7.12. (Local martingales)

An $\mathcal{N}_t$-adapted stochastic process $Z(t) \in \mathbf{R}^n$ is called a *local martingale* with respect to the given filtration $\{\mathcal{N}_t\}$ if there exists an increasing sequence of $\mathcal{N}_t$-stopping times τ_k such that

$$\tau_k \to \infty \quad \text{a.s. as } k \to \infty$$

and

$$Z(t \wedge \tau_k) \qquad \text{is an } \mathcal{N}_t\text{-martingale for all } k\ .$$

a) Show that if $Z(t)$ is a local martingale and there exists a constant $T \le \infty$ such that the family $\{Z(\tau)\}_{\tau \le T}$ is uniformly integrable (Appendix C) then $\{Z(t)\}_{t \le T}$ is a martingale.

1. $\{Z(t)\}_{t \le T}$ is $\mathcal{N}_t$-adapted, i.e., is $\mathcal{N}_t$-measurable for all t.

2. We'll use the following result:

Theorem C.4. *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of real measurable functions on Ω such that*

$$\lim_{k\to\infty} f_k(\omega) = f(\omega) \qquad \text{for a.a. } \omega\ .$$

Then the following are equivalent:

1) $\{f_k\}$ *is uniformly integrable*

2) $f \in L^1(P)$ *and* $f_k \to f$ *in* $L^1(P)$, *i.e.* $\int |f_k - f| dP \to 0$ *as* $k \to \infty$.

Since $\{Z(\tau)\}_{\tau \le T} = \{Z_\tau\}$ is uniformly integrable, and

$$\lim_{k\to\infty} Z_\tau = Z_t \qquad \text{since} \quad \tau = t \wedge \tau_k$$

then $Z_t \in L^1(P)$ and $\lim_{k\to\infty} \int |Z_\tau - Z_t| dP = 0$

i.e., $\mathbb{E}[|Z_t|] < \infty$

3. Since $Z_{t\wedge\tau_k}$ is an $\mathcal{N}_t$-martingale, for $s \le t$,

$$E^x[Z_{\tau_k \wedge t} | \mathcal{N}_s] = Z_{\tau_k \wedge s} \xrightarrow{k\to\infty} Z_s$$

Thus, $\quad \mathbb{E}^x[Z_t | \mathcal{N}_s] = \lim_{k\to\infty} E^x[Z_{\tau_k \wedge t} | \mathcal{N}_s] = Z_s$

b) In particular, if $Z(t)$ is a local martingale and there exists a constant $K < \infty$ such that

$$E[Z^2(\tau)] \leq K$$

for all stopping times $\tau \leq T$, then $\{Z(t)\}_{t \leq T}$ is a martingale.

Since $Z_\tau \to Z_t$ in $L^1(\mathbb{P})$ and $Z_t \in L^1(\mathbb{P})$, $\{Z_t\}$ is uniformly integrable.

By the previous item, the result follows.

c) Show that if $Z(t)$ is a *lower bounded* local martingale, then $Z(t)$ is a supermartingale (Appendix C).

We already verified that $Z(t)$ is $\mathcal{N}_t$-adapted and $\mathbb{E}[|Z(t)|] < \infty$.
Now we need to show that

$$Z(s) \geq \mathbb{E}[Z(t) \mid \mathcal{N}_s], \qquad t > s$$

Notice that, assuming $Z_t \geq 0$,

$$\mathbb{E}[Z_t \mid \mathcal{N}_s] \leq \lim_{k \to \infty} \mathbb{E}[Z_\tau \mid \mathcal{N}_s] = \mathbb{E}[Z_t \mid \mathcal{N}_s] = Z_s$$