7.6. Let $g(x,\omega) = f \circ F(x,t,t+h,\omega)$ be as in the proof of Theorem 7.1.2. Assume that f is continuous.

a) Prove that the map $x \to g(x,\cdot)$ is continuous from $\mathbf{R}^n$ into $L^2(P)$ by using (5.2.9).

Recall that

$$F(x,t,r,\omega) = X_r^{t,x}(\omega) \qquad \text{for } r \geq t\ ,$$

We want to show that

$$\lim_{h \to 0} \mathbb{E}\left[| g(x+h,\cdot) - g(x,\cdot)|^2 \right] = 0$$

Since f is continuous, our task is simplified. What needs to be proved is that

$$\lim_{s \to 0} \mathbb{E}\left[|F(x+s,t,t+h,\omega) - F(x,t,t+h,\omega)|^2 \right] = 0$$

Now remember that

So the function

$$v(t) = E[|X_t - \widehat{X}_t|^2]\ ; \qquad 0 \leq t \leq T$$

satisfies

$$v(t) \leq F + A \int_0^t v(s)ds\ , \tag{5.2.9}$$

where $F = 3E[|Z - \widehat{Z}|^2]$ and $A = 3(1+T)D^2$.

Let

$$\sigma(t) = \mathbb{E}\left[|F(x+s,t,t+h,\omega) - F(x,t,t+h,\omega)|^2 \right]$$

By Gronwall inequality,

$$\sigma(t) \leq F\exp(At)$$

Notice that

$$Z - \hat{Z} = (x+s) - x = s$$

thus

$$\lim_{s \to 0} F = \lim_{s \to 0} 3s^2 = 0$$

Since $\exp(At) < \infty$, it follows that $\lim_{s\to 0} \sigma(t) = 0$, i.e.,

$$\lim_{s \to 0} \mathbb{E}\left[|F(x+s, t, t+h, \omega) - F(x, t, t+h, \omega)|^2\right] = 0$$

$\square$

For simplicity assume that $n = 1$ in the following.

b) Use a) to prove that $(x, \omega) \to g(x, \omega)$ is measurable. (Hint: For each $m = 1, 2, \ldots$ put $\xi_k = \xi_k^{(m)} = k \cdot 2^{-m}$, $k = 1, 2, \ldots$ Then

$$g^{(m)}(x, \cdot) := \sum_k g(\xi_k, \cdot) \cdot \mathcal{X}_{\{\xi_k \leq x < \xi_{k+1}\}}$$

converges to $g(x, \cdot)$ in $L^2(P)$ for each x. Deduce that $g^{(m)} \to g$ in $L^2(dm_R \times dP)$ for all R, where dm_R is Lebesgue measure on $\{|x| \leq R\}$. So a subsequence of $g^{(m)}(x, \omega)$ converges to $g(x, \omega)$ for a.a. (x, ω).)

By the hint and the previous item, $g^m(x, \cdot) \to g(x, \cdot)$ in $L^2(P)$.

Since the function is bounded and composition of continuous and measurable functions is measurable, the result follows.

$\square$