Compute

$$\int_0^T B(t)\,dB(t)$$

Step 1: Let

$$X_n(t) = \sum_{i=0}^{n-1} B_n(t_i)\left[B_n(t_{i+1}) - B_n(t_i)\right] \qquad (1)$$

be a sequence of elementary functions. Notice that $X_n(t)$ converges to $B(t)$.

Step 2: Now notice that

$$\begin{aligned} &B_n(t_i)\left[B_n(t_{i+1}) - B_n(t_i)\right] \\ &= B_n(t_{i+1})\cdot B_n(t_i) - B_n(t_i)^2 + B_n(t_{i+1})^2 - B_n(t_{i+1})^2 \\ &= \frac{1}{2}\left[B_n(t_{i+1})^2 - B_n(t_i)^2 - \left(B_n(t_{i+1}) - B_n(t_i)\right)^2\right] \qquad (2) \end{aligned}$$

Step 3: With (2) and (1), our integral is

$$\begin{aligned} \int_0^T X_n(t)\,dB(t) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[B_n(t_{i+1})^2 - B_n(t_i)^2\right] \\ &\quad - \frac{1}{2}\sum_{i=0}^{n-1}\left(B_n(t_{i+1}) - B_n(t_i)\right)^2 \end{aligned} \qquad (3)$$

Step 4: Since the first sum is a telescopic sum and the second sum converges in probability to T (bc quadratic variation of Brownian motion), we have

$$\int_0^T X_n(t)\,dB(t) = \frac{1}{2}B^2(T) - \cancelto{0}{\frac{1}{2}B^2(0)} - \frac{1}{2}T = \frac{1}{2}B^2(T) - \frac{1}{2}T \qquad (4)$$

Hence, the integral converges in probability to

$$\boxed{\int_0^T B(t)\,dB(t) = \lim_{n\to\infty}\int_0^T X_n(t)\,dB(t) = \frac{1}{2}B^2(T) - \frac{1}{2}T}$$