5.10. Let b, σ satisfy (5.2.1), (5.2.2) and let X_t be the unique strong solution of (5.2.3). Show that

$$E[|X_t|^2] \le K_1 \cdot \exp(K_2 t) \qquad \text{for } t \le T \tag{5.3.2}$$

where $K_1 = 3E[|Z|^2] + 6C^2T(T+1)$ and $K_2 = 6(1+T)C^2$.
(Hint: Use the argument in the proof of (5.2.10)).

First write

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s$$

Using that $(x_1 + \ldots + x_n)^2 \le n(x_1^2 + \ldots + x_n^2)$,

$$E\left[|X_t|^2\right] \le 3E\left[|X_0^2| + \left|\int_0^t b_s ds\right|^2 + \left|\int_0^t \sigma_s dB_s\right|^2\right] \qquad (1)$$

By Cauchy-Schwartz,

$$\left|\int_0^t b_s ds\right|^2 \le \left(\int_0^t |b_s|^2 ds\right)\left(\int_0^t ds\right) = t\int_0^t |b_s|^2 ds \qquad (2)$$

With (2) and Itô's Isometry, (1) becomes

$$E[|X_t|^2] \le 3\left[E[|Z|^2] + tE\left[\int_0^t |b_s|^2 ds\right] + E\left[\int_0^t |\sigma_s|^2 ds\right]\right] \qquad (3)$$

Now, by 5.2.1,

$$|b_s| \le C(1+|x|) \quad \text{and} \quad |\sigma_s| \le C(1+|x|)$$

then

$$\int_0^t |b_s|^2 ds \le \int_0^t |C(1+|x|)|^2 ds = t|C(1+|x|)|^2 \le 2t(C^2 + C^2|x|^2)$$

Therefore,

$$\mathbb{E}\left[|X_t|^2\right] \leq 3\left(\mathbb{E}\left[|Z|^2\right] + t\,\mathbb{E}\left[2t\left(C^2 + C^2|X_t|^2\right)\right] + \mathbb{E}\left[2t\left(C^2 + C^2|X_t|^2\right)\right]\right)$$

$$= 3\left(\mathbb{E}\left[|Z|^2\right] + 2t^2C^2 + 2t^2C^2\,\mathbb{E}\left[|X_t|^2\right] + 2tC^2 + 2tC^2\,\mathbb{E}\left[|X_t|^2\right]\right)$$

$$= 3\,\mathbb{E}\left[|Z|^2\right] + 6C^2t(t+1) + 6C^2t(t+1)\,\mathbb{E}\left[|X_t|^2\right]$$

$$= K_1 + K_2\,\mathbb{E}\left[|X_t|^2\right]$$

Finally, by Gronwall inequality,

$$\mathbb{E}\left[|X_t|^2\right] \leq K_1 \exp(K_2 t)$$

∎