7.9. Let X_t be a geometric Brownian motion, i.e.

$$dX_t = rX_t dt + \alpha X_t dB_t \, , \qquad X_0 = x > 0$$

where $B_t \in \mathbf{R}$; r, α are constants.

a) Find the generator A of X_t and compute $Af(x)$ when $f(x) = x^\gamma$; $x > 0$, γ constant.

The generator is given by

$$A = rx \frac{\partial}{\partial x} + \frac{1}{2} \alpha^2 x^2 \frac{\partial^2}{\partial x^2}$$

Thus,

$$Af(x) = rx\gamma x^{\gamma-1} + \frac{1}{2} \alpha^2 x^2 \gamma(\gamma-1) x^{\gamma-2}$$

$$= \left(r\gamma + \frac{1}{2} \alpha^2 \gamma(\gamma-1) \right) x^\gamma$$

b) If $r < \frac{1}{2}\alpha^2$ then $X_t \to 0$ as $t \to \infty$, a.s. Q^x (Example 5.1.1). But what is the probability p that X_t, when starting from $x < R$, ever hits the value R ? Use Dynkin's formula with $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - \frac{2r}{\alpha^2}$, to prove that

$$p = \left(\frac{x}{R} \right)^{\gamma_1} .$$

First step: define the stopping time.

Take $0 < K < x < R$ and define

- $\tau_R = \inf\{t > 0 : X_t \neq R\}$
- $\tau = \min\{K, \tau_R\}$

Second step: apply Dynkin's formula.

Notice that $f \in C_0^2$ and τ is a stopping time with finite expected value. Then,

$$\mathbb{E}^x\left[f(X_\tau)\right] = f(x) + \mathbb{E}^x\left[\int_0^\tau A f(X_s)\, ds\right]$$

$$= x^{\gamma_1} + \mathbb{E}^x\left[\int_0^\tau \left(r\gamma_1 + \frac{1}{2}\alpha^2 \gamma_1(\gamma_1 - 1)\right) X_s^{\gamma_1}\, ds\right]$$

$$= x^{\gamma_1} + \mathbb{E}^x\left[\int_0^\tau \underbrace{\left(r + \frac{1}{\cancel{2}}\cancel{\alpha^2}\left(-\frac{\cancel{2}r}{\cancel{\alpha^2}}\right)\right)}_{=0}\left(1 - \frac{2r}{\alpha^2}\right) X_s^{1-2r/\alpha^2}\, ds\right]$$

Hence,

$$\mathbb{E}^x\left[f(X_\tau)\right] = f(x)$$

Third step: compute the expected value

Since $\lim \tau < \infty$, by taking $K \to \infty$,

$$\mathbb{E}^x\left[f(X_{\tau_R})\right] = f(x)$$

And notice that $X_\tau \in \{K, R\}$. Define

$$p = \mathbb{P}\left[X_\tau = R\right] \quad \text{and} \quad 1 - p = \mathbb{P}\left[X_\tau = K\right]$$

Then

$$E^x[f(X_\tau)] = f(x) = p\,f(R) + (1-p)\,f(K)$$

$$\iff \quad x^{1-2r/\alpha^2} = pR^{1-2r/\alpha^2} + (1-p)K^{1-2r/\alpha^2} \qquad (*)$$

Taking the limit and noticing that $\gamma_1 > 0$,

$$\lim_{K \downarrow 0} E^x[f(X_\tau)] = pR^{1-2r/\alpha^2} = x^{1-2r/\alpha^2}$$

and thus

$$p = \left(\frac{x}{R}\right)^{\gamma_1}$$

c) If $r > \frac{1}{2}\alpha^2$ then $X_t \to \infty$ as $t \to \infty$, a.s. Q^x. Put

$$\tau = \inf\{t > 0; X_t \geq R\}\ .$$

Use Dynkin's formula with $f(x) = \ln x$, $x > 0$ to prove that

$$E^x[\tau] = \frac{\ln \frac{R}{x}}{r - \frac{1}{2}\alpha^2}\ .$$

(Hint: First consider exit times from (ρ, R), $\rho > 0$ and then let $\rho \to 0$. You need estimates for

$$(1 - p(\rho)) \ln \rho\ ,$$

where

$$p(\rho) = Q^x[X_t \text{ reaches the value } R \text{ before } \rho]\ ,$$

which you can get from the calculations in a), b).)

By Dynkin's formula for $f(x) = \ln x$

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)\,ds\right]$$

Given that

$$A f(x) = r - \frac{1}{2}\alpha^2$$

we have

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau \left(r - \frac{1}{2}\alpha^2\right) ds\right]$$

$$= \ln x + \left(r - \frac{1}{2}\alpha^2\right)\mathbb{E}^x[\tau]$$

Then

$$\mathbb{E}^x[\tau] = \frac{\mathbb{E}^x[f(X_\tau)] - \ln x}{r - \alpha^2/2}$$

and using the given $p(\rho)$,

$$\mathbb{E}^x[\tau] = \frac{p(\rho) f(R) + (1 - p(\rho)) f(\rho) - \ln x}{r - \alpha^2/2} \qquad (**)$$

Taking $(*)$ with $K = \rho$,

$$x^{\gamma_1} = pR^{\gamma_1} + (1-p)\rho^{\gamma_1} \iff p = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}} = \frac{\frac{1}{x^{-\gamma_1}} - \frac{1}{\rho^{-\gamma_1}}}{\frac{1}{R^{-\gamma_1}} - \frac{1}{\rho^{-\gamma_1}}} = \frac{\frac{\rho^{-\gamma_1} - x^{-\gamma_1}}{x^{-\gamma_1}\rho^{-\gamma_1}}}{\frac{\rho^{-\gamma_1} - R^{-\gamma_1}}{R^{-\gamma_1}\rho^{-\gamma_1}}}$$

Thus,

$$p = \frac{(\rho^{-\gamma_1} - x^{-\gamma_1})}{x^{-\gamma_1}\rho^{-\gamma_1}} \cdot \frac{R^{-\gamma_1}\rho^{-\gamma_1}}{(\rho^{-\gamma_1} - R^{-\gamma_1})} = \frac{R^{-\gamma_1}(\rho^{-\gamma_1} - x^{-\gamma_1})}{x^{-\gamma_1}(\rho^{-\gamma_1} - R^{-\gamma_1})}$$

Since $r > \alpha^2/2$, $\gamma_1 < 0$,

$$\lim_{\rho \to 0} p = 1 \qquad \text{and} \qquad \lim_{\rho \to \infty} \left[(1 - p(\rho)) \ln \rho \right] = 0$$

Plugging into (2):

$$E^x[\tau] = \frac{\ln R - \ln x}{r - \alpha^2/2} = \frac{\ln \frac{R}{x}}{r - \alpha^2/2}$$

$\square$