3.1. Prove directly from the definition of Itô integrals (Definition 3.1.6) that

$$\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds\,.$$

(Hint: Note that

$$\sum_j \Delta(s_j B_j) = \sum_j s_j \Delta B_j + \sum_j B_{j+1} \Delta s_j\ .)$$

Our first step is to define an elementary function and prove that it converges to our function $f(s) = s$ in $L^2(P)$.

Let $\varphi_n = \sum_j s_j \chi_{[t_j, t_{j+1})}(s)$. Then,

$$E\left[\int_0^t (\varphi_n - s)^2 ds\right] = E\left[\sum_j \int_{t_j}^{t_{j+1}} (s_j - s)\,ds\right]$$

$$\overset{*}{=} \sum_j \int_{t_j}^{t_{j+1}} (s - t_j)\,ds = \sum_j \frac{1}{2}(t_{j+1} - t_j)^2$$

Hence,

$$\lim_{\Delta t_j \to 0} \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 = 0$$

$*$ Recall that $E[(B_t - B_s)^2] = t - s$

and

$$\int_0^t s\,dB_s = \lim_{\Delta t_j \to 0} \int_0^t \varphi_n\,dB_s = \lim_{\Delta t_j \to 0} \sum_j s_j \Delta B_j$$

Now, using the fact that

$$\sum_j s_j \Delta B_j = \sum_j \Delta(s_j B_j) - \sum_j B_{j+1} \Delta s_j$$

we have

$$\lim_{\Delta t_j \to 0} \sum_j s_j \Delta B_j = \lim_{\Delta t_j \to 0} \sum_j \Delta(s_j B_j) - \lim_{\Delta t_j \to 0} B_{j+1} \Delta s_j$$

$$= tB_t - \int_0^t B_s\,ds$$

$\square$

ØKSENDAL 3.2

We want to prove that

$$\int_0^t B_s^2 \, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

First, let

$$\varphi_n(s) = \sum_j B_j^2 \, \chi_{[t_j, t_{j+1})}(s)$$

Notice that

$$E\left[\int_0^t (\varphi_n - B_s^2)^2 \, ds \right] = E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j^2 - B_s^2)^2 \, ds \right]$$

$$= \sum_j \int_{t_j}^{t_{j+1}} (s^2 - t_j^2) \, ds = \sum_j \frac{1}{3} (t_{j+1} - t_j)^3 \to 0$$

as $\Delta t_j \to 0$.

Hence,

$$\int_0^t B_s^2 \, dB_s = \lim_{\Delta t_j \to 0} \int_0^t \varphi_n(s) \, dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j^2 \Delta B_j$$

Using the identity

$$B_j^2 (B_{j+1} - B_j) = \frac{1}{3}(B_{j+1}^3 - B_j^3) - B_j (B_{j+1} - B_j)^2 - \frac{1}{3}(B_{j+1} - B_j)^3$$

We obtain

$$\lim_{\Delta t_j \to 0} \sum_j B_j^2 (B_{j+1} - B_j) = \lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1}^3 - B_j^3)$$

$$- \lim_{\Delta t_j \to 0} \sum_j B_j (B_{j+1} - B_j)^2$$

$$- \lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1} - B_j)^3$$

Evaluating these limits,

- $$\lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1}^3 - B_j^3) = \frac{1}{3} B_t^3 - \frac{1}{3} B_0^3 = \frac{1}{3} B_t^3$$

- $$\lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1} - B_j)^3 = 0$$
- $$\lim_{\Delta t_j \to 0} \sum_j B_j (B_{j+1} - B_j)^2 = \lim_{\Delta t_j \to 0} \sum_j B_j (B_{j+1} - B_j)$$
$$+ \lim_{\Delta t_j \to 0} \sum_j B_j \left[(B_{j+1} - B_j)^2 - (B_{j+1} - B_j)\right]$$
$$= \int_0^t B_s \, ds + 0$$

Taking all these parts together:

$$\lim_{\Delta t_j \to 0} \sum_j B_j^2 (B_{j+1} - B_j) = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

Thus,

$$\int_0^t B_s^2 \, dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j^2 \Delta B_j$$
$$= \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

∎

Øksendal 3.3

$X_t : \Omega \to \mathbb{R}^n$ stochastic process

$\mathcal{H}_t$ σ-algebra generated by $\{X_s : s \leq t\}$ (filtration of the process)

a) Show that if X_t is a martingale wrt some filtration $\{N_t\}$, then X_t is also a martingale wrt its own filtration $\{\mathcal{H}_t\}$.

Suppose that X_t is a martingale w.r.t. some filtration $\{N_t\}$. That means that

1. X_t is N_t-measurable for all t.
2. $\mathbb{E}[|X_t|] < \infty$
3. $\mathbb{E}[X_t | N_s] = X_s$, $s \leq t$

What we need to show is that X_t is a martingale w.r.t. its natural filtration generated by X_s, $s = 0, 1, \dots, t$

In fact,

1. X_t is $\mathcal{H}_t$-measurable for all t, since $\mathcal{H}_t$ contains all available information at the time t.
2. $\mathbb{E}[|X_t|] < \infty$ remains.
3. Since $\mathcal{H}_t \subseteq N_t$,

$$\mathbb{E}[X_t | \mathcal{H}_s] = \mathbb{E}[\mathbb{E}[X_t | N_s] | \mathcal{H}_s] = \mathbb{E}[X_s | \mathcal{H}_s] = X_s$$

for $s \leq t$.

Hence, X_t is a martingale w.r.t. its own filtration.

$\square$

b) If X_t is a martingale w.r.t. $\mathcal{H}_t$, then

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \qquad \text{for all } t \geq 0$$

Notice that since X_t is a martingale w.r.t. $\mathcal{H}_t$

$$\mathbb{E}[X_t | \mathcal{H}_0] = X_0$$

law of total expectation

However,

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t | \mathcal{H}_0]] = \mathbb{E}[X_0]$$

Hence,

$$\mathbb{E}[X_t] = \mathbb{E}[X_0]$$

c) Example of stochastic process X_t satisfying

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \quad \text{for all } t \geq 0$$

and which is not a martingale w.r.t. its own filtration.

Take Y_t with $Y_0 = 0$ and U a uniform ± 1 r.v. independent from Y_t. Then,

$$\mathbb{E}[UY_t] = \frac{1}{2}[Y_t] + \frac{1}{2}[-Y_t] = 0$$

However,

$$\begin{aligned}
\mathbb{E}[UY_t | \mathcal{F}_s] &= \mathbb{E}[U(Y_t + Y_s - Y_s) | \mathcal{F}_s] \\
&= \mathbb{E}[U(Y_t - Y_s) | \mathcal{F}_s] + \mathbb{E}[UY_s | \mathcal{F}_s] \\
&= \mathbb{E}[U | \mathcal{F}_s] \cdot \mathbb{E}[Y_t - Y_s | \mathcal{F}_s] + \mathbb{E}[UY_s | \mathcal{F}_s] \\
&= Y_s \mathbb{E}[U | \mathcal{F}_s] = 0
\end{aligned}$$

Øksendal 3.4

i) $X_t = B_t + 4t$

1. X_t is $\mathcal{F}_t$-measurable for all t since B_t is.
2. $\mathbb{E}[|X_t|] = \mathbb{E}[|B_t + 4t|] \leq \mathbb{E}[|B_t|] + \mathbb{E}[|4t|] < \infty$

since B_t is a martingale and $|4t|$ is a constant.

3. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad s \leq t$

Notice that

$$\mathbb{E}[B_t + 4t|\mathcal{F}_s] = B_s + 4t \neq X_s \qquad \therefore \text{ Not a martingale}$$

ii) $X_t = B_t^2$ $\qquad B_t^{2(-1)}(U) = \{\omega \in \Omega : B_t^2(\omega) \in U\} \subseteq \mathcal{F}_t$

1. X_t is $\mathcal{F}_t$-measurable
2. $\mathbb{E}[|X_t|] = \mathbb{E}[|B_t^2|]$ may not be finite
3. $\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad s \leq t$

$$\mathbb{E}[B_t^2|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2|\mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s] + 2\mathbb{E}[(B_t - B_s)B_s|\mathcal{F}_s] + \mathbb{E}[B_s^2|\mathcal{F}_s]$$

$$= t - s + B_s^2 \neq B_s^2 \quad (s < t)$$

$\therefore$ Not a martingale

iii) $X_t = t^2 B_t - 2\int_0^t s B_s \, ds$

1. X_t is $\mathcal{F}_t$-measurable.
2. $\mathbb{E}[|X_t|] \leq \mathbb{E}[|t^2 B_t|] + 2\mathbb{E}\left[\int_0^t s B_s ds\right] < \infty$
3. $\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}\left[t^2 B_t - 2\int_0^t s B_s ds \,\middle|\, \mathcal{F}_s\right]$

$$= t^2 B_s - 2\int_0^s s B_s ds - 2\mathbb{E}\left[\int_s^t r B_r dr \,\middle|\, \mathcal{F}_s\right]$$

$$= t^2 B_s - 2\int_0^s s B_s ds - 2\int_s^t r \, \mathbb{E}[B_r|\mathcal{F}_s] ds$$

$$= t^2 B_s - 2\int_0^s s B_s ds - 2\int_s^t r B_s ds$$

$$= t^2 B_s - 2\int_0^s s B_s\,ds - B_s(t^2 - s^2) = s^2 B_s - 2\int_0^s s B_s\,ds$$

$\therefore$ Is a martingale

iv.) $X_t = B_1(t) B_2(t)$ where $(B_1(t), B_2(t))$ is a 2-dim B.m.

1. X_t is $\mathcal{F}_t$-measurable since $B_1(t)$ and $B_2(t)$ is.
2. $\mathbb{E}[|X_t|] = \mathbb{E}[|B_1(t)|] \cdot \mathbb{E}[|B_2(t)|] < \infty$
3. $\mathbb{E}[X_t | \mathcal{F}_s] = B_1(s) \cdot B_2(s) = X_s, \quad s \leq t$

$\therefore$ Is a martingale

Øksendal 3.5

Prove that

$$M_t = B_t^2 - t$$

is an $\mathcal{F}_t$-martingale.

What we need to show is:

1. M_t is $\mathcal{F}_t$-measurable
2. $\mathbb{E}[|M_t|] < \infty$
3. $\mathbb{E}[M_t | \mathcal{F}_s] = M_s, \quad s \le t$

The first is immediate. Since B_t^2 and t are $\mathcal{F}_t$-measurable, M_t also is.

Now notice that

$$\mathbb{E}[|M_t|] = \mathbb{E}[|B_t^2 - t|] \le \mathbb{E}[|B_t^2|] + \mathbb{E}[|t|] < \infty$$

Finally,

$$\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[B_t^2 - t | \mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s + B_s)^2 | \mathcal{F}_s] - \mathbb{E}[t | \mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + 2\mathbb{E}[(B_t - B_s)B_s | \mathcal{F}_s] + \mathbb{E}[B_s^2 | \mathcal{F}_s] - \mathbb{E}[t | \mathcal{F}_s]$$

$$= t - s + B_s^2 - t = B_s^2 - s = M_s$$

for $s < t$ as desired.

3.6. Prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

Let $\mathcal{F}_t$ be the natural filtration of B_t, i.e., generated by $\{B_t : s \leq t\}$

Clearly, N_t is $\mathcal{F}_t$-measurable, since B_t^3 and $3tB_t$ is.

Now we need to check that $\mathbb{E}[|N_t|] < \infty$. In fact,

$$\mathbb{E}[|N_t|] = \mathbb{E}[|B_t^3 - 3tB_t|] \leq \mathbb{E}[|B_t^3|] + \mathbb{E}[|3tB_t|]$$
$$= \mathbb{E}[|B_t|^3] + 3t\,\mathbb{E}[|B_t|] < \infty$$

Finally,

$$\mathbb{E}[N_s|\mathcal{F}_t] = \mathbb{E}[B_s^3 - 3sB_s|\mathcal{F}_t]$$
$$= \mathbb{E}[B_s^3|\mathcal{F}_t] - 3s\,\mathbb{E}[B_s|\mathcal{F}_t]$$

Notice that

$$(B_s - B_t + B_t)^3 = ([B_s - B_t]^2 + 2(B_s - B_t)B_t + B_t^2)\,([B_s - B_t] + B_t)$$
$$= [B_s - B_t]^3 + 3[B_s - B_t]^2 B_t + 3(B_s - B_t)B_t^2 + B_t^3$$

Therefore,

$$\mathbb{E}[B_s^3|\mathcal{F}_t] = \mathbb{E}[(B_s - B_t + B_t)^3|\mathcal{F}_t]$$
$$= \mathbb{E}[(B_s - B_t)^3|\mathcal{F}_t] + 3\,\mathbb{E}[(B_s - B_t)^2 B_t|\mathcal{F}_t]$$
$$+ 3\mathbb{E}[(B_s - B_t)B_t^2|\mathcal{F}_t] + \mathbb{E}[B_t^3|\mathcal{F}_t]$$
$$= 0 + 3B_t\,\mathbb{E}[(B_s - B_t)^2|\mathcal{F}_t] + 3B_t^2\,\mathbb{E}[(B_s - B_t)|\mathcal{F}_t] + B_t^3$$
$$= 3B_t(s - t) + 0 + B_t^3$$

Thus,

$$\mathbb{E}[B_s^3|\mathcal{F}_t] - 3s\,\mathbb{E}[B_s|\mathcal{F}_t] = B_t^3 - 3(t - s)B_t - 3sB_t$$
$$= B_t^3 - 3tB_t$$

as desired.

$\square$

3.7. A famous result of Itô (1951) gives the following formula for n times *iterated Itô integrals*:

$$n! \int \cdots \Big(\int \Big(\int_{0 \le u_1 \le \cdots \le u_n \le t} dB_{u_1} \Big) dB_{u_2} \Big) \cdots dB_{u_n} = t^{\frac{n}{2}} h_n \Big(\frac{B_t}{\sqrt{t}} \Big) \qquad (3.3.8)$$

where h_n is the *Hermite polynomial* of degree n, defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \big(e^{-\frac{x^2}{2}} \big) \; ; \qquad n = 0, 1, 2, \ldots$$

(Thus $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$.)

a) Verify that in each of these n Itô integrals the integrand satisfies the requirements in Definition 3.1.4.

Recall that

Definition 3.1.4. *Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbf{R}$$

such that

(i) $(t,\omega) \to f(t,\omega)$ *is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.*
(ii) $f(t,\omega)$ *is $\mathcal{F}_t$-adapted.*
(iii) $E[\int_S^T f(t,\omega)^2 dt] < \infty$.

Since $f(t,\omega) = 1$, f is $\mathcal{B} \times \mathcal{F}$-measurable and $\mathcal{F}_t$-adapted.
Moreover,

$$E\left[\int_S^T dt \right] = E[t - s] = t - s < \infty$$

b) Verify formula (3.3.8) for $n = 1, 2, 3$ by combining Example 3.1.9 and Exercise 3.2.

For $n = 1$, we have

$$1! \int_{0 \le u_1 \le t} dB_{u_1} = B_t = t^{1/2} h_1 \left(\frac{B_t}{\sqrt{t}} \right) = t^{1/2} \cdot \frac{B_t}{t^{1/2}}$$

For $n = 2$,

$$\int_0^t \int_0^u dB_w \, dB_u = \int_0^t B_u \, dB_u \overset{*}{=} \frac{1}{2} (B_u^2 - u) \Big|_0^t$$

$$= \frac{1}{2} (B_t^2 - t) = \frac{t^{2/2}}{2} \left(\left(\frac{B_t}{\sqrt{t}} \right)^2 - 1 \right)$$

$$= \frac{t^{2/2}}{2!} h_2 \left(\frac{B_t}{\sqrt{t}} \right)$$

* Example 3.1.9

For $n=3$:

$$\int_0^t \int_0^u \int_0^v dB_r\, dB_v\, dB_u = \int_0^t \int_0^u B_v\, dB_v\, dB_u$$

$$= \int_0^t \frac{1}{2}(B_v^2 - v)\Big|_0^u dB_u = \int_0^t \frac{1}{2}(B_u^2 - u)\, dB_u$$

$$= \frac{1}{2}\left[\int_0^t B_u^2\, dB_u - \int_0^t u\, dB_u\right]$$

$$\overset{*}{=} \frac{1}{2}\left[\frac{1}{3}B_t^3 - \int_0^t B_u\, du - \int_0^t u\, dB_u\right] \qquad * \text{ Exercise } 3.2$$

$$\overset{*}{=} \frac{1}{3!}B_t^3 - \frac{tB_t}{2} = \frac{t^{3/2}}{3!}\left[\left(\frac{B_t}{\sqrt{t}}\right)^3 - 3\frac{B_t}{\sqrt{t}}\right]$$

$$= \frac{t^{3/2}}{3!} h_3\left(\frac{B_t}{\sqrt{t}}\right)$$

$*$ Since

$$\int_0^t B_s\, ds + \int_0^t s\, dB_s = tB_t$$

EXERCISE 3.1.

c) Use b) to give a new proof of the statement in Exercise 3.6.

Since

$$\int_0^t (B_s^2 - s)\, dB_s = t h_1\left(\frac{B_t}{\sqrt{t}}\right)$$

is a martingale (the Integral of B.m. is martingale), we have what we wanted.

$\square$

3.8. a) Let Y be a real valued random variable on $(\Omega, \mathcal{F}, P)$ such that

$$E[|Y|] < \infty .$$

Define

$$M_t = E[Y|\mathcal{F}_t] ; \qquad t \geq 0 .$$

Show that M_t is an $\mathcal{F}_t$-martingale.

b) Conversely, let M_t; $t \geq 0$ be a real valued $\mathcal{F}_t$-martingale such that

$$\sup_{t\geq 0} E[|M_t|^p] < \infty \qquad \text{for some } p > 1 .$$

Show that there exists $Y \in L^1(P)$ such that

$$M_t = E[Y|\mathcal{F}_t] .$$

(Hint: Use Corollary C.7.)

a) • $E[M_s \mid \mathcal{F}_t] = E[E[Y \mid \mathcal{F}_s] \mid \mathcal{F}_t]$
$= E[Y \mid \mathcal{F}_t] = M_t$

for all $s \geq t$

• $M_t = E[Y \mid \mathcal{F}_t] = Y$ is $\mathcal{F}_t$-measurable

• $E[|M_t|] = E[|E[Y \mid \mathcal{F}_t]|] = E[|Y|] < \infty$

Thus, M_t is an $\mathcal{F}_t$-martingale.

b)

Corollary C.7. *Let M_t be a continuous martingale such that*

$$\sup_{t>0} E[|M_t|^p] < \infty \qquad \textit{for some } p > 1 .$$

Then there exists $M \in L^1(P)$ such that $M_t \to M$ a.e. (P) and

$$\int |M_t - M| dP \to 0 \qquad \textit{as } t \to \infty .$$

By the Corollary C.7, there exists $M \in L^1(P)$ such that $M_t \to M$ a.e. and

$$\lim_{t\to\infty} \int M_t \, dP = \int M \, dP$$

Given that M_t is a martingale for all $s > t$, for $A \in \mathcal{F}_t$,

$$\int_A M_s \, dP = \int_A E[M_s \mid \mathcal{F}_t] \, dP = \int_A M_t \, dP$$

Taking the limit as $s \to \infty$,

$$\lim_{s\to\infty} \int_A M_s \, dP = \int_A M_t \, dP$$

I.e.,

$$\int_A M \, dP = \int_A \mathbb{E}[M \mid \mathcal{F}_t] \, dP \qquad \forall A \in \mathcal{F}_t$$

Hence,

$$M_t = \mathbb{E}[M \mid \mathcal{F}_t]$$

$\square$

3.9. Suppose $f \in \mathcal{V}(0,T)$ and that $t \to f(t,\omega)$ is continuous for a.a. ω. Then we have shown that

$$\int_0^T f(t,\omega)dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j,\omega)\Delta B_j \qquad \text{in } L^2(P)\,.$$

Similarly we define the *Stratonovich integral* of f by

$$\int_0^T f(t,\omega)\circ dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j^*,\omega)\Delta B_j\,, \quad \text{where } t_j^* = \tfrac{1}{2}(t_j + t_{j+1})\,,$$

whenever the limit exists in $L^2(P)$. In general these integrals are different. For example, compute

$$\int_0^T B_t \circ dB_t$$

and compare with Example 3.1.9.

$$\int_0^T B_t \circ dB_t = \lim_{\Delta t_j \to 0} \sum_j B_{j^*} \Delta B_j, \qquad t_j^* = j^* = \frac{1}{2}(t_j + t_{j+1})$$

Now

$$\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = \left[\left((B_{j+1} - B_{j^*}) + B_{j^*}\right)^2 - \left((B_j - B_{j^*}) + B_{j^*}\right)^2\right]$$
$$= (B_{j+1} - B_{j^*})^2 + 2B_{j^*}(B_{j+1} - B_{j^*}) + B_{j^*}^2$$
$$- \left[(B_j - B_{j^*})^2 + 2B_{j^*}(B_j - B_{j^*}) + B_{j^*}^2\right]$$
$$= (B_{j+1} - B_{j^*})^2 - (B_j - B_{j^*})^2 + 2B_{j^*}(B_{j+1} - B_j)$$

Hence,

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (B_{j+1} - B_{j^*})^2 - \sum_j (B_j - B_{j^*})^2$$
$$+ 2\sum_j B_{j^*}(B_{j+1} - B_j)$$

I.e.,

$$\sum_j B_{j^*}\Delta B_j = \frac{1}{2}B_t^2 + \frac{1}{2}\sum_j (B_j - B_{j^*})^2 - \frac{1}{2}\sum_j (B_{j+1} - B_{j^*})^2$$

Since

$$\sum_j (B_j - B_{j^*})^2 \longrightarrow \frac{t}{2} \quad \text{and} \quad \sum_j (B_{j+1} - B_{j^*})^2 \longrightarrow \frac{t}{2}$$

in $L^2(P)$ as $\Delta t_j \to 0$, we have that

$$\int_0^T B_t \circ dB_t = \frac{1}{2}B_t^2 + \frac{t}{4} - \frac{t}{4} = \frac{1}{2}B_t^2$$

3.10. If the function f in Exercise 3.9 varies "smoothly" with t then in fact the Itô and Stratonovich integrals of f coincide. More precisely, assume that there exists $K < \infty$ and $\epsilon > 0$ such that

$$E[|f(s,\cdot) - f(t,\cdot)|^2] \le K|s-t|^{1+\epsilon}\,; \qquad 0 \le s\,,\ t \le T\,.$$

Prove that then we have

$$\int_0^T f(t,\omega)dB_t = \lim_{\Delta t_j \to 0} \sum_j f(t'_j,\omega)\Delta B_j \qquad (\text{limit in } L^1(P))$$

for any choice of $t'_j \in [t_j, t_{j+1}]$. In particular,

$$\int_0^T f(t,\omega)dB_t = \int_0^T f(t,\omega) \circ dB_t\,.$$

(Hint: Consider $E\big[|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j|\big]$.)

Consider

$$E\left[\left|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j\right|\right]$$

$$\le \sum_j E\left[|f(t_j) - f(t'_j)|\cdot|\Delta B_j|\right]$$

$$\le \sum_j \sqrt{E[|f(t_j) - f(t'_j)|]^2 \cdot E[|\Delta B_j|^2]} \qquad \text{CAUCHY-SCHWARZ}$$

$$\le \sum_j \sqrt{K|t_j - t'_j|^{1+\varepsilon}\cdot|t_j - t'_j|} = \sum_j \sqrt{K}\,|t_j - t'_j|^{\frac{1+\varepsilon}{2}}\cdot|t_j - t'_j|^{1/2}$$

$$= \sqrt{K}\sum_j |t_j - t'_j|^{1+\varepsilon/2} \le T\sqrt{K}\max_j |t_j - t'_j|^{\varepsilon/2} \longrightarrow 0 \quad \text{as } \Delta t_j \to 0$$

Hence,

$$\int_0^T f(t,\omega)dB_t = \lim_{\Delta t_j\to 0}\sum_j f(t_j,\omega)\Delta B_j = \lim_{\Delta t_j \to 0}\sum_j f(t'_j,\omega)\Delta B_j$$

3.11. Let W_t be a stochastic process satisfying (i), (ii) and (iii) (below (3.1.2)). Prove that W_t cannot have continuous paths. (Hint: Consider $E[(W_t^{(N)} - W_s^{(N)})^2]$, where

$$W_t^{(N)} = (-N) \vee (N \wedge W_t),\ N = 1, 2, 3, \dots)\ .$$

Conditions:

(i) $t_1 \neq t_2 \Rightarrow W_{t_1}$ and W_{t_2} are independent.
(ii) $\{W_t\}$ is stationary, i.e. the (joint) distribution of $\{W_{t_1+t}, \dots, W_{t_k+t}\}$ does not depend on t.
(iii) $E[W_t] = 0$ for all t.

