TOPOLOGY OF METRIC SPACES

Adair Antonio da Silva Neto

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Contents

Chapter 1

Metrics

1.1 Basic Definitions

What is a metric space? It is simply a set equipped with a notion of distance.

Definition 1.1.1 (Metric Space)**.** Given a non-empty set M, a **metric** over M is a function

$$\mathrm{d} : \mathrm{M} \times \mathrm{M} \longrightarrow [0, \infty)$$

satisfying, for all $x, y, z \in \mathrm{M}$:

M1. $\mathrm{d}(x,y) = 0 \iff x = y$

M2. $\mathrm{d}(x,y) = \mathrm{d}(y,x)$

M3. $\mathrm{d}(x,y) \leq \mathrm{d}(x,z) + d(z,y)$

Under these conditions, each image $\mathrm{d}(x,y)$ is called the **distance** from x to y, and a set M equipped with a metric d is called **metric space**, which we'll denote by (M, d).

Intuitively, a distance should be positive, only be zero when it is the distance to itself, the 'going' distance should be equal to the 'return' distance, and if you pass somewhere in the way, the distance cannot be lower (triangle inequality).

This metric induces a topology on the set, called **metric topology**. A topological space whose topology comes from some metric is called **metrizable**.

If $\mathrm{S} \subsetneq \mathrm{M}$, then considering the restriction $\mathrm{d}_1 = \mathrm{d}|_{\mathrm{S}}$, we obtain another metric space $(\mathrm{S}, \mathrm{d}_1)$ which is called a **subspace** of M, and the metric d_1 is induced by d over M.

Example 1.1.1 (Discrete (or zero-one) metric)**.** Given any non-empty set M, define

$$\mathrm{d}(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

The space obtained by this metric is called **discrete space**.

Notice that M1. and M2. are immediate. To prove M3., we have two cases. If $x = y$, then

$$d(x,y) = 0 \leq \underbrace{d(x,y)}_{\geq 0} + \underbrace{d(z,y)}_{\geq 0}$$

On the other hand, if $x \neq y$, then either $x \neq z$ or $y \neq z$. Suppose w.l.o.g. $x \neq z$. Then

$$d(x,y) = 1 = d(x,z) \leq \underbrace{d(x,z)}_{\geq 0} + \underbrace{d(z,y)}_{\geq 0}$$

Example 1.1.2 (Real Line)**.** Let $M = \mathbb{R}$ and define $d(x,y) = |x-y|$. This is called the **usual metric on** $\mathbb{R}$.

Before heading on, we'll first need the Cauchy-Schwarz inequality.

Lemma 1.1.1 (Cauchy-Schwarz Inequality)**.** Let $x,y \in \mathbb{R}^n$. Then,

$$\sum_{i=1}^{n} |x_i \cdot y_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \tag{1.1}$$

Proof. If $x_1 = \ldots = x_n = 0$ or $y_1 = \ldots = y_n = 0$, then (1.1) is trivial.

Now notice that, for all $r,s \in \mathbb{R}_{\geq 0}$, we have

$$2rs \leq r^2 + s^2 \tag{1.2}$$

since

$$2rs \leq r^2 + s^2 \iff r^2 - 2rs + s^2 \geq 0 \iff (r-s)^2 \geq 0$$

Fix $l,k \in \mathbf{N}$ satisfying $1 \leq l,k \leq n$, such that x_l and y_k are non-zero. And define

$$p = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \text{ and } q = \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$$

Since both are non-zero, we also define

$$r = \frac{|x_i|}{p} \text{ and } s = \frac{|y_i|}{q}$$

Then, for $1 \leq i \leq n$,

$$2rs = 2\frac{|x_i|}{p}\frac{|y_i|}{q} \leq \frac{|x_i|^2}{p^2} + \frac{|y_i|^2}{q^2}$$

I.e.,

$$2\sum_{i=1}^{n} \frac{|x_i \cdot y_i|}{pq} \leq \sum_{i=1}^{n} \frac{|x_i|^2}{p^2} + \sum_{i=1}^{n} \frac{|y_i|^2}{q^2}$$

Using that $|x_i|^2 = p^2$ and $|y_i|^2 = q^2$, we have

$$\frac{2}{pq}\sum_{i=1}^{n} |x_i \cdot y_i| \leq \frac{p^2}{p^2} + \frac{q^2}{q^2} = 2$$

Hence,

$$\sum_{i=1}^{n} |x_i \cdot y_i| \leq pq$$

□

Example 1.1.3 ($\mathbb{R}^n$ Space)**.** There are three somewhat equivalent metrics on $\mathbb{R}^n$, defined as follows

$$\begin{aligned} \mathrm{d}_1(x,y) &= \sum_{i=1}^{n} |x_i - y_i| \\ \mathrm{d}_2(x,y) &= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \\ \mathrm{d}_\infty(x,y) &= \max_{1 \leq i \leq n} |x_i - y_i| \end{aligned}$$

The metric d_2 is known as the **Euclidean metric**.

Notice that M1. and M2. are verified immediately by these metrics. To show that M3. holds,

$$\begin{aligned} \mathrm{d}_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| &= \sum_{i=1}^{n} |(x_i - z_i) + (z_i - y_i)| \\ &\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| = \mathrm{d}_1(x,z) + \mathrm{d}_1(z,y) \end{aligned}$$

The case for $\mathrm{d}_\infty(x,y)$ is analogous. For $\mathrm{d}_2(x,y)$,

$$\begin{aligned} \mathrm{d}_2^2(x,y) = \sum_{i=1}^{n} (x_i - y_i)^2 &= \sum_{i=1}^{n} [(x_i - z_i) + (z_i - y_i)]^2 \\ &= \sum_{i=1}^{n} [(x_i - z_i)^2 + 2(x_i - z_i)(z_i - y_i) + (z_i - y_i)^2] \\ &\leq \mathrm{d}_2^2(x,y) + \mathrm{d}_2^2(z,y) + 2\sum_{i=1}^{n} |x_i - z_i| \, |z_i - y_i| \\ &= \mathrm{d}_2^2(x,y) + \mathrm{d}_2^2(z,y) + 2\mathrm{d}_2(x,y)\mathrm{d}_2(z,y) \\ &= [\mathrm{d}_2(x,z) + \mathrm{d}_2(z,y)]^2 \end{aligned}$$

Thus,

$$\mathrm{d}_2(x,y) \leq \mathrm{d}_2(x,z) + \mathrm{d}_2(z,y)$$

Lemma 1.1.2 (Relationship between $\mathbb{R}^n$ metrics)**.** For all $x, y \in \mathbb{R}^n$,

$$\mathrm{d}_\infty(x,y) \leq \mathrm{d}_2(x,y) \leq \mathrm{d}_1(x,y) \leq n\mathrm{d}_\infty(x,y)$$

Proof.

$$d_\infty(x,y) = \max_{1\le i\le n} |x_i - y_i| = |x_l - y_l| = \sqrt{(x_l-y_l)^2} \le \sqrt{\sum_{i=1}^n (x_i-y_i)^2} = d_2(x,y)$$

$$d_1^2(x,y) = \left(\sum_{i=1}^n |x_i - y_i|\right)^2 = \sum_{i=1}^n |x_i-y_i|^2 + 2\underbrace{\sum_{1\le j<k\le n} |x_j-y_j|\,|x_k-y_k|}_{\ge 0} \ge d_2^2(x,y)$$

$$d_1(x,y) = \sum_{i=1}^n |x_i-y_i| \le \sum_{i=1}^n \max_{1\le j\le n} |x_j-y_j| = n d_\infty(x,y)$$

□

Proposition 1.1.3 (An additional inequality)**.** Let (M,d) be a metric space and $x,y \in M$. Then

$$|d(x,y) - d(x,z)| \le d(y,z)$$

Proof.

$$|d(x,y)-d(x,z)| \le d(y,z) \iff \begin{cases} d(x,y)-d(x,z) & \le d(y,z) \\ -d(x,y)+d(x,z) & \le d(y,z) \end{cases}$$
$$\iff \begin{cases} d(x,y) \le d(y,z)+d(x,z) \\ d(x,z) \le d(y,z)+d(x,y) \end{cases}$$

which are true since they're both the triangular inequality. □

Definition 1.1.2 (Normed vector spaces)**.** Given a vector space V over $\mathbb{R}$, we say that a function $||\cdot||$ is a **norm** over V if, given $u,v\in V$ and $\lambda\in\mathbb{R}$ we have

N1. $||v|| \ge 0$, and $||v|| = 0 \iff v = 0$.

