Problem 4.3. (*) Let $\widetilde{W}(t) = \int_0^t u(s)ds + W(t)$ and Q be as in Exercise 4.1. Use the generalized Clark–Ocone formula to find the $\mathbb{F}$-adapted process $\widetilde{\varphi}$, such that

$$F = E_Q[F] + \int_0^T \widetilde{\varphi}(t)d\widetilde{W}(t)$$

in the following cases:

(a) $F = W^2(T)$ and $u(t)$, $t \in [0,T]$, is deterministic.

(b) $F = \exp\left\{\int_0^T \lambda(t)dW(t)\right\}$ and the processes $\lambda(t)$ and $u(t)$, $t \in [0,T]$, are deterministic.

(c) F is like in (b) and $u(t) = W(t)$, $t \in [0,T]$.

Theorem 4.5. The Clark–Ocone formula under change of measure. *Suppose $F \in \mathbb{D}_{1,2}$ is $\mathcal{F}_T$-measurable and that*

$$E_Q[|F|] < \infty \tag{4.5}$$

$$E_Q[\int_0^T |D_tF|^2 dt] < \infty \tag{4.6}$$

$$E_Q\Big[|F| \int_0^T \Big(\int_0^T D_t u(s)dW(s) + \int_0^T u(s)D_t u(s)ds\Big)^2 dt\Big] < \infty. \tag{4.7}$$

Then

$$F = E_Q[F] + \int_0^T E_Q\Big[(D_tF - F\int_t^T D_t u(s)d\widetilde{W}(s))|\mathcal{F}_t\Big]d\widetilde{W}(t). \tag{4.8}$$

a) $F = W^2(T)$, $u(t)$ deterministic

Our goal is to find

$$\mathbb{E}_Q\left[D_tF - F\int_t^T D_t u(s)\, d\tilde{W}(s) \,\Big|\, \mathcal{F}_t\right] = \tilde{\varphi}(t)$$

Now, since $D_t W^2(T) = 2W(T)$ and $D_t u(s) = 0$ (deterministic),

$$\tilde{\varphi}(t) = \mathbb{E}_Q[2W(T)\,|\,\mathcal{F}_t] = 2\mathbb{E}_Q\left[\tilde{W}(T) - \int_0^T u(s)\,ds \,\Big|\, \mathcal{F}_t\right]$$

$$= 2\left(\tilde{W}(t) - \int_0^T u(s)\,ds\right)$$

b) $F = \exp\left(\int_0^T \lambda(t)\,dW(t)\right)$, $\lambda(t), u(t)$ deterministic

Again, $D_t u(s) = 0$. Let us compute D_tF. Let

$$g(x) = e^x, \quad G = \int_0^T \lambda(t)\,dW(t), \quad F = g(G)$$

By the Chain Rule,

$$D_t F = g'(G) D_t G = F D_t\left(\int_0^T \lambda(s)\, dW(s)\right) = \lambda(t) F$$

Thus,

$$\tilde{\varphi}(t) = \mathbb{E}_Q\left[\lambda(t) \exp\left(\int_0^T \lambda(s)\, dW(s)\right) \middle| \mathcal{F}_t\right]$$

$$= \mathbb{E}_Q\left[\lambda(t) \exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) - \int_0^T \lambda(s) u(s)\, ds\right) \middle| \mathcal{F}_t\right]$$

$$= \mathbb{E}_Q\left[\lambda(t) \exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) + \frac{1}{2}\int_0^T \lambda^2(s)\, ds - \frac{1}{2}\int_0^T \lambda^2(s)\, ds - \int_0^T \lambda(s) u(s)\, ds\right) \middle| \mathcal{F}_t\right]$$

$$= \lambda(t) \exp\left(\int_0^T \frac{1}{2}\lambda^2(s) - \lambda(s) u(s)\, ds\right) \mathbb{E}_Q\left[\exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) - \frac{1}{2}\int_0^T \lambda^2(s)\, ds\right) \middle| \mathcal{F}_t\right]$$

$$= \lambda(t) \exp\left(\int_0^T \frac{1}{2}\lambda^2(s) - \lambda(s) u(s)\, ds\right) \exp\left(\int_0^t \lambda(s)\, d\tilde{W}(s) - \frac{1}{2}\int_0^t \lambda^2(s)\, ds\right)$$

c) $F = \exp\left(\int_0^T \lambda(t)\, dW(t)\right)$, $\lambda(t)$ deterministic, $u(t) = W(t)$

Now $D_t u(s) = D_t W(s) = 1$ and thus

$$\tilde{\varphi}(t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\int_0^T \lambda(s)\, dW(s)\right)\left(\lambda(t) - \int_t^T 1\, d\tilde{W}(s)\right) \Big| \mathcal{F}_t\right]$$

$$= \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\lambda(t) F \mid \mathcal{F}_t\right]}_{A} - \underbrace{\mathbb{E}_{\mathbb{Q}}\left[F \int_t^T 1\, d\tilde{W}(s)) \Big| \mathcal{F}_t\right]}_{B}$$

