7.7. Let B_t be Brownian motion on $\mathbf{R}^n$ starting at $x \in \mathbf{R}^n$ and let $D \subset \mathbf{R}^n$ be an open ball centered at x.

a) Use Exercise 2.15 to prove that the harmonic measure μ_D^x of B_t is rotation invariant (about x) on the sphere ∂D. Conclude that μ_D^x coincides with normalized surface measure σ on ∂D.

b) Let ϕ be a bounded measurable function on a bounded open set $W \subset \mathbf{R}^n$ and define

$$u(x) = E^x[\phi(B_{\tau_W})] \qquad \text{for } x \in W .$$

Prove that u satisfies the classical mean value property:

$$u(x) = \int_{\partial D} u(y) d\sigma(y)$$

for all balls D centered at x with $\overline{D} \subset W$.

a) First we want to show that

Definition 6.2.6 (Harmonic Measure). The **harmonic measure** of X on ∂G, denoted by μ_G^x, is defined as

$$\mu_G^x(F) = Q^x[X_{\tau_G} \in F], \quad \text{for } F \subset \partial G,\ x \in G$$

$$\mu_D^x(UF) = \mu_D^x(F)$$

where U is a rotation (i.e., orthogonal matrix on $\mathbb{R}^{n\times n}$).

By the exercise 2.15,

2.15. Let B_t be n-dimensional Brownian motion starting at 0 and let $U \in \mathbf{R}^{n\times n}$ be a (constant) orthogonal matrix, i.e. $UU^T = I$. Prove that

$$\widetilde{B}_t := UB_t$$

is also a Brownian motion.

it follows that

$$Q^x[B_{\tau_D} \in F] = Q^x[UB_{\tau_D} \in UF]$$
$$= Q^x[\tilde{B}_{\tau_D} \in UF]$$
$$= Q^x[B_{\tau_D} \in UF]$$

Hence, $\mu_D^x(UF) = \mu_D^x(F)$.

To show that $\mu_D^x = \sigma$ on ∂D, notice that

$$\int_{\partial D} f(x)\, d\sigma(x) = \int_{SO(n)} \int_{\partial D} f(gx)\, d\sigma(x)\, d\mu_D^x(g)$$ since μ_D^x is rot. invariant

$$= \int_{\partial D} \int_{SO(n)} f(gx)\, d\mu_D^x(g)\, d\sigma(x)$$ by Fubini's thm.

$$= \int_{\partial D} \int_{\partial D} f(y)\, d\mu_D^x(y)\, d\sigma(x)$$ using that μ_D^x is rot. invariant

$$= \int_{\partial D} f(y)\, d\mu_D^x(y)$$ since σ is a normalized surf. meas.

Hence, $\sigma = \mu_D^x$ on ∂D as desired.

b) We know that

$$\mathbb{E}^x[f(X_{\tau_H})] = \mathbb{E}^x[\mathbb{E}^{X_{\tau_G}}[f(X_{\tau_H})]] = \int_{\partial G} \mathbb{E}^y[f(X_{\tau_H})] Q^x[X_{\tau_G} \in dy]$$

for $x \in G$.

Plus the fact that $\sigma = \mu_D^x$ on ∂D, it follows that

$$u(x) = \mathbb{E}^x[\varphi(B_{\tau_W})] = \int_{\partial D} \mathbb{E}^y[\varphi(B_{\tau_W})]\, \mu_D^x(dy)$$

$$= \int_{\partial D} u(y)\, d\mu_D^x(y) = \int_{\partial D} u(y)\, d\sigma(y)$$

$\square$