5.1. Verify that the given processes solve the given corresponding stochastic differential equations: (B_t denotes 1-dimensional Brownian motion)

(i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$

By Itô's formula,

$$dX_t = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt = \frac{1}{2} X_t dt + X_t dB_t$$

(ii) $X_t = \frac{B_t}{1+t}$; $B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t \; ; \qquad X_0 = 0$$

$$g(t,x) = \frac{x}{1+t}$$

$$dX_t = \frac{-B_t}{(1+t)^2} dt + \frac{1}{1+t} dB_t = -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t$$

(iii) $X_t = \sin B_t$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves

$$dX_t = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2}\, dB_t \quad \text{for } t < \inf\{s > 0; B_s \notin [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\}$$

$$g(t,x) = \sin x$$

$$dX_t = \cos B_t \, dB_t - \frac{1}{2} \sin B_t \, dt = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2}\, dB_t$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

$g_1(t,x) = t$

$g_2(t,x) = e^t x$

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ e^t B_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^t \end{bmatrix} dB_t = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t \, .$$

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} \sinh B_t \\ \cosh B_t \end{bmatrix} dB_t + \frac{1}{2} \begin{bmatrix} \cosh B_t \\ \sinh B_t \end{bmatrix} dt$$

$$= \frac{1}{2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t$$