5.14. If (B_1, B_2) denotes 2-dimensional Brownian motion we may introduce complex notation and put

$$\mathbf{B}(t) := B_1(t) + iB_2(t) \quad (i = \sqrt{-1}) .$$

$\mathbf{B}(t)$ is called *complex Brownian motion.*

(i) If $F(z) = u(z) + iv(z)$ is an *analytic* function i.e. F satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \qquad z = x + iy$$

and we define

$$Z_t = F(\mathbf{B}(t))$$

prove that

$$dZ_t = F'(\mathbf{B}(t))d\mathbf{B}(t), \tag{5.3.8}$$

where F' is the (complex) derivative of F. (Note that the usual second order terms in the (real) Itô formula are not present in (5.3.8)!)

$$dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t, X)dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X)dX_i dX_j$$

Note that

$$F(B(t)) = u(B(t)) + iv(B(t)) = u(B_1(t) + iB_2(t)) + iv(B_1(t) + iB_2(t))$$

Defining $g(t, z) = u(z) + iv(z)$, for $z = B(t)$ the Itô formula gives

$$dZ_t = dF(B(t)) = \frac{\partial u}{\partial B_1}dB_1 + i\frac{\partial u}{\partial B_2}dB_2 + \frac{1}{2}\left(\cancelto{0}{\frac{\partial^2 u}{\partial B_1^2}dt + \frac{\partial^2 u}{\partial B_2^2}dt}\right)$$

$$\boxed{dB = dB_1 + i\,dB_2}$$

$$+ i\frac{\partial v}{\partial B_1}dB_1 - \frac{\partial v}{\partial B_2}dB_2 + \frac{i}{2}\left(\cancelto{0}{\frac{\partial^2 v}{\partial B_1^2}dt + \frac{\partial^2 v}{\partial B_2^2}dt}\right)$$

$$= \left(\frac{\partial u}{\partial B_1} + i\frac{\partial v}{\partial B_1}\right)dB_1 + i\left(\frac{\partial u}{\partial B_2} + i\frac{\partial v}{\partial B_2}\right)dB_2$$

$$= \left(\frac{\partial u}{\partial B} + i\frac{\partial v}{\partial B}\right)dB = F'(B(t))dB(t)$$

(ii) Solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t d\mathbf{B}(t) \quad \alpha \text{ constant}) \ .$$

For more information about complex stochastic calculus involving analytic functions see e.g. Ubøe (1987).

Let $F(z) = e^{\alpha z}$. Clearly, $F(z)$ is analytic:

$$e^{\alpha z} = e^{\alpha(x+iy)} = e^{\alpha x} \cdot e^{i\alpha y} = \underbrace{e^{\alpha x}(\cos \alpha y}_{u} + \underbrace{i \sin \alpha y}_{v})$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \alpha e^{\alpha x} \cos \alpha y = \dfrac{\partial v}{\partial y} = \alpha e^{\alpha x} \cos \alpha y \\[2em] \dfrac{\partial u}{\partial y} = -\alpha e^{\alpha x} \sin \alpha y = -\dfrac{\partial v}{\partial x} = \alpha e^{\alpha x} \sin \alpha y \end{cases}$$

If $Z_t = F(B(t))$, by item (i),

$$dZ_t = F'(B(t))dB(t) = \alpha e^{\alpha B(t)} dB(t) = \alpha Z_t dB(t)$$