5.17. (The Gronwall inequality)
Let $v(t)$ be a nonnegative function such that

$$v(t) \le C + A\int_0^t v(s)ds \qquad \text{for } 0 \le t \le T$$

for some constants C, A. Prove that

$$v(t) \le C\exp(At) \qquad \text{for } 0 \le t \le T\,. \qquad (5.3.19)$$

Define

$$g(t) = A\exp\left(-\int_0^t A ds\right)\int_0^t v(s)ds = A\exp(-At)\int_0^t v(s)ds$$

and compute

$$g'(t) = A\exp(-At)\left[-A\int_0^t v(s)ds + v(t)\right]$$

$$\le A\exp(-At)\left[-A\int_0^t v(s)ds + C + A\int_0^t v(s)ds\right]$$

$$= AC\exp(-At)$$

Integrating,

$$g(t) \le AC\int_0^t \exp(-As)ds = AC\left[\frac{1}{A}\left(1-e^{-At}\right)\right] = C\left(1-e^{-At}\right)$$

Thus,

$$A\int_0^t v(s)ds = \exp(At)\,g(t) \le e^{At}C\left(1-e^{-At}\right)$$

Finally,

$$v(t) \le C + e^{At}C\left(1-e^{-At}\right) = C\exp(At)$$

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