MONOTONICITY OF OPTION PRICES WITH RESPECT TO VOLATILITY

Adair Antonio da Silva Neto*

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Abstract

This paper explores how option prices behave when the volatility is allowed to vary between two bounds. We show that the call price is a convex function of the stock price and the main result is that the price of a European call is a monotonically increasing function of volatility. Since the Black-Scholes-Merton model assumes constant volatility, this result provides essential information about the option price. To illustrate the result, a numerical simulation is presented.

Keywords and Phrases: probability theory, stochastic calculus, finance, option pricing.

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1 Introduction

The Black-Scholes-Merton model assumes that the price of an asset is the solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

In plain English, the model presumes a constant drift μ and volatility σ. Telling figures indeed, but what do they mean to the asset manager? Notably, when computing the real-life volatility to solve the equation we discover that the volatility is not constant.

Consequently, to make the model more realistic, we consider volatility that fluctuates between two bounds σ_1 and σ_2. The immediate question is how the call price obtained under this model is related to the Black-Scholes-Merton price under the two fixed volatilities. Considering a European call option, we will show that the price is an increasing function of volatility.

One of our results is that the call price is a convex function of the underlying stock price. This fact was proved in a seminal paper by Merton [4], and here we present a different proof. The convexity was explored by Jagannathan [2], who clarified how the volatility of the stock affects the call price. This work was further expanded by Bergman, Grundy, and Wiener [1], who considered what are the effects of changes in the interest rate and volatility on the prices of call options, showing that the Black-Scholes values bound the price at the bounding levels, which is

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also the main theorem of the present work. Our approach is based on [3, Chapter 4, Problem 5], which gives the structure followed here.

Although this is not a new result, the proofs and simulations were made by us. We present simpler proofs with complete calculations to make the subject more accessible. We also show a numerical simulation of the model, computing the price of a call option under it and comparing it with the standard Black-Scholes-Merton and the Cox-Ross-Rubinstein model.

The relevance of the monotonicity of option prices with respect to volatility, as [1] states, is that using it, 'one can then place bounds on the stock position necessary to hedge a given option position using only knowledge of the bounds on the underlying asset's volatility.' These results provide more information to the asset manager, who can use them to find bounds on the option price and then appropriately hedge it. To further study this topic, we could investigate whether monotonicity can reduce the computational cost of trading operations using the boundary values.

The text is organized as follows. We first list the necessary results, and then we give a precise statement of the problem and prove some auxiliary results. With the tools ready, we prove our main result and then present a numerical simulation of the model.

2 Preliminary Concepts

Before heading on, we list some necessary results for our work. We start by presenting the Black-Scholes-Merton model.

Suppose that we have one risky asset S_t and a riskless asset S_t^0 such that

$$dS_t^0 = rS_t^0 dt$$

where $r \geq 0$ is the instantaneous interest rate.

Setting $S_0^0 = 1$, we have $S_t^0 = e^{rt}$. Assume that the following stochastic differential equation determines the behavior of the stock price

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{1}$$

where $\mu > 0$ is the **drift**, and σ is the **volatility** of the stock.

Lemma 2.1. The solution to (1) is

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right) \tag{2}$$

Moreover,

$$\mathbf{E}[S_t] = S_0 e^{\mu t} \quad \text{for} \;\; t \geq 0$$

Proof. See, e.g., [3, Section 3.4.3]. □

Notice that the law of S_t is lognormal and that the hypotheses for this model are the same as the Brownian motion.

Lemma 2.2. Let X and Y be two random variables with values in $(E, \mathfrak{E})$ and $(F, \mathfrak{F})$ respectively. Suppose that X is $\mathfrak{B}$-measurable and that Y is independent of $\mathfrak{B}$. Then, for any non-negative (or bounded) Borel function Ψ on $(E \times F, \mathfrak{E} \otimes \mathfrak{F})$, the function ψ defined by

$$\psi(x) = \mathbf{E}[\Psi(x, Y)], \quad x \in E$$

is a Borel function on $(E, \mathfrak{E})$.