N2. $||\lambda v|| = |\lambda|\,||v||$.

N3. $||u+v|| \le ||u|| + ||v||$.

Then V has a metric given by $d(u,v) = ||u-v||$, called the **metric induced by the norm** $||\cdot||$. And a vector space equipped with a norm is called a **normed vector space**.

Example 1.1.4 ($\mathbb{R}^n$ as a Normed Vector Space)**.** The space $\mathbb{R}^n$ with any of the norms

$$||x||_1 = \sum_{i=1}^n |x_i|$$
$$||x||_2 = \sqrt{\sum_{i=1}^n x_i^2}$$
$$||x||_\infty = \max_{1\le i\le n} |x_i|$$

is a normed vector space. In fact,

$$\mathrm{d}_1(x,y) = \|x-y\|_1;\ \mathrm{d}_2(x,y) = \|x-y\|_2;\ \mathrm{d}_\infty(x,y) = \|x-y\|_\infty$$

Definition 1.1.3 (Inner Product)**.** Let V be a vector space over $\mathbb{R}$. Then a function $\langle\cdot,\cdot\rangle : \mathrm{V}\times\mathrm{V} \longrightarrow \mathbb{R}$ is called a **inner product** if, for any $a,b,c \in \mathrm{V}$ and $\lambda \in \mathbb{R}$, the following conditions hold:

P1. $\langle \lambda a, b\rangle = \lambda\langle a,b\rangle$;

P2. $\langle a+b, c\rangle = \langle a,c\rangle + \langle b,c\rangle$;

P3. $\langle a,b\rangle = \langle b,a\rangle$;

P4. $\langle a,a\rangle > 0$ if $a \neq 0$.

Lemma 1.1.4 (Cauchy-Schwarz Inequality with Inner Product)**.** Let V be a vector space over $\mathbb{R}$ with inner product $\langle\cdot,\cdot\rangle$. Then, for all $u,v \in \mathrm{V}$,

$$|\langle u,v\rangle| \leq \|u\|\|v\|$$

Proof. To prove this inequality, notice that

$$\begin{aligned} 0 \leq \|u+\lambda v\|^2 &= \langle u+\lambda v, u+\lambda v\rangle \\ &= \langle u,u\rangle + \langle u,\lambda v\rangle + \langle \lambda v, u\rangle + \langle \lambda v, \lambda v\rangle \\ &= \|u\|^2 + 2\lambda\langle u,v\rangle + \lambda^2\|v\|^2 \end{aligned}$$

Since this norm is non-negative,

$$\Delta = 4\langle u,v\rangle^2 - 4\|u\|^2\|v\|^2 \leq 0$$

Hence,

$$\langle u,v\rangle^2 \leq \|u\|^2\|v\|^2 \iff |\langle u,v\rangle| \leq \|u\|\|v\|$$

□

Proposition 1.1.5 (Norm induced by inner product)**.** Given a vector space V with inner product $\langle\cdot,\cdot\rangle$, the function $\|\cdot\| : \mathrm{V} \longrightarrow \mathbb{R}$ defined as $\|a\| = \sqrt{\langle a,a\rangle}$, where $a \in \mathrm{V}$, is a norm, which is called the **norm induced by the inner product.**

Proof. N1. $\|u\| = 0 \iff \sqrt{\langle u,u\rangle} = 0 \iff \langle u,u\rangle = 0 \overset{(P4)}{\iff} u = 0$.

N2. $\|\lambda u\| = \sqrt{\langle \lambda u, \lambda v\rangle} = \sqrt{\lambda^2\langle u,u\rangle} = |\lambda|\|u\|$.

N3.

$$\begin{aligned} \|u+v\|^2 = \langle u+v, u+v\rangle &= \|u\|^2 + \|v\|^2 + 2\langle u,v\rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle| \leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

□

Hence, every vector space equipped with inner product is a normed vector space (the converse is not true) and therefore is also a metric space.

$$\underbrace{\langle\cdot,\cdot\rangle}_{\text{Inner Product}} \xrightarrow{\text{induces}} \underbrace{\sqrt{\langle v,v\rangle}}_{\text{Norm } \|v\|} \xrightarrow{\text{induces}} \underbrace{\|u-v\|}_{\text{Metric } \mathrm{d}(u,v)}$$

Proposition 1.1.6 (Bounded real functions)**.** Given a set $X \neq \varnothing$, a function $f :\longrightarrow \mathbb{R}$ is said to be **bounded** if there exists $k \in \mathbb{R}$ such that $|f(x)| < k$ for all $x \in X$. We'll use $\mathscr{B}(X;\mathbb{R})$ to denote the space of bounded functions from X to $\mathbb{R}$.

For any $f, g \in \mathscr{B}(X;\mathbb{R})$ and $c \in \mathbb{R}$, we define

$$\begin{aligned}(f+g)(x) &= f(x)+g(x),\ \forall x \in X\\ (cf)(x) &= cf(x),\ \forall x \in X\\ \|f\| &= \sup\{|f(x)| : x \in X\}\end{aligned}$$

Then the set $\mathscr{B}(X;\mathbb{R})$ is a normed vector space.

Notice that the metric induced by this norm is the function

$$\mathrm{d}(f,g) = \sup\{|f(x)-g(x)| : x \in X\},\ \forall f,g \in \mathscr{B}(X;\mathbb{R})$$

Proof. N1. $0 = \|f\| = \sup\{|f(x)| : x \in X\} \iff |f(x)| \leq 0 \iff f(x) = 0.$

N2. $\|\lambda f\| = \sup\{|\lambda f(x)| : x \in X\} = |\lambda| \sup\{|f(x)| : x \in X\} = |\lambda|\,\|f\|.$

N3. Since

$$|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \sup\{|f(t)| : t \in X\} + \sup\{|g(t) : t \in X\} = \|f\| + \|g\|$$

we have that

$$\|f\| + \|g\| = \sup\{|f(x)+g(x)| : x \in X\} \leq \sup\{\|f\| + \|g\| : x \in X\} = \|f\| + \|g\|$$

□

Proposition 1.1.7 (Continuous real functions on a closed interval)**.** Denote by $\mathscr{C}[a,b]$ the set of continuous real functions defined on $[a,b]$. With respect to the addition of functions and scalar multiplication as defined in the bdd-real-fun, $\mathscr{C}[a,b]$ is a vector space over $\mathbb{R}$ and the function

$$\|f\| = \int_a^b |f(x)|\ \mathrm{d}x$$

is a norm over this space. And we have the following metric:

$$\mathrm{d}(f,g) = \int_a^b |f(x)-g(x)|\ \mathrm{d}x,\ \forall f,g \in \mathscr{C}[a,b]$$

Proof. We begin by the two easiest ones.

N2. $\|\lambda f\| = \int_a^b |\lambda f(x)|\ \mathrm{d}x = |\lambda| \int_a^b |f(x)|\ \mathrm{d}x = |\lambda|\,\|f\|.$

N3. $\|f+g\| = \int_a^b |f(x)-g(x)|\ \mathrm{d}x \leq \int_a^b (|f(x)| + |g(x)|)\ \mathrm{d}x = \|f\| + \|g\|.$

N1. If f is zero, then the property follows trivially. Therefore, suppose that f is non-zero, i.e., there exists $x_0 \in [a,b]$ such that $f(x_0) \neq 0$.

