Problem 5.5. (*) Use the identity (5.28) and Wick calculus to compute the following Skorohod integrals:

(a) $\int_0^T W(T)\delta W(t)$;

$$\int_{\mathbb{R}} Y(t)\delta W(t) = \int_{\mathbb{R}} Y(t) \diamond \overset{\bullet}{W}(t)dt \tag{5.28}$$

Using (5.28),

$$\int_0^T W(T)\,\delta W(t) = \int_0^T W(T) \diamond \dot{W}(t)\,dt$$

$$= W(T) \diamond \int_0^T \dot{W}(t)\,dt = W(T) \diamond W(T) = W^2(T) - T$$

where the last equality is (5.37).

(b) $\int_0^T \int_0^T g(s)dW(s)\delta W(t)$, for the deterministic function $g \in L^2([0,T])$;

$$\int_0^T \int_0^T g(s)\,dW(s)\,\delta W(t) = \int_0^T \left(\int_0^T g(s)\,dW(s)\right) \diamond \dot{W}(t)\,dt$$

$$= \int_0^T g(s)\,dW(s) \diamond W(T) = W(T)\int_0^T g(s)\,dW(s) - \int_0^T g(s)\,ds$$

by (5.62).

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed;

Notice that by (5.65),

$$W^{\diamond 2}(t_0) = t_0 h_2\left(\frac{W(t_0)}{\sqrt{t_0}}\right) = t_0\left(\frac{W^2(t_0)}{t_0} - 1\right) = W^2(t_0) - t_0$$

Thus,

$$\int_0^T W^2(t_0)\,\delta W(t) = \int_0^T (W^{\diamond 2}(t_0) + t_0)\,\delta W(t)$$

$$= \int_0^T (W^{\diamond 2}(t_0) + t_0) \diamond \dot{W}(T)\,dt = (W^2(t_0) + t_0) \diamond W(T)$$

$$= W^{\diamond 2}(t_0) \diamond W(T) + t_0 W(T)$$

$$= W^{\diamond 2}(t_0) \diamond (W(T) - W(t_0)) + W^{\diamond 2}(t_0) \diamond W(t_0) + t_0 W(T)$$

$$= W^{\diamond 2}(t_0)(W(T) - W(t_0)) + W^{\diamond 3}(t_0) + t_0 W(T) \qquad (5.40)$$

Using that $W^{\diamond 2}(t) = W^2(t) - t$ (5.37) in the first term, and that by (5.65),

$$W^{\diamond 3}(t) = t^{3/2} h_3\left(\frac{W(t)}{\sqrt{t}}\right) = t^{3/2}\left(\frac{W^3(t)}{t^{3/2}} - \frac{3W(t)}{t^{1/2}}\right)$$

$$= W^3(t) - 3tW(t)$$

we have

$$(W^2(t_0)-t_0)(W(T)-W(t_0))+W^3(t_0)-3t_0W(t_0)+t_0W(T)$$

$$= W^2(t_0)\,W(T)-2t_0W(t_0)$$

(d) $\int\limits_0^T \exp(W(T))\delta W(t)$.

Compare with your calculations in Problem 2.4!

$$\int_0^T \exp(W(T))\,\delta W(t) = \int_0^T \exp(W(T)) \diamond \dot{W}(t)\,dt$$

$$= \exp W(T) \diamond W(T) \overset{(5.68)}{=} \exp^{\diamond}(W(T)+T/2) \diamond W(T)$$

$$= \sum_{k=0}^{\infty} \frac{(W(T)+T/2)^{\diamond k}}{k!} \diamond W(T) = e^{T/2} \sum_{k=0}^{\infty} \frac{W(T)^{\diamond k+1}}{k!}$$

$$\overset{(5.65)}{=} e^{T/2} \sum_{k=0}^{\infty} \frac{T^{\frac{k+1}{2}}}{k!} h_k\left(\frac{W(T)}{\sqrt{T}}\right)$$