Problem 1.1. **(*)** Let $h_n(x)$, $n = 0, 1, 2, \ldots$, be the Hermite polynomials defined in (1.13).

(a) Prove that

$$\exp\left\{tx - \frac{t^2}{2}\right\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x).$$

[*Hint.* Write $\exp\{tx - \frac{t^2}{2}\} = \exp\{\frac{1}{2}x^2\} \cdot \exp\{-\frac{1}{2}(x-t)^2\}$ and apply the Taylor formula on the last factor.]

Using the hint,

$$\exp(tx - t^2/2) = \exp(1/2\, x^2)\exp(-1/2\,(x-t)^2)$$

$$= e^{1/2x^2} \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\left[\exp(-1/2\,(x-t)^2)\right]\frac{t^n}{n!}$$

Letting $u = x-t$, we have

$$e^{1/2x^2} \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\left[\exp(-1/2\,(x-t)^2)\right]\frac{t^n}{n!} = e^{1/2x^2} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \frac{d^n}{du^n}\left[\exp(-1/2\, u^2)\right]$$

$$= e^{1/2x^2} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \frac{d^n}{dx^n}\left[\exp(-1/2x^2)\right]$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x)$$

(b) Show that if $\lambda > 0$ then

$$\exp\{tx - \frac{t^2\lambda}{2}\} = \sum_{n=0}^{\infty} \frac{t^n\lambda^{\frac{n}{2}}}{n!} h_n(\frac{x}{\sqrt{\lambda}}).$$

Using $u = t\sqrt{\lambda}$ we have

$$\exp\left(tx - \frac{t^2\lambda}{2}\right) = \exp\left(\frac{ux}{\sqrt{\lambda}} - \frac{u^2}{2}\right)$$

By the previous part,

$$\exp\left(\frac{u}{\sqrt{\lambda}}x - \frac{u^2}{2}\right) = \sum_{n=0}^{\infty} \frac{u^n}{n!} h_n\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{n=0}^{\infty} \frac{t^n\lambda^{n/2}}{n!} h_n\left(\frac{x}{\sqrt{\lambda}}\right)$$

(c) Let $g \in L^2([0,T])$. Put

$$\theta = \int_0^T g(s)dW(s).$$

Show that

$$\exp\Big\{\int_0^T g(s)dW(s) - \frac{1}{2}\|g\|^2\Big\} = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n(\frac{\theta}{\|g\|}),$$

where $\|g\| = \|g\|_{L^2([0,T])}$.

Let $t = \|g\|$ and $x = \theta/\|g\|$ and apply item a.

(d) Let $t \in [0,T]$. Show that $\exp\{W(t) - \frac{1}{2}t\} = \sum_{n=0}^{\infty} \frac{t^{n/2}}{n!} h_n(\frac{W(t)}{\sqrt{t}})$.

Apply item b. Use $t = 1$, $\lambda = t$ and $x = W(t)$.