Taking $\varepsilon = |f(x_0)|/2$, by the continuity of f in x_0, there exists $\delta > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \implies |f(x) - f(x_0)| < \varepsilon = \frac{|f(x_0)|}{2}$$

If $x \in [a,b] \cap (x_0 - \delta, x_0 + \delta) = [c,d]$, then

$$|f(x)| = |f(x_0) + (f(x) - f(x_0))| \geq |f(x_0)| - |f(x) - f(x_0)| \geq |f(x_0)| - \frac{|f(x_0)|}{2} = \frac{|f(x_0)|}{2}$$

Hence, for $x \in [c,d]$,

$$\|f\| = \int_a^b |f(x)| \, \mathrm{d}x \geq \int_c^d |f(x)| \, \mathrm{d}x \geq \int_c^d \frac{|f(x_0)|}{2} \, \mathrm{d}x = \frac{|f(x_0)|}{2}(d-c) > 0$$

□

Remark that since every continuous real function is bounded, we have that $\mathscr{C}[a,b]$ is a subset of $\mathscr{B}([a,b];\mathbb{R})$. Then $\mathscr{C}[a,b]$ is also a metric space with respect to the metric

$$\mathrm{d}(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}, \ \forall f,g, \in \mathscr{C}[a,b]$$

Example 1.1.5 (Product Metric). Let $(\mathrm{M}_1, \mathrm{d}_1), \ldots, (\mathrm{M}_n, \mathrm{d}_n)$ be metric spaces. Then we define over $\mathrm{M}_1 \times \cdot \times \mathrm{M}_n$ the **product metric** as

$$\begin{aligned} \mathrm{D}_1(x,y) &= \sum_{i=1}^n \mathrm{d}_i(x_i,y_i) \\ \mathrm{D}_2(x,y) &= \sqrt{\sum_{i=1}^n \mathrm{d}_i^2(x_i,y_i)} \\ \mathrm{D}_\infty(x,y) &= \max_{1 \leq i \leq n} \mathrm{d}_i(x_i,y_i) \end{aligned}$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathrm{M}$.

Exercise. Show that $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_\infty$ are metrics over M.

1.2 Distance between points and sets, Distance between two sets

Recall from elementary geometry that the distance from a point p to a plane π is the measure of the segment pq, where q is the intersection of π and the line passing through p and orthogonal to π.

Definition 1.2.1 (Distance between points and sets). Let (M, d) be a metric space. Given $p \in \mathrm{M}$ and $\mathrm{A} \subset \mathrm{M}$, $\mathrm{A} \neq \varnothing$, we define the **distance** from p to the set A, which we denote by $\mathrm{d}(p,\mathrm{A})$, as the following non-negative real number

$$\mathrm{d}(p,\mathrm{A}) = \inf\{\mathrm{d}(p,x) : x \in \mathrm{A}\}$$

Note that this distance exists, given the fact that the set $d(p,x)$ is lower bounded by zero.

Example 1.2.1. Consider the usual metric over $\mathbb{R}$. If $p = 0$ and $A = \{1, 1/2, 1/3, \ldots\}$, then $d(p,A) = 0$.

This is an ellucidative example because it shows that it is possible to have $d(p,A) = 0$ and $p \notin A$. However, if $p \in A$, then $d(p,A)$.

Proposition 1.2.1. Let (M,d) be a metric space. If $A \subset M$, $A \neq \varnothing$, and $p,q \in M$, then

$$|d(p,A) - d(q,A)| \leq d(p,q)$$

Proof. Let $x \in A$. Then

$$d(p,A) \leq d(p,x) \leq d(p,q) + d(q,x)$$

I.e., $d(p,x) - d(p,q) \leq d(q,x)$.

Since this is valid for every $x \in A$, we obtain that $d(p,x) - d(p,q) \leq d(q,A)$

Hence,

$$d(p,x) - d(q,A) \leq d(p,q)$$

□

Definition 1.2.2 (Distance between sets). Let (M,d) be a metric space. Given two non-empty subsets A, B of M, we define the **distance** between A and B, denoted by $d(A,B)$, as the non-negative real number

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$

Example 1.2.2. Consider $\mathbb{R}^2$ equipped with Euclidean metric. The distance between $A = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ and $B = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$ is zero.

Note that $A \cap B \neq \varnothing$ implies $d(A,B) = 0$. Nonetheless, it is possible to have $d(A,B) = 0$ with $A \cap B = \varnothing$.

Definition 1.2.3 (Bounded set and diameter). Let $A \subseteq M$, $A \neq \varnothing$. If there exists $k \in \mathbb{R}$ such that $d(x,y) < k$ for all $x,y \in A$, then the set A is said to be a **bounded set**.

And its **diameter** $d(A)$ is defined as

$$d(A) = \sup\{d(x,y) : x,y \in A\}$$

If A is not bounded, we define $d(A) = \infty$.

Example 1.2.3. Consider $\mathbb{R}^2$ equipped with Euclidean metric. Then the diameter of $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is equal to two.

To show this, let $p,q \in A$. Note that

$$d_2(p,q) \leq d_2(p,0) + d_2(0,q) = \|p\|_2 + \|q\|_2 < 1 + 1 = 2$$

Define $p_n = \left(-1+1/n, 0\right)$ and $q_n = \left(1-1/n, 0\right) \in \mathrm{A}$. Then

$$\mathrm{d}_2(p_n, q_n) = \sqrt{\left(2-\frac{2}{n}\right)^2} = 2\left(1-\frac{1}{n}\right)$$

Therefore,

$$\begin{aligned} \mathrm{d}(\mathrm{A}) &= \sup\{\mathrm{d}_2(x,y) : x,y \in \mathrm{A}\} \geq \sup\{\mathrm{d}_2(p_n, q_n) : n \in \mathbb{N}\} \\ &= \sup\{2(1-1/n) : n \in \mathbb{N}\} = 2 \end{aligned}$$

Thus, $\mathrm{d}(\mathrm{A}) = 2$.

1.3 Open balls

Definition 1.3.1 (Open and Closed Balls). Let (M, d) be a metric space, $p \in \mathrm{M}$, and r be a positive real number. The **open ball** of center p and radius r, denoted by $\mathrm{B}(p,r)$ or $\mathrm{B}_r^{\mathrm{M}}(p)$, is the following subset of M:

$$\mathrm{B}(p,r) = \{x \in \mathrm{M} : \mathrm{d}(x,p) < r\}$$

And the **closed ball** of center p and radius r is the set

$$\mathrm{B}_r^{\mathrm{M}}[p] = \{x \in \mathrm{M} : \mathrm{d}(x,p) \leq r\}$$

Example 1.3.1. Consider over M the metric zero-one. For $0 < \varepsilon \leq 1$,

$$\mathrm{B}(p,\varepsilon) = \{x \in \mathrm{M} : \mathrm{d}(p,x) < \varepsilon\} = \{p\}$$

However, for $\varepsilon > 1$,

$$\mathrm{B}(p,\varepsilon) = \{x \in \mathrm{M} : \mathrm{d}(p,x) < \varepsilon\} = \mathrm{M}$$

Definition 1.3.2 (Isolated point). A point $p \in \mathrm{M}$ is said to be an **isolated point** of M if there exists $r > 0$ such that $\mathrm{B}(p,r) = \{p\}$.

Example 1.3.2. Consider the metric space $(\mathbb{R}, \mathrm{d})$, where $\mathrm{d}(x,y) = |x-y|$. Then

$$\begin{aligned} \mathrm{d}(p,x) < \varepsilon &\iff |p-x| < \varepsilon \iff -\varepsilon < x-p < \varepsilon \\ &\iff p-\varepsilon < x < p+\varepsilon \iff x \in (p-\varepsilon, p+\varepsilon) \end{aligned}$$

Hence, $\mathrm{B}(p,\varepsilon) = (p-\varepsilon, p+\varepsilon)$.

