4.12. Let $dX_t = u(t,\omega)dt + v(t,\omega)dB_t$ be an Itô process in $\mathbf{R}^n$ such that

$$E\Big[\int_0^t |u(r,\omega)|dr\Big] + E\Big[\int_0^t |vv^T(r,\omega)|dr\Big] < \infty \qquad \text{for all } t \geq 0\,.$$

Suppose X_t is an $\{\mathcal{F}_t^{(n)}\}$-martingale. Prove that

$$u(s,\omega) = 0 \qquad \text{for a.a. } (s,\omega) \in [0,\infty) \times \Omega\,. \tag{4.3.13}$$

Our first step is to show that

If X_t is an $\mathcal{F}_t^{(n)}$-martingale, then deduce that

$$E\Big[\int_t^s u(r,\omega)dr | \mathcal{F}_t^{(n)}\Big] = 0 \qquad \text{for all } s \geq t\,. \tag{1}$$

In order to do that, notice that

$$\overset{\text{martingale}}{0} = \mathbb{E}\left[X_s - X_t \mid \mathcal{F}_t^{(n)}\right] =$$

$$= \mathbb{E}\left[\left(X_0 + \int_0^s u\,dr + \int_0^s v\,dB_r\right) - \left(X_0 + \int_0^t u\,dr + \int_0^t v\,dB_r\right) \Big| \mathcal{F}_t^{(n)}\right]$$

$$= \mathbb{E}\left[\int_0^s u\,dr - \int_0^t u\,dr + \int_0^s v\,dB_r - \int_0^t v\,dB_r \Big| \mathcal{F}_t^{(n)}\right]$$

Thm. 3.2.1 $$= \mathbb{E}\left[\int_t^s u\,dr \Big| \mathcal{F}_t^{(n)}\right] + \underbrace{\cancel{\mathbb{E}\left[\int_t^s v\,dB_r \Big| \mathcal{F}_t^{(n)}\right]}}_{0} = \mathbb{E}\left[\int_t^s u\,dr \Big| \mathcal{F}_t^{(n)}\right]$$

Now, we are going to do the following:

Differentiate w.r.t. s to deduce that

$$E[u(s,\omega)|\mathcal{F}_t^{(n)}] = 0 \qquad \text{a.s., for a.a. } s > t\,.$$

Then let $t \uparrow s$ and apply Corollary C.9.

In fact,

$$\frac{d}{ds}\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \Big| \mathcal{F}_t^{(n)}\right] = \mathbb{E}\left[\frac{d}{ds}\left(\int_t^s u(r,\omega)\,dr\right) \Big| \mathcal{F}_t^{(n)}\right]$$

$$= \mathbb{E}\left[u(s,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] = 0 \quad (2)$$

which is zero by the R.H.S. of (1).

Corollary C.9. *Let $X \in L^1(P)$, let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be an increasing family of σ-algebras, $\mathcal{N}_k \subset \mathcal{F}$ and define $\mathcal{N}_\infty$ to be the σ-algebra generated by $\{\mathcal{N}_k\}_{k=1}^{\infty}$. Then*

$$E[X|\mathcal{N}_k] \to E[X|\mathcal{N}_\infty] \qquad \text{as } k \to \infty ,$$

a.e. P and in $L^1(P)$.

By the Corollary C9. and (2),

$$\lim_{t \uparrow s} \mathbb{E}\left[u(t,\omega) \,\middle|\, \mathcal{F}_t^{(n)} \right] = \mathbb{E}\left[u(s,\omega) \,\middle|\, \mathcal{F}_t^{(n)} \right] = 0$$

Hence,

$$\mathbb{E}\left[u(s,\omega) \,\middle|\, \mathcal{F}_t^{(n)} \right] = u(t,\omega) \quad \text{a.a.}$$

$\square$

$$\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] = \mathbb{E}\left[\int_0^s u(r,\omega)\,dr - \int_0^t u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right]$$

$$= \mathbb{E}\left[\int_0^s u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] - \mathbb{E}\left[\int_0^t u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right]$$

Since u is $\mathcal{F}_t^{(n)}$-measurable,

$$\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] = \int_0^t u(r,\omega)\,dr - \int_0^t u(r,\omega)\,dr = 0 \quad (1)$$

$\triangle$