Monotonicity of Option Prices with Respect to Volatility

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Abstract

This paper explores how European call option prices behave when the volatility is modelled as an adapted process allowed to vary within fixed bounds. We show that the call price is a convex function of the stock price, and the main result is that the price of a European call is a monotonically increasing function of volatility. Since the Black-Scholes-Merton model assumes constant volatility, our result provides essential insight into the option price. Although the results are not new, the proofs are presented in a simpler and more accessible way. A numerical simulation is presented to illustrate the result and compare it with the Black-Scholes-Merton and Cox-Ross-Rubinstein models.

4.1 Introduction

The Black-Scholes-Merton model assumes that the price at time t of an asset S is the solution of the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t,$$

where $\mu > 0$ is called the **drift**, σ is the **volatility** of the stock, and B_t is the Brownian motion.

In plain English, the model presumes a constant drift μ and volatility σ. Telling figures indeed, but what do they mean to the asset manager? Notably, when computing the real-life volatility to solve the equation, we discover that the volatility is not constant.

Consequently, to make the model more realistic, we consider volatility that fluctuates between two bounds σ_1 and σ_2. The immediate question is how the call price obtained under this model is related to the Black-Scholes-Merton price under the two fixed volatilities. Considering a European call option, we will show that the price is an increasing function of volatility, in the sense that the process under higher volatility yields a higher expected payoff in discounted terms. To put it another way, increasing the volatility increases the price of the call. This will be made precise in Section 4.3.

One of our results is that the call price is a convex function of the underlying stock price (Lemma 4.7). This fact was proved in a seminal paper by Merton [Mer73], and here we present a different proof. The convexity was explored by Jagannathan [Jag84], who clarified how the volatility of the stock affects the call price. This work was further expanded by Bergman, Grundy, and Wiener [BGW96], who considered what are the effects of changes in the interest rate and volatility on the prices of call options, showing that the Black-Scholes values bound the price at the bounding levels, which is also the main theorem of the present work. Our approach is based on [LL11, Chapter 4, Problem 5], which gives the structure followed here.

Although this is not a new result, the proofs and simulations were made by us. We present simpler proofs when compared to [Mer73; Jag84; BGW96]. Our approach uses direct stochastic calculus arguments with elementary properties of convex functions. This avoids reliance on general equilibrium considerations or more advanced knowledge of portfolio theory, resulting in self-contained and more accessible proofs.

The relevance of the monotonicity of option prices with respect to volatility, as [BGW96] states, is that using it, 'one can then place bounds on the stock position necessary to hedge a given option position using only knowledge of the bounds on the underlying asset's volatility.' These results provide more information to the asset manager, who can use them to find bounds on the option price and then appropriately hedge it. To further study this topic, we could investigate whether monotonicity can reduce the computational cost of trading operations using the boundary values.

The text is organised as follows. We first list the necessary results, and then we give a precise statement of the problem and prove some auxiliary results. With the tools ready, we prove our main result and then present a numerical simulation of the model.

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4.2 Preliminary concepts

Before heading on, we list some necessary results for our work. We start by presenting the Black-Scholes-Merton model.

Suppose that we have one risky asset S_t and a riskless asset S_t^0 such that

$$dS_t^0 = rS_t^0\, dt,$$

where $r \geqslant 0$ is the instantaneous interest rate.

Setting $S_0^0 = 1$, we have $S_t^0 = e^{rt}$. Assume that the following stochastic differential equation determines the behaviour of the stock price

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t, \tag{4.1}$$

where $\mu > 0$ is the **drift**, and σ is the **volatility** of the stock.

Lemma 4.1. *The solution to (4.1) is*

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right). \tag{4.2}$$

Moreover,

$$\mathbf{E}[S_t] = S_0 e^{\mu t} \quad \textit{for } t \geqslant 0.$$

Proof. See, e.g., [LL11, Section 3.4.3]. □

Notice that the law of S_t is lognormal and that the hypotheses for this model are the same as for the Brownian motion.

Lemma 4.2. *Let X and Y be two random variables with values in $(E, \mathcal{E})$ and $(F, \mathcal{F})$ respectively. Suppose that X is $\mathcal{B}$-measurable and that Y is independent of $\mathcal{B}$. Then, for any non-negative (or bounded) Borel function Ψ on $(E \times F, \mathcal{E} \otimes \mathcal{F})$, the function ψ defined by*

$$\psi(x) = \mathbf{E}[\Psi(x, Y)], \qquad x \in E$$

is a Borel function on $(E, \mathcal{E})$. And we have

$$\mathbf{E}[\Psi(X, Y) \mid \mathcal{B}] = \psi(X) \ \ a.s.$$

Proof. See, e.g., [LL11, Proposition A.2.5., p. 240]. □

An **option** is a financial derivative that gives the holder the right, but not the obligation, to buy or sell a certain amount of an underlying financial asset (e.g., a stock), by a certain date, for a certain price. This price is called the **strike price**, and the **maturity** is the future date by which the option expires. A **European** option can only be exercised at this time, while American options can be exercised anytime before maturity. The option to buy is called a **call** and the option to sell is a **put**. A European call option on the underlying asset S with maturity T and strike price K can be characterized by its **payoff** $h = (S_T - K)_+ = \max\{S_T - K, 0\}$.

