3.15. Suppose $f, g \in \mathcal{V}(S,T)$ and that there exist constants C, D such that

$$C + \int_S^T f(t,\omega)dB_t(\omega) = D + \int_S^T g(t,\omega)dB_t(\omega) \qquad \text{for a.a. } \omega \in \Omega\,.$$

Show that

$$C = D$$

and

$$f(t,\omega) = g(t,\omega) \qquad \text{for a.a. } (t,\omega) \in [S,T] \times \Omega\,.$$

We'll apply Itô Isometry. Notice that

$$\mathbb{E}[(C-D)^2] = \mathbb{E}\left[\left(\int_S^T (g(t,\omega) - f(t,\omega))\, dB_t(\omega)\right)^2\right]$$

$$\overset{\text{Itô Isometry}}{=} \mathbb{E}\left[\int_S^T (g(t,\omega) - f(t,\omega))^2\, dt\right]$$

On the other hand,

odd power?

$$C - D = \mathbb{E}[(C-D)] = \mathbb{E}\left[\int_S^T (g(t,\omega) - f(t,\omega))\, dB_t(\omega)\right] = 0$$

Hence, $C = D$. Moreover, $(C-D)^2 = 0$ and $\mathbb{E}[(C-D)^2] = 0$.
Therefore,

$$g(t,\omega) = f(t,\omega)$$