

4.3. Let X_t, Y_t be Itô processes in \mathbf{R} . Prove that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t.$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s.$$

Let $g(t, x, y) = xy$. By Itô's formula,

$$\begin{aligned} dX_t Y_t &= \frac{\partial g}{\partial t}(t, X_t, Y_t) dt + \frac{\partial g}{\partial x}(t, X_t, Y_t) dX_t + \frac{\partial g}{\partial y}(t, X_t, Y_t) dY_t \\ &\quad + \frac{1}{2} \left[\frac{\partial^2 g}{\partial x \partial y}(t, X_t, Y_t) dX_t dY_t + \frac{\partial^2 g}{\partial x^2}(t, X_t, Y_t) (dX_t)^2 \right. \\ &\quad \left. + \frac{\partial^2 g}{\partial y^2}(t, X_t, Y_t) (dY_t)^2 + \frac{\partial^2 g}{\partial y \partial x}(t, X_t, Y_t) dX_t dY_t \right] \\ &= 0 + Y_t dX_t + X_t dY_t + \frac{1}{2} (dX_t dY_t + dX_t dY_t) \end{aligned}$$

i.e.,

$$dX_t Y_t = Y_t dX_t + X_t dY_t + dX_t dY_t$$

Rearranging

$$X_t dY_t = dX_t Y_t - Y_t dX_t - dX_t dY_t$$

In the integral form,

$$\int_0^+ X_s dY_s = \int_0^+ dX_s Y_s - \int_0^+ Y_s dX_s - \int_0^+ dX_s dY_s$$

Hence,

$$\int_0^+ X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^+ Y_s dX_s - \int_0^+ dX_s dY_s$$