7.13. a) Let $B_t \in \mathbf{R}^2$, $B_0 = x \neq 0$. Fix $0 < \epsilon < R < \infty$ and define

$$X_t = \ln|B_{t\wedge\tau}| \, ; \qquad t \geq 0$$

where

$$\tau = \inf\{t > 0; |B_t| \leq \epsilon \quad \text{or} \quad |B_t| \geq R\} \, .$$

Prove that X_t is an $\mathcal{F}_{t\wedge\tau}$-martingale. (Hint: Use Exercise 4.8.) Deduce that $\ln|B_t|$ is a local martingale (Exercise 7.12).

Let $f(B_t) = \ln|B_t|$. Then $X_t = f(B_{t\wedge\tau})$. Using that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \tfrac{1}{2}\int_0^t \Delta f(B_s) ds \, ,$$

We have

$$f(B_t) = f(B_0) + \int_0^t \left(\frac{(B_1(s), B_2(s))}{|B_s^2|}\right) dB_s$$

$$+ \cancel{\frac{1}{2}\int_0^t \left(\frac{|B_s|^2 - 2B_1^2 + |B_s|^2 - 2B_2^2}{|B_s|^4}\right) ds} \quad 0$$

Thus,

$$f(B_t) = f(B_0) + \int_0^t \frac{B_s}{|B_s^2|} dB_s$$

is a martingale and it follows that X_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.

To show that $\ln|B_t|$ is a local martingale, let

$$\tau_R = \inf\{t > 0 : |B_t| \geq R\}$$

Since $\tau_R \longrightarrow \infty$ as $R \longrightarrow \infty$ and we already showed that $\ln|B_{t\wedge\tau}|$ is a martingale, what remains to be proved is that

$$\tau_\varepsilon = \inf\{t > 0 : |B_t| \leq \varepsilon\}$$

satisfies $\tau_\varepsilon \to \infty$ as $\varepsilon \downarrow 0$. I.e, we need to prove that

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \infty) = \lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \tau_R) = 0$$

Note that

$$\ln|x| = \mathbb{E}^x[\ln|B_{t \wedge \tau}|] = \mathbb{P}^x(\tau_\varepsilon < \tau_R)\ln\varepsilon + \mathbb{P}^x(\tau_\varepsilon > \tau_R)\ln R$$

Hence

$$\ln|x| = \mathbb{P}^x(\tau_\varepsilon < \tau_R)\ln\varepsilon + (1 - \mathbb{P}^x(\tau_\varepsilon < \tau_R))\ln R$$

$$\iff \quad \mathbb{P}^x(\tau_\varepsilon < \tau_R) = \frac{\ln R - \ln|x|}{(\ln R - \ln\varepsilon)}$$

implies that

$$\lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \tau_R) = 0$$

$\square$

b) Let $B_t \in \mathbf{R}^n$ for $n \geq 3$, $B_0 = x \neq 0$. Fix $\epsilon > 0$, $R < \infty$ and define

$$Y_t = |B_{t\wedge\tau}|^{2-n}\ ; \qquad t \geq 0$$

where

$$\tau = \inf\{t > 0;\ |B_t| \leq \epsilon \quad \text{or} \quad |B_t| \geq R\}\ .$$

Prove that Y_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.
Deduce that $|B_t|^{2-n}$ is a local martingale.

Define $f(B_t) = |B_t|^{2-n}$. Then

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s) ds$$

$$= |x|^{2-n} + \int_0^t (2-n)\|B_s\|^{-n} B_s\, dB_s \overset{(*)}{+} \cancel{\frac{1}{2}\int_0^t \Delta f(B_s) ds}$$

Thus,

$$f(B_t) = |x|^{2-n} + \int_0^t (2-n)\|B_s\|^{-n} B_s\, dB_s$$

is a martingale and it follows that Y_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.

To prove that $|B_t|^{2-n}$ is a local martingale we use the same argument as before. Notice that

$$|x|^{2-n} = E^x\left[|B_{t\wedge\tau}|^{2-n}\right] = P^x(\tau_\varepsilon < \tau_R)|\varepsilon|^{2-n} + P^x(\tau_\varepsilon > \tau_R)|R|^{2-n}$$

i.e.,

$$|x|^{2-n} = P^x(\tau_\varepsilon < \tau_R)\,\varepsilon^{2-n} + (1 - P^x(\tau_\varepsilon < \tau_R))R^{2-n}$$

$$\Leftrightarrow \quad P^x(\tau_\varepsilon < \tau_R) = \frac{R^{2-n} - |x|^{2-n}}{(R^{2-n} - \varepsilon^{2-n})} = \frac{1/R^{n-2} - 1/|x|^{n-2}}{1/R^{n-2} - 1/\varepsilon^{n-2}} = \frac{\frac{|x|^{n-2} - R^{n-2}}{R^{n-2}|x|^{n-2}}}{\frac{\varepsilon^{n-2} - R^{n-2}}{R^{n-2}\varepsilon^{n-2}}}$$

$$= \frac{(|x|^{n-2} - R^{n-2})}{|x|^{n-2} R^{n-2}} \cdot \frac{R^{n-2} \varepsilon^{n-2}}{(\varepsilon^{n-2} - R^{n-2})} = \frac{\varepsilon^{n-2}}{|x|^{n-2}} \frac{(|x|^{n-2} - R^{n-2})}{(\varepsilon^{n-2} - R^{n-2})}$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \infty) = \lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \mathbb{P}^x(\tau_\varepsilon < \tau_R) = 0$$

Thus, $|B_t|^{2-n}$ is a local martingale. $\square$

(*)

$$\Delta f(B_s) = \sum_i (2-n)\left[-n \|B_s\|^{-n-2} B_i^2 + \|B_s\|^{-n}\right]$$

$$= \sum_i \frac{(2-n)}{\|B_s\|^n}\left[-n\|B_s\|^{-2} B_i^2 + 1\right]$$