3.14. Show that a function $h(\omega)$ is $\mathcal{F}_t$-measurable if and only if h is a pointwise limit (for a.a. ω) of sums of functions of the form

$$g_1(B_{t_1}) \cdot g_2(B_{t_2}) \cdots g_k(B_{t_k})$$

where $g_1, \ldots, g_k$ are bounded continuous functions and $t_j \le t$ for $j \le k$, $k = 1, 2, \ldots$

Hint: Complete the following steps:

a) We may assume that h is bounded.

b) For $n = 1, 2, \ldots$ and $j = 1, 2, \ldots$ put $t_j = t_j^{(n)} = j \cdot 2^{-n}$. For fixed n let $\mathcal{H}_n$ be the σ-algebra generated by $\{B_{t_j}(\cdot)\}_{t_j \le t}$. Then by Corollary C.9

$$h = E[h|\mathcal{F}_t] = \lim_{n\to\infty} E[h|\mathcal{H}_n] \qquad \text{(pointwise a.e. limit)}$$

c) Define $h_n := E[h|\mathcal{H}_n]$. Then by the Doob-Dynkin lemma (Lemma 2.1.2) we have

$$h_n(\omega) = G_n(B_{t_1}(\omega), \ldots, B_{t_k}(\omega))$$

for some Borel function $G_n: \mathbf{R}^k \to \mathbf{R}$, where $k = \max\{j; j \cdot 2^{-n} \le t\}$. Now use that any Borel function $G: \mathbf{R}^k \to \mathbf{R}$ can be approximated pointwise a.e. by a continuous function $F: \mathbf{R}^k \to \mathbf{R}$ and complete the proof by applying the Stone-Weierstrass theorem.

Assume that h is bounded and put $t_j = t_j^{(n)} = j \cdot 2^{-n}$, $n = 1, 2, \ldots$, $j = 1, 2, \ldots$

For a fixed n, let $\mathcal{H}_n$ be the σ-algebra generated by $\{B_{t_j}\}_{t_j \le t}$. Then, since $\mathcal{H}_n$ is an increasing family of σ-algebras, and $\mathcal{F}_t$ is the σ-algebra generated by $\{\mathcal{H}_n\}_{n=1}^{\infty}$

- The σ-algebra generated by $\{\mathcal{H}_n\}$ is contained in $\mathcal{F}_t$, by the definition of $\mathcal{H}_n$;
- Let $s \le t$ and $r \to s$. By the continuity of B.M., $B_r \to B_s \in \{\mathcal{H}_n\}$, by definition of $\mathcal{H}_n$. Since $B_s = \limsup_r B_r$, we know that $\limsup_r B_r \in \{\mathcal{H}_n\}$.

Now, by definition of $\mathcal{F}_t$ (σ-algebra generated by $B_s : s \le t$), we have that $\mathcal{F}_t \subseteq \{\mathcal{H}_n\}_{n=1}^{\infty}$.

by the Corollary C.9,

$$h = E[h|\mathcal{F}_t] = \lim_{n\to\infty} E[h|\mathcal{H}_n] \qquad \text{pointwise a.e.}$$

Let $h_n = E[h|\mathcal{H}_n]$. By the Doob-Dynkin lemma,

$$h_n(\omega) = G_n\{B_{t_1}(\omega), \ldots, B_{t_k}(\omega)\}$$

for some Borel function G_n, and where $k = \max\{j; j \cdot 2^{-n} \leq t\}$.

Since every Borel function $G: \mathbb{R}^k \to \mathbb{R}$ can be approximated pointwise a.e. by a continuous function $F: \mathbb{R}^k \to \mathbb{R}$, by the Stone-Weierstrass Theorem applied on $[t_1, t_k]$, F can be approximated by a polynomial function g_n, completing the proof.

Corollary C.9. *Let $X \in L^1(P)$, let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be an increasing family of σ-algebras, $\mathcal{N}_k \subset \mathcal{F}$ and define $\mathcal{N}_\infty$ to be the σ-algebra generated by $\{\mathcal{N}_k\}_{k=1}^{\infty}$. Then*

$$E[X|\mathcal{N}_k] \to E[X|\mathcal{N}_\infty] \qquad \text{as } k \to \infty\,,$$

a.e. P and in $L^1(P)$.

Lemma 2.1.2. *If $X, Y : \Omega \to \mathbf{R}^n$ are two given functions, then Y is $\mathcal{H}_X$-measurable if and only if there exists a Borel measurable function $g: \mathbf{R}^n \to \mathbf{R}^n$ such that*

$$Y = g(X)\,.$$