3.16. Let $X\colon \Omega \to \mathbf{R}$ be a random variable such that $E[X^2] < \infty$ and let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra. Show that

$$E\big[(E[X|\mathcal{H}])^2\big] \le E[X^2] \, .$$

(See Lemma 6.1.1. See also the Jensen inequality for conditional expectation (Appendix B).)

By Jensen Inequality for Conditional Expectation,

Theorem B.4 (The Jensen inequality).
If $\phi\colon \mathbf{R} \to \mathbf{R}$ is convex and $E[|\phi(X)|] < \infty$ then

$$\phi(E[X|\mathcal{H}]) \le E[\phi(X)|\mathcal{H}] \, .$$

Since $\phi(x) = x^2$ is convex,

$$E[X|\mathcal{H}]^2 \le E[X^2|\mathcal{H}]$$

Using the Law of Total Expectation, i.e.,

$$E[E[X|\mathcal{H}]] = E[X]$$

and taking the expectations

$$E[(E[X|\mathcal{H}])^2] \le E[E[X^2|\mathcal{H}]] = E[X^2]$$