3.12. As in Exercise 3.9 we let $\circ dB_t$ denote Stratonovich differentials.

(i) Use (3.3.6) to transform the following Stratonovich differential equations into Itô differential equations:

(a) $dX_t = \gamma X_t dt + \alpha X_t \circ dB_t$

(b) $dX_t = \sin X_t \cos X_t dt + (t^2 + \cos X_t) \circ dB_t$

(ii) Transform the following Itô differential equations into Stratonovich differential equations:

(a) $dX_t = rX_t dt + \alpha X_t dB_t$

(b) $dX_t = 2e^{-X_t} dt + X_t^2 dB_t$

Using

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \tfrac{1}{2}\int_0^t \sigma'(s,X_s)\sigma(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s\,, \quad (3.3.6)$$

i.a.) $dX_t = b(t,x)dt + \frac{1}{2}\sigma'(t,x)\sigma(t,x)ds + \sigma(t,x)dB_t$

$dX_t = \gamma X_t dt + \frac{1}{2}\alpha^2 X_t dt + \alpha X_t dB_t$

$\boxed{\begin{gathered}\sigma(t,x) = \alpha x \\ \sigma'(t,x) = \alpha\end{gathered}}$

i. b) $\sigma(t,x) = t^2 + \cos x$

$\sigma'(t,x) = \dfrac{\partial\sigma}{\partial x} = -\operatorname{sen} x$

$dX_t = \sin X_t \cos X_t + \frac{1}{2}\left[-\sin X_t (t^2 + \cos X_t)\right] dt + (t^2 + \cos X_t)dB_t$

$= \sin X_t \cos X_t - \frac{1}{2}\sin X_t (t^2 + \cos X_t) dt + (t^2 + \cos X_t) dB_t$

ii.a) $dX_t = rX_t dt + \alpha X_t dB_t$

$dX_t = rX_t dt - \frac{1}{2}\alpha^2 X_t dt + \alpha X_t \circ dB_t$

ii.b) $dX_t = 2e^{-X_t} dt + X_t^2 dB_t$

$\boxed{\sigma(t,x) = x^2,\ \sigma_x = 2x}$

$dX_t = 2e^{-X_t} dt - X_t^3 dt + X_t^2 \circ dB_t$