3.8. a) Let Y be a real valued random variable on $(\Omega, \mathcal{F}, P)$ such that

$$E[|Y|] < \infty \,.$$

Define

$$M_t = E[Y|\mathcal{F}_t] \,; \qquad t \geq 0 \,.$$

Show that M_t is an $\mathcal{F}_t$-martingale.

b) Conversely, let $M_t;\, t \geq 0$ be a real valued $\mathcal{F}_t$-martingale such that

$$\sup_{t\geq 0} E[|M_t|^p] < \infty \qquad \text{for some } p > 1 \,.$$

Show that there exists $Y \in L^1(P)$ such that

$$M_t = E[Y|\mathcal{F}_t] \,.$$

(Hint: Use Corollary C.7.)

a) • $E[M_s | \mathcal{F}_t] = E[E[Y | \mathcal{F}_s] | \mathcal{F}_t]$
$= E[Y | \mathcal{F}_t] = M_t$

for all $s \geq t$

• $M_t = E[Y | \mathcal{F}_t] = Y$ is $\mathcal{F}_t$-measurable

• $E[|M_t|] = E[|E[Y|\mathcal{F}_t]|] = E[|Y|] < \infty$

Thus, M_t is an $\mathcal{F}_t$-martingale.

b)

Corollary C.7. *Let M_t be a continuous martingale such that*

$$\sup_{t>0} E[|M_t|^p] < \infty \qquad \text{for some } p > 1 \,.$$

Then there exists $M \in L^1(P)$ such that $M_t \to M$ a.e. (P) and

$$\int |M_t - M| dP \to 0 \qquad \text{as } t \to \infty \,.$$

By the Corollary C.7, there exists $M \in L^1(P)$ such that $M_t \to M$ a.e. and

$$\lim_{t\to\infty} \int M_t \, dP = \int M \, dP$$

Given that M_t is a martingale for all $s > t$, for $A \in \mathcal{F}_t$,

$$\int_A M_s \, dP = \int_A E[M_s | \mathcal{F}_t] \, dP = \int_A M_t \, dP$$

Taking the limit as $s \to \infty$,

$$\lim_{s\to\infty} \int_A M_s \, dP = \int_A M_t \, dP$$

i.e.,

$$\int_A m \, dP = \int_A \mathbb{E}[m \mid \mathcal{F}_t] \, dP \qquad \forall A \in \mathcal{F}_t$$

Hence,

$$m_t = \mathbb{E}[m \mid \mathcal{F}_t]$$

$\square$