Proposition 4.3. *The option value V_t can be expressed as $V_t = F(t, S_t)$ in which*

$$F(t, x) = x\Phi(d_1) - Ke^{-r\theta}\Phi(d_2)$$

for a call and

$$F(t, x) = Ke^{-r\theta}\Phi(-d_2) - x\Phi(-d_1)$$

for a put, where $\Phi(x)$, d_1 and d_2 are given by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du$$

and

$$d_1 = \frac{\ln(x/K) + (r + \sigma^2/2)\theta}{\sigma\sqrt{\theta}},$$
$$d_2 = d_1 - \sigma\sqrt{\theta} = \frac{\ln(x/K) + (r - \sigma^2/2)\theta}{\sigma\sqrt{\theta}}.$$

Here, K is the strike price and $\theta = T - t$.

Proof. See, e.g., [LL11, Section 4.3.2]. □

Theorem 4.4 (Girsanov)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbf{P})$ be a filtered probability space and (B_t) be an $(\mathcal{F}_t)$-standard Brownian motion. Also let $(\theta_t)_{0 \leqslant t \leqslant T}$ be an adapted process satisfying*

$$\int_0^T \theta_s^2\, ds < \infty \ \ a.s.$$

and such that the process $(L_t)_{0 \leqslant t \leqslant T}$ given by

$$L_t = \exp\left(-\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds\right)$$

is a martingale.

Then, under the probability $\mathbf{P}^L$ with density L_T with respect to $\mathbf{P}$, the process (W_t) defined by

$$W_t = B_t + \int_0^t \theta_s\, ds$$

is an $(\mathcal{F}_t)$-standard Brownian motion.

Proof. See, e.g., [Shr04, Theorem 5.2.3]. □

Let us use $\mathbf{P}^*$ to denote the probability measure for which W_t is a standard Brownian motion and $\mathbf{E}^*$ to denote the expected value under $\mathbf{P}^*$.

Before stating the last results we will need, we recall some key concepts from finance. A **(trading) strategy** is a stochastic process $\varphi = (\varphi_t) = (H_t^0, H_t)$ adapted to the natural filtration $(\mathcal{F}_t)$ of Brownian motion, where H_t^0 and H_t are the quantities of the riskless and risky asset at time t, respectively. The **value of the portfolio** at time t is given by $V_t(\varphi) = H_t^0 S_t^0 + H_t S_t$.

A trading strategy is called self-financing if it readjusts positions without bringing or consuming wealth. More precisely, a **self-financing strategy** is a pair of $(\mathcal{F}_t)$-adapted processes (H_t^0) and (H_t) satisfying

$$\int_0^T |H_t^0|\,\mathrm{d}t + \int_0^T H_t^2\,\mathrm{d}t < +\infty \quad \text{a.s.}$$

and

$$H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0\,\mathrm{d}S_u^0 + \int_0^t H_u\,\mathrm{d}S_u \quad \text{a.s.}, \quad \forall t \in [0,T].$$

A strategy $\varphi = (H_t^0, H_t)$ is **admissible** if it is self-financing, the discounted value

$$\tilde{V}_t(\varphi) = H_t^0 + H_t \tilde{S}_t$$

of the portfolio is non-negative for all $0 \leqslant t \leqslant T$, and $\sup_{t\in[0,T]} \tilde{V}_t \in L^2(\mathbf{P}^*)$. An option is said to be **replicable** if its payoff at maturity equals the final value of an admissible strategy.

Theorem 4.5. *Any option defined by a non-negative, $\mathcal{F}_T$-measurable random variable h in $L^2(\mathbf{P}^*)$ is replicable (in the Black-Scholes model).*

The value at time t of any replicating portfolio is given by

$$V_t = \mathbf{E}^*[e^{-r(T-t)}h \mid \mathcal{F}_t].$$

Proof. See, e.g., [LL11, Theorem 4.3.2]. □

Hence, the option value at t can be naturally defined by $\mathbf{E}^*[e^{-r(T-t)}h \mid \mathcal{F}_t]$.

Theorem 4.6 (Discounted Feynman-Kac)**.** *Consider the stochastic differential equation*

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t.$$

Let $h(y)$ be a Borel-measurable function and r be a constant. Fix $T > 0$ and let $t \in [0,T]$. Define

$$g(t,x) = \mathbf{E}^{t,x}[e^{-r(T-t)}h(X_T)],$$

which we suppose to satisfy $\mathbf{E}^{t,x}[|h(X_T)|] < \infty$ for all t and x.

