Øksendal 3.3

$X_t : \Omega \to \mathbb{R}^n$ stochastic process

$\mathcal{H}_t$ σ-algebra generated by $\{X_s : s \leq t\}$ (filtration of the process)

a) Show that if X_t is a martingale wrt some filtration $\{\mathcal{N}_t\}$, then X_t is also a martingale wrt its own filtration $\{\mathcal{H}_t\}$.

Suppose that X_t is a martingale w.r.t. some filtration $\{\mathcal{N}_t\}$. That means that

1. X_t is $\mathcal{N}_t$-measurable for all t.
2. $\mathbb{E}[|X_t|] < \infty$
3. $\mathbb{E}[X_t | \mathcal{N}_s] = X_s, \quad s \leq t$

What we need to show is that X_t is a martingale w.r.t. its natural filtration generated by X_s, $s = 0, 1, \dots, t$

In fact,

1. X_t is $\mathcal{H}_t$-measurable for all t, since $\mathcal{H}_t$ contains all available information at the time t.
2. $\mathbb{E}[|X_t|] < \infty$ remains.
3. Since $\mathcal{H}_t \subseteq \mathcal{N}_t$,

$$\mathbb{E}[X_t | \mathcal{H}_s] = \mathbb{E}\big[\mathbb{E}[X_t | \mathcal{N}_s] \,|\, \mathcal{H}_s\big] = \mathbb{E}[X_s | \mathcal{H}_s] = X_s$$

for $s \leq t$.

Hence, X_t is a martingale w.r.t. its own filtration. $\square$

b) If X_t is a martingale w.r.t. $\mathcal{H}_t$, then

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \qquad \text{for all } t \geq 0$$

Notice that since X_t is a martingale w.r.t. $\mathcal{H}_t$

$$\mathbb{E}[X_t | \mathcal{H}_0] = X_0$$

law of total expectation

However,

$$\mathbb{E}[X_t] = \mathbb{E}\big[\mathbb{E}[X_t | \mathcal{H}_0]\big] = \mathbb{E}[X_0]$$

Hence,

$$\mathbb{E}[X_t] = \mathbb{E}[X_0]$$

c) Example of stochastic process X_t satisfying
$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \quad \text{for all } t \geq 0$$
and which is not a martingale w.r.t. its own filtration.

Take Y_t with $Y_0 = 0$ and U a uniform ± 1 r.v. independent from Y_t. Then,
$$\mathbb{E}[UY_t] = \frac{1}{2}[Y_t] + \frac{1}{2}[-Y_t] = 0$$

However,
$$\begin{aligned}
\mathbb{E}[UY_t | \mathcal{F}_s] &= \mathbb{E}[U(Y_t + Y_s - Y_s) | \mathcal{F}_s] \\
&= \mathbb{E}[U(Y_t - Y_s) | \mathcal{F}_s] + \mathbb{E}[UY_s | \mathcal{F}_s] \\
&= \mathbb{E}[U | \mathcal{F}_s] \cdot \mathbb{E}[Y_t - Y_s | \mathcal{F}_s] + \mathbb{E}[UY_s | \mathcal{F}_s] \\
&= Y_s \mathbb{E}[U | \mathcal{F}_s] = 0
\end{aligned}$$