And we have

$$\mathbf{E}[\Psi(X, Y) \mid \mathfrak{B}] = \psi(X) \text{ a.s.}$$

Proof. See, e.g., [3, Proposition A.2.5., p. 240]. □

Proposition 2.3. The option value V_t can be expressed as $V_t = F(t, S_t)$ in which

$$F(t, x) = x\Phi(d_1) - Ke^{-r\theta}\Phi(d_2)$$

for a call and

$$F(t, x) = Ke^{-r\theta}\Phi(-d_2) - x\Phi(-d_1)$$

for a put, where $\Phi(x), d_1$ and d_2 are given by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du$$

and

$$d_1 = \frac{\ln(x/K) + (r + \sigma^2/2)\theta}{\sigma\sqrt{\theta}}, \quad d_2 = d_1 - \sigma\sqrt{\theta} = \frac{\ln(x/K) + (r - \sigma^2/2)\theta}{\sigma\sqrt{\theta}}$$

Proof. See, e.g., [3, Section 4.3.2]. □

Theorem 2.4 (Girsanov)**.** Let (θ_t) be an adapted process satisfying

$$\int_0^T \theta_s^2 \, ds < \infty \text{ a.s.}$$

and such that the process (L_t) given by

$$L_t = \exp\left(-\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right)$$

is a martingale.

Then, under the probability $\mathbf{P}^L$ with density L_T with respect to $\mathbf{P}$, the process (W_t) defined by

$$W_t = B_t + \int_0^t \theta_s \, ds$$

is an $(\mathfrak{F}_t)$-Brownian motion.

Proof. See, e.g., [5, Theorem 5.2.3]. □

Theorem 2.5. Any option defined by a non-negative, $\mathfrak{F}_T$-measurable random variable h in $L^2(\mathbf{P}^*)$ is replicable (in the Black-Scholes model).

The value at time t of any replicating portfolio is given by

$$V_t = \mathbf{E}^*\left[e^{-r(T-t)}h \,\middle|\, \mathfrak{F}_t\right]$$

Proof. See, e.g., [3, Theorem 4.3.2]. □

Hence, option value at t can be naturally defined by $\mathbf{E}^*\left[e^{-r(T-t)}h \mid \mathfrak{F}_t\right]$.

Theorem 2.6 (Discounted Feynman-Kac)**.** Consider the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

Let $h(y)$ be a Borel-measurable function and r be a constant. Fix $T > 0$ and let $t \in [0, T]$. Define

$$g(t,x) = \mathbf{E}^{t,x}[e^{-r(T-t)}h(X_T)]$$

which we suppose to satisfy $\mathbf{E}^{t,x}[|h(X_T)|] < \infty$ for all t and x.

Then $g(t,x)$ satisfies the partial differential equation

$$\frac{\partial g}{\partial t}(t,x) + \mu(t,x)\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 g}{\partial x^2}(t,x) = rg(t,x)$$

and the terminal condition

$$g(T,x) = h(x), \quad \forall\, x$$

Proof. See, e.g., [5, Theorem 6.4.3]. □

3 Analytical Solution

Consider a market consisting of a riskless asset with price $S_t^0 = e^{rt}$ at time t and interest rate r and one risky asset with price S_t at time t. We assume that the stochastic process (S_t) is the solution to

$$dS_t = \mu S_t dt + \sigma(t) S_t dB_t \tag{3}$$

where $\mu \in \mathbf{R}$ and $(\sigma(t))$ is an adapted process with respect to the natural filtration of (B_t) satisfying $\sigma_1 \le \sigma(t) \le \sigma_2$ for all $t \in [0, T]$, with $0 < \sigma_1 < \sigma_2$.

In this market, consider a European call option with maturity T and strike price K. If $\sigma(t) = \sigma$ for all t, then the price of the call at time t is given by $C(t, S_t)$, where the function $C(t,x)$ satisfies

$$\begin{cases} \frac{\partial C}{\partial t}(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 C}{\partial x^2}(t,x) + rx\frac{\partial C}{\partial x}(t,x) - rC(t,x) = 0, & t \in [0,T),\ x > 0 \\ C(T,x) = \max\{x - K,\ 0\} \end{cases} \tag{4}$$

Denote by C_i the function C corresponding to the case $\sigma = \sigma_i$, for $i = 1, 2$. We'll show that the price of the call at time 0 in the model with varying volatility belongs to the interval $[C_1(0, S_0),\ C_2(0, S_0)]$. To show that, we divide the proof into eight steps as follows. The first one is to show that the call prices are convex as a function of the underlying asset.

