5.8. Solve the (2-dimensional) stochastic differential equation

$$dX_1(t) = X_2(t)dt + \alpha dB_1(t)$$
$$dX_2(t) = -X_1(t)dt + \beta dB_2(t)$$

where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion and α, β are constants.
This is a model for a vibrating string subject to a stochastic force. See Example 5.1.3.

In matrix notation,

$$dX_t = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} X_t dt + \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}}_{C} dB_t; \quad X_t = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad B_t = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Multiplying by $\exp(-At)$,

$$\exp(-At)\, dX_t = \exp(-At)\, A X_t\, dt + \exp(-At)\, C\, dB_t \qquad (1)$$

However,

$$d(\exp(-At)X_t) = -A\exp(-At)\, X_t\, dt + \exp(-At)\, dX_t \qquad (2)$$

By (1) and (2)

$$d(\exp(-At)X_t) = \exp(-At)\, C\, dB_t$$

I.e.,

$$\exp(-At)\, X_t = X_0 + \int_0^t \exp(-As)\, C\, dB_s$$

$$\therefore \quad X_t = \exp(At)\left[X_0 + \int_0^t \exp(-As)\, C\, dB_s \right]$$