3.6. Prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

Let $\mathcal{F}_t$ be the natural filtration of B_t, i.e., generated by $\{B_t : s \leq t\}$

Clearly, N_t is $\mathcal{F}_t$-measurable, since B_t^3 and $3tB_t$ is.

Now we need to check that $\mathbb{E}[|N_t|] < \infty$. In fact,

$$\mathbb{E}[|N_t|] = \mathbb{E}[|B_t^3 - 3tB_t|] \leq \mathbb{E}[|B_t^3|] + \mathbb{E}[|3tB_t|]$$
$$= \mathbb{E}[|B_t|^3] + 3t\,\mathbb{E}[|B_t|] < \infty$$

Finally,

$$\mathbb{E}[N_s|\mathcal{F}_t] = \mathbb{E}[B_s^3 - 3sB_s|\mathcal{F}_t]$$
$$= \mathbb{E}[B_s^3|\mathcal{F}_t] - 3s\,\mathbb{E}[B_s|\mathcal{F}_t]$$

Notice that

$$(B_s - B_t + B_t)^3 = ([B_s - B_t]^2 + 2(B_s - B_t)B_t + B_t^2)([B_s - B_t] + B_t)$$
$$= [B_s - B_t]^3 + 3[B_s - B_t]^2 B_t + 3(B_s - B_t)B_t^2 + B_t^3$$

Therefore,

$$\mathbb{E}[B_s^3|\mathcal{F}_t] = \mathbb{E}[(B_s - B_t + B_t)^3|\mathcal{F}_t]$$
$$= \mathbb{E}[(B_s - B_t)^3|\mathcal{F}_t] + 3\,\mathbb{E}[(B_s - B_t)^2 B_t|\mathcal{F}_t]$$
$$+ 3\,\mathbb{E}[(B_s - B_t)B_t^2|\mathcal{F}_t] + \mathbb{E}[B_t^3|\mathcal{F}_t]$$
$$= 0 + 3B_t\,\mathbb{E}[(B_s - B_t)^2|\mathcal{F}_t] + 3B_t^2\,\mathbb{E}[(B_s - B_t)|\mathcal{F}_t] + B_t^3$$
$$= 3B_t(s - t) + 0 + B_t^3$$

Thus,

$$\mathbb{E}[B_s^3|\mathcal{F}_t] - 3s\,\mathbb{E}[B_s|\mathcal{F}_t] = B_t^3 - 3(t - s)B_t - 3sB_t$$
$$= B_t^3 - 3tB_t$$

as desired.

$\square$