7.5. Let the functions b, σ satisfy condition (5.2.1) of Theorem 5.2.1, with a constant C independent of t, i.e.

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|) \qquad \text{for all } x \in \mathbf{R}^n \text{ and all } t \ge 0\,.$$

Let X_t be a solution of

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t\,.$$

Show that

$$E[|X_t|^2] \le (1 + E[|X_0|^2])e^{Kt} - 1$$

for some constant K independent of t.
(Hint: Use Dynkin's formula with $f(x) = |x|^2$ and $\tau = t \wedge \tau_R$, where $\tau_R = \inf\{t > 0;\ |X_t| \ge R\}$, and let $R \to \infty$ to achieve the inequality

$$E[|X_t|^2] \le E[|X_0|^2] + K \cdot \int_0^t (1 + E[|X_s|^2])ds\,,$$

which is of the form (5.2.9).)

Let $\tau_R = \inf\{t>0 : |X_t| \geq R\}$ and $\tau = \min\{t, \tau_R\}$.

By Dynkin's formula applied to $f(x) = |x|^2$,

$$\mathbb{E}[|X_\tau|^2] = \mathbb{E}[|X_0|^2] + \mathbb{E}\left[\int_0^\tau A|X_s|^2 ds\right] \qquad (1)$$

Using the following fact

$$A|X_s|^2 \leq K(1+|X_s|^2) \qquad (*)$$

then

$$\mathbb{E}\left[\int_0^\tau A|X_s|^2 ds\right] \leq \mathbb{E}\left[\int_0^\tau K(1+|X_s|^2)\, ds\right]$$

$$= K\,\mathbb{E}\left[\int_0^\tau (1+|X_s|^2)\, ds\right]$$

$$= K\int_0^\tau (1+\mathbb{E}[|X_s|^2])\, ds \qquad (3)$$

By (1) and (3),

$$\mathbb{E}\left[|X_\tau|^2\right] \leqslant \mathbb{E}\left[|X_0|^2\right] + K\int_0^\tau \left(1+\mathbb{E}\left[|X_s|^2\right]\right) ds$$

To use the Gronwall inequality, write

$$1+\mathbb{E}\left[|X_\tau|^2\right] \leqslant 1+\mathbb{E}\left[|X_0|^2\right] + K\int_0^\tau \left(1+\mathbb{E}\left[|X_s|^2\right]\right) ds$$

Thus,

$$\mathbb{E}\left[|X_\tau|^2\right] \leqslant \left(1+\mathbb{E}\left[|X_0|^2\right]\right) e^{K\tau} - 1$$

$\square$

Proof of (*):

Start by writing

$$Af(x) = 2\sum_{i=1}^{n} b_i(t,x)\,x_i + \sum_{i=1}^{n} \sigma_i^2(t,x)$$

Since

$$2\sum_{i=1}^{n} b_i(t,x)\,x_i \leqslant \sum_{i=1}^{n} |b_i|^2 + \sum_{i=1}^{n} |x_i|^2 = |b|^2 + |x|^2$$

we obtain

$$Af(x) \leqslant |b|^2 + |x|^2 + |\sigma|^2$$

By the conditions (5.2.1.),

$$Af(x) \leqslant C^2(1+|x|)^2 + |x|^2 \quad = C^2(1+2|x|+|x|^2) + |x|^2$$

$$= C^2 + 2C^2|x| + (C^2+1)|x|^2$$

Since $(|x|+1)^2 = |x|^2 + 1 + 2|x|$, we have $2C^2|x| \leqslant C^2|x|^2 + C^2$.
Thus,

$$Af(x) \leqslant 2C^2 + (2C^2+1)|x|^2$$

$$\leqslant K(1+|x|^2)$$

by taking $K > (2C^2+1)$.