5.15. (Population growth in a stochastic, crowded environment)
The nonlinear stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t\,; \qquad X_0 = x > 0 \qquad (5.3.9)$$

is often used as a model for the growth of a population of size X_t in a stochastic, crowded environment. The constant $K > 0$ is called the *carrying capacity* of the environment, the constant $r \in \mathbf{R}$ is a measure of the quality of the environment and the constant $\beta \in \mathbf{R}$ is a measure of the size of the noise in the system.
Verify that

$$X_t = \frac{\exp\{(rK - \frac{1}{2}\beta^2)t + \beta B_t\}}{x^{-1} + r\int\limits_0^t \exp\{(rK - \frac{1}{2}\beta^2)s + \beta B_s\}ds}\,; \qquad t \geq 0 \qquad (5.3.10)$$

is the unique (strong) solution of (5.3.9). (This solution can be found by performing a substitution (change of variables) which reduces (5.3.9) to a linear equation. See Gard (1988), Chapter 4 for details.)

Step 1. Multiply by X_t^{-2}:

$$X_t^{-2}\,dX_t = (rX_t^{-1}K - r)\,dt + \beta X_t^{-1}\,dB_t$$

Step 2. Substitute $Y_t = X_t^{-1}$.

Notice that

$$dY_t = -\frac{dX_t}{X_t^2} + \frac{(dX_t)^2}{X_t^3}$$

Then

$$dY_t = (-rY_tK + r)\,dt - \beta Y_t\,dB_t + \beta^2 Y_t\,dt$$

$$= (-rK + \beta^2)Y_t\,dt - \beta Y_t\,dB_t + r\,dt$$

Step 3. Change of variables.

Let $Z_t := Y_t e^{(rK-\beta^2)t}$. By Itô's formula,

$$dZ_t = (rK - \beta^2)Y_t e^{(rK-\beta^2)t}\,dt + e^{(rK-\beta^2)t}\,dY_t$$

$$= \cancel{(rK-\beta^2)Y_t e^{(rK-\beta^2)t}\,dt} + e^{(rK-\beta^2)t}\left(\cancel{(-rK+\beta^2)Y_t\,dt} - \beta Y_t\,dB_t + r\,dt\right)$$

Thus,

$$dZ_t = r e^{(rK-\beta^2)t} dt - \beta Z_t dB_t$$

Step 4. Use integrating factor to solve the SDE.

Let N_t be such that $dN_t = \theta_t dt + \gamma_t dB_t$. By integration by parts (ex. 4.3),

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \, .$$

Thus,

$$d(N_t Z_t) = N_t dZ_t + Z_t dN_t + dN_t dZ_t$$

$$= N_t r e^{(rK-\beta^2)t} dt - \beta N_t Z_t dB_t + \theta_t Z_t dt + \gamma_t Z_t dB_t - \beta \gamma_t Z_t dt$$

Setting $\theta_t = \beta \gamma_t$ and $\gamma_t = \beta N_t$ we have

$$dN_t = \beta^2 N_t dt + \beta N_t dB_t$$

Then (see ex. 5.3 and example 5.1.2 of my notes),

$$N_t = N_0 \exp\left(\frac{1}{2}\beta^2 t + \beta B_t \right)$$

and

$$d(N_t Z_t) = N_t r e^{(rK-\beta^2)t} = N_0 r \exp\left(\beta B_t + \left(rK - \frac{1}{2}\beta^2 \right) t \right)$$

Choosing $N_0 = 1$,

$$N_t Z_t = Z_0 + \int_0^t r \exp\left(\beta B_s + \left(rK - \frac{1}{2}\beta^2\right)s\right) ds$$

Thus, using that $Z_t := Y_t e^{(rK-\beta^2)t}$ and $Y_t = X_t^{-1}$, $Z_0 = X_0^{-1} = x^{-1}$,

$$X_t = Y_t^{-1} = Z_t^{-1} e^{(rK-\beta^2)t}$$

$$= \frac{e^{(rK-\beta^2)t} N_t}{Z_0 + \int_0^t r \exp\left(\beta B_s + \left(rK - \frac{1}{2}\beta^2\right)s\right) ds}$$

$$= \frac{\exp\left[(rK - 1/2\beta^2)t + \beta B_t\right]}{x^{-1} + r\int_0^t \exp\left[(rK - 1/2\beta^2)s + \beta B_s\right] ds}$$

$\blacksquare$