7.18. a) Let

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t\,; \qquad X_0 = x$$

be a 1-dimensional Itô diffusion with characteristic operator $\mathcal{A}$. Let $f \in C^2(\mathbf{R})$ be a solution of the differential equation

$$\mathcal{A}f(x) = b(x)f'(x) + \tfrac{1}{2}\sigma^2(x)f''(x) = 0\,; \qquad x \in \mathbf{R}\,. \tag{7.5.7}$$

Let $(a,b) \subset \mathbf{R}$ be an open interval such that $x \in (a,b)$ and put

$$\tau = \inf\{t > 0;\ X_t \notin (a,b)\}\,.$$

Assume that $\tau < \infty$ a.s. Q^x and define

$$p = P^x[X_\tau = b]\,.$$

Use Dynkin's formula to prove that if $f(b) \neq f(a)$ then

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}\,. \tag{7.5.8}$$

In other words, the *harmonic measure* $\mu^x_{(a,b)}$ of X on $\partial(a,b) = \{a,b\}$ is given by

$$\mu^x_{(a,b)}(b) = \frac{f(x) - f(a)}{f(b) - f(a)}\,, \quad \mu^x_{(a,b)}(a) = \frac{f(b) - f(x)}{f(b) - f(a)}\,. \tag{7.5.9}$$

Since $\mathcal{A}f(x) = 0$, by Dynkin's formula,

$$\mathbb{E}[f(X_\tau)] = f(x)$$

Using the definition of p,

$$pf(b) + (1-p)f(a) = f(x)$$

I.e.,

$$\frac{f(x) - f(a)}{f(b) - f(a)} = p \qquad (*)$$

b) Now specialize to the process

$$X_t = x + B_t\,; \qquad t \geq 0\,.$$

Prove that

$$p = \frac{x-a}{b-a}\,. \tag{7.5.10}$$

Notice that $\mathcal{A}f(x) = \frac{1}{2}f''(x)$

Taking $f(x) = x$, $\mathcal{A}f(x) = 0$ and the result follows from $(*)$.

c) Find p if

$$X_t = x + \mu t + \sigma B_t \; ; \qquad t \geq 0$$

where $\mu, \sigma \in \mathbf{R}$ are nonzero constants.

Here,

$$Af(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x)$$

Solving the ODE $Af(x)=0$:

Characteristic equation:

$$\frac{1}{2}\sigma^2\lambda^2 + \mu\lambda = 0 \iff \lambda\left(\frac{1}{2}\sigma^2\lambda + \mu\right) = 0$$

$$\iff \lambda = 0 \quad \text{or} \quad \frac{1}{2}\sigma^2\lambda + \mu = 0 \iff \lambda = \frac{-2\mu}{\sigma^2}$$

Let $f(x) = e^{-2\mu x/\sigma^2}$ and notice that, since

$$f'(x) = \frac{-2\mu}{\sigma^2} e^{-2\mu x/\sigma^2} \quad \text{and} \quad f''(x) = \frac{4\mu^2}{\sigma^4} e^{-2\mu x/\sigma^2}$$

$f(x)$ solves the ODE.

Using that,

$$p = \frac{f(x) - f(a)}{f(b) - f(a)} = \frac{e^{-2\mu x/\sigma^2} - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}$$