4.11. Use Itô's formula (for example in the form of Exercise 4.3) to prove that the following stochastic processes are $\{\mathcal{F}_t\}$-martingales:

a) $X_t = e^{\frac{1}{2}t}\cos B_t \qquad (B_t \in \mathbf{R})$

b) $X_t = e^{\frac{1}{2}t}\sin B_t \qquad (B_t \in \mathbf{R})$

c) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t) \qquad (B_t \in \mathbf{R})$.

a) Let $X_t = g(t, B_t)$ where $g(t,x) = e^{1/2t}\cos x$. Then

$$\frac{\partial g}{\partial t} = \frac{1}{2}e^{1/2t}\cos x; \quad \frac{\partial g}{\partial x} = -e^{1/2t}\sin x; \quad \frac{\partial^2 g}{\partial x^2} = -e^{1/2t}\cos x$$

By Itô's formula,

$$dX_t = \frac{1}{2}e^{1/2t}\cos B_t\,dt - e^{1/2t}\sin B_t\,dB_t - \frac{1}{2}e^{1/2t}\cos B_t\,dt$$

Hence,

$$X_t = -\int_0^t e^{\frac{1}{2}s}\sin B_s\,dB_s$$

Therefore, X_t is an $\mathcal{F}_t$-martingale, since it is an Itô's Integral.

b) Let $X_t = g(t, B_t)$, where $g(t,x) = e^{1/2t}\sin x$. By Itô's formula,

$$dX_t = \frac{1}{2}e^{1/2t}\sin B_t\,dt + e^{1/2t}\cos B_t\,dB_t - \frac{1}{2}e^{1/2t}\sin B_t\,dt$$

Thus,

$$X_t = \int_0^t e^{1/2s}\cos B_s\,dB_s \quad \text{is an } \mathcal{F}_t\text{-martingale}$$

c) Let $g(t,x) = (x+t)e^{-x-\frac{1}{2}t}$. Then, $X_t = g(t, B_t)$ and

- $\frac{\partial g}{\partial t} = e^{-x-\frac{1}{2}t} - \frac{(x+t)}{2} e^{-x-\frac{1}{2}t} = \frac{1}{2} e^{-x-\frac{1}{2}t}(2-x-t)$

- $\frac{\partial g}{\partial x} = e^{-x-\frac{1}{2}t} - (x+t)e^{-x-\frac{1}{2}t} = e^{-x-\frac{1}{2}t}(1-x-t)$

- $\frac{\partial^2 g}{\partial x^2} = -e^{-x-\frac{1}{2}t} - e^{-x-\frac{1}{2}t} + (x+t)e^{-x-\frac{1}{2}t} = e^{-x-\frac{1}{2}t}(x+t-2)$

By Itô's formula,

$$dX_t = \frac{1}{2} e^{-B_t-\frac{1}{2}t}(2-B_t-t)dt + e^{-B_t-\frac{1}{2}t}(1-B_t-t)dB_t$$
$$+ \frac{1}{2} e^{-B_t-\frac{1}{2}t}(B_t+t-2)dt$$

Hence,

$$X_t = \int_0^t e^{-B_s-\frac{1}{2}s}(1-B_s-s)\, dB_s$$

is an $\mathcal{F}_t$-martingale. $\square$