4.8. a) Let B_t denote n-dimensional Brownian motion and let $f: \mathbf{R}^n \to \mathbf{R}$ be C^2. Use Itô's formula to prove that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \frac{1}{2} \int_0^t \Delta f(B_s) ds ,$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

By Itô's formula,

$$df(B_t) = \frac{\partial f}{\partial t}(t, B_t)\, dt + \sum_i \frac{\partial f}{\partial x_i}(t, B_t)\, dB_{t_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, B_t)\, dB_{t_i} dB_{t_j}$$

Given that f does not depend on t,

- $\frac{\partial f}{\partial t}(t, B_t) = 0$
- $\sum_i \frac{\partial f}{\partial x_i}(t, B_t) = \nabla f(B_t)$
- $dB_{t_i} dB_{t_j} = \delta_{ij} dt \Rightarrow \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, B_t)\, dB_{t_i} dB_{t_j} = \sum_i \frac{\partial^2 f}{\partial x_i^2}(t, B_t)\, dt$
- $\int_0^t df(B_s) = \int_0^t \frac{df}{dB_s} dB_s = f(B_s)\Big|_0^t = f(B_t) - f(B_0)$

We have that

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \frac{1}{2} \int_0^t \Delta f(B_s) ds$$

$\square$

b) Assume that $g : \mathbf{R} \to \mathbf{R}$ is C^1 everywhere and C^2 outside finitely many points $z_1, \ldots, z_N$ with $|g''(x)| \leq M$ for $x \notin \{z_1, \ldots, z_N\}$. Let B_t be 1-dimensional Brownian motion. Prove that the 1-dimensional version of a) still holds, i.e.

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds \; .$$

(Hint: Choose $f_k \in C^2(\mathbf{R})$ s.t. $f_k \to g$ uniformly, $f_k' \to g'$ uniformly and $|f_k''| \leq M$, $f_k'' \to g''$ outside $z_1, \ldots, z_N$. Apply a) to f_k and let $k \to \infty$).

Let $f_k \in C^2(\mathbb{R})$ such that $f_k \to g$ uniformly, $f_k' \to g'$ uniformly and $|f_k''| \leq M$ and $f_k'' \to g''$ outside $z_1, \ldots, z_N$.
Applying a) to f_k,

$$f_k(B_t) = f_k(B_0) + \int_0^t f_k'(B_s)\,dB_s + \frac{1}{2}\int_0^t f_k''(B_s)\,ds$$

as $k \to \infty$, we have

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)\,dB_s + \frac{1}{2}\int_0^t g''(B_s)\,ds$$

$\square$