Problem 6.5. Generalized Bayes formula. Let $Q(d\omega) = Z(T)P(d\omega)$, where $Z(t), 0 \leq t \leq T$, is the Doleans–Dade exponential in (4.4). Further, let $G \in \mathcal{G}^*$ and assume that $Z(T)G$ belongs to $\mathcal{G}^*$. Show that the following *generalized Bayes formula* holds:

$$E_Q[G \,|\mathcal{F}_t] = \frac{E_Q[Z(T)G \,|\mathcal{F}_t]}{Z(t)}.$$

Let $A \in \mathcal{F}_t$. Then,

$$\mathbb{E}_Q\left[1_A \frac{\mathbb{E}[Z(T)G \mid \mathcal{F}_t]}{Z(t)} \right] = \mathbb{E}\left[1_A \mathbb{E}[Z(T) G \mid \mathcal{F}_t]\right]$$

$$= \mathbb{E}[1_A Z(T) G] = \mathbb{E}_Q[1_A G]$$