Then $g(t,x)$ satisfies the partial differential equation

$$\frac{\partial g}{\partial t}(t,x) + \mu(t,x)\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 g}{\partial x^2}(t,x) = r g(t,x)$$

and the terminal condition

$$g(T,x) = h(x), \quad \forall x.$$

Proof. See, e.g., [Shr04, Theorem 6.4.3]. □

4.3 Analytical solution

Consider a market consisting of a riskless asset with price $S_t^0 = e^{rt}$ at time t and interest rate r, and one risky asset with price S_t at time t. We assume that the stochastic process (S_t) is the solution to

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma(t) S_t\,\mathrm{d}B_t, \tag{4.3}$$

where $\mu \in \mathbf{R}$ and $(\sigma(t))$ is an adapted process with respect to the natural filtration of (B_t) satisfying $\sigma_1 \leqslant \sigma(t) \leqslant \sigma_2$ for all $t \in [0,T]$, with $0 < \sigma_1 < \sigma_2$.

In this market, consider a European call option with maturity T and strike price K. If $\sigma(t) = \sigma$ for all t, then the price of the call at time t is given by $C(t, S_t)$, where the function $C(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,x) + \dfrac{\sigma^2 x^2}{2}\dfrac{\partial^2 C}{\partial x^2}(t,x) \\ \qquad + rx\dfrac{\partial C}{\partial x}(t,x) - rC(t,x) = 0, \\ \qquad\qquad t \in [0,T),\ x > 0, \\ C(T,x) = \max\{x - K, 0\}. \end{cases} \tag{4.4}$$

Denote by C_i the function C corresponding to the case $\sigma = \sigma_i$, for $i = 1,2$. We'll show that the price of the call at time 0 in the model with varying volatility belongs to the interval $[C_1(0,S_0), C_2(0,S_0)]$. To show that, we divide the proof into eight steps as follows. The first one is to show that the call prices are convex as a function of the underlying asset.

Lemma 4.7. *The functions $x \mapsto C_i(t,x)$, for $i = 1,2$, are convex.*

Proof. We aim to show that the gamma, i.e., the second derivative of $C_i(t,x) = F(t,x)$ is positive.

We start by computing the first derivative. By Proposition 4.3,

$$\frac{\partial F}{\partial x} = \Phi(d_1) + x\Phi'(d_1)\frac{\partial d_1}{\partial x} - Ke^{-r\theta}\Phi'(d_2)\frac{\partial d_2}{\partial x}. \tag{4.5}$$

To simplify that identity, remark that

$$\Phi'(d) = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}\,\mathrm{d}x\right)' = \frac{1}{\sqrt{2\pi}}e^{-d^2/2} \tag{4.6}$$

and, since $d_2 = d_1 - \sigma\sqrt{\theta}$,

$$\frac{\partial d_2}{\partial x} = \frac{\partial d_1}{\partial x}. \tag{4.7}$$

Let us evaluate

$$
\begin{aligned}
\Phi'(d_1) &= \Phi'(d_2 + \sigma\sqrt{\theta}) \\
&\overset{(4.6)}{=} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_2+\sigma\sqrt{\theta})^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2}{2} - \frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2}{2}\right) \exp\left(-\frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right) \\
&\overset{(4.6)}{=} \Phi'(d_2) \exp\left(-\frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right).
\end{aligned}
\tag{4.8}
$$

Now notice that

$$
d_2 = \frac{\ln(x/K) + (r - \sigma^2/2)\theta}{\sigma\sqrt{\theta}} = \frac{\ln(xe^{r\theta}/K) - \sigma^2\theta/2}{\sigma\sqrt{\theta}}
$$

is equivalent to

$$
\begin{aligned}
-\ln\left(\frac{xe^{r\theta}}{K}\right) &= -\sigma\sqrt{\theta} d_2 - \frac{\sigma^2\theta}{2} \\
\iff \frac{K}{xe^{r\theta}} &= \exp\left(-\frac{\sigma^2\theta}{2} - d_2\sigma\sqrt{\theta}\right).
\end{aligned}
\tag{4.9}
$$

Using (4.9) in (4.8), we obtain

$$
\Phi'(d_1) = \Phi'(d_2)\frac{K}{xe^{r\theta}} \iff x\Phi'(d_1) = Ke^{-r\theta}\Phi'(d_2).
\tag{4.10}
$$

Now replacing (4.7) and (4.10) in (4.5),

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= \Phi(d_1) + Ke^{-r\theta}\Phi'(d_2)\frac{\partial d_1}{\partial x} - Ke^{-r\theta}\Phi'(d_2)\frac{\partial d_1}{\partial x} \\
&= \Phi(d_1).
\end{aligned}
$$

