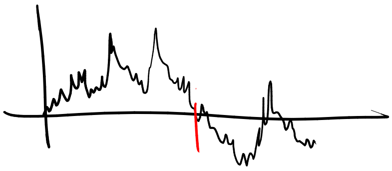
7.17. Let B_t be 1-dimensional and put

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3\,; \qquad t \geq 0\,.$$

Then we have seen in Exercise 4.15 that X_t is a solution of the stochastic differential equation

$$dX_t = \tfrac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t\,; \qquad X_0 = x\,. \tag{7.5.6}$$



Define

$$\tau = \inf\{t > 0; X_t = 0\}$$

and put

$$Y_t = \begin{cases} X_t & \text{for } t \leq \tau \\ 0 & \text{for } t > \tau\,. \end{cases}$$

Prove that Y_t is also a (strong) solution of (7.5.6). Why does not this contradict the uniqueness assertion of Theorem 5.2.1?
(Hint: Verify that

$$Y_t = x + \int_0^t \tfrac{1}{3}Y_s^{1/3}ds + \int_0^t Y_s^{2/3}dB_s$$

for all t by splitting the integrals as follows:

$$\int_0^t = \int_0^{t\wedge\tau} + \int_{t\wedge\tau}^t\,.\;)$$

We'll break into two pieces: $t \leq \tau$ and $t > \tau$.

For $t \leq \tau$,

$$Y_t = X_t = x + \int_0^t \frac{1}{3}X_s^{1/3}ds + \int_0^t X_s^{2/3}dB_s$$

For $t > \tau$,

$$Y_t = X_\tau = x + \int_0^\tau \frac{1}{3}X_s^{1/3}ds + \int_0^\tau X_s^{2/3}dB_s$$

Now, for any $t \geq 0$,

$$Y_t = x + \int_0^{t\wedge\tau} \frac{1}{3}Y_s^{1/3}ds + \int_0^{t\wedge\tau} Y_s^{2/3}dB_s + \int_{t\wedge\tau}^t \frac{1}{3}Y_s^{1/3}ds + \int_{t\wedge\tau}^t Y_s^{2/3}dB_s$$

$$= x + \int_0^t \frac{1}{3}Y_s^{1/3}ds + \int_0^t Y_s^{2/3}dB_s \implies Y_t \text{ is a strong solution}$$

Notice that the functions $1/3\,x_t^{1/3}$ and $x_t^{2/3}$ are not Lipschitz (their derivatives "explode" as $x \to 0$). Hence, the conditions of the theorem aren't met.