Note: this is Problem 2.1. in the corrected printing.

Problem 2.2. Let $u(t), 0 \le t \le T$, be a measurable stochastic process such that

$$E\left[\int_0^T u^2(t)dt\right] < \infty.$$

Show that there exists a sequence of deterministic measurable kernels $f_n(t_1, ..., t_n, t)$ on $[0, T]^{n+1}$ $(n \ge 0)$, with

$$\int_{[0,T]^{n+1}} f_n^2(t_1, ..., t_n, t)dt_1...dt_n dt < \infty$$

such that all f_n are symmetric with respect to the variables $t_1, ..., t_n$ and such that

$$u(t) = u(\omega, t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))(\omega), \qquad \omega \in \Omega, t \in [0, T],$$

with convergence in $L^2(P \times \lambda)$. [*Hint.* Consider approximations of $u(t)$, $t \in [0, T]$, in $L^2(P\times\lambda)$ of the form $\sum_{i=1}^m a_i(\omega)b_i(t)$, $m = 1, 2, ...$, where $a_i \in L^2(P)$ and $b_i \in L^2([0, T])$.]

Consider the approximations of $u(t)$

$$\varphi_m(t) = \sum_{i=1}^{m} a_i(\omega) b_i(t)$$

with $a_i \in L^2(P)$ and $b_i \in L^2([0,T])$.

By the Wiener-Itô Chaos Expansion, for each a_i there exists a unique sequence of functions $g_n^{(i)} \in \tilde{L}^2([0,T]^n)$ such that

$$a_i = \sum_{n=0}^{\infty} I_n(g_n^{(i)})$$

Then

$$\varphi_m(t) = \sum_{i=1}^{m}\sum_{n=0}^{\infty} b_i(t) I_n(g_n^{(i)})$$

$$= \sum_{i=1}^{m}\sum_{n=0}^{\infty} b_i(t) \int_{[0,T]^n} g_n^{(i)}(t_1, \dots, t_n)\, dW(t_1)\cdots dW(t_n)$$

$$= \sum_{n=0}^{\infty} \int_{[0,T]^n} \sum_{i=1}^{m} b_i(t)\, g_n^{(i)}(t_1, \dots, t_n)\, dW(t_1)\cdots dW(t_n)$$

Taking $m \to \infty$, $\varphi_m \to u$. We define

$$f_n(t_1, \dots, t_n, t) = \sum_{i=1}^{\infty} b_i(t)\, g_n^{(i)}(t_1, \dots, t_n)$$

and obtain

$$u(t) = \sum_{n=0}^{\infty} \int_{[0,T]^n} f_n(t_1, \dots, t_n, t)\, dW(t_1) \cdots dW(t_n)$$

$$= \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$$