Let

$$W_t^{(N)} = \max\{-N, \min\{N, W_t\}\}, \quad N \in \mathbb{Z}^+$$

and suppose, by contradiction, that W_t is continuous.
By the Dominated Convergence Theorem, we have

$$\lim_{s \to t} \mathbb{E}\left[(W_t^{(N)} - W_s^{(N)})^2\right] = 0 \qquad (1)$$

and

$$\mathbb{E}\left[W_t^{(N)}\right] = \mathbb{E}\left[W_t\right] = 0 \qquad (2)$$

Since $W_t^{(N)}$ and $W_s^{(N)}$ are i.i.d.,

$$\begin{aligned}
\mathbb{E}\left[(W_t^{(N)} - W_s^{(N)})^2\right] &= \mathbb{E}\left[W_t^{(N)2} - 2W_t^{(N)} W_s^{(N)} + W_s^{(N)2}\right] \\
&= \mathbb{E}\left[(W_t^{(N)})^2\right] - 2\mathbb{E}\left[W_t^{(N)}\right] \cdot \mathbb{E}\left[W_s^{(N)}\right] + \mathbb{E}\left[(W_s^{(N)})^2\right] \\
&= \mathbb{E}\left[(W_t^{(N)})^2\right] - 2\,\mathbb{E}\left[W_t^{(N)}\right]\mathbb{E}\left[W_t^{(N)}\right] + \mathbb{E}\left[(W_t^{(N)})^2\right] \\
&= 2\mathbb{E}\left[(W_t^{(N)})^2\right] - 2\,\mathbb{E}\left[(W_t^{(N)})\right]^2 \\
&= 2\mathrm{Var}\left[W_t^{(N)}\right]
\end{aligned} \qquad (3)$$

By (1), we know that (3) goes to zero. Hence, $W_t^{(N)}$ is constant, i.e.,

$$W_t^{(N)} = \mathbb{E}\left[W_t^{(N)}\right] = \mathbb{E}\left[W_t\right] = 0$$

3.12. As in Exercise 3.9 we let $\circ dB_t$ denote Stratonovich differentials.

(i) Use (3.3.6) to transform the following Stratonovich differential equations into Itô differential equations:

(a) $dX_t = \gamma X_t dt + \alpha X_t \circ dB_t$

(b) $dX_t = \sin X_t \cos X_t dt + (t^2 + \cos X_t) \circ dB_t$

(ii) Transform the following Itô differential equations into Stratonovich differential equations:

(a) $dX_t = rX_t dt + \alpha X_t dB_t$

(b) $dX_t = 2e^{-X_t} dt + X_t^2 dB_t$

Using

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \tfrac{1}{2}\int_0^t \sigma'(s, X_s)\sigma(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s\,, \quad (3.3.6)$$

i.a.) $dX_t = b(t,x)dt + \frac{1}{2}\sigma'(t,x)\sigma(t,x)ds + \sigma(t,x)dB_t$

$dX_t = \gamma X_t dt + \frac{1}{2}\alpha^2 X_t dt + \alpha X_t dB_t$

$\sigma(t,x) = \alpha x$

$\sigma'(t,x) = \alpha$

i.b) $\sigma(t,x) = t^2 + \cos x$

$\sigma'(t,x) = \frac{\partial\sigma}{\partial x} = -\operatorname{sen} x$

$dX_t = \sin X_t \cos X_t + \frac{1}{2}\left[-\sin X_t (t^2 + \cos X_t)\right] dt + (t^2 + \cos X_t) dB_t$

$= \sin X_t \cos X_t - \frac{1}{2}\sin X_t (t^2 + \cos X_t) dt + (t^2 + \cos X_t) dB_t$

ii.a) $dX_t = r X_t dt + \alpha X_t dB_t$

$dX_t = r X_t dt - \frac{1}{2}\alpha^2 X_t dt + \alpha X_t \circ dB_t$

ii.b) $dX_t = 2e^{-X_t} dt + X_t^2 dB_t$

$dX_t = 2e^{-X_t} dt - X_t^3 dt + X_t^2 \circ dB_t$

$\sigma(t,x) = x^2, \quad \sigma_x = 2x$

3.13. A stochastic process $X_t(\cdot)\colon \Omega \to \mathbf{R}$ is *continuous in mean square* if $E[X_t^2] < \infty$ for all t and

$$\lim_{s \to t} E[(X_s - X_t)^2] = 0 \qquad \text{for all } t \geq 0 .$$

a) Prove that Brownian motion B_t is continuous in mean square.

Since

$$\mathbb{E}[B_t^2] = t < \infty, \quad \forall t \in \mathbb{R}$$

and

$$\lim_{s \to t} \mathbb{E}\left[(B_s - B_t)^2\right] = \lim_{s \to t}(s-t) = 0, \quad \forall t \geq 0$$

Hence, B_t is continuous in mean square.

b) Let $f\colon \mathbf{R} \to \mathbf{R}$ be a Lipschitz continuous function, i.e. there exists $C < \infty$ such that

$$|f(x) - f(y)| \leq C|x - y| \qquad \text{for all } x, y \in \mathbf{R} .$$

Prove that

$$Y_t := f(B_t)$$

is continuous in mean square.

By the fact that f is Lipschitz,

$$|Y_t - Y_s| \leq C|t-s|$$

i.e.,

$$\mathbb{E}|Y_t - Y_s|^2 \leq \mathbb{E}\left[C^2|t-s|^2\right]$$

Dominated Convergence

Taking the limit as $t \to s$,

$$\lim_{t \to s} \mathbb{E}|Y_t - Y_s|^2 \leq \lim_{t \to s} \mathbb{E}\left[C^2|t-s|^2\right] = \mathbb{E}\left[\lim_{t \to s}(C^2|t-s|^2)\right] = 0$$

Therefore,

$$\lim_{t \to s} \mathbb{E}\left[|Y_t - Y_s|^2\right] = 0, \quad \forall s \geq 0$$

Moreover,

$$\mathbb{E}[Y_t^2] = \mathbb{E}[f^2(B_T)] < \infty$$

since f is continuous.

c) Let X_t be a stochastic process which is continuous in mean square and assume that $X_t \in \mathcal{V}(S,T)$, $T < \infty$. Show that

$$\int_S^T X_t dB_t = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega) \qquad \text{(limit in } L^2(P))$$

where

$$\phi_n(t,\omega) = \sum_j X_{t_j^{(n)}}(\omega) \mathcal{X}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t), \qquad T < \infty .$$

(Hint: Consider

$$E\Big[\int_S^T (X_t - \phi_n(t))^2 dt\Big] = E\Big[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt\Big]) .$$

Consider

$$\mathbb{E}\left[\int_S^T (X_t - \varphi_n(t))^2 dt\right] = \mathbb{E}\left[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt\right]$$

$$= \sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \mathbb{E}\left[(X_t - X_{t_j^{(n)}})^2\right] dt$$

$$\leq (T-S) \sup_{[t_{j}^{(n)}, t_{j+1}^{(n)}]} \mathbb{E}\left[(X_t - X_{t_j^{(n)}})^2\right]$$

Taking the limit as $n \to \infty$,

$$(T-S) \sup_{[t_{j}^{(n)}, t_{j+1}^{(n)}]} \mathbb{E}\left[(X_t - X_{t_j^{(n)}})^2\right] \longrightarrow 0$$

$\blacksquare$

3.14. Show that a function $h(\omega)$ is $\mathcal{F}_t$-measurable if and only if h is a pointwise limit (for a.a. ω) of sums of functions of the form

$$g_1(B_{t_1}) \cdot g_2(B_{t_2}) \cdots g_k(B_{t_k})$$

where $g_1, \ldots, g_k$ are bounded continuous functions and $t_j \leq t$ for $j \leq k$, $k = 1, 2, \ldots$

Hint: Complete the following steps:

a) We may assume that h is bounded.

b) For $n = 1, 2, \ldots$ and $j = 1, 2, \ldots$ put $t_j = t_j^{(n)} = j \cdot 2^{-n}$. For fixed n let $\mathcal{H}_n$ be the σ-algebra generated by $\{B_{t_j}(\cdot)\}_{t_j \leq t}$. Then by Corollary C.9

$$h = E[h|\mathcal{F}_t] = \lim_{n\to\infty} E[h|\mathcal{H}_n] \qquad \text{(pointwise a.e. limit)}$$

c) Define $h_n := E[h|\mathcal{H}_n]$. Then by the Doob-Dynkin lemma (Lemma 2.1.2) we have

$$h_n(\omega) = G_n(B_{t_1}(\omega), \ldots, B_{t_k}(\omega))$$

for some Borel function $G_n: \mathbf{R}^k \to \mathbf{R}$, where $k = \max\{j; j \cdot 2^{-n} \leq t\}$. Now use that any Borel function $G: \mathbf{R}^k \to \mathbf{R}$ can be approximated pointwise a.e. by a continuous function $F: \mathbf{R}^k \to \mathbf{R}$ and complete the proof by applying the Stone-Weierstrass theorem.

Assume that h is bounded and put $t_j = t_j^{(n)} = j \cdot 2^{-n}$, $n = 1, 2, \ldots$, $j = 1, 2, \ldots$

For a fixed n, let $\mathcal{H}_n$ be the σ-algebra generated by $\{B_{t_j}\}_{t_j \leq t}$. Then, since $\mathcal{H}_n$ is an increasing family of σ-algebras, and $\mathcal{F}_t$ is the σ-algebra generated by $\{\mathcal{H}_n\}_{n=1}^{\infty}$

- The σ-algebra generated by $\{\mathcal{H}_n\}$ is contained in $\mathcal{F}_t$, by the definition of $\mathcal{H}_n$;
- Let $s \leq t$ and $r \to s$. By the continuity of B.M., $B_r \to B_s \in \{\mathcal{H}_n\}$, by definition of $\mathcal{H}_n$. Since $B_s = \limsup_r B_r$, we know that $\limsup_r B_r \in \{\mathcal{H}_n\}$.

Now, by definition of $\mathcal{F}_t$ (σ-algebra generated by $B_s : s \leq t$), we have that $\mathcal{F}_t \subseteq \{\mathcal{H}_n\}_{n=1}^{\infty}$.

by the Corollary C.9.,

$$h = E[h|\mathcal{F}_t] = \lim_{n\to\infty} E[h|\mathcal{H}_n] \qquad \text{pointwise a.e.}$$

Let $h_n = E[h|\mathcal{H}_n]$. By the Doob-Dynkin lemma,

$$h_n(\omega) = G_n\{B_{t_1}(\omega), \ldots, B_{t_k}(\omega)\}$$

for some Borel function G_n, and where $k = \max\{j;\ j \cdot 2^{-n} \leq t\}$.

Since every Borel function $G: \mathbb{R}^k \to \mathbb{R}$ can be approximated pointwise a.e. by a continuous function $F: \mathbb{R}^k \to \mathbb{R}$, by the Stone-Weierstrass Theorem applied on $[t_1, t_k]$, F can be approximated by a polynomial function g_n, completing the proof.

Corollary C.9. *Let $X \in L^1(P)$, let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be an increasing family of σ-algebras, $\mathcal{N}_k \subset \mathcal{F}$ and define $\mathcal{N}_\infty$ to be the σ-algebra generated by $\{\mathcal{N}_k\}_{k=1}^{\infty}$. Then*

$$E[X|\mathcal{N}_k] \to E[X|\mathcal{N}_\infty] \qquad \text{as } k \to \infty\ ,$$

a.e. P and in $L^1(P)$.

Lemma 2.1.2. *If $X, Y: \Omega \to \mathbf{R}^n$ are two given functions,then Y is $\mathcal{H}_X$-measurable if and only if there exists a Borel measurable function $g: \mathbf{R}^n \to \mathbf{R}^n$ such that*

$$Y = g(X)\ .$$

3.15. Suppose $f, g \in \mathcal{V}(S,T)$ and that there exist constants C, D such that

$$C + \int_S^T f(t,\omega)dB_t(\omega) = D + \int_S^T g(t,\omega)dB_t(\omega) \qquad \text{for a.a. } \omega \in \Omega\,.$$

Show that

$$C = D$$

and

$$f(t,\omega) = g(t,\omega) \qquad \text{for a.a. } (t,\omega) \in [S,T] \times \Omega\,.$$

We'll apply Itô Isometry. Notice that

$$\mathbb{E}\left[(C-D)^2\right] = \mathbb{E}\left[\left(\int_S^T \left(g(t,\omega) - f(t,\omega)\right) dB_t(\omega)\right)^2\right]$$

$$\overset{\text{Itô Isometry}}{=} \mathbb{E}\left[\int_S^T \left(g(t,\omega) - f(t,\omega)\right)^2 dt\right]$$

On the other hand,

Martingale + B.M. starting at 0

$$C - D = \mathbb{E}\left[(C-D) \mid \mathcal{F}_S\right] = \mathbb{E}\left[\int_S^T \left(g(t,\omega) - f(t,\omega)\right) dB_t(\omega) \,\middle|\, \mathcal{F}_S\right] = 0$$

Hence, $C = D$. Moreover, $(C-D)^2 = 0$ and $\mathbb{E}[(C-D)^2] = 0$. Therefore,

$$g(t,\omega) = f(t,\omega)$$

3.16. Let $X\colon \Omega \to \mathbf{R}$ be a random variable such that $E[X^2] < \infty$ and let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra. Show that

$$E\big[(E[X|\mathcal{H}])^2\big] \le E[X^2]\,.$$

(See Lemma 6.1.1. See also the Jensen inequality for conditional expectation (Appendix B).)

By Jensen Inequality for Conditional Expectation,

Theorem B.4 (The Jensen inequality).
If $\phi\colon \mathbf{R} \to \mathbf{R}$ is convex and $E[|\phi(X)|] < \infty$ then

$$\phi(E[X|\mathcal{H}]) \le E[\phi(X)|\mathcal{H}]\,.$$

Since $\phi(x) = x^2$ is convex,

$$E[X|\mathcal{H}]^2 \le E[X^2|\mathcal{H}]$$

Using the Law of Total Expectation, i.e.,

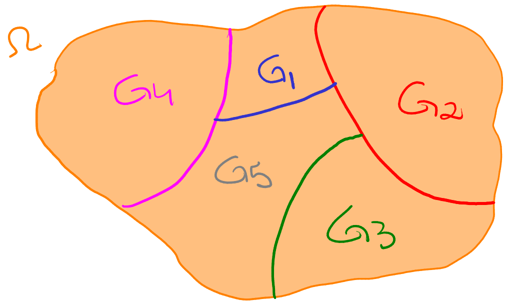
$$E[E[X|\mathcal{H}]] = E[X]$$

and taking the expectations

$$E[(E[X|\mathcal{H}])^2] \le E[E[X^2|\mathcal{H}]] = E[X^2]$$

3.17. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X\colon \Omega \to \mathbf{R}$ be a random variable with $E[|X|] < \infty$. If $\mathcal{G} \subset \mathcal{F}$ is a *finite* σ-algebra, then by Exercise 2.7 there exists a partition $\Omega = \bigcup_{i=1}^{n} G_i$ such that $\mathcal{G}$ consists of $\emptyset$ and unions of some (or all) of $G_1, \ldots, G_n$.

a) Explain why $E[X|\mathcal{G}](\omega)$ is constant on each G_i. (See Exercise 2.7 c).)



2.7. a) Suppose $G_1, G_2, \ldots, G_n$ are disjoint subsets of Ω such that

$$\Omega = \bigcup_{i=1}^{n} G_i \, .$$

Prove that the family $\mathcal{G}$ consisting of $\emptyset$ and all unions of some (or all) of $G_1, \ldots, G_n$ constitutes a σ-algebra on Ω.

b) Prove that any *finite* σ-algebra $\mathcal{F}$ on Ω is of the type described in a).

c) Let $\mathcal{F}$ be a *finite* σ-algebra on Ω and let $X\colon \Omega \to \mathbf{R}$ be $\mathcal{F}$-measurable. Prove that X assumes only finitely many possible values. More precisely, there exists a disjoint family of subsets $F_1, \ldots, F_m \in \mathcal{F}$ and real numbers $c_1, \ldots, c_m$ such that

$$X(\omega) = \sum_{i=1}^{m} c_i \mathcal{X}_{F_i}(\omega) \, .$$

By the exercise 2.7, for $c_i \in \mathbb{R}$, $i = 1, \ldots, n$,

$$E[X|\mathcal{G}](\omega) = \mathbb{E}\Big[\underbrace{\sum_{i=1}^{n} c_i \mathcal{X}_{G_i}(\omega)}_{\text{constant } C}\Big] = \mathbb{E}[C] = C$$

b) Assume that $P[G_i] > 0$. Show that

$$E[X|\mathcal{G}](\omega) = \frac{\int_{G_i} X dP}{P(G_i)} \qquad \text{for } \omega \in G_i \, .$$

$\int\limits_H E[X|\mathcal{H}]dP = \int\limits_H X dP$, *for all* $H \in \mathcal{H}$.

gives

$$\int_{G_i} \mathbb{E}[X|\mathcal{G}]\,d\mathbb{P} = \int_{G_i} X\,d\mathbb{P}$$

however,

$$\int_{G_i} \mathbb{E}[X|\mathcal{G}]\,d\mathbb{P} = \mathbb{E}[X|\mathcal{G}] \int_{G_i} d\mathbb{P} = \mathbb{E}[X|\mathcal{G}]\,\mathbb{P}(G_i)$$

Therefore,

$$\mathbb{E}[X|\mathcal{G}] = \frac{\int_{G_i} X\,d\mathbb{P}}{\mathbb{P}(G_i)}$$

c) Suppose X assumes only finitely many values $a_1, \dots, a_m$. Then from elementary probability theory we know that (see Exercise 2.1)

$$E[X|G_i] = \sum_{k=1}^{m} a_k P[X = a_k|G_i]\ .$$

Compare with b) and verify that

$$E[X|G_i] = E[X|\mathcal{G}](\omega) \qquad \text{for } \omega \in G_i\ .$$

Thus we may regard the conditional expectation as defined in Appendix B as a (substantial) generalization of the conditional expectation in elementary probability theory.

Using the previous item,

$$\mathbb{E}[X|\mathcal{G}] = \frac{\int_{G_i} X\, d\mathbb{P}}{\mathbb{P}(G_i)} = \frac{\int_{G_i} \sum_{k=1}^{m} a_k \chi_{a_k}\, d\mathbb{P}}{\mathbb{P}(G_i)} = \sum_{k=1}^{m} \frac{\int_{G_i} \chi_{G_k}\, d\mathbb{P}}{\mathbb{P}(G_i)}$$

$$= \sum_{k=1}^{m} a_k \frac{\mathbb{P}(G_i \cap G_k)}{\mathbb{P}(G_i)} = \sum_{k=1}^{m} a_k\, \mathbb{P}[X = a_k | G_i]$$

Alternative answer

b) Since

$$\mathbb{E}[f \mid A] = \int f \, d\mu_A, \quad \mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)}$$

We have

$$\mathbb{E}\left[\sum_{i=1}^{n} c_i \chi_{G_i}(\omega) \,\Big|\, G_i\right] = \int_{G_i} \frac{X \, d\mathbb{P}}{\mathbb{P}(G_i)}(\omega)$$

c) Given the definition of a_k, $k = 1, \dots, m$,

$$\mathbb{E}[X \mid G_i] = \sum_{k=1}^{m} a_k \mathbb{P}[X = a_k \mid G_i] = \int X \, d\mathbb{P}_{G_i}$$

By the fact that

$$\mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)}$$

we obtain

$$\mathbb{E}[X \mid G_i] = \int X \, d\mathbb{P}_{G_i} = \int_{G_i} \frac{X \, d\mathbb{P}}{\mathbb{P}(G_i)} = \mathbb{E}[X \mid \mathcal{G}]$$

4.1. Use Itô's formula to write the following stochastic processes X_t on the standard form

$$dX_t = u(t,\omega)dt + v(t,\omega)dB_t$$

for suitable choices of $u \in \mathbf{R}^n$, $v \in \mathbf{R}^{n\times m}$ and dimensions n, m:

a) $X_t = B_t^2$, where B_t is 1-dimensional
b) $X_t = 2 + t + e^{B_t}$ (B_t is 1-dimensional)
c) $X_t = B_1^2(t) + B_2^2(t)$ where (B_1, B_2) is 2-dimensional
d) $X_t = (t_0 + t, B_t)$ (B_t is 1-dimensional)
e) $X_t = (B_1(t)+B_2(t)+B_3(t), B_2^2(t) - B_1(t)B_3(t))$, where (B_1, B_2, B_3) is 3-dimensional.

a) Let $g(t,x) = x^2$. By the Itô's formula,

$$dg(t,X_t) = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2$$

Then,

$$dB_t^2 = 2B_t\,dB_t + dt$$

b) $X_t = 2 + t + e^{B_t}$

Let $g(t,x) = 2 + t + e^x$. Then,

$$dX_t = dt + e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt = \left(1 + \frac{1}{2}e^{B_t}\right)dt + e^{B_t}dB_t$$

c) $X_t = B_1^2(t) + B_2^2(t)$

Let $g(t,x) = x_1^2 + x_2^2$. Since

$$\mathrm{d}Y_k = \frac{\partial g_k}{\partial t}(t,X)\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)\mathrm{d}X_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)\mathrm{d}X_i\mathrm{d}X_j \qquad \mathrm{d}B_i\mathrm{d}B_j = \delta_{ij}\mathrm{d}t$$

We have that

$$dX_t = 2B_1(t)dB_1(t) + 2B_2(t)\,dB_2(t) + \frac{1}{2}\left[2dt + 2dt\right]$$

$$= 2dt + 2B_1(t)dB_1(t) + 2B_2(t)\,dB_2(t)$$

d) $X_t = (t_0 + t, B_t)$

Let $g(t, x) = (t_0 + t, x)$.