Lemma 3.1. The functions $x \longmapsto C_i(t,x)$, for $i = 1, 2$, are convex.

Proof. We aim to show that the gamma, i.e., the second derivative of $C_i(t,x) = F(t,x)$ is positive.

We start by computing the first derivative. By the proposition 2.3,

$$\frac{\partial F}{\partial x} = \Phi(d_1) + x\Phi'(d_1)\frac{\partial d_1}{\partial x} - Ke^{-r\theta}\Phi'(d_2)\frac{\partial d_2}{\partial x} \tag{5}$$

To simplify that identity, remark that

$$\Phi'(d) = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2}\, \mathrm{d}x\right)' = \frac{1}{\sqrt{2\pi}} e^{-d^2/2} \tag{6}$$

and, since $d_2 = d_1 - \sigma\sqrt{\theta}$,

$$\frac{\partial d_2}{\partial x} = \frac{\partial d_1}{\partial x} \tag{7}$$

Let us evaluate

$$\begin{aligned}
\Phi'(d_1) &= \Phi'(d_2 + \sigma\sqrt{\theta}) \\
&\overset{(6)}{=} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_2 + \sigma\sqrt{\theta})^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2}{2} - \frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2}{2}\right) \exp\left(-\frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right) \\
&\overset{(6)}{=} \Phi'(d_2) \exp\left(-\frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right)
\end{aligned} \tag{8}$$

Now notice that

$$d_2 = \frac{\ln(x/\mathrm{K}) + (r - \sigma^2/2)\theta}{\sigma\sqrt{\theta}} = \frac{\ln(xe^{r\theta}/\mathrm{K}) - \sigma^2\theta/2}{\sigma\sqrt{\theta}}$$

is equivalent to

$$-\ln\left(\frac{xe^{r\theta}}{\mathrm{K}}\right) = -\sigma\sqrt{\theta}d_2 - \frac{\sigma^2\theta}{2} \iff \frac{\mathrm{K}}{xe^{r\theta}} = \exp\left(-\frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right) \tag{9}$$

Using (9) in (8), we obtain

$$\Phi'(d_1) = \Phi'(d_2)\frac{\mathrm{K}}{xe^{r\theta}} \iff x\Phi'(d_1) = \mathrm{K}e^{-r\theta}\Phi'(d_2) \tag{10}$$

Now replacing (7) and (10) in (5),

$$\frac{\partial \mathrm{F}}{\partial x} = \Phi(d_1) + \mathrm{K}e^{-r\theta}\Phi'(d_2)\frac{\partial d_1}{\partial x} - \mathrm{K}e^{-r\theta}\Phi'(d_2)\frac{\partial d_1}{\partial x} = \Phi(d_1)$$

Deriving the expression above, we have

$$\frac{\partial^2 \mathrm{F}}{\partial x^2} = \Phi'(d_1)\frac{\partial d_1}{\partial x}$$

Now

$$d_1 = \frac{\ln(xe^{r\theta}/\mathrm{K}) + \sigma^2\theta/2}{\sigma\sqrt{\theta}} \implies \frac{\partial d_1}{\partial x} = \frac{1}{x\sigma\sqrt{\theta}}$$

Hence,

$$\frac{\partial^2 \mathrm{F}}{\partial x^2} = \frac{\Phi'(d_1)}{x\sigma\sqrt{\theta}}$$

Since the expression above is positive, the function is convex. □

The second step is to find the solution of the (3).

Lemma 3.2. The solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma(t) S_t dB_t$$

is

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2} \int_0^t \sigma^2(s)\, ds + \int_0^t \sigma(s)\, dB_s \right)$$

Proof. Let

$$dS_t = S_t dY_t$$

with

$$dY_t = \mu dt + \sigma_t dB_t$$

Notice that

$$Y_t = \int_0^t \mu\, ds + \int_0^t \sigma_s\, dB_s$$

Writing $S_t = g(t, Y_t)$, by Itô's formula,

$$\begin{aligned} g dY_t = dS_t &= \frac{\partial g}{\partial t} dt + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dY_t)^2 + \frac{\partial g}{\partial x} g dY_t \\ &= \left(\frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \sigma_t^2 \right) dt + \frac{\partial g}{\partial x} g dY_t \end{aligned}$$

