4.6. a) For c, α constants, $B_t \in \mathbf{R}$ define

$$X_t = e^{ct+\alpha B_t}\ .$$

Prove that

$$dX_t = (c + \tfrac{1}{2}\alpha^2)X_t dt + \alpha X_t dB_t\ .$$

Let $g(t,x) = e^{ct+\alpha x} = e^{ct}e^{\alpha x}$. Then,

- $\dfrac{\partial g}{\partial t} = c e^{ct+\alpha x}$ • $\dfrac{\partial g}{\partial x} = \alpha e^{ct+\alpha x}$ • $\dfrac{\partial^2 g}{\partial x^2} = \alpha^2 e^{ct+\alpha x}$

By Itô's formula, since $X_t = g(t, B_t)$,

$$dX_t = c e^{ct+\alpha B_t} dt + \alpha e^{ct+\alpha B_t} dB_t + \alpha^2 e^{ct+\alpha x} dt$$

$$\therefore \quad dX_t = \left(c + \alpha^2 \right) X_t dt + \alpha X_t dB_t$$

b) For $c, \alpha_1, \ldots, \alpha_n$ constants, $B_t = (B_1(t), \ldots, B_n(t)) \in \mathbf{R}^n$ define

$$X_t = \exp\left(ct + \sum_{j=1}^{n} \alpha_j B_j(t)\right)\ .$$

Prove that

$$dX_t = \left(c + \tfrac{1}{2}\sum_{j=1}^{n} \alpha_j^2\right) X_t dt + X_t\left(\sum_{j=1}^{n} \alpha_j dB_j\right)\ .$$

By the multidimensional Itô's formula, let $g(t,x) = e^{ct+\alpha x}$

$$X_t = g(t, B_t) = e^{ct+\alpha B_t} = e^{ct+(\alpha_1 B_1 + \ldots + \alpha_n B_n)}$$

and

$$dX_t = c X_t dt + \alpha_1 X_t dB_1 + \ldots + \alpha_n X_t dB_n + \frac{1}{2}\alpha_1^2 X_t dt$$

$$+ \dots + \frac{1}{2} \alpha_n^2 X_t \, dt$$

Hence,

$$dX_t = \left(c + \frac{1}{2} \sum_{j=1}^{n} \alpha_j^2 \right) X_t \, dt + X_t \left(\sum_{j=1}^{n} \alpha_j \, dB_j \right)$$