7.1. Find the generator of the following Itô diffusions:

a) $dX_t = \mu X_t dt + \sigma dB_t$ (The Ornstein-Uhlenbeck process) ($B_t \in \mathbf{R}$; μ, σ constants).

We'll use the formula

$$A f(x) = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^{\mathrm{T}})_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

If $X_t = f(B_t)$,

$$A f(x) = \mu x f'(x) + \frac{1}{2} \sigma^2 f''(x)$$

b) $dX_t = rX_t dt + \alpha X_t dB_t$ (The geometric Brownian motion) ($B_t \in \mathbf{R}$; r, α constants).

$$A f(x) = r x f'(x) + \frac{1}{2} (\alpha x)^2 f''(x)$$

c) $dY_t = r\, dt + \alpha Y_t dB_t$ ($B_t \in \mathbf{R}$; r, α constants)

$$A f(x) = r f'(x) + \frac{1}{2} (\alpha x)^2 f''(x)$$

d) $dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix}$ where X_t is as in a)

$$A f(x) = \frac{\partial f}{\partial x_1} + \mu x \frac{\partial f}{\partial x_2} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x_2^2}$$

e) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t \quad (B_t \in \mathbf{R})$

$$A f(x) = \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}$$

f) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$

$$Af(x) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2} + x_1^2 \frac{\partial^2 f}{\partial x_2^2} \right)$$

g) $X(t) = (X_1, X_2, \cdots, X_n)$, where

$$dX_k(t) = r_k X_k dt + X_k \cdot \sum_{j=1}^{n} \alpha_{kj} dB_j \ ; \qquad 1 \leq k \leq n$$

($(B_1, \cdots, B_n)$ is Brownian motion in $\mathbf{R}^n$, r_k and α_{kj} are constants).

$$Af(x) = \sum_{i=1}^{n} r_i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j \left(\sum_{k=1}^{n} \alpha_{ik} \alpha_{jk} \right) \frac{\partial^2 f}{\partial x_i \partial x_j}$$