Problem 5.8. (a) Solve the stochastic differential equation

$$dX(t) = X(t)\big[\mu dt + \sigma dW(t)\big], \qquad X(0) = x > 0, \tag{5.76}$$

where the parameters μ, σ, and the initial value x are constants, via the following guidelines. First, rewrite the equation in the form

$$\frac{d}{dt}X(t) = X(t) \diamond \big[\mu + \sigma \overset{\bullet}{W}(t)\big], \qquad X(0) = x > 0, \tag{5.77}$$

and regard it as an ordinary differential equation in the $(\mathcal{S})^*$-valued function $X(t)$, $t > 0$. Then use the Wick calculus in $(\mathcal{S})^*$.

(b) Equation (5.77) also makes sense if the initial value $X(0)$ is not constant, but a given element in $(\mathcal{S})^*$ (e.g., $X(0) \in L^2(P)$). What would the solution of (5.77) be in this case?

a) In the integral form,

$$X(t) = X(0) + \int \mu X(t)\,dt + \int \sigma X(t)\,dW_t$$

By 5.70,

$$X(t) = X(0) + \int \mu X(t)\,dt + \int \sigma X(t) \diamond \dot{W}(t)\,dt$$

Differentiating,

$$\frac{dX(t)}{dt} = \mu X(t) + \sigma X(t) \diamond \dot{W}(t)$$

$$= X(t)\left[\mu + \sigma \diamond \dot{W}(t)\right] = X(t) \diamond \left[\mu + \sigma \dot{W}(t)\right]$$

Thus, if we treat it as a separable equation (without "$\diamond$"),

$$\frac{X'(t)}{X(t)} = \mu + \sigma \dot{W}(t) \Rightarrow \log\left(\frac{X(t)}{X(0)}\right) = \mu t + \sigma W(t)$$

i.e.,

$$X(t) = X(0)\exp\left(\mu t + \sigma W(t)\right)$$

But since we have a Wick product,

$$X(t) = X(0)\exp^{\diamond}\left(\mu t + \sigma W(t)\right)$$

Using (5.68),

$$X(t) = X(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right]$$

b) There would appear a Wick product in the solution:

$$X(t) = X(0)\diamond\exp^{\diamond}\left(\mu t + \sigma W(t)\right)$$