Problem 4.2. (*) Verify the Clark–Ocone formula

$$F = E[F] + \int_0^T E[D_t F | \mathcal{F}_t] dW(t)$$

for the following $\mathcal{F}_T$-measurable random variables F:

(a) $F = W(T)$,
(b) $F = \int_0^T W(s)ds$,
(c) $F = W^2(T)$,
(d) $F = W^3(T)$,
(e) $F = \exp W(T)$,
(f) $F = (W(T) + T)\exp\{-W(T) - \frac{1}{2}T\}$.

a) $F = W(T)$

By the exercise 3.2, we have that $D_t F = 1$. Hence,

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[1 | \mathcal{F}_t] \, dW(t) = W(T)$$

b) $F = \int_0^T W(s)\, ds$

We start by computing

$$D_t F = D_t \int_0^T W(s)\, ds = \int_0^T D_t W(s)\, ds = T - t$$

Thus,

$$F = \int_0^T \mathbb{E}[T - t | \mathcal{F}_t] \, dW(t) = \int_0^T (T-t)\, dW(t)$$

(*) Obs. Ex. 3.1

$$= TW(T) - \int_0^T t\, dW(t) \overset{(*)}{=} TW(T) - \left[TW(T) - \int_0^T W(s)\, ds \right]$$

$$= \int_0^T W(s)\, ds = F$$

c) $F = W^2(T)$

Since $E[W^2(T)] = T$ and

> From here, we extract two useful identities. For $n = 1$, we
>
> $$D_t \int_0^T f(s)\, dW(s) = f(t)$$
>
> and for $n > 1$, using induction,
>
> $$D_t \left(\int_0^T f(s)\, dW(s) \right)^n = n \left(\int_0^T f(s)\, dW(s) \right)^{n-1} f(t)$$

$$D_t W^2(T) = D_t \left(\int_0^T 1\, dW(s) \right)^2 = 2W(T)$$

we have

> **Example 3.2.3.** Brownian motion B_t in $\mathbf{R}^n$ is a martingale w.r.t. the σ-algebras $\mathcal{F}_t$ generated by $\{B_s; s \le t\}$, because
>
> $$E[|B_t|]^2 \le E[|B_t|^2] = |B_0|^2 + nt \qquad \text{and if } s \ge t \text{ then}$$
> $$E[B_s|\mathcal{F}_t] = E[B_s - B_t + B_t|\mathcal{F}_t]$$
> $$= E[B_s - B_t|\mathcal{F}_t] + E[B_t|\mathcal{F}_t] = 0 + B_t = B_t .$$
>
> Here we have used that $E[(B_s - B_t)|\mathcal{F}_t] = E[B_s - B_t] = 0$ since $B_s - B_t$ is independent of $\mathcal{F}_t$ (see (2.2.11) and Theorem B.2.d)) and we have used that $E[B_t|\mathcal{F}_t] = B_t$ since B_t is $\mathcal{F}_t$-measurable (see Theorem B.2.c)).

$$\begin{aligned} E[D_t W^2(T) | \mathcal{F}_t] &= 2E[W(T) | \mathcal{F}_t] = 2E[W(T) - W(t) + W(t) | \mathcal{F}_t] \\ &= 2\left(E[W(T) - W(t) | \mathcal{F}_t] + E[W(t) | \mathcal{F}_t] \right) \\ &= 2W(t) \end{aligned}$$

and

$$F = T + 2\int_0^T W(t)\, dW(t) = T + W^2(T) - T = W^2(T)$$

d) $F = W^3(T)$

Since

$$D_t W^3(T) = D_t \left(\int_0^T 1\, dW(s) \right)^3 = 3W^2(T)$$

we have that

$$\begin{aligned} F &= E[W^3(T)] + \int_0^T E[3W^2(T) | \mathcal{F}_t]\, dW(t) \\ &= 0 + 3\int_0^T E[(W(T) - W(t))^2 + 2W(T)W(t) - W^2(t) | \mathcal{F}_t]\, dW(t) \\ &= 3\int_0^T (T - t)\, dW(t) + 6\int_0^T W^2(t)\, dW(t) - 3\int_0^T W^2(t)\, dW(t) \\ &= 3\int_0^T W(t)\, dt - 3\int_0^T W^2(t)\, dW(t) = W^3(T) \end{aligned}$$

