**7.2.** Find an Itô diffusion (i.e. write down the stochastic differential equation for it) whose generator is the following:

a) $Af(x) = f'(x) + f''(x)$; $f \in C_0^2(\mathbf{R})$

b) $Af(t,x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}$ ; $f \in C_0^2(\mathbf{R}^2)$, where $c, \alpha$ are constants.

c) $Af(x_1,x_2) = 2x_2\frac{\partial f}{\partial x_1} + \ln(1+x_1^2+x_2^2)\frac{\partial f}{\partial x_2}$
$+\frac{1}{2}(1+x_1^2)\frac{\partial^2 f}{\partial x_1^2} + x_1\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x_2^2}$ ; $f \in C_0^2(\mathbf{R}^2)$.

a) Let $f(B_t) = X_t$. Then

$$dX_t = dt + \sqrt{2}\, dB_t$$

b) $$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ cX_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \alpha X_2(t) \end{bmatrix} dB_t$$

c) Start by noticing that

$$\sigma\sigma^T = \begin{bmatrix} 1+X_1^2(t) & X_1(t) \\ X_1(t) & 1 \end{bmatrix}$$

To find $\sigma$, we compute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \begin{cases} a^2 + b^2 = 1 + X_1^2(t) \\ ac + bd = X_1(t) \\ c^2 + d^2 = 1 \end{cases}$$

Take $a = X_1(t)$, $b = c = 1$ and $d = 0$, i.e., $\sigma = \begin{bmatrix} X_1(t) & 1 \\ 1 & 0 \end{bmatrix}$

Then, an Itô diffusion for the generator is

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 2X_2(t) \\ \ln(1 + X_1^2(t) + X_2^2(t)) \end{bmatrix} dt + \begin{bmatrix} X_1(t) & 1 \\ 1 & 0 \end{bmatrix} dB_t$$