4.4. (Exponential martingales)
Suppose $\theta(t,\omega) = (\theta_1(t,\omega), \ldots, \theta_n(t,\omega)) \in \mathbf{R}^n$ with $\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for $k = 1, \ldots, n$, where $T \le \infty$. Define

$$Z_t = \exp\Big\{ \underbrace{\int_0^t \theta(s,\omega)dB(s) - \tfrac{1}{2}\int_0^t \theta^2(s,\omega)ds}_{Y_t} \Big\}; \qquad 0 \le t \le T$$

where $B(s) \in \mathbf{R}^n$ and $\theta^2 = \theta \cdot \theta$ (dot product).

a) Use Itô's formula to prove that

$$dZ_t = Z_t\theta(t,\omega)dB(t) .$$

b) Deduce that Z_t is a martingale for $t \le T$, provided that

$$Z_t\theta_k(t,\omega) \in \mathcal{V}[0,T] \qquad \text{for } 1 \le k \le n .$$

a) Let $g(t,x) = e^x$. Then,

$$dZ_t = \frac{\partial g}{\partial t}(t,Y_t)dt + \frac{\partial g}{\partial x}(t,Y_t)\,dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,Y_t)(dY_t)^2$$

$$= 0 + e^{Y_t}dY_t + \frac{1}{2}e^{Y_t}(dY_t)^2 \qquad (1)$$

Where

$$Y_t = \int_0^t \theta(s,\omega)dB(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega)ds$$

Define $h(t,x) = \int_0^t \theta(s,\omega)dx(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega)ds$

then

$$\frac{\partial h}{\partial t}(t,x) = -\frac{1}{2}\theta^2(t,\omega)$$

$$\frac{\partial h}{\partial x}(t,x) = \theta(t,\omega), \qquad \frac{\partial^2 h}{\partial x^2}(t,\omega) = 0$$

Hence, since $Y_t = h(t,B_t)$

$$dY_t = -\frac{1}{2}\theta^2(t,\omega)\,dt + \theta(t,\omega)\,dB_t \qquad (2)$$

Combining (1) and (2):

$$dZ_t = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} (dY_t)^2$$

And noticing that

$$(dY_t)^2 = \theta^2(t,\omega)\, dt$$

We obtain

$$dZ_t = Z_t\left(-\frac{1}{2}\theta^2(t,\omega)\,dt + \theta(t,\omega)\,dB_t\right) + \frac{1}{2} Z_t \theta^2(t,\omega)\,dt$$

i.e.

$$dZ_t = Z_t\,\theta(t,\omega)\,dB_t \qquad (3)$$

b) Given that $\theta_k \in \mathcal{V}[0,T]$, we know that the Itô integral (2) is a martingale.

And given that $Z_t\theta_k \in \mathcal{V}[0,T]$, (3) is a martingale.