4.2. Use Itô's formula to prove that

$$\int_0^t B_s^2 dB_s = \tfrac{1}{3}B_t^3 - \int_0^t B_s ds \, .$$

Recall that

$$df(X_t) = \frac{df(X_t)}{dt}dt + \frac{df}{dx}(X_t)\,dX_t + \frac{1}{2}\frac{d^2f}{dx^2}(X_t)(dB_t)^2$$

Then, let $g(t,x) = x^3$ we know that

$$\frac{dg}{dt}(t,B_t) = 0 \qquad (dB_t)^2 = dt$$

$$\frac{dg}{dx}(t,B_t) = 3B_t^2 \qquad \frac{d^2g}{dx^2} = 6B_t$$

Therefore,

$$dB_t^3 = 3B_t^2 dB_t + \frac{1}{2}6B_t dt = 3B_t^2 dB_t + 3B_t dt$$

I.e.,

$$\frac{1}{3}dB_t^3 = B_t^2 dB_t + B_t dt \iff B_t^2 dB_t = \frac{1}{3}dB_t^3 - B_t dt$$

In the integral form,

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$$