3.10. If the function f in Exercise 3.9 varies "smoothly" with t then in fact the Itô and Stratonovich integrals of f coincide. More precisely, assume that there exists $K < \infty$ and $\epsilon > 0$ such that

$$E[|f(s,\cdot) - f(t,\cdot)|^2] \le K|s-t|^{1+\epsilon}\,; \qquad 0 \le s\,,\ t \le T\,.$$

Prove that then we have

$$\int_0^T f(t,\omega)dB_t = \lim_{\Delta t_j \to 0} \sum_j f(t'_j,\omega)\Delta B_j \qquad (\text{limit in } L^1(P))$$

for any choice of $t'_j \in [t_j, t_{j+1}]$. In particular,

$$\int_0^T f(t,\omega)dB_t = \int_0^T f(t,\omega)\circ dB_t\,.$$

(Hint: Consider $E\big[|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j|\big]$.)

Consider

$$E\Big[\Big|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j\Big|\Big]$$

$$\le \sum_j E\big[|f(t_j) - f(t'_j)|\cdot|\Delta B_j|\big]$$

$$\le \sum_j \sqrt{E[|f(t_j) - f(t'_j)|]^2 \cdot E[|\Delta B_j|^2]} \qquad \text{CAUCHY-SCHWARZ}$$

$$\le \sum_j \sqrt{K|t_j - t'_j|^{1+\varepsilon}\cdot|t_j - t'_j|} = \sum_j \sqrt{K}\,|t_j - t'_j|^{\frac{1+\varepsilon}{2}}\cdot|t_j - t'_j|^{1/2}$$

$$= \sqrt{K}\sum_j |t_j - t'_j|^{1+\varepsilon/2} \le T\sqrt{K}\max_j |t_j - t'_j|^{\varepsilon/2} \longrightarrow 0$$

as $\Delta t_j \to 0$

Hence,

$$\int_0^T f(t,\omega)dB_t = \lim_{\Delta t_j \to 0}\sum_j f(t_j,\omega)\Delta B_j = \lim_{\Delta t_j \to 0}\sum_j f(t'_j,\omega)\Delta B_j$$