For the last term, we have

$$\frac{\partial g}{\partial x} = g \implies g = S_0 e^{b(t)+x}$$

And for the dt term,

$$\frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \sigma_t^2 = 0$$

Computing $b(t)$,

$$\frac{db}{dt} + \frac{1}{2} \sigma_t^2 = 0 \implies b = -\frac{1}{2} \int_0^t \sigma_s^2\, ds$$

Replacing into g, we finish the proof:

$$\begin{aligned} S_t &= S_0 \exp\left(-\frac{1}{2} \int_0^t \sigma_s^2\, ds + \int_0^t \mu\, ds + \int_0^t \sigma_s\, dB_s \right) \\ &= S_0 \exp\left(\int_0^t \left(\mu - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^t \sigma_s\, dB_s \right) \end{aligned}$$

□

Now, we apply Girsanov's Theorem to $\theta = \frac{\mu - r}{\sigma_s}$ to obtain an equivalent probability under which the process below is a standard Brownian motion.

Lemma 3.3. There exists a probability $\mathbf{P}^*$ equivalent to $\mathbf{P}$ under which the process defined by

$$W_t = B_t + \int_0^t \frac{\mu - r}{\sigma_s}\, ds$$

is a standard Brownian motion under $\mathbf{P}^*$.

Proof. Since $\sigma_s > 0$,

$$\int_0^T \left(\frac{\mu - r}{\sigma_s}\right)^2 ds = (\mu - r)^2 \int_0^T \frac{1}{\sigma_s^2}\, ds < \infty \text{ a.s.}$$

Let us verify that

$$L_t = \exp\left(-\int_0^t \frac{\mu - r}{\sigma_s}\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma_s}\right)^2 ds\right)$$

is a martingale.

Let $L_t = e^{Y_t}$ with

$$Y_t = -\int_0^t \frac{\mu - r}{\sigma_s}\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma_s}\right)^2 ds$$

Letting $g(t,x) = e^x$, we have $L_t = g(t, Y_t)$. By Itô's formula,

$$\begin{aligned}
dL_t &= \frac{\partial g}{\partial x}(t,x) dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x)(dY_t)^2 \\
&= g(t,x)\left(-\left(\frac{\mu - r}{\sigma_s}\right) dB_s - \frac{1}{2}\left(\frac{\mu - r}{\sigma_s}\right)^2 ds\right) + g(t,x)\frac{1}{2}\left(\frac{\mu - r}{\sigma_s}\right)^2 ds \\
&= -g(t,x)\left(\frac{\mu - r}{\sigma_s}\right) dB_s
\end{aligned}$$

Hence, L_t is a martingale. By Girsanov's Theorem, there exists a probability $\mathbf{P}^*$, equivalent to $\mathbf{P}$, with density L_T, and such that the process W_t is a standard Brownian motion under $\mathbf{P}^*$. □

Lemma 3.4. The price of the call at time 0 is given by

$$C_0 = \mathbf{E}^*[e^{-rT} \max\{S_T - K,\ 0\}]$$

Proof. Let h be a European call, i.e., $h = \max\{S_T - K,\ 0\}$. Note that h is a non-negative $\mathfrak{F}_T$-measurable random variable in $L^2(\mathbf{P}^*)$.

Thus, by the Theorem 2.5, we have that the option value at t can be naturally defined as $C_t = \mathbf{E}^*\left[e^{-r(T-t)} h \mid \mathfrak{F}_t\right]$.

Taking $t = 0$, the price of the call is

$$\mathbf{E}^*\left[e^{-rT} \max\{S_T - K,\ 0\}\right]$$

□

Lemma 3.5. Let $\tilde{S}_t = e^{-rt} S_t$. Then $\mathbf{E}^*[\tilde{S}_t^2] \le S_0^2 e^{\sigma^2 t}$.