Øksendal Ex. 3.2

e) $F = \exp(W(T))$

Notice that we have a GBM of the form

$$X_t = X_0 \exp\left[(\mu - 1/2\,\sigma^2)t + \sigma W(t)\right]$$

with $\mu = \frac{1}{2}\sigma^2$ and $\sigma = 1$. Thus, $\mu = 1/2$ and

$$\mathbb{E}\left[\exp(W(T))\right] = e^{1/2T}$$

By the chain rule,

$$D_t e^{W(T)} = e^{W(T)} D_t W(T) = e^{W(T)}$$

Hence,

$$F = e^{1/2T} + \int_0^T \mathbb{E}\left[e^{W(T)} \mid \mathcal{F}_t\right] dW(t)$$

$$= e^{1/2T} + \int_0^T \mathbb{E}\left[e^{W(T) - 1/2T} e^{1/2T} \mid \mathcal{F}_t\right] dW(t)$$

$$= e^{1/2T} + e^{1/2T} \int_0^T \mathbb{E}\left[\exp(W(T) - 1/2T) \mid \mathcal{F}_t\right] dW(t)$$

martingale

$$= e^{1/2T} + e^{1/2T} \int_0^T \exp(W(t) - 1/2t)\, dW(t)$$

To compute this integral, we apply Itô's formula to

$$M_t = \exp(W(t) - t/2)$$

which gives
$$dM_t = M_t\, dW_t$$

Thus,
$$\int_0^T M_t\, dW_t = M(T) - M(0)$$

and
$$F = e^{T/2}\left(1 + \exp(W(T) - T/2) - 1\right)$$
$$= \exp(W(T)) = F$$

f) $(W(T)+T)\exp(-W(T) - 1/2\,T)$

We have an GBM with
- $X_0 = W(T) + T$
- $\sigma = -1$
- $\mu - \frac{1}{2}\sigma^2 = -\frac{1}{2} \Leftrightarrow \mu = 0$

Thus, $E[F] = E[X_0]e^{\mu t} = 0$

By the product and chain rules,

$$D_t F = D_t(W(T)+T)\exp(-W(T) - T/2) + (W(T)+T)D_t \exp(-W(T) - T/2)$$
$$= \exp(-W(T) - T/2) - (W(T) + T)\exp(-W(T) - T/2)$$
$$= (1 - W(T) - T)\exp(-W(T) - T/2)$$

Now let

$$Y(t) = (W(t) + t)\exp(-W(t) - t/2)$$

By Itô's formula,

$$dY(t) = \left(1 - \frac{1}{2}(W(t) + t)\right)\exp(-W(t) - t/2)\,dt$$

$$+ \left(1 - (W(t) + t)\right)\exp(-W(t) - t/2)\,dW(t)$$

$$+ \frac{1}{2}\left(-1 - (1 - W(t) - t)\right)\exp(-W(t) - t/2)\,dt$$

$$= \left(1 - (W(t) + t)\right)\exp(-W(t) - t/2)\,dW(t)$$

whence Y_t is a martingale. Hence,

$$F = \int_0^T \mathbb{E}\left[(1 - W(T) - T)\exp(-W(T) - T/2) \mid \mathcal{F}_t\right] dW(t)$$

$$= \int_0^T (1 - W(t) - t)\exp(-W(t) - t/2)\,dW(t)$$

$$= \int_0^T dY(t) = Y(T) - Y(0)$$

$$= (W(T) + T)\exp(-W(T) - 1/2\,T) = F$$