Example 1.3.3. Consider the metric space $(\mathbb{R}^2, \mathrm{d})$ and let $p = (p_1, p_2)$.

If $\mathrm{d} = \mathrm{d}_2$,

$$\mathrm{B}(p,\varepsilon) = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1-p_1)^2 + (x_2-p_2)^2 < \varepsilon^2\}$$

i.e., the open ball $\mathrm{B}(p,\varepsilon)$ is the circle with center p and radius ε.

If $\mathrm{d} = \mathrm{d}_1$,

$$\mathrm{B}(p,\varepsilon) = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1 - p_1| + |x_2 - p_2| < \varepsilon\}$$

If $\mathrm{d} = \mathrm{d}_\infty$,

$$\begin{aligned}\mathrm{B}(p,\varepsilon) &= \{(x_1,x_2) \in \mathbb{R}^2 : \max\{|x_1 - p_1|, |x_2 - p_2|\} < \varepsilon\} \\ &= \{(x_1,x_2) \in \mathbb{R}^2 : |x_1 - p_1| < \varepsilon, \text{ and } |x_2 - p_2| < \varepsilon\} \\ &= (p_1 - \varepsilon, p_1 + \varepsilon) \times (p_2 - \varepsilon, p_2 + \varepsilon)\end{aligned}$$

Proposition 1.3.1. Let $(\mathrm{M}_1, d_1), \ldots, (\mathrm{M}_n, d_n)$ be metric spaces. Consider over $\mathrm{M} = \mathrm{M}_1 \times \ldots \times \mathrm{M}_n$ the metric D_∞. Then for all $a = (a_1, \ldots a_n) \in \mathrm{M}$, we have

$$\mathrm{B}(a,r) = \mathrm{B}(a_1,r) \times \ldots \times \mathrm{B}(a_n,r)$$

Proof. Let $p \in \mathrm{M}$. Then

$$\begin{aligned} p \in \mathrm{B}(a,\varepsilon) &\iff \max\{\mathrm{d}_1(p_1,a_1), \ldots, \mathrm{d}_n(p_n,a_n)\} < \varepsilon \\ &\iff \mathrm{d}_i(p_i,a_i) < \varepsilon, \text{ for all } i = 1,2,\ldots,n \\ &\iff p_i \in \mathrm{B}(a_i,\varepsilon), \text{ for all } i = 1,2,\ldots,n \\ &\iff p \in \mathrm{B}(a_1,\varepsilon) \times \ldots \times \mathrm{B}(a_n,\varepsilon)\end{aligned}$$

□

Theorem 1.3.2 (Basic Properties)**.** Let $\mathrm{B}(p,r)$ be open balls of an arbitrary metric space (M, d).

(P1) Given $\mathrm{B}(p,\varepsilon)$ and $\mathrm{B}(p,\delta)$, if $\varepsilon \le \delta$, then $\mathrm{B}(p,\varepsilon) \subseteq \mathrm{B}(p,\delta)$.

(P2) Given $q \in \mathrm{B}(\mathrm{P},\varepsilon)$, there exists $\delta > 0$ such that

$$\mathrm{B}(q,\delta) \subset \mathrm{B}(p,\varepsilon)$$

(P3) Let $\mathrm{B}(p,\varepsilon)$ and $\mathrm{B}(q,\delta)$ be non-disjoint balls. If $t \in \mathrm{B}(p,\varepsilon) \cap \mathrm{B}(q,\delta)$, then there exists $\lambda > 0$ such that

$$\mathrm{B}(t,\lambda) \subset \mathrm{B}(p,\varepsilon) \cap \mathrm{B}(q,\delta)$$

(P4) Let p and q be two distinct points of M. If $d(p,q) = \varepsilon$, then

$$\mathrm{B}\left(p, \frac{\varepsilon}{2}\right) \cap \mathrm{B}\left(q, \frac{\varepsilon}{2}\right) = \varnothing$$

(P5) Given two open balls $\mathrm{B}(p,\varepsilon)$ and $\mathrm{B}(q,\delta)$, if $\varepsilon + \delta \le d(p,q)$, then

$$\mathrm{B}(p,\varepsilon) \cap \mathrm{B}(q,\delta) = \varnothing$$

(P6) The diameter of a ball $\mathrm{B}(p,\varepsilon)$ is lesser or equal to 2ε, i.e. $d(\mathrm{B}(p,\varepsilon)) \le 2\varepsilon$.

Proof. (P1) Is immediate.

(P2) Take $\delta = \varepsilon - \mathrm{d}(p,q)$. Then $x \in \mathrm{B}(q,\delta) \iff \mathrm{d}(q,x) < \delta$. Now notice that

$$\mathrm{d}(p,x) \le \mathrm{d}(p,q) + \mathrm{d}(q,x) < d(p,q) + \varepsilon - \mathrm{d}(p,q) = \varepsilon$$

i.e., $x \in B(p, \varepsilon)$.

(P3) Follows directly from (P2).

(P4) Suppose, by contradiction, that there exists $x \in B(p, \varepsilon/2) \cap B(q, \varepsilon/2)$. Then, $d(p,x) < \varepsilon/2$ and $d(q,x) < \varepsilon/2$. Therefore,

$$\varepsilon = d(p,q) \leq d(p,x) + d(q,x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(P5) Also arguing by contradiction, suppose that there exists $x \in B(p, \delta) \cap B(q, \varepsilon)$. T hen, $d(p,x) < \delta$ and $d(q,x) < \varepsilon$. Therefore,

$$\delta + \varepsilon \leq d(p,q) \leq d(p,x) + d(q,x) < \delta + \varepsilon$$

(P6) If $x, y \in B(p, \varepsilon)$, then $d(p,x) < \varepsilon$ and $d(p,y) < \varepsilon$. Therefore,

$$d(x,y) \leq d(x,p) + d(p,y) < 2\varepsilon$$

Hence,

$$d(B(p,\varepsilon)) = \sup\{d(x,y) : x,y \in M\} \leq \sup\{d(p,x) + d(p,y) : x,y \in M\} \leq 2\varepsilon$$

□

Proposition 1.3.3. Let $(V, \|\cdot\|)$ be a real normed vector space and $d(x,y) = \|x - y\|$, $x, y \in V$. Then, for all $p \in V$ and $\varepsilon > 0$, $d(B(p,\varepsilon)) = 2\varepsilon$.

Proof. Suppose that $d(B(p,\varepsilon)) = \delta < \varepsilon$ and let $0 < \delta < \lambda < 2\varepsilon$. Take $u \neq 0 \in V$ and define

$$v = p + \frac{\lambda}{2\|u\|} u \text{ and } w = p - \frac{\lambda}{2\|u\|} u$$

Then,

$$d(v,w) = \left\| \frac{\lambda}{\|u\|} u \right\| = \lambda,\ d(p,v) = \left\| \frac{\lambda}{2\|u\|} u \right\| = \frac{\lambda}{2},\ d(p,w) = \frac{\lambda}{2}$$

Hence, $d(p,v) < \varepsilon$ and $d(p,w) < \varepsilon$. I.e., $v, w \in B(p,\varepsilon)$, implying that $d(B(p,\varepsilon)) > \delta$. □

Proposition 1.3.4. Let $(M_1, d_1), \dots, (M_n, d_n)$ be metric spaces and $M = M_1 \times \cdot \times M_n$, and let $D = D_\infty$ the metric given by

$$D(x,y) = \max\{d_1(x_1,y_1), \dots, d_n(x_n,y_n)\}$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in M$. Then for $p \in M$ and $\varepsilon > 0$,

$$B_D(p,\varepsilon) = B_{d_1}(p_1,\varepsilon) \times \cdots \times B_{d_n}(p_n,\varepsilon)$$

Proof. By a simple computation,

$$\begin{aligned}
\mathrm{B_D}(p,\varepsilon) &= \{x \in \mathrm{M} : \mathrm{D}(p,x) < \varepsilon\} \\
&= \{(x_1,\dots,x_n) \in \mathrm{M} : \mathrm{d}_i(p_i,x_i) < \varepsilon, \forall 1 \le i \le n\} \\
&= \{(x_1,\dots,x_n) \in \mathrm{M} : x_i \in \mathrm{B}_{\mathrm{d}_i}(p_i,\varepsilon), \forall 1 \le i \le n\} \\
&= \mathrm{B}_{\mathrm{d}_1}(p_1,\varepsilon) \times \cdots \times \mathrm{B}_{\mathrm{d}_n}(p_n,\varepsilon)
\end{aligned}$$

□

1.4 Equivalent Norms and Metrics

Consider two metrics d and d', not necessarily equal, over the same set M. To avoid confusion, we use $\mathrm{B_d}(p,\varepsilon)$ to denote a ball under the metric d and $\mathrm{B_{d'}}(p,\varepsilon)$ for a ball under the metric d'.

