4.15. Let $x > 0$ be a constant and define

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3 ; \qquad t \geq 0 .$$

Show that

$$dX_t = \tfrac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t ; \qquad X_0 = x .$$

Let $g(t,y) = \left(x^{1/3} + \frac{1}{3}y\right)^3$ and compute

- $\frac{\partial g}{\partial t} = 0$

- $\frac{\partial g}{\partial y} = 3\left(x^{1/3} + \frac{1}{3}y\right)^2 \cdot \frac{1}{3} = \left(x^{1/3} + \frac{1}{3}y\right)^{3 \cdot \frac{2}{3}}$

- $\frac{\partial^2 g}{\partial y^2} = 2\left(x^{1/3} + \frac{1}{3}y\right) \cdot \frac{1}{3} = \frac{2}{3}\left(x^{1/3} + \frac{1}{3}y\right)^{3 \cdot \frac{1}{3}}$

By Itô's formula,

$$dX_t = \left(x^{1/3} + \frac{1}{3}y\right)^{3 \cdot \frac{2}{3}} dB_t + \frac{1}{2}\frac{2}{3}\left(x^{1/3} + \frac{1}{3}y\right)^{3 \cdot \frac{1}{3}} dt$$

$$= X_t^{2/3} dB_t + \frac{1}{3} X_t^{1/3} dt$$