x + W_t$. Then

$$P_T f(\xi) = \mathbb{E}^{\xi}[X_t^3] = \mathbb{E}^{\xi}[x^3 + 3x^2 W_t + 3x W_t^2 + W_t^3] = \xi^3 + 3\xi T$$

we have

$$\mathbb{E}[F] = P_T f(x) = x^3 + 3xT$$

and

$$\varphi(t) = \frac{\partial}{\partial \xi} P_{T-t} f(\xi) = \frac{\partial}{\partial \xi}[\xi^3 + 3(T-t)\xi] = 3\xi^2 + 3(T-t)$$

$$= 3(x+W_t)^2 + 3(T-t)$$

(b.3) Given that $f(\xi) = \xi$, we have

$\boxed{\text{GBM: } \mathbb{E}[X_t] = X_0 e^{\mu t}}$

$$P_T f(\xi) = \mathbb{E}[f(X_T^{\xi})] = \mathbb{E}^{\xi}[X_T] = \xi e^{\rho T} \Rightarrow \mathbb{E}[F] = P_T f(x) = x e^{\rho T}$$

and

$$\varphi(t) = \alpha \xi \frac{\partial}{\partial \xi} P_{T-t} f(\xi) = \alpha \xi \frac{\partial}{\partial \xi}[\xi e^{\rho(T-t)}] = \alpha X_t e^{\rho(T-t)}$$

(c) Extend formula (1.30) to the case when $X(t) \in \mathbb{R}^n$, $t \in [0,T]$, and $f : \mathbb{R}^n \to \mathbb{R}$. In this case, condition (1.28) must be replaced by the *uniform ellipticity* condition

$$\eta^T \sigma^T(x)\sigma(x)\eta \geq \delta|\eta|^2 \quad \text{for all } x \in \mathbb{R}^n, \eta \in \mathbb{R}^n, \tag{1.31}$$

where $\sigma^T(x)$ denotes the transposed of the $m \times n$-matrix $\sigma(x)$.

Using Itô's formula,

$$dY = \frac{\partial g}{\partial t}(t, X_t)\,dt + \sum_i \frac{\partial g}{\partial x_i}(t, X_t)\,dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j}(t, X_t)\,dX_i\,dX_j$$

$$= \cancel{\left(\frac{\partial g}{\partial t} + \sum_i b(X_t)\frac{\partial g}{\partial x_i} + \frac{1}{2}\sum_{i,j} (\sigma\sigma^T)(X_t)\frac{\partial^2 g}{\partial x_i \partial x_j}\right)} + \sum_i \sigma^T(X_t)\frac{\partial g}{\partial x_i}\,dW_i(t)$$

Hence,

$$dY(T) = \sigma^T(X_t)\nabla P_{T-t}f(X_t)\,dW_t$$

and

$$Y(T) = P_T f(x) + \int_0^T \sigma^T(x)\nabla P_{T-t}f(x)\,dW_t$$

$\square$

Note: this is Problem 2.1. in the corrected printing.

Problem 2.2. Let $u(t), 0 \leq t \leq T$, be a measurable stochastic process such that

$$E\left[\int_0^T u^2(t)dt\right] < \infty.$$

Show that there exists a sequence of deterministic measurable kernels $f_n(t_1, ..., t_n, t)$ on $[0,T]^{n+1}$ $(n \geq 0)$, with

$$\int_{[0,T]^{n+1}} f_n^2(t_1, ..., t_n, t)dt_1...dt_n dt < \infty$$

such that all f_n are symmetric with respect to the variables $t_1, ..., t_n$ and such that

$$u(t) = u(\omega, t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))(\omega), \qquad \omega \in \Omega, t \in [0,T],$$

with convergence in $L^2(P \times \lambda)$. [*Hint.* Consider approximations of $u(t)$, $t \in [0,T]$, in $L^2(P\times\lambda)$ of the form $\sum_{i=1}^m a_i(\omega)b_i(t)$, $m = 1, 2, ...$, where $a_i \in L^2(P)$ and $b_i \in L^2([0,T])$.]

Consider the approximations of $u(t)$

$$\varphi_m(t) = \sum_{i=1}^{m} a_i(\omega) b_i(t)$$

with $a_i \in L^2(P)$ and $b_i \in L^2([0,T])$.

By the Wiener-Itô Chaos Expansion, for each a_i there exists a unique sequence of functions $g_n^{(i)} \in \tilde{L}^2([0,T]^n)$ such that

$$a_i = \sum_{n=0}^{\infty} I_n(g_n^{(i)})$$

Then

$$\varphi_m(t) = \sum_{i=1}^{m} \sum_{n=0}^{\infty} b_i(t) I_n(g_n^{(i)})$$

$$= \sum_{i=1}^{m} \sum_{n=0}^{\infty} b_i(t) \int_{[0,T]^n} g_n^{(i)}(t_1, \dots, t_n)\, dW(t_1) \cdots dW(t_n)$$

$$= \sum_{n=0}^{\infty} \int_{[0,T]^n} \sum_{i=1}^{m} b_i(t)\, g_n^{(i)}(t_1, \dots, t_n)\, dW(t_1) \cdots dW(t_n)$$

Taking $m \to \infty$, $\varphi_m \to u$. We define

$$f_n(t_1, \dots, t_n, t) = \sum_{i=1}^{\infty} b_i(t)\, g_n^{(i)}(t_1, \dots, t_n)$$

and obtain

$$u(t) = \sum_{n=0}^{\infty} \int_{[0,T]^n} f_n(t_1, \dots, t_n, t)\, dW(t_1) \cdots dW(t_n)$$

$$= \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$$

Problem 2.2. Prove the linearity of the Skorohod integral.

Let $u, v \in Dom(\delta)$. Then

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n), \qquad \delta(v) = \sum_{n=0}^{\infty} I_{n+1}(\hat{g}_n)$$

By the Problem 1.2,

$$\delta(u) + \delta(v) = \sum_{n=0}^{\infty} I_{n+1}(h_n)$$

where h_n is such that $h_n = \hat{f}_n + \hat{g}_n$ for all n.

Since the sum of symmetric functions is symmetric, we have that h_n is symmetric. Thus,

$$\delta(u) + \delta(v) = \delta(u+v)$$

Now let $u \in Dom(\delta)$ and λ be a scalar. By

Remark 2.3. By (1.17) a stochastic process u belongs to $Dom(\delta)$ if and only if

$$E[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty \,. \tag{2.4}$$

we know that $\lambda u \in Dom(\delta)$. In fact,

$$\delta(\lambda u) = \sum_{n=0}^{\infty} I_{n+1}(\lambda \hat{f}_n) = \lambda \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n) = \lambda\, \delta(u)$$

Proposition 2.6. *For any fixed $t \in [0,T]$ and $u \in Dom(\delta)$ we have $\chi_{(0,t]}u \in Dom(\delta)$ and $\chi_{(t,T]}u \in Dom(\delta)$ and*

$$\int_0^t u(s)\delta W(s) = \int_0^T \chi_{(0,t]}(s)u(s)\delta W(s) \text{ and } \int_t^T u(s)\delta W(s) = \int_0^T \chi_{(t,T]}(s)u(s)\delta W(s),$$

with

$$\int_0^T u(s)\delta W(s) = \int_0^t u(s)\delta W(s) + \int_t^T u(s)\delta W(s).$$

Since $u \in Dom(\delta)$,

$$\mathbb{E}[\delta(u\chi_{(0,t]})^2] = \sum_{n=0}^{\infty} (n+1)! \, \| \tilde{f}_n \chi_{(0,t]} \|^2_{L^2([0,T]^{n+1})}$$

$$\leq \sum_{n=0}^{\infty} (n+1)! \, \| \tilde{f}_n \|^2_{L^2([0,T]^{n+1})} = \mathbb{E}[\delta(u)^2] < \infty$$

Similarly for $\chi_{(t,T]}$. Hence, $\chi_{(0,t]}u$ and $\chi_{(t,T]}u$ belong to $Dom(\delta)$.

Note that

$$\int_0^t u(s) \, \delta W(s) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n \cdot \chi_{(0,t]}(s)) = \int_0^T \chi_{(0,t]}(s) \, u(s) \, \delta W(s)$$

and

$$\int_t^T u(s) \, \delta W(s) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n \cdot \chi_{(t,T]}(s)) = \int_0^T \chi_{(t,T]}(s) \, u(s) \, \delta W(s)$$

Since the Skorohod integral is linear,

$$\int_0^T u(s)\,\delta W(s) = \int_0^T \left(\chi_{(0,t]} + \chi_{(t,T]}\right) u(s)\ \delta W(s)$$

$$= \int_0^T \chi_{(0,t]}(s)\,u(s)\,\delta W(s) + \int_0^T \chi_{(t,T]}(s)\,u(s)\,\delta W(s)$$

$$= \int_0^t u(s)\,\delta W(s) + \int_t^T u(s)\,\delta W(s)$$

Problem 2.5. (*) Compute the following Skorohod integrals:

(a) $\int_0^T W(t)\delta W(t)$,

Note that $W(t)$ is $\mathbb{F}$-adapted. Thus, the Skorohod and Itô integrals are the same:

$$\delta(W(t)) = \int_0^T W(t)\,\delta W(t) = \int_0^T W(t)\,dW(t) = \frac{1}{2}(W_T^2 - T)$$

(b) $\int_0^T \left(\int_0^T g(s)dW(s)\right)\delta W(t)$, for a given function $g \in L^2([0,T])$,

First step: write the Wiener-Itô chaos expansion of the integrand

$$\int_0^T g(s)\,dW(s) = I_1(g(s)), \qquad f_i = 0, \ \forall i \in \mathbb{Z}_{\geq 0}\setminus\{1\}$$

Second step: compute the symmetrization of f_i

$$f_1(t_1,t) = g(t_1) \Rightarrow \tilde{f}_1(t_1,t) = \frac{1}{2}\left[f_1(t_1,t) + f_1(t,t_1)\right] = \frac{1}{2}\left[g(t_1) + g(t)\right]$$

Third step: write the Skorohod integral

$$\delta\left(\int_0^T g(s)\,dW(s)\right) = I_2(\tilde{f}_1) = 2\int_0^T\int_0^{t_2} \tilde{f}_1(t_1,t_2)\,dW(t_1)\,dW(t_2)$$

$$= 2\int_0^T\int_0^{t_2} \frac{1}{2}\left[g(t_1) + g(t_2)\right]dW(t_1)\,dW(t_2)$$

$$= \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T \int_0^{t_2} g(t_2)\, dW(t_1)\, dW(t_2)$$

$$= \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T g(t_2) \int_0^{t_2} dW(t_1)\, dW(t_2)$$

$$= \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T g(t_2)\, W(t_2)\, dW(t_2)$$

Using

4.3. Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t\ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s\ .$$

$$\left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) = \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) + \int_0^T g(t_2)\, W(t_2)\, dW(t_2) + \int_0^T g(t)\, dt$$

Replacing we obtain

$$\delta\left(\int_0^T g(s)\, dW(s) \right) = \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2)$$

$$+ \left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) - \int_0^T \int_0^{t_2} g(t_1)\, dW(t_1)\, dW(t_2) - \int_0^T g(t)\, dt$$

Hence,

$$\int_0^T \left(\int_0^T g(s)\, dW(s) \right) \delta W(t) = \left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) - \int_0^T g(t)\, dt$$

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed,

First step: write the Wiener-Itô chaos expansion of the integrand

$$W^2(t_0) = I_2(\chi_{[0,t_0]}(t_1,t_2)) + t_0 \qquad \text{(see exercise 1.3c)}$$

Second step: compute the symmetrization of f_i

Since $f_2(t_1,t_2,t) = \chi_{[0,t_0]}(t_1,t_2)$, we have

(1)
$$\begin{aligned}
\tilde{f}_2(t_1,t_2) &= \frac{1}{3}\left[f_2(t_1,t_2,t) + f_2(t_2,t,t_1) + f_2(t_1,t,t_2) \right] \\
&= \frac{1}{3}\left[\chi_{[0,t_0]}(t_1,t_2) + \chi_{[0,t_0]}(t_2,t) + \chi_{[0,t_0]}(t_1,t) \right] \\
&= \chi_{\{t,t_1,t_2<t_0\}} + \frac{1}{3}\left[\chi_{\{t_1,t_2<t_0<t\}} + \chi_{\{t_2,t<t_0<t_1\}} + \chi_{\{t_1,t<t_0<t_2\}} \right]
\end{aligned}$$

