4.14. In each of the cases below find the process $f(t,\omega) \in \mathcal{V}[0,T]$ such that (4.3.6) holds, i.e.

$$F(\omega) = E[F] + \int_0^T f(t,\omega)dB_t(\omega) .$$

a) $F(\omega) = B_T(\omega)$

Let $f(t,\omega) = 1$. Then, since $\mathbb{E}[B_t(\omega)] = 0$, we have

$$\int_0^T dB_t(\omega) = B_T(\omega) = F(\omega)$$

b) $F(\omega) = \int_0^T B_t(\omega)dt$

Recall that

$$\int_0^t s\,dB_s = tB_t - \int_0^t B_s ds$$

Note that

$$\mathbb{E}[F(\omega)] = \mathbb{E}\left[\int_0^T B_t dt\right] = \int_0^T \mathbb{E}[B_t]dt = 0$$

Therefore,

$$\int_0^T B_t dt = TB_T - \int_0^T t\,dB_t = T\int_0^T dB_t - \int_0^T t\,dB_t$$

$$= \int_0^T (T-t)dB_t \Rightarrow \underline{f(t,\omega) = T-t}$$

c) $F(\omega) = B_T^2(\omega)$

Recall that

$$\int_0^t B_t dB_t = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

and that $\mathbb{E}[B_t^2] = t$. Hence, taking $f(t,\omega) = B_t(\omega)\cdot 2$,

$$B_T^2(\omega) = T + 2\int_0^T B_t \, dB_t(\omega)$$

d) $F(\omega) = B_T^3(\omega)$

Recall that $\mathbb{E}[B_t^3] = 0$ and

$$\int_0^t B_s^2 \, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

$$= \frac{1}{3} B_t^3 - \int_0^t (t-s) \, dB_s$$

Hence,

$$B_T^3 = 3\left(\int_0^T B_t^2 \, dB_t + \int_0^T (T-t) \, dB_t \right)$$

$$= 3\int_0^T (B_t^2 + T - t) \, dB_t$$

and $f(t,\omega) = 3(B_t^2 + T - t)$

e) $F(\omega) = e^{B_T(\omega)}$

By Itô's formula,

$$d e^{B_T(\omega)} = e^{B_T(\omega)} dB_T + \frac{1}{2} e^{B_T(\omega)} dt = e^{B_T(\omega)} \left(\frac{1}{2} dt + dB_T \right)$$

and

$$d\left(e^{B_T(\omega) - \frac{1}{2}T}\right) = -\frac{1}{2} e^{B_T - \frac{1}{2}T} dt + e^{B_T - \frac{1}{2}T} dB_T + \frac{1}{2} e^{B_T - \frac{1}{2}T} dt$$

$$= e^{B_T - \frac{1}{2}T} dB_T$$

Let $U(t) = e^{B_t - \frac{1}{2}t}$ and notice that we have the following SDE:

$$dU_t = U_t dB_t, \qquad U_0 = 1$$

In the integral form,

$$U_T - U_0 = \int_0^T U_t \, dB_t \iff e^{B_T - \frac{1}{2}T} = 1 + \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$

$$\iff e^{B_T} = e^{1/2T} + \int_0^T e^{B_t - \frac{1}{2}t} e^{1/2T} dB_t$$

$$= e^{1/2T} + \int_0^T e^{B_t + \frac{1}{2}(T-t)} dB_t$$

f) $F(\omega) = \sin B_T(\omega)$

From the exercise 4.11,

$$d\left(e^{1/2t} \sin B_t\right) = e^{1/2t} \cos B_t \, dB_t$$

Hence,

$$e^{1/2t} \sin B_t = \int_0^t e^{1/2s} \cos B_s \, dB_s$$

and,

$$\sin B_t = \int_0^t e^{1/2(s-t)} \cos B_s \, dB_s$$

$$\therefore \quad f(t,\omega) = e^{1/2(t,T)} \cos B_t$$