ØKSENDAL 3.2

We want to prove that
$$\int_0^t B_s^2 \, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

First, let
$$\varphi_n(s) = \sum_j B_j^2 \, \chi_{[t_j, t_{j+1})}(s)$$

Notice that
$$E\left[\int_0^t (\varphi_n - B_s^2)^2 \, ds\right] = E\left[\sum_j \int_{t_j}^{t_{j+1}} (B_j^2 - B_s^2)^2 \, ds\right]$$
$$= \sum_j \int_{t_j}^{t_{j+1}} (s^2 - t_j^2) \, ds = \sum_j \frac{1}{3}(t_{j+1} - t_j)^3 \to 0$$

as $\Delta t_j \to 0$.

Hence,
$$\int_0^t B_s^2 \, dB_s = \lim_{\Delta t_j \to 0} \int_0^t \varphi_n(s) \, dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j^2 \Delta B_j$$

Using the identity
$$B_j^2 (B_{j+1} - B_j) = \frac{1}{3}(B_{j+1}^3 - B_j^3) - B_j (B_{j+1} - B_j)^2 - \frac{1}{3}(B_{j+1} - B_j)^3$$

We obtain
$$\lim_{\Delta t_j \to 0} \sum_j B_j^2 (B_{j+1} - B_j) = \lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1}^3 - B_j^3)$$
$$- \lim_{\Delta t_j \to 0} \sum_j B_j (B_{j+1} - B_j)^2$$
$$- \lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1} - B_j)^3$$

Evaluating these limits,
$$\lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1}^3 - B_j^3) = \frac{1}{3} B_t^3 - \frac{1}{3} B_0^3 = \frac{1}{3} B_t^3$$

- $$\lim_{\Delta t_j \to 0} \sum_j \frac{1}{3}(B_{j+1} - B_j)^3 = 0$$
- $$\lim_{\Delta t_j \to 0} \sum_j B_j (B_{j+1} - B_j)^2 = \lim_{\Delta t_j \to 0} \sum_j B_j (B_{j+1} - B_j)$$
$$+ \lim_{\Delta t_j \to 0} \sum_j B_j \left[(B_{j+1} - B_j)^2 - (B_{j+1} - B_j)\right]$$
$$= \int_0^t B_s \, ds + 0$$

Taking all these parts together:

$$\lim_{\Delta t_j \to 0} \sum_j B_j^2 (B_{j+1} - B_j) = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

Thus,

$$\int_0^t B_s^2 \, dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j^2 \, \Delta B_j$$
$$= \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

$\blacksquare$