Third step: write the Skorohod integral

Using the linearity of the Skorohod integral and the fact that t_0 is $\mathbb{F}$-adapted,

(2)
$$\begin{aligned}
\int_0^T W^2(t_0)\,\delta W(t) &= \int_0^T \left(I_2(\chi_{[0,t_0]}(t_1,t_2)) + t_0 \right) \delta W(t) \\
&= \int_0^T I_2(\chi_{[0,t_0]}(t_1,t_2))\,\delta W(t) + t_0 W(T) \\
&= t_0 W(T) + I_3(\tilde{f}_2)
\end{aligned}$$

Using (1) to compute $I_3(\tilde{f}_2)$,

(3) $$I_3(\tilde{f}_2) = 6J_3(\tilde{f}_2) = 6\int_0^T\int_0^{t_3}\int_0^{t_2} \tilde{f}_2(t_1,t_2,t)\,dW(t_1)\,dW(t_2)\,dW(t_3)$$

$$= 6\int_0^T\int_0^{t_3}\int_0^{t_2} \chi_{\{t,t_1,t_2<t_0\}}\,dW(t_1)\,dW(t_2)\,dW(t_3)$$

$$+ 6\int_0^T\int_0^{t_3}\int_0^{t_2} \frac{1}{3}\chi_{\{t_1,t_2<t_0<t_3\}}\,dW(t_1)\,dW(t_2)\,dW(t_3)$$

$$\overset{(*)}{=} t_0^{3/2}\,h_3\left(\frac{W(t_0)}{\sqrt{t_0}}\right) + 2\underline{\int_{t_0}^T\int_0^{t_0}\int_0^{t_2}}\,dW(t_1)\,dW(t_2)\,dW(t_3)$$

?

Since $h^3(x) = x^3 - 3x$, (3) equals

(4) $$t_0^{3/2}\left(\frac{W^3(t_0)}{t_0^{3/2}} - \frac{3W(t_0)}{\sqrt{t_0}}\right) + 2\int_{t_0}^T \frac{1}{2}(W^2(t_0) - t_0)\,dW(t_3)$$

Replacing (4) into (2),

$$\int_0^T W^2(t_0)\,\delta W(t) = t_0 W(T) + W^3(t_0) - 3t_0 W(t_0)$$
$$+ (W^2(t_0) - t_0)(W(T) - W(t_0))$$

Thus,

$$\boxed{\int_0^T W^2(t_0)\,\delta W(t) = W^2(t_0)\,W(T) - 2t_0 W(t_0)}$$

(d) $\int_0^T \exp\{W(T)\}\delta W(t)$ [*Hint*. Use Problem 1.3.],

By the problem 1.3.d,

$$\exp\left[\int_0^T g(s)\,dW(s)\right] = \sum_{n=0}^{\infty} I_n\left[\frac{g^{\otimes n}\exp(1/2\,\|g\|^2)}{n!}\right]$$

Thus, taking $g(s)=1$,

$$\int_0^T \exp(W(T))\,\delta W(t) = \int_0^T \sum_{n=0}^{\infty} \frac{1}{n!}\exp(T/2)\, I_n[1]\,\delta W(t)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\exp(T/2)\, I_{n+1}[1] \overset{(*)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} e^{T/2}\, T^{\frac{(n+1)}{2}} h_{n+1}\left(\frac{W(T)}{\sqrt{T}}\right)$$

$(*)$

$$I_n(g^{\otimes n}) = n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)\,dW(t_1)\cdots dW(t_n) = \|g\|^n h_n\left(\frac{\int_0^T g(t)\,dW(t)}{\|g\|}\right)$$

(e) $\int_0^T F\delta W(t)$, where $F = \int_0^T g(s)W(s)ds$, with $g \in L^2([0,T])$ [*Hint*. Use Problem 1.3].

By the exercise 1.3.e, $\displaystyle\int_0^T g(s)\,W(s)\,ds = I_1\left[\int_t^T g(s)\,ds\right]$

Letting $f_1(t_1,t) = \int_t^T g(s)\,ds$, we have

$$\tilde{f}_1(t_1,t_2) = \frac{1}{2}\left[f_1(t_1,t_2) + f_1(t_2,t_1)\right] = \frac{1}{2}\left[\int_{t_2}^T g(s)\,ds + \int_{t_1}^T g(s)\,ds\right]$$

Thus, the Skorohod Integral is

$$\int_0^T \int_0^T g(s) W(s) ds \, \delta W(t) = I_2[\tilde{f}_1]$$

$$= \int_0^T \int_0^{t_2} \left[\int_{t_2}^T g(s) ds + \int_{t_1}^T g(s) ds \right] dW(t_1) dW(t_2)$$

$$= \int_0^T \int_0^{t_2} \int_{t_2}^T g(s) ds \, dW(t_1) dW(t_2) + \int_0^T \int_0^{t_2} \int_{t_1}^T g(s) ds \, dW(t_1) dW(t_2)$$

$$= \int_0^T \int_0^{t_2} dW(t_1) \int_{t_2}^T g(s) ds \, dW(t_2) + \int_0^T \int_0^{t_2} \int_{t_1}^T g(s) ds \, dW(t_1) dW(t_2)$$

$$= \int_0^T 2W(t_2) \int_{t_2}^T g(s) ds \, dW(t_2) + \int_0^T \int_0^{t_2} g(s) W(s) ds \, dW(t_2)$$

Since

$$\int_0^T \int_0^{t_2} \int_{t_1}^T g(s) ds \, dW(t_1) dW(t_2) =$$

$$\int_0^T W(t_2) \int_{t_2}^T g(s) ds \, dW(t_2) + \int_0^T \int_0^{t_2} g(s) W(s) ds \, dW(t_2)$$

Problem 3.1. Let ξ, ζ be orthonormal functions in $L^2([0,T])$. Using the properties of Hermite polynomials compute directly the following:

(a) $I_1(\xi)I_2(\zeta^{\otimes 2})$
(b) $I_3(\xi\hat{\otimes}\zeta^{\otimes 2})$
(c) $D_t I_3(\xi\hat{\otimes}\zeta^{\otimes 2})$ [*Hint.* Use (1.14), (3.5)–(3.9)].

Using the chain rule compute:

(d) $D_t(I_1(\xi)I_2(\zeta^{\otimes 2}))$.

Compare the results in (c) and (d).

$h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = x^2 - 1,\ h_3(x) = x^3 - 3x,$
$h_4(x) = x^4 - 6x^2 + 3,\ h_5(x) = x^5 - 10x^3 + 15x, \dots .$

a)

$$I_1(\xi) = \int_0^T \xi(t)\, dW(t)$$

$$I_2(\zeta^{\otimes 2}) = \|\zeta^{\otimes 2}\|^2 h_2\left(\frac{\int_0^T \zeta^{\otimes 2} dW(t)}{\|\zeta^{\otimes 2}\|}\right)$$

Since $\|\zeta^{\otimes 2}\| = 1$ and $h_2(x) = x^2 - 1$,

$$I_2(\zeta^{\otimes 2}) = \left(\int_0^T \zeta^{\otimes 2}\, dW(t)\right)^2 - 1$$

b) We use the following result

Proposition 1.8. *If $\xi_1, \xi_2, \dots$ are orthonormal functions in $L^2([0,T])$, we have that*

$$I_n(\xi_1^{\otimes \alpha_1}\hat{\otimes}\cdots\hat{\otimes}\xi_m^{\otimes \alpha_m}) = \prod_{k=1}^{m} h_{\alpha_k}\Big(\int_0^T \xi_k(t) W(t)\Big), \tag{1.14}$$

with $\alpha_1 + \cdots + \alpha_m = n$. Here $\otimes$ denotes the tensor power *and $\alpha_k \in \{0,1,2,\dots\}$ for all k.*

Then

$$I_3(\xi\hat{\otimes}\zeta^{\otimes 2}) = h_1\left(\int_0^T \xi(t)\, dW(t)\right) h_2\left(\int_0^T \zeta(t)\, dW(t)\right)$$

$$= \left(\int_0^T \xi(t)\, dW(t)\right)\left(\left(\int_0^T \zeta(t)\, dW(t)\right)^2 - 1\right)$$

c) Notice that

$$D_t I_3(\xi \hat{\otimes} \zeta^{\otimes 2}) = 3\|\xi \hat{\otimes} \zeta^{\otimes 2}\|^2 h_2\left(\frac{\int_0^T \xi \hat{\otimes} \zeta^{\otimes 2}\, dW(t)}{\|\xi \hat{\otimes} \zeta^{\otimes 2}\|}\right) \xi \hat{\otimes} \zeta^{\otimes 2}(t)$$

$$= 3\|\xi \hat{\otimes} \zeta^{\otimes 2}\|^2 \left(\left(\frac{\int_0^T \xi \hat{\otimes} \zeta^{\otimes 2}\, dW(t)}{\|\xi \hat{\otimes} \zeta^{\otimes 2}\|}\right)^2 - 1\right) \xi \hat{\otimes} \zeta^{\otimes 2}(t)$$

$$= 3\, \xi \hat{\otimes} \zeta^{\otimes 2}(t) \left[\left(\int_0^T \xi \hat{\otimes} \zeta^{\otimes 2}\, dW(t)\right)^2 - \|\xi \hat{\otimes} \zeta^{\otimes 2}\|^2\right]$$

d) On the other hand,

$$D_t\left(\int_0^T \xi(t)\, dW(t)\right)\left(\left(\int_0^T \zeta(t)\, dW(t)\right)^2 - 1\right)$$

$$= \left(\left(\int_0^T \zeta(t)\, dW(t)\right)^2 - 1\right) D_t\left(\int_0^T \xi(t)\, dW(t)\right)$$

$$+ \left(\int_0^T \xi(t)\, dW(t)\right) D_t\left(\left(\int_0^T \zeta(t)\, dW(t)\right)^2 - 1\right)$$

$$= \left(\left(\int_0^T \zeta(t)\, dW(t)\right)^2 - 1\right) \xi(t)$$

$$+ \left(\int_0^T \xi(t)\, dW(t)\right) \cdot 2\, \zeta(t) \int_0^T \zeta(t)\, dW(t)$$

Problem 3.2. (*) Find the Malliavin derivative D_tF of the following random variables:

(a) $F = W(T)$.

Since the Chaos Expansion of F is $F = I_1(1)$, and

(1) $$D_t \int_0^T f(s)\, dW(s) = f(t)$$

we have that

$$D_t F = D_t \int_0^T 1\, dW(s) = 1$$

(b) $F = \int\limits_0^T s^2 dW(s)$.

Using (1) again,

$$D_t F = D \int_0^T s^2\, dW(s) = s^2$$

(c) $F = \int\limits_0^T \int\limits_0^{t_2} \cos(t_1 + t_2) dW(t_1) dW(t_2)$.

By definition,

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t))$$

Since $F = J_2(\cos(t_1 + t_2))$, we have that

$$D_t F = D_t J_2(\cos(t_1+t_2)) = D_t \frac{1}{2} I_2(\cos(t_1+t_2))$$

$$= \frac{1}{2} 2 I_1(\cos(t_1+t_2)) = \int_0^T \cos(t_1+t_2)\, dW(t_1)$$

(d) $F = 3W(s_0)W^2(t_0) + \log(1 + W^2(s_0))$, for given $s_0, t_0 \in [0, T]$.

