3.7. A famous result of Itô (1951) gives the following formula for n times *iterated Itô integrals*:

$$n! \int \cdots \Big(\int \Big(\int_{0 \le u_1 \le \cdots \le u_n \le t} dB_{u_1} \Big) dB_{u_2} \Big) \cdots dB_{u_n} = t^{\frac{n}{2}} h_n\Big(\frac{B_t}{\sqrt{t}}\Big) \qquad (3.3.8)$$

where h_n is the *Hermite polynomial* of degree n, defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\big(e^{-\frac{x^2}{2}}\big)\ ; \qquad n = 0, 1, 2, \ldots$$

(Thus $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$.)

a) Verify that in each of these n Itô integrals the integrand satisfies the requirements in Definition 3.1.4.

Recall that

Definition 3.1.4. *Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbf{R}$$

such that

(i) $(t,\omega) \to f(t,\omega)$ *is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.*
(ii) $f(t,\omega)$ *is $\mathcal{F}_t$-adapted.*
(iii) $E[\int_S^T f(t,\omega)^2 dt] < \infty$.

Since $f(t,\omega) = 1$, f is $\mathcal{B} \times \mathcal{F}$-measurable and $\mathcal{F}_t$-adapted.

Moreover,

$$E\Big[\int_S^T dt\Big] = E[t - s] = t - s < \infty$$

b) Verify formula (3.3.8) for $n = 1, 2, 3$ by combining Example 3.1.9 and Exercise 3.2.

For $n = 1$, we have

$$1! \int_{0 \le u_1 \le t} dB_{u_1} = B_t = t^{1/2} h_1\Big(\frac{B_t}{\sqrt{t}}\Big) = t^{1/2} \cdot \frac{B_t}{t^{1/2}}$$

For $n = 2$,

$$\int_0^t \int_0^u dB_w \, dB_u = \int_0^t B_u \, dB_u \overset{*}{=} \frac{1}{2}(B_u^2 - u)\Big|_0^t$$

* Example 3.1.9

$$= \frac{1}{2}(B_t^2 - t) = \frac{t^{2/2}}{2}\Big(\Big(\frac{B_t}{\sqrt{t}}\Big)^2 - 1\Big)$$

$$= \frac{t^{2/2}}{2!} h_2\Big(\frac{B_t}{\sqrt{t}}\Big)$$

For $n=3$:

$$\int_0^t\int_0^u\int_0^v dB_r\,dB_v\,dB_u = \int_0^t\int_0^u B_v\,dB_v\,dB_u$$

$$= \int_0^t \frac{1}{2}(B_v^2 - v)\Big|_0^u dB_u = \int_0^t \frac{1}{2}(B_u^2 - u)\,dB_u$$

$$= \frac{1}{2}\left[\int_0^t B_u^2\,dB_u - \int_0^t u\,dB_u\right]$$

$$\overset{*}{=} \frac{1}{2}\left[\frac{1}{3}B_t^3 - \int_0^t B_u\,du - \int_0^t u\,dB_u\right] \qquad * \text{ Exercise 3.2}$$

$$\overset{*}{=} \frac{1}{3!}B_t^3 - \frac{tB_t}{2} = \frac{t^{3/2}}{3!}\left[\left(\frac{B_t}{\sqrt{t}}\right)^3 - 3\frac{B_t}{\sqrt{t}}\right]$$

$$= \frac{t^{3/2}}{3!}h_3\left(\frac{B_t}{\sqrt{t}}\right)$$

* Since

$$\int_0^t B_s\,ds + \int_0^t s\,dB_s = tB_t$$

EXERCISE 3.1.

c) Use b) to give a new proof of the statement in Exercise 3.6.

Since

$$\int_0^t (B_s^2 - s)\,dB_s = t h_1\left(\frac{B_t}{\sqrt{t}}\right)$$

is a martingale (the Integral of B.m. is martingale), we have what we wanted.

$\square$