Deriving the expression above, we have

$$
\frac{\partial^2 F}{\partial x^2} = \Phi'(d_1)\frac{\partial d_1}{\partial x}.
$$

Now

$$
d_1 = \frac{\ln(xe^{r\theta}/K) + \sigma^2\theta/2}{\sigma\sqrt{\theta}} \implies \frac{\partial d_1}{\partial x} = \frac{1}{x\sigma\sqrt{\theta}}.
$$

Hence,

$$
\frac{\partial^2 F}{\partial x^2} = \frac{\Phi'(d_1)}{x\sigma\sqrt{\theta}}.
$$

Since the expression above is positive, the function is convex. □

The second step is to find the solution of (4.3).

Lemma 4.8. *The solution of the stochastic differential equation*

$$
dS_t = \mu S_t\, dt + \sigma(t) S_t\, dB_t
$$

is

$$
S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t \sigma^2(s)\, ds + \int_0^t \sigma(s)\, dB_s\right).
$$

Proof. Let

$$
dS_t = S_t\, dY_t
$$

with

$$
dY_t = \mu\, dt + \sigma_t\, dB_t.
$$

Notice that

$$
Y_t = \int_0^t \mu\, ds + \int_0^t \sigma_s\, dB_s.
$$

Writing $S_t = g(t, Y_t)$, by Itô's formula,

$$
\begin{aligned}
g\, dY_t = dS_t &= \frac{\partial g}{\partial t} dt + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dY_t)^2 + \frac{\partial g}{\partial x} g\, dY_t \\
&= \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\sigma_t^2\right) dt + \frac{\partial g}{\partial x} g\, dY_t.
\end{aligned}
$$

For the last term, we have

$$
\frac{\partial g}{\partial x} = g \implies g = S_0 e^{b(t)+x}.
$$

And for the dt term,

$$
\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\sigma_t^2 = 0.
$$

Computing $b(t)$,

$$
\frac{db}{dt} + \frac{1}{2}\sigma_t^2 = 0 \implies b = -\frac{1}{2}\int_0^t \sigma_s^2\, ds.
$$

Replacing into g, we finish the proof:

$$
\begin{aligned}
S_t &= S_0 \exp\left(-\frac{1}{2}\int_0^t \sigma_s^2\, ds + \int_0^t \mu\, ds + \int_0^t \sigma_s\, dB_s\right) \\
&= S_0 \exp\left(\int_0^t \left(\mu - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s\, dB_s\right).
\end{aligned}
$$

□

Now, we apply Girsanov's Theorem to $\theta = (\mu - r)/\sigma_s$ to obtain an equivalent probability under which the process below is a standard Brownian motion.

Lemma 4.9. *There exists a probability* $\mathbf{P}^*$ *equivalent to* $\mathbf{P}$ *under which the process defined by*

$$
W_t = B_t + \int_0^t \frac{\mu - r}{\sigma_s}\, ds
$$

is a standard Brownian motion.

Proof. Since $\sigma_s > 0$,

$$
\int_0^T \left(\frac{\mu - r}{\sigma_s}\right)^2 ds = (\mu - r)^2 \int_0^T \frac{1}{\sigma_s^2}\, ds < \infty \text{ a.s.}
$$

Let us verify that

$$
L_t = \exp\left(-\int_0^t \frac{\mu - r}{\sigma_s}\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma_s}\right)^2 ds\right)
$$

is a martingale. Let $L_t = e^{Y_t}$ with

$$
Y_t = -\int_0^t \frac{\mu - r}{\sigma_s}\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma_s}\right)^2 ds.
$$

Letting $g(t,x) = e^x$, we have $L_t = g(t, Y_t)$. By Itô's formula,

$$\begin{aligned} \mathrm{d}L_t &= \frac{\partial g}{\partial x}(t,x)\,\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x)(\mathrm{d}Y_t)^2 \\ &= g(t,x)\Big(-\Big(\frac{\mu - r}{\sigma_s}\Big)\mathrm{d}B_s - \frac{1}{2}\Big(\frac{\mu - r}{\sigma_s}\Big)^2\mathrm{d}s\Big) \\ &\qquad + g(t,x)\frac{1}{2}\Big(\frac{\mu - r}{\sigma_s}\Big)^2\mathrm{d}s \\ &= -g(t,x)\Big(\frac{\mu - r}{\sigma_s}\Big)\mathrm{d}B_s. \end{aligned}$$

Hence, L_t is a martingale. By Girsanov's Theorem, there exists a probability $\mathbf{P}^*$, equivalent to $\mathbf{P}$, with density L_T, and such that the process W_t is a standard Brownian motion under $\mathbf{P}^*$. □

