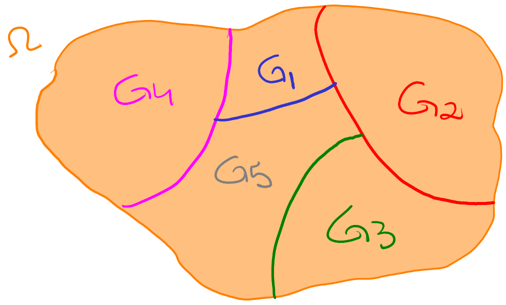
3.17. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X\colon \Omega \to \mathbf{R}$ be a random variable with $E[|X|] < \infty$. If $\mathcal{G} \subset \mathcal{F}$ is a *finite* σ-algebra, then by Exercise 2.7 there exists a partition $\Omega = \bigcup_{i=1}^{n} G_i$ such that $\mathcal{G}$ consists of $\emptyset$ and unions of some (or all) of $G_1, \dots, G_n$.

a) Explain why $E[X|\mathcal{G}](\omega)$ is constant on each G_i. (See Exercise 2.7 c).)



2.7. a) Suppose $G_1, G_2, \dots, G_n$ are disjoint subsets of Ω such that

$$\Omega = \bigcup_{i=1}^{n} G_i \, .$$

Prove that the family $\mathcal{G}$ consisting of $\emptyset$ and all unions of some (or all) of $G_1, \dots, G_n$ constitutes a σ-algebra on Ω.

b) Prove that any *finite* σ-algebra $\mathcal{F}$ on Ω is of the type described in a).

c) Let $\mathcal{F}$ be a *finite* σ-algebra on Ω and let $X\colon \Omega \to \mathbf{R}$ be $\mathcal{F}$-measurable. Prove that X assumes only finitely many possible values. More precisely, there exists a disjoint family of subsets $F_1, \dots, F_m \in \mathcal{F}$ and real numbers $c_1, \dots, c_m$ such that

$$X(\omega) = \sum_{i=1}^{m} c_i \mathcal{X}_{F_i}(\omega) \, .$$

By the exercise 2.7, for $c_i \in \mathbb{R}$, $i = 1, \dots, n$,

$$E[X|\mathcal{G}](\omega) = E\Big[\underbrace{\sum_{i=1}^{n} c_i \mathcal{X}_{G_i}(\omega)}_{\text{constant } C}\Big] = E[C] = C$$

b) Assume that $P[G_i] > 0$. Show that

$$E[X|\mathcal{G}](\omega) = \frac{\int_{G_i} X dP}{P(G_i)} \qquad \text{for } \omega \in G_i \, .$$

$\int\limits_H E[X|\mathcal{H}]dP = \int\limits_H XdP$, *for all* $H \in \mathcal{H}$.

gives

$$\int_{G_i} E[X|\mathcal{G}]\, dP = \int_{G_i} X\, dP$$

however,

$$\int_{G_i} E[X|\mathcal{G}]\, dP = E[X|\mathcal{G}] \int_{G_i} dP = E[X|\mathcal{G}]\, P(G_i)$$

Therefore,

$$E[X|\mathcal{G}] = \frac{\int_{G_i} X\, dP}{P(G_i)}$$

c) Suppose X assumes only finitely many values $a_1, \ldots, a_m$. Then from elementary probability theory we know that (see Exercise 2.1)

$$E[X|G_i] = \sum_{k=1}^{m} a_k P[X = a_k | G_i] \; .$$

Compare with b) and verify that

$$E[X|G_i] = E[X|\mathcal{G}](\omega) \qquad \text{for } \omega \in G_i \; .$$

Thus we may regard the conditional expectation as defined in Appendix B as a (substantial) generalization of the conditional expectation in elementary probability theory.

Using the previous item,

$$\mathbb{E}[X|\mathcal{G}] = \frac{\int_{G_i} X \, d\mathbb{P}}{\mathbb{P}(G_i)} = \frac{\int_{G_i} \sum_{k=1}^{m} a_k \mathcal{X}_{a_k} \, d\mathbb{P}}{\mathbb{P}(G_i)} = \sum_{k=1}^{m} \frac{\int_{G_i} \mathcal{X}_{a_k} d\mathbb{P}}{\mathbb{P}(G_i)}$$

$$= \sum_{k=1}^{m} a_k \frac{\mathbb{P}(G_i \cap G_k)}{\mathbb{P}(G_i)} = \sum_{k=1}^{m} a_k \mathbb{P}[X = a_k | G_i]$$

Alternative answer

b) Since

$$\mathbb{E}[f \mid A] = \int f \, d\mu_A, \quad \mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)}$$

We have

$$\mathbb{E}\left[\sum_{i=1}^{n} c_i \chi_{G_i}(\omega) \mid G_i\right] = \int_{G_i} \frac{X \, d\mathbb{P}(\omega)}{\mathbb{P}(G_i)}$$

c) Given the definition of a_k, $k = 1, \dots, m$,

$$\mathbb{E}[X \mid G_i] = \sum_{k=1}^{m} a_k \mathbb{P}[X = a_k \mid G_i] = \int X \, d\mathbb{P}_{G_i}$$

By the fact that

$$\mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)}$$

we obtain

$$\mathbb{E}[X \mid G_i] = \int X \, d\mathbb{P}_{G_i} = \int_{G_i} \frac{X \, d\mathbb{P}}{\mathbb{P}(G_i)} = \mathbb{E}[X \mid \mathcal{G}]$$