4.3. Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \ .$$

Let $g(t,x,y) = xy$. By Itô's formula,

$$dX_tY_t = \frac{\partial g}{\partial t}(t, X_t, Y_t)\,dt + \frac{\partial g}{\partial x}(t, X_t, Y_t)\,dX_t + \frac{\partial g}{\partial y}(t, X_t, Y_t)\,dY_t$$

$$+ \frac{1}{2}\left[\frac{\partial^2 g}{\partial x \partial y}(t, X_t, Y_t)\,dX_t dY_t + \frac{\partial^2 g}{\partial x^2}(t, X_t, Y_t)(dX_t)^2 + \frac{\partial^2 g}{\partial y^2}(t, X_t, Y_t)(dY_t)^2 + \frac{\partial^2 g}{\partial y \partial x}(t, X_t, Y_t)\,dX_t dY_t\right]$$

$$= 0 + Y_t dX_t + X_t dY_t + \frac{1}{2}\left(dX_t dY_t + dX_t dY_t\right)$$

I.e.,

$$dX_tY_t = Y_t dX_t + X_t dY_t + dX_t dY_t$$

Rearranging

$$X_t dY_t = dX_tY_t - Y_t dX_t - dX_t\, dY_t$$

In the integral form,

$$\int_0^t X_s dY_s = \int_0^t dX_sY_s - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s$$

Hence,

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s$$