5.6. Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha Y_t dB_t$$

where r, α are real constants, $B_t \in \mathbf{R}$.
(Hint: Multiply the equation by the 'integrating factor'

$$F_t = \exp\left(-\alpha B_t + \tfrac{1}{2}\alpha^2 t \right) .)$$

By the exercise 4.3.

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \, .$$

Then

$$d(F_t Y_t) = F_t dY_t + Y_t dF_t + dF_t dY_t \qquad (1)$$

Since

$$dF_t = \frac{1}{2}\alpha^2 F_t dt - \alpha F_t dB_t + \frac{1}{2}\alpha^2 F_t dt$$

$$= F_t \left(\alpha^2 dt - \alpha\, dB_t \right) \qquad (2)$$

then

$$dF_t dY_t = F_t \left(\alpha^2 dt - \alpha\, dB_t \right)\left(r\,dt + \alpha Y_t dB_t \right)$$

$$= F_t \left(-\alpha^2 Y_t\, dt \right) \qquad (3)$$

Now we can rewrite (1) as

$$d(F_t Y_t) = F_t\left(r\,dt + \alpha Y_t dB_t \right) + Y_t F_t \left(\alpha^2 dt - \alpha\, dB_t \right) + F_t \left(-\alpha^2 Y_t\, dt \right)$$

$$= r F_t dt + \alpha F_t Y_t dB_t + \alpha^2 F_t Y_t dt - \alpha F_t Y_t dB_t - \alpha^2 F_t Y_t dt$$

$$= r F_t dt$$

Therefore,

$$F_t Y_t = Y_0 + \int_0^t r F_s \, ds$$

$$\therefore \quad Y_t = F_t^{-1} \left[Y_0 + \int_0^t r F_s \, ds \right]$$

$$= \exp\left(\alpha B_t - 1/2\, \alpha^2 t\right) \left[Y_0 + r \int_0^t \exp\left(-\alpha B_t + 1/2\, \alpha^2 t\right) ds \right]$$