7.14. (Doob's h-transform)

Let B_t be n-dimensional Brownian motion, $D \subset \mathbf{R}^n$ a bounded open set and $h > 0$ a harmonic function on D (i.e. $\Delta h = 0$ in D). Let X_t be the solution of the stochastic differential equation

$$dX_t = \nabla(\ln h)(X_t)dt + dB_t$$

More precisely, choose an increasing sequence $\{D_k\}$ of open subsets of D such that $\overline{D}_k \subset D$ and $\bigcup\limits_{k=1}^{\infty} D_k = D$. Then for each k the equation above can be solved (strongly) for $t < \tau_{D_k}$. This gives in a natural way a solution for $t < \tau := \lim\limits_{k\to\infty} \tau_{D_k}$.

a) Show that the generator A of X_t satisfies

$$Af = \frac{\Delta(hf)}{2h} \qquad \text{for } f \in C_0^2(D) .$$

In particular, if $f = \frac{1}{h}$ then $\mathcal{A}f = 0$.

$$Af(x) = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j} (\sigma\sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

$$= \sum_i \frac{\partial(\ln h)}{\partial x_i}(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial x_i^2}(x)$$

$$= \sum_i \frac{1}{h}\frac{\partial h}{\partial x_i}(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \Delta f(x)$$

$$= \frac{1}{h} \nabla h(x) \nabla f(x) + \frac{1}{2}\Delta f(x)$$

$$= \frac{2\nabla h(x)\nabla f(x) + h\Delta f(x)}{2h} = \frac{\Delta(hf)}{2h}$$

Obs: $\Delta(hf) = \cancel{f\Delta h} + 2\nabla h \nabla f + h\Delta f$

b) Use a) to show that if there exists $x_0 \in \partial D$ such that

$$\lim_{x \to y \in \partial D} h(x) = \begin{cases} 0 & \text{if } y \neq x_0 \\ \infty & \text{if } y = x_0 \end{cases}$$

(i.e. h is a *kernel function*), then

$$\lim_{t \to \tau} X_t = x_0 \text{ a.s.}$$

(Hint: Consider $E^x[f(X_T)]$ for suitable stopping times T and with $f = \frac{1}{h}$)

In other words, we have imposed a drift on B_t which causes the process to exit from D at the point x_0 only. This can also be formulated as follows: X_t is obtained by *conditioning* B_t *to exit from* D *at* x_0. See Doob (1984).

DRAFT

Define the stopping time

Let $T_k = \inf\{t > 0 : X_t \notin D_k\}$ and notice that as $k \to \infty$ we have that

$$T_k \longrightarrow T = \inf\{t > 0 : X_t \notin D\} = \tau$$

More than that, by hypothesis, $\lim_{k \to \infty} T_k < \infty$.

Apply Dynkin's formula with $f = 1/h$

$$E^x[f(X_T)] = f(x) + E^x\left[\int_0^T Af(s)\,ds\right] = f(x)$$

Open the expected value:

$$f(x) = \mathbb{P}[X_t \in D]\, f(x) + (1 - \mathbb{P}[X_t \in D])\, f(y)$$

$$\mathbb{P}[X_t \in D] = \frac{f(x) - f(y)}{f(x) - f(y)} \xrightarrow[x \to y]{} 1 \quad \text{since } f(y) = 1/h(y)$$