Proof. Using that $dS_t = \mu S_t dt + \sigma_t S_t dB_t$, by Itô's formula we let $g(t,x) = e^{-rt}x$, $\tilde{S}_t = g(t, S_t)$ and obtain

$$
\begin{aligned}
d\tilde{S}_t &= -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= -re^{-rt}S_t dt + e^{-rt}(\mu S_t dt + \sigma_t S_t dB_t) \\
&= (\mu - r)e^{-rt}S_t dt + \sigma_t e^{-rt}S_t dB_t \\
&= \tilde{S}_t[(\mu - r)dt + \sigma_t dB_t]
\end{aligned}
\tag{11}
$$

Using that $W_t = B_t + \int_0^t \frac{(\mu-r)u}{\sigma_u}du$,

$$
dW_t = dB_t + \frac{(\mu - r)t}{\sigma_t}dt \iff \sigma_t dW_t = \sigma_t dB_t + (\mu - r)dt \tag{12}
$$

Putting (12) into (11),

$$
d\tilde{S}_t = \tilde{S}_t \sigma_t dW_t
$$

Thus, $\tilde{S}_t$ is a $\mathbf{P}^*$-martingale.

By Itô's formula,

$$
\tilde{S}_t^2 = S_0^2 + 2\int_0^t S_u \, dS_u + \int_0^t d[S_u, S_u]
$$

Since $d[S_t, S_t] = S_t^2 \sigma_t^2 dt$, we have

$$
\mathbf{E}^*[\tilde{S}_t^2] = S_0^2 + \int_0^t \mathbf{E}^*[S_u^2]\sigma_u^2 \, du
$$

By Gronwall's inequality,

$$
\mathbf{E}^*[\tilde{S}_t^2] \leq S_0^2 e^{\int_0^t \sigma_u^2 \, du}
$$

Using that $\sigma_t < \sigma$, the result follows. $\square$

Lemma 3.6. The process defined by

$$
M_t = \int_0^t e^{-ru}\frac{\partial C_1}{\partial x}(u, S_u)\sigma_u S_u \, dW_u
$$

is a martingale under probability $\mathbf{P}^*$.

Proof. Notice that e^{-ru} is bounded because $u \geq 0$ and $r \geq 0$, σ_u is bounded by hypothesis, and S_u is bounded by the previous lemma. Now, from the 3.1, we know that $\frac{\partial C_1}{\partial x} = \Phi(d_1)$ is bounded. Thus, M_t is an Itô integral, and M_t is a $\mathbf{P}^*$-martingale.

Here, an alternative way to prove that $\frac{\partial C_1}{\partial x}$ is bounded is presented.

We know that $C_1(t, S_t) = \mathbf{E}^*[e^{-r(T-t)}f(S_T) \mid \mathfrak{F}_t]$, where $f(x) = \max\{x - K,\ 0\}$.

Using the solution (2) and that $B_t = W_t - \frac{(\mu-r)}{\sigma_1}t$,

$$
S_t = S_0 e^{\left((\mu - \frac{1}{2}\sigma_1^2)t + \sigma_1 B_t\right)} = S_0 e^{\left((r - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_t\right)}
$$

Replacing S_T in $C_1(t,S_t)$, we obtain

$$\begin{aligned} C_1(t,S_t) &= \mathbf{E}^*[e^{-r(T-t)}f(S_0 e^{((r-\frac{1}{2}\sigma_1^2)T+\sigma_1 W_T)}) \mid \mathfrak{F}_t] \\ &= \mathbf{E}^*\left[e^{-r(T-t)}f\left(S_t e^{r(T-t)}e^{\sigma_1(W_T-W_t)}e^{-(\sigma_1^2/2)(T-t)}\right) \Big| \mathfrak{F}_t\right] \end{aligned}$$

Since S_t is $\mathfrak{F}_t$-measurable and, under $\mathbf{P}^*$, $W_T - W_t$ is independent of $\mathfrak{F}_t$, by the lemma 2.2 we have that $C_1(t,S_t) = F(t,S_t)$, where

$$F(t,x) = \mathbf{E}^*\left[e^{-r(T-t)}f\left(xe^{r(T-t)}e^{\sigma_1(W_T-W_t)}e^{-(\sigma_1^2/2)(T-t)}\right)\right]$$

We know that $(W_T - W_t)$ has a normal distribution with mean zero and variance $T-t$ under the probability $\mathbf{P}^*$. Thus,

$$F(t,x) = e^{-r(T-t)}\int_{\mathbf{R}} f\left(xe^{(r-\sigma_1^2/2)(T-t)+\sigma_1 y\sqrt{T-t}}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

Notice that we can simplify this expression as follows

$$F(t,x) = e^{-r(T-t)}\int_{\mathbf{R}} f(y)p(x,y,T-t)\,dy$$

where $p(x,y,t)$ is the transition density of the process from x to y over a time interval of length t.

