3.13. A stochastic process $X_t(\cdot): \Omega \to \mathbf{R}$ is *continuous in mean square* if $E[X_t^2] < \infty$ for all t and

$$\lim_{s \to t} E[(X_s - X_t)^2] = 0 \qquad \text{for all } t \geq 0 .$$

a) Prove that Brownian motion B_t is continuous in mean square.

Since

$$E[B_t^2] = t < \infty, \quad \forall t \in \mathbb{R}$$

and

$$\lim_{s \to t} E[(B_s - B_t)^2] = \lim_{s \to t} (s - t) = 0, \quad \forall t \geq 0$$

Hence, B_t is continuous in mean square.

b) Let $f: \mathbf{R} \to \mathbf{R}$ be a Lipschitz continuous function, i.e. there exists $C < \infty$ such that

$$|f(x) - f(y)| \leq C|x - y| \qquad \text{for all } x, y \in \mathbf{R} .$$

Prove that

$$Y_t := f(B_t)$$

is continuous in mean square.

By the fact that f is Lipschitz,

$$|Y_t - Y_s| \leq C|t-s|$$

i.e.,

$$E|Y_t - Y_s|^2 \leq E[C^2|t-s|^2]$$

Dominated Convergence

Taking the limit as $t \to s$,

$$\lim_{t \to s} E|Y_t - Y_s|^2 \leq \lim_{t \to s} E[C^2|t-s|^2] = E[\lim_{t \to s}(C^2|t-s|^2)] = 0$$

Therefore,

$$\lim_{t \to s} E[|Y_t - Y_s|^2] = 0, \quad \forall s \geq 0$$

Moreover,

$$\mathbb{E}[Y_t^2] = \mathbb{E}[f^2(B_T)] < \infty$$

since f is continuous.

c) Let X_t be a stochastic process which is continuous in mean square and assume that $X_t \in \mathcal{V}(S,T)$, $T < \infty$. Show that

$$\int_S^T X_t dB_t = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega) \qquad \text{(limit in } L^2(P))$$

where

$$\phi_n(t,\omega) = \sum_j X_{t_j^{(n)}}(\omega)\mathcal{X}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t) \,, \qquad T < \infty \,.$$

(Hint: Consider

$$E\Big[\int_S^T (X_t - \phi_n(t))^2 dt\Big] = E\Big[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt\Big]) \,.$$

Consider

$$\mathbb{E}\left[\int_S^T (X_t - \varphi_n(t))^2 dt\right] = \mathbb{E}\left[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt\right]$$

$$= \sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \mathbb{E}[(X_t - X_{t_j^{(n)}})^2] dt$$

$$\leq (T-S) \sup_{[t_j^{(n)}, t_{j+1}^{(n)}]} \mathbb{E}[(X_t - X_{t_j^{(n)}})^2]$$

Taking the limit as $n \to \infty$,

$$(T-S) \sup_{[t_j^{(n)}, t_{j+1}^{(n)}]} \mathbb{E}[(X_t - X_{t_j^{(n)}})^2] \longrightarrow 0$$

$\blacksquare$