7.8. Let $\{\mathcal{N}_t\}$ be a right-continuous family of σ-algebras of subsets of Ω, containing all sets of measure zero.

a) Let τ_1, τ_2 be stopping times (w.r.t. $\mathcal{N}_t$). Prove that $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ are stopping times.

$$\bullet \{\tau_1 \wedge \tau_2 \leq t\} = \{\omega : \tau_1 \wedge \tau_2(\omega) \leq t\}$$

$$= \{\omega : \tau_1(\omega) \leq t \text{ or } \tau_2(\omega) \leq t\}$$

$$= \{\omega : \tau_1(\omega) \leq t\} \cup \{\omega : \tau_2(\omega) \leq t\}$$

$$= \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{N}_t$$

$$\bullet \{\tau_1 \vee \tau_2 \leq t\} = \{\omega : \tau_1(\omega) \leq t \text{ and } \tau_2(\omega) \leq t\}$$

$$= \{\omega : \tau_1(\omega) \leq t\} \cap \{\omega : \tau_2(\omega) \leq t\}$$

$$= \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{N}_t$$

b) If $\{\tau_n\}$ is a decreasing family of stopping times prove that $\tau := \lim_n \tau_n$ is a stopping time.

We need to show that $\{\tau \leq t\} \in \mathcal{N}_t$. In fact, since $\{\tau_n\}$ is decreasing,

$$\{\tau \leq t\} = \bigcap_n \{\tau_n \leq t\} = \bigcap_n \{\tau_n > t\}^c = \left(\bigcup_n \{\tau_n > t\}\right)^c \in \mathcal{N}_t$$

Since $\{\tau_n \leq t\} \in \mathcal{N}_t$ implies that $\{\tau_n \leq t\}^c = \{\tau_n > t\} \in \mathcal{N}_t$.

c) If X_t is an Itô diffusion in $\mathbf{R}^n$ and $F \subset \mathbf{R}^n$ is closed, prove that τ_F is a stopping time w.r.t. $\mathcal{M}_t$. (Hint: Consider open sets decreasing to F).

Let $\{F_n\}$ be a family of open sets decreasing to F. Now we define $\{\tau_{F_n}\}$ as the family of stopping times given by

$$\{\tau_{F_n}\} = \inf\{t > 0 : X_t \notin F_n\}$$

Since each F_n is open, by the Example 7.2.2, each τ_{F_n} is a stopping time w.r.t. $\mathcal{M}_t$. By the previous item, its limit τ_F is a stopping time w.r.t. $\mathcal{M}_t$.