5.17. (The Gronwall inequality)
Let $v(t)$ be a nonnegative function such that

$$v(t) \le C + A \int_0^t v(s)ds \qquad \text{for } 0 \le t \le T$$

for some constants C, A. Prove that

$$v(t) \le C \exp(At) \qquad \text{for } 0 \le t \le T\,. \tag{5.3.19}$$

Define

$$g(t) = A \exp\left(-\int_0^t A ds\right) \int_0^t v(s)ds = A \exp(-At) \int_0^t v(s)ds$$

and compute

$$g'(t) = A \exp(-At)\left[-A \int_0^t v(s)ds + v(t)\right]$$

$$\le A \exp(-At)\left[-A \int_0^t v(s)ds + C + A \int_0^t v(s)ds\right]$$

$$= AC \exp(-At)$$

Integrating,

$$g(t) \le AC \int_0^t \exp(-As)\, ds = AC\left[\frac{1}{A}\left(1 - e^{-At}\right)\right] = C\left(1 - e^{-At}\right)$$

Thus,

$$A \int_0^t v(s)ds = \exp(At)\, g(t) \le e^{At} C \left(1 - e^{-At}\right)$$

Finally,

$$v(t) \le C + e^{At} C \left(1 - e^{-At}\right) = C \exp(At)$$

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