

3.9. Suppose $f \in \mathcal{V}(0, T)$ and that $t \rightarrow f(t, \omega)$ is continuous for a.a. ω . Then we have shown that

$$\int_0^T f(t, \omega) dB_t(\omega) = \lim_{\Delta t_j \rightarrow 0} \sum_j f(t_j, \omega) \Delta B_j \quad \text{in } L^2(P).$$

Similarly we define the *Stratonovich integral* of f by

$$\int_0^T f(t, \omega) \circ dB_t(\omega) = \lim_{\Delta t_j \rightarrow 0} \sum_j f(t_j^*, \omega) \Delta B_j, \quad \text{where } t_j^* = \frac{1}{2}(t_j + t_{j+1}),$$

whenever the limit exists in $L^2(P)$. In general these integrals are different. For example, compute

$$\int_0^T B_t \circ dB_t$$

and compare with Example 3.1.9.

$$\int_0^T B_t \circ dB_t = \lim_{\Delta t_j \rightarrow 0} \sum_j B_{j^*} \Delta B_j, \quad t_j^* = t_j^* = \frac{1}{2}(t_j + t_{j+1})$$

Now

$$\begin{aligned} \Delta(B_j^2) &= B_{j+1}^2 - B_j^2 = [((B_{j+1} - B_{j^*}) + B_{j^*})^2 - ((B_j - B_{j^*}) + B_{j^*})^2] \\ &= (B_{j+1} - B_{j^*})^2 + 2B_{j^*}(B_{j+1} - B_{j^*}) + B_{j^*}^2 \\ &\quad - [(B_j - B_{j^*})^2 + 2B_{j^*}(B_j - B_{j^*}) + B_{j^*}^2] \\ &= (B_{j+1} - B_{j^*})^2 - (B_j - B_{j^*})^2 + 2B_{j^*}(B_{j+1} - B_j) \end{aligned}$$

Hence,

$$\begin{aligned} B_+^2 &= \sum_j \Delta(B_j^2) = \sum_j (B_{j+1} - B_{j^*})^2 - \sum_j (B_j - B_{j^*})^2 \\ &\quad + 2 \sum_j B_{j^*} (B_{j+1} - B_j) \end{aligned}$$

i.e.,

$$\sum_j B_{j^*} \Delta B_j = \frac{1}{2} B_+^2 + \frac{1}{2} \sum_j (B_j - B_{j^*})^2 - \frac{1}{2} \sum_j (B_{j+1} - B_{j^*})^2$$

Since

$$\sum_j (B_j - B_{j^*})^2 \rightarrow \frac{1}{2} \quad \text{and} \quad \sum_j (B_{j+1} - B_{j^*})^2 \rightarrow \frac{1}{2}$$

in $L^2(P)$ as $\Delta t_j \rightarrow 0$, we have that

$$\int_0^T B_+ \circ dB_+ = \frac{1}{2} B_+^2 + \frac{1}{4} - \frac{1}{4} = \frac{1}{2} B_+^2$$