5.13. As a model for the horizontal slow drift motions of a moored floating platform or ship responding to incoming irregular waves John Grue (1989) introduced the equation

$$x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t')\eta W_t\,, \tag{5.3.5}$$

where W_t is 1-dimensional white noise, a_0, w, T_0, α_0 and η are constants.

(i) Put $X_t = \begin{bmatrix} x_t \\ x_t' \end{bmatrix}$ and rewrite the equation in the form

$$dX_t = AX_t dt + KX_t dB_t + M dB_t\,,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -w^2 & -a_0 \end{bmatrix}, \quad K = \alpha_0 \eta \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M = T_0 \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Show that X_t satisfies the integral equation

$$X_t = \int_0^t e^{A(t-s)} K X_s dB_s + \int_0^t e^{A(t-s)} M dB_s \qquad \text{if } X_0 = 0\,.$$

Multiplying the equation by $\exp(-At)$,

$$\exp(-At)\, dX_t = \exp(-At) AX_t dt + \exp(-At)[KX_t + M]\, dB_t$$

Applying Itô's formula to $d(\exp(-At) X_t)$,

$$d(\exp(-At) X_t) = -A\exp(-At) X_t dt + \exp(-At)\, dX_t$$

$$= \exp(-At)[KX_t + M]\, dB_t$$

Therefore,

$$\exp(-At) X_t = X_0 + \int_0^t \exp(-As)[KX_s + M]\, dB_s$$

If $X_0 = 0$,

$$\boxed{X_t = \int_0^t e^{A(t-s)} K X_s dB_s + \int_0^t e^{A(t-s)} M dB_s}$$

(iii) Verify that

$$e^{At} = \frac{e^{-\lambda t}}{\xi}\{(\xi \cos \xi t + \lambda \sin \xi t)I + A \sin \xi t\}$$

where $\lambda = \frac{a_0}{2}, \xi = (w^2 - \frac{a_0^2}{4})^{\frac{1}{2}}$ and use this to prove that

$$x_t = \eta \int_0^t (T_0 - \alpha_0 y_s) g_{t-s} dB_s \tag{5.3.6}$$

and

$$y_t = \eta \int_0^t (T_0 - \alpha_0 y_s) h_{t-s} dB_s\ , \qquad \text{with } y_t := x_t'\ , \tag{5.3.7}$$

where

$$g_t = \frac{1}{\xi}\mathrm{Im}(e^{\zeta t})$$
$$h_t = \frac{1}{\xi}\mathrm{Im}(\zeta e^{\bar{\zeta} t})\ , \qquad \zeta = -\lambda + i\xi \quad (i = \sqrt{-1})\ .$$

So we can solve for y_t first in (5.3.7) and then substitute in (5.3.6) to find x_t.

Idea: write $A = PDP^{-1}$ and then use that $e^{At} = P e^{Dt} P^{-1}$

We start by computing the caracteristic polynomial of A:

$$c_A(x) = \begin{vmatrix} x & -1 \\ w^2 & x + a_0 \end{vmatrix} = x(x+a_0) + w^2 = x^2 + x a_0 + w^2$$

Setting $c_A(x) = 0$,

$$x = \frac{-a_0 \pm \sqrt{a_0^2 - 4w^2}}{2} = \frac{-a_0}{2} \pm i\sqrt{w^2 - a_0^2/4} = -\lambda \pm i\xi$$

Finding the eigenvectors associated with $-\lambda + i\xi$:

$$\begin{bmatrix} \lambda - i\xi & 1 \\ -w^2 & -a_0 + \lambda - i\xi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{cases} (\lambda - i\xi)x + y = 0 \\ -w^2 x + (-a_0 + \lambda - i\xi) y = 0 \end{cases}$$

Then $y = -(\lambda - i\xi)x$. Let $x = 1$ and then $y = -\lambda + i\xi$.
For $-\lambda - i\xi$, the same calculation yields $x = 1$ and $-\lambda - i\xi$.
Thus, we can write

using the idea

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -\lambda+i\xi & -\lambda-i\xi \end{bmatrix} \cdot \begin{bmatrix} e^{(-\lambda+i\xi)t} & 0 \\ 0 & e^{(-\lambda-i\xi)t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\lambda+i\xi & -\lambda-i\xi \end{bmatrix}^{-1} \qquad (1)$$

Now let us compute the inverse above.

$$\begin{bmatrix} 1 & 1 \\ -\lambda+i\xi & -\lambda-i\xi \end{bmatrix}^{-1} = \frac{-1}{2\xi i} \begin{bmatrix} -\lambda-i\xi & -1 \\ \lambda-i\xi & 1 \end{bmatrix}$$

