Problem 3.6. Let $u = u(t)$, $t \in [0,T]$, be a stochastic process such that

$$E\Big[\int_0^T u^2(t)dt\Big] < \infty.$$

Suppose that there exists a constant K such that

$$\Big|E\Big[\int_0^T D_t F u(t)dt\Big]\Big| \le K\|F\|_{L^2(P)}, \qquad \text{for all } F \in \mathbb{D}_{1,2}.$$

Show that u is Skorohod integrable.

Note that we cannot apply the Duality Formula. Let

$$u(t) = \sum_{k=0}^{\infty} I_k(g_k(\cdot,t)), \qquad F = \sum_{n=0}^{\infty} I_n(f_n)$$

be the Chaos Expansions of $u(t)$ and F, respectively.

Then

$$E\Big[\int_0^T u(t) D_t F\,dt\Big] = E\Big[\int_0^T \Big(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\Big)\Big(\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot,t))\Big)dt\Big]$$

$$= \int_0^T \sum_{k=0}^{\infty} E\big[(k+1) I_k(f_{k+1}(\cdot,t)) I_k(g_k(\cdot,t))\big]dt$$

for all $g \in \widetilde{L}^2([0,T]^n)$. Moreover, if $g \in \widetilde{L}^2([0,T]^m)$ and $h \in \widetilde{L}^2([0,T]^n)$, we have

$$E[I_m(g)I_n(h)] = \begin{cases} 0 & , n \neq m \\ (g,h)_{L^2([0,T]^n)} & , n = m \end{cases} \qquad (m,n = 1,2,\ldots),$$

with $(g,h)_{L^2([0,T]^n)} = n!(g,h)_{L^2(S_n)}$.

$$= \int_0^T \sum_{k=0}^{\infty} (k+1)k!\,\langle f_{k+1}(\cdot,t), g_k(\cdot,t)\rangle_{L^2([0,T]^k)}\,dt$$

$$= \sum_{k=0}^{\infty} (k+1)!\,\langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}$$

Hence, we have that

(1) $$\Big|\sum_{k=0}^{\infty} (k+1)!\,\langle f_{k+1}, g_k\rangle_{L^2([0,T]^{k+1})}\Big| = \Big|E\Big[\int_0^T u(t) D_t F\,dt\Big]\Big| \le K\|F\|_{L^2(P)}$$

We'll use the following remark:

Remark 2.3. By (1.17) a stochastic process u belongs to $Dom(\delta)$ if and only if

$$E[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty \,. \tag{2.4}$$

If we have that

$$\mathbb{E}\left[\delta(u)^2\right] \leq \left| \sum_{k=0}^{\infty} (k+1)! \langle f_{k+1}, g_k \rangle_{L^2([0,T]^{k+1})} \right| \leq K \|F\|_{L^2(\mathbb{P})} < \infty$$

we finish the proof.

Take F such that $f_{k+1} = g_k$, i.e.,

$$F = \sum_{k=1}^{\infty} I_{k+1}(g_{k-1}) \in \mathbb{D}_{1,2} \text{ by the restriction on } u(t) \; ?$$

Then we can write (1) as

$$\mathbb{E}\left[\delta(u)^2\right] = \sum_{k=0}^{\infty} (k+1)! \|\tilde{g}_k\|^2_{L^2([0,T]^{k+1})} \leq K \|F\|_{L^2(\mathbb{P})}$$

where we used that $\langle f_{k+1}, \tilde{g}_k \rangle_{L^2([0,T]^{k+1})} = \langle f_{k+1}, g_k \rangle_{L^2([0,T]^{k+1})}$ (see the proof of Thm. 3.14).