Notice that, since $u(t) = W(t)$,

$$\tilde{W}(t) = W(t) + \int_0^t W(s)\, ds \Rightarrow d\tilde{W}(t) = dW(t) + W(t)\, dt$$

Multiplying by e^t,

$$d(e^t W(t)) = e^t W(t)\, dt + e^t\, dW(t) = e^t\, d\tilde{W}(t)$$

i.e.,

$$W(t) = e^{-t} \int_0^t e^s\, d\tilde{W}(s)$$

Differentiating,

$$dW(t) = -t e^{-t} \int_0^t e^s\, d\tilde{W}(s)\, dt + e^{-t} e^t\, d\tilde{W}(t)$$

$$= -t e^{-t} \int_0^t e^s\, d\tilde{W}(s)\, dt + d\tilde{W}(t)$$

With this expression, we can write

$$F = \exp\left(\int_0^T \lambda(s)\, dW(s)\right)$$

$$= \exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) - \int_0^T \lambda(u) e^{-u} \int_0^u e^{s}\, d\tilde{W}(s)\, du\right)$$

$$= \exp\left(\int_0^T \lambda(s)\, d\tilde{W}(s) - \int_0^T \int_0^T \lambda(u) e^{-u} du\ e^{-s}\, d\tilde{W}(s)\right)$$

$$= K(T) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\, ds\right)$$

where

$$\zeta(s) = \lambda(s) - e^{s}\int_0^T \lambda(u) e^{-u}\, du$$

and

$$K(t) = \exp\left(\int_0^t \zeta(s)\, d\tilde{W}(s) - \frac{1}{2}\int_0^t \zeta^2(s)\, ds\right)$$

Now we can compute

$$A = \mathbb{E}_Q\left[\lambda(t) F \mid \mathcal{F}_t\right] = \lambda(t) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\, ds\right) \mathbb{E}_Q\left[K(T) \mid \mathcal{F}_t\right]$$

(1) $$= \lambda(t) \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\, ds\right) K(t)$$

and

$$B = \mathbb{E}_Q\left[F \int_t^T 1 \, d\tilde{w}(s)\bigg) \bigg| \mathcal{F}_t\right] = \mathbb{E}_Q[F(\tilde{w}(T) - \tilde{w}(t)) \mid \mathcal{F}_t]$$

$$= \underbrace{\exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right)}_{H} \mathbb{E}_Q[K(T)(\tilde{w}(T) - \tilde{w}(t)) \mid \mathcal{F}_t]$$

$$= HK(t)\mathbb{E}_Q\left[\exp\left(\int_t^T \zeta(s)\,d\tilde{w}(s) - \frac{1}{2}\int_t^T \zeta^2(s)\,ds\right)(\tilde{w}(T) - \tilde{w}(t)) \bigg| \mathcal{F}_t\right]$$

(2) $$= HK(t)\mathbb{E}\left[\exp\left(\int_t^T \zeta(s)\,dw(s) - \frac{1}{2}\int_t^T \zeta^2(s)\,ds\right)(w(T) - w(t)) \bigg| \mathcal{F}_t\right]$$

To compute this expectation, let

$$X_t = \exp\left(\int_{t_0}^t \zeta(s)\,dw(s) - \frac{1}{2}\int_{t_0}^t \zeta^2(s)\,ds\right)$$

and $Y_t = X_t(w(t) - w(t_0))$. Then,

$$dY_t = X_t\,dw(t) + (w(t) - w(t_0))\,dX_t + dX_t\,dw_t$$

$$= X_t(1 + (w(t) - w(t_0))\zeta(t))\,dw_t + \zeta(t)X_t\,dt$$

Thus,

$$\mathbb{E}[Y(T)] = \mathbb{E}[Y(t_0)] + \mathbb{E}\left[\int_{t_0}^T \zeta(s)\,X(s)\,ds\right]$$

$$= \int_{t_0}^T \zeta(s)\mathbb{E}[X(s)]\,ds = \int_{t_0}^T \zeta(s)\,ds$$

Hence,

$$\tilde{\varphi}(t) = \lambda(t)\exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right)K(t) - \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right)K(t)\int_{t_0}^T \zeta(s)\,ds$$

$$= \exp\left(\frac{1}{2}\int_0^T \zeta^2(s)\,ds\right)\exp\left(\int_0^t \zeta(s)\,d\tilde{W}(s) - \frac{1}{2}\int_0^t \zeta^2(s)\,ds\right)\left(\lambda(t) - \int_t^T \zeta(s)\,ds\right)$$