7.10. Let X_t be the geometric Brownian motion

$$dX_t = rX_t dt + \alpha X_t dB_t\ .$$

Find $E^x[X_T|\mathcal{F}_t]$ for $t \le T$ by
a) using the Markov property

By the Markov property,

$$\mathbb{E}^x[X_T | \mathcal{F}_t] = \mathbb{E}^{X_t}[X_{T-t}]$$

Given that X_t is of the form

$$X_t = x + \int_0^t rX_s ds + \int_0^t \alpha X_s dB_s$$

applying the expectation operator gives

$$\mathbb{E}^x[X_t] = x + r\int_0^t \mathbb{E}^x[X_s] ds$$

Note that this function $\mathbb{E}^x[X_t]$ evaluated at zero must return x. This inspires us to build the following initial value problem.

Let

$$f(t) = x + r\int_0^t \mathbb{E}^x[X_s] ds = x + r\int_0^t f(s) ds$$

then our problem is

$$f'(t) = rf(t), \quad \text{with } f(0) = x$$

with solution

$$\mathbb{E}^x[X_t] = f(t) = x e^{rt}$$

Thus,

$$\mathbb{E}^{X_t}[X_{T-t}] = X_t e^{r(T-t)}$$

and

b) writing $X_t = x\, e^{rt} M_t$, where

$$M_t = \exp(\alpha B_t - \tfrac{1}{2}\alpha^2 t) \qquad \text{is a martingale .}$$

Notice that

$$\begin{aligned}
\mathbb{E}^x[X_T | \mathcal{F}_t] &= \mathbb{E}^x[x e^{rT} M_T | \mathcal{F}_t] \\
&= x e^{rT}\, \mathbb{E}^x[M_T | \mathcal{F}_t] \\
&= x e^{rT} M_t \\
&= x e^{rT} \frac{1}{x e^{rt}} X_t \\
&= X_t e^{r(T-t)}
\end{aligned}$$