5.4. Solve the following stochastic differential equations:

(i) $$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$$

Notice that if $X_1 = X_{1_0} + t + B_1(t)$, then $dX_1(t) = dt + dB_1(t)$

Similarly, $dX_2(t) = X_1 dB_2(t) = (X_1(0) + t + B_1)\, dB_2$. Thus

$$X_2(t) = X_2(0) + X_1(0) B_2 + \int_0^t s\, dB_2 + \int_0^t B_1\, dB_2$$

(ii) $dX_t = X_t dt + dB_t$
(Hint: Multiply both sides with "the integrating factor" e^{-t} and compare with $d(e^{-t}X_t)$)

Multiplying by e^{-t}:

$$e^{-t} dX_t = e^{-t} X_t dt + e^{-t} dB_t$$

On the other hand,

$$d(e^{-t} X_t) = -e^{-t} X_t dt + e^{-t} dX_t$$

Thus,

$$d(e^{-t} X_t) = -e^{-t} X_t dt + e^{-t} X_t dt + e^{-t} dB_t = e^{-t} dB_t$$

I.e.,

$$e^{-t} X_t = X_0 + \int_0^t e^{-s} dB_s \iff \boxed{X_t = e^{t} X_0 + \int_0^t e^{(t-s)} dB_s}$$

(iii) $dX_t = -X_t dt + e^{-t} dB_t.$

Consider $X_t = e^{-t} B_t$. Then

$$dX_t = -e^{-t} B_t \, dt + e^{-t} dB_t = -X_t \, dt + e^{-t} dB_t$$

Thus, taking $X_0 = B_0 = 0$,

$$X_t = e^{-t} B_t, \quad X_0 = 0$$

solves the SDE.