Then,

$$dX_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \end{bmatrix} dt = \begin{bmatrix} dt \\ dB_t \end{bmatrix}$$

e) $X_t = (B_1 + B_2 + B_3, B_2^2 - B_1 B_3)$

Let $g(t, x) = (x_1 + x_2 + x_3, x_2^2 - x_1 x_3)$. Then,

$$dX_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 \\ -B_3 \end{bmatrix} dB_1 + \begin{bmatrix} 1 \\ 2B_2 \end{bmatrix} dB_2 + \begin{bmatrix} 1 \\ -B_1 \end{bmatrix} dB_3$$

$$+ \frac{1}{2} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \end{bmatrix} dt \right)$$

$$= \begin{bmatrix} dB_1 + dB_2 + dB_3 \\ -B_3 dB_1 + 2B_2 dB_2 - B_1 dB_3 + dt \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_3 & 2B_2 & -B_1 \end{bmatrix} \cdot \begin{bmatrix} dB_1 \\ dB_2 \\ dB_3 \end{bmatrix}$$

4.2. Use Itô's formula to prove that

$$\int_0^t B_s^2 dB_s = \tfrac{1}{3}B_t^3 - \int_0^t B_s ds \, .$$

Recall that

$$df(X_t) = \frac{df(X_t)}{dt}dt + \frac{df}{dx}(X_t)\, dX_t + \frac{1}{2}\frac{d^2 f}{dx^2}(X_t)(dB_t)^2$$

Then, let $g(t, x) = x^3$ we know that

$\frac{dg}{dt}(t, B_t) = 0$ $\qquad (dB_t)^2 = dt$

$\frac{dg}{dx}(t, B_t) = 3B_t^2$ $\qquad \frac{d^2 g}{dx^2} = 6B_t$

Therefore,

$$dB_t^3 = 3B_t^2 dB_t + \frac{1}{2}6B_t dt = 3B_t^2 dB_t + 3B_t dt$$

I.e.,

$$\frac{1}{3}dB_t^3 = B_t^2 dB_t + B_t dt \iff B_t^2 dB_t = \frac{1}{3}dB_t^3 - B_t dt$$

In the integral form,

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$$

4.3. Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \ .$$

Let $g(t,x,y) = xy$. By Itô's formula,

$$dX_tY_t = \frac{\partial g}{\partial t}(t,X_t,Y_t)\,dt + \frac{\partial g}{\partial x}(t,X_t,Y_t)\,dX_t + \frac{\partial g}{\partial y}(t,X_t,Y_t)\,dY_t$$
$$+ \frac{1}{2}\Big[\frac{\partial^2 g}{\partial x \partial y}(t,X_t,Y_t)\,dX_t dY_t + \frac{\partial^2 g}{\partial x^2}(t,X_t,Y_t)(dX_t)^2$$
$$+ \frac{\partial^2 g}{\partial y^2}(t,X_t,Y_t)(dY_t)^2 + \frac{\partial^2 g}{\partial y \partial x}(t,X_t,Y_t)\,dX_t dY_t\Big]$$
$$= 0 + Y_t dX_t + X_t dY_t + \frac{1}{2}\left(dX_t dY_t + dX_t dY_t\right)$$

I.e.,

$$dX_tY_t = Y_t dX_t + X_t dY_t + dX_t dY_t$$

Rearranging

$$X_t dY_t = dX_tY_t - Y_t dX_t - dX_t dY_t$$

In the integral form,

$$\int_0^t X_s dY_s = \int_0^t dX_sY_s - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s$$

Hence,

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s$$

4.4. (Exponential martingales)
Suppose $\theta(t,\omega) = (\theta_1(t,\omega), \dots, \theta_n(t,\omega)) \in \mathbf{R}^n$ with $\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for $k = 1, \dots, n$, where $T \le \infty$. Define

$$Z_t = \exp\Bigg\{ \underbrace{\int_0^t \theta(s,\omega)dB(s) - \tfrac{1}{2}\int_0^t \theta^2(s,\omega)ds}_{Y_t} \Bigg\}; \qquad 0 \le t \le T$$

where $B(s) \in \mathbf{R}^n$ and $\theta^2 = \theta \cdot \theta$ (dot product).

a) Use Itô's formula to prove that

$$dZ_t = Z_t \theta(t,\omega) dB(t) \, .$$

b) Deduce that Z_t is a martingale for $t \le T$, provided that

$$Z_t \theta_k(t,\omega) \in \mathcal{V}[0,T] \qquad \text{for } 1 \le k \le n \, .$$

a) Let $g(t,x) = e^x$. Then,

$$dZ_t = \frac{\partial g}{\partial t}(t, Y_t)dt + \frac{\partial g}{\partial x}(t,Y_t)\,dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,Y_t)(dY_t)^2$$

$$= 0 + e^{Y_t} dY_t + \frac{1}{2} e^{Y_t}(dY_t)^2 \qquad (1)$$

Where

$$Y_t = \int_0^t \theta(s,\omega)\,dB(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega)\,ds$$

Define $h(t,x) = \int_0^t \theta(s,\omega)\,dx(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega)\,ds$

then

$$\frac{\partial h}{\partial t}(t,x) = -\frac{1}{2}\theta^2(t,\omega)$$

$$\frac{\partial h}{\partial x}(t,x) = \theta(t,\omega), \qquad \frac{\partial^2 h}{\partial x^2}(t,\omega) = 0$$

Hence, since $Y_t = h(t, B_t)$

$$dY_t = -\frac{1}{2}\theta^2(t,\omega)\,dt + \theta(t,\omega)\,dB_t \qquad (2)$$

Combining (1) and (2):

$$dZ_t = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} (dY_t)^2$$

And noticing that

$$(dY_t)^2 = \theta^2(t,\omega)\,dt$$

We obtain

$$dZ_t = Z_t\left(-\frac{1}{2}\theta^2(t,\omega)\,dt + \theta(t,\omega)\,dB_t\right) + \frac{1}{2} Z_t \theta^2(t,\omega)\,dt$$

i.e.

$$dZ_t = Z_t\,\theta(t,\omega)\,dB_t \qquad (3)$$

b) Given that $\theta_k \in \mathcal{V}[0,T]$, we know that the Itô integral (2) is a martingale.

And given that $Z_t\theta_k \in \mathcal{V}[0,T]$, (3) is a martingale.

4.5. Let $B_t \in \mathbf{R}$, $B_0 = 0$. Define

$$\beta_k(t) = E[B_t^k]\,; \qquad k = 0, 1, 2, \dots;\ t \geq 0\,.$$

Use Itô's formula to prove that

$$\beta_k(t) = \tfrac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds\,; \qquad k \geq 2\,.$$

Deduce that

$$E[B_t^4] = 3t^2 \qquad \text{(see (2.2.14))}$$

and find

$$E[B_t^6]\,.$$

Notice that:

$$K=0: \quad \beta_0(t) = \mathbb{E}[B_t^0] = 1$$
$$K=1: \quad \beta_1(t) = \mathbb{E}[B_t]$$

Let $g_k(t,x) = x^k$ Then, if

$$\alpha_k(t) = g_k(t, B_t), \quad \beta_k(t) = \mathbb{E}[\alpha_k(t)]$$

By Itô's formula,

$$d\alpha_k = k B_t^{k-1} dB_t + \frac{1}{2}k(k-1)B_t^{k-2}\,dt$$

Hence,

$$\alpha_k = \int_0^t k B_s^{k-1} dB_s + \frac{1}{2}\int_0^t k(k-1) B_s^{k-2} ds$$

and

$$\beta_k = \mathbb{E}\left[\int_0^t k B_s^{k-1} dB_s + \frac{1}{2}\int_0^t k(k-1) B_s^{k-2} ds\right]$$

$$= \underbrace{k\mathbb{E}\left[\int_0^t B_s^{k-1} dB_s\right]}_{0} + \frac{1}{2}k(k-1)\,\mathbb{E}\left[\int_0^t B_s^{k-2} ds\right]$$

$$= \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_s^{k-2}]\,ds = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}\,ds$$

With that

$$\mathbb{E}[B_t^4] = 6\int_0^t \mathbb{E}[B_s^2]\,ds = 6\int_0^t s\,ds = 6\frac{t^2}{2} = 3t^2$$

and

$$\mathbb{E}[B_t^6] = 15\int_0^t \mathbb{E}[B_s^4]\,ds = 15\int_0^t 3s^2\,ds$$

$$= 15 \cdot 3\frac{t^3}{3} = 15t^3$$

4.6. a) For c, α constants, $B_t \in \mathbf{R}$ define

$$X_t = e^{ct+\alpha B_t} .$$

Prove that

$$dX_t = (c + \tfrac{1}{2}\alpha^2)X_t dt + \alpha X_t dB_t .$$

Let $g(t,x) = e^{ct+\alpha x} = e^{ct} e^{\alpha x}$. Then,

- $\frac{\partial g}{\partial t} = c e^{ct+\alpha x}$
- $\frac{\partial g}{\partial x} = \alpha e^{ct+\alpha x}$
- $\frac{\partial^2 g}{\partial x^2} = \alpha^2 e^{ct+\alpha x}$

By Itô's formula, since $X_t = g(t, B_t)$,

$$dX_t = c e^{ct+\alpha B_t} dt + \alpha e^{ct+\alpha B_t} dB_t + \alpha^2 e^{ct+\alpha x} dt$$

$$\therefore \quad dX_t = \left(c + \alpha^2 \right) X_t dt + \alpha X_t dB_t$$

b) For $c, \alpha_1, \ldots, \alpha_n$ constants, $B_t = (B_1(t), \ldots, B_n(t)) \in \mathbf{R}^n$ define

$$X_t = \exp\left(ct + \sum_{j=1}^{n} \alpha_j B_j(t) \right) .$$

Prove that

$$dX_t = \left(c + \tfrac{1}{2} \sum_{j=1}^{n} \alpha_j^2 \right) X_t dt + X_t \left(\sum_{j=1}^{n} \alpha_j dB_j \right) .$$

By the multidimensional Itô's formula, let $g(t,x) = e^{ct+\alpha x}$

$$X_t = g(t, B_t) = e^{ct+\alpha B_t} = e^{ct+(\alpha_1 B_1 + \ldots + \alpha_n B_n)}$$

and

$$dX_t = c X_t dt + \alpha_1 X_t dB_1 + \ldots + \alpha_n X_t dB_n + \frac{1}{2} \alpha_1^2 X_t dt$$

$$+ \dots + \frac{1}{2}\alpha_n^2 X_t \, dt$$

Hence,

$$dX_t = \left(c + \frac{1}{2}\sum_{j=1}^{n}\alpha_j^2\right)X_t\,dt + X_t\left(\sum_{j=1}^{n}\alpha_j\,dB_j\right)$$

4.7. Let X_t be an Itô integral

$$dX_t = v(t,\omega)dB_t(\omega) \quad \text{where } v \in \mathbf{R}^n,\ v \in \mathcal{V}(0,T),\ B_t \in \mathbf{R}^n,\ 0 \le t \le T\,.$$

a) Give an example to show that X_t^2 is not in general a martingale.

Take $v = 1$. Then $X_t = B_t$ and $X_t^2 = B_t^2$ is not a martingale.

b) Prove that if v is bounded then

$$M_t := X_t^2 - \int_0^t |v_s|^2 ds \quad \text{is a martingale}\,.$$

The process $\langle X, X\rangle_t := \int_0^t |v_s|^2 ds$ is called the *quadratic variation process* of the martingale X_t. For general processes X_t it is defined by

$$\langle X, X\rangle_t = \lim_{\Delta t_k \to 0} \sum_{t_k \le t} |X_{t_{k+1}} - X_{t_k}|^2 \quad \text{(limit in probability)} \quad (4.3.11)$$

where $0 = t_1 < t_2 \cdots < t_n = t$ and $\Delta t_k = t_{k+1} - t_k$. The limit can be shown to exist for continuous square integrable martingales X_t. See e.g. Karatzas and Shreve (1991).

- M_t is $\mathcal{M}_t$-measurable since X_t^2 is and also $\int_0^t |v_s|^2 ds$

- $E[|M_t|] < \infty$ for all t

$$\boxed{\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s\,.}$$

By Integration by Parts (4.3), taking $X_t = Y_t$,

$$X_t^2 = X_0^2 + \int_0^t X_s\, dX_s + \int_0^t X_s\, dX_s + \int_0^t (dX_s)^2$$

$$= X_0^2 + 2\int_0^t X_s\, dX_s + \int_0^t |v_s|^2 ds \qquad \leftarrow \langle X, X\rangle_t$$

$$= X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s + \int_0^t |\sigma_s|^2 ds$$

Therefore,

$$M_t = X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s$$

and

$$\mathbb{E}[|M_t|] = \mathbb{E}\left[\left|X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s\right|\right]$$
$$\leq X_0^2 + 2\,\mathbb{E}\left[\int_0^t |X_s \sigma_s| \, dB_s\right]$$

Since σ_s is bounded, there exists C such that $\sigma_s \leq C$. By Itô Isometry,

$$\mathbb{E}\left[\left(\int_0^t |X_s \sigma_s| \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t |X_s \sigma_s|^2 ds\right]$$
$$\leq C^2 \,\mathbb{E}\left[\int_0^t |X_s|^2 ds\right]$$

Using that $dX_t = \sigma(t,\omega)\, dB_t$ and Fubini's theorem

$$C^2 \mathbb{E}\left[\int_0^t |X_s|^2 ds\right] = C^2 \int_0^t \mathbb{E}\left[\left|\int_0^s \sigma_r \, dB_r\right|^2\right] ds$$
$$= C^2 \int_0^t \mathbb{E}\left[\int_0^s |\sigma_r|^2 dr\right] ds$$
$$\leq C^4 \int_0^t \mathbb{E}\left[\int_0^s dr\right] ds$$
$$= C^4 \int_0^t s \, ds = \frac{C^4 t^2}{2} < \infty$$

- $\mathbb{E}[M_s \mid \mathcal{M}_t] = M_t, \quad \forall s \geq t$

$$\mathbb{E}\left[X_0^2 + 2\int_0^s X_r \sigma_r \, dB_r \,\middle|\, \mathcal{M}_t\right] = X_0^2 + 2\,\mathbb{E}\left[\int_0^s X_r \sigma_r \, dB_r \,\middle|\, \mathcal{M}_t\right]$$

$$= X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s$$

4.8. a) Let B_t denote n-dimensional Brownian motion and let $f: \mathbf{R}^n \to \mathbf{R}$ be C^2. Use Itô's formula to prove that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \tfrac{1}{2} \int_0^t \Delta f(B_s) ds \ ,$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

By Itô's formula,

$$df(B_t) = \frac{\partial f}{\partial t}(t, B_t)\,dt + \sum_i \frac{\partial f}{\partial x_i}(t, B_t)\,dB_{t_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, B_t)\,dB_{t_i} dB_{t_j}$$

Given that f does not depend on t,

- $\frac{\partial f}{\partial t}(t, B_t) = 0$
- $\sum_i \frac{\partial f}{\partial x_i}(t, B_t) = \nabla f(B_t)$
- $dB_{t_i} dB_{t_j} = \delta_{ij} dt \Rightarrow \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, B_t)\,dB_{t_i} dB_{t_j} = \sum_i \frac{\partial^2 f}{\partial x_i^2}(t, B_t)\,dt$
- $\int_0^t df(B_s) = \int_0^t \frac{df}{dB_s} dB_s = f(B_s)\Big|_0^t = f(B_t) - f(B_0)$

We have that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \frac{1}{2} \int_0^t \Delta f(B_s) ds$$

$\square$

b) Assume that $g : \mathbf{R} \to \mathbf{R}$ is C^1 everywhere and C^2 outside finitely many points $z_1, \ldots, z_N$ with $|g''(x)| \le M$ for $x \notin \{z_1, \ldots, z_N\}$. Let B_t be 1-dimensional Brownian motion. Prove that the 1-dimensional version of a) still holds, i.e.

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds\ .$$

(Hint: Choose $f_k \in C^2(\mathbf{R})$ s.t. $f_k \to g$ uniformly, $f_k' \to g'$ uniformly and $|f_k''| \le M$, $f_k'' \to g''$ outside $z_1, \ldots, z_N$. Apply a) to f_k and let $k \to \infty$).

Let $f_k \in C^2(\mathbb{R})$ such that $f_k \to g$ uniformly, $f_k' \to g'$ uniformly and $|f_k''| \leq M$ and $f_k'' \to g''$ outside $z_1, \ldots, z_N$.

Applying a) to f_k,

$$f_k(B_t) = f_k(B_0) + \int_0^t f_k'(B_s)\,dB_s + \frac{1}{2}\int_0^t f_k''(B_s)\,ds$$

as $k \to \infty$, we have

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)\,dB_s + \frac{1}{2}\int_0^t g''(B_s)\,ds$$

$\square$

4.9. Prove that we may assume that g and its first two derivatives are bounded in the proof of the Itô formula (Theorem 4.1.2) by proceeding as follows: For fixed $t \geq 0$ and $n = 1, 2, \dots$ choose g_n as in the statement such that $g_n(s,x) = g(s,x)$ for all $s \leq t$ and all $|x| \leq n$. Suppose we have proved that (4.1.9) holds for each g_n. Define the stochastic time

$$\tau_n = \tau_n(\omega) = \inf\{s > 0; |X_s(\omega)| \geq n\}$$

(τ_n is called a *stopping* time (See Chapter 7)) and prove that

$$\left(\int_0^t v \frac{\partial g_n}{\partial x}(s, X_s) \mathcal{X}_{s \leq \tau_n} dB_s := \right)$$
$$\int_0^{t\wedge\tau_n} v \frac{\partial g_n}{\partial x}(s, X_s) dB_s = \int_0^{t\wedge\tau_n} v \frac{\partial g}{\partial x}(s, X_s) dB_s$$

for each n. This gives that

$$g(t \wedge \tau_n, X_{t\wedge\tau_n}) = g(0, X_0) + \int_0^{t\wedge\tau_n} \left(\frac{\partial g}{\partial s} + u \frac{\partial g}{\partial x} + \tfrac{1}{2} v^2 \frac{\partial^2 g}{\partial x^2} \right) ds + \int_0^{t\wedge\tau_n} v \frac{\partial g}{\partial x} dB_s$$

and since

$$P[\tau_n > t] \to 1 \qquad \text{as } n \to \infty$$

we can conclude that (4.1.9) holds (a.s.) for g.

derivative of open sets: ok

the problem is on the border

show that $\frac{\partial g_n}{\partial x} = \frac{\partial g}{\partial x}$

$$g(t, X_t) = g(0, X_0) + \int_0^t \left(\frac{\partial g}{\partial s}(s, X_s) + u_s \frac{\partial g}{\partial x}(s, X_s) + \tfrac{1}{2} v_s^2 \cdot \frac{\partial^2 g}{\partial x^2}(s, X_s) \right) ds$$
$$+ \int_0^t v_s \cdot \frac{\partial g}{\partial x}(s, X_s) dB_s \quad \text{where } u_s = u(s,\omega),\ v_s = v(s,\omega)\,. \qquad (4.1.9)$$

Let $t \geq 0$ be fixed and $g_n(s,x) = g(s,x) \in C^2([0,\infty)\times\mathbb{R})$ for all $s \leq t$ and all $|x| \leq n$.

Suppose that (4.1.9) holds for each g_n. Define

$$\tau_n(\omega) = \inf\{ s > 0 : |X_s(\omega)| \geq n \}$$

Claim: $\displaystyle \int_0^{t\wedge\tau_n} v \frac{\partial g_n}{\partial x}(s, X_s) dB_s = \int_0^{t\wedge\tau_n} v \frac{\partial g}{\partial x}(s, X_s) dB_s$

Proof. Since (4.1.9) holds for each g_n,

$$g_n(t, X_t) = g_n(0, X_0) + \int_0^t \left(\frac{\partial g_n}{\partial s}(s, X_s) + u_s \frac{\partial g_n}{\partial x}(s, X_s) + \frac{1}{2} v_s^2 \frac{\partial^2 g_n}{\partial x^2}(s, X_s) \right) ds$$
$$+ \int_0^t v_s \frac{\partial g}{\partial x}(s, X_s) dB_s$$

Replacing t by $t\wedge\tau_n$, we have that $s \leq t\wedge\tau_n$ and $|X_t| \leq n$. Hence, $g_n = g$ and the claim holds.

$\boxed{\text{Claim: } \mathbb{P}[\tau_n > t] \longrightarrow 1 \quad \text{as} \quad n \to \infty}$

Proof. As $n \to \infty$, $\inf\{s > 0 : |X_s| \geq \infty\} = \infty = \tau_n$. Then

$$\mathbb{P}[\infty > t] = 1$$

since $t < \infty$.

Conclusion. Hence,

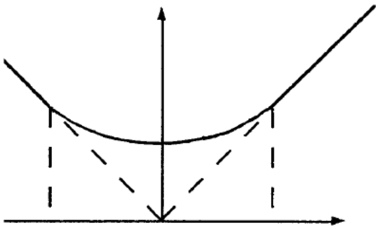
$$g_n(t \wedge \tau_n, X_{t \wedge \tau_n}) = g(t \wedge \tau_n, X_{t \wedge \tau_n}) = g(t, X_t)$$

And (4.1.9.) holds a.s. for g.

4.10. (Tanaka's formula and local time).
What happens if we try to apply the Itô formula to $g(B_t)$ when B_t is 1-dimensional and $g(x) = |x|$? In this case g is not C^2 at $x = 0$, so we modify $g(x)$ near $x = 0$ to $g_\epsilon(x)$ as follows:

$$g_\epsilon(x) = \begin{cases} |x| & \text{if} \quad |x| \geq \epsilon \\ \frac{1}{2}(\epsilon + \frac{x^2}{\epsilon}) & \text{if} \quad |x| < \epsilon \end{cases}$$

where $\epsilon > 0$.



a) Apply Exercise 4.8 b) to show that

$$g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g'_\epsilon(B_s)dB_s + \frac{1}{2\epsilon} \cdot |\{s \in [0,t]; B_s \in (-\epsilon, \epsilon)\}|$$

where $|F|$ denotes the Lebesgue measure of the set F.

Since

- g_ε is C^1 everywhere
- g_ε is C^2 outside 0
- $|g''_\varepsilon(x)| = \begin{cases} 0, & |x| \geqslant \varepsilon \\ 1/\varepsilon, & |x| < \varepsilon \end{cases}$ (*)

Applying 4.8. b),

$$\boxed{g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds\ .}$$

$$g_\varepsilon(B_t) = g_\varepsilon(B_0) + \int_0^t g'_\varepsilon(B_s)\,dB_s + \frac{1}{2}\int_0^t g''_\varepsilon(B_s)\,ds$$

Given our expression for $g''_\varepsilon(x)$ in (*) we see that

$$\int_0^t g''(B_s)\,ds = \frac{1}{\varepsilon}|\{s \in [0,t] : B_s \in (-\varepsilon,\varepsilon)\}|$$

Hence,

$$g_\varepsilon(B_t) = g_\varepsilon(B_0) + \int_0^t g'_\varepsilon(B_s)\,dB_s + \frac{1}{2\varepsilon}|\{s \in [0,t] : B_s \in (-\varepsilon,\varepsilon)\}|$$

∎

b) Prove that

$$\int_0^t g_\epsilon'(B_s)\cdot\mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s = \int_0^t \frac{B_s}{\epsilon}\cdot\mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s \to 0$$

in $L^2(P)$ as $\epsilon\to 0$.
(Hint: Apply the Itô isometry to

$$E\Big[\Big(\int_0^t \frac{B_s}{\epsilon}\cdot\mathcal{X}_{B_s\in(-\epsilon,\epsilon)}dB_s\Big)^2\Big].$$

$(*)$
$$g_\varepsilon'(x) = \begin{cases} \frac{x}{|x|}, & |x|\geq\varepsilon \\ \frac{x}{\varepsilon}, & |x|<\varepsilon \end{cases}$$

By the expression for g_ε' in $(*)$, we have

$$\int_0^t g_\varepsilon'(B_s)\cdot\chi_{B_s\in(-\varepsilon,\varepsilon)}dB_s = \int_0^t \frac{B_s}{\varepsilon}\chi_{B_s\in(-\varepsilon,\varepsilon)}dB_s$$

Using the Itô's Isometry,

$$\mathbb{E}\left[\left(\int_0^t \frac{B_s}{\varepsilon}\chi_{B_s\in(-\varepsilon,\varepsilon)}dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t\left(\frac{B_s}{\varepsilon}\right)^2\chi_{B_s\in(-\varepsilon,\varepsilon)}ds\right]$$

$$= \mathbb{E}\left[\frac{1}{\varepsilon^2}\int_0^t B_s^2\chi_{B_s\in(-\varepsilon,\varepsilon)}ds\right] \leq \mathbb{E}\left[\int_0^t\chi_{B_s\in(-\varepsilon,\varepsilon)}ds\right]$$

$$= \int_0^t\mathbb{E}\left[\chi_{B_s\in(-\varepsilon,\varepsilon)}\right]ds = \int_0^t\mathbb{P}\left[B_s\in(-\varepsilon,\varepsilon)\right]ds \longrightarrow 0$$

as $\varepsilon\to 0$. ∎

c) By letting $\epsilon \to 0$ prove that

$$|B_t| = |B_0| + \int_0^t \operatorname{sign}(B_s) dB_s + L_t(\omega) \,, \qquad (4.3.12)$$

where

$$L_t = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \cdot |\{s \in [0,t]; B_s \in (-\epsilon, \epsilon)\}| \quad (\text{limit in } L^2(P))$$

and

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for} \quad x \leq 0 \\ 1 & \text{for} \quad x > 0 \end{cases} .$$

L_t is called the *local time* for Brownian motion at 0 and (4.3.12) is the *Tanaka formula* (for Brownian motion). (See e.g. Rogers and Williams (1987)).

