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MONOTONICITY OF OPTION PRICES WITH RESPECT TO VOLATILITY

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Introduction

This work explores how option prices behave when the volatility is allowed to vary between two bounds. The main result is that the price of a European call is an increasing function of volatility.

Metodology

Consider a market of a riskless asset with price $S_t^0 = e^{rt}$ at time t and interest rate r, and one risky asset with price S_t. We assume that the stochastic process (S_t) is the solution to

$$dS_t = \mu S_t dt + \sigma(t) S_t dB_t \tag{1}$$

In this market, we consider a European call option with maturity T and strike price K. If $\sigma(t) = \sigma$ for all t, then the price of the call at time t is given by $C(t, S_t)$, where the function $C(t,x)$ satisfies

$$\begin{cases} \frac{\partial C}{\partial t}(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 C}{\partial x^2}(t,x) + rx\frac{\partial C}{\partial x}(t,x) - rC(t,x) = 0 \\ C(T,x) = \max\{x - K,\ 0\} \end{cases} \tag{2}$$

Denote by C_i the function C corresponding to the case $\sigma = \sigma_i$. We'll show that $C_0 \in [C_1(0,S_0),\ C_2(0,S_0)]$. We divide the proof into eight steps as follows.

Lemma 1. The functions $x \longmapsto C_i(t,x)$, for $i = 1,2$, are convex.

Lemma 2. The solution of the stochastic differential equation $dS_t = \mu S_t dt + \sigma(t) S_t dB_t$ is

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t \sigma^2(s)\ ds + \int_0^t \sigma(s)\ dB_s\right)$$

Lemma 3. There exists a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the process defined by $W_t = B_t + \int_0^t \frac{\mu - r}{\sigma_s}\, ds$ is a standard Brownian motion.

Lemma 4. The price of the call at time 0 is given by $C_0 = \mathbb{E}^*[e^{-rT}\max\{S_T - K,\ 0\}]$.

Lemma 5. Let $\tilde{S}_t = e^{-rt}S_t$. Then $\mathbb{E}^*[\tilde{S}_t^2] \le S_0^2 e^{\sigma^2 t}$.

Lemma 6. The process defined by

$$M_t = \int_0^t e^{-ru}\frac{\partial C_1}{\partial x}(u, S_u)\sigma_u S_u\ dW_u$$

is a martingale under probability $\mathbb{P}^*$.

Theorem 7. The process $(e^{-rt}C_1(t,S_t))$ is a submartingale under the probability measure $\mathbb{P}^*$. Furthermore, $C_1(0,S_0) \le C_0$.

Proof. Let $g(t,x) = e^{-rt}C_1(t,x)$, $X_t = g(t,S_t)$, and apply Itô's formula.

Replace $dS_t = \mu S_t dt + \sigma_t S_t dB_t$ and $(dS_t)^2 = \sigma_t^2 S_t^2 dt$.

From the proof of the Lemma 5, we know that $\sigma_t dB_t = \sigma_t dW_t - (\mu - r)dt$.

By the equation (2), this simplifies to

$$dX_t = \frac{1}{2}e^{-rt}(\sigma_t^2 - \sigma_1^2)S_t^2\frac{\partial^2 C_1}{\partial x^2}dt + e^{-rt}\frac{\partial C_1}{\partial x}\sigma_t S_t dW_t \tag{3}$$

Since $C_1(t,x)$ is convex as function of x (from the Lemma 1), $\frac{\partial^2 C_1}{\partial x^2} > 0$, and using that $\sigma(t) > \sigma_1$, we have $(\sigma_t^2 - \sigma_1^2) > 0$. Therefore,

$$\frac{1}{2}e^{-rt}(\sigma_t^2 - \sigma_1^2)S_t^2\frac{\partial^2 C_1}{\partial x^2} > 0$$

By the Lemma 6, $\int_0^t e^{-ru}\frac{\partial C_1}{\partial x}(u,S_u)\sigma_u S_u\ dW_u$ is a $\mathbb{P}^*$-martingale. Thus, $\mathbb{E}^*[X_t \mid \mathfrak{F}_u] \ge X_u$.

