4.7. Let X_t be an Itô integral

$$dX_t = v(t,\omega)dB_t(\omega) \quad \text{where } v \in \mathbf{R}^n,\ v \in \mathcal{V}(0,T),\ B_t \in \mathbf{R}^n,\ 0 \le t \le T\ .$$

a) Give an example to show that X_t^2 is not in general a martingale.

Take $v=1$. Then $X_t = B_t$ and $X_t^2 = B_t^2$ is not a martingale.

b) Prove that if v is bounded then

$$M_t := X_t^2 - \int_0^t |v_s|^2 ds \quad \text{is a martingale}\ .$$

The process $\langle X,X\rangle_t := \int_0^t |v_s|^2 ds$ is called the *quadratic variation process* of the martingale X_t. For general processes X_t it is defined by

$$\langle X,X\rangle_t = \lim_{\Delta t_k \to 0} \sum_{t_k \le t} |X_{t_{k+1}} - X_{t_k}|^2 \quad \text{(limit in probability)} \quad (4.3.11)$$

where $0 = t_1 < t_2 \cdots < t_n = t$ and $\Delta t_k = t_{k+1} - t_k$. The limit can be shown to exist for continuous square integrable martingales X_t. See e.g. Karatzas and Shreve (1991).

- M_t is $\mathcal{M}_t$-measurable since X_t^2 is and also $\int_0^t |v_s|^2 ds$

- $E[|M_t|] < \infty$ for all t

$$\boxed{\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s\ .}$$

By Integration by Parts (4.3), taking $X_t = Y_t$,

$$X_t^2 = X_0^2 + \int_0^t X_s\, dX_s + \int_0^t X_s\, dX_s + \int_0^t (dX_s)^2$$

$$= X_0^2 + 2\int_0^t X_s\, dX_s + \int_0^t |v_s|^2 ds \qquad \langle X, X\rangle_t$$

$$= X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s + \int_0^t |\sigma_s|^2 \, ds$$

Therefore,

$$M_t = X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s$$

and

$$\mathbb{E}[|M_t|] = \mathbb{E}\left[\left| X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s \right|\right]$$

$$\leq X_0^2 + 2\,\mathbb{E}\left[\int_0^t |X_s \sigma_s| \, dB_s\right]$$

Since σ_s is bounded, there exists C such that $\sigma_s \leq C$. By Itô Isometry,

$$\mathbb{E}\left[\left(\int_0^t |X_s \sigma_s| \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t |X_s \sigma_s|^2 \, ds\right]$$

$$\leq C^2\,\mathbb{E}\left[\int_0^t |X_s|^2 \, ds\right]$$

Using that $dX_t = \sigma(t,\omega)\, dB_t$ and Fubini's theorem

$$C^2\,\mathbb{E}\left[\int_0^t |X_s|^2 \, ds\right] = C^2 \int_0^t \mathbb{E}\left[\left|\int_0^s \sigma_r \, dB_r\right|^2\right] ds$$

$$= C^2 \int_0^t \mathbb{E}\left[\int_0^s |\sigma_r|^2 \, dr\right] ds$$

$$\leq C^4 \int_0^t \mathbb{E}\left[\int_0^s dr\right] ds$$

$$= C^4 \int_0^t s \, ds = \frac{C^4 t^2}{2} < \infty$$

- $\mathbb{E}[M_s \mid \mathcal{M}_t] = M_t, \quad \forall s \geq t$

$$\mathbb{E}\left[X_0^2 + 2\int_0^s X_r \sigma_r \, dB_r \,\middle|\, \mathcal{M}_t\right] = X_0^2 + 2\,\mathbb{E}\left[\int_0^s X_r \sigma_r \, dB_r \,\middle|\, \mathcal{M}_t\right]$$

$$= X_0^2 + 2\int_0^t X_s \sigma_s \, dB_s$$