Øksendal 3.5

Prove that

$$M_t = B_t^2 - t$$

is an $\mathcal{F}_t$-martingale.

What we need to show is:

1. M_t is $\mathcal{F}_t$-measurable
2. $\mathbb{E}[|M_t|] < \infty$
3. $\mathbb{E}[M_t | \mathcal{F}_s] = M_s, \quad s \leq t$

The first is immediate. Since B_t^2 and t are $\mathcal{F}_t$-measurable, M_t also is.

Now notice that

$$\mathbb{E}[|M_t|] = \mathbb{E}[|B_t^2 - t|] \leq \mathbb{E}[|B_t^2|] + \mathbb{E}[|t|] < \infty$$

Finally,

$$\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[B_t^2 - t | \mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s + B_s)^2 | \mathcal{F}_s] - \mathbb{E}[t | \mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + 2\mathbb{E}[(B_t - B_s)B_s | \mathcal{F}_s] + \mathbb{E}[B_s^2 | \mathcal{F}_s] - \mathbb{E}[t | \mathcal{F}_s]$$

$$= t - s + B_s^2 - t = B_s^2 - s = M_s$$

for $s < t$ as desired.