Lemma 4.10. *The price of the call at time 0 is given by*

$$C_0 = \mathbf{E}^*[e^{-rT}\max\{S_T - K, 0\}].$$

Proof. Let h be a European call, i.e., $h = \max\{S_T - K, 0\}$. Note that h is a non-negative $\mathcal{F}_T$-measurable random variable in $L^2(\mathbf{P}^*)$. Thus, by Theorem 4.5, we have that the option value at t can be naturally defined as $C_t = \mathbf{E}^*[e^{-r(T-t)}h \mid \mathcal{F}_t]$. Taking $t = 0$, the price of the call is

$$\mathbf{E}^*[e^{-rT}\max\{S_T - K, 0\}].$$ □

Lemma 4.11. *Let $\tilde{S}_t = e^{-rt}S_t$. Then $\mathbf{E}^*[\tilde{S}_t^2] \leqslant S_0^2 e^{\sigma^2 t}$.*

Proof. Using that $\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}B_t$, by Itô's formula we let $g(t,x) = e^{-rt}x$, $\tilde{S}_t = g(t, S_t)$ and obtain

$$\begin{aligned} \mathrm{d}\tilde{S}_t &= -re^{-rt}S_t\,\mathrm{d}t + e^{-rt}\,\mathrm{d}S_t \\ &= -re^{-rt}S_t\,\mathrm{d}t + e^{-rt}(\mu S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}B_t) \\ &= (\mu - r)e^{-rt}S_t\,\mathrm{d}t + \sigma_t e^{-rt}S_t\,\mathrm{d}B_t \\ &= \tilde{S}_t[(\mu - r)\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t]. \end{aligned} \tag{4.11}$$

Using that $W_t = B_t + \int_0^t \frac{(\mu - r)u}{\sigma_u}\,\mathrm{d}u$,

$$\begin{aligned} \mathrm{d}W_t &= \mathrm{d}B_t + \frac{(\mu - r)t}{\sigma_t}\mathrm{d}t \\ &\iff \sigma_t\,\mathrm{d}W_t = \sigma_t\,\mathrm{d}B_t + (\mu - r)\,\mathrm{d}t. \end{aligned} \tag{4.12}$$

Putting (4.12) into (4.11),

$$\mathrm{d}\tilde{S}_t = \tilde{S}_t\sigma_t\,\mathrm{d}W_t.$$

Thus, $\tilde{S}_t$ is a $\mathbf{P}^*$-martingale. By Itô's formula,

$$\tilde{S}_t^2 = S_0^2 + 2\int_0^t S_u\,\mathrm{d}S_u + \int_0^t \mathrm{d}[S_u, S_u].$$

Since $\mathrm{d}[S_t, S_t] = S_t^2\sigma_t^2\,\mathrm{d}t$, we have

$$\mathbf{E}^*[\tilde{S}_t^2] = S_0^2 + \int_0^t \mathbf{E}^*[S_u^2]\sigma_u^2\,\mathrm{d}u.$$

By Gronwall's inequality,

$$\mathbf{E}^*[\tilde{S}_t^2] \leqslant S_0^2 e^{\int_0^t \sigma_u^2\,\mathrm{d}u}.$$

Using that $\sigma_t < \sigma$, the result follows. □

Lemma 4.12. *The process defined by*

$$M_t = \int_0^t e^{-ru}\frac{\partial C_1}{\partial x}(u, S_u)\sigma_u S_u\,\mathrm{d}W_u$$

is a martingale under the probability measure $\mathbf{P}^*$.

Proof. Notice that e^{-ru} is bounded because $u \geqslant 0$ and $r \geqslant 0$, σ_u is bounded by hypothesis, and S_u is bounded by the previous lemma. Now, from Lemma 4.7, we know that $\frac{\partial C_1}{\partial x} = \Phi(d_1)$ is bounded. Thus, M_t is an Itô integral, and M_t is a $\mathbf{P}^*$-martingale.

Here, an alternative way to prove that $\frac{\partial C_1}{\partial x}$ is bounded is presented. We know that $C_1(t, S_t) = \mathbf{E}^*[e^{-r(T-t)}f(S_T) \mid \mathcal{F}_t]$, where $f(x) = \max\{x - K, 0\}$. Using the solution (4.2) and that $B_t = W_t - \frac{\mu - r}{\sigma_1}t$,

$$S_t = S_0 e^{((\mu - \frac{1}{2}\sigma_1^2)t + \sigma_1 B_t)} = S_0 e^{((r - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_t)}.$$

Replacing S_T in $C_1(t, S_t)$, we obtain

$$\begin{aligned} C_1(t, S_t) &= \mathbf{E}^*\big[e^{-r(T-t)}f\big(S_0 e^{((r - \frac{1}{2}\sigma_1^2)T + \sigma_1 W_T)}\big) \mid \mathcal{F}_t\big] \\ &= \mathbf{E}^*\big[e^{-r(T-t)}f\big(S_t e^{r(T-t)}e^{\sigma_1(W_T - W_t)}e^{-(\sigma_1^2/2)(T-t)}\big) \mid \mathcal{F}_t\big]. \end{aligned}$$