Thus

$$e^{At} = \frac{-1}{2\xi i} \begin{bmatrix} 1 & 1 \\ -\lambda+i\xi & -\lambda-i\xi \end{bmatrix} \cdot \begin{bmatrix} e^{(-\lambda+i\xi)t} & 0 \\ 0 & e^{(-\lambda-i\xi)t} \end{bmatrix} \begin{bmatrix} -\lambda-i\xi & -1 \\ \lambda-i\xi & 1 \end{bmatrix}$$

$$= \frac{-1}{2\xi i} \begin{bmatrix} 1 & 1 \\ -\lambda+i\xi & -\lambda-i\xi \end{bmatrix} \cdot \begin{bmatrix} -e^{(-\lambda+i\xi)t}(\lambda+i\xi) & -e^{(-\lambda+i\xi)t} \\ e^{(-\lambda-i\xi)t}(\lambda-i\xi) & e^{(-\lambda-i\xi)t} \end{bmatrix}$$

$$= \frac{-1}{2\xi i} \begin{bmatrix} F & G \\ H & I \end{bmatrix} \qquad (2)$$

where

$$F = e^{(-\lambda-i\xi)t}(\lambda-i\xi) - e^{(-\lambda+i\xi)t}(\lambda+i\xi)$$

$$G = e^{(-\lambda-i\xi)t} - e^{(-\lambda+i\xi)t}$$

$$H = (-\xi^2-\lambda^2)e^{(-\lambda-i\xi)t} - (\xi^2-\lambda^2)e^{(-\lambda+i\xi)t}$$

$$I = (-\lambda+i\xi)(-e^{(-\lambda+i\xi)t}) + (-\lambda-i\xi)(e^{(-\lambda-i\xi)t})$$

Simplifying,

$$F = e^{(-\lambda-i\xi)t}(\lambda-i\xi) - e^{(-\lambda+i\xi)t}(\lambda+i\xi)$$

$$= e^{-\lambda t}(\lambda-i\xi)e^{-i\xi t} - e^{-\lambda t}(\lambda+i\xi)e^{i\xi t}$$

$$= e^{-\lambda t}(\lambda-i\xi)(\cos(\xi t) - i\sin(\xi t)) - e^{-\lambda t}(\lambda+i\xi)(\cos(\xi t)+i\sin(\xi t))$$

$$= e^{-\lambda t}\Big[\lambda\big(\cancel{\cos(\xi t)} - i\sin(\xi t)\big) - i\xi\big(\cos(\xi t) \cancel{- i\sin(\xi t)}\big) - \lambda\big(\cancel{\cos(\xi t)} + i\sin(\xi t)\big) - i\xi\big(\cos(\xi t) \cancel{+ i\sin(\xi t)}\big)\Big]$$

$$= -2ie^{-\lambda t}\left[\lambda\sin(\xi t) + \xi\cos(\xi t)\right] \tag{3}$$

$$G = e^{(-\lambda-i\xi)t} - e^{(-\lambda+i\xi)t} = e^{-\lambda t}\left[e^{-i\xi t} - e^{i\xi t}\right]$$

$$= -2ie^{-\lambda t}\sin(\xi t) \tag{4}$$

$$H = (-\xi^2-\lambda^2)e^{(-\lambda-i\xi)t} - (\xi^2-\lambda^2)e^{(-\lambda+i\xi)t}$$

$$= -\omega^2\left(e^{(-\lambda-i\xi)t} - e^{(-\lambda+i\xi)t}\right) \overset{(4)}{=} -\omega^2\left(-2ie^{-\lambda t}\sin(\xi t)\right) \tag{5}$$

where we used that $\lambda^2+\xi^2=\omega^2$.

$$I = (-\lambda+i\xi)\left(-e^{(-\lambda+i\xi)t}\right) + (-\lambda-i\xi)\left(e^{(-\lambda-i\xi)t}\right)$$

$$= -(-\lambda+i\xi)e^{-\lambda t}e^{i\xi t} + e^{-\lambda t}(-\lambda-i\xi)e^{-i\xi t}$$

$$= e^{-\lambda t}\Big[(\lambda - i\xi)\big(\cos(\xi t) + i\sin(\xi t)\big)$$
$$+ (-\lambda - i\xi)\big(\cos(\xi t) - i\sin(\xi t)\big)\Big]$$

$$= e^{-\lambda t}\Big[\lambda\big(\cancel{\cos(\xi t)} + i\sin(\xi t)\big) - i\xi\big(\cos(\xi t) + \cancel{i\sin(\xi t)}\big)$$
$$- \lambda\big(\cancel{\cos(\xi t)} - i\sin(\xi t)\big) - i\xi\big(\cos(\xi t) - \cancel{i\sin(\xi t)}\big)\Big]$$

$$= e^{-\lambda t}\big[2i\lambda\sin(\xi t) - 2i\xi\cos(\xi t)\big]$$

$$= -2i e^{-\lambda t}\big[\lambda\sin(\xi t) + \xi\cos(\xi t) - 2\lambda\sin(\xi t)\big] \qquad (6)$$

