5.16. The technique used in Exercise 5.6 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t\,, \qquad X_0 = x \tag{5.3.11}$$

where $f\colon \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and $c\colon \mathbf{R}\to\mathbf{R}$ are given continuous (deterministic) functions. Proceed as follows:

a) Define the 'integrating factor'

$$F_t = F_t(\omega) = \exp\Big(-\int_0^t c(s)dB_s + \tfrac{1}{2}\int_0^t c^2(s)ds\Big)\,. \tag{5.3.12}$$

Show that (5.3.11) can be written

$$d(F_t X_t) = F_t \cdot f(t, X_t)dt\,. \tag{5.3.13}$$

We start by applying Itô's formula to F_t.

If $g(t, X_t) = \exp\left(-\int_0^t c(s)dX_t + \frac{1}{2}\int_0^t c^2(s)ds\right)$, $F_t = g(t, B_t)$,

- $\frac{\partial g}{\partial t} = \frac{1}{2}c^2(t)\cdot g(t, X_t)$
- $\frac{\partial^2 g}{\partial x^2} = c^2(t)\, g(t, X_t)$
- $\frac{\partial g}{\partial x} = -c(t)\cdot g(t, X_t)$

Thus,

$$dF_t = \frac{1}{2}c^2(t)\cdot g(t, B_t)\,dt - c(t)\cdot g(t, B_t)\,dB_t + \frac{1}{2}c^2(t)\cdot g(t, B_t)\,dt$$

$$= c^2(t)\,g(t, B_t)\,dt - c(t)\cdot g(t, B_t)\,dB_t \tag{1}$$

Using Integration by Parts (ex. 4.3),

$$d(F_t X_t) = F_t\,dX_t + X_t\,dF_t + dF_t\,dX_t \tag{2}$$

Computing $dF_t dX_t$,

$$dF_t\,dX_t = \left(c^2(t)\,g(t,B_t)\,dt - c(t)\cdot g(t,B_t)\,dB_t\right)\left(f(t,X_t)\,dt + c(t)X_t\,dB_t\right)$$

$$= -c^2(t)\,g(t,B_t)\,X_t\,dt \qquad (3)$$

By (2),

$$d(F_tX_t) = g(t,B_t)\left(f(t,X_t)\,dt + \cancel{c(t)X_t\,dB_t}\right)$$

$$+ X_t\left(\cancel{c^2(t)\,g(t,B_t)\,dt} - \cancel{c(t)\cdot g(t,B_t)\,dB_t}\right)$$

$$- \cancel{c^2(t)\,g(t,B_t)\,X_t\,dt}$$

$$= F_t\,f(t,X_t)\,dt \qquad (4)$$

b) Now define

$$Y_t(\omega) = F_t(\omega)X_t(\omega) \qquad (5.3.14)$$

so that

$$X_t = F_t^{-1}Y_t\ . \qquad (5.3.15)$$

Deduce that equation (5.3.13) gets the form

$$\frac{dY_t(\omega)}{dt} = F_t(\omega)\cdot f(t, F_t^{-1}(\omega)Y_t(\omega))\ ; \qquad Y_0 = x\ . \qquad (5.3.16)$$

Note that this is just a *deterministic* differential equation in the function $t \to Y_t(\omega)$, for each $\omega \in \Omega$. *We can therefore solve (5.3.16) with ω as a parameter to find $Y_t(\omega)$ and then obtain $X_t(\omega)$ from (5.3.15).*

Notice that

$$dY_t = d(F_tX_t) = F_t\,f(t,X_t)\,dt$$

Thus,

$$\frac{dY_t}{dt} = F_t\,f(t, F_t^{-1}Y_t) \qquad (5)$$

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t \,; \qquad X_0 = x > 0 \qquad (5.3.17)$$

where α is constant.

Define

$$F_t = \exp\left(-\int_0^t \alpha \, dB_s + \frac{1}{2}\int_0^t \alpha^2 \, ds\right) = \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right)$$

By (5), if $Y_t = F_t X_t$,

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t) = F_t \cdot \frac{1}{F_t^{-1} Y_t} = \left[\exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right)\right]^2 Y_t^{-1}$$

$$= \frac{1}{Y_t}\exp\left(\alpha^2 t - 2\alpha B_t\right)$$

Solving by separable equation, we can write $Y_t \, dY_t - F_t^2 \, dt = 0$.

Let

$$G(Y_t) = \int Y_t \, dY_t = \frac{Y_t^2}{2} + c$$

$$F(t) = \int F_t^2 \, dt = \int e^{-2\alpha B_t} \cdot e^{\alpha^2 t} \, dt = \frac{e^{\alpha^2 t - 2\alpha B_t}}{\alpha^2} + c$$

Thus,

$$G(Y_t) - F(x) = \frac{Y_t^2}{2} - \int e^{\alpha^2 s - 2\alpha B_s} \, ds = C$$

and

$$Y_t^2 = 2C + 2\int_0^t e^{\alpha^2 s - 2\alpha B_s} \, ds$$

i.e.,

$$Y_t = \sqrt{Y_0^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha B_s}\, ds}$$

Since $X_t = F_t^{-1} Y_t$,

$$X_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t} \sqrt{x^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha B_s}\, ds}$$

d) Apply the method to study the solutions of the stochastic differential equation

$$dX_t = X_t^{\gamma} dt + \alpha X_t dB_t\ ; \qquad X_0 = x > 0 \tag{5.3.18}$$

where α and γ are constants.
For what values of γ do we get explosion?

We'll use F_t as defined in the previous item.
By (5), if $Y_t = F_t X_t$,

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t) = F_t (F_t^{-1} Y_t)^{\gamma} = F_t^{1-\gamma} Y_t^{\gamma}$$

Solving by separable equation, we can write $Y_t^{-\gamma} dY_t - F_t^{1-\gamma} dt = 0$.
Let

$$G(Y_t) = \int Y_t^{-\gamma} dY_t = \frac{Y_t^{-\gamma+1}}{-\gamma+1} + c$$

$$F(t) = \int \exp\left(-\alpha B_t + \frac{1}{2}\alpha^2 t\right)^{1-\gamma} dt$$

$$= \int \exp\left(-(1-\gamma)\alpha B_t + (1-\gamma)\frac{\alpha^2}{2} t\right) dt$$

Therefore,

$$G(Y_t) - F(t) = \frac{Y_t^{-\gamma+1}}{-\gamma+1} - \int \exp\left(-(1-\gamma)\alpha B_t + (1-\gamma)\frac{\alpha^2}{2}t\right) dt = C$$

and,

$$Y_t^{1-\gamma} = (-\gamma+1)C + (1-\gamma)\int_0^t \exp\left(-(1-\gamma)\alpha B_s + (1-\gamma)\frac{\alpha^2}{2}s\right) ds$$

i.e.,

$$Y_t = \left(Y_0^{1-\gamma} + (1-\gamma)\int_0^t \exp\left(-(1-\gamma)\alpha B_s + (1-\gamma)\frac{\alpha^2}{2}s\right) ds\right)^{\frac{1}{1-\gamma}}$$

Since $X_t = F_t^{-1} Y_t$,

$$X_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}\left(x^{1-\gamma} + (1-\gamma)\int_0^t \exp\left(-(1-\gamma)\alpha B_s + (1-\gamma)\frac{\alpha^2}{2}s\right) ds\right)^{\frac{1}{1-\gamma}}$$