Since S_t is $\mathcal{F}_t$-measurable and, under $\mathbf{P}^*$, $W_T - W_t$ is independent of $\mathcal{F}_t$, by Lemma 4.2 we have that $C_1(t, S_t) = F(t, S_t)$, where

$$F(t,x) = \mathbf{E}^*\big[e^{-r(T-t)}f\big(xe^{r(T-t)}e^{\sigma_1(W_T - W_t)}e^{-(\sigma_1^2/2)(T-t)}\big)\big].$$

We know that $(W_T - W_t)$ has a normal distribution with mean zero and variance $T - t$ under the probability $\mathbf{P}^*$. Thus,

$$F(t,x) = e^{-r(T-t)}\int_{\mathbf{R}} f\big(xe^{(r - \sigma_1^2/2)(T-t) + \sigma_1 y\sqrt{T-t}}\big)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y.$$

Notice that we can simplify this expression as follows

$$F(t,x) = e^{-r(T-t)}\int_{\mathbf{R}} f(y)p(x, y, T-t)\,\mathrm{d}y,$$

where $p(x, y, t)$ is the transition density of the process from x to y over a time interval of length t. □

4.4 Main result

With the previous lemmas, we are ready to prove our main result. In words, it states that, at time zero, the price of a European call option under any volatility process bounded between σ_1 and σ_2 lies between the prices obtained from the Black-Scholes-Merton model with constant volatilities $0 < \sigma_1 < \sigma_2$.

Theorem 4.13. *Under the probability measure $\mathbf{P}^*$, the processes $(e^{-rt}C_1(t, S_t))$ and $(e^{-rt}C_2(t, S_t))$ are, respectively, a submartingale and a supermartingale. Furthermore, we have that*

$$C_1(0, S_0) \leqslant C_0 \leqslant C_2(0, S_0).$$

Proof. Our first goal is to prove that

$$\mathbf{E}^*[e^{-rt}C_1(t,S_t)\mid\mathcal{F}_u]\geqslant e^{-ru}C_1(u,S_u),\qquad t>u.$$

Let $g(t,x)=e^{-rt}C_1(t,x)$ and $X_t=g(t,S_t)$. By Itô's formula,

$$\begin{aligned}\mathrm{d}X_t=&\left(-re^{-rt}C_1(t,S_t)+e^{-rt}\frac{\partial C_1}{\partial t}(t,S_t)\right)\mathrm{d}t\\&+e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\,\mathrm{d}S_t\\&+\frac{1}{2}e^{-rt}\frac{\partial^2C_1}{\partial x^2}(t,S_t)(\mathrm{d}S_t)^2.\end{aligned}$$

Replacing $\mathrm{d}S_t=\mu S_t\,\mathrm{d}t+\sigma_tS_t\,\mathrm{d}B_t$ and $(\mathrm{d}S_t)^2=\sigma_t^2S_t^2\,\mathrm{d}t$,

$$\begin{aligned}\mathrm{d}X_t=&\left(-re^{-rt}C_1(t,S_t)+e^{-rt}\frac{\partial C_1}{\partial t}(t,S_t)\right)\mathrm{d}t\\&+e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)(\mu S_t\,\mathrm{d}t+\sigma_tS_t\,\mathrm{d}B_t)\\&+\frac{1}{2}e^{-rt}\frac{\partial^2C_1}{\partial x^2}(t,S_t)\sigma_t^2S_t^2\,\mathrm{d}t.\end{aligned}$$

Organizing yields

$$\begin{aligned}\mathrm{d}X_t=e^{-rt}\Bigg(&-rC_1(t,S_t)+\frac{\partial C_1}{\partial t}(t,S_t)\\&+\mu S_t\frac{\partial C_1}{\partial x}(t,S_t)+\frac{1}{2}\sigma_t^2S_t^2\frac{\partial^2C_1}{\partial x^2}(t,S_t)\Bigg)\mathrm{d}t\\&+e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_tS_t\,\mathrm{d}B_t.\end{aligned}\tag{4.13}$$

From the proof of Lemma 4.11, we know that $\sigma_t\,\mathrm{d}B_t=\sigma_t\,\mathrm{d}W_t-(\mu-r)\,\mathrm{d}t$. Thus,

$$\begin{aligned}&e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)S_t\sigma_t\,\mathrm{d}B_t\\&=e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)S_t(\sigma_t\,\mathrm{d}W_t-(\mu-r)\,\mathrm{d}t)\\&=e^{-rt}\sigma_tS_t\frac{\partial C_1}{\partial x}(t,S_t)\,\mathrm{d}W_t-e^{-rt}S_t(\mu-r)\frac{\partial C_1}{\partial x}(t,S_t)\,\mathrm{d}t.\end{aligned}\tag{4.14}$$

