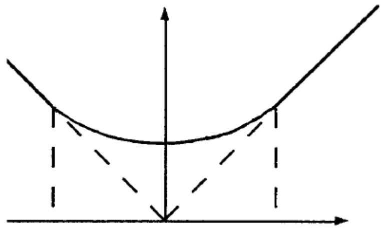
4.10. (Tanaka's formula and local time).
What happens if we try to apply the Itô formula to $g(B_t)$ when B_t is 1-dimensional and $g(x) = |x|$? In this case g is not C^2 at $x = 0$, so we modify $g(x)$ near $x = 0$ to $g_\epsilon(x)$ as follows:

$$g_\epsilon(x) = \begin{cases} |x| & \text{if } \quad |x| \geq \epsilon \\ \frac{1}{2}(\epsilon + \frac{x^2}{\epsilon}) & \text{if } \quad |x| < \epsilon \end{cases}$$

where $\epsilon > 0$.



a) Apply Exercise 4.8 b) to show that

$$g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g'_\epsilon(B_s)dB_s + \frac{1}{2\epsilon} \cdot |\{s \in [0,t]; B_s \in (-\epsilon, \epsilon)\}|$$

where $|F|$ denotes the Lebesgue measure of the set F.

Since

- g_ε is C^1 everywhere
- g_ε is C^2 outside 0
- $|g''_\varepsilon(x)| = \begin{cases} 0, & |x| \geqslant \varepsilon \\ 1/\varepsilon, & |x| < \varepsilon \end{cases}$ $(*)$

Applying 4.8. b),

$$\boxed{g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds\,.}$$

$$g_\varepsilon(B_t) = g_\varepsilon(B_0) + \int_0^t g'_\varepsilon(B_s)\,dB_s + \frac{1}{2}\int_0^t g''_\varepsilon(B_s)\,ds$$

Given our expression for $g''_\varepsilon(x)$ in $(*)$ we see that

$$\int_0^t g''(B_s)\,ds = \frac{1}{\varepsilon}|\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\}|$$

Hence,

$$g_\varepsilon(B_t) = g_\varepsilon(B_0) + \int_0^t g'_\varepsilon(B_s)\,dB_s + \frac{1}{2\varepsilon}|\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\}|$$

∎

b) Prove that

$$\int_0^t g'_\epsilon(B_s)\cdot \mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s = \int_0^t \frac{B_s}{\epsilon}\cdot \mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s \to 0$$

in $L^2(P)$ as $\epsilon \to 0$.
(Hint: Apply the Itô isometry to

$$E\Big[\Big(\int_0^t \frac{B_s}{\epsilon}\cdot \mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s\Big)^2\Big].$$

$$(*)\quad g'_\varepsilon(x) = \begin{cases} \dfrac{x}{|x|}, & |x| \geq \varepsilon \\ \dfrac{x}{\varepsilon}, & |x| < \varepsilon \end{cases}$$

By the expression for g'_ε in $(*)$, we have

$$\int_0^t g'_\varepsilon(B_s)\cdot \chi_{B_s\in(-\varepsilon,\varepsilon)}\,dB_s = \int_0^t \frac{B_s}{\varepsilon}\chi_{B_s\in(-\varepsilon,\varepsilon)}\,dB_s$$

Using the Itô's Isometry,

$$\mathbb{E}\left[\left(\int_0^t \frac{B_s}{\varepsilon}\chi_{B_s\in(-\varepsilon,\varepsilon)}\,dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t\left(\frac{B_s}{\varepsilon}\right)^2\chi_{B_s\in(-\varepsilon,\varepsilon)}\,ds\right]$$

$$= \mathbb{E}\left[\frac{1}{\varepsilon^2}\int_0^t B_s^2\,\chi_{B_s\in(-\varepsilon,\varepsilon)}\,ds\right] \leq \mathbb{E}\left[\int_0^t \chi_{B_s\in(-\varepsilon,\varepsilon)}\,ds\right]$$

$$= \int_0^t \mathbb{E}\left[\chi_{B_s\in(-\varepsilon,\varepsilon)}\right]ds = \int_0^t \mathbb{P}\left[B_s\in(-\varepsilon,\varepsilon)\right]ds \longrightarrow 0$$

as $\varepsilon \to 0$. ∎

c) By letting $\epsilon \to 0$ prove that

$$|B_t| = |B_0| + \int_0^t \operatorname{sign}(B_s)dB_s + L_t(\omega) , \qquad (4.3.12)$$

where

$$L_t = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \cdot |\{s \in [0,t]; B_s \in (-\epsilon, \epsilon)\}| \quad (\text{limit in } L^2(P))$$

and

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for} \quad x \le 0 \\ 1 & \text{for} \quad x > 0 \end{cases} .$$

L_t is called the *local time* for Brownian motion at 0 and (4.3.12) is the *Tanaka formula* (for Brownian motion). (See e.g. Rogers and Williams (1987)).

Using the previous items,

$$|B_t| = \lim_{\varepsilon \downarrow 0} g_\varepsilon(B_t) =$$

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(B_0) + \lim_{\varepsilon \downarrow 0} \int_0^t g_\varepsilon'(B_s)\,dB_s + \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} |\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\}|$$

$$= |B_0| + \lim_{\varepsilon \downarrow 0} \int_0^t \frac{B_s}{|B_s|} \chi_{|B_s| \geq \varepsilon}\,dB_s + \cancel{\lim_{\varepsilon \downarrow 0} \int_0^t \frac{B_s}{\varepsilon} \chi_{B_s \in (-\varepsilon, \varepsilon)}\,dB_s}_{\to 0} + L_t$$

$$= |B_0| + \int_0^t \operatorname{sign}(B_s)\,dB_s + L_t$$