7.3. Let B_t be Brownian motion on $\mathbf{R}$, $B_0 = 0$ and define

$$X_t = X_t^x = x \cdot e^{ct+\alpha B_t},$$

where c, α are constants. Prove directly from the definition that X_t is a Markov process.

Denote the σ-algebra generated by the B.M. as $\mathcal{F}_t$.

Start by noticing that

$$\mathbb{E}^x\left[X_{t+h} \mid \mathcal{F}_t\right] = \mathbb{E}^x\left[x e^{c(t+h)+\alpha B_{t+h}} \mid \mathcal{F}_t\right]$$

$$= \mathbb{E}^{B_t}\left[x e^{c(t+h)+\alpha B_h}\right]$$

Therefore, we can write

$$\mathbb{E}^x\left[X_{t+h} \mid X_t\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[x e^{c(t+h)+\alpha B_{t+h}} \mid \mathcal{F}_t\right] \mid X_t\right]$$

$$= \mathbb{E}^x\left[\mathbb{E}^{B_t}\left[x e^{c(t+h)+\alpha B_h}\right] \mid X_t\right]$$

$$= \mathbb{E}^{B_t}\left[x e^{c(t+h)+\alpha B_h}\right]$$

$$= \mathbb{E}^x\left[X_{t+h} \mid \mathcal{F}_t\right]$$