Definition 1.4.1 (Equivalent metrics)**.** The metrics d and d', over the same set M, are said to be **equivalent metrics** if, for all $p \in \mathrm{M}$,

1. For any open ball $\mathrm{B_d}(p,\varepsilon)$, there exists $\lambda > 0$ such that $\mathrm{B_{d'}}(p,\lambda) \subseteq \mathrm{B_d}(p,\varepsilon)$.
2. For any open ball $\mathrm{B_{d'}}(p,\varepsilon)$, there exists $\lambda > 0$ such that $\mathrm{B_d}(p,\lambda) \subseteq \mathrm{B_{d'}}(p,\varepsilon)$.

If d and d' are equivalent, we write $\mathrm{d} \sim \mathrm{d}'$.

Note that, by definition and the property (P2), it follows that if d and d' are equivalent metrics, then every ball $\mathrm{B_d}(p,\varepsilon)$ is an union of balls $\mathrm{B_{d'}}(p_i,\varepsilon_i)$ and vice-versa.

Proposition 1.4.1. Consider two metrics d and d' over M. If there exists two positive real numbers r, s satisfying

$$r\mathrm{d}(x,y) \le \mathrm{d}'(x,y) \le s\mathrm{d}(x,y)$$

for all $x, y \in \mathrm{M}$, then $\mathrm{d} \sim \mathrm{d}'$.

Proof. Let $\varepsilon > 0$ and $p \in \mathrm{M}$. We want to show that

$$\mathrm{B_{d'}}(p,\varepsilon r) \subseteq \mathrm{B_d}(p,\varepsilon) \text{ and } \mathrm{B_d}(p,\varepsilon/s) \subseteq \mathrm{B_{d'}}(p,\varepsilon)$$

Take $x \in \mathrm{B_{d'}}(p,\varepsilon r)$. Then $\mathrm{d}'(p,x) < \varepsilon r$, i.e.,

$$r\mathrm{d}(p,x) < \varepsilon r \iff \mathrm{d}(p,x) < \varepsilon$$

Hence, $x \in \mathrm{B_d}(p,\varepsilon)$.

Now take $x \in \mathrm{B_d}(p,\varepsilon/s)$. Then $\mathrm{d}(p,x) < \varepsilon/s$, i.e.,

$$\mathrm{d}'(p,x) \le s d(p,x) < \varepsilon$$

Therefore, $x \in \mathrm{B_{d'}}(p,\varepsilon)$. □

Example 1.4.1. The Euclidean metric and the maximum metric in $\mathbb{R}^2$ are equivalent.

Definition 1.4.2 (Equivalent norms)**.** Two norms $\|\cdot\|$ and $\|\cdot\|'$ over the same vector space V are said to be **equivalent** if the metrics induced by these norms are equivalent.

Proposition 1.4.2. Two norms $\|\cdot\|$ and $\|\cdot\|'$ over the same vector space V are equivalent if, and only if, there exist $r, s \in \mathbb{R}_{\geq 0}$ such that

$$r\|u\| \leq \|u\|' \leq s\|u\|$$

for all $u \in \mathrm{V}$.

Proof. One way is directly guaranteed by the previous result. Now suppose that both norms are equivalent. Then there exists $\lambda > 0$ such that

$$\mathrm{B}_{\mathrm{d}'}(0, \lambda) \subseteq \mathrm{B}_{\mathrm{d}}(0, 1)$$

Let $u \neq 0 \in \mathrm{V}$ and $0 < r < \lambda$. Notice that for $u = 0$ the claim is trivial. Then,

$$\left\| \frac{r}{\|u\|'} u \right\|' = r < \lambda \implies \frac{r}{\|u\|'} u \in \mathrm{B}_{\mathrm{d}'}(0, \lambda) \implies \frac{r}{\|u\|'} u \in \mathrm{B}_{\mathrm{d}}(0, 1)$$

I.e.,

$$\left\| \frac{r}{\|u\|'} u \right\| < 1 \iff \frac{r}{\|u\|'} \|u\| < 1 \iff r\|u\| < \|u\|'$$

Conversely, let $\lambda > 0$ such that

$$\mathrm{B}_{\mathrm{d}}(0, \lambda) \subseteq \mathrm{B}_{\mathrm{d}'}(0, 1)$$

and let $u \neq 0 \in \mathrm{V}$ and $0 < r < \lambda$. Then,

$$\left\| \frac{r}{\|u\|} u \right\| = r < \lambda \implies \frac{r}{\|u\|} u \in \mathrm{B}_{\mathrm{d}}(0, \lambda) \implies \frac{r}{\|u\|} u \in \mathrm{B}_{\mathrm{d}'}(0, 1)$$

I.e.,

$$\left\| \frac{r}{\|u\|} u \right\|' < 1 \iff \frac{r}{\|u\|} \|u\|' < 1 \iff \|u\|' < \frac{1}{r} \|u\|$$

Taking $s = 1/r$ the proof is completed. □

Example 1.4.2. The norms given by

$$\|f\| = \sup\{|f(x)| : x \in [0, 1]\}$$

$$\|f\|' = \int_0^1 |f(x)| \, \mathrm{d}x$$

defined over $\mathscr{C}[0, 1]$ are not equivalent.

1.5 Sequences in Metric Spaces

1.5.1 The Limit of a Sequence

Recall that a sequence is a family $(x_n)_{n\in\mathbb{N}}$ of points of M.

Definition 1.5.1 (Limit of a sequence)**.** Let (M, d) be a metric space. A point $p \in \mathrm{M}$ is the **limit** of a sequence (x_n) if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies x_n \in \mathrm{B}(p, \varepsilon)$$

Then we say that (x_n) is a **convergent sequence** and that (x_n) **converges** to p. We denote this by $\lim x_n = p$ or $x_n \longrightarrow p$.

This definition can be written in the following equivalent way.

Proposition 1.5.1. A sequence $(x_n) \in \mathrm{M}$ converges to $p \in \mathrm{M}$ if, and only if, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies \mathrm{d}(x_n, p) < \varepsilon$$

Proof.

$$x_n \in \mathrm{B}(p, \varepsilon) \iff \mathrm{d}(x_n, p) < \varepsilon$$

□

Example 1.5.1. Consider the set of continuous functions $\mathscr{C}[a,b]$ with the supremum metric, and let $f, f_n : [a,b] \longrightarrow \mathbb{R}$ be defined as follows:

$$f_n(x) = \frac{1}{n} \text{ and } f(x) = 0,\ \forall x \in [a,b]$$

Then

$$\begin{aligned} \mathrm{d}(f, f_n) &= \sup\{\mathrm{d}(f(x), f_n(x)) : x \in [a,b]\} \\ &= \sup\{|f(x) - f_n(x) : x \in [a,b]\} \\ &= \frac{1}{n} \longrightarrow 0 \text{ as } n \to \infty \end{aligned}$$

Example 1.5.2 (Stationary sequence)**.** Let $\mathrm{M} \neq \varnothing$ equipped with the zero-one metric and suppose that (x_n) converges to $p \in \mathrm{M}$.