If $t \in [0, s_0]$, we can take $g(t) = 3tW^2(t_0) + \log(1+t^2)$. Then

$$g'(t) = 3W^2(t_0) + \frac{2t}{1+t^2} \quad \text{is bounded}$$

By the Chain Rule, $D_t\, g(G) = g'(G) D_t G$. Since $F = g(W(s_0))$ and

$$D_t W(s_0) = D_t \int_0^T \chi_{[0,s_0]}(s)\, dW(s) = \chi_{[0,s_0]}(t)$$

we have

$$D_t F = \left[3W^2(t_0) + \frac{2W(s_0)}{1+W^2(s_0)} \right] \chi_{[0,s_0]}(t)$$

Now, if $t \in [0, t_0]$, let $g(t) = 3W(s_0)t^2 + \log(1 + W^2(s_0))$.

Then $g'(t) = 6W(s_0)t$ is bounded, $F = g(W(t_0))$, and

$$D_t W(t_0) = D_t \int_0^T \chi_{[0,t_0]}(s)\, dW(s) = \chi_{[0,t_0]}(t)$$

Thus,

$$D_t F = 6W(s_0)W(t_0)\, \chi_{[0,t_0]}(t)$$

Hence, in the general case,

$$D_t F = \left[3W^2(t_0) + \frac{2W(s_0)}{1+W^2(s_0)} \right] \chi_{[0,s_0]}(t) + 6W(s_0)W(t_0)\, \chi_{[0,t_0]}(t)$$

(e) $F = \int_0^T W(t_0)\delta W(t)$, for a given $t_0 \in [0, T]$. [*Hint*. Use Problem 2.5 (b).]

By the Problem 2.5b.,

$$\boxed{\int_0^T \left(\int_0^T g(s)\, dW(s) \right) \delta W(t) = \left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) - \int_0^t g(t)\, dt}$$

Then

$$F = \int_0^T \int_0^T \chi_{[0,t_0]}(s)\, \delta W(s)\, \delta W(t) = W(T) \int_0^T \chi_{[0,t_0]}(s)\, dW(s) - \int_0^T \chi_{[0,t_0]}(t)\, dt$$

Using Integration by Parts,

$$F = \int_0^T W(T) \chi_{[0,t_0]}(s)\, dW(s)$$

and by the Fundamental Theorem of Calculus,

$$D_t F = \int_0^T D_t W(T) \chi_{[0,t_0]}(s)\, \delta W(s) + W(T) \chi_{[0,t_0]}(t)$$

$$= W(t_0) + W(T) \chi_{[0,t_0]}(t)$$

Problem 3.3. (*)

(a) Find the Malliavin derivative D_tF, when

$$F = e^G \quad \text{with} \quad G = \int_0^T g(s)dW(s), \quad g \in L^2([0,T]),$$

by using that $F = \sum_{n=0}^{\infty} I_n[f_n]$, with

$$f_n(t_1, \ldots, t_n) = \frac{1}{n!} \exp\left\{\frac{1}{2}\|g\|^2_{L^2([0,T])}\right\} g(t_1) \ldots g(t_n)$$

(see Problem 1.1 and Problem 1.3 (d)).

Using the definition, we have that

$$D_tF = \sum_{n\geq 1}^{\infty} n I_{n-1}[f_n(\cdot, t)]$$

$$= \sum_{n\geq 1}^{\infty} I_{n-1}\left[\frac{1}{(n-1)!} \exp\left(\frac{1}{2}\|g\|^2_{L^2([0,T])}\right) g(t_1) \cdots g(t_{n-1})\, g(t)\right]$$

$$= \sum_{n\geq 0}^{\infty} I_n\left[\frac{1}{n!} \exp\left(\frac{1}{2}\|g\|^2_{L^2([0,T])}\right) g(t_1) \cdots g(t_n)\right] g(t) = g(t)F$$

(b) Verify that the result in (a) can be expressed in terms of the chain rule: $D_te^G = e^G D_tG$.

Since

$$D_tG = D_t \int_0^T g(s)\, dW(s) = g(t)$$

we have

$$e^G D_tG = \sum_{n\geq 0}^{\infty} I_n\left[\frac{1}{n!} \exp\left(\frac{1}{2}\|g\|^2_{L^2([0,T])}\right) g(t_1) \cdots g(t_n)\right] g(t)$$

$$= D_tF$$

(c) Find the Malliavin derivative of $F = e^G$ with $G = W(t_0)$, for a given $t_0 \in [0, T]$.

For $G = W(t_0)$, we have

$$G = \int_0^T \chi_{[0,t_0]}(s)\, dW(s)$$

and

$$D_t G = D_t \int_0^T \chi_{[0,t_0]}(s)\, dW(s) = \chi_{[0,t_0]}(t)$$

By the problem 1.3.d, its Chaos Expansion is

$$e^G = \sum_{n=0}^{\infty} I_n\left[\frac{1}{n!} \exp\left(\frac{1}{2} \|\chi_{[0,t_0]}\|^2_{L^2([0,T])} \right) \chi_{[0,t_0]}(t_1) \cdots \chi_{[0,t_0]}(t_n) \right]$$

Using that

$$\|\chi_{[0,t_0]}\|^2_{L^2([0,T])} = \int_0^T \chi^2_{[0,t_0]}(s)\, ds = \int_0^{t_0} ds = t_0$$

we obtain

$$D_t F = e^G D_t G = \sum_{n=0}^{\infty} I_n\left[\frac{1}{n!} e^{t_0/2} \chi_{[0,t_0]}(t_1) \cdots \chi_{[0,t_0]}(t_n) \right] \chi_{[0,t_0]}(t)$$

Problem 3.4. Use the integration by parts formula (Theorem 3.15) to compute the Skorohod integrals

$$\int_0^T F\delta W(t),$$

for the random variables F given in Problem 3.2 and in Problem 3.3.

Recall that

Theorem 3.15. Integration by parts. *Let $u(t)$, $t \in [0,T]$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then*

$$F\int_0^T u(t)\delta W(t) = \int_0^T Fu(t)\delta W(t) + \int_0^T u(t)D_tF dt. \qquad (3.21)$$

(a) $F = W(T)$.

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= F\cdot W(T) - \int_0^T 1\,dt$$

$$= W^2(T) - T$$

(b) $F = \int\limits_0^T s^2 dW(s)$.

$$\int_0^T\int_0^T s^2 dW(s)\,\delta W(t) = \int_0^T s^2 dW(s)\int_0^T \delta W(t) - \int_0^T t^2\,dt$$

$$= W(T)\left[T^2 W(T) - \int_0^T W(s)2s\,ds\right] - \frac{T^3}{3}$$

where we used

Theorem 4.1.5 (Integration by parts). *Suppose $f(s,\omega) = f(s)$ only depends on s and that f is continuous and of bounded variation in $[0,t]$. Then*

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s\,.$$

to obtain that $\int_0^T s^2 dW(s) = T^2 W(T) - \int_0^T W(s)2s\,ds$

(c) $F = \int\limits_0^T \int\limits_0^{t_2} \cos(t_1 + t_2) dW(t_1) dW(t_2)$.

$$\int_0^T F \delta W(t) = F W(T) - \int_0^T \int_0^T \cos(t_1 + t_2)\, dW(t_1) dt$$

(d) $F = 3W(s_0)W^2(t_0) + \log(1 + W^2(s_0))$, for given $s_0, t_0 \in [0, T]$.

$$\int_0^T F \delta W(t) = F W(T)$$

$$- \int_0^T \left(\left[3W^2(t_0) + \frac{2W(s_0)}{1 + W^2(s_0)} \right] \chi_{[0,s_0]}(t) + 6W(s_0)W(t_0)\, \chi_{[0,t_0]}(t) \right) dt$$

(e) $F = \int\limits_0^T W(t_0) \delta W(t)$, for a given $t_0 \in [0, T]$. [*Hint*. Use Problem 2.5 (b).]

$$\int_0^T F \delta W(t) = F \int_0^T \delta W(t) - \int_0^T D_t F\, dt$$

$$= W(T) \int_0^T W(T)\, \chi_{[0,t_0]}(s)\, dW(s) - \int_0^T \left(W(t_0) + W(T)\, \chi_{[0,t_0]}(t) \right) dt$$

$$= W^2(T)\, W(t_0) - W(t_0)\, T - W(T)\, t_0$$

$$F = e^G \quad \text{with} \quad G = \int_0^T g(s)dW(s), \quad g \in L^2([0,T]),$$

$$\int_0^T F \,\delta W(t) = W(T)e^G - \int_0^T e^G g(t)\,dt$$
$$= e^G\left(W(T) - \int_0^T g(t)\,dt\right)$$

$$F = e^G \text{ with } G = W(t_0)$$

$$\int_0^T F \,\delta W(t) = W(T)e^G - \int_0^T e^G \chi_{[0,t_0]}(t)\,dt = e^G\left(W(T) - t_0\right)$$

Problem 3.5. Use the integration by parts formula to compute the Skorohod integrals in Problem 2.5.

(a) $\int_0^T W(t)\delta W(t)$,

By the integration by parts formula with $u = 1$ and $F = W(t)$,

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= W(t)\,W(T) - T$$

Since

$$D_t F = D_t \int_0^T \chi_{[0,t]}(s)\,dW(s) = \chi_{[0,t]}(t) = 1$$

(b) $\int_0^T \left(\int_0^T g(s)dW(s)\right)\delta W(t)$, for a given function $g \in L^2([0,T])$,

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= \left(\int_0^T g(s)\,dW(s)\right) W(T) - \int_0^T g(t)\,dt$$

Since

$$D_t \int_0^T g(s)\,dW(s) = g(t)$$

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed,

Recall that

$$\int_0^{t_0} W(t)\,dW(t) = \frac{1}{2}\left(W^2(t_0) - t_0\right) \Rightarrow W^2(t_0) = 2\int_0^{t_0} W(t)\,dW(t) + t_0$$

Hence

$$D_t W^2(t_0) = D_t\left(2\int_0^{t_0} W(s)\,dW(s) + t_0\right) = 2W(t)$$

and

$$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

$$= \boxed{W^2(t_0)\,W(T) - \int_0^T 2W(t)\,dt}$$

not necessary → Using that

$$\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds\,.$$

Øksendal, Integration by Parts

We can write

$$-2\int_0^T W(t)\,dt = 2\left(\int_0^T t\,dW(t) - TW(T)\right)$$

Hence,

$$\int_0^T F\,\delta W(t) = W^2(t_0)\,W(T) + 2\left(\int_0^T t\,dW(t) - TW(T)\right)$$

(d) $\int_0^T \exp\{W(T)\}\delta W(t)$ [*Hint*. Use Problem 1.3.], $F\int_0^T u(t)\,\delta W(t) = \int_0^T F u(t)\,\delta W(t) + \int_0^T u(t) D_t F\, dt$

By the Integration by Parts formula:

$$\int_0^T \exp(W(T))\,\delta W(t) = \exp(W(T))\,W(T) - \int_0^T D_t \exp(W(T))\,dt$$

By the problem 1.3.d, taking $g(s)=1$,

$$\exp\left[\int_0^T g(s)\,dW(s)\right] = \sum_{n=0}^{\infty} I_n\left[\frac{g^{\otimes n}\exp(1/2\,\|g\|^2)}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\exp(T/2)\, I_n[1]$$

Using the definition of the Malliavin derivative,

$$D_t \exp(W(T)) = \sum_{n=1}^{\infty} n \cdot \frac{1}{(n-1)!}\exp(T/2)\, I_{n-1}[1]$$

$$= \sum_{n=1}^{\infty} \frac{n}{(n-1)!} e^{T/2} \cdot T^{(n-1)/2} h_{n-1}\left(\frac{W(T)}{\sqrt{T}}\right) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e^{T/2}\, T^{(n-1)/2} h_n'\left(\frac{W(T)}{\sqrt{T}}\right)$$

Hence,

$$\int_0^T \exp(W(T))\,\delta W(t) = W(T)\sum_{n=0}^{\infty}\frac{1}{n!} e^{T/2}\, T^{n/2} h_n\left(\frac{W(T)}{\sqrt{T}}\right)$$

$$- \int_0^T \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e^{T/2}\, T^{(n-1)/2} h_n'\left(\frac{W(T)}{\sqrt{T}}\right) dt$$

change $n-1 \to n$, integrate?

(e) $\int_0^T F\delta W(t)$, where $F = \int_0^T g(s)W(s)ds$, with $g \in L^2([0,T])$ [*Hint*. Use Problem 1.3].