□

4 Main Result

With the previous lemmas, we are ready to prove our main result. First we show that the process $(e^{-rt}C_1(t,S_t))$ is a submartingale and $C_1(0,S_0) \le C_0$. Very similarly, we also show that $(e^{-rt}C_2(t,S_t))$ is a supermartingale and $C_0 \le C_2(0,S_0)$. Whence it follows that that $C_1(0,S_0) \le C_0 \le C_2(0,S_0)$, as desired.

Theorem 4.1. The process $(e^{-rt}C_1(t,S_t))$ is a submartingale under the probability measure $\mathbf{P}^*$. Furthermore, at time zero, the call price under varying volatility is greater or equal to the call price under the Black-Scholes-Merton model and volatility σ_1, i.e., $C_1(0,S_0) \le C_0$.

Proof. Our first goal is to prove that

$$\mathbf{E}^*[e^{-rt}C_1(t,S_t) \mid \mathfrak{F}_u] \ge e^{-ru}C_1(u,S_u), \quad t > u$$

Let $g(t,x) = e^{-rt}C_1(t,x)$ and $X_t = g(t,S_t)$. By Itô's formula,

$$dX_t = \left(-re^{-rt}C_1(t,S_t) + e^{-rt}\frac{\partial C_1}{\partial t}(t,S_t)\right)dt + e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)dS_t + \frac{1}{2}e^{-rt}\frac{\partial^2 C_1}{\partial x^2}(t,S_t)(dS_t)^2$$

Replacing $dS_t = \mu S_t dt + \sigma_t S_t dB_t$ and $(dS_t)^2 = \sigma_t^2 S_t^2 dt$,

$$\begin{aligned} dX_t = &\left(-re^{-rt}C_1(t,S_t) + e^{-rt}\frac{\partial C_1}{\partial t}(t,S_t)\right)dt + e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)(\mu S_t dt + \sigma_t S_t dB_t) \\ &+ \frac{1}{2}e^{-rt}\frac{\partial^2 C_1}{\partial x^2}(t,S_t)\sigma_t^2 S_t^2 dt \end{aligned}$$

Organizing yields

$$dX_t = e^{-rt}\left(-rC_1(t,S_t) + \frac{\partial C_1}{\partial t}(t,S_t) + \mu S_t\frac{\partial C_1}{\partial x}(t,S_t) + \frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 C_1}{\partial x^2}(t,S_t)\right)dt + e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_t S_t dB_t \tag{13}$$

From the proof of the Lemma 3.5, we know that $\sigma_t dB_t = \sigma_t dW_t - (\mu - r)dt$. Thus,

$$\begin{aligned} e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)S_t\sigma_t dB_t &= e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)S_t(\sigma_t dW_t - (\mu - r)dt) \\ &= e^{-rt}\sigma_t S_t\frac{\partial C_1}{\partial x}(t,S_t)dW_t - e^{-rt}S_t(\mu - r)\frac{\partial C_1}{\partial x}(t,S_t)dt \end{aligned} \tag{14}$$

Replacing (14) into (13) and simplifying

$$dX_t = e^{-rt}\left(-rC_1(t,S_t) + \frac{\partial C_1}{\partial t}(t,S_t) + rS_t\frac{\partial C_1}{\partial x}(t,S_t) + \frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 C_1}{\partial x^2}(t,S_t)\right)dt + e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_t S_t dW_t$$

Now we write

$$dX_t = e^{-rt}\left(-rC_1(t,S_t) + \frac{\partial C_1}{\partial t}(t,S_t) + rS_t\frac{\partial C_1}{\partial x}(t,S_t) + \frac{1}{2}\sigma_1^2 S_t^2\frac{\partial^2 C_1}{\partial x^2}(t,S_t)\right)dt + \frac{1}{2}e^{-rt}(\sigma_t^2 - \sigma_1^2)S_t^2\frac{\partial^2 C_1}{\partial x^2}(t,S_t)dt + e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_t S_t dW_t$$

By the equation (4), this simplifies to

$$dX_t = \frac{1}{2}e^{-rt}(\sigma_t^2 - \sigma_1^2)S_t^2\frac{\partial^2 C_1}{\partial x^2}(t,S_t)dt + e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_t S_t dW_t \tag{15}$$