Replacing (4.14) into (4.13) and simplifying

$$\begin{aligned}\mathrm{d}X_t=e^{-rt}\Bigg(&-rC_1(t,S_t)+\frac{\partial C_1}{\partial t}(t,S_t)\\&+rS_t\frac{\partial C_1}{\partial x}(t,S_t)+\frac{1}{2}\sigma_t^2S_t^2\frac{\partial^2C_1}{\partial x^2}(t,S_t)\Bigg)\mathrm{d}t\\&+e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_tS_t\,\mathrm{d}W_t.\end{aligned}$$

Now we write

$$\begin{aligned}\mathrm{d}X_t=e^{-rt}\Bigg(&-rC_1(t,S_t)+\frac{\partial C_1}{\partial t}(t,S_t)\\&+rS_t\frac{\partial C_1}{\partial x}(t,S_t)+\frac{1}{2}\sigma_1^2S_t^2\frac{\partial^2C_1}{\partial x^2}(t,S_t)\Bigg)\mathrm{d}t\\&+\frac{1}{2}e^{-rt}(\sigma_t^2-\sigma_1^2)S_t^2\frac{\partial^2C_1}{\partial x^2}(t,S_t)\,\mathrm{d}t\\&+e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_tS_t\,\mathrm{d}W_t.\end{aligned}$$

By equation (4.4), this simplifies to

$$\begin{aligned}\mathrm{d}X_t=&\frac{1}{2}e^{-rt}(\sigma_t^2-\sigma_1^2)S_t^2\frac{\partial^2C_1}{\partial x^2}(t,S_t)\,\mathrm{d}t\\&+e^{-rt}\frac{\partial C_1}{\partial x}(t,S_t)\sigma_tS_t\,\mathrm{d}W_t.\end{aligned}\tag{4.15}$$

Since $C_1(t,x)$ is convex as function of x (from Lemma 4.7), $\frac{\partial^2C_1}{\partial x^2}(t,S_t)>0$, and using that $\sigma(t)>\sigma_1$, we have $(\sigma_t^2-\sigma_1^2)>0$. Therefore,

$$\frac{1}{2}e^{-rt}(\sigma_t^2-\sigma_1^2)S_t^2\frac{\partial^2C_1}{\partial x^2}(t,S_t)>0.$$

By Lemma 4.12, $\int_0^t e^{-ru}\frac{\partial C_1}{\partial x}(u,S_u)\sigma_uS_u\,\mathrm{d}W_u$ is a $\mathbf{P}^*$-martingale. Thus,

$$\mathbf{E}^*[X_t\mid\mathcal{F}_u]\geqslant X_u.$$

Finally, to show that $C_1(0,S_0)\leqslant C_0$, we use that $C_1(0,S_0)$ is a submartingale. With Lemma 4.10, the expression inside the expectation is exactly C_0.

$$\begin{aligned}e^{-r0}C_1(0,S_0)&=C_1(0,S_0)\\&\leqslant\mathbf{E}^*[e^{-rT}C_1(T,S_T)\mid\mathcal{F}_0]=C_0.\end{aligned}$$

The proof that $(e^{-rt}C_2(t,S_t))$ is a supermartingale and $C_0\leqslant C_2(0,S_0)$ is similar. Define $X_t=e^{-rt}C_2(t,S_t)$. By equation (4.15), we have

$$\begin{aligned}\mathrm{d}X_t=&\frac{1}{2}e^{-rt}(\sigma_t^2-\sigma_2^2)S_t^2\frac{\partial^2C_2}{\partial x^2}(t,S_t)\,\mathrm{d}t\\&+e^{-rt}\frac{\partial C_2}{\partial x}(t,S_t)\sigma_tS_t\,\mathrm{d}W_t.\end{aligned}$$

Since $(\sigma_t^2-\sigma_2^2)<0$, the same argument from the previous proof implies that

$$\mathbf{E}^*[X_t\mid\mathcal{F}_u]\leqslant X_u.$$

Thus, X_t is a supermartingale, and it follows that

$$C_0=\mathbf{E}^*[e^{-rT}C_2(T,S_T)\mid\mathcal{F}_0]\leqslant C_2(0,S_0).\qquad\square$$

Remark 4.14. The inequality $C_1(0,S_0)\leqslant C_0\leqslant C_2(0,S_0)$ makes precise the intuition that higher volatility leads to higher option prices. In particular, the case of fixed volatility $\sigma(t)\equiv\sigma\in[\sigma_1,\sigma_2]$ is included as a special case of our setting. An immediate consequence of the main result is that $C_1(0,S_0)\leqslant C_2(0,S_0)$.

