4.5. Let $B_t \in \mathbf{R}$, $B_0 = 0$. Define

$$\beta_k(t) = E[B_t^k]\,; \qquad k = 0, 1, 2, \ldots; \ t \geq 0\,.$$

Use Itô's formula to prove that

$$\beta_k(t) = \tfrac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds\,; \qquad k \geq 2\,.$$

Deduce that

$$E[B_t^4] = 3t^2 \qquad \text{(see (2.2.14))}$$

and find

$$E[B_t^6]\,.$$

Notice that:

$$K=0: \quad \beta_0(t) = \mathbb{E}[B_t^0] = 1$$

$$K=1: \quad \beta_1(t) = \mathbb{E}[B_t]$$

Let $g_K(t,x) = x^K$ Then, if

$$\alpha_K(t) = g_K(t, B_t), \quad \beta_K(t) = \mathbb{E}[\alpha_K(t)]$$

By Itô's formula,

$$d\alpha_K = K B_t^{K-1} dB_t + \frac{1}{2} K(K-1) B_t^{K-2} dt$$

Hence,

$$\alpha_K = \int_0^t K B_s^{K-1} dB_s + \frac{1}{2}\int_0^t K(K-1) B_s^{K-2} ds$$

and

$$\beta_K = \mathbb{E}\left[\int_0^t K B_s^{K-1} dB_s + \frac{1}{2}\int_0^t K(K-1) B_s^{K-2} ds\right]$$

$$= K\,\underbrace{\mathbb{E}\left[\int_0^t B_s^{K-1} dB_s\right]}_{0} + \frac{1}{2}K(K-1)\,\mathbb{E}\left[\int_0^t B_s^{K-2} ds\right]$$

$$= \frac{1}{2}K(K-1)\int_0^t \mathbb{E}[B_s^{K-2}]\,ds = \frac{1}{2}K(K-1)\int_0^t \beta_{K-2}\,ds$$

With that

$$\mathbb{E}[B_t^4] = 6\int_0^t \mathbb{E}[B_s^2]\,ds = 6\int_0^t s\,ds = 6\frac{t^2}{2} = 3t^2$$

and

$$\mathbb{E}[B_t^6] = 15\int_0^t \mathbb{E}[B_s^4]\,ds = 15\int_0^t 3s^2\,ds$$

$$= 15 \cdot 3\frac{t^3}{3} = 15t^3$$