Using the previous items,

$$|B_t| = \lim_{\varepsilon \downarrow 0} g_\varepsilon(B_t) =$$

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(B_0) + \lim_{\varepsilon \downarrow 0} \int_0^t g_\varepsilon'(B_s)\, dB_s + \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} |\{ s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\}|$$

$$= |B_0| + \lim_{\varepsilon \downarrow 0} \int_0^t \frac{B_s}{|B_s|} \chi_{|B_s| \geq \varepsilon}\, dB_s + \cancelto{0}{\lim_{\varepsilon \downarrow 0} \int_0^t \frac{B_s}{\varepsilon} \chi_{B_s \in (-\varepsilon, \varepsilon)}\, dB_s} + L_t$$

$$= |B_0| + \int_0^t \operatorname{sign}(B_s)\, dB_s + L_t$$

4.11. Use Itô's formula (for example in the form of Exercise 4.3) to prove that the following stochastic processes are $\{\mathcal{F}_t\}$-martingales:

a) $X_t = e^{\frac{1}{2}t}\cos B_t \qquad (B_t \in \mathbf{R})$

b) $X_t = e^{\frac{1}{2}t}\sin B_t \qquad (B_t \in \mathbf{R})$

c) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t) \qquad (B_t \in \mathbf{R})$.

a) Let $X_t = g(t, B_t)$ where $g(t,x) = e^{1/2t}\cos x$. Then

$$\frac{\partial g}{\partial t} = \frac{1}{2}e^{1/2t}\cos x; \quad \frac{\partial g}{\partial x} = -e^{1/2t}\sin x; \quad \frac{\partial^2 g}{\partial x^2} = -e^{1/2t}\cos x$$

By Itô's formula,

$$dX_t = \frac{1}{2}e^{1/2t}\cos B_t\,dt - e^{1/2t}\sin B_t\,dB_t - \frac{1}{2}e^{1/2t}\cos B_t\,dt$$

Hence,

$$X_t = -\int_0^t e^{\frac{1}{2}s}\sin B_s\,dB_s$$

Therefore, X_t is an $\mathcal{F}_t$-martingale, since it is an Itô's Integral.

b) Let $X_t = g(t, B_t)$, where $g(t,x) = e^{1/2t}\sin x$. By Itô's formula,

$$dX_t = \frac{1}{2}e^{1/2t}\sin B_t\,dt + e^{1/2t}\cos B_t\,dB_t - \frac{1}{2}e^{1/2t}\sin B_t\,dt$$

Thus,

$$X_t = \int_0^t e^{1/2s}\cos B_s\,dB_s \quad \text{is an } \mathcal{F}_t\text{-martingale}$$

c) Let $g(t,x) = (x+t)e^{-x-\frac{1}{2}t}$. Then, $X_t = g(t, B_t)$ and

- $\frac{\partial g}{\partial t} = e^{-x-\frac{1}{2}t} - \frac{(x+t)}{2}e^{-x-\frac{1}{2}t} = \frac{1}{2}e^{-x-\frac{1}{2}t}(2-x-t)$

- $\frac{\partial g}{\partial x} = e^{-x-\frac{1}{2}t} - (x+t)e^{-x-\frac{1}{2}t} = e^{-x-\frac{1}{2}t}(1-x-t)$

- $\frac{\partial^2 g}{\partial x^2} = -e^{-x-\frac{1}{2}t} - e^{-x-\frac{1}{2}t} + (x+t)e^{-x-\frac{1}{2}t} = e^{-x-\frac{1}{2}t}(x+t-2)$

By Itô's formula,

$$dX_t = \frac{1}{2}e^{-B_t-\frac{1}{2}t}(2-B_t-t)dt + e^{-B_t-\frac{1}{2}t}(1-B_t-t)dB_t$$
$$+ \frac{1}{2}e^{-B_t-\frac{1}{2}t}(B_t+t-2)dt$$

Hence,

$$X_t = \int_0^t e^{-B_s-\frac{1}{2}s}(1-B_s-s)\, dB_s$$

is an $\mathcal{F}_t$-martingale. $\square$

4.12. Let $dX_t = u(t,\omega)dt + v(t,\omega)dB_t$ be an Itô process in $\mathbf{R}^n$ such that

$$E\Big[\int_0^t |u(r,\omega)|dr\Big] + E\Big[\int_0^t |vv^T(r,\omega)|dr\Big] < \infty \qquad \text{for all } t \geq 0 .$$

Suppose X_t is an $\{\mathcal{F}_t^{(n)}\}$-martingale. Prove that

$$u(s,\omega) = 0 \qquad \text{for a.a. } (s,\omega) \in [0,\infty) \times \Omega . \qquad (4.3.13)$$

Our first step is to show that

If X_t is an $\mathcal{F}_t^{(n)}$-martingale, then deduce that

$$E\Big[\int_t^s u(r,\omega)dr|\mathcal{F}_t^{(n)}\Big] = 0 \qquad \text{for all } s \geq t . \qquad (1)$$

In order to do that, notice that

martingale

$$0 = \mathbb{E}\left[X_s - X_t \mid \mathcal{F}_t^{(n)}\right] =$$

$$= \mathbb{E}\left[\left(X_0 + \int_0^s u\,dr + \int_0^s v\,dB_r\right) - \left(X_0 + \int_0^t u\,dr + \int_0^t v\,dB_r\right) \Big| \mathcal{F}_t^{(n)}\right]$$

$$= \mathbb{E}\left[\int_0^s u\,dr - \int_0^t u\,dr + \int_0^s v\,dB_r - \int_0^t v\,dB_r \,\Big|\, \mathcal{F}_t^{(n)}\right]$$

Thm. 3.2.1 $$= \mathbb{E}\left[\int_t^s u\,dr \,\Big|\, \mathcal{F}_t^{(n)}\right] + \underbrace{\cancel{\mathbb{E}\left[\int_t^s v\,dB_r \,\Big|\, \mathcal{F}_t^{(n)}\right]}}_{0} = \mathbb{E}\left[\int_t^s u\,dr \,\Big|\, \mathcal{F}_t^{(n)}\right]$$

Now, we are going to do the following:

Differentiate w.r.t. s to deduce that

$$E[u(s,\omega)|\mathcal{F}_t^{(n)}] = 0 \qquad \text{a.s., for a.a. } s > t .$$

Then let $t \uparrow s$ and apply Corollary C.9.

In fact,

$$\frac{d}{ds}\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \,\Big|\, \mathcal{F}_t^{(n)}\right] = \mathbb{E}\left[\frac{d}{ds}\left(\int_t^s u(r,\omega)\,dr\right) \Big|\, \mathcal{F}_t^{(n)}\right]$$

$$= \mathbb{E}\left[u(s,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] = 0 \quad (2)$$

which is zero by the R.H.S. of (1).

Corollary C.9. *Let $X \in L^1(P)$, let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be an increasing family of σ-algebras, $\mathcal{N}_k \subset \mathcal{F}$ and define $\mathcal{N}_\infty$ to be the σ-algebra generated by $\{\mathcal{N}_k\}_{k=1}^{\infty}$. Then*

$$E[X|\mathcal{N}_k] \to E[X|\mathcal{N}_\infty] \qquad \text{as } k \to \infty\,,$$

a.e. P and in $L^1(P)$.

By the Corollary C9. and (2),

$$\lim_{t \uparrow s} \mathbb{E}\left[u(t,\omega) \,\middle|\, \mathcal{F}_t^{(n)} \right] = \mathbb{E}\left[u(s,\omega) \,\middle|\, \mathcal{F}_t^{(n)} \right] = 0$$

Hence,

$$\mathbb{E}\left[u(s,\omega) \,\middle|\, \mathcal{F}_t^{(n)} \right] = u(t,\omega) \quad \text{a.a.}$$

$\square$

$$\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] = \mathbb{E}\left[\int_0^s u(r,\omega)\,dr - \int_0^t u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right]$$

$$= \mathbb{E}\left[\int_0^s u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] - \mathbb{E}\left[\int_0^t u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right]$$

Since u is $\mathcal{F}_t^{(n)}$-measurable,

$$\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \,\middle|\, \mathcal{F}_t^{(n)} \right] = \int_0^t u(r,\omega)\,dr - \int_0^t u(r,\omega)\,dr = 0 \quad (1)$$

$\triangle$

4.13. Let $dX_t = u(t,\omega)dt + dB_t$ $(u \in \mathbf{R},\ B_t \in \mathbf{R})$ be an Itô process and assume for simplicity that u is bounded. Then from Exercise 4.12 we know that unless $u = 0$ the process X_t is not an $\mathcal{F}_t$-martingale. However, it turns out that we can construct an $\mathcal{F}_t$-martingale from X_t by multiplying by a suitable exponential martingale. More precisely, define

$$Y_t = X_t M_t$$

where

$$M_t = \exp\left(-\int_0^t u(r,\omega)dB_r - \frac{1}{2}\int_0^t u^2(r,\omega)dr\right).$$

Use Itô's formula to prove that

$$Y_t \quad \text{is an } \mathcal{F}_t\text{-martingale}.$$

Let

$$Z_t := -\int_0^t u(r,\omega)\,dB_r - \frac{1}{2}\int_0^t u^2(r,\omega)\,dr$$

i.e.

$$dZ_t = -u(t,\omega)dB_t - \frac{1}{2}u^2(t,\omega)dt \qquad (1)$$

With this, it is possible to write $M_t = e^{Z_t}$ and apply Itô's formula:

$$dM_t = \frac{\partial M_t}{\partial t}dt + \frac{\partial M_t}{\partial Z_t}dZ_t + \frac{1}{2}\frac{\partial^2 M_t}{\partial Z_t^2}(dZ_t)^2$$

$$= e^{Z_t}dZ_t + \frac{1}{2}e^{Z_t}(dZ_t)^2 \qquad (2)$$

Expanding $(dZ_t)^2$,

$$(dZ_t)^2 = \left(u(t,\omega)dB_t - \frac{1}{2}u^2(t,\omega)dt\right)^2 = u^2 dt$$

Hence,

$$dM_t = e^{Z_t}dZ_t + \frac{1}{2}e^{Z_t}u^2 dt = M_t\left(\frac{1}{2}u^2 dt + dZ_t\right) \qquad (3)$$

Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \, .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \, .$$

Now, by Integration by Parts,

$$d(X_t M_t) = X_t dM_t + M_t dX_t + dX_t dM_t \qquad (4)$$

Since

- $X_t dM_t \overset{(3)}{=} X_t M_t \left(\frac{1}{2} u^2 dt + dZ_t \right) \overset{(1)}{=} X_t M_t \left(-u(t,\omega) dB_t \right)$

- $M_t dX_t = M_t (u dt + dB_t)$

- $dX_t dM_t \overset{(3)}{=} (u dt + dB_t) M_t \left(\frac{1}{2} u^2 dt + dZ_t \right)$

$$\overset{(1)}{=} M_t (u dt + dB_t)\left(-u(t,\omega) dB_t \right)$$

$$= -u M_t dt$$

We have that (4) can be written as

$$d(X_t M_t) = X_t M_t \left(-u(t,\omega) dB_t \right) + M_t (u dt + dB_t) - u M_t dt$$

$$= -u_t X_t M_t dB_t + M_t dB_t = M_t (1 - u_t X_t) dB_t$$

Therefore,

$$Y_t = Y_0 + \int_0^t M_s (1 - u_s X_s) dB_s \quad \text{is a martingale.}$$

4.14. In each of the cases below find the process $f(t,\omega) \in \mathcal{V}[0,T]$ such that (4.3.6) holds, i.e.

$$F(\omega) = E[F] + \int_0^T f(t,\omega)dB_t(\omega) .$$

a) $F(\omega) = B_T(\omega)$

Let $f(t,\omega) = 1$. Then, since $\mathbb{E}[B_t(\omega)] = 0$, we have

$$\int_0^T dB_t(\omega) = B_T(\omega) = F(\omega)$$

b) $F(\omega) = \int_0^T B_t(\omega)dt$

Recall that

$$\int_0^t s\,dB_s = tB_t - \int_0^t B_s ds$$

Note that

$$\mathbb{E}[F(\omega)] = \mathbb{E}\left[\int_0^T B_t dt\right] = \int_0^T \mathbb{E}[B_t]dt = 0$$

Therefore,

$$\int_0^T B_t dt = TB_T - \int_0^T t\,dB_t = T\int_0^T dB_t - \int_0^T t\,dB_t$$

$$= \int_0^T (T-t)dB_t \Rightarrow \underline{f(t,\omega) = T-t}$$

c) $F(\omega) = B_T^2(\omega)$

Recall that

$$\int_0^t B_t dB_t = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

and that $\mathbb{E}[B_t^2] = t$. Hence, taking $f(t,\omega) = B_t(\omega)\cdot 2$,

$$B_T^2(\omega) = T + 2\int_0^T B_t \, dB_t(\omega)$$

d) $F(\omega) = B_T^3(\omega)$

Recall that $\mathbb{E}[B_t^3] = 0$ and

$$\int_0^t B_s^2 \, dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$$
$$= \frac{1}{3}B_t^3 - \int_0^t (t-s) dB_s$$

Hence,

$$B_T^3 = 3\left(\int_0^T B_t^2 dB_t + \int_0^T (T-t) dB_t\right)$$

$$= 3\int_0^T (B_t^2 + T - t) dB_t$$

and $f(t,\omega) = 3(B_t^2 + T - t)$

e) $F(\omega) = e^{B_T(\omega)}$

By Itô's formula,

$$d e^{B_T(\omega)} = e^{B_T(\omega)} dB_T + \frac{1}{2} e^{B_T(\omega)} dt = e^{B_T(\omega)} \left(\frac{1}{2} dt + dB_T \right)$$

and

$$d\left(e^{B_T(\omega) - \frac{1}{2}T}\right) = -\frac{1}{2} e^{B_T - \frac{1}{2}T} dt + e^{B_T - \frac{1}{2}T} dB_T + \frac{1}{2} e^{B_T - \frac{1}{2}T} dt$$

$$= e^{B_T - \frac{1}{2}T} dB_T$$

Let $U(t) = e^{B_t - \frac{1}{2}t}$ and notice that we have the following SDE:

$$dU_t = U_t dB_t, \qquad U_0 = 1$$

In the integral form,

$$U_T - U_0 = \int_0^T U_t \, dB_t \iff e^{B_T - \frac{1}{2}T} = 1 + \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$

$$\iff e^{B_T} = e^{1/2T} + \int_0^T e^{B_t - \frac{1}{2}t} e^{1/2T} dB_t$$

$$= e^{1/2T} + \int_0^T e^{B_t + \frac{1}{2}(T-t)} dB_t$$

f) $F(\omega) = \sin B_T(\omega)$

From the exercise 4.11,

$$d\left(e^{1/2t}\sin B_t\right) = e^{1/2t}\cos B_t \, dB_t$$

Hence,

$$e^{1/2t}\sin B_t = \int_0^t e^{1/2s}\cos B_s \, dB_s$$

and,

$$\sin B_t = \int_0^t e^{1/2(s-t)}\cos B_s \, dB_s$$

$$\therefore \quad f(t,\omega) = e^{1/2(t,T)}\cos B_t$$

4.15. Let $x > 0$ be a constant and define

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3\,; \qquad t \geq 0\,.$$

Show that

$$dX_t = \tfrac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t\,; \qquad X_0 = x\,.$$

Let $g(t,y) = \left(x^{1/3} + \frac{1}{3}y\right)^3$ and compute

- $\frac{\partial g}{\partial t} = 0$

- $\frac{\partial g}{\partial y} = 3\left(x^{1/3} + \frac{1}{3}y\right)^2 \cdot \frac{1}{3} = \left(x^{1/3} + \frac{1}{3}y\right)^{3\cdot\frac{2}{3}}$

- $\frac{\partial^2 g}{\partial y^2} = 2\left(x^{1/3} + \frac{1}{3}y\right) \cdot \frac{1}{3} = \frac{2}{3}\left(x^{1/3} + \frac{1}{3}y\right)^{3\cdot\frac{1}{3}}$

By Itô's formula,

$$dX_t = \left(x^{1/3} + \frac{1}{3}y\right)^{3\cdot\frac{2}{3}} dB_t + \frac{1}{2}\,\frac{2}{3}\left(x^{1/3} + \frac{1}{3}y\right)^{3\cdot\frac{1}{3}} dt$$

$$= X_t^{2/3}\,dB_t + \frac{1}{3}X_t^{1/3}\,dt$$

5.1. Verify that the given processes solve the given corresponding stochastic differential equations: (B_t denotes 1-dimensional Brownian motion)

(i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$

By Itô's formula,

$$dX_t = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt = \frac{1}{2} X_t dt + X_t dB_t$$

(ii) $X_t = \frac{B_t}{1+t}$; $B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t ; \qquad X_0 = 0$$

$$g(t,x) = \frac{x}{1+t}$$

$$dX_t = \frac{-B_t}{(1+t)^2} dt + \frac{1}{1+t} dB_t = -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t$$

(iii) $X_t = \sin B_t$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves

$$dX_t = -\tfrac{1}{2} X_t dt + \sqrt{1 - X_t^2}\, dB_t \ \text{ for } \ t < \inf\{s > 0; B_s \notin [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\}$$

$$g(t,x) = \sin x$$

$$dX_t = \cos B_t \, dB_t - \frac{1}{2} \sin B_t \, dt = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2}\, dB_t$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

$g_1(t,x) = t$

$g_2(t,x) = e^t x$

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ e^t B_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^t \end{bmatrix} dB_t = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t \ .$$

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} \sinh B_t \\ \cosh B_t \end{bmatrix} dB_t + \frac{1}{2} \begin{bmatrix} \cosh B_t \\ \sinh B_t \end{bmatrix} dt$$

$$= \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t$$

5.2. A natural candidate for what we could call *Brownian motion on the ellipse*

$$\left\{(x,y); \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\} \qquad \text{where } a>0, b>0$$

is the process $X_t=(X_1(t),X_2(t))$ defined by

$$X_1(t)=a\cos B_t\ , \quad X_2(t)=b\sin B_t$$

where B_t is 1-dimensional Brownian motion. Show that X_t is a solution of the stochastic differential equation

$$dX_t=-\tfrac{1}{2}X_t dt+MX_t dB_t$$

where $M=\begin{bmatrix}0 & -\frac{a}{b}\\ \frac{b}{a} & 0\end{bmatrix}$.

Applying Itô's formula,

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} -a\sin B_t \\ b\cos B_t \end{bmatrix} dB_t + \frac{1}{2}\begin{bmatrix} -a\cos B_t \\ -b\sin B_t \end{bmatrix} dt$$

$$= -\frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 & -a/b \\ b/a & 0 \end{bmatrix} \cdot \begin{bmatrix} a\cos B_t \\ b\sin B_t \end{bmatrix} dB_t$$

$$= -\frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 & -a/b \\ b/a & 0 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dB_t$$

Thus,

$$dX_t = -\frac{1}{2} X_t\, dt + M X_t dB_t$$

5.3. Let $(B_1, \ldots, B_n)$ be Brownian motion in $\mathbf{R}^n$, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation

$$dX_t = rX_t dt + X_t\Big(\sum_{k=1}^{n} \alpha_k dB_k(t)\Big); \qquad X_0 > 0 .$$

(This is a model for exponential growth with several independent white noise sources in the relative growth rate).

Let

$$X_t = X_0 \exp\left(\left(r - \frac{1}{2}\sum_{k=1}^{n} a_k^2\right)t + \sum_{k=1}^{n} a_k B_k(t)\right)$$

Then by Itô's formula,

$$dX_t = \left(r - \frac{1}{2}\sum_{k=1}^{n} a_k^2\right) X_t dt + \sum_{k=1}^{n} a_k X_t dB_k(t)$$

$$+ \frac{1}{2}\sum_{k=1}^{n} a_k^2 X_t dt$$

Hence,

$$dX_t = rX_t dt + X_t \sum_{k=1}^{n} a_k dB_k(t)$$

5.4. Solve the following stochastic differential equations:

(i) $$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$$

Notice that if $X_1 = X_{1_0} + t + B_1(t)$, then $dX_1(t) = dt + dB_1(t)$

Similarly, $dX_2(t) = X_1 dB_2(t) = (X_1(0) + t + B_1)\, dB_2$. Thus

$$X_2(t) = X_2(0) + X_1(0) B_2 + \int_0^t s\, dB_2 + \int_0^t B_1\, dB_2$$

(ii) $dX_t = X_t dt + dB_t$
(Hint: Multiply both sides with "the integrating factor" e^{-t} and compare with $d(e^{-t}X_t)$)

Multiplying by e^{-t}:

$$e^{-t} dX_t = e^{-t} X_t dt + e^{-t} dB_t$$

On the other hand,

$$d(e^{-t}X_t) = -e^{-t} X_t dt + e^{-t} dX_t$$

Thus,

$$d(e^{-t}X_t) = -e^{-t} X_t dt + e^{-t} X_t dt + e^{-t} dB_t = e^{-t} dB_t$$

I.e.,

$$e^{-t} X_t = X_0 + \int_0^t e^{-s} dB_s \iff \boxed{X_t = e^{t} X_0 + \int_0^t e^{(t-s)} dB_s}$$

(iii) $dX_t = -X_t dt + e^{-t} dB_t$.

Consider $X_t = e^{-t} B_t$. Then

$$dX_t = -e^{-t} B_t\, dt + e^{-t} dB_t = -X_t\, dt + e^{-t} dB_t$$

Thus, taking $X_0 = B_0 = 0$,

$$X_t = e^{-t} B_t, \quad X_0 = 0$$

solves the SDE.

5.5. a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t dt + \sigma dB_t$$

where μ, σ are real constants, $B_t \in \mathbf{R}$.
The solution is called the *Ornstein-Uhlenbeck process*. (Hint: See Exercise 5.4 (ii).)

Multiplying by the integrating factor $e^{-\mu t}$,

$$e^{-\mu t} dX_t = e^{-\mu t} \mu X_t dt + e^{-\mu t} \sigma dB_t \qquad (1)$$

Now notice that

$$d(e^{-\mu t} X_t) = -\mu e^{-\mu t} X_t dt + e^{-\mu t} dX_t \qquad (2)$$

With (1) and (2)

$$d(e^{-\mu t} X_t) = e^{-\mu t} \sigma dB_t$$

Therefore,

$$e^{-\mu t} X_t = X_0 + \sigma \int_0^t e^{-\mu s} dB_s$$

And hence,

$$\boxed{X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{-\mu (s-t)} dB_s}$$

b) Find $E[X_t]$ and $\text{Var}[X_t] := E[(X_t - E[X_t])^2]$.

- $$\mathbb{E}[X_t] = \mathbb{E}\left[e^{\mu t}X_0 + \sigma\int_0^t e^{-\mu(s-t)}\,dB_s\right] = \mathbb{E}\left[e^{\mu t}X_0\right]$$

$$= e^{\mu t}\,\mathbb{E}[X_0]$$

- Notice that

$$\mathbb{E}[X_t^2] = \mathbb{E}\left[e^{2\mu t}X_0^2 + \sigma^2\int_0^t e^{-2\mu(s-t)}\,ds + 2e^{\mu t}X_0\sigma\int_0^t e^{-\mu(s-t)}\,dB_s\right]$$

$$= e^{2\mu t}\mathbb{E}[X_0^2] + \sigma^2\,\mathbb{E}\left[\int_0^t e^{-2\mu(s-t)}\,ds\right]$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \sigma^2\int_0^t e^{-2\mu(s-t)}\,ds$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right) \qquad (*)$$

Hence,

$$\text{Var}[X_t] = \mathbb{E}[X_t^2] - \mathbb{E}^2[X_t]$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right) - e^{2\mu t}\,\mathbb{E}^2[X_0]$$

$$= e^{2\mu t}\left(\mathbb{E}[X_0^2] - \mathbb{E}^2[X_0]\right) + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$$

(*)

Obs: $e^{2\mu t}\int_0^t e^{-2\mu s}\,ds = e^{2\mu t}\cdot\left[\dfrac{e^{-2\mu s}}{-2\mu}\right]_0^t = \dfrac{-e^{2\mu t}}{2\mu}\left(e^{-2\mu t}-1\right)$

$= -\dfrac{1}{2\mu}\left(1-e^{2\mu t}\right) = \dfrac{1}{2\mu}\left(e^{2\mu t}-1\right)$

5.6. Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha Y_t dB_t$$

where r, α are real constants, $B_t \in \mathbf{R}$.
(Hint: Multiply the equation by the 'integrating factor'

$$F_t = \exp\left(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\right).)$$

By the exercise 4.3.

