3.9. Suppose $f \in \mathcal{V}(0,T)$ and that $t \to f(t,\omega)$ is continuous for a.a. ω. Then we have shown that

$$\int_0^T f(t,\omega)dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j,\omega)\Delta B_j \qquad \text{in } L^2(P)\,.$$

Similarly we define the *Stratonovich integral* of f by

$$\int_0^T f(t,\omega)\circ dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j^*,\omega)\Delta B_j\,, \quad \text{where } t_j^* = \tfrac{1}{2}(t_j + t_{j+1})\,,$$

whenever the limit exists in $L^2(P)$. In general these integrals are different. For example, compute

$$\int_0^T B_t \circ dB_t$$

and compare with Example 3.1.9.

$$\int_0^T B_t \circ dB_t = \lim_{\Delta t_j \to 0} \sum_j B_{j^*} \Delta B_j, \qquad t_j^* = j^* = \frac{1}{2}(t_j + t_{j+1})$$

Now

$$\begin{aligned}
\Delta(B_j^2) = B_{j+1}^2 - B_j^2 &= \left[\left((B_{j+1} - B_{j^*}) + B_{j^*}\right)^2 - \left((B_j - B_{j^*}) + B_{j^*}\right)^2\right] \\
&= (B_{j+1} - B_{j^*})^2 + 2B_{j^*}(B_{j+1} - B_{j^*}) + B_{j^*}^2 \\
&\quad - \left[(B_j - B_{j^*})^2 + 2B_{j^*}(B_j - B_{j^*}) + B_{j^*}^2\right] \\
&= (B_{j+1} - B_{j^*})^2 - (B_j - B_{j^*})^2 + 2B_{j^*}(B_{j+1} - B_j)
\end{aligned}$$

Hence,

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (B_{j+1} - B_{j^*})^2 - \sum_j (B_j - B_{j^*})^2 + 2\sum_j B_{j^*}(B_{j+1} - B_j)$$

I.e.,

$$\sum_j B_{j^*} \Delta B_j = \frac{1}{2}B_t^2 + \frac{1}{2}\sum_j (B_j - B_{j^*})^2 - \frac{1}{2}\sum_j (B_{j+1} - B_{j^*})^2$$

Since

$$\sum_j (B_j - B_{j^*})^2 \longrightarrow \frac{t}{2} \quad \text{and} \quad \sum_j (B_{j+1} - B_{j^*})^2 \longrightarrow \frac{t}{2}$$

in $L^2(P)$ as $\Delta t_j \to 0$, we have that

$$\int_0^T B_t \circ dB_t = \frac{1}{2}B_t^2 + \frac{t}{4} - \frac{t}{4} = \frac{1}{2}B_t^2$$