Since $C_1(t,x)$ is convex as function of x (from the Lemma 3.1), $\frac{\partial^2 C_1}{\partial x^2}(t,S_t) > 0$, and using that $\sigma(t) > \sigma_1$, we have $(\sigma_t^2 - \sigma_1^2) > 0$. Therefore,

$$\frac{1}{2}e^{-rt}(\sigma_t^2 - \sigma_1^2)S_t^2\frac{\partial^2 C_1}{\partial x^2}(t,S_t) > 0$$

By the Lemma 3.6, $\int_0^t e^{-ru}\frac{\partial C_1}{\partial x}(u,S_u)\sigma_u S_u \, dW_u$ is a $\mathbf{P}^*$-martingale. Thus,

$$\mathbf{E}^*[X_t \mid \mathfrak{F}_u] \geq X_u$$

Finally, to show that $C_1(0,S_0) \leq C_0$, we use that $C_1(0,S_0)$ is a submartingale. With the Lemma 3.4, the expression inside the expectation is exactly C_0.

$$e^{-r0}C_1(0,S_0) = C_1(0,S_0) \leq \mathbf{E}^*[e^{-rT}C_1(T,S_T) \mid \mathfrak{F}_0] = C_0$$

$\square$

Theorem 4.2. The process $(e^{-rt}C_2(t,S_t))$ is a supermartingale under the probability measure $\mathbf{P}^*$. Moreover, at time zero, the call price under varying volatility is lesser or equal to the call price under Black-Scholes-Merton model and volatility σ_2, i.e., $C_0 \le C_2(0,S_0)$.

Proof. This result will be analogous to the last proof. Define $X_t = e^{-rt}C_2(t,S_t)$. By the equation (15), we have

$$dX_t = \frac{1}{2}e^{-rt}(\sigma_t^2 - \sigma_2^2)S_t^2\frac{\partial^2 C_2}{\partial x^2}(t,S_t)dt + e^{-rt}\frac{\partial C_2}{\partial x}(t,S_t)\sigma_t S_t dW_t$$

Since $(\sigma_t^2 - \sigma_2^2) < 0$, the same argument from the previous proof implies that

$$\mathbf{E}^*[X_t \mid \mathfrak{F}_u] \le X_u$$

Thus, X_t is a supermartingale and it follows that

$$C_0 = \mathbf{E}^*[e^{-rT}C_2(T,S_T) \mid \mathfrak{F}_0] \le C_2(0,S_0)$$

□

5 Numerical Simulation

To illustrate the result, we compute the call price of an underlying asset with stock price $S = 100$, strike price $K = 100$, interest rate $r = 0.06$, maturity $T = 1$ year, and volatility $\sigma = 0.06$. These simulations were made using MatLab.

Figure 1 presents two simulations (the first with volatility $\sigma_1 = 0.09$, and the second with $\sigma_2 = 0.03$) and their average. In Figure 2, the volatility σ is an array of uniformly distributed random numbers between σ_1 and σ_2.

Under these models, it is possible to price a call option as presented in Table 1. The first two rows give the call price under the Cox-Ross-Rubinstein model with ten steps (see [3]) and the standard Black and Scholes, respectively.

In the third and fourth rows, we compute the price in the standard Black and Scholes model with two volatilities: first with $\sigma_1 = 0.09$, and then with $\sigma_2 = 0.03$, as in Figure 1.

Lastly, we compute the stock price under a varying volatility model, in which the volatility σ is an array of uniformly distributed random numbers between $\sigma_1 = 0.09$ and $\sigma_2 = 0.03$, as in Figure 2. We can see that the price under varying volatility, 6.376718, belongs to the interval $(5.848264, 7.142684)$ as desired.

