5.11. (The Brownian bridge).

For fixed $a, b \in \mathbf{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t \; ; \qquad 0 \le t < 1 \, , \; Y_0 = a \, . \tag{5.3.3}$$

Verify that

$$Y_t = a(1-t) + bt + (1-t) \int_0^t \frac{dB_s}{1-s} \; ; \qquad 0 \le t < 1 \tag{5.3.4}$$

solves the equation and prove that $\lim_{t \to 1} Y_t = b$ a.s. The process Y_t is called *the Brownian bridge* (from a to b). For other characterizations of Y_t see Rogers and Williams (1987, pp. 86–89).

Let $g(t,x) = a(1-t) + bt + (1-t) \int_0^t \frac{dx}{1-s}$

By Itô's formula,

$$dY_t = \left(-a + b - \int_0^t \frac{dB_s}{1-s} \right) dt + (1-t) \cdot \frac{1}{1-t} dB_t = \left(\frac{b - Y_t}{1-t} \right) dt + dB_t$$

Hence Y_t solves the equation.

To show that

$$\lim_{t \to 1} Y_t = \lim_{t \to 1} \left[a(1-t) + bt + (1-t) \int_0^t \frac{dB_s}{1-s} \right] = b$$

notice that the deterministic part $a(1-t) + bt$ is immediate.

Now, using Itô's isometry,

$$\mathbb{E}\left[\left((1-t) \int_0^t \frac{dB_s}{1-s} \right)^2 \right] = \mathbb{E}\left[(1-t)^2 \int_0^t \frac{ds}{(1-s)^2} \right]$$

$$= (1-t)^2 \int_0^t \frac{ds}{(1-s)^2} = (1-t)^2 \cdot \left[\frac{1}{(1-x)} \right]_0^t = (1-t)^2 \left[\frac{1}{(1-t)} - 1 \right]$$

$$= (1-t) - (1-t)^2 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 1$$

Thus, $\lim_{t \to 1} Y_t = b$ a.s.