4.13. Let $dX_t = u(t,\omega)dt + dB_t$ $(u \in \mathbf{R},\ B_t \in \mathbf{R})$ be an Itô process and assume for simplicity that u is bounded. Then from Exercise 4.12 we know that unless $u = 0$ the process X_t is not an $\mathcal{F}_t$-martingale. However, it turns out that we can construct an $\mathcal{F}_t$-martingale from X_t by multiplying by a suitable exponential martingale. More precisely, define

$$Y_t = X_t M_t$$

where

$$M_t = \exp\Big(-\int_0^t u(r,\omega)dB_r - \tfrac{1}{2}\int_0^t u^2(r,\omega)dr \Big) .$$

Use Itô's formula to prove that

$$Y_t \quad \text{is an } \mathcal{F}_t\text{-martingale} .$$

Let

$$Z_t := -\int_0^t u(r,\omega)\,dB_r - \frac{1}{2}\int_0^t u^2(r,\omega)\,dr$$

i.e.

$$dZ_t = -u(t,\omega)\,dB_t - \frac{1}{2}u^2(t,\omega)\,dt \qquad (1)$$

With this, it is possible to write $M_t = e^{Z_t}$ and apply Itô's formula:

$$dM_t = \frac{\partial M_t}{\partial t}dt + \frac{\partial M_t}{\partial Z_t}dZ_t + \frac{1}{2}\frac{\partial^2 M_t}{\partial Z_t^2}(dZ_t)^2$$

$$= e^{Z_t}dZ_t + \frac{1}{2}e^{Z_t}(dZ_t)^2 \qquad (2)$$

Expanding $(dZ_t)^2$,

$$(dZ_t)^2 = \Big(u(t,\omega)dB_t - \frac{1}{2}u^2(t,\omega)dt \Big)^2 = u^2 dt$$

Hence,

$$dM_t = e^{Z_t}dZ_t + \frac{1}{2}e^{Z_t}u^2 dt = M_t\Big(\frac{1}{2}u^2 dt + dZ_t \Big) \qquad (3)$$

Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t\ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s\ .$$

Now, by Integration by Parts,

$$d(X_t M_t) = X_t dM_t + M_t dX_t + dX_t dM_t \qquad (4)$$

Since

- $X_t dM_t \overset{(3)}{=} X_t M_t \left(\frac{1}{2} u^2 dt + dZ_t \right) \overset{(1)}{=} X_t M_t \left(-u(t,\omega) dB_t \right)$

- $M_t dX_t = M_t \left(u dt + dB_t \right)$

- $dX_t dM_t \overset{(3)}{=} \left(u dt + dB_t \right) M_t \left(\frac{1}{2} u^2 dt + dZ_t \right)$

$$\overset{(1)}{=} M_t \left(u dt + dB_t \right) \left(-u(t,\omega) dB_t \right)$$

$$= -u M_t dt$$

We have that (4) can be written as

$$d(X_t M_t) = X_t M_t \left(-u(t,\omega) dB_t \right) + M_t \left(u dt + dB_t \right) - u M_t dt$$

$$= -u_t X_t M_t dB_t + M_t dB_t = M_t (1 - u_t X_t) dB_t$$

Therefore,

$$Y_t = Y_0 + \int_0^t M_s (1 - u_s X_s) dB_s \quad \text{is a martingale.}$$