3.1. Prove directly from the definition of Itô integrals (Definition 3.1.6) that

$$\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds\,.$$

(Hint: Note that

$$\sum_j \Delta(s_jB_j) = \sum_j s_j\Delta B_j + \sum_j B_{j+1}\Delta s_j\,.)$$

Our first step is to define an elementary function and prove that it converges to our function $f(s)=s$ in $L^2(P)$.

Let $\varphi_n = \sum_j s_j \chi_{[t_j,t_{j+1})}(s)$. Then,

$$\mathbb{E}\left[\int_0^t (\varphi_n - s)^2 ds\right] = \mathbb{E}\left[\sum_j \int_{t_j}^{t_{j+1}} (s_j - s)\,ds\right]$$

$$\overset{*}{=} \sum_j \int_{t_j}^{t_{j+1}} (s-t_j)\,ds = \sum_j \frac{1}{2}(t_{j+1}-t_j)^2$$

Hence,

$$\lim_{\Delta t_j \to 0} \sum_j \frac{1}{2}(t_{j+1}-t_j)^2 = 0$$

$*$ Recall that $\mathbb{E}[(B_t - B_s)^2] = t-s$

and

$$\int_0^t s\,dB_s = \lim_{\Delta t_j\to 0} \int_0^t \varphi_n\,dB_s = \lim_{\Delta t_j \to 0} \sum_j s_j \Delta B_j$$

Now, using the fact that

$$\sum_j s_j\Delta B_j = \sum_j \Delta(s_jB_j) - \sum_j B_{j+1}\Delta s_j$$

we have

$$\lim_{\Delta t_j\to 0} \sum_j s_j \Delta B_j = \lim_{\Delta t_j \to 0}\sum_j \Delta(s_jB_j) - \lim_{\Delta t_j\to 0} B_{j+1}\Delta s_j$$

$$= tB_t - \int_0^t B_s\,ds$$

$\square$