7.15. Let B_t be 1-dimensional and define

$$F(\omega) = (B_T(\omega) - K)^+$$

where $K > 0$, $T > 0$ are constants.
By the Itô representation theorem (Theorem 4.3.3) we know that there exists $\phi \in \mathcal{V}(0, T)$ such that

$$F(\omega) = E[F] + \int_0^T \phi(t, \omega) dB_t \, .$$

How do we find ϕ explicitly? This problem is of interest in mathematical finance, where ϕ may be regarded as the replicating portfolio for the contingent claim F (see Chapter 12). Using the Clark-Ocone formula (see Karatzas and Ocone (1991) or Øksendal (1996)) one can deduce that

$$\phi(t, \omega) = E[\mathcal{X}_{[K,\infty)}(B_T)|\mathcal{F}_t] \, ; \qquad t < T \, . \tag{7.5.3}$$

Use (7.5.3) and the Markov property of Brownian motion to prove that for $t < T$ we have

$$\phi(t, \omega) = \frac{1}{\sqrt{2\pi(T-t)}} \int_K^\infty \exp\Big(- \frac{(x - B_t(\omega))^2}{2(T-t)} \Big) dx \, . \tag{7.5.4}$$

Recall the Markov Property:

Theorem 6.2.1 (Markov Property for Itô Diffusions). Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a bounded Borel function. For $t, h \geq 0$,

$$\mathbb{E}^x[f(X_{t+h}) \mid \mathfrak{F}_t^{(m)}](\omega) = \mathbb{E}^{X_t(\omega)}[f(X_h)] \tag{6.2}$$

where $\mathbb{E}^x$ denotes the expected value w.r.t. the probability measure Q^x, i.e., $\mathbb{E}^x[f(X_h)] = [f(X_h^x)]$ w.r.t. the probability measure P^0.

Using it, for $t<T$, we have

$$\varphi(t,\omega) = E\left[\mathcal{X}_{[K,\infty)}(B_T) \mid \mathcal{F}_t\right] = E^{B_t}\left[\mathcal{X}_{[K,\infty)}(B_{T-t})\right]$$

By the exercise 2.8.c.

$$E[f(B_t)] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbf{R}} f(x) e^{-\frac{x^2}{2t}} dx$$

it follows that

$$E^{B_t}\left[\mathcal{X}_{[K,\infty)}(B_{T-t})\right] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} \mathcal{X}_{[K,\infty)}(x) \exp\left(-\frac{(x-B_t)^2}{2(T-t)}\right) dx$$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \int_K^\infty \exp\left(-\frac{(x-B_t)^2}{2(T-t)}\right) dx$$