With (3)-(6) in the matrix (2) we have

$$e^{At} = \frac{e^{-\lambda t}}{\xi}\begin{bmatrix} \lambda\sin(\xi t) + \xi\cos(\xi t) & \sin(\xi t) \\ -\omega^2\sin(\xi t) & \lambda\sin(\xi t) + \xi\cos(\xi t) - 2\lambda\sin(\xi t) \end{bmatrix}$$

Hence,

$$\boxed{e^{At} = \frac{e^{-\lambda t}}{\xi}\Big[\big(\lambda\sin(\xi t) + \xi\cos(\xi t)\big)I + A\sin(\xi t)\Big]}$$

Now let $y_t = x_t'$, and define $\zeta = -\lambda + i\xi$,

$$g_t = \frac{1}{\xi}\operatorname{Im}(e^{\zeta t}) = \frac{1}{\xi}e^{-\lambda t}\sin(\xi t)$$

and

$$h_t = \frac{1}{\xi}\operatorname{Im}(\zeta e^{\zeta t}) = \frac{1}{\xi}e^{-\lambda t}\big(\xi\cos(\xi t) - \lambda\sin(\xi t)\big)$$

We'll the solution with the expression for e^{At} above. For this, let us compute

$$e^{A(t-s)} K X_s = \frac{e^{-\lambda(t-s)}}{\xi}\left[\left(\lambda \sin(\xi(t-s)) + \xi \cos(\xi(t-s))\right) I + A \sin(\xi(t-s))\right]$$

$$\cdot\, a_0 \eta \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_s \\ x_s' \end{bmatrix}$$

$$= \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} \lambda \sin(\xi(t-s)) + \xi \cos(\xi(t-s)) & \sin(\xi(t-s)) \\ -\omega^2 \sin(\xi(t-s)) & (\lambda - a_0)\sin(\xi(t-s)) + \xi \cos(\xi(t-s)) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -a_0\eta \end{bmatrix} \begin{bmatrix} x_s \\ x_s' \end{bmatrix}$$

$$= -a_0 \eta \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} 0 & \sin(\xi(t-s)) \\ 0 & \xi\cos(\xi(t-s)) - \lambda \sin(\xi(t-s)) \end{bmatrix} \begin{bmatrix} x_s \\ x_s' \end{bmatrix}$$

(7)
$$= \begin{bmatrix} \left(-a_0 \eta e^{-\lambda(t-s)}/\xi\right) \sin(\xi(t-s))\, y_s \\ \left(-a_0 \eta e^{-\lambda(t-s)}/\xi\right)\left(\xi\cos(\xi(t-s)) - \lambda\sin(\xi(t-s))\right) y_s \end{bmatrix} = \begin{bmatrix} -a_0 \eta g_{t-s} y_s \\ -a_0 \eta h_{t-s} y_s \end{bmatrix}$$

and

$$e^{A(t-s)} m = \frac{e^{-\lambda(t-s)}}{\xi}\left[\left(\lambda \sin(\xi(t-s)) + \xi \cos(\xi(t-s))\right) I + A \sin(\xi(t-s))\right]$$

$$T_0 \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(8)

$$= T_0 \eta \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} \sin(\xi(t-s)) \\ \xi\cos(\xi(t-s)) - \lambda\sin(\xi(t-s)) \end{bmatrix} = \begin{bmatrix} \eta T_0 \, g_{t-s} \\ \eta T_0 \, h_{t-s} \end{bmatrix}$$

Finally, using the solution from the item (ii), it follows that

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \int_0^t \begin{bmatrix} -a_0 \eta \, g_{t-s} Y_s \\ -a_0 \eta \, h_{t-s} Y_s \end{bmatrix} dB_s + \int_0^t \begin{bmatrix} \eta T_0 \, g_{t-s} \\ \eta T_0 \, h_{t-s} \end{bmatrix} dB_s$$

In particular,

$$X_t = \eta \int_0^t (T_0 - a_0 Y_s) g_{t-s} \, dB_s$$

and

$$Y_t = \eta \int_0^t (T_0 - a_0 Y_s) h_{t-s} \, dB_s$$