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t\ .$$

Then

$$d(F_t Y_t) = F_t dY_t + Y_t dF_t + dF_t dY_t \qquad (1)$$

Since

$$dF_t = \frac{1}{2}\alpha^2 F_t dt - \alpha F_t dB_t + \frac{1}{2}\alpha^2 F_t dt$$

$$= F_t(\alpha^2 dt - \alpha\, dB_t) \qquad (2)$$

then

$$dF_t dY_t = F_t(\alpha^2 dt - \alpha\, dB_t)(r\,dt + \alpha Y_t dB_t)$$

$$= F_t(-\alpha^2 Y_t\, dt) \qquad (3)$$

Now we can rewrite (1) as

$$d(F_t Y_t) = F_t(r\,dt + \alpha Y_t dB_t) + Y_t F_t(\alpha^2 dt - \alpha\, dB_t) + F_t(-\alpha^2 Y_t\, dt)$$

$$= rF_t dt + \alpha F_t Y_t dB_t + \alpha^2 F_t Y_t dt - \alpha F_t Y_t dB_t - \alpha^2 F_t Y_t dt$$

$$= rF_t dt$$

Therefore,

$$F_t Y_t = Y_0 + \int_0^t r F_s \, ds$$

$$\therefore \quad Y_t = F_t^{-1} \left[Y_0 + \int_0^t r F_s \, ds \right]$$

$$= \exp\left(\alpha B_t - 1/2\, \alpha^2 t\right) \left[Y_0 + r \int_0^t \exp\left(-\alpha B_t + 1/2\, \alpha^2 t\right) ds \right]$$

5.7. The *mean-reverting Ornstein-Uhlenbeck process* is the solution X_t of the stochastic differential equation

$$dX_t = (m - X_t)dt + \sigma dB_t$$

where m, σ are real constants, $B_t \in \mathbf{R}$.

a) Solve this equation by proceeding as in Exercise 5.5 a).

b) Find $E[X_t]$ and $\mathrm{Var}[X_t] := E[(X_t - E[X_t])^2]$.

a) Let $Y_t = X_t - m$. By Itô's formula, $dY_t = dX_t$. Hence

$$dY_t = -Y_t dt + \sigma dB_t$$

Applying 5.5. $(\mu = -1)$

$$Y_t = e^{-t} Y_0 + \sigma \int_0^t e^{(s-t)} \, dB_s$$

and finally,

$$\boxed{X_t = m + e^{-t} Y_0 + \sigma \int_0^t e^{(s-t)} \, dB_s}$$

b) By 5.5.,

$$\mathbb{E}[X_t] = m + \mathbb{E}[Y_t] = m + e^{\mu t}\mathbb{E}[X_0] = m + e^{-t}\mathbb{E}[X_0]$$

$$\mathrm{Var}[X_t] = e^{2\mu t}\left(\mathbb{E}[X_0^2] - \mathbb{E}^2[X_0]\right) + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$$

$$= e^{-2t}\left(\mathbb{E}[X_0^2] - \mathbb{E}^2[X_0]\right) - \frac{\sigma^2}{2}\left(e^{-2t} - 1\right)$$

5.8. Solve the (2-dimensional) stochastic differential equation

$$dX_1(t) = X_2(t)dt + \alpha dB_1(t)$$
$$dX_2(t) = -X_1(t)dt + \beta dB_2(t)$$

where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion and α, β are constants.
This is a model for a vibrating string subject to a stochastic force. See Example 5.1.3.

In matrix notation,

$$dX_t = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} X_t dt + \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}}_{C} dB_t; \quad X_t = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad B_t = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Multiplying by $\exp(-At)$,

$$\exp(-At)\, dX_t = \exp(-At)\, A X_t\, dt + \exp(-At)\, C\, dB_t \qquad (1)$$

However,

$$d\left(\exp(-At) X_t\right) = -A \exp(-At)\, X_t\, dt + \exp(-At)\, dX_t \qquad (2)$$

By (1) and (2)

$$d\left(\exp(-At) X_t\right) = \exp(-At)\, C\, dB_t$$

I.e.,

$$\exp(-At)\, X_t = X_0 + \int_0^t \exp(-As)\, C\, dB_s$$

$$\therefore \quad X_t = \exp(At)\left[X_0 + \int_0^t \exp(-As)\, C\, dB_s \right]$$

5.9. Show that there is a unique strong solution X_t of the 1-dimensional stochastic differential equation

$$dX_t = \ln(1 + X_t^2)dt + \mathcal{X}_{\{X_t>0\}} X_t dB_t \,, \qquad X_0 = a \in \mathbf{R} \,.$$

First we need to show that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \quad x \in \mathbf{R}^n,\ t \in [0,T]$$

In fact, $|\mathcal{X}_{\{x>0\}} x| \leq |x|$ and

$$\begin{aligned} |\ln(1+x^2)| &= \ln(1+x^2) \\ &\leq \ln(x^2 - 2|x| + 1) && \text{since } 1+x^2 \leq x^2 - 2|x| + 1 \\ &= \ln(|x|-1)^2 \\ &= 2\ln(|x|-1) \\ &\leq 2|x| && \text{since } \ln(x) \leq x-1 \end{aligned}$$

Hence,

$$|\ln(1+x^2)| + |\mathcal{X}_{\{x>0\}} x| \leq 2|x| + |x| < 3(1+|x|)$$

Now we need to show that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|, \quad x,y \in \mathbf{R}^n,\ t \in [0,T]$$

However, by the Mean Value Theorem,

$$|\ln(1+x^2) - \ln(1+y^2)| = \left|\frac{2z}{1+z^2}\right| |x-y|$$

Given that

$$-\frac{(x-1)^2}{1+x^2} \leq 0 \iff \frac{2x - x^2 - 1}{1+x^2} \leq 0 \iff \frac{2x}{1+x^2} \leq \frac{1+x^2}{1+x^2} = 1$$

We obtain

$$|\ln(1+x^2) - \ln(1+y^2)| \leq |x-y|$$

Thus, b is Lipschitz. Also

$$|\chi_{\{x>0\}} x - \chi_{\{y>0\}} y| \leq |x-y|$$

Hence, a strong solution does exist.

5.10. Let b, σ satisfy (5.2.1), (5.2.2) and let X_t be the unique strong solution of (5.2.3). Show that

$$E[|X_t|^2] \leq K_1 \cdot \exp(K_2 t) \qquad \text{for } t \leq T \qquad (5.3.2)$$

where $K_1 = 3E[|Z|^2] + 6C^2T(T+1)$ and $K_2 = 6(1+T)C^2$.
(Hint: Use the argument in the proof of (5.2.10)).

First write

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s$$

Using that $(x_1 + \ldots + x_n)^2 \leq n(x_1^2 + \ldots + x_n^2)$,

$$E\left[|X_t|^2\right] \leq 3E\left[|X_0^2| + \left|\int_0^t b_s ds\right|^2 + \left|\int_0^t \sigma_s dB_s\right|^2\right] \qquad (1)$$

By Cauchy-Schwartz,

$$\left|\int_0^t b_s ds\right|^2 \leq \left(\int_0^t |b_s|^2 ds\right)\left(\int_0^t ds\right) = t\int_0^t |b_s|^2 ds \qquad (2)$$

With (2) and Itô's Isometry, (1) becomes

$$E[|X_t|^2] \leq 3\left[E[|Z|^2] + tE\left[\int_0^t |b_s|^2 ds\right] + E\left[\int_0^t |\sigma_s|^2 ds\right]\right] \qquad (3)$$

Now, by 5.2.1,

$$|b_s| \leq C(1+|x|) \text{ and } |\sigma_s| \leq C(1+|x|)$$

then

$$\int_0^t |b_s|^2 ds \leq \int_0^t |C(1+|x|)|^2 ds = t|C(1+|x|)|^2 \leq 2t(C^2 + C^2|x|^2)$$

Therefore,

$$\mathbb{E}[|X_t|^2] \leq 3\left(\mathbb{E}[|Z|^2] + t\,\mathbb{E}\left[2t(C^2 + C^2|X_t|^2)\right] + \mathbb{E}\left[2t(C^2 + C^2|X_t|^2)\right]\right)$$

$$= 3\left(\mathbb{E}[|Z|^2] + 2t^2C^2 + 2t^2C^2\,\mathbb{E}[|X_t|^2] + 2tC^2 + 2tC^2\,\mathbb{E}[|X_t|^2]\right)$$

$$= 3\,\mathbb{E}[|Z|^2] + 6C^2t(t+1) + 6C^2t(t+1)\,\mathbb{E}[|X_t|^2]$$

$$= K_1 + K_2\,\mathbb{E}[|X_t|^2]$$

Finally, by Gronwall inequality,

$$\mathbb{E}[|X_t|^2] \leq K_1 \exp(K_2 t)$$

∎

5.11. (The Brownian bridge).
For fixed $a, b \in \mathbf{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t}dt + dB_t \; ; \qquad 0 \le t < 1\,, \;\; Y_0 = a\,. \qquad (5.3.3)$$

Verify that

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s} \; ; \qquad 0 \le t < 1 \qquad (5.3.4)$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s. The process Y_t is called *the Brownian bridge* (from a to b). For other characterizations of Y_t see Rogers and Williams (1987, pp. 86–89).

Let $g(t,x) = a(1-t) + bt + (1-t)\int_0^t \frac{dx}{1-s}$

By Itô's formula,

$$dY_t = \left(-a + b - \int_0^t \frac{dB_s}{1-s}\right)dt + (1-t)\cdot\frac{1}{1-t}dB_t = \left(\frac{b - Y_t}{1-t}\right)dt + dB_t$$

Hence Y_t solves the equation.
To show that

$$\lim_{t\to 1} Y_t = \lim_{t\to 1}\left[a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}\right] = b$$

notice that the deterministic part $a(1-t)+bt$ is immediate.
Now, using Itô's isometry,

$$\mathbb{E}\left[\left((1-t)\int_0^t \frac{dB_s}{1-s}\right)^2\right] = \mathbb{E}\left[(1-t)^2\int_0^t \frac{ds}{(1-s)^2}\right]$$

$$= (1-t)^2\int_0^t \frac{ds}{(1-s)^2} = (1-t)^2\cdot\left[\frac{1}{(1-x)}\right]_0^t = (1-t)^2\left[\frac{1}{(1-t)} - 1\right]$$

$$= (1-t) - (1-t)^2 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 1$$

Thus, $\lim_{t\to 1} Y_t = b$ a.s.

5.12. To describe the motion of a pendulum with small, random perturbations in its environment we try an equation of the form

$$y''(t) + (1 + \epsilon W_t)y = 0\,; \qquad y(0), y'(0) \text{ given },$$

where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise, $\epsilon > 0$ is constant.

a) Discuss this equation, for example by proceeding as in Example 5.1.3.

b) Show that $y(t)$ solves a *stochastic Volterra equation* of the form

$$y(t) = y(0) + y'(0)\cdot t + \int_0^t a(t,r)y(r)dr + \int_0^t \gamma(t,r)y(r)dB_r$$

where $a(t,r) = r - t$, $\gamma(t,r) = \epsilon(r-t)$.

a) Define

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

Therefore,

$$\begin{cases} X_1'(t) = X_2(t) \\ X_2' = -(1+\varepsilon W_t)X_1 \end{cases}$$

In matrix notation,

$$dX_t = AX_t dt + CX_t dB_t$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ -\varepsilon & 0 \end{bmatrix}$$

Now,

$$\exp(-At)dX_t = \exp(-At)AX_t dt + \exp(-At)CX_t dB_t$$

By Itô's formula,

$$d(\exp(-At)X_t) = -A\exp(-At)X_t dt + \exp(-At)dX_t$$

Hence,

$$d(\exp(-At)X_t) = \exp(-At)CX_t dB_t$$

I.e.,

$$\exp(-At)X_t = X_0 + \int_0^t \exp(-As)CX_s\,dB_s$$

$$\iff \quad X_t = \exp(At)\left[X_0 + \int_0^t \exp(-As)CX_s\,dB_s\right]$$

b) Rewriting the solution

$$\boxed{\frac{d}{dx}\left(\int_a^x f(x,t)dt\right) = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,t)dt}$$

Using the Leibniz rule,

$$y'(t) = y'(0) - \int_0^t y(r)dr - \varepsilon\int_0^t y(r)\,dB_r$$

Since $W_t = \frac{dB_t}{dt}$, we have

$$y''(t) = -y(t) - \varepsilon\, y(t)W_t = -y(t)(1+\varepsilon W_t)$$

I.e.,

$$y''(t) + (1+\varepsilon W_t)\,y(t) = 0$$

Another way: notice that

$$Y_t = y_0 + y_0' t + \int_0^t (r-t)y_r\,dr + \int_0^t \varepsilon(r-t)y_r\,\underline{\omega_r dr}$$

$dB_t = \omega_t\,dt$

$$= y_0 + y_0' t + \int_0^t r\underbrace{(1+\varepsilon\omega_r)y_r}_{-y_r''}dr - \int_0^t t\underbrace{(1+\varepsilon\omega_r)y_r}_{-y_r''}dr$$

$$= y_0 + y_0' t - r y_r' \Big|_0^t + \int_0^t y_r' \, dr + t y_r' \Big|_0^t$$

$$= \cancel{y_0} + \cancel{y_0' t} - \cancel{t y_t'} + y_t - \cancel{y_0} + \cancel{t y_t'} - \cancel{t y_0'} = y_t \quad ✓$$

5.13. As a model for the horizontal slow drift motions of a moored floating platform or ship responding to incoming irregular waves John Grue (1989) introduced the equation

$$x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t')\eta W_t \,, \qquad (5.3.5)$$

where W_t is 1-dimensional white noise, a_0, w, T_0, α_0 and η are constants.

(i) Put $X_t = \begin{bmatrix} x_t \\ x_t' \end{bmatrix}$ and rewrite the equation in the form

$$dX_t = AX_t dt + KX_t dB_t + MdB_t \,,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -w^2 & -a_0 \end{bmatrix}, \quad K = \alpha_0 \eta \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M = T_0 \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Show that X_t satisfies the integral equation

$$X_t = \int_0^t e^{A(t-s)} K X_s dB_s + \int_0^t e^{A(t-s)} M dB_s \qquad \text{if } X_0 = 0 \,.$$

Multiplying the equation by $\exp(-At)$,

$$\exp(-At)\, dX_t = \exp(-At)\, AX_t dt + \exp(-At)\left[KX_t + M\right] dB_t$$

Applying Itô's formula to $d(\exp(-At)\, X_t)$,

$$d(\exp(-At)\, X_t) = -A\exp(-At) X_t dt + \exp(-At)\, dX_t$$

$$= \exp(-At)\left[KX_t + M\right] dB_t$$

Therefore,

$$\exp(-At)\, X_t = X_0 + \int_0^t \exp(-As)\left[KX_s + M\right] dB_s$$

If $X_0 = 0$,

$$\boxed{X_t = \int_0^t e^{A(t-s)} K X_s dB_s + \int_0^t e^{A(t-s)} M dB_s}$$

(iii) Verify that

$$e^{At} = \frac{e^{-\lambda t}}{\xi}\{(\xi\cos\xi t + \lambda\sin\xi t)I + A\sin\xi t\}$$

where $\lambda = \frac{a_0}{2}, \xi = (w^2 - \frac{a_0^2}{4})^{\frac{1}{2}}$ and use this to prove that

$$x_t = \eta\int_0^t (T_0 - \alpha_0 y_s)g_{t-s}dB_s \qquad (5.3.6)$$

and

$$y_t = \eta\int_0^t (T_0 - \alpha_0 y_s)h_{t-s}dB_s\ , \qquad \text{with } y_t := x_t'\ , \qquad (5.3.7)$$

where

$$g_t = \frac{1}{\xi}\mathrm{Im}(e^{\zeta t})$$
$$h_t = \frac{1}{\xi}\mathrm{Im}(\zeta e^{\bar{\zeta} t})\ , \qquad \zeta = -\lambda + i\xi \quad (i = \sqrt{-1})\ .$$

So we can solve for y_t first in (5.3.7) and then substitute in (5.3.6) to find x_t.

Idea: write $A = PDP^{-1}$ and then use that $e^{At} = Pe^{Dt}P^{-1}$

We start by computing the caracteristic polynomial of A:

$$C_A(x) = \begin{vmatrix} x & -1 \\ w^2 & x + a_0 \end{vmatrix} = x(x+a_0) + w^2 = x^2 + x a_0 + w^2$$

Setting $C_A(x) = 0$,

$$x = \frac{-a_0 \pm \sqrt{a_0^2 - 4w^2}}{2} = \frac{-a_0}{2} \pm i\sqrt{w^2 - a_0^2/4} = -\lambda \pm i\xi$$

Finding the eigenvectors associated with $-\lambda + i\xi$:

$$\begin{bmatrix} \lambda - i\xi & 1 \\ -w^2 & -a_0 + \lambda - i\xi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{cases} (\lambda - i\xi)x + y = 0 \\ -w^2 x + (-a_0 + \lambda - i\xi)y = 0 \end{cases}$$

Then $y = -(\lambda - i\xi)x$. Let $x = 1$ and then $y = -\lambda + i\xi$.
For $-\lambda - i\xi$, the same calculation yields $x = 1$ and $-\lambda - i\xi$.
Thus, we can write

using the idea

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -\lambda + i\xi & -\lambda - i\xi \end{bmatrix} \cdot \begin{bmatrix} e^{(-\lambda+i\xi)t} & 0 \\ 0 & e^{(-\lambda-i\xi)t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\lambda + i\xi & -\lambda - i\xi \end{bmatrix}^{-1} \qquad (1)$$

Now let us compute the inverse above.

$$\begin{bmatrix} 1 & 1 \\ -\lambda + i\xi & -\lambda - i\xi \end{bmatrix}^{-1} = \frac{-1}{2\xi i} \begin{bmatrix} -\lambda - i\xi & -1 \\ \lambda - i\xi & 1 \end{bmatrix}$$

Thus

$$e^{At} = \frac{-1}{2\xi i} \begin{bmatrix} 1 & 1 \\ -\lambda + i\xi & -\lambda - i\xi \end{bmatrix} \cdot \begin{bmatrix} e^{(-\lambda+i\xi)t} & 0 \\ 0 & e^{(-\lambda-i\xi)t} \end{bmatrix} \begin{bmatrix} -\lambda - i\xi & -1 \\ \lambda - i\xi & 1 \end{bmatrix}$$

$$= \frac{-1}{2\xi i} \begin{bmatrix} 1 & 1 \\ -\lambda + i\xi & -\lambda - i\xi \end{bmatrix} \cdot \begin{bmatrix} -e^{(-\lambda+i\xi)t}(\lambda + i\xi) & -e^{(-\lambda+i\xi)t} \\ e^{(-\lambda-i\xi)t}(\lambda - i\xi) & e^{(-\lambda-i\xi)t} \end{bmatrix}$$

$$= \frac{-1}{2\xi i} \begin{bmatrix} F & G \\ H & I \end{bmatrix} \qquad (2)$$

where

$$F = e^{(-\lambda-i\xi)t}(\lambda - i\xi) - e^{(-\lambda+i\xi)t}(\lambda + i\xi)$$

$$G = e^{(-\lambda-i\xi)t} - e^{(-\lambda+i\xi)t}$$

$$H = (-\xi^2 - \lambda^2)\, e^{(-\lambda-i\xi)t} - (\xi^2 - \lambda^2)\, e^{(-\lambda+i\xi)t}$$

$$I = (-\lambda + i\xi)(-e^{(-\lambda+i\xi)t}) + (-\lambda - i\xi)(e^{(-\lambda-i\xi)t})$$

Simplifying,

$$F = e^{(-\lambda-i\xi)t}(\lambda - i\xi) - e^{(-\lambda+i\xi)t}(\lambda+i\xi)$$

$$= e^{-\lambda t}(\lambda - i\xi)e^{-i\xi t} - e^{-\lambda t}(\lambda+i\xi)e^{i\xi t}$$

$$= e^{-\lambda t}(\lambda - i\xi)(\cos(\xi t) - i\sin(\xi t)) - e^{-\lambda t}(\lambda + i\xi)(\cos(\xi t) + i\sin(\xi t))$$

$$= e^{-\lambda t}\Big[\lambda(\cancel{\cos(\xi t)} - i\sin(\xi t)) - i\xi(\cos(\xi t) - \cancel{i\sin(\xi t)})$$
$$- \lambda(\cancel{\cos(\xi t)} + i\sin(\xi t)) - i\xi(\cos(\xi t) + \cancel{i\sin(\xi t)})\Big]$$

$$= -2ie^{-\lambda t}\left[\lambda\sin(\xi t) + \xi\cos(\xi t)\right] \qquad (3)$$

$$G = e^{(-\lambda-i\xi)t} - e^{(-\lambda+i\xi)t} = e^{-\lambda t}\left[e^{-i\xi t} - e^{i\xi t}\right]$$

$$= -2ie^{-\lambda t}\sin(\xi t) \qquad (4)$$

$$H = (-\xi^2 - \lambda^2)e^{(-\lambda-i\xi)t} - (\xi^2 - \lambda^2)e^{(-\lambda+i\xi)t}$$

$$= -\omega^2\left(e^{(-\lambda-i\xi)t} - e^{(-\lambda+i\xi)t}\right) \overset{(4)}{=} -\omega^2\left(-2ie^{-\lambda t}\sin(\xi t)\right) \qquad (5)$$

where we used that $\lambda^2 + \xi^2 = \omega^2$.

$$I = (-\lambda + i\xi)\left(-e^{(-\lambda+i\xi)t}\right) + (-\lambda - i\xi)\left(e^{(-\lambda-i\xi)t}\right)$$

$$= -(-\lambda + i\xi)e^{-\lambda t}e^{i\xi t} + e^{-\lambda t}(-\lambda - i\xi)e^{-i\xi t}$$

$$= e^{-\lambda t}\Big[(\lambda - i\xi)\big(\cos(\xi t) + i\sin(\xi t)\big) + (-\lambda - i\xi)\big(\cos(\xi t) - i\sin(\xi t)\big)\Big]$$

$$= e^{-\lambda t}\Big[\lambda\big(\cancel{\cos(\xi t)} + i\sin(\xi t)\big) - i\xi\big(\cos(\xi t) + \cancel{i\sin(\xi t)}\big) - \lambda\big(\cancel{\cos(\xi t)} - i\sin(\xi t)\big) - i\xi\big(\cos(\xi t) - \cancel{i\sin(\xi t)}\big)\Big]$$

$$= e^{-\lambda t}\big[2i\lambda\sin(\xi t) - 2i\xi\cos(\xi t)\big]$$

$$= -2ie^{-\lambda t}\big[\lambda\sin(\xi t) + \xi\cos(\xi t) - 2\lambda\sin(\xi t)\big] \qquad (6)$$

With (3)-(6) in the matrix (2) we have

$$e^{At} = \frac{e^{-\lambda t}}{\xi}\begin{bmatrix} \lambda\sin(\xi t) + \xi\cos(\xi t) & \sin(\xi t) \\ -\omega^2\sin(\xi t) & \lambda\sin(\xi t) + \xi\cos(\xi t) - 2\lambda\sin(\xi t) \end{bmatrix}$$

