Problem 6.2. The Wick chain rule. Let F be Malliavin differentiable and let $n \in \mathbb{N}$. Show that

$$D_t(F^{\diamond n}) = nF^{\diamond(n-1)} \diamond D_t F.$$

Write $F^{\diamond n}(\omega) = \langle \omega, f \rangle^{\diamond n}$ and suppose that $\|f\|_{L^2(\mathbb{R})} = 1$.
Since $\langle \omega, f \rangle^{\diamond n} = h_n(\langle \omega, f \rangle)$, we have

$$\frac{1}{\varepsilon}\left[F^{\diamond n}(\omega + \varepsilon\gamma) - F^{\diamond n}(\omega)\right] = \frac{1}{\varepsilon}\left[\langle \omega + \varepsilon\gamma, f\rangle^{\diamond n} - \langle \omega, f\rangle^{\diamond n}\right]$$

$$= \frac{1}{\varepsilon}\left[h_n\langle \omega + \varepsilon\gamma, f\rangle - h_n\langle \omega, f\rangle\right]$$

$$= \frac{1}{\varepsilon}\left[h_n(\langle \omega, f\rangle + \varepsilon\langle \gamma, f\rangle) - h_n\langle \omega, f\rangle\right]$$

Taking the limit as $\varepsilon \to 0$,

$$D_t(F^{\diamond n}) = h_n'(\langle \omega, f\rangle)\langle \gamma, f\rangle = n h_{n-1}(\langle \omega, f\rangle)\langle \gamma, f\rangle$$

$$= nF^{\diamond(n-1)}(\omega) \diamond D_t F$$

By induction on n. For $n=1,2$, it is immediate, since

$$D_+(F\diamond G) = F\diamond D_+G + G\diamond D_+F$$

Suppose that the result holds for $n-1$. Then,

$$D_+(F^{\diamond n}) = D_+(F^{\diamond(n-1)}\diamond F) = F^{\diamond(n-1)}\diamond D_+F + F\diamond D_+F^{\diamond(n-1)}$$

$$= F^{\diamond(n-1)}\diamond D_+F + F\diamond\left((n-1)\cdot F^{\diamond(n-2)}\diamond D_+F\right)$$

$$= F^{\diamond(n-1)}\diamond D_+F + (n-1)F^{\diamond(n-1)}\diamond D_+F$$

$$= nF^{\diamond(n-1)}\diamond D_+F$$