(1) $$\int_0^T F\,\delta W(t) = F\int_0^T \delta W(t) - \int_0^T D_t F\,dt$$

Using the problem 1.3,

(2) $$\int_0^T g(s)W(s)ds = I_1\left[\int_t^T g(s)ds\right]$$

Thus,

(3) $$D_t \int_0^T g(s)W(s)ds = I_0\left[\int_t^T g(s)ds\right] = \int_t^T g(s)ds$$

Using (2) and (3) in (1),

$$\int_0^T F\,\delta W(t) = W(T)\int_0^T g(s)W(s)\,ds - \int_0^T\int_t^T g(s)ds\,dt$$

$$= W(T)\int_0^T\int_t^T g(s)ds\,dW(t) - \int_0^T\int_t^T g(s)ds\,dt$$

Problem 3.6. Let $u = u(t)$, $t \in [0,T]$, be a stochastic process such that

$$E\Big[\int_0^T u^2(t)dt\Big] < \infty.$$

Suppose that there exists a constant K such that

$$\Big|E\Big[\int_0^T D_t F u(t) dt\Big]\Big| \le K\|F\|_{L^2(P)}, \qquad \text{for all } F \in \mathbb{D}_{1,2}.$$

Show that u is Skorohod integrable.

Note that we cannot apply the Duality Formula. Let

$$u(t) = \sum_{k=0}^{\infty} I_k(g_k(\cdot,t)), \qquad F = \sum_{n=0}^{\infty} I_n(f_n)$$

be the Chaos Expansions of $u(t)$ and F, respectively.

Then

$$E\Big[\int_0^T u(t) D_t F\, dt\Big] = E\Big[\int_0^T \Big(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\Big)\Big(\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot,t))\Big) dt\Big]$$

$$= \int_0^T \sum_{k=0}^{\infty} E\big[(k+1) I_k(f_{k+1}(\cdot,t)) I_k(g_k(\cdot,t))\big] dt$$

for all $g \in \widetilde{L}^2([0,T]^n)$. Moreover, if $g \in \widetilde{L}^2([0,T]^m)$ and $h \in \widetilde{L}^2([0,T]^n)$, we have

$$E[I_m(g)I_n(h)] = \begin{cases} 0 & , n \neq m \\ (g,h)_{L^2([0,T]^n)} & , n = m \end{cases} \qquad (m,n = 1,2,\ldots),$$

with $(g,h)_{L^2([0,T]^n)} = n!(g,h)_{L^2(S_n)}$.

$$= \int_0^T \sum_{k=0}^{\infty} (k+1)k! \langle f_{k+1}(\cdot,t), g_k(\cdot,t)\rangle_{L^2([0,T]^k)} dt$$

$$= \sum_{k=0}^{\infty} (k+1)! \langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}$$

Hence, we have that

(1) $$\Big|\sum_{k=0}^{\infty} (k+1)! \langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}\Big| = \Big|E\Big[\int_0^T u(t) D_t F\, dt\Big]\Big| \le K\|F\|_{L^2(P)}$$

We'll use the following remark:

Remark 2.3. By (1.17) a stochastic process u belongs to $Dom(\delta)$ if and only if

$$E[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty\,. \tag{2.4}$$

If we have that

$$\mathbb{E}\left[\delta(u)^2\right] \leq \left|\sum_{k=0}^{\infty}(k+1)!\,\langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}\right| \leqslant K\|F\|_{L^2(\mathbb{P})} < \infty$$

we finish the proof.

Take F such that $f_{k+1} = g_k$, i.e.,

$$F = \sum_{k=1}^{\infty} I_{k+1}(g_{k-1}) \in \mathbb{D}_{1,2} \text{ by the restriction on } u(t) \; ?$$

Then we can write (1) as

$$\mathbb{E}\left[\delta(u)^2\right] = \sum_{k=0}^{\infty}(k+1)!\,\|\tilde{g}_k\|^2_{L^2([0,T]^{k+1})} \leqslant K\|F\|_{L^2(\mathbb{P})}$$

where we used that $\langle f_{k+1}, \tilde{g}_k\rangle_{L^2([0,T]^{k+1})} = \langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}$ (see the proof of Thm. 3.14).

Problem 4.1. **(*)** Recall the *Girsanov theorem* (see, e.g., [178, Theorem 8.26]). Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form

$$dY(t) = \beta(t)dt + \gamma(t)dW(t), \qquad t \leq T,$$

where $\beta(t) \in \mathbb{R}^n$, $\gamma(t) \in \mathbb{R}^{n\times m}$, and $t \in [0,T]$, are $\mathbb{F}$-adapted and $W(t)$, $t \in [0,T]$, is an m-dimensional Wiener process. Suppose there exist $\mathbb{F}$-adapted processes $u(t) \in \mathbb{R}^m$ and $\alpha(t) \in \mathbb{R}^n$, $t \in [0,T]$, such that

$$\gamma(t)u(t) = \beta(t) - \alpha(t)$$

and such that the Novikov condition

$$E\Big[\exp\Big\{\frac{1}{2}\int_0^T u^2(s)ds\Big\}\Big] < \infty$$

holds. Put

$$Z(t) = \exp\Big\{-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u^2(s)ds\Big\}, \qquad t \leq T,$$

and define a measure Q on $\mathcal{F}_T$ by

$$dQ = Z(T)dP.$$

Then

$$\widetilde{W}(t) := \int_0^t u(s)ds + W(t), \qquad 0 \leq t \leq T$$

is a Wiener process with respect to Q, and in terms of $\widetilde{W}$ the process Y has the stochastic integral representation

$$dY(t) = \alpha(t)dt + \gamma(t)d\widetilde{W}(t).$$

(a) Show that $\widetilde{W}$ is an $\mathbb{F}$-martingale with respect to Q. [*Hint.* Apply Itô formula to $Y(t) := Z(t)\widetilde{W}(t)$.]

By the product rule,

(1) $$dY(t) = Z(t)\,d\tilde{W}(t) + \tilde{W}(t)\,dZ(t) + d[Z, \tilde{W}](t)$$

Using

Corollary 4.8. *Let Q and Z be as in (4.3) and (4.4) respectively. Suppose $G \in L^1(Q)$. Then*

$$E_Q[G|\mathcal{F}_t] = \frac{E[Z(T)G|\mathcal{F}_t]}{Z(t)}. \tag{4.10}$$

we have that

(2) $$E_Q[\tilde{W}(s)\,|\mathcal{F}_t] = Z^{-1}(t)\,\mathbb{E}[Z(T)\,\tilde{W}(s)\,|\mathcal{F}_t]$$

$$= Z^{-1}(t)\,\mathbb{E}\big[E[Z(T)\,\tilde{W}(s)\,|\mathcal{F}_s]\,|\mathcal{F}_t\big] = Z^{-1}(t)\,\mathbb{E}[\tilde{W}(s)\,Z(s)\,|\mathcal{F}_t]$$

$Z(t)$ is an exp. martingale

Applying Itô's formula to $Z(t)$,

(3) $$dZ(t) = Z(t)\frac{1}{2}v^2(t)\,dt - Z(t)\,v(t)\,dw(t) - \frac{1}{2}v^2(t)Z(t)\,dt$$
$$= -v(t)\,Z(t)\,dw(t)$$

With this, we can compute

(4) $$dZ(t)\,d\tilde{w}(t) = -v(t)Z(t)\,dt$$

Replacing (2), (3) and (4) in (1) yields

$$dY(t) = Z(t)\left[dw(t) + v(t)dt\right] - \tilde{w}(t)\,v(t)\,Z(t)\,dw(t) - v(t)Z(t)\,dt$$
$$= Z(t)\,dw(t) - v(t)\,\tilde{w}(t)\,Z(t)\,dw(t)$$
$$= Z(t)\left[1 - v(t)\tilde{w}(t)\right]dw(t)$$

It follows that Y_t is a martingale w.r.t. $\mathbb{P}$. Using (2) again,

$$\mathbb{E}_{\mathbb{Q}}\left[\tilde{w}(s)\,|\,\mathcal{F}_t\right] = Z^{-1}(t)\,\mathbb{E}\left[\tilde{w}(s)Z(s)\,|\,\mathcal{F}_t\right]$$
$$= Z^{-1}(t)\,\mathbb{E}\left[Y(s)\,|\,\mathcal{F}_t\right]$$
$$= Z^{-1}(t)Y(t) = \tilde{w}(t)$$

whence $\tilde{w}(t)$ is a martingale w.r.t. $\mathbb{Q}$.

(b) Suppose $X(t) = at + W(t) \in \mathbb{R}$, $t \leq T$, where $a \in \mathbb{R}$ is a constant. Find a probability measure Q on $\mathcal{F}_T$ such that X is a Wiener process with respect to Q.

Let

$$v(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)} = a$$

Then

$$X(t) = at + W(t) = at + \tilde{W}(t) - \int_0^t a\,ds = \tilde{W}(t)$$

Hence, $X(t)$ is a martingale w.r.t. Q.

(c) Let $a, b, c \neq 0$ be real constants and define

$$dY(t) = bY(t)dt + cY(t)dW(t).$$

Find a probability measure Q and a Wiener process $\widetilde{W}$ with respect to Q such that

$$dY(t) = aY(t)dt + cY(t)d\widetilde{W}(t).$$

$$v(t) = \frac{(b-a)}{c} \qquad \tilde{W}(t) = W(t) + \int_0^t v(s)\,ds$$

$$dY(t) = bY(t)dt - \cancel{c}Y(t)\frac{(b-a)}{\cancel{c}}dt + cY(t)d\tilde{W}(t)$$

$$= aY(t)dt + cY(t)d\tilde{W}(t)$$

Problem 4.2. **(*)** Verify the Clark–Ocone formula

$$F = E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t)$$

for the following $\mathcal{F}_T$-measurable random variables F:

(a) $F = W(T)$,
(b) $F = \int_0^T W(s)ds$,
(c) $F = W^2(T)$,
(d) $F = W^3(T)$,
(e) $F = \exp W(T)$,
(f) $F = (W(T) + T)\exp\{-W(T) - \frac{1}{2}T\}$.

a) $F = W(T)$

By the exercise 3.2, we have that $D_t F = 1$. Hence,

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[1 | \mathcal{F}_t] \, dW(t) = W(T)$$

b) $F = \int_0^T W(s)\, ds$

We start by computing

$$D_t F = D_t \int_0^T W(s)\, ds = \int_0^T D_t W(s)\, ds = T - t$$

Thus,

$$F = \int_0^T \mathbb{E}[T-t | \mathcal{F}_t] \, dW(t) = \int_0^T (T-t)\, dW(t)$$

(*) Obs.
Ex. 3.1

$$= T W(T) - \int_0^T t \, dW(t) \overset{(*)}{=} T W(T) - \left[T W(T) - \int_0^T W(s)\, ds \right]$$

$$= \int_0^T W(s)\, ds = F$$

From here, we extract two useful identities. For $n = 1$, we

$$D_t \int_0^T f(s)\, dW(s) = f(t)$$

and for $n > 1$, using induction,

$$D_t \left(\int_0^T f(s)\, dW(s) \right)^n = n \left(\int_0^T f(s)\, dW(s) \right)^{n-1} f(t)$$

c) $F = W^2(T)$

Since $\mathbb{E}[W^2(T)] = T$ and

$$D_t W^2(T) = D_t \left(\int_0^T 1\, dW(s) \right)^2 = 2W(T)$$

Example 3.2.3. Brownian motion B_t in $\mathbf{R}^n$ is a martingale w.r.t. the σ-algebras $\mathcal{F}_t$ generated by $\{B_s; s \le t\}$, because

$$E[|B_t|]^2 \le E[|B_t|^2] = |B_0|^2 + nt \qquad \text{and if } s \ge t \text{ then}$$
$$E[B_s|\mathcal{F}_t] = E[B_s - B_t + B_t|\mathcal{F}_t]$$
$$= E[B_s - B_t|\mathcal{F}_t] + E[B_t|\mathcal{F}_t] = 0 + B_t = B_t\ .$$

