5.2. A natural candidate for what we could call *Brownian motion on the ellipse*

$$\left\{(x,y); \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\} \qquad \text{where } a>0, b>0$$

is the process $X_t = (X_1(t), X_2(t))$ defined by

$$X_1(t) = a\cos B_t\ , \quad X_2(t) = b\sin B_t$$

where B_t is 1-dimensional Brownian motion. Show that X_t is a solution of the stochastic differential equation

$$dX_t = -\tfrac{1}{2}X_t dt + MX_t dB_t$$

where $M = \begin{bmatrix} 0 & -\frac{a}{b} \\ \frac{b}{a} & 0 \end{bmatrix}$.

Applying Itô's formula,

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} -a\sin B_t \\ b\cos B_t \end{bmatrix} dB_t + \frac{1}{2}\begin{bmatrix} -a\cos B_t \\ -b\sin B_t \end{bmatrix} dt$$

$$= -\frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 & -a/b \\ b/a & 0 \end{bmatrix} \cdot \begin{bmatrix} a\cos B_t \\ b\sin B_t \end{bmatrix} dB_t$$

$$= -\frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 & -a/b \\ b/a & 0 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dB_t$$

Thus,

$$dX_t = -\frac{1}{2} X_t\, dt + M X_t\, dB_t$$