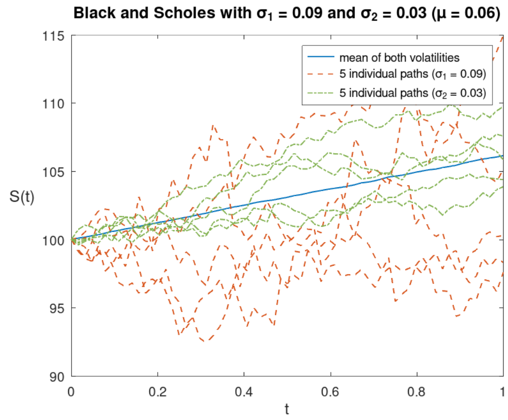
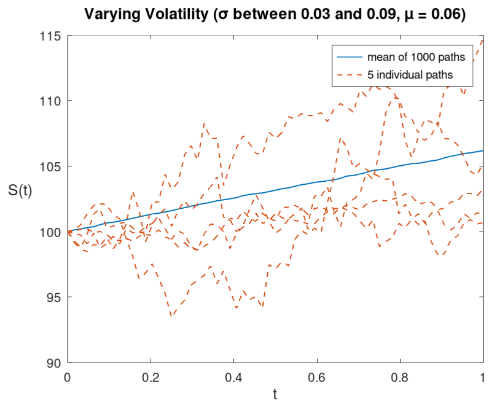
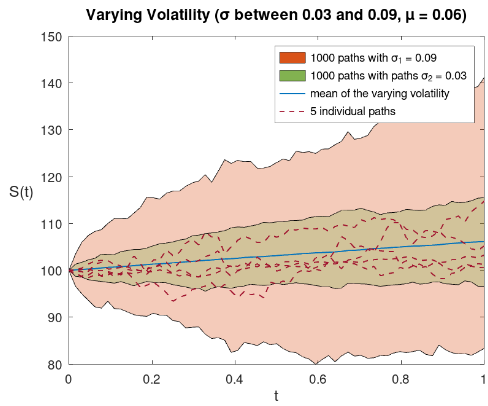


Figure 4.1: Black and Scholes with $\sigma_1 = 0.09$ and $\sigma_2 = 0.03$



(a) Five individual paths and their mean.



(b) Shaded area with simulations under σ_1 and σ_2.

Figure 4.2: Model with varying volatility.

4.5 Numerical simulation

To illustrate the result, in this section we present a numerical simulation of the model, computing the price of a call option under it and comparing it with the standard Black-Scholes-Merton and the Cox-Ross-Rubinstein models. The latter model, also known as the binomial options pricing model, is a binomial lattice model for option pricing that converges to the Black-Scholes-Merton model. Because of its simplicity, it is more elucidative and computationally more efficient. For a primer, see [CRR79].

Here, we compute the call price of an underlying asset with stock price $S = 100$, strike price $K = 100$, interest rate $r = 0.06$, maturity $T = 1$ year, and volatility $\sigma = 0.06$. These simulations were made using MATLAB. Figure 4.1 presents two simulations (the first with volatility $\sigma_1 = 0.09$, and the second with $\sigma_2 = 0.03$) and their average. In Figure 4.2, the volatility σ is an array of uniformly distributed random numbers between σ_1 and σ_2.

Under these models, it is possible to price a call option as presented in Table 4.3. The first two rows give the call price under the Cox-Ross-Rubinstein model with ten steps (see [LL11]) and the standard Black and Scholes, respectively.

In the third and fourth rows, we compute the price in the standard Black and Scholes model with two volatilities: first with $\sigma_1 = 0.09$, and then with $\sigma_2 = 0.03$, as in Figure 4.1.

Lastly, we compute the stock price under a varying volatility model, in which the volatility σ is an array of uniformly distributed random numbers between $\sigma_1 = 0.09$ and $\sigma_2 = 0.03$, as in Figure 4.2. We can see that the price under varying volatility, 6.376718, belongs to the interval $(5.848264, 7.142684)$ as desired. In Figure 4.2, the blue line represents the empirical mean of a thousand simulations of asset prices $S(t)$ when the volatility is uniformly distributed between σ_1 and σ_2.

Table 4.3: Call prices

Method	**Call Price**
Cox-Ross-Rubinstein	6.333048
Black-Scholes (0.06)	6.308527
Black-Scholes (0.09)	7.142684
Black-Scholes (0.03)	5.848264
Varying volatility	6.376718

4.6 Conclusion

We present a more general model than Black-Scholes-Merton in which the volatility is not constant but can vary between two bounds. We've shown that the price of the European call option at time zero in this varying volatility model belongs to the interval $[C_1(0, S_0), C_2(0, S_0)]$ defined by the standard Black-Scholes-Merton prices. More than that, it is an increasing function of volatility. At the end of the work, we presented an example of the result with numerical simulations and a computation of call prices under four different models.

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