5.9. Show that there is a unique strong solution X_t of the 1-dimensional stochastic differential equation

$$dX_t = \ln(1 + X_t^2)dt + \mathcal{X}_{\{X_t>0\}} X_t dB_t\ , \qquad X_0 = a \in \mathbf{R}\ .$$

First we need to show that

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|), \quad x \in \mathbf{R}^n,\ t \in [0,T]$$

In fact, $|\mathcal{X}_{\{x>0\}} x| \le |x|$ and

$$\begin{aligned} |\ln(1+x^2)| &= \ln(1+x^2) \\ &\le \ln(x^2 - 2|x| + 1) && \text{since } 1+x^2 \le x^2 - 2|x| + 1 \\ &= \ln(|x|-1)^2 \\ &= 2\ln(|x|-1) \\ &\le 2|x| && \text{since } \ln(x) \le x-1 \end{aligned}$$

Hence,

$$|\ln(1+x^2)| + |\mathcal{X}_{\{x>0\}} x| \le 2|x| + |x| < 3(1+|x|)$$

Now we need to show that

$$|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le D|x-y|, \quad x,y \in \mathbf{R}^n,\ t \in [0,T]$$

However, by the Mean Value Theorem,

$$|\ln(1+x^2) - \ln(1+y^2)| = \left|\frac{2z}{1+z^2}\right| |x-y|$$

Given that

$$-\frac{(x-1)^2}{1+x^2} \le 0 \iff \frac{2x - x^2 - 1}{1+x^2} \le 0 \iff \frac{2x}{1+x^2} \le \frac{1+x^2}{1+x^2} = 1$$

We obtain

$$|\ln(1+x^2) - \ln(1+y^2)| \le |x-y|$$

Thus, b is Lipschitz. Also

$$\left| \chi_{\{x>0\}} x - \chi_{\{y>0\}} y \right| \leq |x-y|$$

Hence, a strong solution does exist.