3.11. Let W_t be a stochastic process satisfying (i), (ii) and (iii) (below (3.1.2)). Prove that W_t cannot have continuous paths. (Hint: Consider $E[(W_t^{(N)} - W_s^{(N)})^2]$, where

$$W_t^{(N)} = (-N) \vee (N \wedge W_t),\ N = 1, 2, 3, \dots)\ .$$

Conditions:

(i) $t_1 \neq t_2 \Rightarrow W_{t_1}$ and W_{t_2} are independent.
(ii) $\{W_t\}$ is stationary, i.e. the (joint) distribution of $\{W_{t_1+t}, \dots, W_{t_k+t}\}$ does not depend on t.
(iii) $E[W_t] = 0$ for all t.

Let

$$W_t^{(N)} = \max\{-N, \min\{N, W_t\}\}, \quad N \in \mathbb{Z}^+$$

and suppose, by contradiction, that W_t is continuous. By the Dominated Convergence Theorem, we have

$$\lim_{s \to t} \mathbb{E}\left[(W_t^{(N)} - W_s^{(N)})^2\right] = 0 \qquad (1)$$

and

$$\mathbb{E}\left[W_t^{(N)}\right] = \mathbb{E}\left[W_t\right] = 0 \qquad (2)$$

Since $W_t^{(N)}$ and $W_s^{(N)}$ are i.i.d.,

$$\begin{aligned}
\mathbb{E}\left[(W_t^{(N)} - W_s^{(N)})^2\right] &= \mathbb{E}\left[W_t^{(N)2} - 2W_t^{(N)} W_s^{(N)} + W_s^{(N)2}\right] \\
&= \mathbb{E}\left[(W_t^{(N)})^2\right] - 2\mathbb{E}\left[W_t^{(N)}\right] \cdot \mathbb{E}\left[W_s^{(N)}\right] + \mathbb{E}\left[(W_s^{(N)})^2\right] \\
&= \mathbb{E}\left[(W_t^{(N)})^2\right] - 2\mathbb{E}\left[W_t^{(N)}\right]\mathbb{E}\left[W_t^{(N)}\right] + \mathbb{E}\left[(W_t^{(N)})^2\right] \\
&= 2\mathbb{E}\left[(W_t^{(N)})^2\right] - 2\,\mathbb{E}\left[(W_t^{(N)})\right]^2 \\
&= 2\operatorname{Var}\left[W_t^{(N)}\right]
\end{aligned} \qquad (3)$$

By (1), we know that (3) goes to zero. Hence, $W_t^{(N)}$ is constant, i.e.,

$$W_t^{(N)} = \mathbb{E}\left[W_t^{(N)}\right] = \mathbb{E}\left[W_t\right] = 0$$