Øksendal 3.4

i) $X_t = B_t + 4t$

1. X_t is $\mathcal{F}_t$-measurable for all t since B_t is.
2. $\mathbb{E}[|X_t|] = \mathbb{E}[|B_t + 4t|] \leq \mathbb{E}[|B_t|] + \mathbb{E}[|4t|] < \infty$

since B_t is a martingale and $|4t|$ is a constant.

3. $\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad s \leq t$

Notice that

$$\mathbb{E}[B_t + 4t | \mathcal{F}_s] = B_s + 4t \neq X_s \qquad \therefore \text{Not a martingale}$$

ii) $X_t = B_t^2 \qquad B_t^{2(-1)}(U) = \{\omega \in \Omega : B_t^2(\omega) \in U\} \subseteq \mathcal{F}_t$

1. X_t is $\mathcal{F}_t$-measurable
2. $\mathbb{E}[|X_t|] = \mathbb{E}[|B_t^2|]$ may not be finite
3. $\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad s \leq t$

$$\mathbb{E}[B_t^2 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2 | \mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + 2\mathbb{E}[(B_t - B_s)B_s | \mathcal{F}_s] + \mathbb{E}[B_s^2 | \mathcal{F}_s]$$

$$= t - s + B_s^2 \neq B_s^2 \quad (s < t)$$

$\therefore$ Not a martingale

iii) $X_t = t^2 B_t - 2\int_0^t s B_s \, ds$

1. X_t is $\mathcal{F}_t$-measurable.
2. $\mathbb{E}[|X_t|] \leq \mathbb{E}[|t^2 B_t|] + 2\mathbb{E}\left[\int_0^t s B_s \, ds\right] < \infty$
3. $\mathbb{E}[X_t | \mathcal{F}_s] = \mathbb{E}\left[t^2 B_t - 2\int_0^t s B_s \, ds \,\middle|\, \mathcal{F}_s\right]$

$$= t^2 B_s - 2\int_0^s s B_s \, ds - 2\mathbb{E}\left[\int_s^t r B_r \, dr \,\middle|\, \mathcal{F}_s\right]$$

$$= t^2 B_s - 2\int_0^s s B_s \, ds - 2\int_s^t r\, \mathbb{E}[B_r | \mathcal{F}_s] \, ds$$

$$= t^2 B_s - 2\int_0^s s B_s \, ds - 2\int_s^t r B_s \, ds$$

$$= t^2 B_s - 2\int_0^s sB_s\,ds - B_s(t^2 - s^2) = s^2 B_s - 2\int_0^s sB_s\,ds$$

$\therefore$ Is a martingale

iv.) $X_t = B_1(t)B_2(t)$ where $(B_1(t), B_2(t))$ is a 2-dim B.m.

1. X_t is $\mathcal{F}_t$-measurable since $B_1(t)$ and $B_2(t)$ is.
2. $\mathbb{E}[|X_t|] = \mathbb{E}[|B_1(t)|] \cdot \mathbb{E}[|B_2(t)|] < \infty$
3. $\mathbb{E}[X_t | \mathcal{F}_s] = B_1(s) \cdot B_2(s) = X_s,\quad s \leq t$

$\therefore$ Is a martingale