7.19. Let B_t^x be 1-dimensional Brownian motion starting at $x > 0$. Define

$$\tau = \tau(x, \omega) = \inf\{t > 0;\ B_t^x(\omega) = 0\}\ .$$

From Exercise 7.4 we know that

$$\tau < \infty \quad \text{a.s. } P^x \text{ and } E^x[\tau] = \infty\ .$$

What is the distribution of the random variable $\tau(\omega)$?

a) To answer this, first find the *Laplace transform*

$$g(\lambda) := E^x[e^{-\lambda\tau}] \qquad \text{for } \lambda > 0\ .$$

(Hint: Let $M_t = \exp(-\sqrt{2\lambda}\, B_t - \lambda t)$. Then

$$\{M_{t\wedge\tau}\}_{t\geq 0} \quad \text{is a bounded martingale}\ .$$

[Solution: $g(\lambda) = \exp(-\sqrt{2\lambda}\, x)$.])

Notice that

$$E^x[M_t] = E^x[M_0] = e^{-\sqrt{2\lambda}\,x}$$

and

$$M_\tau = e^{-\sqrt{2\lambda}\,\cancel{B_\tau}^{\,0}} \cdot e^{-\lambda\tau} = e^{-\lambda\tau}$$

Thus

$$g(\lambda) = E[e^{-\lambda\tau}] = E[M_\tau] = E[M_0] = e^{-\sqrt{2\lambda}\,x}$$

b) To find the density $f(t)$ of τ it suffices to find $f(t) = f(t, x)$ such that

$$\int_0^\infty e^{-\lambda t} f(t)dt = \exp(-\sqrt{2\lambda}\, x) \qquad \text{for all } \lambda > 0$$

i.e. to find the *inverse* Laplace transform of $g(\lambda)$. Verify that

$$f(t, x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right); \qquad t > 0\ .$$

Define $g(\lambda, x) = \int_0^\infty e^{-\lambda t} f(t)\,dt$

and notice that it satisfies the ODE $g'' = 2\lambda g$ with solution

$$c_1 e^{\sqrt{2\lambda}\,x} + c_2 e^{-\sqrt{2\lambda}\,x}$$

Showing that $c_1 = 0$ and $c_2 = 1$, the result follows.