7.11. Let X_t be an Itô diffusion in $\mathbf{R}^n$ and let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a function such that

$$E^x\Big[\int_0^\infty |f(X_t)|dt\Big] < \infty \qquad \text{for all } x \in \mathbf{R}^n .$$

Let τ be a stopping time. Use the strong Markov property to prove that

$$E^x\Big[\int_\tau^\infty f(X_t)dt\Big] = E^x[g(X_\tau)] ,$$

where

$$g(y) = E^y\Big[\int_0^\infty f(X_t)dt\Big] .$$

We begin using a change of variables

$$\int_\tau^\infty f(X_t)\,dt = \int_0^\infty f(X_{\tau+s})\,ds$$

And notice that by the law of the iterated expectation,

$$\mathbb{E}^x\left[\int_0^\infty f(X_{\tau+s})\,ds\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[\int_0^\infty f(X_{\tau+s})\,ds \,\Big|\, \mathcal{F}_t\right]\right]$$

Then, using the Strong Markov Property,

$$\mathbb{E}^x\left[\mathbb{E}^x\left[\int_0^\infty f(X_{\tau+s})\,ds \,\Big|\, \mathcal{F}_\tau\right]\right] = \mathbb{E}^x\left[\mathbb{E}^{X_\tau}\left[\int_0^\infty f(X_s)\,ds\right]\right]$$

$$= \mathbb{E}^x\left[g(X_\tau)\right]$$

$\square$