Here we have used that $E[(B_s - B_t)|\mathcal{F}_t] = E[B_s - B_t] = 0$ since $B_s - B_t$ is independent of $\mathcal{F}_t$ (see (2.2.11) and Theorem B.2.d)) and we have used that $E[B_t|\mathcal{F}_t] = B_t$ since B_t is $\mathcal{F}_t$-measurable (see Theorem B.2.c)).

we have

$$\begin{aligned}
\mathbb{E}[D_t W^2(T) \mid \mathcal{F}_t] &= 2\mathbb{E}[W(T) \mid \mathcal{F}_t] = 2\,\mathbb{E}[W(T) - W(t) + W(t) \mid \mathcal{F}_t] \\
&= 2\big(\mathbb{E}[W(T) - W(t) \mid \mathcal{F}_t] + \mathbb{E}[W(t) \mid \mathcal{F}_t]\big) \\
&= 2W(t)
\end{aligned}$$

and

$$F = T + 2\int_0^T W(t)\, dW(t) = T + W^2(T) - T = W^2(T)$$

d) $F = W^3(T)$

Since

$$D_t W^3(T) = D_t \left(\int_0^T 1\, dW(s) \right)^3 = 3W^2(T)$$

we have that

$$\begin{aligned}
F &= \mathbb{E}[W^3(T)] + \int_0^T \mathbb{E}[3W^2(T) \mid \mathcal{F}_t]\, dW(t) \\
&= 0 + 3\int_0^T \mathbb{E}[(W(T) - W(t))^2 + 2W(T)W(t) - W^2(t) \mid \mathcal{F}_t]\, dW(t) \\
&= 3\int_0^T (T - t)\, dW(t) + 6\int_0^T W^2(t)\, dW(t) - 3\int_0^T W^2(t)\, dW(t) \\
&= 3\int_0^T W(t)\, dt - 3\int_0^T W^2(t)\, dW(t) = W^3(T)
\end{aligned}$$

Øksendal Ex. 3.2

e) $F = \exp(W(T))$

Notice that we have a GBM of the form

$$X_t = X_0 \exp\left[(\mu - 1/2\,\sigma^2)t + \sigma W(t)\right]$$

with $\mu = \frac{1}{2}\sigma^2$ and $\sigma = 1$. Thus, $\mu = 1/2$ and

$$\mathbb{E}\left[\exp(W(T))\right] = e^{1/2T}$$

By the chain rule,

$$D_t e^{W(T)} = e^{W(T)} D_t W(T) = e^{W(T)}$$

Hence,

$$F = e^{1/2T} + \int_0^T \mathbb{E}\left[e^{W(T)} \mid \mathcal{F}_t\right] dW(t)$$

$$= e^{1/2T} + \int_0^T \mathbb{E}\left[e^{W(T) - 1/2T} e^{1/2T} \mid \mathcal{F}_t\right] dW(t)$$

$$= e^{1/2T} + e^{1/2T} \int_0^T \mathbb{E}\left[\exp(W(T) - 1/2T) \mid \mathcal{F}_t\right] dW(t)$$

martingale

$$= e^{1/2T} + e^{1/2T} \int_0^T \exp(W(t) - 1/2t)\, dW(t)$$

To compute this integral, we apply Itô's formula to

$$M_t = \exp(W(t) - t/2)$$

which gives

$$dM_t = M_t \, dW_t$$

Thus,

$$\int_0^T M_t \, dW_t = M(T) - M(0)$$

and

$$F = e^{T/2}\left(1 + \exp(W(T) - T/2) - 1\right)$$

$$= \exp(W(T)) = F$$

f) $(W(T)+T)\exp(-W(T) - 1/2T)$

We have an GBM with

- $X_0 = W(T) + T$
- $\sigma = -1$
- $\mu - \frac{1}{2}\sigma^2 = -\frac{1}{2} \iff \mu = 0$

Thus, $E[F] = E[X_0]e^{\mu t} = 0$

By the product and chain rules,

$$D_t F = D_t(W(T)+T)\exp(-W(T) - T/2) + (W(T)+T) D_t \exp(-W(T) - T/2)$$

$$= \exp(-W(T) - T/2) - (W(T) + T)\exp(-W(T) - T/2)$$

$$= (1 - W(T) - T)\exp(-W(T) - T/2)$$

Now let

$$Y(t) = (W(t) + t)\exp(-W(t) - t/2)$$

By Itô's formula,

$$dY(t) = \left(1 - \frac{1}{2}(W(t) + t)\right)\exp(-W(t) - t/2)\,dt$$

$$+ \left(1 - (W(t) + t)\right)\exp(-W(t) - t/2)\,dW(t)$$

$$+ \frac{1}{2}\left(-1 - (1 - W(t) - t)\right)\exp(-W(t) - t/2)\,dt$$

$$= \left(1 - (W(t) + t)\right)\exp(-W(t) - t/2)\,dW(t)$$

whence Y_t is a martingale. Hence,

$$F = \int_0^T \mathbb{E}\left[(1 - W(T) - T)\exp(-W(T) - T/2) \mid \mathcal{F}_t\right] dW(t)$$

$$= \int_0^T (1 - W(t) - t)\exp(-W(t) - t/2)\,dW(t)$$

$$= \int_0^T dY(t) = Y(T) - Y(0)$$

$$= (W(T) + T)\exp(-W(T) - 1/2\,T) = F$$

Problem 4.3. (*) Let $\widetilde{W}(t) = \int_0^t u(s)ds + W(t)$ and Q be as in Exercise 4.1. Use the generalized Clark–Ocone formula to find the $\mathbb{F}$-adapted process $\widetilde{\varphi}$, such that

$$F = E_Q[F] + \int_0^T \widetilde{\varphi}(t)d\widetilde{W}(t)$$

in the following cases:

(a) $F = W^2(T)$ and $u(t)$, $t \in [0,T]$, is deterministic.

(b) $F = \exp\{\int_0^T \lambda(t)dW(t)\}$ and the processes $\lambda(t)$ and $u(t)$, $t \in [0,T]$, are deterministic.

(c) F is like in (b) and $u(t) = W(t)$, $t \in [0,T]$.

Theorem 4.5. The Clark–Ocone formula under change of measure. *Suppose $F \in \mathbb{D}_{1,2}$ is $\mathcal{F}_T$-measurable and that*

$$E_Q[|F|] < \infty \tag{4.5}$$

$$E_Q[\int_0^T |D_tF|^2 dt] < \infty \tag{4.6}$$

$$E_Q\Big[|F| \int_0^T \Big(\int_0^T D_t u(s)dW(s) + \int_0^T u(s)D_t u(s)ds\Big)^2 dt\Big] < \infty. \tag{4.7}$$

Then

$$F = E_Q[F] + \int_0^T E_Q\Big[(D_tF - F\int_t^T D_t u(s)d\widetilde{W}(s))\big|\mathcal{F}_t\Big]d\widetilde{W}(t). \tag{4.8}$$

a) $F = W^2(T)$, $u(t)$ deterministic

Our goal is to find

$$E_Q\left[D_tF - F\int_t^T D_t u(s)\,d\tilde{W}(s) \,\Big|\, \mathcal{F}_t\right] = \tilde{\varphi}(t)$$

Now, since $D_t W^2(T) = 2W(T)$ and $D_t u(s) = 0$ (deterministic),

$$\tilde{\varphi}(t) = E_Q[2W(T)\,|\,\mathcal{F}_t] = 2E_Q\left[\tilde{W}(T) - \int_0^T u(s)\,ds \,\Big|\, \mathcal{F}_t\right]$$

$$= 2\left(\tilde{W}(t) - \int_0^T u(s)\,ds\right)$$

b) $F = \exp\left(\int_0^T \lambda(t)\,dW(t)\right)$, $\lambda(t)$, $u(t)$ deterministic

Again, $D_t u(s) = 0$. Let us compute D_tF. Let

$$g(x) = e^x, \quad G = \int_0^T \lambda(t)\,dW(t), \quad F = g(G)$$

By the Chain Rule,

$$D_t F = g'(G) D_t G = F D_t\left(\int_0^T \lambda(s)\,dW(s)\right) = \lambda(t) F$$

Thus,

$$\tilde{\varphi}(t) = \mathbb{E}_Q\left[\lambda(t)\exp\left(\int_0^T \lambda(s)\,dW(s)\right)\middle|\mathcal{F}_t\right]$$

$$= \mathbb{E}_Q\left[\lambda(t)\exp\left(\int_0^T \lambda(s)\,d\tilde{W}(s) - \int_0^T \lambda(s)u(s)\,ds\right)\middle|\mathcal{F}_t\right]$$

$$= \mathbb{E}_Q\left[\lambda(t)\exp\left(\int_0^T \lambda(s)\,d\tilde{W}(s) + \frac{1}{2}\int_0^T \lambda^2(s)\,ds - \frac{1}{2}\int_0^T \lambda^2(s)\,ds - \int_0^T \lambda(s)u(s)\,ds\right)\middle|\mathcal{F}_t\right]$$

$$= \lambda(t)\exp\left(\int_0^T \frac{1}{2}\lambda^2(s) - \lambda(s)u(s)\,ds\right)\mathbb{E}_Q\left[\exp\left(\int_0^T \lambda(s)\,d\tilde{W}(s) - \frac{1}{2}\int_0^T \lambda^2(s)\,ds\right)\middle|\mathcal{F}_t\right]$$

$$= \lambda(t)\exp\left(\int_0^T \frac{1}{2}\lambda^2(s) - \lambda(s)u(s)\,ds\right)\exp\left(\int_0^t \lambda(s)\,d\tilde{W}(s) - \frac{1}{2}\int_0^t \lambda^2(s)\,ds\right)$$

c) $F = \exp\left(\int_0^T \lambda(t)\, dW(t)\right)$, $\lambda(t)$ deterministic, $u(t) = W(t)$

Now $D_t u(s) = D_t W(s) = 1$ and thus

$$\tilde{\varphi}(t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\int_0^T \lambda(s)\, dW(s)\right)\left(\lambda(t) - \int_t^T 1\, d\tilde{W}(s)\right) \Bigg| \mathcal{F}_t\right]$$

$$= \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\lambda(t) F \,\middle|\, \mathcal{F}_t\right]}_{A} - \underbrace{\mathbb{E}_{\mathbb{Q}}\left[F \int_t^T 1\, d\tilde{W}(s)\Bigg) \Bigg| \mathcal{F}_t\right]}_{B}$$

Notice that, since $u(t) = W(t)$,

$$\tilde{W}(t) = W(t) + \int_0^t W(s)\, ds \implies d\tilde{W}(t) = dW(t) + W(t)\, dt$$

Multiplying by e^t,

$$d(e^t W(t)) = e^t W(t)\, dt + e^t\, dW(t) = e^t\, d\tilde{W}(t)$$

i.e.,

$$W(t) = e^{-t} \int_0^t e^s\, d\tilde{W}(s)$$

Differentiating,

$$dW(t) = -te^{-t} \int_0^t e^s\, d\tilde{W}(s)\, dt + e^{-t} e^t\, d\tilde{W}(t)$$

$$= -te^{-t} \int_0^t e^s\, d\tilde{W}(s)\, dt + d\tilde{W}(t)$$

With this expression, we can write

$$F = \exp\left(\int_0^T \lambda(s)\, dW(s)\right)$$

$$= \exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) - \int_0^T \lambda(u) e^{-u} \int_0^u e^{s}\, d\tilde{W}(s)\, du\right)$$

$$= \exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) - \int_0^T \int_0^T \lambda(u) e^{-u} du\ e^{-s}\, d\tilde{W}(s)\right)$$

$$= K(T) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\, ds\right)$$

where

$$\zeta(s) = \lambda(s) - e^{s}\int_0^T \lambda(u) e^{-u}\, du$$

and

$$K(t) = \exp\left(\int_0^t \zeta(s)\, d\tilde{W}(s) - \frac{1}{2}\int_0^t \zeta^2(s)\, ds\right)$$

