

3.13. A stochastic process $X_t(\cdot): \Omega \rightarrow \mathbf{R}$ is *continuous in mean square* if $E[X_t^2] < \infty$ for all t and

$$\lim_{s \rightarrow t} E[(X_s - X_t)^2] = 0 \quad \text{for all } t \geq 0.$$

a) Prove that Brownian motion B_t is continuous in mean square.

Since

$$E[B_t^2] = t < \infty, \quad \forall t \in \mathbf{R}$$

and

$$\lim_{s \rightarrow t} E[(B_s - B_t)^2] = \lim_{s \rightarrow t} (s - t) = 0, \quad \forall t \geq 0$$

Hence, B_t is continuous in mean square.

b) Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a Lipschitz continuous function, i.e. there exists $C < \infty$ such that

$$|f(x) - f(y)| \leq C|x - y| \quad \text{for all } x, y \in \mathbf{R}.$$

Prove that

$$Y_t := f(B_t)$$

is continuous in mean square.

By the fact that f is Lipschitz,
 $|Y_t - Y_s| \leq C|t - s|$

i.e.,

$$E|Y_t - Y_s|^2 \leq E[C^2|t - s|^2]$$

Taking the limit as $t \rightarrow s$,

$$\lim_{t \rightarrow s} E|Y_t - Y_s|^2 \leq \lim_{t \rightarrow s} E[C^2|t - s|^2] = E\left[\lim_{t \rightarrow s} (C^2|t - s|^2)\right] = 0$$

Dominated
Convergence

Therefore,

$$\lim_{t \rightarrow s} E|Y_t - Y_s|^2 = 0, \quad \forall s \geq 0$$

Moreover,

$$\mathbb{E}[Y_T^2] = \mathbb{E}[f^2(B_T)] < \infty$$

since f is continuous.

c) Let X_t be a stochastic process which is continuous in mean square and assume that $X_t \in \mathcal{V}(S, T)$, $T < \infty$. Show that

$$\int_S^T X_t dB_t = \lim_{n \rightarrow \infty} \int_S^T \phi_n(t, \omega) dB_t(\omega) \quad (\text{limit in } L^2(P))$$

where

$$\phi_n(t, \omega) = \sum_j X_{t_j^{(n)}}(\omega) \chi_{[t_j^{(n)}, t_{j+1}^{(n)})}(t), \quad T < \infty.$$

(Hint: Consider

$$E \left[\int_S^T (X_t - \phi_n(t))^2 dt \right] = E \left[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt \right].$$

Consider

$$\mathbb{E} \left[\int_S^T (X_t - \phi_n(t))^2 dt \right] = \mathbb{E} \left[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt \right]$$

Taking the limit as $n \rightarrow \infty$ (i.e. $\Delta t_j \rightarrow 0$), by continuity in mean square

$$\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \lim_{\Delta t_j \rightarrow 0} \mathbb{E} (X_t - X_{t_j^{(n)}})^2 dt = \sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} 0 \cdot dt = 0$$