3.15. Suppose $f, g \in \mathcal{V}(S,T)$ and that there exist constants C, D such that

$$C + \int_S^T f(t,\omega)dB_t(\omega) = D + \int_S^T g(t,\omega)dB_t(\omega) \qquad \text{for a.a. } \omega \in \Omega\,.$$

Show that

$$C = D$$

and

$$f(t,\omega) = g(t,\omega) \qquad \text{for a.a. } (t,\omega) \in [S,T] \times \Omega\,.$$

We'll apply Itô Isometry. Notice that

$$\mathbb{E}\left[(C-D)^2\right] = \mathbb{E}\left[\left(\int_S^T (g(t,\omega)-f(t,\omega))\,dB_t(\omega)\right)^2\right]$$

Itô Isometry

$$= \mathbb{E}\left[\int_S^T (g(t,\omega)-f(t,\omega))^2\,dt\right]$$

Martingale + B.M. starting at 0

On the other hand,

$$C-D = \mathbb{E}\left[(C-D)\,|\,\mathcal{F}_S\right] = \mathbb{E}\left[\int_S^T (g(t,\omega)-f(t,\omega))\,dB_t(\omega)\,\middle|\,\mathcal{F}_S\right] = 0$$

Hence, $C=D$. Moreover, $(C-D)^2=0$ and $\mathbb{E}[(C-D)^2]=0$. Therefore,

$$g(t,\omega) = f(t,\omega)$$