5.3. Let $(B_1, \ldots, B_n)$ be Brownian motion in $\mathbf{R}^n$, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation

$$dX_t = rX_t dt + X_t\Big(\sum_{k=1}^{n} \alpha_k dB_k(t)\Big); \qquad X_0 > 0 .$$

(This is a model for exponential growth with several independent white noise sources in the relative growth rate).

Let

$$X_t = X_0 \exp\left(\left(r - \frac{1}{2}\sum_{k=1}^{n} a_k^2\right)t + \sum_{k=1}^{n} a_k B_k(t)\right)$$

Then by Itô's formula,

$$dX_t = \left(r - \frac{1}{2}\sum_{k=1}^{n} a_k^2\right) X_t dt + \sum_{k=1}^{n} a_k X_t dB_k(t)$$

$$+ \frac{1}{2}\sum_{k=1}^{n} a_k^2 X_t dt$$

Hence,

$$dX_t = rX_t dt + X_t \sum_{k=1}^{n} a_k dB_k(t)$$