Hence,

$$\boxed{e^{At} = \frac{e^{-\lambda t}}{\xi}\Big[\big(\lambda\sin(\xi t) + \xi\cos(\xi t)\big)I + A\sin(\xi t)\Big]}$$

Now let $y_t = x_t'$, and define $\zeta = -\lambda + i\xi$,

$$g_t = \frac{1}{\xi}\operatorname{Im}(e^{\zeta t}) = \frac{1}{\xi}e^{-\lambda t}\sin(\xi t)$$

and

$$h_t = \frac{1}{\xi}\operatorname{Im}(\zeta e^{\zeta t}) = \frac{1}{\xi}e^{-\lambda t}\big(\xi\cos(\xi t) - \lambda\sin(\xi t)\big)$$

We'll the solution with the expression for e^{At} above. For this, let us compute

$$e^{A(t-s)} K X_s = \frac{e^{-\lambda(t-s)}}{\xi}\left[\left(\lambda \sin(\xi(t-s)) + \xi \cos(\xi(t-s))\right) I + A \sin(\xi(t-s))\right]$$

$$\cdot\, a_0 \eta \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_s \\ x_s' \end{bmatrix}$$

$$= \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} \lambda \sin(\xi(t-s)) + \xi \cos(\xi(t-s)) & \sin(\xi(t-s)) \\ -\omega^2 \sin(\xi(t-s)) & (\lambda - a_0)\sin(\xi(t-s)) + \xi \cos(\xi(t-s)) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -a_0\eta \end{bmatrix} \begin{bmatrix} x_s \\ x_s' \end{bmatrix}$$

$$= -a_0 \eta \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} 0 & \sin(\xi(t-s)) \\ 0 & \xi\cos(\xi(t-s)) - \lambda \sin(\xi(t-s)) \end{bmatrix} \begin{bmatrix} x_s \\ x_s' \end{bmatrix}$$

(7) $$= \begin{bmatrix} \left(-a_0 \eta e^{-\lambda(t-s)}/\xi\right) \sin(\xi(t-s))\, y_s \\ \left(-a_0 \eta e^{-\lambda(t-s)}/\xi\right)\left(\xi \cos(\xi(t-s)) - \lambda \sin(\xi(t-s))\right) y_s \end{bmatrix} = \begin{bmatrix} -a_0 \eta g_{t-s} y_s \\ -a_0 \eta h_{t-s} y_s \end{bmatrix}$$

and

$$e^{A(t-s)} m = \frac{e^{-\lambda(t-s)}}{\xi}\left[\left(\lambda \sin(\xi(t-s)) + \xi \cos(\xi(t-s))\right) I + A \sin(\xi(t-s))\right]$$

$$T_0 \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(8) $$= T_0 \eta \frac{e^{-\lambda(t-s)}}{\xi} \begin{bmatrix} \sin(\xi(t-s)) \\ \xi\cos(\xi(t-s)) - \lambda\sin(\xi(t-s)) \end{bmatrix} = \begin{bmatrix} \eta T_0 \, g_{t-s} \\ \eta T_0 \, h_{t-s} \end{bmatrix}$$

Finally, using the solution from the item (ii), it follows that

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \int_0^t \begin{bmatrix} -a_0 \eta \, g_{t-s} Y_s \\ -a_0 \eta \, h_{t-s} Y_s \end{bmatrix} dB_s + \int_0^t \begin{bmatrix} \eta T_0 \, g_{t-s} \\ \eta T_0 \, h_{t-s} \end{bmatrix} dB_s$$

In particular,

$$X_t = \eta \int_0^t (T_0 - a_0 Y_s) \, g_{t-s} \, dB_s$$

and

$$Y_t = \eta \int_0^t (T_0 - a_0 Y_s) \, h_{t-s} \, dB_s$$

5.14. If (B_1, B_2) denotes 2-dimensional Brownian motion we may introduce complex notation and put

$$\mathbf{B}(t) := B_1(t) + iB_2(t) \quad (i = \sqrt{-1})\,.$$

$\mathbf{B}(t)$ is called *complex Brownian motion*.

(i) If $F(z) = u(z) + iv(z)$ is an *analytic* function i.e. F satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,; \qquad z = x + iy$$

and we define

$$Z_t = F(\mathbf{B}(t))$$

prove that

$$dZ_t = F'(\mathbf{B}(t))d\mathbf{B}(t)\,, \tag{5.3.8}$$

where F' is the (complex) derivative of F. (Note that the usual second order terms in the (real) Itô formula are not present in (5.3.8)!)

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j$$

Note that

$$F(B(t)) = u(B(t)) + iv(B(t)) = u(B_1(t) + iB_2(t)) + iv(B_1(t) + iB_2(t))$$

Defining $g(t,z) = u(z) + iv(z)$, for $z = B(t)$ the Itô formula gives

$$dZ_t = dF(B(t)) = \frac{\partial u}{\partial B_1}dB_1 + i\frac{\partial u}{\partial B_2}dB_2 + \frac{1}{2}\cancel{\left(\frac{\partial^2 u}{\partial B_1^2}dt + \frac{\partial^2 u}{\partial B_2^2}dt\right)}_{0}$$

$$\boxed{dB = dB_1 + i\,dB_2}$$

$$+ i\frac{\partial v}{\partial B_1}dB_1 - \frac{\partial v}{\partial B_2}dB_2 + \frac{i}{2}\cancel{\left(\frac{\partial^2 v}{\partial B_1^2}dt + \frac{\partial^2 v}{\partial B_2^2}dt\right)}_{0}$$

$$= \left(\frac{\partial u}{\partial B_1} + i\frac{\partial v}{\partial B_1}\right)dB_1 + i\left(\frac{\partial u}{\partial B_2} + i\frac{\partial v}{\partial B_2}\right)dB_2$$

$$= \left(\frac{\partial u}{\partial B} + i\frac{\partial v}{\partial B}\right)dB = F'(B(t))dB(t)$$

(ii) Solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t d\mathbf{B}(t) \quad \alpha \text{ constant}) .$$

For more information about complex stochastic calculus involving analytic functions see e.g. Ubøe (1987).

Let $F(z) = e^{\alpha z}$. Clearly, $F(z)$ is analytic:

$$e^{\alpha z} = e^{\alpha(x+iy)} = e^{\alpha x} \cdot e^{i\alpha y} = \underbrace{e^{\alpha x}(\cos\alpha y}_{u} + \underbrace{i \sin\alpha y}_{v})$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \alpha e^{\alpha x} \cos\alpha y = \dfrac{\partial v}{\partial y} = \alpha e^{\alpha x} \cos\alpha y \\[2ex] \dfrac{\partial u}{\partial y} = -\alpha e^{\alpha x} \sin\alpha y = -\dfrac{\partial v}{\partial x} = \alpha e^{\alpha x} \sin\alpha y \end{cases}$$

If $Z_t = F(B(t))$, by Item (i),

$$dZ_t = F'(B(t))dB(t) = \alpha e^{\alpha B(t)} dB(t) = \alpha Z_t dB(t)$$

5.15. (Population growth in a stochastic, crowded environment)
The nonlinear stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t\,; \qquad X_0 = x > 0 \qquad (5.3.9)$$

is often used as a model for the growth of a population of size X_t in a stochastic, crowded environment. The constant $K > 0$ is called the *carrying capacity* of the environment, the constant $r \in \mathbf{R}$ is a measure of the quality of the environment and the constant $\beta \in \mathbf{R}$ is a measure of the size of the noise in the system.
Verify that

$$X_t = \frac{\exp\{(rK - \frac{1}{2}\beta^2)t + \beta B_t\}}{x^{-1} + r\int\limits_0^t \exp\{(rK - \frac{1}{2}\beta^2)s + \beta B_s\}ds}\,; \qquad t \geq 0 \qquad (5.3.10)$$

is the unique (strong) solution of (5.3.9). (This solution can be found by performing a substitution (change of variables) which reduces (5.3.9) to a linear equation. See Gard (1988), Chapter 4 for details.)

Step 1. Multiply by X_t^{-2}:

$$X_t^{-2} dX_t = (rX_t^{-1}K - r)dt + \beta X_t^{-1} dB_t$$

Step 2. Substitute $Y_t = X_t^{-1}$.

Notice that

$$dY_t = -\frac{dX_t}{X_t^2} + \frac{(dX_t)^2}{X_t^3}$$

Then

$$dY_t = (-rY_tK + r)dt - \beta Y_t dB_t + \beta^2 Y_t dt$$

$$= (-rK + \beta^2)Y_t dt - \beta Y_t dB_t + r dt$$

Step 3. Change of variables.

Let $Z_t := Y_t e^{(rK-\beta^2)t}$. By Itô's formula,

$$dZ_t = (rK - \beta^2)Y_t e^{(rK-\beta^2)t} dt + e^{(rK-\beta^2)t} dY_t$$

$$= \cancel{(rK-\beta^2)Y_t e^{(rK-\beta^2)t} dt} + e^{(rK-\beta^2)t}\left(\cancel{(-rK+\beta^2)Y_t dt} - \beta Y_t dB_t + r dt\right)$$

Thus,

$$dZ_t = r e^{(rK-\beta^2)t} dt - \beta Z_t dB_t$$

Step 4. Use integrating factor to solve the SDE.

Let N_t be such that $dN_t = \theta_t dt + \gamma_t dB_t$. By integration by parts (ex. 4.3),

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \ .$$

Thus,

$$d(N_tZ_t) = N_t dZ_t + Z_t dN_t + dN_t dZ_t$$

$$= N_t r e^{(rK-\beta^2)t} dt - \beta N_t Z_t dB_t + \theta_t Z_t dt + \gamma_t Z_t dB_t - \beta \gamma_t Z_t dt$$

Setting $\theta_t = \beta \gamma_t$ and $\gamma_t = \beta N_t$ we have

$$dN_t = \beta^2 N_t dt + \beta N_t dB_t$$

Then (see ex. 5.3 and example 5.1.2 of my notes),

$$N_t = N_0 \exp\left(\frac{1}{2}\beta^2 t + \beta B_t \right)$$

and

$$d(N_tZ_t) = N_t r e^{(rK-\beta^2)t} = N_0 r \exp\left(\beta B_t + \left(rK - \frac{1}{2}\beta^2 \right) t \right)$$

Choosing $N_0 = 1$,

$$N_t Z_t = Z_0 + \int_0^t r \exp\left(\beta B_s + \left(rK - \frac{1}{2}\beta^2\right)s\right) ds$$

Thus, using that $Z_t := Y_t e^{(rK-\beta^2)t}$ and $Y_t = X_t^{-1}$, $Z_0 = X_0^{-1} = x^{-1}$,

$$X_t = Y_t^{-1} = Z_t^{-1} e^{(rK-\beta^2)t}$$

$$= \frac{e^{(rK-\beta^2)t} N_t}{Z_0 + \int_0^t r \exp\left(\beta B_s + \left(rK - \frac{1}{2}\beta^2\right)s\right) ds}$$

$$= \frac{\exp\left[(rK - 1/2\beta^2)t + \beta B_t\right]}{x^{-1} + r\int_0^t \exp\left[(rK - 1/2\beta^2)s + \beta B_s\right] ds}$$

$\blacksquare$

5.16. The technique used in Exercise 5.6 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t\ , \qquad X_0 = x \tag{5.3.11}$$

where $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $c: \mathbf{R} \to \mathbf{R}$ are given continuous (deterministic) functions. Proceed as follows:

a) Define the 'integrating factor'

$$F_t = F_t(\omega) = \exp\Big(- \int_0^t c(s)dB_s + \tfrac{1}{2}\int_0^t c^2(s)ds \Big)\ . \tag{5.3.12}$$

Show that (5.3.11) can be written

$$d(F_t X_t) = F_t \cdot f(t, X_t)dt\ . \tag{5.3.13}$$

We start by applying Itô's formula to F_t.

If $g(t, X_t) = \exp\left(- \int_0^t c(s)dX_t + \frac{1}{2}\int_0^t c^2(s)ds \right)$, $F_t = g(t, B_t)$,

- $\frac{\partial g}{\partial t} = \frac{1}{2}c^2(t) \cdot g(t, X_t)$
- $\frac{\partial g}{\partial x} = -c(t) \cdot g(t, X_t)$
- $\frac{\partial^2 g}{\partial x^2} = c^2(t)\, g(t, X_t)$

Thus,

$$dF_t = \frac{1}{2}c^2(t) \cdot g(t, B_t)\,dt - c(t) \cdot g(t, B_t)\,dB_t + \frac{1}{2}c^2(t) \cdot g(t, B_t)\,dt$$

$$= c^2(t)\, g(t, B_t)\,dt - c(t) \cdot g(t, B_t)\,dB_t \qquad (1)$$

Using Integration by Parts (ex. 4.3),

$$d(F_t X_t) = F_t dX_t + X_t dF_t + dF_t dX_t \qquad (2)$$

Computing $dF_t dX_t$,

$$dF_t\, dX_t = \left(c^2(t)\, g(t, B_t)\, dt - c(t)\cdot g(t, B_t)\, dB_t\right)\left(f(t, X_t)\, dt + c(t) X_t\, dB_t\right)$$

$$= -c^2(t)\, g(t, B_t) X_t\, dt \qquad (3)$$

By (2),

$$d(F_t X_t) = g(t, B_t)\left(f(t, X_t)\, dt + \cancel{c(t) X_t\, dB_t}\right)$$

$$+ X_t\left(\cancel{c^2(t)\, g(t, B_t)\, dt} - \cancel{c(t)\cdot g(t, B_t)\, dB_t}\right)$$

$$- \cancel{c^2(t)\, g(t, B_t) X_t\, dt}$$

$$= F_t f(t, X_t)\, dt \qquad (4)$$

b) Now define

$$Y_t(\omega) = F_t(\omega) X_t(\omega) \qquad (5.3.14)$$

so that

$$X_t = F_t^{-1} Y_t\ . \qquad (5.3.15)$$

Deduce that equation (5.3.13) gets the form

$$\frac{dY_t(\omega)}{dt} = F_t(\omega)\cdot f(t, F_t^{-1}(\omega) Y_t(\omega))\ ; \qquad Y_0 = x\ . \qquad (5.3.16)$$

Note that this is just a *deterministic* differential equation in the function $t \to Y_t(\omega)$, for each $\omega \in \Omega$. *We can therefore solve (5.3.16) with ω as a parameter to find $Y_t(\omega)$ and then obtain $X_t(\omega)$ from (5.3.15).*

Notice that

$$dY_t = d(F_t X_t) = F_t f(t, X_t)\, dt$$

Thus,

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t) \qquad (5)$$

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t} dt + \alpha X_t dB_t \; ; \qquad X_0 = x > 0 \qquad (5.3.17)$$

where α is constant.

Define

$$F_t = \exp\left(- \int_0^t \alpha \, dB_s + \frac{1}{2} \int_0^t \alpha^2 ds \right) = \exp\left(- \alpha B_t + \frac{1}{2} \alpha^2 t \right)$$

By (5), if $Y_t = F_t X_t$,

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t) = F_t \cdot \frac{1}{F_t^{-1} Y_t} = \left[\exp\left(- \alpha B_t + \frac{1}{2} \alpha^2 t \right) \right]^2 Y_t^{-1}$$

$$= \frac{1}{Y_t} \exp\left(\alpha^2 t - 2\alpha B_t \right)$$

Solving by separable equation, we can write $Y_t dY_t - F_t^2 dt = 0$.

Let

$$G(Y_t) = \int Y_t \, dY_t = \frac{Y_t^2}{2} + c$$

$$F(t) = \int F_t^2 dt = \int e^{-2\alpha B_t} \cdot e^{\alpha^2 t} dt = \frac{e^{\alpha^2 t - 2\alpha B_t}}{\alpha^2} + c$$

Thus,

$$G(Y_t) - F(x) = \frac{Y_t^2}{2} - \int e^{\alpha^2 s - 2\alpha B_s} ds = C$$

and

$$Y_t^2 = 2C + 2 \int_0^t e^{\alpha^2 s - 2\alpha B_s} ds$$

i.e.,

$$Y_t = \sqrt{Y_0^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha B_s}\, ds}$$

Since $X_t = F_t^{-1} Y_t$,

$$X_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t} \sqrt{x^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha B_s}\, ds}$$

d) Apply the method to study the solutions of the stochastic differential equation

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t \,; \qquad X_0 = x > 0 \qquad (5.3.18)$$

where α and γ are constants.
For what values of γ do we get explosion?

We'll use F_t as defined in the previous item.
By (5), if $Y_t = F_t X_t$,

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t) = F_t (F_t^{-1} Y_t)^\gamma = F_t^{1-\gamma} Y_t^\gamma$$

Solving by separable equation, we can write $Y_t^{-\gamma} dY_t - F_t^{1-\gamma} dt = 0$.
Let

$$G(Y_t) = \int Y_t^{-\gamma} dY_t = \frac{Y_t^{-\gamma+1}}{-\gamma+1} + c$$

$$F(t) = \int \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right)^{1-\gamma} dt$$

$$= \int \exp\left(-(1-\gamma)\alpha B_t + (1-\gamma)\frac{\alpha^2}{2} t\right) dt$$

Therefore,

$$G(Y_t) - F(t) = \frac{Y_t^{-\gamma+1}}{-\gamma+1} - \int \exp\left(-(1-\gamma)\alpha B_t + (1-\gamma)\frac{\alpha^2}{2}t\right) dt = C$$

and,

$$Y_t^{1-\gamma} = (-\gamma+1)C + (1-\gamma)\int_0^t \exp\left(-(1-\gamma)\alpha B_s + (1-\gamma)\frac{\alpha^2}{2}s\right) ds$$

i.e.,

$$Y_t = \left(Y_0^{1-\gamma} + (1-\gamma)\int_0^t \exp\left(-(1-\gamma)\alpha B_s + (1-\gamma)\frac{\alpha^2}{2}s\right) ds\right)^{\frac{1}{1-\gamma}}$$

Since $X_t = F_t^{-1} Y_t$,

$$X_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}\left(x^{1-\gamma} + (1-\gamma)\int_0^t \exp\left(-(1-\gamma)\alpha B_s + (1-\gamma)\frac{\alpha^2}{2}s\right) ds\right)^{\frac{1}{1-\gamma}}$$

5.17. (The Gronwall inequality)
Let $v(t)$ be a nonnegative function such that

$$v(t) \le C + A \int_0^t v(s)ds \qquad \text{for } 0 \le t \le T$$

for some constants C, A. Prove that

$$v(t) \le C\exp(At) \qquad \text{for } 0 \le t \le T\,. \tag{5.3.19}$$

Define

$$g(t) = A\exp\left(-\int_0^t A\,ds\right)\int_0^t v(s)ds = A\exp(-At)\int_0^t v(s)ds$$

and compute

$$g'(t) = A\exp(-At)\left[-A\int_0^t v(s)ds + v(t)\right]$$

$$\le A\exp(-At)\left[-A\int_0^t v(s)ds + C + A\int_0^t v(s)ds\right]$$

$$= AC\exp(-At)$$

Integrating,

$$g(t) \le AC\int_0^t \exp(-As)\,ds = AC\left[\frac{1}{A}\left(1-e^{-At}\right)\right] = C\left(1-e^{-At}\right)$$

Thus,

$$A\int_0^t v(s)ds = \exp(At)\,g(t) \le e^{At}\,C\left(1-e^{-At}\right)$$

Finally,

$$v(t) \le C + e^{At}\,C\left(1-e^{-At}\right) = C\exp(At)$$

■

7.1. Find the generator of the following Itô diffusions:

a) $dX_t = \mu X_t dt + \sigma dB_t$ (The Ornstein-Uhlenbeck process) ($B_t \in \mathbf{R}$; μ, σ constants).

We'll use the formula

$$A f(x) = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^{\mathrm{T}})_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

If $X_t = f(B_t)$,

$$A f(x) = \mu x f'(x) + \frac{1}{2} \sigma^2 f''(x)$$

b) $dX_t = rX_t dt + \alpha X_t dB_t$ (The geometric Brownian motion) ($B_t \in \mathbf{R}$; r, α constants).

$$A f(x) = r x f'(x) + \frac{1}{2} (\alpha x)^2 f''(x)$$

c) $dY_t = r\,dt + \alpha Y_t dB_t$ ($B_t \in \mathbf{R}$; r, α constants)

$$A f(x) = r f'(x) + \frac{1}{2} (\alpha x)^2 f''(x)$$

d) $dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix}$ where X_t is as in a)

$$A f(x) = \frac{\partial f}{\partial x_1} + \mu x \frac{\partial f}{\partial x_2} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x_2^2}$$

e) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t \quad (B_t \in \mathbf{R})$

$$A f(x) = \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}$$

f) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$

$$A f(x) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2} + x_1^2 \frac{\partial^2 f}{\partial x_2^2} \right)$$

g) $X(t) = (X_1, X_2, \cdots, X_n)$, where

$$dX_k(t) = r_k X_k dt + X_k \cdot \sum_{j=1}^{n} \alpha_{kj} dB_j \; ; \qquad 1 \le k \le n$$

($(B_1, \cdots, B_n)$ is Brownian motion in $\mathbf{R}^n$, r_k and α_{kj} are constants).

$$A f(x) = \sum_{i=1}^{n} r_i x_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j \left(\sum_{k=1}^{n} \alpha_{ik} \alpha_{jk} \right) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

7.2. Find an Itô diffusion (i.e. write down the stochastic differential equation for it) whose generator is the following:

a) $Af(x) = f'(x) + f''(x)\,;\ \ f \in C_0^2(\mathbf{R})$

b) $Af(t,x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}\,;\ \ f \in C_0^2(\mathbf{R}^2)$,
where c, α are constants.

c) $Af(x_1,x_2) = 2x_2\frac{\partial f}{\partial x_1} + \ln(1+x_1^2+x_2^2)\frac{\partial f}{\partial x_2}$
$\quad + \frac{1}{2}(1+x_1^2)\frac{\partial^2 f}{\partial x_1^2} + x_1\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x_2^2}\,;\ \ f \in C_0^2(\mathbf{R}^2)$.

a) Let $f(B_t) = X_t$. Then

$$dX_t = dt + \sqrt{2}\, dB_t$$

b) $$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ cX_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \alpha X_2(t) \end{bmatrix} dB_t$$

c) Start by noticing that

$$\sigma\sigma^T = \begin{bmatrix} 1+X_1^2(t) & X_1(t) \\ X_1(t) & 1 \end{bmatrix}$$

To find σ, we compute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \begin{cases} a^2 + b^2 = 1 + X_1^2(t) \\ ac + bd = X_1(t) \\ c^2 + d^2 = 1 \end{cases}$$

Take $a = X_1(t)$, $b = c = 1$ and $d = 0$, i.e., $\sigma = \begin{bmatrix} X_1(t) & 1 \\ 1 & 0 \end{bmatrix}$

Then, an Itô diffusion for the generator is

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 2X_2(t) \\ \ln(1+X_1^2(t)+X_2^2(t)) \end{bmatrix} dt + \begin{bmatrix} X_1(t) & 1 \\ 1 & 0 \end{bmatrix} dB_t$$

7.3. Let B_t be Brownian motion on $\mathbf{R}$, $B_0 = 0$ and define

$$X_t = X_t^x = x \cdot e^{ct + \alpha B_t} ,$$

where c, α are constants. Prove directly from the definition that X_t is a Markov process.

Denote the σ-algebra generated by the B.M. as $\mathcal{F}_t$.

