5.7. The *mean-reverting Ornstein-Uhlenbeck process* is the solution X_t of the stochastic differential equation

$$dX_t = (m - X_t)dt + \sigma dB_t$$

where m, σ are real constants, $B_t \in \mathbf{R}$.

a) Solve this equation by proceeding as in Exercise 5.5 a).

b) Find $E[X_t]$ and $\mathrm{Var}[X_t] := E[(X_t - E[X_t])^2]$.

a) Let $Y_t = X_t - m$. By Itô's formula, $dY_t = dX_t$. Hence

$$dY_t = -Y_t dt + \sigma dB_t$$

Applying 5.5. $(\mu = -1)$

$$Y_t = e^{-t} Y_0 + \sigma \int_0^t e^{(s-t)} \, dB_s$$

and finally,

$$\boxed{X_t = m + e^{-t} Y_0 + \sigma \int_0^t e^{(s-t)} \, dB_s}$$

b) By 5.5.,

$$\mathbb{E}[X_t] = m + \mathbb{E}[Y_t] = m + e^{\mu t}\mathbb{E}[X_0] = m + e^{-t}\mathbb{E}[X_0]$$

$$\mathrm{Var}[X_t] = e^{2\mu t}\left(\mathbb{E}[X_0^2] - \mathbb{E}^2[X_0]\right) + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$$

$$= e^{-2t}\left(\mathbb{E}[X_0^2] - \mathbb{E}^2[X_0]\right) - \frac{\sigma^2}{2}\left(e^{-2t} - 1\right)$$