For $\varepsilon = 1/2$, there exists $r \in \mathbb{N}$ such that

$$n \geq r \implies \mathrm{d}(x_n, p) < \varepsilon = \frac{1}{2}$$

I.e., $x_n = p$ for all $n \geq r$. This sequence is called **stationary**.

Proposition 1.5.2 (Uniqueness of the limit)**.** If a sequence (x_n) over a metric space M converges, then its limit is unique.

Proof. By contradiction, suppose that $\lim x_n = p$ and $\lim x_n = q$, with $p \neq q$. Then take $\varepsilon = \mathrm{d}(p,q)/2 > 0$. Then there exists $r, s \in \mathbb{N}$ such that

$$n \geq r \implies \mathrm{d}(x_n, p) < \varepsilon$$

$$n \geq s \implies \mathrm{d}(x_n, q) < \varepsilon$$

Take $t = \max\{r, s\}$. Then, for $n \geq t$,

$$\mathrm{d}(p,q) \leq \mathrm{d}(p, x_n) + \mathrm{d}(x_n, q) < 2\varepsilon = \mathrm{d}(p,q)$$

□

Proposition 1.5.3. Let d and d′ be equivalent metrics over M. Then a sequence (x_n) of points in M converges in the space (M, d) to $p \in \mathrm{M}$ if, and only if, this sequence also converges to p in the space $(\mathrm{M}, \mathrm{d}')$.

Proof. Suppose that $x_n \to p$ in (M, d) and let $\varepsilon > 0$. Since $d \sim d'$, there exists $\delta > 0$ such that

$$\mathrm{B}_{\mathrm{d}}(p, \delta) \subset \mathrm{B}_{\mathrm{d}'}(p, \varepsilon)$$

Given that x_n converges to p in (M, d), there exists $r \in \mathbb{N}$ such that $n \geq r$ implies $\mathrm{d}(x_n, p) < \delta$, i.e.,

$$x_n \in \mathrm{B}_{\mathrm{d}}(p, \delta) \subset \mathrm{B}_{\mathrm{d}'}(p, \varepsilon) \implies \mathrm{d}'(x_n, p) < \varepsilon$$

Therefore, $x_n \to p$ in $(\mathrm{M}, \mathrm{d}')$. The reciprocal is analogous. □

Proposition 1.5.4 (Subsequences converge to the same point)**.** If a sequence $(x_n) \in \mathrm{M}$ converges to $p \in \mathrm{M}$, then every subsequence of (x_n) also converges to p.

Proof. Let $(x_{r_1}, x_{r_2}, \ldots)$ be a subsequence of the given sequence, and let $\varepsilon > 0$. From the hypothesis that $\lim x_n = p$, we know that there exists k such that

$$n \geq k \implies \mathrm{d}(x_n, p) < \varepsilon$$

Note that each $r_i \in \mathbb{N}$ and $r_1 < r_2 < \ldots$. Hence, there exists $r_t > k$ and, therefore,

$$r_i \geq r_t \implies \mathrm{d}(x_{r_i}, p) < \varepsilon$$

□

Definition 1.5.2 (Bounded sequence)**.** A sequence $(x_n) \in \mathrm{M}$ is **bounded** if there exists $k > 0$ such that $\mathrm{d}(x_r, x_s) < k$, for any x_r, x_s in the set of terms $\{x_n\}$.

Proposition 1.5.5. Every convergent sequence is bounded.

Proof. Consider the convergent sequence $(x_n) \longrightarrow p$. Given the ball $\mathrm{B}(p, 1)$, there exists $r \in \mathbb{N}$ such that

$$n \geq r \implies x_n \in \mathrm{B}(p, 1)$$

Let $k > \max\{\mathrm{d}(x_i, p) : i = 1, \ldots, r-1\}$ and consider the ball $\mathrm{B}(p, \varepsilon)$, where $\varepsilon = \max\{1, k\}$.

Then every term of the sequence is contained in this ball and, therefore, for any x_i, x_j in the sequence
$$\mathrm{d}(x_i, x_j) \leq \mathrm{d}(x_i, p) + \mathrm{d}(p, x_j) < 2\varepsilon$$
□

1.5.2 Sequences in a Product Space

Given a sequence of points of the form
$$((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \ldots)$$
how can we study the convergence of $((x_n, y_n))$ in terms of each sequence (x_n) and (y_n)?

Proposition 1.5.6. A sequence $((x_n, y_n))$ of points in the product space $\mathrm{M} \times \mathrm{N}$ converges to $(p, q) \in \mathrm{M} \times \mathrm{N}$ if, and only if, $x_n \longrightarrow p$ in M and $y_n \longrightarrow q$ in N.

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. There exists $r \in \mathbb{N}$ such that
$$n \geq r \implies \mathrm{D}_1((x_n, y_n), (p, q)) = \mathrm{d}(x_n, p) + \mathrm{d}(y_n, q) < \varepsilon$$
Hence,
$$\mathrm{d}(x_n, p) < \varepsilon \text{ and } \mathrm{d}(y_n, q) < \varepsilon$$
i.e., $\lim x_n = p$ and $\lim y_n = q$.

($\Leftarrow$) Let $\varepsilon > 0$. There exists $r, s \in \mathbb{N}$ such that
$$n \geq r \implies \mathrm{d}(x_n, p) < \frac{\varepsilon}{2} \text{ and } n \geq s \implies \mathrm{d}(y_n, q) < \frac{\varepsilon}{2}$$
Take $t = \max\{r, s\}$. Then
$$n \geq t \implies \mathrm{D}_1((x_n, y_n), (p, q)) = \mathrm{d}(x_n, p) + \mathrm{d}(y_n, q) < \varepsilon$$
i.e., $(x_n, y_n) \longrightarrow (p, q)$. □

Note that this proposition can be immediately generalized to n metric spaces.

Example 1.5.3. The following sequence
$$\left((1, 2), \left(\frac{1}{2}, 2\right), \left(\frac{1}{3}, 2\right), \ldots\right)$$
converges in $\mathbb{R}^2$ to $(0, 2)$.

Example 1.5.4. The sequence
$$\left((1, 2), \left(\frac{1}{2}, 1\right), \left(\frac{1}{3}, 2\right), \left(\frac{1}{4}, 1\right), \ldots\right)$$
does not converge in $\mathbb{R}^2$, since the sequence of the second terms $(2, 1, 2, 1, \ldots)$ does not converge in $\mathbb{R}$.

1.5.3 Sequences in Normed Vector Spaces

Proposition 1.5.7. Every increasing or strictly increasing sequence such that the set of terms is upper bounded converges to the supremum of this set.

Analogously, every decreasing or strictly decreasing sequence such that the set of terms is lower bounded converges to the infimum of this set.

Proposition 1.5.8 (Sign conservation)**.** Let (x_n) be a sequence in $\mathbb{R}$.

- If $\lim x_n = p > 0$, then there exists an index r and a constant $c > 0$ such that $x_n > c$ for all $n \geq r$.
- If $\lim x_n = p < 0$, then there exists an index r and a constant $c < 0$ such that $x_n < c$ for all $n \geq r$.

Proposition 1.5.9. Let (x_n) be a sequence in a normed vector space V which converges to $p \in \mathrm{V}$. Then there exists a ball centered in origin containing all terms of the sequence.

Definition 1.5.3. Let $f = (x_n)$ and $g = (y_n)$ be sequences in a normed vector space V. We define the **sum** of f and g as the sequence

$$f + g = (x_1 + y_1, \ldots, x_n + y_n, \ldots)$$

If $k = (c_n)$ is a sequence of elements in $\mathbb{R}$, then the **product** $k \cdot f$ is defined as

$$k \cdot f = (c_1 x_1, \ldots, c_n x_n, \ldots)$$

Proposition 1.5.10. Let (x_n) and (y_n) be two sequences in a normed vector space V. If $\lim x_n = p$ and $\lim y_n = q$, then $\lim(x_n + y_n) = p + q$.

