Problem 3.2. (*) Find the Malliavin derivative $D_t F$ of the following random variables:

(a) $F = W(T)$.

Since the Chaos Expansion of F is $F = I_1(1)$, and

(1) $$D_t \int_0^T f(s)\, dW(s) = f(t)$$

we have that

$$D_t F = D_t \int_0^T 1 \, dW(s) = 1$$

(b) $F = \int\limits_0^T s^2 dW(s)$.

Using (1) again,

$$D_t F = D \int_0^T s^2 \, dW(s) = s^2$$

(c) $F = \int\limits_0^T \int\limits_0^{t_2} \cos(t_1 + t_2) dW(t_1) dW(t_2)$.

By definition,

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t))$$

Since $F = J_2(\cos(t_1 + t_2))$, we have that

$$D_t F = D_t J_2(\cos(t_1+t_2)) = D_t \frac{1}{2} I_2(\cos(t_1+t_2))$$

$$= \frac{1}{2} 2 I_1(\cos(t_1+t_2)) = \int_0^T \cos(t_1+t_2) \, dW(t_1)$$

(d) $F = 3W(s_0)W^2(t_0) + \log(1 + W^2(s_0))$, for given $s_0, t_0 \in [0, T]$.

If $t \in [0, s_0]$, we can take $g(t) = 3tW^2(t_0) + \log(1+t^2)$. Then

$$g'(t) = 3W^2(t_0) + \frac{2t}{1+t^2} \text{ is bounded}$$

By the Chain Rule, $D_t g(G) = g'(G) D_t G$. Since $F = g(W(s_0))$ and

$$D_t W(s_0) = D_t \int_0^T \chi_{[0,s_0]}(s)\, dW(s) = \chi_{[0,s_0]}(t)$$

we have

$$D_t F = \left[3W^2(t_0) + \frac{2W(s_0)}{1+W^2(s_0)} \right] \chi_{[0,s_0]}(t)$$

Now, if $t \in [0, t_0]$, let $g(t) = 3W(s_0)t^2 + \log(1 + W^2(s_0))$.

Then $g'(t) = 6W(s_0)t$ is bounded, $F = g(W(t_0))$, and

$$D_t W(t_0) = D_t \int_0^T \chi_{[0,t_0]}(s)\, dW(s) = \chi_{[0,t_0]}(t)$$

Thus,

$$D_t F = 6W(s_0)W(t_0)\chi_{[0,t_0]}(t)$$

Hence, in the general case,

$$D_t F = \left[3W^2(t_0) + \frac{2W(s_0)}{1+W^2(s_0)} \right] \chi_{[0,s_0]}(t) + 6W(s_0)W(t_0)\chi_{[0,t_0]}(t)$$

(e) $F = \int_0^T W(t_0)\delta W(t)$, for a given $t_0 \in [0, T]$. [*Hint*. Use Problem 2.5 (b).]

By the Problem 2.5b.,

$$\boxed{\int_0^T \left(\int_0^T g(s)\, dW(s) \right) \delta W(t) = \left(\int_0^T g(t_1)\, dW(t_1) \right) W(T) - \int_0^t g(t)\, dt}$$

Then

$$F = \int_0^T \int_0^T \chi_{[0,t_0]}(s)\, \delta W(s)\, \delta W(t) = W(T) \int_0^T \chi_{[0,t_0]}(s)\, dW(s) - \int_0^T \chi_{[0,t_0]}(t)\, dt$$

Using Integration by Parts,

$$F = \int_0^T W(T)\chi_{[0,t_0]}(s)\, dW(s)$$

and by the Fundamental Theorem of Calculus,

$$D_t F = \int_0^T D_t W(T)\chi_{[0,t_0]}(s)\, \delta W(s) + W(T)\chi_{[0,t_0]}(t)$$

$$= W(t_0) + W(T)\chi_{[0,t_0]}(t)$$