4.1. Use Itô's formula to write the following stochastic processes X_t on the standard form

$$dX_t = u(t,\omega)dt + v(t,\omega)dB_t$$

for suitable choices of $u \in \mathbf{R}^n$, $v \in \mathbf{R}^{n\times m}$ and dimensions n, m:

a) $X_t = B_t^2$, where B_t is 1-dimensional
b) $X_t = 2 + t + e^{B_t}$ (B_t is 1-dimensional)
c) $X_t = B_1^2(t) + B_2^2(t)$ where (B_1, B_2) is 2-dimensional
d) $X_t = (t_0 + t, B_t)$ (B_t is 1-dimensional)
e) $X_t = (B_1(t)+B_2(t)+B_3(t), B_2^2(t)-B_1(t)B_3(t))$, where (B_1, B_2, B_3) is 3-dimensional.

a) Let $g(t,x) = x^2$. By the Itô's formula,

$$dg(t,X_t) = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2$$

Then,

$$dB_t^2 = 2B_t\,dB_t + dt$$

b) $X_t = 2+t+e^{B_t}$

Let $g(t,x) = 2+t+e^x$. Then,

$$dX_t = dt + e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt = \left(1+\frac{1}{2}e^{B_t}\right)dt + e^{B_t}dB_t$$

c) $X_t = B_1^2(t) + B_2^2(t)$

Let $g(t,x) = x_1^2 + x_2^2$. Since

$$\mathrm{d}Y_k = \frac{\partial g_k}{\partial t}(t,X)\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)\mathrm{d}X_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)\mathrm{d}X_i\mathrm{d}X_j \qquad \mathrm{d}B_i\mathrm{d}B_j = \delta_{ij}\mathrm{d}t$$

We have that

$$dX_t = 2B_1(t)dB_1(t) + 2B_2(t)\,dB_2(t) + \frac{1}{2}\left[2dt + 2dt\right]$$

$$= 2dt + 2B_1(t)dB_1(t) + 2B_2(t)\,dB_2(t)$$

d) $X_t = (t_0 + t,\ B_t)$

Let $g(t, x) = (t_0 + t,\ x)$.

Then,

$$dX_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} dt = \begin{bmatrix} dt \\ dB_t \end{bmatrix}$$

e) $X_t = (B_1 + B_2 + B_3,\ B_2^2 - B_1 B_3)$

Let $g(t, x) = (x_1 + x_2 + x_3,\ x_2^2 - x_1 x_3)$. Then,

$$dX_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 \\ -B_3 \end{bmatrix} dB_1 + \begin{bmatrix} 1 \\ 2B_2 \end{bmatrix} dB_2 + \begin{bmatrix} 1 \\ -B_1 \end{bmatrix} dB_3$$

$$+ \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \end{bmatrix} dt \right)$$

$$= \begin{bmatrix} dB_1 + dB_2 + dB_3 \\ -B_3\, dB_1 + 2B_2\, dB_2 - B_1\, dB_3 + dt \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_3 & 2B_2 & -B_1 \end{bmatrix} \cdot \begin{bmatrix} dB_1 \\ dB_2 \\ dB_3 \end{bmatrix}$$