Table 1: Call prices

Method	**Call Price**
Cox-Ross-Rubinstein	6.333048
Black-Scholes (0.06)	6.308527
Black-Scholes (0.09)	7.142684
Black-Scholes (0.03)	5.848264
Varying volatility	6.376718

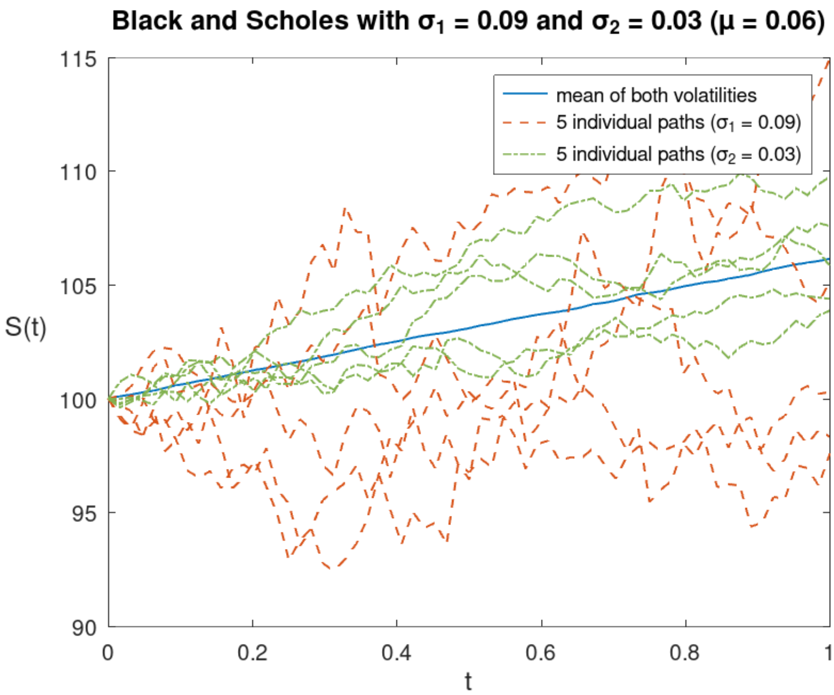
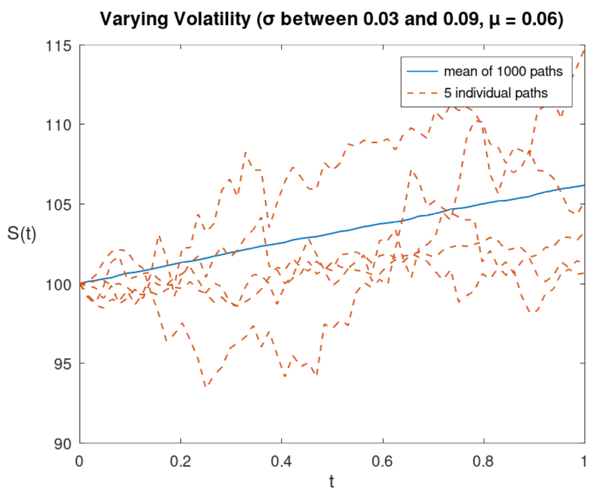
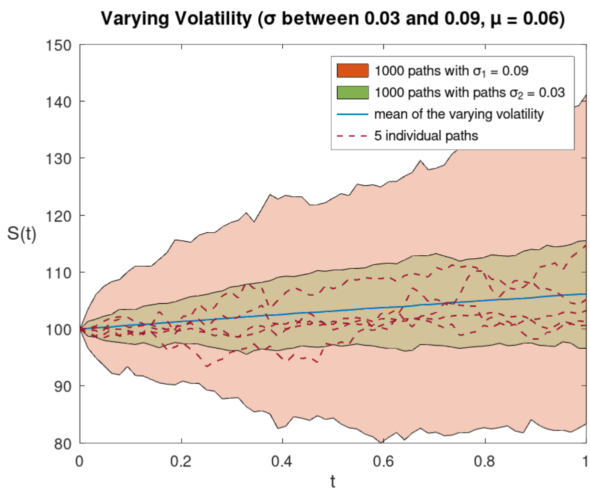


Figure 1: Black and Scholes with $\sigma_1 = 0.09$ and $\sigma_2 = 0.03$



(a) Five individual paths and their mean



(b) Shaded area with simulations under σ_1 and σ_2

Figure 2: Model with varying volatility

6 Conclusion

We present a more general model than Black-Scholes-Merton in which the volatility is not constant but can vary between two bounds. We've shown that the price of the European call option at time zero in this varying volatility model belongs to the interval $[C_1(0, S_0),\ C_2(0, S_0)]$ defined by the standard Black-Scholes-Merton prices. More than that, it is an increasing function of volatility. At the end of the work, we presented an example of the result with numerical simulations and a computation of call prices under four different models.

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