3.16. Let $X: \Omega \to \mathbf{R}$ be a random variable such that $E[X^2] < \infty$ and let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra. Show that

$$E\big[(E[X|\mathcal{H}])^2\big] \le E[X^2]\,.$$

(See Lemma 6.1.1. See also the Jensen inequality for conditional expectation (Appendix B).)

By Jensen Inequality for Conditional Expectation,

Theorem B.4 (The Jensen inequality).
If $\phi: \mathbf{R} \to \mathbf{R}$ is convex and $E[|\phi(X)|] < \infty$ then

$$\phi(E[X|\mathcal{H}]) \le E[\phi(X)|\mathcal{H}]\,.$$

Since $\phi(x) = x^2$ is convex,

$$\mathbb{E}[X|\mathcal{H}]^2 \le \mathbb{E}[X^2|\mathcal{H}]$$

Using the Law of Total Expectation, i.e.,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]] = \mathbb{E}[X]$$

and taking the expectations

$$\mathbb{E}[(\mathbb{E}[X|\mathcal{H}])^2] \le \mathbb{E}[\mathbb{E}[X^2|\mathcal{H}]] = \mathbb{E}[X^2]$$