7.4. Let B_t^x be 1-dimensional Brownian motion starting at $x \in \mathbf{R}^+$. Put

$$\tau = \inf\{t > 0;\ B_t^x = 0\}\ .$$

a) Prove that $\tau < \infty$ a.s. P^x for all $x > 0$. (Hint: See Example 7.4.2, second part).

b) Prove that $E^x[\tau] = \infty$ for all $x > 0$. (Hint: See Example 7.4.2, first part).

a) Let $K > x$ and define the stopping times

$$\tau_0 = \tau \quad \text{and} \quad \tau_K = \inf\{t > 0 : B_t = K\},$$

Also let $T_K = \tau_0 \wedge \tau_K$.

Since the Brownian motion has continuous paths,

$$\lim_{K \to \infty} \tau_K = +\infty$$

and note that

$$\mathbb{E}^x[B_{T_K}] = \mathbb{E}^x[B_0] = x$$

Then,

$$x = \mathbb{E}^x[B_{T_K}] = 0 \cdot \mathbb{P}^x[\tau_0 < \tau_K] + K\,\mathbb{P}^x[\tau_0 \geq \tau_K]$$

Rewriting

$$\mathbb{P}^x[\tau_0 < \tau_K] = 1 - \mathbb{P}^x[\tau_0 \geq \tau_K] = 1 - \frac{x}{K}$$

I.e,

$$\mathbb{P}^x[\tau_0 < \infty] = \lim_{K \to \infty} \mathbb{P}^x[\tau_0 < \tau_K] = 1$$

b) Let $f(x) = x^2$. By Itô's formula,

$$B_t^2 = 0 + 2\int_0^t B_s \, dB_s + \int_0^t ds$$

Thus,

$$\mathbb{E}\left[B_{t\wedge \tau_K}^2\right] = \mathbb{E}\left[t \wedge \tau_K\right]$$

Now notice that

$$\begin{aligned}
\mathbb{E}[\tau_0] &= \lim_{K\to\infty} \mathbb{E}[\tau_K] \\
&= \lim_{K\to\infty} \mathbb{E}[B_{\tau_K}^2] \\
&= \lim_{K\to\infty} b^2 \, \mathbb{P}[\tau_0 \geq \tau_K] \\
&= \lim_{K\to\infty} K^2 \frac{x}{K} = \infty
\end{aligned}$$