Now we can compute

$$A = \mathbb{E}_Q\left[\lambda(t) F \mid \mathcal{F}_t\right] = \lambda(t) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\, ds\right) \mathbb{E}_Q\left[K(T) \mid \mathcal{F}_t\right]$$

(1) $$= \lambda(t) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\, ds\right) K(t)$$

and

$$B = \mathbb{E}_Q\left[F \int_t^T 1\, d\tilde{w}(s)\Bigg| \mathcal{F}_t\right] = \mathbb{E}_Q[F(\tilde{w}(T)-\tilde{w}(t))\,|\mathcal{F}_t]$$

$$= \underbrace{\exp\left(\frac{1}{2}\int_0^T \xi^2(s)\,ds\right)}_{H} \mathbb{E}_Q[K(T)(\tilde{w}(T)-\tilde{w}(t))\,|\mathcal{F}_t]$$

$$= HK(t)\mathbb{E}_Q\left[\exp\left(\int_t^T \xi(s)\,d\tilde{w}(s) - \frac{1}{2}\int_t^T \xi^2(s)\,ds\right)(\tilde{w}(T)-\tilde{w}(t))\Bigg|\mathcal{F}_t\right]$$

(2) $$= HK(t)\mathbb{E}\left[\exp\left(\int_t^T \xi(s)\,dw(s) - \frac{1}{2}\int_t^T \xi^2(s)\,ds\right)(w(T)-w(t))\Bigg|\mathcal{F}_t\right]$$

To compute this expectation, let

$$X_t = \exp\left(\int_{t_0}^t \xi(s)\,dw(s) - \frac{1}{2}\int_{t_0}^t \xi^2(s)\,ds\right)$$

and $Y_t = X_t(w(t)-w(t_0))$. Then,

$$dY_t = X_t\,dw(t) + (w(t)-w(t_0))\,dX_t + dX_t\,dw_t$$

$$= X_t(1+(w(t)-w(t_0))\xi(t))\,dw_t + \xi(t)X_t\,dt$$

Thus,

$$\mathbb{E}[Y(T)] = \mathbb{E}[Y(t_0)] + \mathbb{E}\left[\int_{t_0}^T \xi(s)\,X(s)\,ds\right]$$

$$= \int_{t_0}^T \xi(s)\mathbb{E}[X(s)]\,ds = \int_{t_0}^T \xi(s)\,ds$$

Hence,

$$\tilde{\varphi}(t) = \lambda(t) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right) K(t) - \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right) K(t) \int_{t_0}^T \xi(s)\,ds$$

$$= \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right) \exp\left(\int_0^t \xi(s)\,d\tilde{W}(s) - \frac{1}{2}\int_0^t \zeta^2(s)\,ds\right)\left(\lambda(t) - \int_t^T \xi(s)\,ds\right)$$

Problem 4.4. (*) Suppose we have a market with two investments of type (4.14) and (4.15). Find the initial fortune $V^\theta(0)$ and the number of units $\theta_1(t)$, which must be invested at time t in the risky investment to produce the terminal value $V^\theta(T) = F = W(T)$ when $\rho(t) = \rho > 0$ is constant and the price $S_1(t)$, $t \in [0, T]$, of the risky investment is given by:

(a) $dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t)$; μ, σ constants ($\sigma \neq 0$). This is the case of the geometric Brownian motion.

(b) $dS_1(t) = c dW(t)$; $c \neq 0$ constant.

(c) $dS_1(t) = \mu S_1(t)dt + c dW(t)$; μ, c constants. This is the case of the Ornstein–Uhlenbeck process. [*Hint.* $S_1(t) = e^{\mu t}S_1(0) + c\int_0^T e^{\mu(t-s)}dW(s)$.]

(a) A risk-less asset (e.g., a bond), with price dynamics

$$\begin{cases} dS_0(t) = \rho(t)S_0(t)dt, \\ S_0(0) = 1. \end{cases} \tag{4.14}$$

(b) A risky asset (e.g., a stock), with price dynamics

$$\begin{cases} dS_1(t) = \mu(t)S_1(t)dt + \sigma(t)S_1(t)dW(t) \\ S_1(0) > 0. \end{cases} \tag{4.15}$$

a) $dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)\,dW(t)$

Recall that the solution to the SDE above is given by

$$S_1(t) = S_1(0)\exp\left(\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right)t\right)$$

(see Example 5.1.5 in the previous research notes).

By definition

$$V^\theta(t) = \theta_0(t)S_0(t) + \theta_1(t)S_1(t)$$

and, by self financing hypothesis,

(1) $$dV^\theta(t) = \theta_0(t)\,dS_0(t) + \theta_1(t)\,dS_1(t)$$

Using that

$$\theta_0(t) = \frac{V^\theta(t) - \theta_1(t)S_1(t)}{S_0(t)}$$

and 4.14 in (1) yields

(2) $$dV^\theta(t) = \rho\left(V^\theta(t) - \theta_1(t)S_1(t)\right)dt + \theta_1(t)\,dS_1(t)$$

Replacing $dS_1(t)$,

$$dV^\theta(t) = \left[(\mu-\rho)\theta_1(t)S_1(t) + \rho V^\theta(t)\right]dt + \sigma\theta_1(t)S_1(t)\,dW(t)$$

Let $u(t) = (\mu-\rho)/\sigma$. By Girsanov theorem,

$$dV^\theta(t) = \rho V^\theta(t)\,dt + \sigma\theta_1(t)S_1(t)\,d\tilde{W}(t)$$

where

$$\tilde{W}(t) = W(t) + \int_0^t u(s)\,ds$$

Defining

$$U^\theta(t) = e^{-\rho t}V^\theta(t)$$

we obtain

$$dU^\theta(t) = -\rho e^{-\rho t}V^\theta(t)\,dt + e^{-\rho t}\,dV^\theta(t)$$

$$= e^{-\rho t}\sigma\theta_1(t)S_1(t)\,d\tilde{W}(t)$$

Thus

$$e^{-\rho t}V^\theta(t) = V^\theta(0) + \int_0^t e^{-\rho s}\sigma\theta_1(s)S_1(s)\,d\tilde{W}(s)$$

We now apply Clark-Ocone to $G = e^{-\rho T}F$:

(3) $$G = E_Q[G] + \int_0^T E_Q\left[D_tG - G\int_t^T D_t u(s)\,d\tilde{W}(s)\,\middle|\,\mathcal{F}_t\right]d\tilde{W}(t)$$

Notice that

$$V^{\theta}(0) = \mathbb{E}_{\mathbb{Q}}[G] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\rho T} W(T)\right]$$

$$= e^{-\rho T}\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{W}(T) - \int_0^T \upsilon(s)\,ds\right]$$

$$= e^{-\rho T}\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{W}(t) - \upsilon T\right] = \underline{-e^{-\rho T}(\mu-\rho)T/\sigma}$$

Since $\upsilon(t)$ is constant, $D_t \upsilon(s) = 0$ and, by the Chain Rule,

$$D_t G = D_t e^{-\rho T} W(T) = e^{-\rho T}$$

we have

$$\theta_1(t) = e^{\rho t}\sigma^{-1} S_1^{-1}(t)\,\mathbb{E}_{\mathbb{Q}}\left[D_t G - G\int_t^T D_t \upsilon(s)\,d\tilde{W}(s)\,\middle|\,\mathcal{F}_t\right]$$

$$= e^{\rho t}\sigma^{-1} S_1^{-1}(t)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\rho T}\,\middle|\,\mathcal{F}_t\right] = \underline{e^{\rho(t-T)}\sigma^{-1} S_1^{-1}(t)}$$

b) $dS_1(t) = c\,dW(t), \; c \neq 0$

Using (2),

$$dV^{\theta}(t) = \rho\left(V^{\theta}(t) - \theta_1(t)S_1(t)\right)dt + c\,\theta_1(t)\,dW(t)$$

Define

$$u(t) = -\frac{\rho S_1(t)}{c}$$

By Girsanov,

$$dV^{\theta}(t) = \rho V^{\theta}(t)\,dt + c\,\theta_1(t)\,d\tilde{W}(t)$$

Again, we let

$$U^{\theta}(t) = e^{-\rho t}V^{\theta}(t)$$

and obtain

$$dU^{\theta}(t) = -\rho e^{-\rho t}V^{\theta}(t)\,dt + e^{-\rho t}\,dV^{\theta}(t)$$

$$= c e^{-\rho t}\theta_1(t)\,d\tilde{W}(t)$$

Thus

$$e^{-\rho t}V^{\theta}(t) = V^{\theta}(0) + \int_0^t c e^{-\rho s}\theta_1(s)\,d\tilde{W}(s)$$

By (3) we have

$$V^{\theta}(0) = \mathbb{E}_Q[G] = \mathbb{E}_Q[e^{-\rho T} W(T)]$$

$$= e^{-\rho T} \mathbb{E}_Q\left[\tilde{W}(T) - \int_0^T u(s)\,ds\right]$$

$$= e^{-\rho T} \mathbb{E}_Q\left[\tilde{W}(T) + \frac{\rho}{c}\int_0^T S_1(s)\,ds\right] \overset{?}{=} 0$$

Also,

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[D_t G - G\int_t^T D_t u(s)\,d\tilde{W}(s) \,\middle|\, \mathcal{F}_t\right]$$

Now, since $D_t G = e^{-\rho T}$ and

$$D_t - \frac{\rho S_1(s)}{c} = -\frac{\rho}{c} D_t S_1(s) = -\frac{\rho}{c} D_t\left[S_1(0) + \int_0^s c\,dW\right] = -\rho$$

we have

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[e^{-\rho T} + e^{-\rho T} W(T)\int_t^T \rho\,d\tilde{W}(s) \,\middle|\, \mathcal{F}_t\right]$$

$$= c^{-1} \mathbb{E}_Q\left[1 + \rho W(T)[\tilde{W}(T) - \tilde{W}(t)] \,\middle|\, \mathcal{F}_t\right] = \frac{1}{c}$$

c) $dS_1(t) = \mu S_1(t)\,dt + c\,dW(t)$

By (2),

$$dV^\theta(t) = \rho\left(V^\theta(t) - \theta_1(t)S_1(t)\right)dt + \theta_1(t)\,dS_1(t)$$

$$= \left[\rho V^\theta(t) + (\mu-\rho)\theta_1(t)S_1(t)\right]dt + c\,\theta_1(t)\,dW(t)$$

We apply Girsanov with

$$u(t) = \frac{(\mu-\rho)S_1(t)}{c}$$

and get

$$dV^\theta(t) = \rho V^\theta(t)\,dt + c\,\theta_1(t)\,d\tilde{W}(t)$$

Again, let $U^\theta(t) = e^{-\rho t}V^\theta(t)$. Then

$$dU^\theta(t) = -\rho e^{-\rho t}V^\theta(t)\,dt + e^{-\rho t}\,dV^\theta(t)$$

$$= c e^{-\rho t}\theta_1(t)\,d\tilde{W}(t)$$

Thus,

$$e^{-\rho t}V^\theta(t) = V^\theta(0) + \int_0^t c e^{-\rho s}\theta_1(s)\,d\tilde{W}(s)$$

Using (3),

$$V^\theta(0) = E_Q[G] = E_Q\left[e^{-\rho T}W(T)\right] = e^{-\rho T}E_Q\left[\tilde{W}(T) - \int_0^T u(s)\,ds\right]$$

$$= e^{-\rho T}E_Q\left[\tilde{W}(T) - \frac{(\mu-\rho)}{c}\int_0^T S_1(s)\,ds\right] \overset{?}{=} -e^{-\rho T}\frac{(\mu-\rho)}{c}e^{\mu T}S_1(0)$$

Also,

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[D_t G - G \int_t^T D_t u(s)\, d\tilde{W}(s) \,\Big|\, \mathcal{F}_t \right]$$

