3.11. Let W_t be a stochastic process satisfying (i), (ii) and (iii) (below (3.1.2)). Prove that W_t cannot have continuous paths. (Hint: Consider $E[(W_t^{(N)} - W_s^{(N)})^2]$, where

$$W_t^{(N)} = (-N) \vee (N \wedge W_t), \ N = 1, 2, 3, \dots) .$$

Conditions:

(i) $t_1 \neq t_2 \Rightarrow W_{t_1}$ and W_{t_2} are independent.
(ii) $\{W_t\}$ is stationary, i.e. the (joint) distribution of $\{W_{t_1+t}, \dots, W_{t_k+t}\}$ does not depend on t.
(iii) $E[W_t] = 0$ for all t.

Let

$$W_t^{(N)} = \max\{-N, \min\{N, W_t\}\}, \quad N \in \mathbb{Z}^+$$

and suppose, by contradiction, that W_t is continuous.

By the Dominated Convergence Theorem,

Theorem A.10 (Lebesgue Dominated Convergence Theorem). Let $(S, \mathfrak{S}, \mu)$ be a measure space, and let (f_n) be a sequence of integrable functions that converges almost everywhere to a measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and

$$\int f \, d\mu = \lim \int f_n \, d\mu$$

we have

$$\lim_{s \to t} \mathbb{E}[(W_t^{(N)} - W_s^{(N)})^2] = \mathbb{E}[0] = 0$$

Now, applying Dominated Convergence Theorem to $\mathbb{E}[W_t^{(N)}]$, we have

$$\mathbb{E}[W_t^{(N)}] = \mathbb{E}[W_t] = 0$$

Since $W_t^{(N)}$ and $W_s^{(N)}$ are i.i.d., we have that

$$\begin{aligned}
\mathbb{E}[(W_t^{(N)} - W_s^{(N)})^2] &= \mathbb{E}[W_t^{(N)2} - 2W_t^{(N)} W_s^{(N)} + W_s^{(N)2}] \\
&= \mathbb{E}[(W_t^{(N)})^2] - 2\mathbb{E}[W_t^{(N)}]\mathbb{E}[W_s^{(N)}] + \mathbb{E}[(W_s^{(N)})^2] \\
&= \mathbb{E}[(W_t^{(N)})^2] - 2\mathbb{E}[W_t^{(N)}]\mathbb{E}[W_t^{(N)}] + \mathbb{E}[(W_t^{(N)})^2] \\
&= 2\mathbb{E}[(W_t^{(N)})^2] - \underbrace{\cancel{2\mathbb{E}[(W_t^{(N)})]^2}}_{0} = 2\mathrm{Var}[W_t^{(N)}]
\end{aligned}$$

However, $\mathrm{Var}[W_t^{(N)}] = 0$. Hence,

$$W_t^{(N)} = \mathbb{E}[W_t^{(N)}] \quad \text{a.s.}$$???

Suppose that prob. of W_T^N cont. < 1, say $1-\varepsilon$,

By independence,

$$P(W_{t_1}^{(N)}, W_{t_2}^{(N)}, \dots, W_{t_K}^{(N)} \text{ cont.}) = (1-\varepsilon)^K \to 0 \quad \text{as } K \to \infty$$

W_t cont $\Rightarrow$ W_{t_N} cont $\leadsto$ W_t not cont.