4.9. Prove that we may assume that g and its first two derivatives are bounded in the proof of the Itô formula (Theorem 4.1.2) by proceeding as follows: For fixed $t \geq 0$ and $n = 1, 2, \ldots$ choose g_n as in the statement such that $g_n(s,x) = g(s,x)$ for all $s \leq t$ and all $|x| \leq n$. Suppose we have proved that (4.1.9) holds for each g_n. Define the stochastic time

$$\tau_n = \tau_n(\omega) = \inf\{s > 0; |X_s(\omega)| \geq n\}$$

(τ_n is called a *stopping* time (See Chapter 7)) and prove that

$$\left(\int_0^t v\frac{\partial g_n}{\partial x}(s,X_s)\mathcal{X}_{s\leq\tau_n}dB_s := \right)$$

$$\int_0^{t\wedge\tau_n} v\frac{\partial g_n}{\partial x}(s,X_s)dB_s = \int_0^{t\wedge\tau_n} v\frac{\partial g}{\partial x}(s,X_s)dB_s$$

for each n. This gives that

$$g(t\wedge\tau_n, X_{t\wedge\tau_n}) = g(0,X_0) + \int_0^{t\wedge\tau_n}\left(\frac{\partial g}{\partial s} + u\frac{\partial g}{\partial x} + \tfrac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}\right)ds + \int_0^{t\wedge\tau_n} v\frac{\partial g}{\partial x}dB_s$$

and since

$$P[\tau_n > t] \to 1 \qquad \text{as } n\to\infty$$

we can conclude that (4.1.9) holds (a.s.) for g.

$$g(t,X_t) = g(0,X_0) + \int_0^t\left(\frac{\partial g}{\partial s}(s,X_s) + u_s\frac{\partial g}{\partial x}(s,X_s) + \tfrac{1}{2}v_s^2\cdot\frac{\partial^2 g}{\partial x^2}(s,X_s)\right)ds + \int_0^t v_s\cdot\frac{\partial g}{\partial x}(s,X_s)dB_s \quad \text{where } u_s = u(s,\omega),\ v_s = v(s,\omega)\,. \tag{4.1.9}$$

Let $t \geq 0$ be fixed and $g_n(s,x) = g(s,x) \in C^2([0,\infty)\times\mathbb{R})$ for all $s \leq t$ and all $|x| \leq n$.

Suppose that (4.1.9) holds for each g_n. Define

$$\tau_n(\omega) = \inf\{s > 0 : |X_s(\omega)| \geq n\}$$

Claim: $\displaystyle\int_0^{t\wedge\tau_n} v\frac{\partial g_n}{\partial x}(s,X_s)dB_s = \int_0^{t\wedge\tau_n} v\frac{\partial g}{\partial x}(s,X_s)dB_s$

Proof. Since (4.1.9) holds for each g_n,

$$g_n(t,X_t) = g_n(0,X_0) + \int_0^t\left(\frac{\partial g_n}{\partial s}(s,X_s) + u_s\frac{\partial g_n}{\partial x}(s,X_s) + \frac{1}{2}v_s^2\frac{\partial^2 g_n}{\partial x^2}(s,X_s)\right)ds + \int_0^t v_s\frac{\partial g}{\partial x}(s,X_s)dB_s$$

Replacing t by $t\wedge\tau_n$, we have that $s \leq t\wedge\tau_n$ and $|X_t| \leq n$. Hence, $g_n = g$ and the claim holds.

$\boxed{\text{Claim: } \mathbb{P}[\tau_n > t] \longrightarrow 1 \quad \text{as} \quad n \to \infty}$

Proof. As $n \to \infty$, $\inf\{s > 0 : |X_s| \geq \infty\} = \infty = \tau_n$. Then

$$\mathbb{P}[\infty > t] = 1$$

since $t < \infty$.

Conclusion. Hence,

$$g_n(t \wedge \tau_n, X_{t \wedge \tau_n}) = g(t \wedge \tau_n, X_{t \wedge \tau_n}) = g(t, X_t)$$

And (4.1.9.) holds a.s. for g.