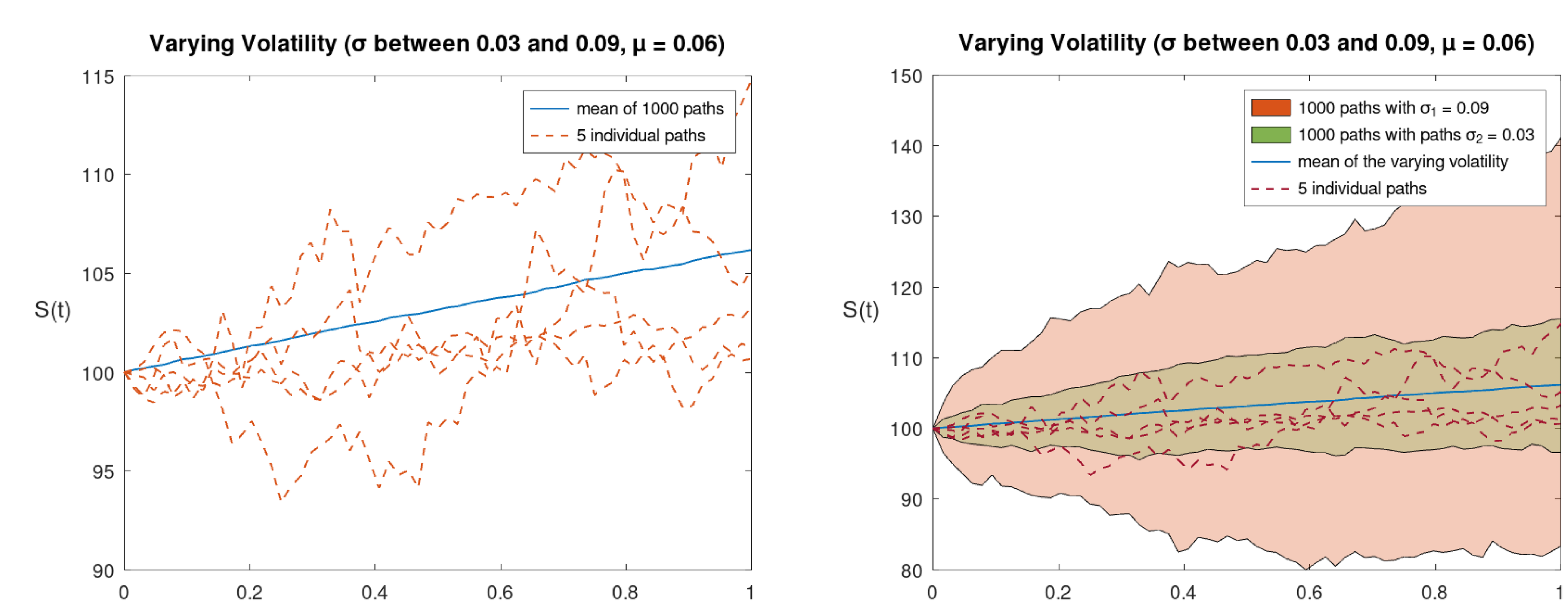
Using that $C_1(0,S_0)$ is a submartingale and Lemma 4, it follows that $C_1(0,S_0) \le C_0$. □

Theorem 8. $C_0 \le C_2(0,S_0)$.

Results and Discussion

To illustrate the result, we compute the call price of an underlying asset with stock price $S = 100$, strike price $K = 100$, interest rate $r = 0.06$, maturity $T = 1$ year, and volatility $\sigma = 0.06$.

Figure 1 presents a simulation in which the volatility σ is an array of uniformly distributed numbers between σ_1 and σ_2.



(a) Five individual paths and their mean   (b) Shaded area with simulations under σ_1 and σ_2

Figure 1: Model with varying volatility

Pricing the option under this and other models, we obtain the values presented in Table 1. We can see that the price under varying volatility belongs to the desired interval.

Table 1: Call prices

Method	Call Price
Cox-Ross-Rubinstein	6.333048
Black-Scholes (0.06)	6.308527
Black-Scholes (0.09)	7.142684
Black-Scholes (0.03)	5.848264
Varying volatility	6.376718

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