Now, since $D_t G = e^{-\rho T}$ and

$$D_t\left[\frac{(\mu-\rho)S_1(s)}{c}\right] = \frac{(\mu-\rho)}{c} D_t S_1(s) = (\mu-\rho) e^{\mu(s-t)} \chi_{[0,s]}(t)$$

we have

$$\theta_1(t) = c^{-1} e^{\rho t} \mathbb{E}_Q\left[e^{-\rho T} - e^{-\rho T} W(T) \int_t^T (\mu-\rho) e^{\mu(s-t)}\, d\tilde{W}(s) \,\Big|\, \mathcal{F}_t \right]$$

$$= e^{\rho(t-T)} c^{-1} \left(1 - (\mu-\rho) \mathbb{E}_Q\left[W(T) \int_t^T e^{\mu(s-t)}\, d\tilde{W}(s) \,\Big|\, \mathcal{F}_t \right]\right)$$

Since

$$d\tilde{W}(t) = dW(t) + \frac{(\mu-\rho)S_1(t)}{c}\, dt$$

$$= dW(t) + \frac{(\mu-\rho)}{c}\left[e^{\mu t} S(0) + c \int_0^t e^{\mu(t-r)}\, dW(r) \right] dt$$

we have

$$e^{-\mu t} d\tilde{W}(t) = e^{-\mu t} dW(t) + \left[\frac{(\mu-\rho)}{c} S(0) + (\mu-\rho) \int_0^t e^{-\mu r}\, dW(r) \right] dt$$

Defining
$$X(t) = \int_0^t e^{-\mu r}\, dw(r), \quad \tilde{X}(t) = \int_0^t e^{-\mu r}\, d\tilde{w}(r)$$

we rewrite the previous expression as

$$d\tilde{X}(t) = dX(t) + \frac{(\mu-\rho)}{c} S(0)\, dt + (\mu-\rho) X(t)\, dt$$

$$\Rightarrow d\left(e^{(\mu-\rho)t} X(t)\right) = e^{(\mu-\rho)t} d\tilde{X}(t) - \frac{(\mu-\rho)}{c} S(0)\, e^{(\mu-\rho)t}\, dt$$

$$\Rightarrow X(t) = e^{(\rho-\mu)t} \int_0^t e^{-\rho s}\, d\tilde{w}(s) - \frac{(\mu-\rho)}{c} S(0)\, e^{(\rho-\mu)t} \int_0^t e^{(\mu-\rho)s}\, ds$$

$$= e^{(\rho-\mu)t} \int_0^t e^{-\rho s}\, d\tilde{w}(s) - \frac{S(0)}{c}\left(1 - e^{(\rho-\mu)t}\right)$$

Thus,

$$e^{-\mu t}\, dw(t) = e^{(\rho-\mu)t} e^{-\rho t}\, d\tilde{w}(t) + (\rho-\mu) e^{(\rho-\mu)t} \int_0^t e^{-\rho s}\, d\tilde{w}(s)\, dt + \frac{S(0)}{c}(\rho-\mu)\, e^{(\rho-\mu)t}\, dt$$

and

$$dw(t) = d\tilde{w}(t) + (\rho-\mu) e^{\rho t} \int_0^t e^{-\rho s}\, d\tilde{w}(s)\, dt + \frac{S(0)}{c}(\rho-\mu)\, e^{\rho t}\, dt$$

$$\Rightarrow w(T) = \tilde{w}(T) + (\rho-\mu) \int_0^T e^{\rho s} \int_0^s e^{-\rho r}\, d\tilde{w}(r)\, ds + \frac{S(0)}{\rho c}(\rho-\mu)(e^{\rho T} - 1)$$

Replacing into $\theta_1(t)$,

$$\theta_1(t) = e^{\rho(t-T)} c^{-1} \Bigg(1 - (\mu-\rho)\, \mathbb{E}_Q \left[\tilde{\omega}(T) \int_t^T e^{\mu(s-t)}\, d\tilde{\omega}(s) \,\middle|\, \mathcal{F}_t \right]$$

$$+ (\mu-\rho)^2\, \mathbb{E}_Q \left[\int_0^T e^{\rho s} \int_0^s e^{-\rho r}\, d\tilde{\omega}(r)\, ds \int_t^T e^{\mu(s-t)}\, d\tilde{\omega}(s) \,\middle|\, \mathcal{F}_t \right]$$

$$= e^{\rho(t-T)} c^{-1} \Bigg(1 - (\mu-\rho) \int_t^T e^{\mu(s-t)}\, ds$$

$$+ (\mu-\rho)^2 \int_0^T e^{\rho s}\, \mathbb{E}_Q \left[\int_0^s e^{-\rho r}\, d\tilde{\omega}(r) \int_t^T e^{\mu(r-t)}\, d\tilde{\omega}(r) \,\middle|\, \mathcal{F}_t \right] ds \Bigg)$$

$$= e^{\rho(t-T)} c^{-1} \left(1 - \frac{(\mu-\rho)}{\mu} \left(e^{\mu(T-t)} - 1 \right) + (\mu-\rho)^2 \int_t^T e^{\rho r} \int_t^s e^{-\rho r + \mu(r-t)}\, dr\, ds \right)$$

$$= e^{\rho(t-T)} c^{-1} \left(1 - \frac{(\mu-\rho)}{\rho} \left(e^{\rho(T-t)} - 1 \right) \right)$$

Problem 5.1. Prove equation (5.4). [*Hint.* First consider step functions ϕ of the form $\phi(t) = \sum_i e_i \chi_{(a_i,a_{i+1}]}(t), \quad t \in \mathbb{R}$.]

$$w_\phi(\omega) = \int_{\mathbb{R}} \phi(t) dW(t,\omega), \quad \omega \in \Omega, \qquad \phi \in L^2(\mathbb{R}), \tag{5.4}$$

Let

$$\varphi(t) = \sum_i e_i \chi_{[a_i, a_{i+1}]}(t)$$

Then, since $\tilde{W}(t) = \langle \omega, \chi_{[0,t]} \rangle$,

$$w_\varphi(\omega) = \langle \omega, \varphi \rangle = \left\langle \omega, \sum_i e_i \chi_{[a_i, a_{i+1}]}(t) \right\rangle$$

$$= \sum_i e_i \langle \omega, \chi_{[a_i, a_{i+1}]} \rangle$$

$$= \sum_i e_i [W(a_{i+1}) - W(a_i)]$$

$$= \int_{\mathbb{R}} \varphi(t)\, dW(t)$$

Hence, the result holds for step functions. Since these functions are dense in the space of Itô integrable functions, we have the result.

Problem 5.2. Prove equation (5.22), that is, that

$$\frac{d}{dt}W(t) = \overset{\bullet}{W}(t),$$

where the derivative exists in $(\mathcal{S})^*$.

We saw (Example 5.3) that

$$W(t) = \sum_k \left(\int_0^t e_k(y)\, dy \right) H_{\varepsilon(k)}$$

and recall that

$$\dot{W}(t) := \sum_k e_k(t) H_{\varepsilon(k)}$$

Therefore,

$$\frac{d}{dt} W(t) = \frac{d}{dt} \sum_k \left(\int_0^t e_k(y)\, dy \right) H_{\varepsilon(k)}$$

$$= \sum_k e_k(t) H_{\varepsilon(k)}$$

Problem 5.4. Prove (5.26), that is, that

$$(\overset{\bullet}{W})^{\diamond 2}(t) \in (\mathcal{S})^*, \qquad t \in \mathbb{R}.$$

Since $\overset{\bullet}{W} \in (S)^*$ (Example 5.7.b) and the fact that the product of elements of $(S)^*$ is in $(S)^*$, the result follows.

Problem 5.5. (*) Use the identity (5.28) and Wick calculus to compute the following Skorohod integrals:

(a) $\int_0^T W(T)\delta W(t)$;

$$\int_{\mathbb{R}} Y(t)\delta W(t) = \int_{\mathbb{R}} Y(t) \diamond \overset{\bullet}{W}(t)dt \tag{5.28}$$

Using (5.28),

$$\int_0^T W(T)\,\delta W(t) = \int_0^T W(T) \diamond \dot{W}(t)\,dt$$

$$= W(T) \diamond \int_0^T \dot{W}(t)\,dt = W(T) \diamond W(T) = W^2(T) - T$$

where the last equality is (5.37).

(b) $\int_0^T \int_0^T g(s)dW(s)\delta W(t)$, for the deterministic function $g \in L^2([0,T])$;

$$\int_0^T \int_0^T g(s)\,dW(s)\,\delta W(t) = \int_0^T \left(\int_0^T g(s)\,dW(s) \right) \diamond \dot{W}(t)\,dt$$

$$= \int_0^T g(s)\,dW(s) \diamond W(T) = W(T) \int_0^T g(s)\,dW(s) - \int_0^T g(s)\,ds$$

by (5.62).

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed;

Notice that by (5.65),

$$W^{\diamond 2}(t_0) = t_0 h_2\left(\frac{W(t_0)}{\sqrt{t_0}}\right) = t_0\left(\frac{W^2(t_0)}{t_0} - 1\right) = W^2(t_0) - t_0$$

Thus,

$$\int_0^T W^2(t_0)\,\delta W(t) = \int_0^T (W^{\diamond 2}(t_0) + t_0)\,\delta W(t)$$

$$= \int_0^T (W^{\diamond 2}(t_0) + t_0) \diamond \dot{W}(T)\,dt = (W^2(t_0) + t_0) \diamond W(T)$$

$$= W^{\diamond 2}(t_0) \diamond W(T) + t_0 W(T)$$

$$= W^{\diamond 2}(t_0) \diamond (W(T) - W(t_0)) + W^{\diamond 2}(t_0) \diamond W(t_0) + t_0 W(T)$$

$$= W^{\diamond 2}(t_0)(W(T) - W(t_0)) + W^{\diamond 3}(t_0) + t_0 W(T) \qquad (5.40)$$

Using that $W^{\diamond 2}(t) = W^2(t) - t$ (5.37) in the first term, and that by (5.65),

$$W^{\diamond 3}(t) = t^{3/2} h_3\left(\frac{W(t)}{\sqrt{t}}\right) = t^{3/2}\left(\frac{W^3(t)}{t^{3/2}} - \frac{3W(t)}{t^{1/2}}\right)$$

$$= W^3(t) - 3tW(t)$$

we have

$$(W^2(t_0)-t_0)(W(T)-W(t_0))+W^3(t_0)-3t_0W(t_0)+t_0W(T)$$

$$= W^2(t_0)W(T)-2t_0W(t_0)$$

(d) $\int_0^T \exp(W(T))\delta W(t)$.

Compare with your calculations in Problem 2.4!

$$\int_0^T \exp(W(T))\,\delta W(t) = \int_0^T \exp(W(T)) \diamond \dot{W}(t)\,dt$$

$$= \exp W(T) \diamond W(T) \overset{(5.68)}{=} \exp^{\diamond}(W(T)+T/2) \diamond W(T)$$

$$= \sum_{k=0}^{\infty} \frac{(W(T)+T/2)^{\diamond k}}{k!} \diamond W(T) = e^{T/2} \sum_{k=0}^{\infty} \frac{W(T)^{\diamond k+1}}{k!}$$

$$\overset{(5.65)}{=} e^{T/2} \sum_{k=0}^{\infty} \frac{T^{\frac{k+1}{2}}}{k!} h_k\left(\frac{W(T)}{\sqrt{T}}\right)$$

Problem 5.7. (a) Let $f \in L^2(\mathbb{R})$ be deterministic. Prove that

$$\frac{d}{dt}\int_{-\infty}^{t} f(s)dW(s) = f(t)\overset{\bullet}{W}(t) \quad \text{in} \quad (\mathcal{S}^*).$$

Using (5.70) and the fact that f is deterministic,

$$\int_{-\infty}^{t} f(s)\, dW(s) = \int_{-\infty}^{t} f(s) \diamond \dot{W}(s)\, ds = \int_{-\infty}^{t} f(s)\, \dot{W}(s)\, ds$$

Thus,

$$\frac{d}{dt}\int_{-\infty}^{t} f(s)\, dW(s) = \int_{-\infty}^{t} \frac{d}{ds}\left(f(s)\, \dot{W}(s)\right) ds$$

$$= f(t)\, \dot{W}(t)$$

(b) Let u be a Skorohod integrable process. Prove that

$$\frac{d}{dt}\int_{-\infty}^{t} u(s)\delta W(s) = u(t) \diamond \overset{\bullet}{W}(t) \quad \text{in} \quad (\mathcal{S}^*).$$

Follows from approximation. Let (φ_n) be a sequence of elementary functions such that $\varphi_n \uparrow f$. By the previous result and monotone convergence,

$$\frac{d}{dt}\int_{-\infty}^{t} u(s)\, \delta W(s) = \int_{-\infty}^{t} \frac{d}{ds}\left(u(s) \diamond \dot{W}(s)\right) ds = u(t) \diamond \dot{W}(t)$$

Problem 5.8. (a) Solve the stochastic differential equation

$$dX(t) = X(t)\big[\mu dt + \sigma dW(t)\big], \qquad X(0) = x > 0, \tag{5.76}$$

where the parameters μ, σ, and the initial value x are constants, via the following guidelines. First, rewrite the equation in the form

$$\frac{d}{dt}X(t) = X(t) \diamond \big[\mu + \sigma \dot{W}(t)\big], \qquad X(0) = x > 0, \tag{5.77}$$

and regard it as an ordinary differential equation in the $(\mathcal{S})^*$-valued function $X(t)$, $t > 0$. Then use the Wick calculus in $(\mathcal{S})^*$.