Start by noticing that

$$\mathbb{E}^x[X_{t+h} \mid \mathcal{F}_t] = \mathbb{E}^x\left[x e^{c(t+h) + \alpha B_{t+h}} \mid \mathcal{F}_t\right]$$

$$= \mathbb{E}^{B_t}\left[x e^{c(t+h) + \alpha B_h}\right]$$

Therefore, we can write

$$\mathbb{E}^x[X_{t+h} \mid X_t] = \mathbb{E}^x\left[\mathbb{E}^x\left[x e^{c(t+h) + \alpha B_{t+h}} \mid \mathcal{F}_t\right] \mid X_t\right]$$

$$= \mathbb{E}^x\left[\mathbb{E}^{B_t}\left[x e^{c(t+h) + \alpha B_h}\right] \mid X_t\right]$$

$$= \mathbb{E}^{B_t}\left[x e^{c(t+h) + \alpha B_h}\right]$$

$$= \mathbb{E}^x[X_{t+h} \mid \mathcal{F}_t]$$

7.4. Let B_t^x be 1-dimensional Brownian motion starting at $x \in \mathbf{R}^+$. Put

$$\tau = \inf\{t > 0;\ B_t^x = 0\}\ .$$

a) Prove that $\tau < \infty$ a.s. P^x for all $x > 0$. (Hint: See Example 7.4.2, second part).
b) Prove that $E^x[\tau] = \infty$ for all $x > 0$. (Hint: See Example 7.4.2, first part).

a) Let $K > x$ and define the stopping times

$$\tau_0 = \tau \quad \text{and} \quad \tau_K = \inf\{t > 0 : B_t = K\},$$

Also let $T_K = \tau_0 \wedge \tau_K$.

Since the Brownian motion has continuous paths,

$$\lim_{K \to \infty} \tau_K = +\infty$$

and note that

$$\mathbb{E}^x[B_{T_K}] = \mathbb{E}^x[B_0] = x$$

Then,

$$x = \mathbb{E}^x[B_{T_K}] = 0 \cdot \mathbb{P}^x[\tau_0 < \tau_K] + K\,\mathbb{P}^x[\tau_0 \geq \tau_K]$$

Rewriting

$$\mathbb{P}^x[\tau_0 < \tau_K] = 1 - \mathbb{P}^x[\tau_0 \geq \tau_K] = 1 - \frac{x}{K}$$

I.e,

$$\mathbb{P}^x[\tau_0 < \infty] = \lim_{K \to \infty} \mathbb{P}^x[\tau_0 < \tau_K] = 1$$

b) Let $f(x)=x^2$. By Itô's formula,

$$B_t^2 = 0 + 2\int_0^t B_s\, dB_s + \int_0^t ds$$

Thus,

$$\mathbb{E}[B^2_{t\wedge \tau_K}] = \mathbb{E}[t \wedge \tau_K]$$

Now notice that

$$\mathbb{E}[\tau_0] = \lim_{K\to\infty} \mathbb{E}[\tau_K]$$
$$= \lim_{K\to\infty} \mathbb{E}[B^2_{\tau_K}]$$
$$= \lim_{K\to\infty} b^2 \mathbb{P}[\tau_0 \geq \tau_K]$$
$$= \lim_{K\to\infty} K^2 \frac{x}{K} = \infty$$

7.5. Let the functions b, σ satisfy condition (5.2.1) of Theorem 5.2.1, with a constant C independent of t, i.e.

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|) \qquad \text{for all } x \in \mathbf{R}^n \text{ and all } t \geq 0\,.$$

Let X_t be a solution of

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t\,.$$

Show that

$$E[|X_t|^2] \leq (1+E[|X_0|^2])e^{Kt} - 1$$

for some constant K independent of t.
(Hint: Use Dynkin's formula with $f(x) = |x|^2$ and $\tau = t \wedge \tau_R$, where $\tau_R = \inf\{t > 0;\, |X_t| \geq R\}$, and let $R \to \infty$ to achieve the inequality

$$E[|X_t|^2] \leq E[|X_0|^2] + K \cdot \int_0^t (1 + E[|X_s|^2])ds\,,$$

which is of the form (5.2.9).)

Let $\tau_R = \inf\{t > 0 : |X_t| \geq R\}$ and $\tau = \min\{t, \tau_R\}$.

By Dynkin's formula applied to $f(x) = |x|^2$,

$$\mathbb{E}[|X_\tau|^2] = \mathbb{E}[|X_0|^2] + \mathbb{E}\left[\int_0^\tau A|X_s|^2 ds\right] \qquad (1)$$

Using the following fact

$$A|X_s|^2 \leq K(1+|X_s|^2) \qquad (*)$$

then

$$\mathbb{E}\left[\int_0^\tau A|X_s|^2 ds\right] \leq \mathbb{E}\left[\int_0^\tau K(1+|X_s|^2)\, ds\right]$$

$$= K\,\mathbb{E}\left[\int_0^\tau (1+|X_s|^2)\, ds\right]$$

$$= K\int_0^\tau (1+\mathbb{E}[|X_s|^2])\, ds \qquad (3)$$

By (1) and (3),

$$\mathbb{E}\left[|X_\tau|^2\right] \leqslant \mathbb{E}\left[|X_0|^2\right] + K\int_0^\tau \left(1+\mathbb{E}[|X_s|^2]\right) ds$$

To use the Gronwall inequality, write

$$1+\mathbb{E}\left[|X_\tau|^2\right] \leqslant 1+\mathbb{E}\left[|X_0|^2\right] + K\int_0^\tau \left(1+\mathbb{E}[|X_s|^2]\right) ds$$

Thus,

$$\mathbb{E}\left[|X_\tau|^2\right] \leqslant \left(1+\mathbb{E}\left[|X_0|^2\right]\right) e^{K\tau} - 1$$

$\square$

Proof of (*):

Start by writing

$$Af(x) = 2\sum_{i=1}^{n} b_i(t,x)\,x_i + \sum_{i=1}^{n} \sigma_i^2(t,x)$$

Since

$$2\sum_{i=1}^{n} b_i(t,x)\,x_i \leq \sum_{i=1}^{n} |b_i|^2 + \sum_{i=1}^{n} |x_i|^2 = |b|^2 + |x|^2$$

we obtain

$$Af(x) \leq |b|^2 + |x|^2 + |\sigma|^2$$

By the conditions (5.2.1.),

$$Af(x) \leq C^2(1+|x|)^2 + |x|^2 \quad = C^2(1+2|x|+|x|^2) + |x|^2$$

$$= C^2 + 2C^2|x| + (C^2+1)|x|^2$$

Since $(|x|+1)^2 = |x|^2 + 1 + 2|x|$, we have $2C^2|x| \leq C^2|x|^2 + C^2$.
Thus,

$$Af(x) \leq 2C^2 + (2C^2+1)|x|^2$$

$$\leq K(1+|x|^2)$$

by taking $K > (2C^2+1)$.

7.6. Let $g(x,\omega) = f \circ F(x,t,t+h,\omega)$ be as in the proof of Theorem 7.1.2. Assume that f is continuous.

a) Prove that the map $x \to g(x,\cdot)$ is continuous from $\mathbf{R}^n$ into $L^2(P)$ by using (5.2.9).

Recall that

$$F(x,t,r,\omega) = X_r^{t,x}(\omega) \qquad \text{for } r \geq t\ ,$$

We want to show that

$$\lim_{h \to 0} \mathbb{E}[|g(x+h,\cdot) - g(x,\cdot)|^2] = 0$$

Since f is continuous, our task is simplified. What needs to be proved is that

$$\lim_{s \to 0} \mathbb{E}[|F(x+s,t,t+h,\omega) - F(x,t,t+h,\omega)|^2] = 0$$

Now remember that

So the function

$$v(t) = E[|X_t - \widehat{X}_t|^2]\ ; \qquad 0 \leq t \leq T$$

satisfies

$$v(t) \leq F + A\int_0^t v(s)ds\ , \tag{5.2.9}$$

where $F = 3E[|Z - \widehat{Z}|^2]$ and $A = 3(1+T)D^2$.

Let

$$\sigma(t) = \mathbb{E}[|F(x+s,t,t+h,\omega) - F(x,t,t+h,\omega)|^2]$$

By Gronwall inequality,

$$\sigma(t) \leq F\exp(At)$$

Notice that

$$Z - \hat{Z} = (x+s) - x = s$$

thus

$$\lim_{s\to 0} F = \lim_{s \to 0} 3s^2 = 0$$

Since $\exp(At) < \infty$, it follows that $\lim_{s\to 0} \sigma(t) = 0$, i.e.,

$$\lim_{s\to 0} \mathbb{E}\left[|F(x+s, t, t+h, \omega) - F(x, t, t+h, \omega)|^2 \right] = 0$$

$\square$

For simplicity assume that $n = 1$ in the following.

b) Use a) to prove that $(x, \omega) \to g(x, \omega)$ is measurable. (Hint: For each $m = 1, 2, \ldots$ put $\xi_k = \xi_k^{(m)} = k \cdot 2^{-m}$, $k = 1, 2, \ldots$ Then

$$g^{(m)}(x, \cdot) := \sum_k g(\xi_k, \cdot) \cdot \mathcal{X}_{\{\xi_k \le x < \xi_{k+1}\}}$$

converges to $g(x, \cdot)$ in $L^2(P)$ for each x. Deduce that $g^{(m)} \to g$ in $L^2(dm_R \times dP)$ for all R, where dm_R is Lebesgue measure on $\{|x| \le R\}$. So a subsequence of $g^{(m)}(x, \omega)$ converges to $g(x, \omega)$ for a.a. (x, ω).)

By the hint and the previous item, $g^m(x, \cdot) \to g(x, \cdot)$ in $L^2(P)$.

Since the function is bounded and composition of continuous and measurable functions is measurable, the result follows.

$\square$

7.7. Let B_t be Brownian motion on $\mathbf{R}^n$ starting at $x \in \mathbf{R}^n$ and let $D \subset \mathbf{R}^n$ be an open ball centered at x.

a) Use Exercise 2.15 to prove that the harmonic measure μ_D^x of B_t is rotation invariant (about x) on the sphere ∂D. Conclude that μ_D^x coincides with normalized surface measure σ on ∂D.

b) Let ϕ be a bounded measurable function on a bounded open set $W \subset \mathbf{R}^n$ and define

$$u(x) = E^x[\phi(B_{\tau_W})] \qquad \text{for } x \in W .$$

Prove that u satisfies the classical mean value property:

$$u(x) = \int_{\partial D} u(y) d\sigma(y)$$

for all balls D centered at x with $\overline{D} \subset W$.

a) First we want to show that

Definition 6.2.6 (Harmonic Measure). The **harmonic measure** of X on ∂G, denoted by μ_G^x, is defined as

$$\mu_G^x(F) = Q^x[X_{\tau_G} \in F], \quad \text{for } F \subset \partial G,\ x \in G$$

$$\mu_D^x(UF) = \mu_D^x(F)$$

where U is a rotation (i.e., orthogonal matrix on $\mathbb{R}^{n\times n}$).

By the exercise 2.15,

2.15. Let B_t be n-dimensional Brownian motion starting at 0 and let $U \in \mathbf{R}^{n\times n}$ be a (constant) orthogonal matrix, i.e. $UU^T = I$. Prove that

$$\widetilde{B}_t := UB_t$$

is also a Brownian motion.

it follows that

$$\begin{aligned} Q^x[B_{\tau_D} \in F] &= Q^x[UB_{\tau_D} \in UF] \\ &= Q^x[\tilde{B}_{\tau_D} \in UF] \\ &= Q^x[B_{\tau_D} \in UF] \end{aligned}$$

Hence, $\mu_D^x(UF) = \mu_D^x(F)$.

To show that $\mu_D^x = \sigma$ on ∂D, notice that

$$\int_{\partial D} f(x)\, d\sigma(x) = \int_{SO(n)} \int_{\partial D} f(gx)\, d\sigma(x)\, d\upsilon(g)$$

where υ is left-inv. measure on $SO(n)$

$$= \int_{\partial D} \int_{SO(n)} f(gx)\, d\upsilon(g)\, d\sigma(x)$$

by Fubini's thm.

$$= \int_{\partial D} \int_{\partial D} f(y)\, d\mu_D^x(y)\, d\sigma(x)$$

since μ_D^x is rot. invariant

$$= \int_{\partial D} f(y)\, d\mu_D^x(y)$$

since σ is a normalized surf. meas.

Since f is arbitrary, $\sigma = \mu_D^x$ on ∂D as desired.

b) We know that

$$\mathbb{E}^x[f(X_{\tau_H})] = \mathbb{E}^x[\mathbb{E}^{X_{\tau_G}}[f(X_{\tau_H})]] = \int_{\partial G} \mathbb{E}^y[f(X_{\tau_H})] Q^x[X_{\tau_G} \in dy]$$

for $x \in G$.

Plus the fact that $\sigma = \mu_D^x$ on ∂D, it follows that

$$u(x) = \mathbb{E}^x[\varphi(B_{\tau_W})] = \int_{\partial D} \mathbb{E}^y[\varphi(B_{\tau_W})]\, \mu_D^x(dy)$$

$$= \int_{\partial D} u(y)\, d\mu_D^x(y) = \int_{\partial D} u(y)\, d\sigma(y)$$

$\square$

7.8. Let $\{\mathcal{N}_t\}$ be a right-continuous family of σ-algebras of subsets of Ω, containing all sets of measure zero.

a) Let τ_1, τ_2 be stopping times (w.r.t. $\mathcal{N}_t$). Prove that $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ are stopping times.

- $\{\tau_1 \wedge \tau_2 \leq t\} = \{\omega : \tau_1 \wedge \tau_2(\omega) \leq t\}$

$$= \{\omega : \tau_1(\omega) \leq t \text{ or } \tau_2(\omega) \leq t\}$$

$$= \{\omega : \tau_1(\omega) \leq t\} \cup \{\omega : \tau_2(\omega) \leq t\}$$

$$= \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{N}_t$$

- $\{\tau_1 \vee \tau_2 \leq t\} = \{\omega : \tau_1(\omega) \leq t \text{ and } \tau_2(\omega) \leq t\}$

$$= \{\omega : \tau_1(\omega) \leq t\} \cap \{\omega : \tau_2(\omega) \leq t\}$$

$$= \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{N}_t$$

b) If $\{\tau_n\}$ is a decreasing family of stopping times prove that $\tau := \lim_n \tau_n$ is a stopping time.

We need to show that $\{\tau \leq t\} \in \mathcal{N}_t$. In fact, since $\{\tau_n\}$ is decreasing,

$$\{\tau \leq t\} = \bigcap_n \{\tau_n \leq t\} = \bigcap_n \{\tau_n > t\}^c = \left(\bigcup_n \{\tau_n > t\}\right)^c \in \mathcal{N}_t$$

Since $\{\tau_n \leq t\} \in \mathcal{N}_t$ implies that $\{\tau_n \leq t\}^c = \{\tau_n > t\} \in \mathcal{N}_t$.

c) If X_t is an Itô diffusion in $\mathbf{R}^n$ and $F \subset \mathbf{R}^n$ is closed, prove that τ_F is a stopping time w.r.t. $\mathcal{M}_t$. (Hint: Consider open sets decreasing to F).

Let $\{F_n\}$ be a family of open sets decreasing to F. Now we define $\{\tau_{F_n}\}$ as the family of stopping times given by

$$\{\tau_{F_n}\} = \inf\{t > 0 : X_t \notin F_n\}$$

Since each F_n is open, by the Example 7.2.2, each τ_{F_n} is a stopping time w.r.t. $\mathcal{M}_t$. By the previous item, its limit τ_F is a stopping time w.r.t. $\mathcal{M}_t$.

7.9. Let X_t be a geometric Brownian motion, i.e.

$$dX_t = rX_t dt + \alpha X_t dB_t\ , \qquad X_0 = x > 0$$

where $B_t \in \mathbf{R}$; r, α are constants.

a) Find the generator A of X_t and compute $Af(x)$ when $f(x) = x^\gamma$; $x > 0$, γ constant.

The generator is given by

$$A = rx\frac{\partial}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2}{\partial x^2}$$

Thus,

$$Af(x) = rx\gamma x^{\gamma-1} + \frac{1}{2}\alpha^2 x^2 \gamma(\gamma-1)x^{\gamma-2}$$

$$= \left(r\gamma + \frac{1}{2}\alpha^2\gamma(\gamma-1)\right)x^\gamma$$

b) If $r < \frac{1}{2}\alpha^2$ then $X_t \to 0$ as $t \to \infty$, a.s. Q^x (Example 5.1.1). But what is the probability p that X_t, when starting from $x < R$, ever hits the value R ? Use Dynkin's formula with $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - \frac{2r}{\alpha^2}$, to prove that

$$p = \left(\frac{x}{R}\right)^{\gamma_1}.$$

First step: define the stopping time.

Take $0 < K < x < R$ and define

- $\tau_R = \inf\{t > 0 : X_t \neq R\}$

- $\tau = \min\{K, \tau_R\}$

Second step: apply Dynkin's formula.

Notice that $f \in C_0^2$ and τ is a stopping time with finite expected value. Then,

$$\mathbb{E}^x\left[f(X_\tau)\right] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)\,ds\right]$$

$$= x^{\gamma_1} + \mathbb{E}^x\left[\int_0^\tau \left(r\gamma_1 + \frac{1}{2}\alpha^2\gamma_1(\gamma_1 - 1)\right) X_s^{\gamma_1}\,ds\right]$$

$$= x^{\gamma_1} + \mathbb{E}^x\left[\int_0^\tau \underbrace{\left(r + \frac{1}{\cancel{2}}\cancel{\alpha^2}\left(-\frac{\cancel{2}r}{\cancel{\alpha^2}}\right)\right)}_{=0}\left(1 - \frac{2r}{\alpha^2}\right) X_s^{1-2r/\alpha^2}\,ds\right]$$

Hence,

$$\mathbb{E}^x\left[f(X_\tau)\right] = f(x)$$

Third step: compute the expected value

Since $\lim \tau < \infty$, by taking $K \to \infty$,

$$\mathbb{E}^x\left[f(X_{\tau_R})\right] = f(x)$$

And notice that $X_\tau \in \{K, R\}$. Define

$$p = \mathbb{P}\left[X_\tau = R\right] \quad \text{and} \quad 1 - p = \mathbb{P}\left[X_\tau = K\right]$$

Then

$$E^x[f(X_\tau)] = f(x) = p f(R) + (1-p) f(K)$$

$$\iff \quad x^{1-2r/\alpha^2} = pR^{1-2r/\alpha^2} + (1-p)K^{1-2r/\alpha^2} \qquad (*)$$

Taking the limit and noticing that $\gamma_1 > 0$,

$$\lim_{K \downarrow 0} E^x[f(X_\tau)] = pR^{1-2r/\alpha^2} = x^{1-2r/\alpha^2}$$

and thus

$$p = \left(\frac{x}{R}\right)^{\gamma_1}$$

c) If $r > \frac{1}{2}\alpha^2$ then $X_t \to \infty$ as $t \to \infty$, a.s. Q^x. Put

$$\tau = \inf\{t > 0; X_t \geq R\} .$$

Use Dynkin's formula with $f(x) = \ln x$, $x > 0$ to prove that

$$E^x[\tau] = \frac{\ln \frac{R}{x}}{r - \frac{1}{2}\alpha^2} .$$

(Hint: First consider exit times from (ρ, R), $\rho > 0$ and then let $\rho \to 0$. You need estimates for

$$(1 - p(\rho)) \ln \rho ,$$

where

$$p(\rho) = Q^x[X_t \text{ reaches the value } R \text{ before } \rho] ,$$

which you can get from the calculations in a), b).)

By Dynkin's formula for $f(x) = \ln x$

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)\,ds\right]$$

Given that

$$\mathcal{A} f(x) = r - \frac{1}{2}\alpha^2$$

we have

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau \left(r - \frac{1}{2}\alpha^2\right) ds\right]$$

$$= \ln x + \left(r - \frac{1}{2}\alpha^2\right)\mathbb{E}^x[\tau]$$

Then

$$\mathbb{E}^x[\tau] = \frac{\mathbb{E}^x[f(X_\tau)] - \ln x}{r - \alpha^2/2}$$

and using the given $p(\rho)$,

$$\mathbb{E}^x[\tau] = \frac{p(\rho) f(R) + (1-p(\rho)) f(\rho) - \ln x}{r - \alpha^2/2} \qquad (**)$$

Taking $(*)$ with $K = \rho$,

$$x^{\gamma_1} = pR^{\gamma_1} + (1-p)\rho^{\gamma_1} \iff p = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}} = \frac{\frac{1}{x^{-\gamma_1}} - \frac{1}{\rho^{-\gamma_1}}}{\frac{1}{R^{-\gamma_1}} - \frac{1}{\rho^{-\gamma_1}}} = \frac{\frac{\rho^{-\gamma_1} - x^{-\gamma_1}}{x^{-\gamma_1}\rho^{-\gamma_1}}}{\frac{\rho^{-\gamma_1} - R^{-\gamma_1}}{R^{-\gamma_1}\rho^{-\gamma_1}}}$$

Thus,

$$p = \frac{(\rho^{-\gamma_1} - x^{-\gamma_1})}{x^{-\gamma_1}\rho^{-\gamma_1}} \cdot \frac{R^{-\gamma_1}\rho^{-\gamma_1}}{(\rho^{-\gamma_1} - R^{-\gamma_1})} = \frac{R^{-\gamma_1}(\rho^{-\gamma_1} - x^{-\gamma_1})}{x^{-\gamma_1}(\rho^{-\gamma_1} - R^{-\gamma_1})}$$

Since $r > \alpha^2/2$, $\gamma_1 < 0$,

$$\lim_{\rho \to 0} p = 1 \qquad \text{and} \qquad \lim_{\rho \to \infty} \left[(1 - p(\rho)) \ln \rho \right] = 0$$

Plugging into (2):

$$\mathbb{E}^x[\tau] = \frac{\ln R - \ln x}{r - \alpha^2/2} = \frac{\ln \frac{R}{x}}{r - \alpha^2/2}$$

$\square$

7.10. Let X_t be the geometric Brownian motion

$$dX_t = rX_t dt + \alpha X_t dB_t \ .$$

Find $E^x[X_T|\mathcal{F}_t]$ for $t \leq T$ by
a) using the Markov property

By the Markov property,

$$\mathbb{E}^x[X_T | \mathcal{F}_t] = \mathbb{E}^{X_t}[X_{T-t}]$$

Given that X_t is of the form

$$X_t = x + \int_0^t rX_s ds + \int_0^t \alpha X_s dB_s$$

applying the expectation operator gives

$$\mathbb{E}^x[X_t] = x + r\int_0^t \mathbb{E}^x[X_s] ds$$

Note that this function $\mathbb{E}^x[X_t]$ evaluated at zero must return x. This inspires us to build the following initial value problem.

Let

$$f(t) = x + r\int_0^t \mathbb{E}^x[X_s] ds = x + r\int_0^t f(s) ds$$

then our problem is

$$f'(t) = rf(t), \quad \text{with } f(0) = x$$

with solution

$$\mathbb{E}^x[X_t] = f(t) = x e^{rt}$$

Thus,

$$\mathbb{E}^{X_t}[X_{T-t}] = X_t e^{r(T-t)}$$

and

b) writing $X_t = x\, e^{rt} M_t$, where

$$M_t = \exp(\alpha B_t - \tfrac{1}{2}\alpha^2 t) \qquad \text{is a martingale .}$$

Notice that

$$\mathbb{E}^x[X_T | \mathcal{F}_t] = \mathbb{E}^x[x e^{rT} M_T | \mathcal{F}_t]$$

$$= x e^{rT} \mathbb{E}^x[M_T | \mathcal{F}_t]$$

$$= x e^{rT} M_t$$

$$= x e^{rT} \frac{1}{x e^{rt}} X_t$$

$$= X_t e^{r(T-t)}$$

7.11. Let X_t be an Itô diffusion in $\mathbf{R}^n$ and let $f: \mathbf{R}^n \to \mathbf{R}$ be a function such that

$$E^x\left[\int_0^\infty |f(X_t)|dt\right] < \infty \qquad \text{for all } x \in \mathbf{R}^n .$$

Let τ be a stopping time. Use the strong Markov property to prove that

$$E^x\left[\int_\tau^\infty f(X_t)dt\right] = E^x[g(X_\tau)] ,$$

where

$$g(y) = E^y\left[\int_0^\infty f(X_t)dt\right] .$$

We begin using a change of variables

$$\int_\tau^\infty f(X_t)\,dt = \int_0^\infty f(X_{\tau+s})\,ds$$

And notice that by the law of the iterated expectation,

$$\mathbb{E}^x\left[\int_0^\infty f(X_{\tau+s})\,ds\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[\int_0^\infty f(X_{\tau+s})\,ds \,\middle|\, \mathcal{F}_t\right]\right]$$

Then, using the Strong Markov Property,

$$\mathbb{E}^x\left[\mathbb{E}^x\left[\int_0^\infty f(X_{\tau+s})\,ds \,\middle|\, \mathcal{F}_\tau\right]\right] = \mathbb{E}^x\left[\mathbb{E}^{X_\tau}\left[\int_0^\infty f(X_s)\,ds\right]\right]$$

$$= \mathbb{E}^x\left[g(X_\tau)\right]$$

$\square$

7.12. (Local martingales)

An $\mathcal{N}_t$-adapted stochastic process $Z(t) \in \mathbf{R}^n$ is called a *local martingale* with respect to the given filtration $\{\mathcal{N}_t\}$ if there exists an increasing sequence of $\mathcal{N}_t$-stopping times τ_k such that

$$\tau_k \to \infty \quad \text{a.s. as } k \to \infty$$

and

$$Z(t \wedge \tau_k) \qquad \text{is an } \mathcal{N}_t\text{-martingale for all } k\ .$$

a) Show that if $Z(t)$ is a local martingale and there exists a constant $T \leq \infty$ such that the family $\{Z(\tau)\}_{\tau \leq T}$ is uniformly integrable (Appendix C) then $\{Z(t)\}_{t \leq T}$ is a martingale.

