Problem 3.3. (*)

(a) Find the Malliavin derivative $D_t F$, when

$$F = e^G \quad \text{with} \quad G = \int_0^T g(s)dW(s), \quad g \in L^2([0,T]),$$

by using that $F = \sum_{n=0}^{\infty} I_n[f_n]$, with

$$f_n(t_1, \dots, t_n) = \frac{1}{n!} \exp\left\{\frac{1}{2}\|g\|^2_{L^2([0,T])}\right\} g(t_1) \dots g(t_n)$$

(see Problem 1.1 and Problem 1.3 (d)).

Using the definition, we have that

$$D_t F = \sum_{n \geq 1}^{\infty} n I_{n-1}[f_n(\cdot, t)]$$

$$= \sum_{n \geq 1}^{\infty} I_{n-1}\left[\frac{1}{(n-1)!} \exp\left(\frac{1}{2}\|g\|^2_{L^2([0,T])}\right) g(t_1) \cdots g(t_{n-1}) g(t)\right]$$

$$= \sum_{n=0}^{\infty} I_n\left[\frac{1}{n!} \exp\left(\frac{1}{2}\|g\|^2_{L^2([0,T])}\right) g(t_1) \cdots g(t_n)\right] g(t)$$

(b) Verify that the result in (a) can be expressed in terms of the chain rule: $D_t e^G = e^G D_t G$.

Since

$$D_t G = D_t \int_0^T g(s)\, dW(s) = g(t)$$

we have

$$e^G D_t G = \sum_{n=0}^{\infty} I_n\left[\frac{1}{n!} \exp\left(\frac{1}{2}\|g\|^2_{L^2([0,T])}\right) g(t_1) \cdots g(t_n)\right] g(t)$$

$$= D_t F$$

(c) Find the Malliavin derivative of $F = e^G$ with $G = W(t_0)$, for a given $t_0 \in [0, T]$.

For $G = W(t_0)$, we have

$$G = \int_0^T \chi_{[0,t_0]}(s)\, dW(s)$$

and

$$D_t G = D_t \int_0^T \chi_{[0,t_0]}(s)\, dW(s) = \chi_{[0,t_0]}(t)$$

By the problem 1.3.d, its Chaos Expansion is

$$e^G = \sum_{n=0}^{\infty} I_n\left[\frac{1}{n!} \exp\left(\frac{1}{2} \|\chi_{[0,t_0]}\|^2_{L^2([0,T])} \right) \chi_{[0,t_0]}(t_1) \cdots \chi_{[0,t_0]}(t_n) \right]$$

Using that

$$\|\chi_{[0,t_0]}\|^2_{L^2([0,T])} = \int_0^T \chi^2_{[0,t_0]}(s)\, ds = \int_0^{t_0} ds = t_0$$

we obtain

$$D_t F = e^G D_t G = \sum_{n=0}^{\infty} I_n\left[\frac{1}{n!} e^{t_0/2} \chi_{[0,t_0]}(t_1) \cdots \chi_{[0,t_0]}(t_n) \right] \chi_{[0,t_0]}(t)$$