5.5. a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t dt + \sigma dB_t$$

where μ, σ are real constants, $B_t \in \mathbf{R}$.
The solution is called the *Ornstein-Uhlenbeck process*. (Hint: See Exercise 5.4 (ii).)

Multiplying by the integrating factor $e^{-\mu t}$,

$$e^{-\mu t} dX_t = e^{-\mu t} \mu X_t dt + e^{-\mu t} \sigma dB_t \qquad (1)$$

Now notice that

$$d(e^{-\mu t} X_t) = -\mu e^{-\mu t} X_t dt + e^{-\mu t} dX_t \qquad (2)$$

With (1) and (2)

$$d(e^{-\mu t} X_t) = e^{-\mu t} \sigma dB_t$$

Therefore,

$$e^{-\mu t} X_t = X_0 + \sigma \int_0^t e^{-\mu s} dB_s$$

And hence,

$$\boxed{X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{-\mu(s-t)} dB_s}$$

b) Find $E[X_t]$ and $\text{Var}[X_t] := E[(X_t - E[X_t])^2]$.

- $$\mathbb{E}[X_t] = \mathbb{E}\left[e^{\mu t}X_0 + \sigma\int_0^t e^{-\mu(s-t)}\,dB_s\right] = \mathbb{E}\left[e^{\mu t}X_0\right]$$

$$= e^{\mu t}\,\mathbb{E}[X_0]$$

- Notice that

$$\mathbb{E}[X_t^2] = \mathbb{E}\left[e^{2\mu t}X_0^2 + \sigma^2\int_0^t e^{-2\mu(s-t)}\,ds + 2e^{\mu t}X_0\sigma\int_0^t e^{-\mu(s-t)}\,dB_s\right]$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \sigma^2\,\mathbb{E}\left[\int_0^t e^{-2\mu(s-t)}\,ds\right]$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \sigma^2\int_0^t e^{-2\mu(s-t)}\,ds$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right) \qquad (\ast)$$

Hence,

$$\text{Var}[X_t] = \mathbb{E}[X_t^2] - \mathbb{E}^2[X_t]$$

$$= e^{2\mu t}\,\mathbb{E}[X_0^2] + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right) - e^{2\mu t}\,\mathbb{E}^2[X_0]$$

$$= e^{2\mu t}\left(\mathbb{E}[X_0^2] - \mathbb{E}^2[X_0]\right) + \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$$

(*)

Obs: $$e^{2\mu t}\int_0^t e^{-2\mu s}\,ds = e^{2\mu t}\cdot\left[\frac{e^{-2\mu s}}{-2\mu}\right]_0^t = -\frac{e^{2\mu t}}{2\mu}\left(e^{-2\mu t}-1\right)$$

$$= -\frac{1}{2\mu}\left(1-e^{2\mu t}\right) = \frac{1}{2\mu}\left(e^{2\mu t}-1\right)$$