1. $\{Z(t)\}_{t \leq T}$ is $\mathcal{N}_t$-adapted, i.e., is $\mathcal{N}_t$-measurable for all t.

2. We'll use the following result:

Theorem C.4. *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of real measurable functions on Ω such that*

$$\lim_{k \to \infty} f_k(\omega) = f(\omega) \qquad \textit{for a.a. } \omega\ .$$

Then the following are equivalent:
1) *$\{f_k\}$ is uniformly integrable*
2) *$f \in L^1(P)$ and $f_k \to f$ in $L^1(P)$, i.e. $\int |f_k - f| dP \to 0$ as $k \to \infty$.*

Since $\{Z(\tau)\}_{\tau \leq T} = \{Z_\tau\}$ is uniformly integrable, and

$$\lim_{k \to \infty} Z_\tau = Z_t \qquad \text{since} \quad \tau = t \wedge \tau_k$$

then $Z_t \in L^1(P)$ and $\displaystyle\lim_{k \to \infty} \int |Z_\tau - Z_t| dP = 0$

i.e., $\mathbb{E}[|Z_t|] < \infty$

3. Since $Z_{t \wedge \tau_k}$ is an $\mathcal{N}_t$-martingale, for $s \leq t$,

$$E^x[Z_{\tau_k \wedge t} | \mathcal{N}_s] = Z_{\tau_k \wedge s} \xrightarrow{k \to \infty} Z_s$$

Thus, $\qquad \mathbb{E}^x[Z_t | \mathcal{N}_s] = \lim_{k \to \infty} E^x[Z_{\tau_k \wedge t} | \mathcal{N}_s] = Z_s$

b) In particular, if $Z(t)$ is a local martingale and there exists a constant $K < \infty$ such that

$$E[Z^2(\tau)] \leq K$$

for all stopping times $\tau \leq T$, then $\{Z(t)\}_{t \leq T}$ is a martingale.

Since $Z_\tau \to Z_t$ in $L^1(\mathbb{P})$ and $Z_t \in L^1(\mathbb{P})$, $\}Z_t\{$ is uniformly integrable.

By the previous item, the result follows.

c) Show that if $Z(t)$ is a *lower bounded* local martingale, then $Z(t)$ is a supermartingale (Appendix C).

We already verified that $Z(t)$ is $\mathcal{N}_t$-adapted and $\mathbb{E}[|Z(t)|] < \infty$. Now we need to show that

$$Z(s) \geq \mathbb{E}[Z(t) | \mathcal{N}_s], \qquad t > s$$

Notice that, assuming $Z_t \geq 0$,

$$\mathbb{E}[Z_t | \mathcal{N}_s] \leq \lim_{k \to \infty} \mathbb{E}[Z_\tau | \mathcal{N}_s] = \mathbb{E}[Z_t | \mathcal{N}_s] = Z_s$$

7.13. a) Let $B_t \in \mathbf{R}^2$, $B_0 = x \neq 0$. Fix $0 < \epsilon < R < \infty$ and define

$$X_t = \ln |B_{t\wedge\tau}| \; ; \qquad t \geq 0$$

where

$$\tau = \inf\{t > 0; |B_t| \leq \epsilon \quad \text{or} \quad |B_t| \geq R\} .$$

Prove that X_t is an $\mathcal{F}_{t\wedge\tau}$-martingale. (Hint: Use Exercise 4.8.) Deduce that $\ln|B_t|$ is a local martingale (Exercise 7.12).

Let $f(B_t) = \ln|B_t|$. Then $X_t = f(B_{t\wedge\tau})$. Using that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \tfrac{1}{2}\int_0^t \Delta f(B_s) ds ,$$

We have

$$f(B_t) = f(B_0) + \int_0^t \left(\frac{(B_1(s), B_2(s))}{|B_s^2|} \right) dB_s$$

$$\cancel{+ \frac{1}{2}\int_0^t \left(\frac{|B_s|^2 - 2B_1^2 + |B_s|^2 - 2B_2^2}{|B_s|^4} \right) ds} \quad 0$$

Thus,

$$f(B_t) = f(B_0) + \int_0^t \frac{B_s}{|B_s^2|} dB_s$$

is a martingale and it follows that X_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.

To show that $\ln|B_t|$ is a local martingale, let

$$\tau_R = \inf\{t > 0 : |B_t| \geq R\}$$

Since $\tau_R \to \infty$ as $R \to \infty$ and we already showed that $\ln|B_{t\wedge\tau}|$ is a martingale, what remains to be proved is that

$$\tau_\varepsilon = \inf\{t>0 : |B_t| \leq \varepsilon\}$$

satisfies $\tau_\varepsilon \to \infty$ as $\varepsilon \downarrow 0$. I.e, we need to prove that

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \infty) = \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \tau_R) = 0$$

Note that

$$\ln|x| = \mathbb{E}^x\left[\ln|B_{t\wedge\tau}|\right] = \mathbb{P}^x(\tau_\varepsilon < \tau_R)\ln\varepsilon + \mathbb{P}^x(\tau_\varepsilon > \tau_R)\ln R$$

Hence

$$\ln|x| = \mathbb{P}^x(\tau_\varepsilon < \tau_R)\ln\varepsilon + (1-\mathbb{P}^x(\tau_\varepsilon < \tau_R))\ln R$$

$$\iff \quad \mathbb{P}^x(\tau_\varepsilon < \tau_R) = \frac{\ln R - \ln|x|}{(\ln R - \ln\varepsilon)}$$

implies that

$$\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \tau_R) = 0$$

$\square$

b) Let $B_t \in \mathbf{R}^n$ for $n \geq 3$, $B_0 = x \neq 0$. Fix $\epsilon > 0$, $R < \infty$ and define

$$Y_t = |B_{t\wedge\tau}|^{2-n}\,; \qquad t \geq 0$$

where

$$\tau = \inf\{t > 0;\, |B_t| \leq \epsilon \quad \text{or} \quad |B_t| \geq R\}\,.$$

Prove that Y_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.
Deduce that $|B_t|^{2-n}$ is a local martingale.

Define $f(B_t) = |B_t|^{2-n}$. Then

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s) ds$$

$$= |x|^{2-n} + \int_0^t (2-n)\|B_s\|^{-n} B_s\, dB_s \overset{(*)}{+} \cancel{\frac{1}{2}\int_0^t \Delta f(B_s) ds}$$

Thus,

$$f(B_t) = |x|^{2-n} + \int_0^t (2-n)\|B_s\|^{-n} B_s\, dB_s$$

is a martingale and it follows that Y_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.

To prove that $|B_t|^{2-n}$ is a local martingale we use the same argument as before. Notice that

$$|x|^{2-n} = \mathbb{E}^x\left[|B_{t\wedge\tau}|^{2-n}\right] = \mathbb{P}^x(\tau_\varepsilon < \tau_R)|\varepsilon|^{2-n} + \mathbb{P}^x(\tau_\varepsilon > \tau_R)|R|^{2-n}$$

i.e.,

$$|x|^{2-n} = \mathbb{P}^x(\tau_\varepsilon < \tau_R)\, \varepsilon^{2-n} + (1-\mathbb{P}^x(\tau_\varepsilon < \tau_R))R^{2-n}$$

$$\Longleftrightarrow \quad \mathbb{P}^x(\tau_\varepsilon < \tau_R) = \frac{R^{2-n} - |x|^{2-n}}{(R^{2-n} - \varepsilon^{2-n})} = \frac{1/R^{n-2} - 1/|x|^{n-2}}{1/R^{n-2} - 1/\varepsilon^{n-2}} = \frac{\frac{|x|^{n-2} - R^{n-2}}{R^{n-2}|x|^{n-2}}}{\frac{\varepsilon^{n-2} - R^{n-2}}{R^{n-2}\varepsilon^{n-2}}}$$

$$= \frac{(|x|^{n-2} - R^{n-2})}{|x|^{n-2} R^{n-2}} \cdot \frac{R^{n-2} \varepsilon^{n-2}}{(\varepsilon^{n-2} - R^{n-2})} = \frac{\varepsilon^{n-2}}{|x|^{n-2}} \frac{(|x|^{n-2} - R^{n-2})}{(\varepsilon^{n-2} - R^{n-2})}$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \infty) = \lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \tau_R) = 0$$

Thus, $|B_t|^{2-n}$ is a local martingale. $\square$

(*)
$$\Delta f(B_s) = \sum_i (2-n)\left[-n \|B_s\|^{-n-2} B_i^2 + \|B_s\|^{-n}\right]$$

$$= \sum_i \frac{(2-n)}{\|B_s\|^n}\left[-n\|B_s\|^{-2} B_i^2 + 1\right]$$

7.14. (Doob's h-transform)

Let B_t be n-dimensional Brownian motion, $D \subset \mathbf{R}^n$ a bounded open set and $h > 0$ a harmonic function on D (i.e. $\Delta h = 0$ in D). Let X_t be the solution of the stochastic differential equation

$$dX_t = \nabla(\ln h)(X_t)dt + dB_t$$

More precisely, choose an increasing sequence $\{D_k\}$ of open subsets of D such that $\overline{D}_k \subset D$ and $\bigcup_{k=1}^{\infty} D_k = D$. Then for each k the equation above can be solved (strongly) for $t < \tau_{D_k}$. This gives in a natural way a solution for $t < \tau := \lim_{k\to\infty} \tau_{D_k}$.

a) Show that the generator A of X_t satisfies

$$Af = \frac{\Delta(hf)}{2h} \qquad \text{for } f \in C_0^2(D) .$$

In particular, if $f = \frac{1}{h}$ then $\mathcal{A}f = 0$.

$$Af(x) = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

$$= \sum_i \frac{\partial(\ln h)}{\partial x_i}(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_i \frac{\partial^2 f}{\partial x_i^2}(x)$$

$$= \sum_i \frac{1}{h} \frac{\partial h}{\partial x_i}(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \Delta f(x)$$

$$= \frac{1}{h} \nabla h(x) \nabla f(x) + \frac{1}{2} \Delta f(x)$$

$$= \frac{2\nabla h(x) \nabla f(x) + h\Delta f(x)}{2h} = \frac{\Delta(hf)}{2h}$$

Obs: $\Delta(hf) = \cancel{f\Delta h} + 2\nabla h \nabla f + h\Delta f$

b) Use a) to show that if there exists $x_0 \in \partial D$ such that

$$\lim_{x \to y \in \partial D} h(x) = \begin{cases} 0 & \text{if } y \neq x_0 \\ \infty & \text{if } y = x_0 \end{cases}$$

(i.e. h is a *kernel function*), then

$$\lim_{t \to \tau} X_t = x_0 \text{ a.s.}$$

(Hint: Consider $E^x[f(X_T)]$ for suitable stopping times T and with $f = \frac{1}{h}$)

In other words, we have imposed a drift on B_t which causes the process to exit from D at the point x_0 only. This can also be formulated as follows: X_t is obtained by *conditioning B_t to exit from D at x_0*. See Doob (1984).

DRAFT

Define the stopping time

Let $T_k = \inf\{t>0 : X_t \notin D_k\}$ and notice that as $k \to \infty$ we have that

$$T_k \longrightarrow T = \inf\{t>0 : X_t \notin D\} = \tau$$

More than that, by hypothesis, $\lim_{k \to \infty} T_k < \infty$.

Apply Dynkin's formula with $f = 1/h$

$$E^x[f(X_T)] = f(x) + E^x\left[\int_0^T Af(s)\,ds\right] = f(x)$$

Open the expected value:

$$f(x) = \mathbb{P}[X_t \in D]\, f(x) + (1 - \mathbb{P}[X_t \in D])\, f(y)$$

$$\mathbb{P}[X_t \in D] = \frac{f(x) - f(y)}{f(x) - f(y)} \xrightarrow[x \to y]{} 1 \quad \text{since } f(y) = 1/h(y)$$

7.15. Let B_t be 1-dimensional and define

$$F(\omega) = (B_T(\omega) - K)^+$$

where $K > 0$, $T > 0$ are constants.
By the Itô representation theorem (Theorem 4.3.3) we know that there exists $\phi \in \mathcal{V}(0,T)$ such that

$$F(\omega) = E[F] + \int_0^T \phi(t,\omega) dB_t \, .$$

How do we find ϕ explicitly? This problem is of interest in mathematical finance, where ϕ may be regarded as the replicating portfolio for the contingent claim F (see Chapter 12). Using the Clark-Ocone formula (see Karatzas and Ocone (1991) or Øksendal (1996)) one can deduce that

$$\phi(t,\omega) = E[\mathcal{X}_{[K,\infty)}(B_T)|\mathcal{F}_t] \, ; \qquad t < T \, . \tag{7.5.3}$$

Use (7.5.3) and the Markov property of Brownian motion to prove that for $t < T$ we have

$$\phi(t,\omega) = \frac{1}{\sqrt{2\pi(T-t)}} \int_K^\infty \exp\Big(-\frac{(x - B_t(\omega))^2}{2(T-t)}\Big) dx \, . \tag{7.5.4}$$

Recall the Markov Property:

Theorem 6.2.1 (Markov Property for Itô Diffusions). Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a bounded Borel function. For $t, h \geq 0$,

$$\mathbb{E}^x[f(X_{t+h}) \mid \mathfrak{F}_t^{(m)}](\omega) = \mathbb{E}^{X_t(\omega)}[f(X_h)] \tag{6.2}$$

where $\mathbb{E}^x$ denotes the expected value w.r.t. the probability measure Q^x, i.e., $\mathbb{E}^x[f(X_h)] = [f(X_h^x)]$ w.r.t. the probability measure P^0.

Using it, for $t<T$, we have

$$\varphi(t,\omega) = E[\mathcal{X}_{[K,\infty)}(B_T) \mid \mathcal{F}_t] = E^{B_t}[\mathcal{X}_{[K,\infty)}(B_{T-t})]$$

By the exercise 2.8.c.

$$E[f(B_t)] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbf{R}} f(x) e^{-\frac{x^2}{2t}} dx$$

it follows that

$$E^{B_t}[\mathcal{X}_{[K,\infty)}(B_{T-t})] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} \mathcal{X}_{[K,\infty)}(x) \exp\left(-\frac{(x-B_t)^2}{2(T-t)}\right) dx$$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_K^\infty \exp\left(-\frac{(x-B_t)^2}{2(T-t)}\right) dx$$

7.16. Let B_t be 1-dimensional and let $f: \mathbf{R} \to \mathbf{R}$ be a bounded function. Prove that if $t < T$ then

$$E^x[f(B_T)|\mathcal{F}_t] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbf{R}} f(x) \exp\Big(-\frac{(x - B_t(\omega))^2}{2(T-t)}\Big) dx . \tag{7.5.5}$$

(Compare with (7.5.4).)

By the Markov property,

$$E^x[f(B_T)|\mathcal{F}_t] = E^{B_t}[f(B_{T-t})]$$

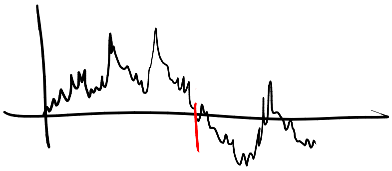
$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} f(x) \exp\Big(-\frac{(x-B_t)^2}{2(T-t)}\Big) dx$$

7.17. Let B_t be 1-dimensional and put

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3 \, ; \qquad t \geq 0 \, .$$

Then we have seen in Exercise 4.15 that X_t is a solution of the stochastic differential equation

$$dX_t = \tfrac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t \, ; \qquad X_0 = x \, . \tag{7.5.6}$$



Define

$$\tau = \inf\{t > 0; X_t = 0\}$$

and put

$$Y_t = \begin{cases} X_t & \text{for } t \leq \tau \\ 0 & \text{for } t > \tau \, . \end{cases}$$

Prove that Y_t is also a (strong) solution of (7.5.6). Why does not this contradict the uniqueness assertion of Theorem 5.2.1?
(Hint: Verify that

$$Y_t = x + \int_0^t \tfrac{1}{3}Y_s^{1/3}ds + \int_0^t Y_s^{2/3}dB_s$$

for all t by splitting the integrals as follows:

$$\int_0^t = \int_0^{t\wedge\tau} + \int_{t\wedge\tau}^t \, . \,)$$

We'll break into two pieces: $t \leq \tau$ and $t > \tau$.

For $t \leq \tau$,

$$Y_t = X_t = x + \int_0^t \frac{1}{3}X_s^{1/3}ds + \int_0^t X_s^{2/3}dB_s$$

For $t > \tau$,

$$Y_t = X_\tau = x + \int_0^\tau \frac{1}{3}X_s^{1/3}ds + \int_0^\tau X_s^{2/3}dB_s$$

Now, for any $t \geq 0$,

$$Y_t = x + \int_0^{t\wedge\tau} \frac{1}{3}Y_s^{1/3}ds + \int_0^{t\wedge\tau} Y_s^{2/3}dB_s + \int_{t\wedge\tau}^t \frac{1}{3}Y_s^{1/3}ds + \int_{t\wedge\tau}^t Y_s^{2/3}dB_s$$

$$= x + \int_0^t \frac{1}{3}Y_s^{1/3}ds + \int_0^t Y_s^{2/3}dB_s \implies Y_t \text{ is a strong solution}$$

Notice that the functions $1/3\,x_t^{1/3}$ and $x_t^{2/3}$ are not Lipschitz (their derivatives "explode" as $x \to 0$). Hence, the conditions of the theorem aren't met.

7.18. a) Let

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t\,; \qquad X_0 = x$$

be a 1-dimensional Itô diffusion with characteristic operator $\mathcal{A}$. Let $f \in C^2(\mathbf{R})$ be a solution of the differential equation

$$\mathcal{A}f(x) = b(x)f'(x) + \tfrac{1}{2}\sigma^2(x)f''(x) = 0\,; \qquad x \in \mathbf{R}\,. \tag{7.5.7}$$

Let $(a,b) \subset \mathbf{R}$ be an open interval such that $x \in (a,b)$ and put

$$\tau = \inf\{t > 0;\ X_t \notin (a,b)\}\,.$$

Assume that $\tau < \infty$ a.s. Q^x and define

$$p = P^x[X_\tau = b]\,.$$

Use Dynkin's formula to prove that if $f(b) \neq f(a)$ then

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}\,. \tag{7.5.8}$$

In other words, the *harmonic measure* $\mu^x_{(a,b)}$ of X on $\partial(a,b) = \{a,b\}$ is given by

$$\mu^x_{(a,b)}(b) = \frac{f(x) - f(a)}{f(b) - f(a)}\,, \quad \mu^x_{(a,b)}(a) = \frac{f(b) - f(x)}{f(b) - f(a)}\,. \tag{7.5.9}$$

Since $Af(x) = 0$, by Dynkin's formula,

$$\mathbb{E}[f(X_\tau)] = f(x)$$

Using the definition of p,

$$pf(b) + (1-p)f(a) = f(x)$$

i.e.,

$$\frac{f(x) - f(a)}{f(b) - f(a)} = p \qquad (*)$$

b) Now specialize to the process

$$X_t = x + B_t\,; \qquad t \geq 0\,.$$

Prove that

$$p = \frac{x-a}{b-a}\,. \tag{7.5.10}$$

Notice that $Af(x) = \frac{1}{2}f''(x)$

Taking $f(x) = x$, $Af(x) = 0$ and the result follows from $(*)$.

c) Find p if

$$X_t = x + \mu t + \sigma B_t\,; \qquad t \geq 0$$

where $\mu, \sigma \in \mathbf{R}$ are nonzero constants.

Here,

$$Af(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x)$$

Solving the ODE $Af(x)=0$:

Characteristic equation:

$$\frac{1}{2}\sigma^2\lambda^2 + \mu\lambda = 0 \iff \lambda\left(\frac{1}{2}\sigma^2\lambda + \mu\right) = 0$$

$$\iff \lambda = 0 \quad \text{or} \quad \frac{1}{2}\sigma^2\lambda + \mu = 0 \iff \lambda = \frac{-2\mu}{\sigma^2}$$

Let $\underline{f(x) = e^{-2\mu x/\sigma^2}}$ and notice that, since

$$f'(x) = \frac{-2\mu}{\sigma^2} e^{-2\mu x/\sigma^2} \quad \text{and} \quad f''(x) = \frac{4\mu^2}{\sigma^4} e^{-2\mu x/\sigma^2}$$

$f(x)$ solves the ODE.

Using that,

$$p = \frac{f(x) - f(a)}{f(b) - f(a)} = \frac{e^{-2\mu x/\sigma^2} - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}$$

7.19. Let B_t^x be 1-dimensional Brownian motion starting at $x > 0$. Define

$$\tau = \tau(x, \omega) = \inf\{t > 0;\ B_t^x(\omega) = 0\}\ .$$

From Exercise 7.4 we know that

$$\tau < \infty \quad \text{a.s. } P^x \text{ and } E^x[\tau] = \infty\ .$$

What is the distribution of the random variable $\tau(\omega)$?

a) To answer this, first find the *Laplace transform*

$$g(\lambda) := E^x[e^{-\lambda\tau}] \qquad \text{for } \lambda > 0\ .$$

(Hint: Let $M_t = \exp(-\sqrt{2\lambda}\, B_t - \lambda t)$. Then

$$\{M_{t\wedge\tau}\}_{t\geq 0} \quad \text{is a bounded martingale}\ .$$

[Solution: $g(\lambda) = \exp(-\sqrt{2\lambda}\, x)$.]

Notice that

$$E^x[M_t] = E^x[M_0] = e^{-\sqrt{2\lambda}\,x}$$

and

$$M_\tau = e^{-\sqrt{2\lambda}\,\cancel{B_\tau}^{\,0}} \cdot e^{-\lambda\tau} = e^{-\lambda\tau}$$

Thus

$$g(\lambda) = E[e^{-\lambda\tau}] = E[M_\tau] = E[M_0] = e^{-\sqrt{2\lambda}\,x}$$

b) To find the density $f(t)$ of τ it suffices to find $f(t) = f(t,x)$ such that

$$\int_0^\infty e^{-\lambda t} f(t)dt = \exp(-\sqrt{2\lambda}\, x) \qquad \text{for all } \lambda > 0$$

i.e. to find the *inverse* Laplace transform of $g(\lambda)$. Verify that

$$f(t,x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right); \qquad t > 0\ .$$

Define $g(\lambda, x) = \int_0^\infty e^{-\lambda t} f(t)\, dt$

and notice that it satisfies the ODE $g'' = 2\lambda g$ with solution

$$c_1 e^{\sqrt{2\lambda}\,x} + c_2 e^{-\sqrt{2\lambda}\,x}$$

Showing that $c_1 = 0$ and $c_2 = 1$, the result follows.