Corollary 1.5.11. If (x_n) and (y_n) are convergent sequences of real numbers satisfying $x_n \leq y_n$ from a given index r, then $\lim x_n \leq \lim y_n$.

Proposition 1.5.12. Let $(x_n) \longrightarrow p$ in a vector space V. If (c_n) is a sequence of real numbers such that $\lim c_n = c \in \mathbb{R}$, then $\lim c_n x_n = cp$.

Example 1.5.5. Let a be a real number such that $0 < a < 1$. We show that the sequence $(a, a^2, a^3, \ldots)$ converges to 0.

Since the sequence is strictly decreasing and the set of its terms is lower bounded by zero, the sequence converges in $\mathbb{R}$ and $\lim a^n = p = \inf\{a^n : n = 1, 2, \ldots\}$.

Remark that

$$(a, a^2, a^3, \ldots) = (a, a, a, \ldots) \cdot (1, a, a^2, \ldots)$$

have the same limit. By the previous proposition,

$$p = ap \iff p(1 - a) = 0 \iff p = 0$$

This result can be generalized as follows. If $a \in \mathbb{R}$ and $|a| < 1$, then $\lim |a|^n = 0$.

Lemma 1.5.13. If a sequence (x_n) in a normed vector space converges to p, then the sequence $(\|x_n\|)$ converges to $\|p\|$.

Proposition 1.5.14. Let (c_n) be a sequence in $\mathbb{R}$ satisfying $\lim c_n = c \neq 0$. Then the sequence (d_n) defined by $d_n = 0$ if $c_n = 0$ and $d_n = 1/a_n$ for all $a_n \neq 0$, converges to $1/c$.

Chapter 2

The Topology of Metric Spaces

2.1 Open and Closed Sets

In this section, we introduce some topological concepts.

Definition 2.1.1 (Topology). A **topology** on a set X is a family τ of subsets of X with the following properties:

- $\emptyset, X \in \tau$;
- $A, B \in \tau \implies A \cap B \in \tau$;
- $\mathscr{U} \subseteq \tau \implies \bigcup \mathscr{U} \in \tau$.

The pair (X, τ) is called a **topological space** and its members are called the **open sets** of X with respect to τ.

To put it another way, a topological space is closed under finite intersections and arbitrary unions.

Definition 2.1.2 (Open and Closed Sets). Let (M, d) be a metric space. A subset $A \subset M$ is said to be **open** if, for all $p \in A$, there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subset A$.

A set $F \subset M$ is said to be **closed** if $F^c = M \setminus F$ is open.

Intuitively, a set is open if all of its points fit loosely inside it.

Remark that if $A \neq \emptyset$ is an open set, then A is a union of open balls. And if A is a union of open balls, then A is an open set.

The next proposition asserts that the collection of open sets of a metric space is a topology over M, called the **topology induced by a metric**.

Proposition 2.1.1. Let $\mathscr{A}$ be the collection of all open sets of a metric space (M, d). Then

1. $\emptyset, M \in \mathscr{A}$;
2. $X, Y \in \mathscr{A} \implies X \cap Y \in \mathscr{A}$;

3. If (X_i) is a family of open sets of M, i.e., each $X_i \in \mathscr{A}$, then $\bigcup X_i \in \mathscr{A}$.

Proof. 1. $\emptyset, M \in \mathscr{A}$ trivially.

2. Let $X, Y \in \mathscr{A}$ and $p \in X \cap Y$.

   Since X is open, there exists $\varepsilon_1 > 0$ such that $B(p, \varepsilon_1) \subset X$.

   Since Y is open, there exists $\varepsilon_2 > 0$ such that $B(p, \varepsilon_2) \subset Y$.

   Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Given that $B(p, \varepsilon) \subset B(p, \varepsilon_1) \cap B(p, \varepsilon_2)$, we know that $B(p, \varepsilon) \subset X \cap Y$. Hence, $X \cap Y$ is an open set of $\mathscr{A}$.

3. Let $\{X_i\}_{i \in I} \subset \mathscr{A}$ and $X = \bigcup_{i \in I} X_i$. Taking $p \in X$, there exists $i \in I$ such that $p \in X_i$. Therefore, there is $\varepsilon > 0$ such that $B(p, \varepsilon) \subset X_1 \subset X$. Thus, X is an open set of $\mathscr{A}$.

□

Equivalent metrics determine the same topological structure.

Proposition 2.1.2. Let d and d′ be equivalent metrics over M. If $\mathscr{A}$ is a collection of open sets of (M, d) and $\mathscr{A}'$ is a collection of open sets of (M, d'), then $\mathscr{A} = \mathscr{A}'$.

Proof. Let $\mathscr{A} \subset \mathscr{A}'$ and $A \in \mathscr{A}$. Given $p \in A$, there exists $\varepsilon > 0$ such that $B_d(p, \varepsilon) \subset A$.

Since $d \sim d'$, there exists $\delta > 0$ such that

$$B_{d'}(p, \delta) \subset B_d(p, \varepsilon)$$

Hence, $B_{d'}(p, \delta) \subset A$ and, therefore, $A \in \mathscr{A}'$. □

Definition 2.1.3 (Interior Point)**.** Let (M, d) be a metric space and $A \subset M$. A point $p \in A$ is called an **interior point** of A if there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subset A$. The set of interior points of A is called the **interior** of A.

With this definition, an equivalent notion of open sets follows: a set A is open iff. every point of A is interior.

Proposition 2.1.3. Let $\mathscr{F}$ be the collection of all closed sets of a metric space (M, d). Then

1. $\emptyset, M \in \mathscr{F}$;
2. $X, Y \in \mathscr{F} \implies X \cup Y \in \mathscr{F}$;
3. If (X_i) is a family of open sets of M, i.e., each $X_i \in \mathscr{F}$, then $\bigcap X_i \in \mathscr{F}$.

Proof. 1. $\emptyset, M \in \mathscr{F}$ trivially.

2. $X, Y \in \mathscr{F} \implies X^c, Y^c \in \mathscr{A} \implies X^c \cap Y^c \in \mathscr{A} \implies (X \cup Y)^c \in \mathscr{A} \implies X \cup Y \in \mathscr{F}$.

3. $\{X_i\}_{i \in I} \subset \mathscr{F} \iff \{X_i^c\}_{i \in I} \subset \mathscr{A} \implies \bigcup_{i \in I} X_i^c \in \mathscr{A} \implies \bigcap_{i \in I} X_i \in \mathscr{F}$.

□

2.2 Adherence, Accumulation and Closure

This section investigates when a point is ‘close’ to a set.

Definition 2.2.1 (Adherent point)**.** Let $A \subset M$. A point $p \in M$ is called an **adherent point** of A if, for all $\varepsilon > 0$, we have

$$B(p,\varepsilon) \cap A \neq \emptyset$$

The set of all adherent points of A is called the **closure** of A and is denoted by $\overline{A}$.

The intuition behind this definition is that every neighborhood of p contains at least one point of A.

Proposition 2.2.1. The complement of the closure equals the interior of the complement, i.e.,

$$(\overline{A})^c = \operatorname{int}(A^c)$$

Proof.

$$\begin{aligned} p \in (\overline{A})^c &\iff p \notin \overline{A} \\ &\iff \exists \varepsilon > 0 : B(p,\varepsilon) \cap A = \emptyset \\ &\iff \exists \varepsilon > 0 : B(p,\varepsilon) \subset A^c \\ &\iff p \in \operatorname{int}(A^c) \end{aligned}$$

□

Corollary 2.2.2. A set is closed iff. it is equal to its closure.

Proof. We know that F is closed iff. F^c is open, which means that $\operatorname{int}(F^c) = F^c$. By the preceeding proposition, $(\overline{F})^c = F^c$, i.e., $\overline{F} = F$. □

Proposition 2.2.3. Let (M,d) be a metric space. If $p \in M$ and $A \subset M$, then $d(p,A) = 0$ iff. $p \in \overline{A}$.