(b) Equation (5.77) also makes sense if the initial value $X(0)$ is not constant, but a given element in $(\mathcal{S})^*$ (e.g., $X(0) \in L^2(P)$). What would the solution of (5.77) be in this case?

a) In the integral form,

$$X(t) = X(0) + \int \mu X(t)\,dt + \int \sigma X(t)\,dW_t$$

By 5.70,

$$X(t) = X(0) + \int \mu X(t)\,dt + \int \sigma X(t) \diamond \dot{W}(t)\,dt$$

Differentiating,

$$\frac{dX(t)}{dt} = \mu X(t) + \sigma X(t) \diamond \dot{W}(t)$$

$$= X(t)\left[\mu + \sigma \diamond \dot{W}(t)\right] = X(t) \diamond \left[\mu + \sigma \dot{W}(t)\right]$$

Thus, if we treat it as a separable equation (without "$\diamond$"),

$$\frac{X'(t)}{X(t)} = \mu + \sigma \dot{W}(t) \Rightarrow \log\left(\frac{X(t)}{X(0)}\right) = \mu t + \sigma W(t)$$

i.e.,

$$X(t) = X(0) \exp(\mu t + \sigma W(t))$$

But since we have a Wick product,

$$X(t) = X(0) \exp^{\diamond}(\mu t + \sigma W(t))$$

Using (5.68),

$$X(t) = X(0) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right]$$

b) There would appear a Wick product in the solution:

$$X(t) = X(0) \diamond \exp^{\diamond}(\mu t + \sigma W(t))$$

Problem 6.1. Prove $L^2(P) \subseteq Dom(D_t) \subseteq (\mathcal{S})^*$. See Remark 6.6.

It is immediate from the definition that $Dom(D_t) \subseteq (S)^*$.

If $F \in L^2(\mathbb{P})$, then $F \in (S)^*$. Hence, there exists q_0 such that

$$\|F\|^2_{-q_0} = \sum_{\alpha \in J} \alpha! c_\alpha^2 (2\mathbb{N})^{-\alpha q_0} < \infty \Rightarrow \alpha! c_\alpha^2 (2\mathbb{N})^{-\alpha q_0} \leq J, \ \forall \alpha \in J$$

Our goal is to show that

$$D_t F = \sum_{\alpha \in J} \sum_{k=1}^{\infty} c_\alpha \alpha_k e_k(t) H_{\alpha - \varepsilon^{(k)}} \in (S)^*$$

i.e., we need to check that there exists q_1 such that

$$(\alpha - \varepsilon^{(k)})! \, c_\alpha^2 \alpha_k^2 e_k^2 (2\mathbb{N})^{-(\alpha - \varepsilon^{(k)})(q_0 + q_1)} \leq J$$

Since $\{e_k(t)\}$ is bounded,

$$(\alpha - \varepsilon^{(k)})! \, c_\alpha^2 \alpha_k^2 e_k^2 (2\mathbb{N})^{-(\alpha - \varepsilon^{(k)})(q_0 + q_1)}$$

$$\leq C_1 (\alpha - \varepsilon^{(k)})! \, c_\alpha^2 \alpha_k^2 (2\mathbb{N})^{-(\alpha - \varepsilon^{(k)})(q_0 + q_1)}$$

$$= C_1 \alpha! \frac{(\alpha_k - 1)}{\alpha_k} c_\alpha^2 \alpha_k^2 (2\mathbb{N})^{-q_0(\alpha - \varepsilon^{(k)})} (2\mathbb{N})^{-q_1(\alpha - \varepsilon^{(k)})}$$

$$= C_1 \alpha! \, c_\alpha^2 (2\mathbb{N})^{-q_0(\alpha - \varepsilon^{(k)})} (\alpha_k - 1)\alpha_k (2\mathbb{N})^{-q_1(\alpha - \varepsilon^{(k)})}$$

$$\leq C_1 \alpha! \, c_\alpha^2 (2\mathbb{N})^{-q_0 \alpha} (2k)^{q_0} \left[(\alpha_k - 1)(2(k-1))^{-q_1 \alpha_{k-1}}\right]\left[\alpha_k (2k)^{-q_1(\alpha_k - 1)}\right]$$

$$\leq C_1 \alpha! \, c_\alpha^2 \left[(\alpha_k - 1) 2^{-q_1 \alpha_{k-1}}\right]\left[\alpha_k (2k)^{-q_1(\alpha_k - 1) + q_0}\right] \chi_{\alpha_k > 1} \leq J$$

Problem 6.2. The Wick chain rule. Let F be Malliavin differentiable and let $n \in \mathbb{N}$. Show that

$$D_t(F^{\diamond n}) = nF^{\diamond(n-1)} \diamond D_tF.$$

Write $F^{\diamond n}(\omega) = \langle \omega, f \rangle^{\diamond n}$ and suppose that $\|f\|_{L^2(\mathbb{R})} = 1$.
Since $\langle \omega, f \rangle^{\diamond n} = h_n(\langle \omega, f \rangle)$, we have

$$\frac{1}{\varepsilon}\left[F^{\diamond n}(\omega + \varepsilon\gamma) - F^{\diamond n}(\omega)\right] = \frac{1}{\varepsilon}\left[\langle \omega + \varepsilon\gamma, f \rangle^{\diamond n} - \langle \omega, f \rangle^{\diamond n}\right]$$

$$= \frac{1}{\varepsilon}\left[h_n \langle \omega + \varepsilon\gamma, f \rangle - h_n \langle \omega, f \rangle\right]$$

$$= \frac{1}{\varepsilon}\left[h_n(\langle \omega, f \rangle + \varepsilon\langle \gamma, f \rangle) - h_n\langle \omega, f \rangle\right]$$

Taking the limit as $\varepsilon \to 0$,

$$D_t(F^{\diamond n}) = h_n'(\langle \omega, f \rangle)\langle \gamma, f \rangle = n h_{n-1}(\langle \omega, f \rangle)\langle \gamma, f \rangle$$

$$= nF^{\diamond(n-1)}(\omega) \diamond D_t F$$

By induction on n. For $n=1,2$, it is immediate, since

$$D_+(F \diamond G) = F \diamond D_+ G + G \diamond D_+ F$$

Suppose that the result holds for $n-1$. Then,

$$D_+(F^{\diamond n}) = D_+(F^{\diamond(n-1)} \diamond F) = F^{\diamond(n-1)} \diamond D_+ F + F \diamond D_+ F^{\diamond(n-1)}$$

$$= F^{\diamond(n-1)} \diamond D_+ F + F \diamond \left((n-1) \cdot F^{\diamond(n-2)} \diamond D_+ F \right)$$

$$= F^{\diamond(n-1)} \diamond D_+ F + (n-1) F^{\diamond(n-1)} \diamond D_+ F$$

$$= n F^{\diamond(n-1)} \diamond D_+ F$$

Problem 6.3. Verify that if $g \in L^2(\mathbb{R})$, $n \in \mathbb{N}$, then

$$W^n(T) \diamond \int_0^T g(t)\delta W(t) = W^n(T)\int_0^T g(t)\delta W(t) - nW^{n-1}(T)\int_0^T g(t)dt.$$

We need to compute $D_\gamma W^n(T)$.

$$D_\gamma W^n(T) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\langle \omega + \varepsilon\gamma, \chi_{[0,T]}\rangle^n - \langle \omega, \chi_{[0,T]}\rangle^n\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\left(\langle \omega, \chi_{[0,T]}\rangle + \varepsilon\langle \gamma, \chi_{[0,T]}\rangle\right)^n - \langle \omega, \chi_{[0,T]}\rangle^n\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\sum_{k=0}^{n}\binom{n}{k}\langle \omega, \chi_{[0,T]}\rangle^{n-k}\cdot\left(\varepsilon\langle \gamma, \chi_{[0,T]}\rangle\right)^k - \langle \omega, \chi_{[0,T]}\rangle^n\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[\langle \omega, \chi_{[0,T]}\rangle^n + n\langle \omega, \chi_{[0,T]}\rangle^{n-1}\varepsilon\langle \gamma, \chi_{[0,T]}\rangle + \cdots +$$

$$\varepsilon^n\langle \gamma, \chi_{[0,T]}\rangle^n - \langle \omega, \chi_{[0,T]}\rangle^n\Big]$$

$$= n\langle \omega, \chi_{[0,T]}\rangle^{n-1}\langle \gamma, \chi_{[0,T]}\rangle = nW^{n-1}(T)\int_0^T \gamma(t)\,dt$$

Thus, $D_t F = nW^{n-1}(T)$. Now, using (6.9),

$$W^n(T) \diamond \int_{\mathbb{R}} g(t)\,dW(t) = W^n(T)\int_{\mathbb{R}} g(t)\,dW(t) - nW^{n-1}(T)\int_{\mathbb{R}} g(t)\,dt$$

Problem 6.4. (*) Show that the singular noise $\overset{\bullet}{W}$ does not belong to the domain of the Hida–Malliavin derivative as given in (6.8).

Since

$$\dot{W}(t) = \sum_{k} e_k(t) H_{\varepsilon^{(k)}} \in (S)^*$$

we have that

$$D_t \dot{W}(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} e_k(s) e_j(t) H_{\varepsilon^{(k)} - \varepsilon^{(j)}}$$

$$= \sum_{k=1}^{\infty} e_k(s) e_k(t) \quad \text{does not converge in } (S)^*$$

where we used that

$$H_{\varepsilon^{(k)} - \varepsilon^{(j)}} = \begin{cases} 1, & \text{if } k=j \\ 0, & \text{otherwise} \end{cases}$$

Notice that $D_t \dot{W}(s)$ does not converge in $(S)^*$ since

$$\left\| \sum_{k=1}^{\infty} e_k(s) e_k(t) \right\| \leq \sum_{k=1}^{\infty} \| e_k(s) e_k(t) \| = \sum_{k=1}^{\infty} | e_k(s) | \cdot \| e_k(t) \|$$

and $\{e_k\}$ is a orthonormal basis of $L^2(\mathbb{R})$.

Problem 6.5. Generalized Bayes formula. Let $Q(d\omega) = Z(T)P(d\omega)$, where $Z(t), 0 \leq t \leq T$, is the Doleans–Dade exponential in (4.4). Further, let $G \in \mathcal{G}^*$ and assume that $Z(T)G$ belongs to $\mathcal{G}^*$. Show that the following *generalized Bayes formula* holds:

$$E_Q[G \,|\mathcal{F}_t] = \frac{E_Q[Z(T)G \,|\mathcal{F}_t]}{Z(t)}.$$

Let $A \in \mathcal{F}_t$. Then,

$$\mathbb{E}_Q\left[1_A \frac{\mathbb{E}[Z(T)G \,|\mathcal{F}_t]}{Z(t)} \right] = \mathbb{E}\left[1_A \mathbb{E}[Z(T)G \,|\mathcal{F}_t]\right]$$

$$= \mathbb{E}[1_A Z(T) G] = \mathbb{E}_Q[1_A G]$$