5.12. To describe the motion of a pendulum with small, random perturbations in its environment we try an equation of the form

$$y''(t) + (1 + \epsilon W_t)y = 0\,; \qquad y(0), y'(0) \text{ given },$$

where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise, $\epsilon > 0$ is constant.

a) Discuss this equation, for example by proceeding as in Example 5.1.3.

b) Show that $y(t)$ solves a *stochastic Volterra equation* of the form

$$y(t) = y(0) + y'(0) \cdot t + \int_0^t a(t,r)y(r)dr + \int_0^t \gamma(t,r)y(r)dB_r$$

where $a(t,r) = r - t$, $\gamma(t,r) = \epsilon(r - t)$.

a) Define

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

Therefore,

$$\begin{cases} X_1'(t) = X_2(t) \\ X_2' = -(1+\varepsilon W_t) X_1 \end{cases}$$

In matrix notation,

$$dX_t = AX_t dt + CX_t dB_t$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ -\varepsilon & 0 \end{bmatrix}$$

Now,

$$\exp(-At)dX_t = \exp(-At)AX_t dt + \exp(-At)CX_t dB_t$$

By Itô's formula,

$$d(\exp(-At)X_t) = -A\exp(-At)X_t dt + \exp(-At)dX_t$$

Hence,

$$d(\exp(-At)X_t) = \exp(-At)CX_t dB_t$$

I.e.,

$$\exp(-At)X_t = X_0 + \int_0^t \exp(-As)CX_s\,dB_s$$

$$\iff \quad X_t = \exp(At)\left[X_0 + \int_0^t \exp(-As)CX_s\,dB_s\right]$$

b) Rewriting the solution

$$\boxed{\frac{d}{dx}\left(\int_a^x f(x,t)dt\right) = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,t)dt}$$

Using the Leibniz rule,

$$y'(t) = y'(0) - \int_0^t y(r)\,dr - \varepsilon\int_0^t y(r)\,dB_r$$

Since $W_t = \frac{dB_t}{dt}$, we have

$$y''(t) = -y(t) - \varepsilon y(t)W_t = -y(t)(1+\varepsilon W_t)$$

I.e.,

$$y''(t) + (1+\varepsilon W_t)y(t) = 0$$

Another way: notice that

$$Y_t = y_0 + y_0' t + \int_0^t (r-t)y_r\,dr + \int_0^t \varepsilon(r-t)y_r\,\underline{\omega_r\,dr}$$

$dB_t = \omega_t dt$

$$= y_0 + y_0' t + \int_0^t r\underbrace{(1+\varepsilon\omega_r)y_r}_{-y_r''}\,dr - \int_0^t t\underbrace{(1+\varepsilon\omega_r)y_r}_{-y_r''}\,dr$$

$$= y_0 + y_0' t - r y_r' \Big|_0^t + \int_0^t y_r' \, dr + t y_r' \Big|_0^t$$

$$= \cancel{y_0} + \cancel{y_0' t} - \cancel{t y_t'} + y_t - \cancel{y_0} + \cancel{t y_t'} - \cancel{t y_0'} = y_t \quad \checkmark$$