Proof. ($\Rightarrow$) Apply the definition of distance from a point to a set and equal it to zero. Show that there is $a \in A$ such that $a \in B(p,\varepsilon)$.

($\Leftarrow$) Suppose that the distance equals $\varepsilon > 0$. By hypothesis, there exists $a \in A$ such that $d(a,p) < \varepsilon$ and derive a contradiction. □

Proposition 2.2.4. For all non-empty subsets A of M, we have $d(A) = d(\overline{A})$.

Proof. Since $A \subseteq \overline{A}$, it follows that $d(A) \leq d(\overline{A})$. Take $\varepsilon > 0$ and $x, y \in \overline{A}$.

Thus, there exists $a, b \in A$ such that

$$a \in B(x,\varepsilon/2) \text{ and } b \in B(y,\varepsilon/2)$$

I.e.,

$$d(a,x) < \frac{\varepsilon}{2} \text{ and } d(y,b) < \frac{\varepsilon}{2}$$

By the triangular inequality,

$$\begin{aligned} d(x,y) &\le d(x,a) + d(a,y) \\ &\le d(x,a) + d(a,b) + d(b,y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + d(a,b) \\ &< d(A) + \varepsilon \end{aligned}$$

Therefore, $d(\overline{A}) < d(A) + \varepsilon$, which means that $d(\overline{A}) \le d(A)$. Hence, $d(A) = d(\overline{A})$. □

A closure is a set plus its 'boundary'. Naturally, the boundary of a set A is formed by the points $p \in M$ such that every open ball centered on p contains at least one point of A and one point of the complement $M \setminus A$.

Definition 2.2.2 (Boundary). Let (M, d) be a metric space and $A \subset M$. The **boundary** (or **frontier**) of A is the set

$$\partial A = \{p \in M : B(p,\varepsilon) \cap A \neq \emptyset \text{ and } B(p,\varepsilon) \cap (M \setminus A) \neq \emptyset,\ \forall \varepsilon > 0\}$$

Proposition 2.2.5. Let (M, d) be a metric space and $A \subset M$. Then $\overline{A} = A \cup \partial A$.

Proof. Let $a \in \overline{A}$ and $a \notin A$. Using the definition of adherence point, we know that $a \in \partial A$.

Let $a \in A \cup \partial A$. If $a \in A$, then $a \in \overline{A}$. If $a \notin A$, then $a \in \partial A$, hence $a \in \overline{A}$. □

Proposition 2.2.6. Let (M, d) be a metric space. If $A \subset M$ and $p \in \overline{A}$, then there exists a sequence (x_n) of points in A such that $\lim x_n = p$.

Proof. For each $n \in \mathbb{N}$, let $x_n \in B(p, 1/n) \cap A$. I.e., $d(p, x_n) < 1/n$. Given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$. Then,

$$n \ge n_0 \implies d(p, x_n) < \frac{1}{n} \le \frac{1}{n_0} < \varepsilon$$

□

Definition 2.2.3 (Dense). Given a metric space (M, d), a subset $A \subset M$ is said to be **dense** in M if $\overline{A} = M$.

The idea here is that for all points $p \in M$ there exists another point $a \in A$ arbitrarily close to p. E.g. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proposition 2.2.7. Let M be a metric space. If $A \subset M$ is dense in M, then $G \cap A \neq \emptyset$ for all open set $G \neq \emptyset$ of this space.

Proof. Let G be an open set and $p \in G$. There exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subset G$. Since A is dense in M,

$$A \cap B(p,\varepsilon) \neq \emptyset \implies G \cap A \supset A \cap B(p,\varepsilon) \neq \emptyset$$

Hence, $G \cap A \neq \emptyset$. □

The following definition adds a restriction to our definition of adherence point. This will be useful when studying limits and continuity.

Definition 2.2.4 (Accumulation point). Let $A \subset M$. A point $p \in M$ is called an **accumulation point** of A if p is an adherent point of $A \setminus \{p\}$, i.e., for all $\varepsilon > 0$, we have that

$$B(p,\varepsilon) \cap A \setminus \{p\} \neq \emptyset$$

The set of accumulation points of A is called **derived set** of A and is denoted by A'.

Example 2.2.1.

1. $A \subset B \implies A' \subset B'$ and $(a,b)' = [a,b]$.
2. $A = [0,1] \cup \{2\}$. Then $A' = [0,1]$.
3. $A = (0,1) \cap \mathbb{Q}$. Then $A' = [0,1]$.

Proposition 2.2.8. Let p be an accumulation point of a set A. Then every ball centered at p has infinitely many points of A.

Proof. Suppose that there exists $\varepsilon > 0$ such that $B(p,\varepsilon) \cap A = \{x_1, \dots, x_n\}$, i.e., is a finite set.

Let
$$(B(p,\varepsilon) \setminus \{p\}) \cap A = \{y_1, \dots, y_m\}$$
where $y_i \neq y_j$ for $i \neq j$. And define

$$\delta = \min\{d(p,y_j) : 1 \leq j \leq m\} > 0$$

Therefore,
$$(B(p,\delta) \setminus \{p\}) \cap A = \emptyset$$
implying that p is not an accumulation point of A, which contradicts our hypothesis. □

Example 2.2.2. Let $M \neq \emptyset$ equipped with the zero-one metric. Let $A \subset M$ and $p \in M$. Then

$$B(p,1) = \{p\} \text{ and } B(p,1) \setminus \{p\} = \emptyset$$

Hence, p is not an accumulation point of A, i.e., $A' = \emptyset$.

Remark. Let (M,d) be a metric space, $A \subset M$ and A finite. Then $A' = \emptyset$.

Proposition 2.2.9. Let (x_n) be a sequence of (M,d) such that $A = \{x_n : n \in \mathbb{N}\}$ has an accumulation point p. Then there exists a subsequence of $\{x_n\}$ which converges to p.

Proof. For each $j \in \mathbb{N}$ define
$$C_j = B(p,1/j) \cap A$$
and notice that C_j is infinite. Choose $x_{n_1} \in C_1$, $x_{n_2} \in C_2$ such that $x_{n_2} \neq x_{n_1}$ and $n_2 > n_1$.

Suppose that we already chose $x_{n_k} \in C_k$ such that $x_{n_i} \neq x_{n_l}$ for all $i \neq l$ and $n_1 < n_2 < \dots < n_{k-1} < n_k$.

We now choose $x_{n_{k+1}}$ such that $x_{n_{k+1}} \in C_{k+1}$. $x_{n_{k+1}} \notin \{x_{n_1}, \dots, x_{n_k}\}$ and $n_{k+1} > n_k$.

By the induction principle, we construct a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of (x_n).

And we have that

$$x_{n_j} \in C_j \iff d(p, x_{n_j}) < \frac{1}{j} \implies x_{n_j} \longrightarrow p \text{ as } j \to \infty$$

$\square$

Proposition 2.2.10. A set $F \subset M$ is closed iff. $F' \subset F$.

Proof. ($\Rightarrow$) Suppose, by contradiction, that F is closed, but $F' \not\subset F$. Therefore, there exists $p \in F'$ such that $p \notin F$. Then $p \in F' \setminus F = F' \cap F^c$.

Since F^c is open, there exists $\varepsilon > 0$ such that $B(p,\varepsilon) \subset F^c$. I.e., $B(p,\varepsilon) \cap F = \emptyset$. Hence, $p \notin F'$.

($\Leftarrow$) Let $p \in F^c$. Then $p \notin F$ and $p \notin F'$. Then there exists $\varepsilon > 0$ such that

$$(B(p,\varepsilon) \setminus \{p\}) \cap F = \emptyset$$

Since $p \notin F$,

$$B(p,\varepsilon) \cap F = \emptyset \implies B(p,\varepsilon) \subset F^c$$

Hence, F^c is open, i.e., F is closed. $\square$